LINEAR TRANSFORMATIONS ON VECTOR SPACES

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PALSave

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This is stupid. A pegasus can fly. An alicorn can fly.
Everyone knows that *unicorns can't fly*.

– anonymous linear algebra student

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Preface

Welcome to Linear Algebra! Don't worry so much about unicorns right now. Let us first focus on the Linear Algebra part. Unicorns will, of course, appear later on naturally.

To All Readers

This text was originally written for the Linear Algebra course offered at Butler University, but we have worked to make it appropriate for use elsewhere or for self-study. The order of the topics differs from many other Linear Algebra texts, but this was what we found worked best to help our students gain a deep understanding of the topic. We encourage you to follow the sections in order. We start with a quick, preliminary discussion of sets and functions, since these are central to the course. Then Chapters 1 and 2 focus on vector spaces. In Chapter 3, we study linear transformations, beginning with the definition and building to the connection with matrices. Finally, Chapters 4 and 5 are devoted to matrix theory.

If you would prefer to start the course with a discussion of matrices, we suggest the first part of Section 3.4, followed by Section 4.1 and all but the last subsection of Section 4.2. This covers the definition of matrices and their connections to solving systems of equations. One could then pick up at the beginning of the text and use the techniques just learned whenever solving a system of equations.

Some of the sections can be viewed as optional. In particular, Sections 1.5, 2.6, 3.6, 4.7, and 4.8 can be omitted without causing problems when covering future sections. After completing Sections 5.1, 5.2, and 5.3, each of Sections 5.4, 5.5, and 5.6 could serve independently as a capstone section with some additional applications in Section 5.7.

The text is written to be read. The tone is conversational (and sometimes a little silly); there is a unicorn theme, so don't be surprised to see pictures of them throughout. The mathematics is written thoroughly, with most results proven in either the section or the Appendix. There are many examples and "explorations" to engage readers as they encounter the material, written specifically with the idea that the reader work through the explorations as they read. We leave it to the instructor's discretion whether to provide solutions or work some of these during class meetings. We've also attempted to make it lighthearted and fun in places because there's no reason it shouldn't be. We hope you enjoy reading it, but mostly, we hope you learn from it.

A Note about Unicorns

While you may have heard other myths about how to summon a unicorn, we're happy to reveal what happened to work for us. To summon a unicorn, you should include one in an example while writing a mathematics textbook. Apparently, unicorns love the study of mathematics so much, they reveal themselves to any textbook authors who appear to be receptive to their input. The ones who approached us initially were Ricky and Bubbles. We decided a nice way to include these muses was to allow them to narrate our side notes. They did get a bit carried away adding side notes of their own. Hopefully, you will not find them too distracting. It turns out it is difficult to *unsummon* a unicorn.

A Message for Students

Here are some goals to keep in mind when using this book:

- **Learn the Content.** Linear Algebra is inarguably one of the most applicable areas in modern mathematics. It is foundational to advanced mathematics courses, but is also widely used in statistics, computer science, physics, chemistry, and a host of other fields. Note here that we have used the word "applicable" rather than "applied." You are not learning an applied version of Linear Algebra. The goal of this text is to give you a strong foundation in this topic so that you can recognize the applications in your own field as you encounter them. Thus, we begin with the conceptual definitions, but we build towards the application side as the book progresses.

- **Improve Independent Learning Skills.** Realistically, this is the overall goal of undergraduate education. Yes, you will learn specific content in courses you take, but there will always be things you still need to learn as you continue on beyond coursework. The main goal of college is to prepare you for life after college, and a large part of that is giving you the tools to tackle challenges and master new concepts outside of a classroom setting. This may be a goal in all your classes, but it is not necessarily always addressed directly. This text helps facilitate this goal by giving you ways to interact as you read. We encourage you to rework examples and also attempt all the explorations as you go through the reading. Uncovering areas in which your conceptual understanding can be refined is a valuable step in the learning process.

- **Transition to Advanced Coursework.** Linear Algebra is a prerequisite for many upper level mathematics courses and also courses in other departments. While some of this is due to its content, part of this prerequisite is also the experience of the course itself. For many students, this content will push you to think about more abstract mathematics topics than you may have in your previous experiences, and it will help you to see connections between different areas of mathematics, particularly algebra and geometry.

This content will be both challenging and time consuming, but it will also be rewarding. We hope you each find joy in the learning of this mathematics.

Best wishes,
Drs. Kaschner and Russell

Contents

0 Functions on Sets

Functions are a ubiquitous tool in mathematics. They've been used by so many fields of thought and over such a vast swath of time, many forms of notation have evolved. Before we proceed, we should probably all get on the same page with functions. The following crash course in function theory is a deep dive both theoretically and notationally. Be sure to take the time to think critically about all of the definitions; this section is the foundation for all that comes later.

0.1 Sets

We will begin with some basic definitions and notation. It's imperative that the notation is understood since that is essentially the language we will be speaking.[1]

Definition 0.1.1 *A **set** is an unordered collection of objects we call **elements**.*

Let's see a few familiar examples of sets.

- The set of integers, $\mathbb{Z}$, is the set of counting numbers, negative counting numbers, and 0. That is,

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

Note that we use brackets, $\{$ and $\}$, to enclose our set, and we list enough elements to see the pattern of what this set contains. We use the ellipsis, "…," at the beginning or end of our list (both in this case) to indicate that this pattern continues. Let us see a seemingly similar example.

1: Well, and English. We will also speak English.

Hey, Ricky, what about telepathy?

Sorry, Bubbles. No telepathy… in this chapter.

- We don't have a standard name for this one, so we will use C to denote this set.

$$C = \{-2, -1, 0, 1, 2\}$$

While this looks similar to $\mathbb{Z}$ above, C is what's called a finite set. This set has only those 5 elements listed. We would read this as "C is the set that contains the elements -2, -1, 0, 1, and 2."

- The set of rational numbers, $\mathbb{Q}$, is the set of well-defined[2] ratios of integers. That is,

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}.$$

  We will also adopt the usual convention that two rational numbers are equal (that is, the *same* rational number) if they represent the same quantity. For example, $2/4 = 1/2$, so $2/4$ and $1/2$ are the same rational number.

2: Ironically, "well-defined" is not defined well here. We just mean you can't divide by zero in your ratio.

Here, we again used brackets, $\{$ and $\}$, to bound our set description, but this time we were not able to list the elements. There are way too many! Instead, we used a general form with variables, $\frac{p}{q}$, with conditions on the variables, p and q. The colon is read as the phrase "such that" and separates the general form from the conditions. In some places (not this book), you will see a bar, $|$, used instead of the colon, but it means the same thing. The symbol, $\in$, is used to indicate when an element is in a set and is read as "is an element of" or "is in." The conditions in the above definition of $\mathbb{Q}$ says p and q are both elements of $\mathbb{Z}$, so they are integers.

- Let's see another example of this notation.

$$D = \left\{ \frac{n}{2} : n \in \mathbb{Z} \right\}$$

  This is read as "D is the set of all elements of the form $\frac{n}{2}$ such that n is an integer."

Note that since 2 is a nonzero integer, every element in this set, D, is also a rational number, so every element of D is in $\mathbb{Q}$.

Definition 0.1.2 *A* ***subset*** *of a set A is a subcollection of the elements of A; that is, B is a subset of A, written as $B \subseteq A$, if and only if every element of B is an element of A.*

From our discussion before, D is a subset of $\mathbb{Q}$, and we would denote this as $D \subseteq \mathbb{Q}$. Sometimes, the notation $\subset$ is used instead when we know that there are elements of the larger set that are not in the subset. This would be appropriate here. For example, $\frac{1}{3} \in \mathbb{Q}$ but $\frac{1}{3} \notin D$,[3] so we could alternatively say $D \subset \mathbb{Q}$.

3: It is common that a slash through a symbol means "not that symbol's meaning."

For example, ☎ clearly means "not a phone."

Definition 0.1.3 *A* ***is equal as a set to*** *B, written as $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.*

Example 0.1.1 ► It may seem obvious that $C = \{-2, -1, 0, 1, 2\}$ is not equal to $\mathbb{Z}$ just from their clearly different definitions. However, there are many (more subtle) situations in which this would not be quite so clear; such situations require a concrete definition. Let's practice using Definition 0.1.3 by verifying that C is not equal to Z. The intuition is that we need to show there is an element that is not common to both sets. Formally, note that $3 \in \mathbb{Z}$, but $3 \notin C$. Thus, $\mathbb{Z} \nsubseteq C$ and $C \neq \mathbb{Z}$.

► Consider these two sets

$$A = \{2n + 1\colon n \in \mathbb{Z}\} \qquad B = \{2m + 3\colon m \in \mathbb{Z}\}.$$

After writing out several elements for each set, one might begin to believe that both A and B are the set of odd integers and, therefore, equal. Let's use the definition of set equality to show that $A = B$. First, let $a \in A$. Then there is some $n \in \mathbb{Z}$ such that

$$\begin{aligned} a &= 2n + 1 \\ &= 2n - 2 + 2 + 1 \\ &= 2(n - 1) + 3. \end{aligned}$$

Since $n - 1 \in \mathbb{Z}$ for any $n \in \mathbb{Z}$, we know $a \in B$. This shows us that $A \subseteq B$. Now, we just need to start with $b \in B$ and argue that $b \in A$. This feels like a good place for an exploration!

Exploration 1 In the example above, we defined sets A and B and that $A \subseteq B$. Show that $B \subseteq A$ to complete the argument that these two sets are actually equal.

Exploration 2 Use Definition 0.1.3 to show $\mathbb{Z} \neq \mathbb{Q}$.

Definition 0.1.4 *Let A and B be sets and $B \subseteq A$. The **set difference of** A **and** B, denoted $A \backslash B$, is the set of elements in A and not in B. Specifically,*

$$A \backslash B = \{a\colon a \in A \text{ and } a \notin B\}.$$

Example 0.1.2 Let $A = \{n \in \mathbb{Z}\colon n < 0\}$, so A is the set of negative integers. Then $\mathbb{Z} \backslash A$ is the set of all integers except the ones in A; that is, the set of integers that *aren't* negative. Using set notation, we have

$$\mathbb{Z} \backslash A = \{n \in \mathbb{Z}\colon n \geq 0\}.$$

Definition 0.1.5 *Let A and B be sets. The **Cartesian product of** A **and** B, denoted $A \times B$, is the set of ordered pairs with the first component from A and the second from B. Specifically,*

$$A \times B = \{(a, b)\colon a \in A \text{ and } b \in B\}.$$

Example 0.1.3 Let's see an example of this. Let $E = \{1, 2\}$ and $F = \{a, b\}$. Then we have

$$E \times F = \{(1, a), (1, b), (2, a), (2, b)\}.$$

In this case, both E and F are finite sets, so $E \times F$ is also finite. Let's replace F here with a set that is not finite, say

$$G = \{1, 4, 7, 10, 13 \dots\}.$$

Then

$$E \times G = \{(1, 1), (1, 4), (1, 7), (1, 10), \dots (2, 1), (2, 4), (2, 7), (2, 10), \dots\}.$$

There is a very familiar example of a Cartesian product that we have not discussed. We call it $\mathbb{R}^2$, and it should have played a large role in many of the courses leading up to this one. First, let $\mathbb{R}$ denote the set of real numbers.[4] Then we can let both sets in the Cartesian product be $\mathbb{R}$ and form $\mathbb{R} \times \mathbb{R}$. In set notation, this is

$$\mathbb{R} \times \mathbb{R} = \{(x, y)\colon x, y \in \mathbb{R}\}.$$

It will be convenient to refer to this as $\mathbb{R}^2$, mainly so we don't have to write as much!

Let's consider $E \times G$ above again for just a bit. Note that $E \subseteq \mathbb{R}$ and $G \subseteq \mathbb{R}$. It follows that $E \times G \subseteq \mathbb{R}^2$. We have a special name for this too!

Definition 0.1.6 *A **relation from** A **to** B, r, is a subset of $A \times B$; that is, $r \subseteq A \times B$.*

Let's take a step back now and explore why a subset might be termed a "relation." Suppose we were to ask a set of people what their favorite colors were. For privacy's sake, we will just use each person's first initial in our set of people:

$$P = \{I, S, O, W, R\}.$$

Our set of colors will be

$$C = \{\text{Red, Orange, Yellow, Blue, Green, Purple, Pink}\}$$

Now, we can use each person's response to form a subset of $P \times C$. Then

$$\begin{aligned} r \quad = \quad & \{(I, \text{Red}), (S, \text{Pink}), (O, \text{Red}), (W, \text{Green}), \\ & (R, \text{Yellow}), (R, \text{Blue}), (R, \text{Green}), (R, \text{Purple})\}, \end{aligned}$$

and r is a relation. This word choice seems to fit this scenario very well. We are able to relate people and colors in this natural way and record it mathematically![5]

This idea of relating elements to each other probably seems pretty vague. That's actually a feature of this particular definition. It gives us flexibility so that we can define a relation by whatever goofy condition we want.

4: We will talk about this set quite a bit, so definitely note that notation.

I see what you did there. Very cute.

5: Note that in this example, R really only claimed the first three colors as his favorite and Purple as a secret favorite. There was no easy way to reflect this mathematically, though.

Example 0.1.4 Here are a couple of sets and a relation on them.

$$\begin{aligned} A &= \{1,2,3,4,5,6,7\} \\ B &= \{1,2,3,4,5,6,7\} \\ r &= \{(1,3),(2,4),(3,5),(4,6),(5,7)\} \end{aligned}$$

Let's describe this relation in words. Wait. Before reading the answer, do you see the pattern? The relation r relates elements in A to elements in B that are two greater.

Exploration 3 Now it's your turn! Let $A = \{1,3,5\}$ and $B = \{2,4,6,8\}$. Define a relation between the sets A and B that uses every element of A but not every element of B.

Example 0.1.5 Great! Now let's consider infinite sets. An example of a relation from $\mathbb{Z}$ to $\mathbb{Q}$ would be:

$$r = \left\{\left(z, \frac{z}{2}\right) : z \in \mathbb{Z}\right\}.$$

Let's list out a few of the elements in this relation to see what it looks like.

$$(-2,-1), \left(-1,-\frac{1}{2}\right), (0,0), \left(1,\frac{1}{2}\right), (2,1), \left(3,\frac{3}{2}\right)$$

Note that the only elements of $\mathbb{Q}$ appearing here are actually elements in the subset D from earlier! Not every element in $\mathbb{Q}$ is paired with an element of $\mathbb{Z}$ here. Also note that each element of $\mathbb{Z}$ is paired with a distinct element of $\mathbb{Q}$. These are properties of relations that will be important for us soon.

Exploration 4 Now, define a relation between $\mathbb{Z}$ and $\mathbb{Q}$ yourself. Use set notation like above but also list a few elements of your relation.

Exercises for Section 0.1

0.1.1. All of the following statements are false. Give specific examples illustrating why they are false.

(a) $\mathbb{Q} \subset \mathbb{Z}$

(b) $\mathbb{R} \subset \mathbb{Q}$

(c) If $A \subseteq B$, then it's not possible for $A = B$.

0.1.2. Determine whether $x \in A$.

(a) $x = \dfrac{6}{5}$ and $A = \left\{\dfrac{a+1}{a} \colon a \in \mathbb{Z}\right\}$

(b) $x = \dfrac{5}{3}$ and $A = \left\{\dfrac{a+1}{a} \colon a \in \mathbb{Z}\right\}$

(c) $x = \dfrac{1}{2}$ and $A = \left\{\dfrac{a}{4} \colon a \in \mathbb{Z}\right\}$

(d) $x = \dfrac{1}{6}$ and $A = \left\{\dfrac{a}{2} + \dfrac{b}{3} \colon a, b \in \mathbb{Z}\right\}$

(e) $x = 3$ and $A = \{2a + 1 \colon a \in \mathbb{Z}\}$

(f) $x = 4$ and $A = \{2a + 1 \colon a \in \mathbb{Z}\}$

(g) $x = 4$ and $A = \{2a + 1 \colon a \in \mathbb{R}\}$

0.1.3. Consider the sets below:

$$\begin{aligned} A =& \{a + b \colon a, b \in \mathbb{R}\} \text{ and} \\ B =& \{c - d \colon c, d \in \mathbb{R}\}. \end{aligned}$$

(a) Show $A \subseteq B$ by writing any sum of two real numbers as a difference.

(b) Show $B \subseteq A$ by writing any difference of two real numbers as a sum.

Congratulations! You've just proven that $A = B$!

0.1.4. Are these sets equal? Justify using the definition of set equality.

(a) $A = \{3n + 1 \colon n \in \mathbb{Z}\}$ and $B = \{3m + 4 \colon m \in \mathbb{Z}\}$

(b) $A = \{2n + 1 \colon n \in \mathbb{Z}\}$ and $B = \{2m + 2 \colon m \in \mathbb{Z}\}$

(c) $A = \left\{\dfrac{a}{2} \colon a \in \mathbb{Z}\right\}$ and $\mathbb{Q}$

(d) $A = \left\{\dfrac{a}{2} \colon a \in \mathbb{Q}\right\}$ and $\mathbb{Q}$

(e) $A = \left\{\dfrac{a}{b} \colon a, b \in \mathbb{Q}\right\}$ and $\mathbb{Q}$

(f) $A = \{3x + 1 \colon x \in \mathbb{R}\}$ and $\mathbb{R}$

(g) $A = \left\{x^2 \colon x \in \mathbb{R}\right\}$ and $\mathbb{R}$

(h) $A = \left\{x^2 \colon x \in \mathbb{R}\right\}$ and $B = \{x \colon x \in \mathbb{R}, x \geq 0\}$

0.1.5. Define a relation between the sets $A = \{1, 2, 3, 4, 5\}$ and $B = \{x, y, z\}$.

0.1.6. Find all relations between the sets $C = \{1, 2\}$ and $D = \{a, b\}$.

0.2 Functions

We have all the pieces now for a nice, formal definition of functions. Let's see if we can make sense of this and relate it (pun intended) to our intuition about what a function is.

Definition 0.2.1 *Let A and B be sets. A* ***function*** ***from*** *A* ***to*** *B, often written $f\colon A \to B$, is a relation f from A to B such that*

$$\text{if } (a, b_1) \in f \text{ and } (a, b_2) \in f, \text{ then } b_1 = b_2.$$

The element $(a, b) \in f$ is often written $f(a) = b$.

Before we barrel[6] forward, let's see an example of this in action.

6: One of the two authors rejects the use of "barrel" as a verb.

Example 0.2.1 Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Then the relation f given by

$$f = \{(1, 2), (3, 4), (5, 6), (7, 8), (9, 10)\}$$

is a function. We could also describe this function by

$$f(1) = 2,\ f(3) = 4,\ f(5) = 6,\ f(7) = 8, \text{ and } f(9) = 10.$$

Example 0.2.2 Let's see a relation that is not a function with this definition. Again, let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Define the relation r by

$$r = \{(1, 2), (1, 4), (3, 6), (5, 8), (7, 10)\}.$$

This is *not* a function because $(1, 2)$ and $(1, 4)$ are both in r, but $2 \neq 4$.

The notational convention at the end of Definition 0.2.1 (we write $(a, b) \in f$ as $f(a) = b$) allows us to rewrite the condition

"if $(a, b_1) \in f$ and $(a, b_2) \in f$, then $b_1 = b_2$" as
"if $f(a) = b_1$ and $f(a) = b_2$, then $b_1 = b_2$"

Furthermore, since $(a, b) \in f$ is taken to mean the same thing as $f(a) = b$, it is common to see a function defined by an algebraic formula. For example, if we were to define a function $f\colon A \to B$, where $A = B = \mathbb{Z}$, by the relation

$$f = \{(a, a + 3)\colon a \in A\},$$

we see that for any element $a \in A$, we have $(a, a + 3) \in f$, where $a + 3 \in B$. Thus, we could just as well define this relation by

$$f(a) = a + 3.$$

That probably looks a *lot* more familiar,[7] but what *is* the equation $f(a) = a + 3$? Is it the function? No. The function's name is f. The equation $f(a) = a + 3$ is an equation that defines the relation f; it tells us in practice how f relates elements from A to elements from B. This may seem like a trivial point, but a great deal of confusion springs forth from this subtlety in terminology. Here is a concrete breakdown of the terminology:

7: Notice that in this notation, the function from Example 0.2.1 would be described using $f(a) = a + 1$.

- The name of the function is f;
- a is an element of the first set A;
- $f(a)$ is an element of the second set B; and

- $f(a) = a + 3$ is an equation that indicates how to compute the element of B related to a by f.

These four things are *not* interchangeable. Understanding the difference between them will be particularly important for some of the discussion later in this text.

It's also worth underscoring that a function need not be defined by a formula. Indeed, if either the domain or the codomain is not a set of numbers, then a formula becomes quite impractical.

Exploration 5 Let r_1 and r_2 be relations from the set A to the set B given below:

$$\begin{aligned} A &= \{2, \sigma, +\}, \\ B &= \{1, 2, 3, 4, 5, 6, 7\}, \\ r_1 &= \{(2,3), (\sigma,3), (+,3), (2,6), (\sigma,6), (+,6)\}, \text{ and} \\ r_2 &= \{(2,3), (\sigma,3), (+,3)\}. \end{aligned}$$

Which of r_1 and r_2 are functions?

Exploration 6 Let r be the relation from $\mathbb{Q}$ to $\mathbb{Q}$ given by

$$r = \{(x^2, x) \colon x \in \mathbb{Q}\}.$$

Is r a function? Justify your response.

Definition 0.2.2 *Let $f \colon A \to B$ be a function. The* ***domain of*** *f is the set*

$$\begin{aligned} \operatorname{dom}(f) &= \{a \in A \colon \textit{there exists } b \in B \textit{ such that } (a,b) \in f\} \\ &= \{a \in A \colon \textit{there exists } b \in B \textit{ such that } f(a) = b\} \end{aligned}$$

Definition 0.2.3 *Let $f \colon A \to B$ be a function. The* ***range of*** *f is the set*

$$\begin{aligned} \operatorname{ran}(f) &= \{b \in B \colon \textit{there exists } a \in A \textit{ such that } (a,b) \in f\} \\ &= \{b \in B \colon \textit{there exists } a \in A \textit{ such that } f(a) = b\} \end{aligned}$$

Definition 0.2.4 *Let $f \colon A \to B$ be a function. The* ***codomain of*** *f, written* $\operatorname{codom}(f)$*, is the set B.*

This definition clears up a common ambiguity. If $f \colon A \to B$ is a function, must f relate *every* element of A to an element of B? Well, no. However, in practice, people often restrict the definition of their function to eliminate all the elements of A not related to an element of B by f. This is not always the case, though. Perhaps you have encountered functions like $f \colon \mathbb{Q} \to \mathbb{Q}$, defined by the equation $f(q) = 1/q$ where the the domain is incorrectly referred to as

$\mathbb{Q}$. Have you spotted the problem? While $0 \in \mathbb{Q}$, 0 cannot be related to some other rational number by the given equation. Thus, $\operatorname{dom}(f)$ in this example is actually $\mathbb{Q}\backslash\{0\}$, all rational numbers except 0.

Here's the takeaway from all of this. Given a function $f\colon A \to B$,

$$\begin{aligned}\operatorname{dom}(f) &\subseteq A, \text{ and}\\ \operatorname{ran}(f) &\subseteq B = \operatorname{codom}(f).\end{aligned}$$

If you've never heard the word "codomain" before, *don't panic*. It's just another name for B, the second set in the relation f. Given an arbitrary function f, there's no reason to expect it to relate *all* the elements of A to *all* the elements of B, so we have these subsets $\operatorname{dom}(f)$ and $\operatorname{ran}(f)$ of A and B, respectively.

Lastly, it is common to think of elements of $\operatorname{dom}(f)$ as "inputs" and elements of $\operatorname{ran}(f)$ as "outputs" and to refer to the relating f does from A to B as "mapping." For example, given $f\colon \mathbb{Z} \to \mathbb{Z}$ by $f(z) = 9$, one could say the input -2 is mapped to the output 9. In fact, this very boring constant function maps all inputs to the output 9. If this makes you happy, then so be it. It does fit nicely with the notation $f\colon A \to B$.

Example 0.2.3 Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be given by $\{(z, z^2)\colon z \in \mathbb{Z}\}$. This would be written as $f(z) = z^2$. Let's do the easy one first: $\operatorname{codom}(f) = \mathbb{Z}$. Now note that $\operatorname{dom}(f) = \mathbb{Z}$, and this is given explicitly in the set notation. However, using the notation $f(z) = z^2$, we must rely on the fact that there are no integers that cannot be squared in order to determine that $\operatorname{dom}(f) = \mathbb{Z}$. What about the range? Formally, we have

$$\operatorname{ran}(f) = \{z^2\colon z \in \mathbb{Z}\} = \{0, 1, 4, 9, 16, 25, \dots\},$$

that is, $\operatorname{ran}(f)$ is the set of all square integers. Lastly, note that our function f maps $\mathbb{Z}$ to $\mathbb{Z}$, so in the equation $f(z) = z^2$,

- f is the name of the function,
- $f(z) = z^2$ is an equation defining the relation between $\mathbb{Z}$ and $\mathbb{Z}$,
- z is an integer, and
- $f(z)$ is an integer (in $\operatorname{ran}(f)$); in particular, $f(z)$ is a squared integer.

Example 0.2.4 Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be given by $f(z) = z^2/z^2$. You may be tempted to "simplify," whatever that means. Do not. Now, what's the domain of f? Are there integers that cannot be "plugged in" for z here? Yep! Our function, as it is defined, cannot handle 0. Thus, the domain is actually $\mathbb{Z}\backslash\{0\}$. Also, the range here is not all of $\mathbb{Z}$. What does f map 5 to for example? This function maps everything to 1. Thus, $\operatorname{ran}(f) = \{1\}$.

Exploration 7 Define the floor function $f\colon \mathbb{Q} \to \mathbb{Q}$ by $f(q) = \lfloor q \rfloor$, where $\lfloor q \rfloor$ is the greatest integer less than or equal to q. You can think of this function as the function that takes rational numbers and rounds them down to the nearest integer. Find the domain, range, and codomain for the function f. Note that these are all sets!

Operations on Sets

Now that we've talked about sets, Cartesian products, and functions, we can talk about something even more fun, binary operations on sets.

Definition 0.2.5 *Let A be a set. A* ***binary operation*** *on a set A is a function $f\colon A \times A \to A$, where the domain is $A \times A$.*

Some familiar examples are addition, subtraction, and multiplication.[8] If we write these in our standard function notation, we would have

$$\begin{aligned} +&\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \\ -&\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \text{ and} \\ \times&\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}. \end{aligned}$$

8: Yes, these stars of elementary school mathematics are about to get the full, formal treatment.

However, it is much more convenient and common to write $+(a, b)$ as $a + b$, and so on for the other symbols.[9]

9: Note that division is not listed here. That's because the old "can't divide by 0" rule makes this what's called a "partial binary operation."

An important question about operations is whether they are valid operations on a specific set. For example, let $A = \{x \in \mathbb{R} : x < 0\}$; that is, A is the set of negative real numbers. Then the operation of multiplication is not well-defined for this set. The product of two negative real numbers is a positive real number, which is not an element of A. This is not how we like our operations to behave; we would say in this case that A is **not closed under multiplication**.

Definition 0.2.6 *If $a * b \in A$ for any $a, b \in A$, we say the set A is* ***closed under the operation*** $*$.

Example 0.2.5 Let's consider the set $P = \{x \in \mathbb{R} : x \geq 0\}$. That is, P is the set of nonnegative real numbers. Since P is a subset of $\mathbb{R}$, all of our operations above make sense, but they may not give us an output that is also in P. Thus, we have to ask whether our set P is closed under these standard operations.

Addition:
To check this, we need to start with two elements of our set, so suppose $a, b \in P$. That means a and b are nonnegative real numbers. Must $a + b$ still be a nonnegative real number? Yes, yes it must. We know the sum of two real numbers is again a real number, and the sum of two positive numbers will again be positive. Also, 0 added to a positive number will just still be that positive number.

Subtraction:
For this one, the operation is not closed. We can see this by giving an example where the difference of two positive real numbers is no longer positive. First, note that $5, 10 \in P$. Then consider $5 - 10 = -5$. Since $-5 \notin P$, our set is not closed under this operation.

Exploration 8 Is our set P above closed under the operation of multiplication of real numbers?

Exploration 9 Can you think of a subset of $\mathbb{R}$ that is closed under addition but not under multiplication?

Exercises for Section 0.2

0.2.1. All of the following statements are false. Give specific examples illustrating why they are false.

(a) Every relation from a set A to a set B is a function.

(b) It's not possible for a function to have the same domain and codomain.

(c) It's not possible for a function to have the same codomain and range.

0.2.2. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 8, 16\}$. Determine whether each of the following is a relation from A to B, a function from A to B, or neither.

(a) $r_1 = \{(1, 2), (2, 4), (3, 8), (4, 16), (5, 16)\}$

(b) $r_2 = \{(1, 16), (1, 8), (2, 16), (2, 8), (3, 16), (3, 8)\}$

(c) $r_3 = \{(1, 2), (2, 3), (3, 5), (4, 4)\}$

(d) $r_4 = \{(1, 4), (2, 4), (3, 4), (4, 4), (5, 4)\}$

0.2.3. Let $f\colon \mathbb{Q} \to \mathbb{Z}$ be defined by $f\left(\frac{a}{b}\right) = a$ for any $\frac{a}{b} \in \mathbb{Q}$. Is this a function?

0.2.4. Let $g\colon \mathbb{Q} \to \mathbb{Q}$ be defined by $g\left(\frac{a}{b}\right) = \frac{a+b}{b}$ for any $\frac{a}{b} \in \mathbb{Q}$. Is this a function?

0.2.5. Consider $A = \{1, 2, 3\}$ and $B = \{1, 3, 4, 5\}$. Is there a function $f\colon A \to B$ with $\operatorname{ran}(f) = B$? Give the function or explain why it does not exist.

0.2.6. For each function below, state the domain, codomain, and range of the function.

(a) $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$ for all $x \in \mathbb{R}$

(b) $f\colon \mathbb{Z} \to \mathbb{Q}$ defined by $f(n) = \frac{n}{2}$ for all $n \in \mathbb{Z}$

(c) $f\colon \mathbb{Z} \to \mathbb{Q}$ defined by $f(n) = \frac{n+1}{3}$ for all $n \in \mathbb{Z}$

(d) $f\colon \mathbb{Q} \to \mathbb{Q}$ defined by $f(r) = \frac{r}{2}$ for all $r \in \mathbb{Q}$

(e) $f\colon \mathbb{Q} \to \mathbb{Q}$ defined by $f(r) = \frac{2}{r}$ for all $r \in \mathbb{Q}\backslash\{0\}$

(f) $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x$ for all $x \in \mathbb{R}$

(g) $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5$ for all $x \in \mathbb{R}$

(h) $f\colon \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $f(x) = (5, 5x)$ for all $x \in \mathbb{R}$

0.2.7. Determine whether the set A is closed under the given operation $\odot$.

(a) $A = \mathbb{Q}$ and $a \odot b = ab$ for any $a, b \in A$

(b) $A = \{n^2 : n \in \mathbb{Z}\}$ and $a \odot b = ab$ for any $a, b \in A$

(c) $A = \{n : n \in \mathbb{Z}, n \leq 0\}$ and $a \odot b = ab$ for any $a, b \in A$

(d) $A = \{n : n \in \mathbb{Z}, n \leq 0\}$ and $a \odot b = a + b$ for any $a, b \in A$

(e) $A = \{n : n \in \mathbb{Z}, n \leq 0\}$ and $a \odot b = a - b$ for any $a, b \in A$

(f) $A = \{x : x \in \mathbb{R}, x \geq 0\}$ and $a \odot b = a - b$ for any $a, b \in A$

(g) $A = \{x : x \in \mathbb{R}, x \geq 0\}$ and $a \odot b = ab$ for any $a, b \in A$

(h) $A = \{x : x \in \mathbb{R}, x > 0\}$ and $a \odot b = ab$ for any $a, b \in A$

(i) $A = \{x : x \in \mathbb{R}, 0 \leq x \leq 1\}$ and $a \odot b = ab$ for any $a, b \in A$

0.2.8. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function with the property that $f(ab) = af(b)$ for any $a, b \in \mathbb{R}$.

(a) What must $f(0)$ be?

(b) There are only two options for $\operatorname{ran}(f)$. Can you name them?

0.2.9. Suppose $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has the related property that $g(ab, ac) = ag(b, c)$. To what must g map $(0, 0)$?

0.2.10. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $f(a, b) = a + b + ab$.

(a) Find an (a, b) such that $f(a, b) = 5$.

(b) Now, for any real number, x, find an appropriate (a, b) such that $f(a, b) = x$ for any real number x. This will show that $\operatorname{ran}(f) = \mathbb{R}$.

(c) Is the (a, b) from part (a) unique? If so, why? If not, find a different (a, b) such that $f(a, b) = 5$.

0.2.11. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(a) = 2a + a^2$.

(a) Find an a so that $f(a) = -1$.

(b) Is there an a such that $f(a) = -2$?

(c) Suppose $f(a) = f(b)$. Must $a = b$?

1 Vector Spaces

Since Linear Algebra is the study of linear transformations on vector spaces, the topic of vector spaces is certainly a reasonable place to start. One could easily argue that it's actually the only reasonable place to start.[1] As we saw in the previous chapter, you cannot construct a function without clearly defining its domain. Linear transformations are functions, and we'll eventually see that vector spaces are used as their domains. Thus, we should begin here with one of the two main characters of this story, vector spaces.

1: Wha? Oh, yes. That *is* my gauntlet on the ground!

1.1 Vector Spaces

$\mathbb{R}$? Yep, $\mathbb{R}$. It turns out the set of real numbers, $\mathbb{R}$, is a vector space. This seems to be an excellent place to start our discussion.

There are many ways to construct the real numbers.[2] That said, we're not going to spend any time constructing the real numbers, $\mathbb{R}$, any more than you would in a typical calculus course. We're going to step over this particular mathematical duck by simply saying the following dissatisfying thing:

2: They are all super interesting, very cool, and usually discussed in a Real Analysis course.

$$\mathbb{R} = (-\infty, \infty). \tag{1.1}$$

Note that the authors have chosen to number this equation. This is so that if future equations seem mystifying, you, the reader, can be referred back to this thing for some perspective.

The goal of this section is not to define $\mathbb{R}$ but to understand the properties of $\mathbb{R}$ with typical operations (adding and multiplying) that we really like. Hey, speaking of operations, let's recall some in case any of you skipped over Chapter 0.

Definition 1.1.1 ***Addition*** *is the function* $(+)\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *defined by relating two real numbers to their sum.* ***Multiplication*** *is the function* $(\cdot)\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *defined by relating two real numbers to their product.*

You should be very familiar with both of these operations and all of the properties they enjoy; you probably know them all by name. Nevertheless, here's a nice comprehensive list:

Theorem 1.1.1 (Field Axioms for Real Numbers) *Let $+$ and $\cdot$ be used for the usual operations of addition and multiplication, respectively. There are elements $0, 1 \in \mathbb{R}$ such that for any $a, b, c \in \mathbb{R}$ and any nonzero $d \in \mathbb{R}$,*

Commutativity of Addition	$a + b = b + a$,
Associativity of Addition	$(a + b) + c = a + (b + c)$,
Additive Identity	$a + 0 = 0 + a = a$,
Additive Inverses	*there is an element $-a \in \mathbb{R}$ such that $a + (-a) = (-a) + a = 0$,*
Commutativity of Multiplication	$a \cdot b = b \cdot a$,
Associativity of Multiplication	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
Multiplicative Identity	$a \cdot 1 = 1 \cdot a = a$,
Multiplicative Inverses	*there is an element $d^{-1} \in \mathbb{R}$ such that $d \cdot d^{-1} = d^{-1} \cdot d = 1$, and*
Distributive Property	$a \cdot (b + c) = a \cdot b + a \cdot c$.

The real numbers are pretty great, though. We built a whole sequence of calculus courses using them! Asking that any old set do all nine of these things is a bit much. Let's see what happens when we change it up a bit.

Example 1.1.1 Let's first consider the set $\mathbb{Z}$. Which of these properties hold and which fail for $\mathbb{Z}$? First of all, since we're using a different set ($\mathbb{Z}$ instead of $\mathbb{R}$), we need to be sure that $\mathbb{Z}$ is closed under the operations of addition and multiplication as well; indeed, $\mathbb{Z}$ is closed under addition and multiplication. Moreover, since these are actually the same operations as the ones in $\mathbb{R}$, the following properties hold in $\mathbb{Z}$ because they hold in $\mathbb{R}$:

- Commutativity of Addition: $a + b = b + a$ for $a, b \in \mathbb{Z}$
- Associativity of Addition: $(a+b)+c = a+(b+c)$ for $a, b, c \in \mathbb{Z}$
- Commutativity of Multiplication: $a \cdot b = b \cdot a$ for $a, b \in \mathbb{Z}$
- Associativity of Multiplication: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for $a, b, c \in \mathbb{Z}$
- Distributive Property: $a \cdot (b + c) = a \cdot b + a \cdot c$ for $a, b, c \in \mathbb{Z}$

These were all properties of the operations $+$ and $\cdot$ instead of properties of the set. The properties of the set will require a bit more thought from us.

- Additive Identity: Since this is the same addition as for $\mathbb{R}$, our element 0 is still the additive identity. The condition we are now checking is whether this element is in $\mathbb{Z}$, which it is! Thus, this one also holds for $\mathbb{Z}$.
- Additive Inverses: Again, since any $a \in \mathbb{Z}$ is also in $\mathbb{R}$, the additive inverse of a is still the same as in $\mathbb{R}$, so for any $a \in \mathbb{Z}$, we need to see that $-a \in \mathbb{Z}$, which it is! Thus, this one also holds for $\mathbb{Z}$.
- Multiplicative Identity: Same as these others, we need only note that $1 \in \mathbb{Z}$ to see this one holds as well.

- Multiplicative Inverses: Again, the multiplicative inverses are the same as the ones in $\mathbb{R}$, so we need only check that if $a \in \mathbb{Z}$, then $\frac{1}{a} \in \mathbb{Z}$. But this one fails! In particular, $2 \in \mathbb{Z}$ but $\frac{1}{2} \notin \mathbb{Z}$.

Exploration 10 Can you name another set with the operations of addition and multiplication where all these properties hold? (Hint: Many answers do exist.)

Exploration 11 Find a property that fails for $\mathbb{R}$ if we replace $+$ with $-$.

The real numbers work super well. Having two operations is pretty great. We get all those extra rules to make sure they both work correctly and play nice together. We get to put them in a list and memorize them. It's all very satisfying.

Let's scrap it all and start over. Don't memorize anything yet. Well, maybe we'll keep that addition bit. Most people rather like that one. But multiplication? Be honest. Hasn't "repeated addition" always felt a little scammy to you as an operation?[3]

3: Total pyramid scheme!

Vector Spaces, by Definition

In all seriousness, though, what was just described is roughly what's about to happen. The set of real numbers is a very complicated set. You do get a lot of very nice properties with addition and multiplication on $\mathbb{R}$, but the properties themselves, when taken all together, can be restrictive in some ways. The idea is to give up some of the properties we enjoyed from Theorem 1.1.1 so that we can enjoy that smaller collection of properties on a wider variety of sets.

It's not as bad as it sounds. In fact, we'll still have two operations!

Definition 1.1.2 *A **vector space** over $\mathbb{R}$ is a set V (whose elements we call **vectors**) together with two operations that satisfies all the properties listed below.*

- *The first operation, called **vector addition** $(+)\colon V \times V \to V$, relates two vectors $\vec{v}$ and $\vec{w}$ to a third vector, commonly written as $\vec{v} + \vec{w}$, and called the sum of these two vectors.*
- *The second operation, called **scalar multiplication** $(\cdot)\colon \mathbb{R} \times V \to V$ relates any scalar $a \in \mathbb{R}$ and any vector $\vec{v} \in V$ to another vector $a\vec{v}$.*

There are elements $\vec{0} \in V$ and $1 \in \mathbb{R}$ such that for all $\vec{u}, \vec{v}, \vec{w} \in V$ and all $a, b \in \mathbb{R}$,

Closure under Addition	$\vec{u}+\vec{v} \in V$,
Associativity of Addition	$\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$,
Commutativity of Addition	$\vec{u}+\vec{v} = \vec{v}+\vec{u}$,
Additive Identity	$\vec{v}+\vec{0} = \vec{0}+\vec{v} = \vec{v}$,
Additive Inverses	*there exists an element* $-\vec{v} \in V$ *such that* $\vec{v}+(-\vec{v}) = (-\vec{v})+\vec{v} = \vec{0}$,
Closure under Scalar Multiplication	$a\vec{v} \in V$,
Scalar and Real Multiplication	$a(b\vec{v}) = (ab)\vec{v}$,
Multiplicative Identity	$1\vec{v} = \vec{v}$,
Distributivity Over Vector Addition	$a(\vec{u}+\vec{v}) = a\vec{u}+a\vec{v}$, *and*
Distributivity Over Real Addition	$(a+b)\vec{v} = a\vec{v}+b\vec{v}$.

Note that before you can even call something a vector space and start talking about all the cool stuff it can or can't do, you need a set and two operations. Therefore, if someone says to you, "Hey! Check out this cool set! I think it's a vector space." Your immediate reaction should be roughly, "Oh yeah? With what operations?" If the other person runs away, they were probably wrong about their set. If they produce two reasonable operations that could be called vector addition and scalar multiplication, then you've found a worthy companion to assist in verifying every single one of the axioms neatly listed in Definition 1.1.2. Then, and only then, should you dare declare your set a vector space.

Fun fact: Fun fact: If there's some other field you prefer, you can use that one instead of $\mathbb{R}$ here, and define vector spaces over that field instead. Wild, right? Anywho, we're gonna stick with vector spaces over $\mathbb{R}$ in this text.

Vector spaces over other fields are often used in physics and computer science and are a topic in abstract algebra courses-

-ahem! We're gonna stick with vector spaces over $\mathbb{R}$ in this text.

Fun fact: $\mathbb{R}$ is a vector space.[4] This is not surprising at all; it was, after all, the muse which begat this fun new definition.[5] Now, let's take a bunch of Cartesian products!

$$\mathbb{R}^n = \mathbb{R}\times\cdots\times\mathbb{R} = \{(x_1,\ldots,x_n) \colon x_i \in \mathbb{R} \text{ for } i = 1,\ldots,n\}$$

People use all sorts of goofy notaton for $\mathbb{R}^n$. The preceding one is respectable enough. Your calculus book might use something like

$$\mathbb{R}^n = \{\langle x_1,\ldots,x_n\rangle \colon x_i \in \mathbb{R} \text{ for } i = 1,\ldots,n\}.$$

We assume the common use of this notation is due to its pointy-ness and general sinister appearance. We'll more often use the seemingly obnoxious notation

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R} \text{ for } 1 \le i \le n \right\}. \tag{1.2}$$

What a neat set! Did you know that $\mathbb{R}^n$ is a vector space?

Ahem.

We're waiting. . .

4: Oh yeah? With what operations?

Addition and multiplication. You know, the regular ones.

5: Yes,"begat." It's a perfectly good word. Challenge: Find a way to use this word in a sentence of your own today.

Oh? Operations? Yes, of course. Vector addition is done componentwise. For any $\vec{x}, \vec{y} \in \mathbb{R}^n$, where

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \text{ we define } \vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}. \tag{1.3}$$

For scalar multiplication, just multiply the scalar to each component of the given vector. That is, for any $\vec{x} \in \mathbb{R}^n$ and any $a \in \mathbb{R}$, we define

$$a\vec{x} = a \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} ax_1 \\ \vdots \\ ax_n \end{bmatrix}. \tag{1.4}$$

Very nice, eh? Perfectly good vector addition and scalar multiplication. We defined these using what's called a *general element* of $\mathbb{R}^n$. We should talk more about this.

Many people find the most difficult part in verifying the axioms of a vector space (besides the tedium) to be showing each axiom works *for all* potential vectors in the given potential vector space. Almost all the real vector spaces we will see end up having an infinite number of vectors, so verifying each property for literally all vectors is not feasible. What one needs is a *general vector* that could represent any vector in the set. Nothing too specific, this vector needs to be generic enough that it satisfies all the properties required to be in the set *and nothing else*. Let's look at $\mathbb{R}^3$.

$$\mathbb{R}^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_i \in \mathbb{R} \text{ for } i = 1, \ldots, 3 \right\} \tag{1.5}$$

Now what does it mean exactly to be "in $\mathbb{R}^3$?" While we've settled on the notation above, we also saw two others. Specifically,

$$\mathbb{R}^3 = \{(x_1, x_2, x_3) \colon x_1, x_2, x_3 \in \mathbb{R}\} \text{ or } \mathbb{R}^3 = \{\langle x_1, x_2, x_3 \rangle \colon x_1, x_2, x_3 \in \mathbb{R}\}.$$

It's instructive now to consider what they all have in common. In each notation, we would use three real numbers; each could be any real number, so we used a variable x_i (where $1 \le i \le 3$) for each. These sequential subscripts also suggest these entries are ordered. That's a lot of information that's easy to look past! Evidently, whether we write them vertically or horizontally does not matter; we choose the former for reasons that will be clear later.

To make a general vector in $\mathbb{R}^3$, we need three real *variables* in order. Thus,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is a general vector in } \mathbb{R}^3;$$

$$\begin{bmatrix} 8 \\ -1.3 \\ 1/2 \end{bmatrix} \text{ is not. It is a } \textit{specific} \text{ vector in } \mathbb{R}^3.$$

Most of the axioms in Definition 1.1.2 involve more than one vector, so we'd actually need two general vectors. It's standard to just use a different letter for the variable, so for $\vec{x}, \vec{y} \in \mathbb{R}^3$, we would write

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

These are both general vectors in $\mathbb{R}^3$. If we can show something is true with these, then it can be assumed to be true for *any* vector in $\mathbb{R}^3$. It's sometimes not hard at all to find a general vector for a vector space; they are often given in the definition of the set, as in Equation 1.5.

Let's show that $\mathbb{R}^2$ satisfies all the requirements to be a vector space given in Definition 1.1.2.[6] We'll make careful use of general vectors to do so.

Example 1.1.2 Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^2$ and $a, b \in \mathbb{R}$. Then,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \text{ and } \vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

for some $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$.

- **Closure under Addition:** Note that

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}.$$

  Since $x_i, y_i \in \mathbb{R}$ for each $i = 1, 2$ and we know $\mathbb{R}$ is closed under addition, each $x_i + y_i \in \mathbb{R}$. Thus, $\vec{x} + \vec{y} \in \mathbb{R}^2$ and $\mathbb{R}^2$ is closed under this vector addition.

- **Associativity of Addition:**

$$\begin{aligned}
\vec{x} + (\vec{y} + \vec{z}) &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 + z_1 \\ y_2 + z_2 \end{bmatrix} = \begin{bmatrix} x_1 + (y_1 + z_1) \\ x_2 + (y_2 + z_2) \end{bmatrix} \\
&= \begin{bmatrix} (x_1 + y_1) + z_1 \\ (x_2 + y_2) + z_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\
&= (\vec{x} + \vec{y}) + \vec{z}.
\end{aligned}$$

  Here we used the associative property for $\mathbb{R}$ in each component.

- **Commutativity of Addition:**

$$\begin{aligned}
\vec{x} + \vec{y} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} \\
&= \begin{bmatrix} y_1 + x_1 \\ y_2 + x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{y} + \vec{x}
\end{aligned}$$

  Here, we used the commutative property of real numbers to see that $x_1 + y_1 = y_1 + x_1$ and $x_2 + y_2 = y_2 + x_2$.

- **Additive Identity:** For this, we need to identify the additive identity $\vec{0}$, the special vector in $\mathbb{R}^2$ such that $\vec{x} + \vec{0} = \vec{x}$ for any $\vec{x} \in \mathbb{R}^2$. There's something pretty obvious to guess, but that won't always be the case. We'll do this one the "long way." Suppose our identity is the vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Then we know $\vec{x} + \vec{v} = \vec{x}$, which means

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

6: We could just as easily show $\mathbb{R}^3$ or $\mathbb{R}^{100}$, but it'll take up less space if we do $\mathbb{R}^2$.

We can use the definition of our vector addition then to get

$$\begin{bmatrix} x_1 + v_1 \\ x_2 + v_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

so $x_1 + v_1 = x_1$, which says $v_1 = 0$. Similarly, $x_2 + v_2 = x_2$, so $v_2 = 0$ as well. This says our $\vec{v} = \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Fortunately, $\vec{0} \in \mathbb{R}^2$ and this calculation works for any vector $\vec{x}$, so our set contains an additive identity!

- **Additive Inverses:** Now that we know what the additive identity is, we can find the additive inverse of a vector $\vec{x} \in \mathbb{R}^2$. This is the special vector $\vec{v}$ such that $\vec{x} + \vec{v} = \vec{0}$, so if $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, we have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then we see that

$$\begin{bmatrix} x_1 + v_1 \\ x_2 + v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, $x_1 + v_1 = 0$ and $v_1 = -x_1$. We also see $v_2 = -x_2$, so the additive inverse of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is $\begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$, which is also in $\mathbb{R}^2$. Thus, every element in $\mathbb{R}^2$ has an additive inverse in $\mathbb{R}^2$.

- **Closure under Scalar Multiplication:** We need to argue why $a\vec{x} \in \mathbb{R}^2$ for any real number a and any vector $\vec{x} \in \mathbb{R}^2$. Note that

$$a\vec{x} = a\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 \\ ax_2 \end{bmatrix}.$$

Then since $a, x_1, x_2 \in \mathbb{R}$ and $\mathbb{R}$ is closed under multiplication, we know $ax_1, ax_2 \in \mathbb{R}$. This means $a\vec{x} \in \mathbb{R}^2$.

- **Scalar and Real Multiplication:** This property is essentially guaranteeing our definition of scalar multiplication works well with the usual idea of multiplication in $\mathbb{R}$. Formally, we need to show $(ab)\vec{x} = a(b\vec{x})$. Using the definition of scalar multiplication and the associative property for multiplication of real numbers, we have

$$(ab)\vec{x} = (ab)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (ab)x_1 \\ (ab)x_2 \end{bmatrix} = \begin{bmatrix} a(bx_1) \\ a(bx_2) \end{bmatrix}$$
$$= a\begin{bmatrix} bx_1 \\ bx_2 \end{bmatrix} = a\left(b\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = a(b\vec{x}).$$

- **Multiplicative Identity:** In the case of the additive identity, we had to find it and be sure it was in the set. Here, we know the multiplicative identity has to be 1, but we just need to check that it does what it should with this scalar multiplication definition. Let's see what $1\vec{x}$ is then.

$$1\vec{x} = 1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1x_1 \\ 1x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{x}.$$

Yay! It works!

- **Distributivity Over Vector Addition:** For this, we are checking that scalar multiplication distributes across vector addition, so we need to see $a(\vec{x}+\vec{y}) = a\vec{x}+a\vec{y}$ for any real number a and any vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$. Using the distributive property for real numbers, we get

$$\begin{aligned} a(\vec{x}+\vec{y}) = a\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) &= a\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix} \\ &= \begin{bmatrix} a(x_1+y_1) \\ a(x_2+y_2) \end{bmatrix} = \begin{bmatrix} ax_1+ay_1 \\ ax_2+ay_2 \end{bmatrix} \\ &= \begin{bmatrix} ax_1 \\ ax_2 \end{bmatrix} + \begin{bmatrix} ay_1 \\ ay_2 \end{bmatrix} = a\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + a\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = a\vec{x}+a\vec{y}. \end{aligned}$$

- **Distributivity Over Real Addition:** This last axiom is checking that our new operations work well with our classic addition in $\mathbb{R}$. That is, we need to verify that $(a+b)\vec{x} = a\vec{x}+b\vec{x}$ for any real numbers a and b and any vector $\vec{x} \in \mathbb{R}^2$. Again using the distributive property for real numbers, we get

$$\begin{aligned} (a+b)\vec{x} = (a+b)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} (a+b)x_1 \\ (a+b)x_2 \end{bmatrix} = \begin{bmatrix} ax_1+bx_1 \\ ax_2+bx_2 \end{bmatrix} \\ &= \begin{bmatrix} ax_1 \\ ax_2 \end{bmatrix} + \begin{bmatrix} bx_1 \\ bx_2 \end{bmatrix} = a\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a\vec{x}+b\vec{x}. \end{aligned}$$

Great! Now, you've seen how a set with operations of vector addition and scalar multiplication can be shown to be a vector space! Note that for each of the axioms, we relied upon established knowledge about $\mathbb{R}$, the set of real numbers.[7] However, we only showed $\mathbb{R}^2$ is a vector space. We *claimed* that $\mathbb{R}^n$ is a vector space for any positive integer n. Well, that proof is very similar. Let's see how by focusing on closure of addition for $\mathbb{R}^n$. That is, given any $\vec{x}, \vec{y} \in \mathbb{R}^n$, we want to show $\vec{x}+\vec{y} \in \mathbb{R}^n$. Note that

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

are general vectors in $\mathbb{R}^n$. Then by definition of vector addition in $\mathbb{R}^n$ (Equation 1.3), we have

$$\vec{x}+\vec{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{bmatrix}. \tag{1.6}$$

It remains to show that the vector at the end of Equation 1.6 is a vector in $\mathbb{R}^n$. Since $\mathbb{R}$ is closed under addition, we know that $x_i + y_i \in \mathbb{R}$ for $i = 1, \ldots, n$. Thus,

$$\begin{bmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{bmatrix} \in \mathbb{R}^n.$$

It follows that $\vec{x}+\vec{y} \in \mathbb{R}^n$, so $\mathbb{R}^n$ is closed under vector addition.

7: This is what we do in mathematics! We build logically upon what is already known.

Exploration 12 Following arguments similar to Example 1.1.2 and the one above, show that $\mathbb{R}^n$ is a vector space when n is any positive integer. Well, that might take a while. At least show two of the axioms hold, so you get a feel for how this goes.

Remark: There is a small hiccup with your favorite[8] vector space, $\mathbb{R}$. Since we're usually dealing with vector spaces over $\mathbb{R}$, when we talk about the vector space $\mathbb{R}$, it can be unclear whether a real number is a vector or a scalar. We resolve this ambiguity by using bold font for vectors, so if you see $\vec{5} \in \mathbb{R}$, you should interpret this as a vector in the vector space $\mathbb{R}$. If you see $5 \in \mathbb{R}$, you should interpret this as the scalar 5. Thus, if you want to rescale the vector $\vec{5} \in \mathbb{R}$ by the scalar 5, you would have

$$5\vec{5} = \vec{25} \in \mathbb{R}.$$

Exactly what just happened there? Well, the vector $\vec{5}$ rescaled by the scalar 5 gives us the vector $\vec{25}$. It's potentially confusing because the real number 25 is being interpreted as a vector. We concede this is both weird and annoying, but it cannot be avoided. Again, *we use bold font to indicate when mathematical objects are vectors*; this will always serve to answer whether or not something is a vector.[9]

8: My favorite is actually $\mathbb{R}^2$.

9: Unless there is a typo. No guarantees.

Other Vector Spaces

One of the most powerful aspects of vector spaces is the wide variety of sets that can be made into vector spaces by providing appropriate operations. We'll now discuss an example of a standard vector space that is not $\mathbb{R}^n$, and you'll see other examples in the exercises.

The set of polynomials of degree n or less is also a very nice set we can make into a vector space:

$$\mathbb{P}_n = \{a_0 + a_1x + a_2x^2 + \cdots a_nx^n \colon a_i \in \mathbb{R} \text{ for } i = 1, \ldots, n\}. \tag{1.7}$$

The "vectors" in this set are polynomials, so we often write strange-looking things like

$$\vec{p} = 27 - 2x + x^n.$$

Here we're saying that $\vec{p}$ is the polynomial in the set $\{a_0 + a_1x + a_2x^2 + \cdots a_nx^n \colon a_i \in \mathbb{R} \text{ for } i = 1, \ldots, n\}$, where $a_0 = 27$, $a_1 = -2$, $a_3 = \cdots = a_{n-1} = 0$, and $a_n = 1$.

Given general $\vec{p}, \vec{q} \in \mathbb{P}_n$, which we may write as

$$\begin{aligned} \vec{p} &= a_0 + a_1 x + \cdots a_n x^n \text{ and} \\ \vec{q} &= b_0 + b_1 x + \cdots b_n x^n, \end{aligned}$$

where $a_i, b_i \in \mathbb{R}$, and a scalar $c \in \mathbb{R}$, we define vector addition as the usual polynomial addition

$$\begin{aligned} \vec{p} + \vec{q} &= (a_0 + a_1 x + \cdots a_n x^n) + (b_0 + b_1 x + \cdots b_n x^n) \\ &= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n, \end{aligned}$$

and we define scalar multiplication as

$$c\vec{p} = c(a_0 + a_1 x + \cdots a_n x^n) = ca_0 + ca_1 x + \cdots + ca_n x^n.$$

Example 1.1.3 We won't spoil all the fun, but let's show $\mathbb{P}_n$ satisfies at least one of the axioms to be a vector space with these operations.

- **Commutativity of Vector Addition:** Let $\vec{p}, \vec{q} \in \mathbb{P}_n$ for some positive integer n. Then

  $\vec{p} = a_0 + a_1 x + \cdots a_n x^n$ and $\vec{q} = b_0 + b_1 x + \cdots b_n x^n$

  for some real numbers $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$. Using the definitions above and the property of commutativity for addition of real numbers, we have

$$\begin{aligned} \vec{p} + \vec{q} &= [a_0 + a_1 x + \cdots a_n x^n] + [b_0 + b_1 x + \cdots b_n x^n] \\ &= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n \\ &= (b_0 + a_0) + (b_1 + a_1)x + \cdots + (b_n + a_n)x^n \\ &= [b_0 + b_1 x + \cdots b_n x^n] + [a_0 + a_1 x + \cdots a_n x^n] = \vec{q} + \vec{p}. \end{aligned}$$

  Yay! Vector addition commutes!

Exploration 13 We should really verify all the axioms in Definition 1.1.2 for $\mathbb{P}_n$. We've done one together, so that leaves nine more! For the sake of brevity, though, let's walk through a modified version of this exercise.

- What is the additive identity in $\mathbb{P}_n$?

- What is the additive inverse of the polynomial $5x - 3x^3 + 4x^7$ in $\mathbb{P}_8$?

- What is the additive inverse of $\vec{p} = a_0 + a_1 x + \cdots a_n x^n$ in $\mathbb{P}^n$?

- Show that $12(5x - 3x^3 + 4x^7) = 2(6(5x - 3x^3 + 4x^7))$ by simplifying both sides separately.

- Following the specific example above, show that $ab\vec{p} = a(b\vec{p})$ for any $\vec{p} \in \mathbb{P}_n$ and any $a, b \in \mathbb{R}$.

Great! That's a total of four of the axioms down. Just six more to go!

Note that in the exploration above, we sometimes asked that you work a specific example before taking on the task of verifying the more general statement. This is a worthwhile problem-solving strategy, and you may find it a useful starting place whenever you are tasked thusly. Just be sure that you do not confuse it with showing the general statement. *An affirmative example will not let you conclude the statement holds in general*. And yet, what about when the statement does not hold?

Example 1.1.4 We have now seen some examples of sets with operations of vector addition and scalar multiplication that we've at least claimed are vector spaces. What does it look like when the set is *not* a vector space? Consider again the set $\mathbb{Z}$ of integers. We can use the regular addition of integers for our vector addition and use multiplication of real numbers for our scalar multiplication. This is not a vector space. Let's see a property that fails:

- **Closure under Scalar Multiplication:** When we multiply an integer by a real number, we have no reason to expect the outcome need be an integer. While this statement is true, a much stronger statement would involve an actual example where this property fails. Consider $2 \in \mathbb{Z}$ and $1/3 \in \mathbb{R}$. Note that $(1/3)\mathbf{2} \notin \mathbb{Z}$, so $\mathbb{Z}$ is not closed under this scalar multiplication.

The example of failure used above is a technique called "providing a counterexample," and it is a very useful way to show one of our properties of a vector space fails.

Section Highlights

This section is where we encounter one of the two main topics of this text for the first time, a *vector space*.

- A *vector space* is a set with elements we call *vectors* together with the operations of *vector addition* and *scalar multiplication*.
- There are 10 properties that a set and operations must satisfy in order for it to be a vector space. See Definition 1.1.2.
- To show one of these properties holds, one must use a general elements from the set. See Example 1.1.2.
- To show one of these properties fails, a specific example of it failing using elements from the set is sufficient. See Example 1.1.4.

► A few important examples of real vector spaces:
$\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{P}_n$ for any positive integer, n.

Exercises for Section 1.1

1.1.1. For each of the 10 vector space properties, define an operation of vector addition and/or scalar multiplication on $\mathbb{R}$ for which the property fails. Give an explicit example to show the failure.

1.1.2. Give the complete argument that $\mathbb{P}_n$ is a vector space.

1.1.3. The complex numbers $\mathbb{C}$ are $\{a + bi\colon a, b \in \mathbb{R}\}$ where $i = \sqrt{-1}$. Verify all the vector space axioms to show that $\mathbb{C}$ is a vector space over the field $\mathbb{R}$.

1.1.4. Below are several operations on $\mathbb{R}$. To keep from confusing them with the standard operations, we'll use the symbol $\boxplus$ to denote them. Determine whether these operations obey the commutative and associative properties.

(a) $a \boxplus b = a + 2b$ for any $a, b \in \mathbb{R}$

(b) $a \boxplus b = ab$ for any $a, b \in \mathbb{R}$

(c) $a \boxplus b = a + b + ab$ for any $a, b \in \mathbb{R}$

(d) $a \boxplus b = a + b - 3$ for any $a, b \in \mathbb{R}$

(e) $a \boxplus b = a + b - ab$ for any $a, b \in \mathbb{R}$

1.1.5. Below are several operations on the given set V. To keep from confusing them with the standard operations, we'll use the symbol $\boxplus$ to denote them. For each of these operations, determine whether there is an additive identity (a.k.a zero vector) $\vec{z}$ such that $\vec{x} \boxplus \vec{z} = \vec{z} \boxplus \vec{x} = \vec{x}$ for any $\vec{x} \in V$. If it does exist, what is it?

(a) Let $V = \mathbb{R}$. Define $\vec{a} \boxplus \vec{b} = a + 2b$ for any $\vec{a}, \vec{b} \in \mathbb{R}$.

(b) Let $V = \mathbb{R}$. Define $\vec{a} \boxplus \vec{b} = ab$ for any $\vec{a}, \vec{b} \in \mathbb{R}$.

(c) Let $V = \mathbb{R}$. Define $\vec{a} \boxplus \vec{b} = a + b + ab$ for any $\vec{a}, \vec{b} \in \mathbb{R}$.

(d) Let $V = \mathbb{R}$. Define $\vec{a} \boxplus \vec{b} = a + b - 3$ for any $\vec{a}, \vec{b} \in \mathbb{R}$.

(e) Let $V = \mathbb{R}^2$. Define

$$\begin{bmatrix} a \\ b \end{bmatrix} \boxplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a + d \\ b + c \end{bmatrix}$$

for any $a, b, c, d \in \mathbb{R}$.

(f) Let $V = \mathbb{R}^2$. Define

$$\begin{bmatrix} a \\ b \end{bmatrix} \boxplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a + c - 3 \\ b + d + 4 \end{bmatrix}$$

for any $a, b, c, d \in \mathbb{R}$.

(g) Let $V = \mathbb{R}^2$. Define

$$\begin{bmatrix} a \\ b \end{bmatrix} \boxplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac \\ b+d \end{bmatrix}$$

for any $a, b, c, d \in \mathbb{R}$.

1.1.6. Note that the additive inverse of $\vec{x} \in V$ with the operation $\boxplus$ is the unique element $\vec{i}$ such that $\vec{x} \boxplus \vec{i} = \vec{i} \boxplus \vec{x} = \vec{z}$ where $\vec{z}$ is the additive identity. Thus, in order for an element to have an additive inverse, the set must first have an additive identity. Use your work from the previous exercise to determine whether the additive identity for the given vectors exists.

(a) Let $V = \mathbb{R}$. Define $\vec{a} \boxplus \vec{b} = ab$ for any $\vec{a}, \vec{b} \in \mathbb{R}$. If possible, find the additive inverse for $\vec{5}$ and $\vec{0}$.

(b) Let $V = \mathbb{R}$. Define $\vec{a} \boxplus \vec{b} = a + b + ab$ for any $\vec{a}, \vec{b} \in \mathbb{R}$. If possible, find the additive inverse for $\vec{5}$ and $\vec{-1}$.

(c) Let $V = \mathbb{R}$. Define $\vec{a} \boxplus \vec{b} = a + b - 3$ for any $\vec{a}, \vec{b} \in \mathbb{R}$. If possible, find the additive inverse for $\vec{5}$ and $\vec{0}$.

1.1.7. Define $\vec{a} \boxplus \vec{b} = a + b - 3$ for any $\vec{a}, \vec{b} \in \mathbb{R}$. In each of the parts below, we also define a new operation $\odot$ for our scalar multiplication. With this notation, the two distributive properties become:

Distributivity Across Vector Addition $\quad k \odot (\vec{x} \boxplus \vec{y}) = (k \odot \vec{x}) \boxplus (k \odot \vec{y})$

Distributivity Across Scalar Addition $\quad (j + k) \odot \vec{x} = (j \odot \vec{x}) \boxplus (k \odot \vec{x})$.

Verify whether these properties hold for this vector addition $\boxplus$ and the given $\odot$.

(a) Define $k \odot \vec{a} = ka$, for any $k \in \mathbb{R}$, $\vec{a} \in \mathbb{R}$.

(b) Define $k \odot \vec{a} = ka - 3$, for any $k \in \mathbb{R}$, $\vec{a} \in \mathbb{R}$.

(c) Define $k \odot \vec{a} = ka - 3k + 3$, for any $k \in \mathbb{R}$, $\vec{a} \in \mathbb{R}$.

1.1.8. Let's consider $\mathbb{R}$, with a new operation. Let's replace the usual vector addition with the operation "$\boxplus$" defined by $a \boxplus b = a + b + ab$ for any $a, b \in \mathbb{R}$. Determine whether $\mathbb{R}$ is still a vector space with the operation $\boxplus$ for vector addition and the usual scalar multiplication. If it is not, which properties fail?

1.1.9. Let's consider $\mathbb{R}$, with two new operations. Let's replace the usual vector addition with the operation "$\boxplus$" defined by $a \boxplus b = a + b - 3$ for any $a, b \in \mathbb{R}$ and scalar multiplication defined by $k \odot a = ka - 3k + 3$ for any $k \in \mathbb{R}$ and $a \in \mathbb{R}$. Determine whether $\mathbb{R}$ is still a vector space with the operation $\boxplus$ for vector addition and the operation $\odot$ for scalar multiplication. If it is not, which properties fail?

1.1.10. Let's consider $\mathbb{R}^2$ with a new operation as well. Let's vector addition with "$\boxplus$" defined by

$$\begin{bmatrix} a \\ b \end{bmatrix} \boxplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+d \\ b+c \end{bmatrix}$$

for any $a, b, c, d \in \mathbb{R}$. Determine whether $\mathbb{R}^2$ is still a vector space with the operation $\boxplus$ and the usual scalar multiplication. If it is not, which properties fail?

1.1.11. Let's consider $\mathbb{P}_2$ with a new operation. Replace vector addition with "$\boxplus$" defined by

$$(a + bx + cx^2) \boxplus (d + ex + fx^2) = (b + 2cx) + (e + 2fx) = (b + e) + 2(c + f)x.$$

Determine whether $\mathbb{P}_2$ is still a vector space with the operation $\boxplus$ and the usual scalar multiplication. If it is not, which properties fail?

1.1.12. Let $V = \{a \colon a \in \mathbb{R}, a > 0\} = (0, \infty)$. Verify V is a vector space over $\mathbb{R}$ with vector addition given by $\vec{a} \boxplus \vec{b} = \vec{ab}$ (vector addition is defined as real multiplication as positive numbers) and scalar multiplication given by $k\vec{a} = \mathbf{a^k}$ (scalar multiplication is defined as real exponentiation) for any $k \in \mathbb{R}$, $\vec{a}, \vec{b} \in V$.

1.1.13. Let $V = \{a \colon a \in \mathbb{R}, a > 0\} = (0, \infty)$. As in the previous exercise, let vector addition be given by $\vec{a} \boxplus \vec{b} = \vec{ab}$, but now, let scalar multiplication be given by $k\vec{a} = \mathbf{k^a}$ for any $k \in \mathbb{R}$, $\vec{a}, \vec{b} \in V$. Determine whether this is a vector space.

1.1.14. Let V be the interval $(0, 1)$ on the real numbers. Define vector addition as $\vec{a} \boxplus \vec{b} = \vec{ab}$ for $\vec{a}, \vec{b} \in V$ and scalar multiplication as $k\vec{a} = \mathbf{a^k}$ for any $k \in \mathbb{R}$ and $\vec{a} \in V$. This is *not* a vector space over $\mathbb{R}$. List each of the properties of a vector space that fail.

1.1.15. Let $X = \mathbb{R} \cup ☺$, where ☺ is a new number referred to as “unity face.” For all $x \in X$, define $x + ☺ = ☺ + x = ☺$ and $x(☺) = ☺(x) = ☺$. With the usual addition and multiplication operations, is X a vector space?

1.2 Arrow Vectors and $\mathbb{R}^n$ for Small n

Many of you have seen vectors before, whether in physics, calculus, or perhaps your favorite animated movie. However, the definition you saw was perhaps a bit different. You likely learned that a vector is some quantity with both magnitude and direction, such as velocity. Well, does this match up with what we've said here about vector spaces? Indeed, it would be very embarrassing if it did not.[10]

10: Hey! What are all these pictures doing in our margin?

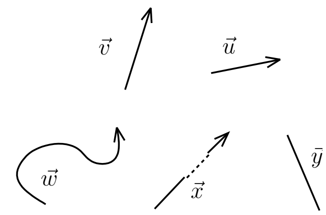


Figure 1.1: Here $\vec{v}$ and $\vec{u}$ are arrow vectors. However, $\vec{w}$, $\vec{x}$, and $\vec{y}$ are not.

If we let a vector be a quantity with magnitude and direction, it is natural to represent this with an arrow. The arrow points in the direction desired, and it has a length that can be used to represent its magnitude. For now, let's call this an **arrow vector**. See Figure 1.1 to the side.

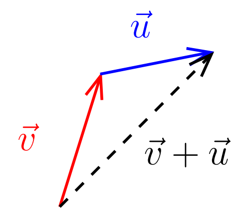


Figure 1.2: Here $\vec{v}$ and $\vec{u}$ are added to form the new arrow vector $\vec{v}+\vec{u}$, identified with the dashed arrow line.

Thus, we would like to establish that an arrow vector is actually an element in some vector space over $\mathbb{R}$, as we defined in the previous section. For these arrow vectors to form a vector space over $\mathbb{R}$ so that the set, V, we are dealing with is the set of all arrow vectors, we need to have a way to add them, and we need to have a concept of scalar multiplication with scalars from $\mathbb{R}$.

First of all, since all these arrows are doing is recording magnitude and direction, their placement on this page does not matter. Thus, an arrow vector can be moved to a new location without changing the arrow vector itself. Thus, a natural addition of the vectors $\vec{v}$ and $\vec{u}$ is to first follow along $\vec{v}$ and then from there, follow along $\vec{u}$. As seen in Figure 1.2, the sum is then the arrow vector drawn from where you started to where you ended.

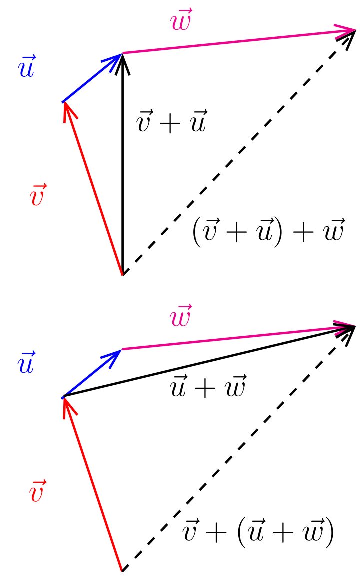


Figure 1.3: Here $\vec{v}$, $\vec{u}$, and $\vec{w}$ are added to form the new arrow vector $\vec{v}+\vec{u}+\vec{w}$, identified with the dashed arrow line, two ways to show this vector addition is associative.

Now, does this addition satisfy our axioms for addition in a vector space? We see quickly that it is closed since the result is a new arrow vector. It is also associative and commutative from the diagrams in Figures 1.3 and 1.4 respectively. What about inverses and the zero vector? The inverse of a vector should just reverse direction or put the arrow on the other end as in Figure 1.5. The zero vector is pictured in Figure 1.6; it is very hard to see as it has magnitude 0. If you zoom in a lot, you might think that you'll be able to see it. However, even when you zoom in a lot, it still has magnitude 0, so it will be very hard (yes, impossible) to see. Some people like to use a dot for the zero vector, but we find our convention to be more accurate.

What should scalar multiplication be? Well, we know it must satisfy repeated addition such as $2\vec{v} = \vec{v} + \vec{v}$. The right hand side of this gives an arrow vector that is twice as long as $\vec{v}$ but still in the same direction. Since our scalar multiplication must agree with this, we will define it to be a scaling of the length. Thus, $\alpha\vec{v}$ is an arrow vector in the direction of $\vec{v}$, but of length α times the length of $\vec{v}$ as in Figure 1.7. Convince yourself that this satisfies the rest of the axioms.

Connection to $\mathbb{R}^n$ (for small n).

Now that we've established that our arrow vectors actually form a vector space, how does this relate to $\mathbb{R}^n$ for small[11] n? Well, if we include the restriction that our arrow vectors must begin at the origin in $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^3$, then it becomes fairly straightforward to show that this is equivalent to how we've already defined the vector spaces $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$!

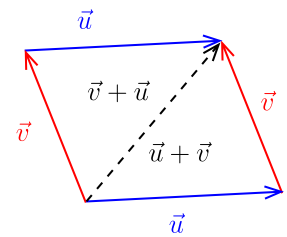


Figure 1.4: Here vector addition is shown to be commutative; this is sometimes, unimaginatively and with overstated importance, called the Parallelogram Law.

Figure 1.5: This is a vector $\vec{v}$ and its inverse $-\vec{v}$.

Figure 1.6: The zero vector is the arrow with no magnitude; it is pictured above. It is very hard to see as it has magnitude 0.

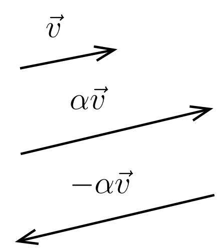


Figure 1.7: Here $\vec{v}$ is scalar multiplied by α and $-\alpha$ for some positive scalar α.

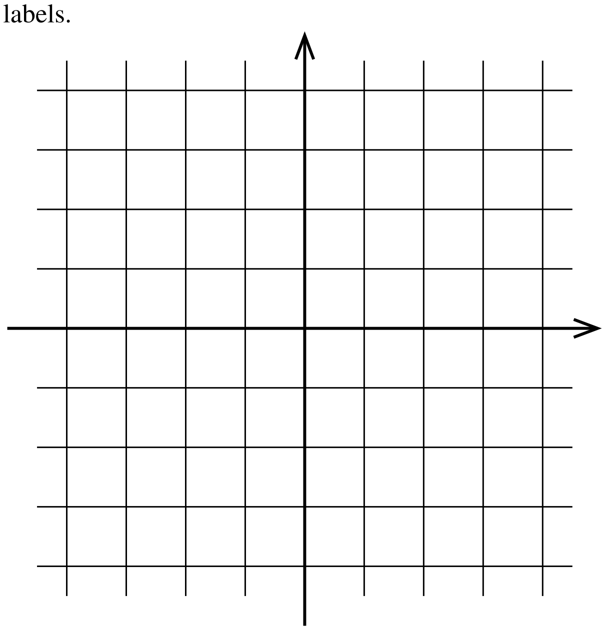
Let us try this with $\mathbb{R}^2$, for example. If the arrow vector $\vec{v}$ begins at the origin and extends to the point $(x, y) \in \mathbb{R}^2$, then we can call this the column vector

$$\begin{bmatrix} x \\ y \end{bmatrix}.$$

Since all vectors begin at the origin, the tip of the arrow vector determines the vector itself, so this naturally defines a relation from the set of points in $\mathbb{R}^2$ to the set of arrows in the plane beginning at the origin. Similarly, one could define a relation from the set of arrows in the plane beginning at the origin to the set of points in $\mathbb{R}^2$. We'll get into the extent to which these sets are "the same" later, but for now, we strongly suspect you'll agree that these sets are similar enough to think of them interchangeably.

Exploration 14 Let's do an example here to see how these arrow vectors agree with $\mathbb{R}^2$.

- ► Draw the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as described above on the grid below. Include labels.



- ► Then, from the arrow tip of the vector you just drew, go up one square and to the left 2.
- ► Draw the arrow vector from the origin to the place you ended above. This is the sum of the vectors $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, so you should have drawn the vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Did you?

Exploration 15 We haven't mentioned scalar multiplication. Let's do an example with that one.

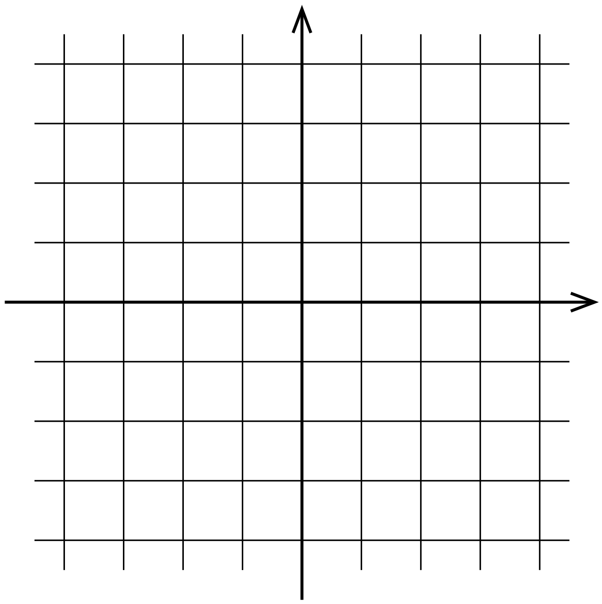
- Draw the vector $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ on the grid provided.

- Now, draw a line from the tip of the arrow down to the positive x-axis. This gives you a right triangle, and you can find the length of $\vec{v}$ using the Pythagorean theorem.[12]

- Now use the Pythagorean theorem again and compute the length of $2\vec{v} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$. Did you get twice the length of $\vec{v}$?

12: If you have forgotten what knowledge our good friend Pythagoras imparted, you should Google it. Then write an essay about how wonderful it is so that you never forget again.

Or maybe a tattoo would be better...



In that last exploration, we used the Pythagorean theorem to find the length of our vectors when viewed as vectors in $\mathbb{R}^2$. Well, what about when they're in $\mathbb{R}^n$? Although we no longer have our handy arrow vectors for visualization in $\mathbb{R}^n$ for $n \geq 4$, we do actually have a way to discuss distances and lengths, so that some of the geometry that feels natural in $\mathbb{R}^2$ and $\mathbb{R}^3$ can be extended to these other cases. The next section will set this up for us.

More Geometry with $\mathbb{R}^n$

As you'll recall, we gave up our notion of multiplication between two vectors in favor of scalar multiplication when we defined vector spaces in Definition 1.1.2. That doesn't stop people from trying to "multiply" two vectors anyway; there are a couple of different notions of "multiplication of vectors" out there. At least one of them ends up being pretty useful:

Definition 1.2.1 *The* ***inner product*** *is the function* $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *defined by relating two vectors to the real number given by summing the products of like components of the two vectors. That is, given* $\vec{v}, \vec{u} \in \mathbb{R}^n$*, we denote* *the inner product of $\vec{v}$ and $\vec{u}$ as $\vec{v}\cdot\vec{u}$, given by*

$$\vec{v}\cdot\vec{u} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = v_1u_1 + \cdots + v_nu_n = \sum_{i=1}^{n} v_iu_i.$$

Inner product is also synonymously called **scalar product**[13] and **dot product**.[14]

Exploration 16 Let's see this in action! Let

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \quad \text{and } \vec{w} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

be vectors in $\mathbb{R}^3$. Then $\vec{v}\cdot\vec{u} = 0+0+6 = 6$ and $\vec{u}\cdot\vec{w} = -2+0+12 = 10$. Find $\vec{v}\cdot\vec{w}$.

Exploration 17 Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. Why doesn't $\vec{v}\cdot\vec{u}$ make sense?

Exploration 18 Is the inner product commutative? That is, for vectors $\vec{v}$ and $\vec{u}$ in $\mathbb{R}^n$, is it always true that $\vec{v}\cdot\vec{u} = \vec{u}\cdot\vec{v}$? Compute an example to illustrate your conclusion.

It turns out inner product has a lot of nice properties. Since it's used to define length, that's probably a good thing.

Theorem 1.2.1 *Let $\vec{v}$, $\vec{u}$, and $\vec{w}$ be vectors in the $\mathbb{R}^n$, and let a be a scalar. Then*

(a) $\vec{v}\cdot\vec{u} = \vec{u}\cdot\vec{v}$

(b) $(\vec{v}+\vec{u})\cdot\vec{w} = \vec{v}\cdot\vec{w} + \vec{u}\cdot\vec{w}$

(c) $(a\vec{v})\cdot\vec{u} = a(\vec{v}\cdot\vec{u}) = \vec{v}\cdot(a\vec{u})$

(d) $\vec{u}\cdot\vec{u} \geq 0$ *with* $\vec{u}\cdot\vec{u} = 0$ *if and only if* $\vec{u} = \vec{0}$.

Exploration 19 Let's walk through the proof of Theorem 1.2.1. We will need general forms of the vectors $\vec{v}$, $\vec{u}$, and $\vec{w}$ for this, so let

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \text{and } \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}.$$

13: This was either named by codomain enthusiasts or Professor Ignacious J. Scalar whose surname is derived from the Spanish infinitive that translates roughly as "to cover with scales, weigh, and then climb".

14: This term was coined by. . . um. . . some one named Dot?

Do better, Ricky.

(a) First, let's show that, as you suspected, the inner product is commutative.

$$\begin{aligned}\vec{v}\cdot\vec{u} =& v_1u_1 + \cdots + v_nu_n \\ =& u_1v_1 + \cdots + u_nv_n = \vec{u}\cdot\vec{v}\end{aligned}$$

(b) For this one, first, compute $(\vec{v}+\vec{u})\cdot\vec{w}$.

Now, compute $\vec{v}\cdot\vec{w}+\vec{u}\cdot\vec{w}$. (Then, they should be the same!)

(c) Note that $c\vec{v} = \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix}$. Compute $(c\vec{v})\cdot\vec{u}$.

Now, compute $c(\vec{v}\cdot\vec{u})$.

Lastly, compute $\vec{v}\cdot(c\vec{u})$.

(d) Note that $\vec{u}\cdot\vec{u} = u_1^2+\cdots+u_n^2$. Why must this always be nonnegative?

Now, for the last part, suppose $\vec{u}\cdot\vec{u} = u_1^2+\cdots+u_n^2 = 0$. Then, each of the u_i must be zero for $1 \leq i \leq n$. Thus, $\vec{u} = \vec{0}$. Also, if we compute $\vec{0}\cdot\vec{0}$, we see this must be 0.

Definition 1.2.2 ***Length*** *(or* ***norm****) is the function* $\|\cdot\|\colon \mathbb{R}^n \to \mathbb{R}$ *defined for any* $\vec{v}\in\mathbb{R}^n$ *as*

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2+\cdots+v_n^2}.$$

A vector $\vec{v}\in\mathbb{R}^n$ *is said to be a* ***unit vector*** *(or to have* ***unit length****) if* $\|\vec{v}\| = 1$.

If we're thinking of our vectors in $\mathbb{R}^n$ as having the two properties, magnitude and direction, then the inner product gives us a way to identify the magnitude (that is, length) of a vector. Unit vectors, having length 1, are a nice way to look at just the direction of a vector. The process of making a nonzero vector into a unit vector by dividing it by its length is sometimes called **normalizing a vector**.

Theorem 1.2.2 *For any nonzero vector* $\vec{v}\in\mathbb{R}^n$, $\dfrac{\vec{v}}{\|\vec{v}\|}$ *is a unit vector.*

PROOF.

$$\left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| = \sqrt{\frac{\vec{v}}{\|\vec{v}\|} \cdot \frac{\vec{v}}{\|\vec{v}\|}} = \frac{\sqrt{\vec{v} \cdot \vec{v}}}{\sqrt{\|\vec{v}\|^2}} = \frac{\|\vec{v}\|}{\|\vec{v}\|} = 1.$$

$\square$

Exploration 20 Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

- Find the length of the vector $\vec{v}$, denoted by $\|\vec{v}\|$.
- Find a vector with the same direction as $\vec{v}$ but with length 1.
- Find a vector with the same direction as $\vec{v}$ but with length 5.

We can think of distance between points in $\mathbb{R}^n$ (for $n \leq 3$). There are some formulae you may have seen:

- For $x, y \in \mathbb{R}$, the distance between x and y is
$$d(x, y) = \sqrt{(x-y)^2} = |x - y|,$$
which is just the usual absolute value in $\mathbb{R}$.
- For $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, we have
$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$
This is sometimes called “the distance formula”.
- For $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$, we have
$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

Yeah, there’s a pattern there. This is because these are all specific versions of the same formula. We state it below for vectors, rather than for points.

Definition 1.2.3 ***Distance*** *is the function $dist\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by relating two vectors to the length of their difference. That is, given $\vec{v}, \vec{u} \in \mathbb{R}^n$, we denote the distance between $\vec{v}$ and $\vec{u}$ as* $\operatorname{dist}(\vec{v}, \vec{u})$ *given by*

$$\operatorname{dist}(\vec{v}, \vec{u}) = \|\vec{v} - \vec{u}\|.$$

One of the immediate benefits of Definition 1.2.3 is that it works in $\mathbb{R}^n$ for larger[15] n. Indeed, it is difficult to imagine what distance looks like or means in $\mathbb{R}^7$. We invite you to try, just don’t spend too much time trying. A better use of your time would be to make sure Definition 1.2.3 is really the same thing as our notion of distance between points in $\mathbb{R}^n$ for small n.

15: We define “larger” in this context to be an integer greater than 3.

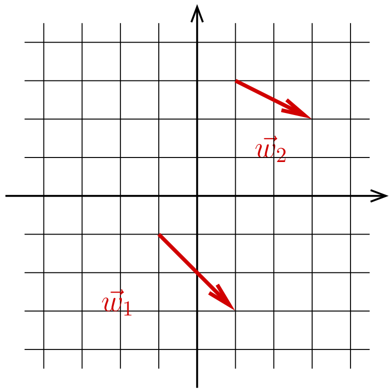
Section Highlights

The main idea of this section is to talk about some geometric properties for the real vector spaces $\mathbb{R}^n$.

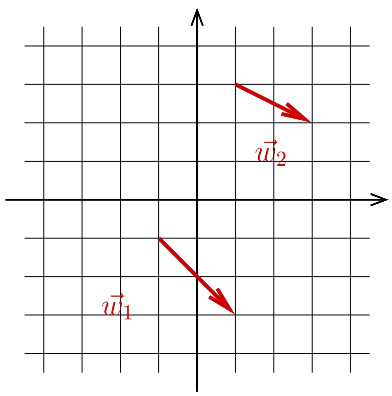
- A vector space can be formed from a set of arrows in either $\mathbb{R}^2$ or $\mathbb{R}^3$ using carefully chosen definitions for vector addition (Figure 1.2) and scalar multiplication (Figure 1.7).
- An arrow vector starting at the origin in $\mathbb{R}^2$ can be associated with the vector in $\mathbb{R}^2$ defined by the coordinates of the point at the tip of the arrow. The same can be done in $\mathbb{R}^3$. These can be used to view arrow vectors as a graphical representation of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- The length of a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is the actual length of the associated arrow vector.
- The distance between two vectors in $\mathbb{R}^2$ (or $\mathbb{R}^3$) is the distance between the tips of the two arrow vectors starting at the origin.
- This geometry can be generalized to $\mathbb{R}^n$ with the help of an *inner product* (*dot product*). See Definition 1.2.1.
- The dot product allows us to extend geometric concepts like the length of a vector (Definition 1.2.2) and the distance between two vectors (Definition 1.2.3) to the vector space $\mathbb{R}^n$.

Exercises for Section 1.2

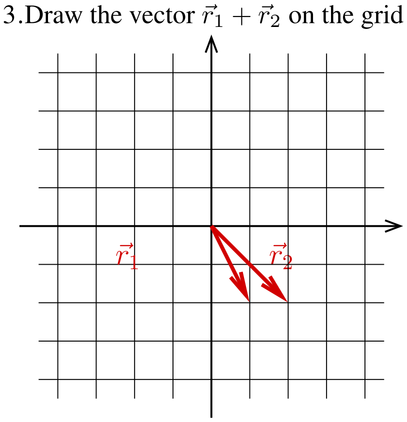
1.2.1. Draw the vector $\vec{w}_1 + \vec{w}_2$ on the grid below.



1.2.2. Draw the vector $\vec{w}_1 - \vec{w}_2$ on the grid below.

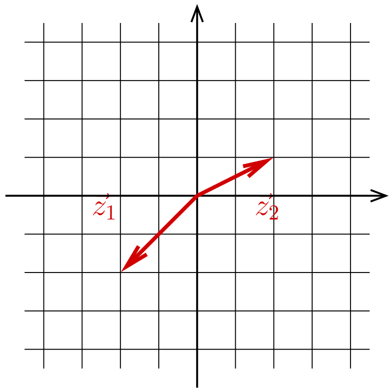


1.2.3. Draw the vector $\vec{r}_1 + \vec{r}_2$ on the grid below.



Then, find the column vector representations of $\vec{r}_1$ and $\vec{r}_2$ in $\mathbb{R}^2$. Use these to find $\vec{r}_1 + \vec{r}_2$. Does this agree with what you drew?

1.2.4. Draw the vector $\vec{z}_1 - \vec{z}_2$ on the grid below.



Then, find the column vector representations of $\vec{z}_1$ and $\vec{z}_2$ in $\mathbb{R}^2$. Use these to find $\vec{z}_1 - \vec{z}_2$. Does this agree with what you drew?

1.2.5. Let

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}, \qquad \text{and } \vec{c} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}.$$

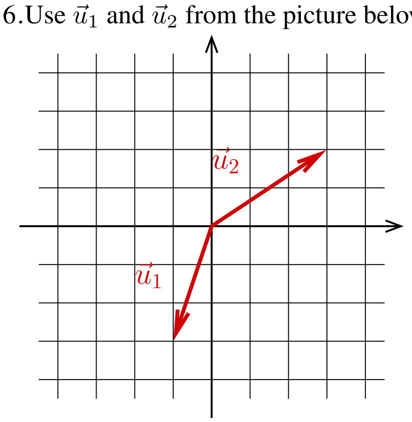
Note that $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^2$, which is a vector space. Simplify the following expressions down to a single vector, and indicate which properties of Definition 1.1.2 you use at each step.

(a) $3(\vec{a} - 2\vec{b}) + 2(\vec{b} + \vec{c})$

(b) $5(\vec{c} + 2\vec{b}) - 2(\vec{b} - 3\vec{a}) + 3(\vec{a} - 3\vec{b} - 2\vec{c})$

Sketch each term of each expression (for example, $3(\vec{a} - 2\vec{b})$ and $2(\vec{b} + \vec{c})$ in part a) on the same grid with the simplified vector.

1.2.6. Use $\vec{u}_1$ and $\vec{u}_2$ from the picture below to answer the questions.

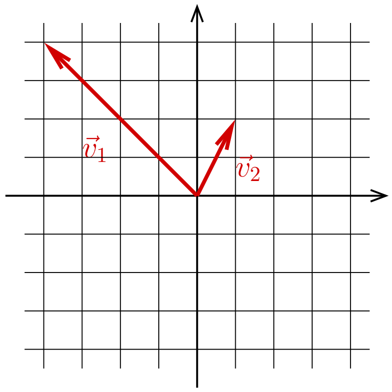


(a) Find $||\vec{u}_1||$ and $||\vec{u}_2||$.

(b) Find a unit vector in $\mathbb{R}^2$ in the direction of $\vec{u}_1$.

(c) Find a vector in $\mathbb{R}^2$ in the direction of $\vec{u}_1$ with length 7.

1.2.7. Use $\vec{v}_1$ and $\vec{v}_2$ from the picture below to answer the questions.



(a) Find $||\vec{v}_1||$ and $||\vec{v}_2||$.

(b) Find a unit vector in $\mathbb{R}^2$ in the direction of $\vec{v}_2$.

(c) Find a vector in $\mathbb{R}^2$ in the direction of $\vec{v}_2$ with length 2.

1.2.8. Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

(a) Find $\vec{v} \cdot \vec{u}$.

(b) Find $\vec{v} \cdot (2\vec{u})$.

(c) Find $(2\vec{v}) \cdot \vec{u}$.

(d) Find a nonzero vector $\vec{w}$ for which $\vec{v} \cdot \vec{w} = 0$.

1.2.9. Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$.

(a) Find $\vec{v} \cdot \vec{u}$.

(b) Find a nonzero vector $\vec{w}$ for which $\vec{v} \cdot \vec{w} = 0$.

1.2.10. Let's define a different product between vectors in $\mathbb{R}^3$. Let this product, denoted by $\boxtimes$, be given by

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \boxtimes \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 2v_1u_1 + 4v_2u_2 + 2v_3u_3.$$

Determine whether each property in Theorem 1.2.1 holds for $\boxtimes$.

1.2.11. Let's again define a different product between vectors in $\mathbb{R}^3$. Let this product, denoted by ☹, be given by

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \frownie \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = v_1 u_3 + v_2 u_2 + v_3 u_1.$$

Determine whether each property in Theorem 1.2.1 holds for ☹.

1.3 Linear Independence and Span

Let's detour for a minute and talk about chess. A chessboard is an 8 by 8 grid, and there are several different pieces with rules about how they can move. For instance, the rook can move forward, backward, left or right, but not diagonally. The pawn can only move forward or diagonally forward if it is capturing another piece. Now, we come to the reason for our detour. Which spaces on the board can be reached by moving any piece using its specific rules? For the rook, we can reach any space on the board by moving up and over in the grid pattern. However, for the pawn, the spaces behind its starting space are unobtainable since it can only move forward.

Now, let's go back to vector spaces. Suppose we start at a vector $\vec{v}$ in a vector space V, and we are allowed to use our operations of scalar multiplication and vector addition to "move around" the vector space with $\vec{v}$. What other vectors can we obtain this way? What if we are only allowed to add certain other vectors from V? This sounds fun like chess, right?[16]

16: One of the authors finds this much more fun than chess.

Linear Combinations and Span

Since vector spaces are closed under vector addition and scalar multiplication, we can do both of these operations as many times as we want to vectors in a vector space V and still end up with a vector in V. This is so convenient that it gets its own name and definition:

Definition 1.3.1 *Let V be a vector space, $\vec{v}_1, \ldots, \vec{v}_p \in V$, and $a_1, \ldots, a_p \in \mathbb{R}$. The vector in V*

$$a_1\vec{v}_1 + \cdots + a_p\vec{v}_p \tag{1.8}$$

is called a ***linear combination*** *of the vectors $\vec{v}_1, \ldots, \vec{v}_p$ with* ***weights*** *(or scalars) $a_1, \ldots, a_p$.*

Exploration 21 Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Now pick your two favorite real numbers. Did you pick 3 and 4? Great! Here's a linear combination of $\vec{v}_1$ and $\vec{v}_2$:

$$3\vec{v}_1 + 4\vec{v}_2 = 3\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+4 \\ 6+0 \\ -3+4 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 1 \end{bmatrix}.$$

► Compute the linear combination $4\vec{v}_1 + 3\vec{v}_2$.

Suppose we are instead given a vector and asked whether or not it is a linear combination of some set of vectors.[17] How should this be handled? Well, we

17: This is just like our chess discussion! Which spaces can be reached with the allowed "moves?"

would need to find the appropriate scalars to make it a linear combination or show that no such scalars are possible. Let's see an example of this.

- Is $\vec{x} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

  Well, if it is, then there exist $a, b \in \mathbb{R}$ such that $a\vec{v}_1 + b\vec{v}_2 = \vec{x}$. This means

  $$a \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$$

  Using the definitions of scalar multiplication and vector addition for $\mathbb{R}^3$, we can see that this means

  $$\begin{bmatrix} a+b \\ 2a \\ -a+b \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}.$$

  This gives us equations $a + b = 3$, $2a = 4$ and $-a + b = -1$. These solve to get $a = 2$ and $b = 1$. Thus,

  $$2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix},$$

  which is true!

- Is $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

  Again, if it is, there exist $a, b \in \mathbb{R}$ such that $a\vec{v}_1 + b\vec{v}_2 = \vec{y}$. This means

  $$a \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

  Just like before, this gives us equations. Here, they are $a + b = 1$, $2a = 1$ and $-a + b = 1$. However, unlike last time, these equations have no common solution; note that from the second equation, we have $a = 1/2$, which implies $b = 1/2$ from the first equation, but these values do not work in the third equation. Thus, we conclude that $\vec{y}$ is *not* a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

- Is $\vec{z} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

Example 1.3.1 Now that we've seen examples in $\mathbb{R}^3$, let's see how things change in $\mathbb{P}_3$. Consider the vectors

$$\vec{p}_1 = 1 + x, \quad \vec{p}_2 = x, \quad \vec{p}_3 = x^3$$

in $\mathbb{P}_3$.

- Is $\vec{q} = 7 - x + 9x^3$ a linear combination of $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$? Yes! To see that, we should solve for $a, b, c \in \mathbb{R}$ such that $a\vec{p}_1 + b\vec{p}_2 + c\vec{p}_3 = \vec{q}$. That is,

  $$a(1 + x) + b(x) + c(x^3) = 7 - x + 9x^3.$$

  Rearranging the left hand side gives us

  $$a + (a + b)x + cx^3 = 7 - x + 9x^3.$$

  Solving for these scalars then gives us $a = 7$, $b = -8$, and $c = 9$. Then

  $$\vec{q} = 7 - x + 9x^3 = 7\vec{p}_1 - 8\vec{p}_2 + 9\vec{p}_3$$

  is a linear combination of the three vectors $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$.
- Now, let's consider

  $$\vec{v} = 7 - x + x^2 + 9x^3.$$

  Is $\vec{v}$ a linear combination of $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$? Nope! No sum or rescaling of $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$ will produce the x^2 term in $\vec{v}$.

We've spent a bit of time now asking whether or not a vector is a linear combination of some set of vectors, so let's just formalize this a bit.

Definition 1.3.2 *Let V be a vector space and $\{\vec{v}_1, \ldots, \vec{v}_p\} \subseteq V$. The* ***span of*** *$\vec{v}_1, \ldots, \vec{v}_p$, denoted $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$, is the set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_p$. That is,*

(1.9)
$$\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_p\} = \{a_1\vec{v}_1 + \cdots + a_p\vec{v}_p \colon a_i \in \mathbb{R} \textit{ for } i = 1, \ldots, p\}.$$

Exploration 22 Is $\vec{x} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$ in $\operatorname{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$?
Hint: This is another way to ask a question we've already asked, so you should be able to answer it by looking back at a previous example.

Example 1.3.2 Let's revisit the vectors in Example 1.3.1. That is, consider again the vectors

$$\vec{p}_1 = 1 + x, \quad \vec{p}_2 = x, \quad \vec{p}_3 = x^3$$

in $\mathbb{P}_3$. Now, what is $\operatorname{Span}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$? Well, let's figure it out.

$$\begin{aligned} \operatorname{Span}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\} &= \{a\vec{p}_1 + b\vec{p}_2 + c\vec{p}_3 \colon a, b, c \in \mathbb{R}\} \\ &= \{a(1 + x) + b(x) + c(x^3) \colon a, b, c \in \mathbb{R}\} \\ &= \{a + (a + b)x + cx^3 \colon a, b, c \in \mathbb{R}\}. \end{aligned}$$

Note here that we have coefficients a, $a+b$, and c. There is obviously some relationship between a and $a+b$, but in this case, the relation doesn't really matter. Because b is not related to a or c and can be any real number, we could actually replace $a+b$ with a new variable $d = a+b$ that can be any real number. Then we get

$$\text{Span}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\} = \{a + dx + cx^3 \colon a, d, c \in \mathbb{R}\}.$$

This makes it clear that the span is all polynomials in $\mathbb{P}_3$ without an x^2 term.

We've seen in these explorations and examples how to determine whether a specific vector is or is not in the span of some set of vectors. Then, this last example gave us some idea about how to compute span algebraically. Now, let's talk a bit about the geometry and the bigger picture of what's in a span.

Example 1.3.3 First of all, what does the span of a single vector "look like?" Well, to picture anything, we should really think about the case of $\mathbb{R}^2$ or $\mathbb{R}^3$. Since $\mathbb{R}^2$ is much easier to draw, let's start there. Let's look at the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $\text{Span}\{\vec{v}\}$ is just the set of scalar multiples of $\vec{v}$, which really forms the line that contains the vector $\vec{v}$. See Figure 1.8.

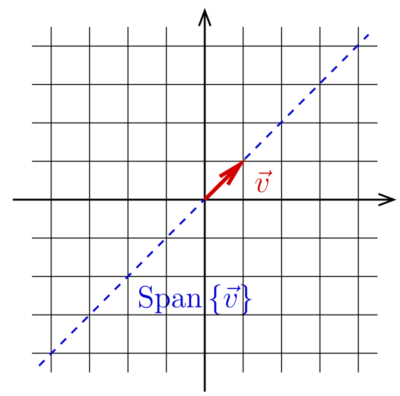


Figure 1.8: The single vector $\vec{v}$ is shown with a solid arrow line, and its span, $\text{Span}\{\vec{v}\}$ is shown with a dashed arrow line.

In the example above, note that we chose a nonzero vector $\vec{v}$. What's $\text{Span}\left\{\vec{0}\right\}$? Well, it's just $\vec{0}$ since any scalar multiple of $\vec{0}$ is just again $\vec{0}$. Now, let's see an example with two nonzero vectors in $\mathbb{R}^3$.

Example 1.3.4 Here are two vectors

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \text{ and } \vec{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^3$. Then

$$\begin{aligned} \text{Span}\{\vec{u}_1, \vec{u}_2\} &= \{a_1\vec{u}_1 + a_2\vec{u}_2 \colon a_i \in \mathbb{R}\} \\ &= \left\{ a_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} : a_i \in \mathbb{R} \right\} \end{aligned}$$

- First, note that a valid choice for a_2 is 0. Then $\text{Span}\{\vec{u}_1, \vec{u}_2\}$ contains any scalar multiple of $\vec{u}_1$, including $\vec{u}_1$ itself. Geometrically, this is the line containing $\vec{u}_1$. It's also $\text{Span}\{\vec{u}_1\}$! The same can be done for $\vec{u}_2$; refer to Figure 1.9.
- What should we expect for the span of two vectors then? Good question. Start with $\text{Span}\{\vec{u}_1\} = \{a_1\vec{u}_1 \colon a_1 \in \mathbb{R}\}$; to get $\text{Span}\{\vec{u}_1, \vec{u}_2\} = \{a_1\vec{u}_1 + a_2\vec{u}_2 \colon a_i \in \mathbb{R}\}$, we need to add any scalar multiple of $\vec{u}_2$ to any scalar multiple of $\vec{u}_1$. This means we're going to get $\text{Span}\{\vec{u}_2\}$ through any point of $\text{Span}\{\vec{u}_1\}$; in Figure 1.9, this is the red lines through any point on one of the blue lines, yielding the plane containing the two vectors $\vec{u}_1$ and $\vec{u}_2$.

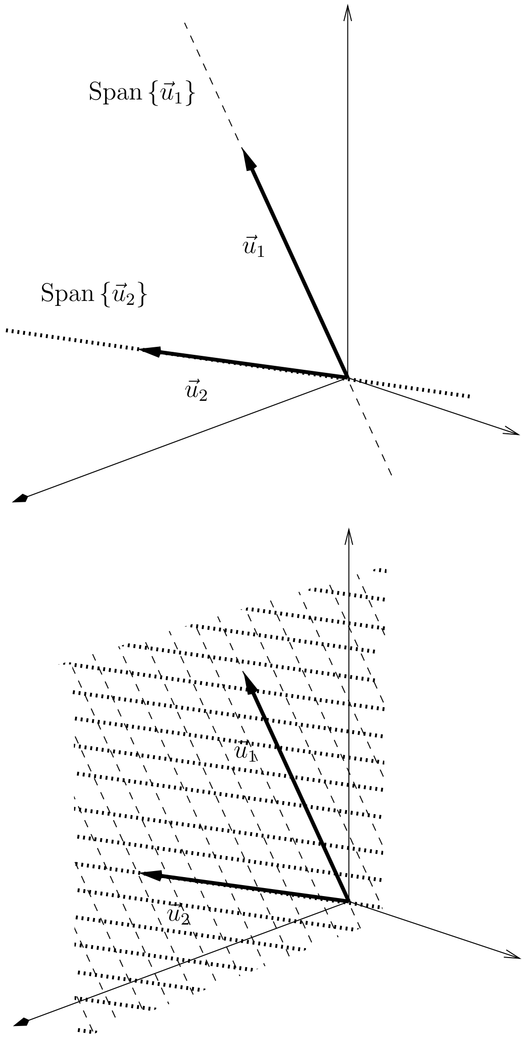


Figure 1.9: In the first image, Span $\{\vec{u}_1\}$ is the dashed line, and Span $\{\vec{u}_2\}$ is the dotted line; the second is Span $\{\vec{u}_1, \vec{u}_2\}$
.

Here's another fun question. Are either of the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

in Span $\{\vec{u}_1, \vec{u}_2\}$? Nope. Note that any linear combination of $\vec{u}_1$ and $\vec{u}_2$ will have a zero in the second component. Neither of the given vectors have zero in the second component, so neither is a linear combination of $\vec{u}_1$ and $\vec{u}_2$. Thus, neither is in Span $\{\vec{u}_1, \vec{u}_2\}$.

Is the span of two vectors in $\mathbb{R}^3$ always a plane? Again, good question. If one of your vectors is the zero vector, then you just get the span of the other vector, so the answer is firmly "no." Ok. Fine. What about the span of two *nonzero* vectors in $\mathbb{R}^3$? Is that always a plane?

Example 1.3.5 Here are two vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}.$$

in $\mathbb{R}^3$. Note that $\vec{v}_2$ is a scalar multiple of $\vec{v}_1$. Specifically, $2\vec{v}_1 = \vec{v}_2$. Watch what happens now to the span of these two vectors:

$$\begin{aligned}
\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} &= \{a_1\vec{v}_1 + a_2\vec{v}_2 : a_1, a_2 \in \mathbb{R}\} \\
&= \{a_1\vec{v}_1 + a_2(2\vec{v}_1) : a_1, a_2 \in \mathbb{R}\} \\
&= \{a_1\vec{v}_1 + 2a_2\vec{v}_1 : a_1, a_2 \in \mathbb{R}\} \\
&= \{(a_1 + 2a_2)\vec{v}_1 : a_1, a_2 \in \mathbb{R}\} \\
&= \{a\vec{v}_1 : a \in \mathbb{R}\} = \operatorname{Span}\{\vec{v}_1\}.
\end{aligned}$$

Here, we used the fact that *any* $a \in \mathbb{R}$ can be realized as $a_1 + 2a_2$ for $a_1, a_2 \in \mathbb{R}$. Note that this is equivalent to showing $\{a_1 + 2a_2 : a_1, a_2 \in \mathbb{R}\}$ is equal to $\mathbb{R}$.
What does this mean for the span? Since $\vec{v}_2$ is a scalar multiple of $\vec{v}_1$, there is redundancy in the span of the vectors. Thus, the span of these vectors will form a line, not a plane.

In the example above, the fact that one vector was a scalar multiple of the other gave us redundancy, so we were able to more efficiently write $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$ as $\operatorname{Span}\{\vec{v}_1\}$. We could also have written it as $\operatorname{Span}\{\vec{v}_2\}$. When can we not do this? We definitely can't remove all the vectors, so is there a condition that says you can't remove a vector?

Linear Independence

Yes! This one! This one! Notions of dependence and independence between vectors can be used to detect the kind redundancy (or the lack of it) we saw in the previous example.

Definition 1.3.3 *A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$ is said to be* ***linearly independent*** *if*

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0} \tag{1.10}$$

only when $a_1 = \cdots = a_n = 0$. The set $\{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$ is said to be ***linearly dependent*** *if there are scalars $a_1, \ldots, a_n \in \mathbb{R}$, not all 0, such that*

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}. \tag{1.11}$$

Before we explore how this affects the span of a set of vectors, let's spend some time getting comfortable with the definition.

Example 1.3.6 Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Is the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly independent or linearly dependent? To answer this, suppose there exist scalars $a, b, c \in \mathbb{R}$ such that $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{0}$. If we can find nonzero a, b, and c, then we know they are dependent. If we cannot, then they are independent. Let's try to find them!

$$a \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This leads us to the equations $2a + b - c = 0$, $b + c = 0$, and $a = 0$. These equations have the unique solution $a = b = c = 0$, so the set is linearly independent.

Example 1.3.7 Let's replace $\vec{v}_3$ above with a new vector we'll call $\vec{v}_4$ and consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{v}_4 = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}.$$

Now, is the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ linearly independent or linearly dependent? Suppose that $a, b, c \in \mathbb{R}$ are scalars such that $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_4 = \vec{0}$. That is,

$$a \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives us the equations $2a + b + 4c = 0$, $b + 2c = 0$, and $a + c = 0$. The latter two simplify to give $b = -2c$ and $a = -c$. Substituting these into the first equation gives us $0 = 0$, which is of course true but seemingly unhelpful. However, recall that if we get an equation like $0 = 0$, this often suggests that there are multiple solutions, and we can try to find one by choosing a value for one of the variables. If we choose $c = 1$, we will have $a = -1$ and $b = -2$. We can quickly check that this works to give us a solution to $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_4 = \vec{0}$ where the scalars are nonzero, and therefore, the set is linearly dependent. Note here that we could have made a different choice for c and found a different solution.

Exploration 23 Now it's your turn! Here are three vectors. Are they linearly independent or linearly dependent?

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Exploration 24 Let's explore the situation when we have a set with two vectors in it.

► Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now, determine whether the following sets are linearly independent or linearly dependent. Hint: exactly two of these sets are linearly dependent.

(a) $\{\vec{v}_1, \vec{v}_2\}$

(b) $\{\vec{v}_1, \vec{v}_3\}$

(c) $\{\vec{v}_1, \vec{v}_4\}$

► Let V be a vector space and $\vec{u}, \vec{v} \in V$. Is there an easy way to tell if $\{\vec{u}, \vec{v}\}$ is a linearly independent set? In other words, is there an advantage to only dealing with two vectors when determining linear independence?

► Above, the vector $\vec{v}_4$ is the zero vector in $\mathbb{R}^3$. Can the zero vector ever be included in a linearly independent set?

Exploration 25 Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

► We can write $\vec{v}_1$ as a linear combination of $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$. To do this, solve for $a, b, c \in \mathbb{R}$ so that

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

- ▶ Now, rearrange to see that $\vec{v}_1 - a\vec{v}_2 - b\vec{v}_3 - c\vec{v}_4 = \vec{0}$. What does this tell us about the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$?

Did you say the vectors are linearly dependent? That is correct!![18] This works in general and gives us another way to think about linear dependence... Now, let's prove it!

Theorem 1.3.1 *A set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ of two or more vectors, is linearly dependent if and only if one of the vectors is a linear combination of the other vectors.*

The "if and only if" bit means this is a biconditional statement. P if and only if Q means precisely the following two things: P implies Q *and* Q implies P.[19] This means that these conditions are logically equivalent. so we can use this as a test of dependence/independence. This is fairly common in mathematics; an "if and only if" is usually a vehicle for an alternative way of thinking about something.

PROOF. Suppose $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a linearly dependent set. Then there are weights $a_1, \ldots, a_p \in \mathbb{R}$ not all zero such that

$$a_1\vec{v}_1 + \cdots + a_p\vec{v}_p = \vec{0}.$$

We can assume without loss of generality[20] that it is the first weight that is nonzero, so $a_1 \neq 0$. Thus,

$$a_1\vec{v}_1 = -a_2\vec{v}_2 - \cdots - a_p\vec{v}_p.$$

Since $a_1 \neq 0$, we may multiply both sides of the equation by $1/a_1$, so

$$\vec{v}_1 = \frac{-a_2}{a_1}\vec{v}_2 + \cdots + \frac{-a_p}{a_1}\vec{v}_p.$$

Thus, v_1 is a linear combination of the other vectors.

Now suppose one of the vectors in the set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a linear combination of the other vectors. Again, we may assume without loss of generality that the guilty vector is the first one; that is, v_1 is a linear combination of the other vectors. Then

$$\vec{v}_1 = a_2\vec{v}_2 + \cdots + a_n\vec{v}_p.$$

We may rewrite this as

$$\vec{v}_1 - a_2\vec{v}_2 - \cdots - a_n\vec{v}_p = \vec{0}.$$

The weight on $\vec{v}_1$ is not zero, so by definition, $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a linearly dependent set. □

With this theorem in hand, we are ready now to talk about how linear independence affects our computations of the span of a set of vectors.

18: Confetti!! Picture lots of colorful confetti!

19: Biconditional statements are great. It's a logical two-for-one deal. In one statement, you get two logical implications!

20: "Without loss of generality" (sometimes abbreviated WLOG) means we are making a new, specific assumption that does effect the generality of the proof process. It often involves simply reordering or relabelling things.

Good point, RIcky, but did you notice why we were able to assume one of our scalars was nonzero?

More Span

Let's start small, but not *too* small. Three vectors should be enough. Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly dependent set of vectors in a vector space V. Then by Theorem 1.3.1, we know one of these vectors can be written as a linear combination of the others. For our purposes, it's okay to suppose that $\vec{v}_1$ is a linear combination of $\vec{v}_2$ and $\vec{v}_3$. That is,

$$\vec{v}_1 = b\vec{v}_2 + c\vec{v}_3$$

for some $b, c \in \mathbb{R}$.[21] Now, let's see how this connects to span.

21: It'll be clear in just a second why we skipped a...

$$\begin{aligned}
\operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} &= \{a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 \colon a_1, a_2, a_3 \in \mathbb{R}\} \\
&= \{a_1(b\vec{v}_2 + c\vec{v}_3) + a_2\vec{v}_2 + a_3\vec{v}_3 \colon a_1, a_2, a_3 \in \mathbb{R}\} \\
&= \{(a_1 b + a_2)\vec{v}_2 + (a_1 c + a_3)\vec{v}_3 \colon a_1, a_2, a_3 \in \mathbb{R}\} \\
&= \{d_1\vec{v}_2 + d_2\vec{v}_3 \colon d_1, d_2 \in \mathbb{R}\} \\
&= \operatorname{Span}\{\vec{v}_2, \vec{v}_3\}.
\end{aligned}$$

Here, we use the fact that for any $a_1, b, c \in \mathbb{R}$, the sets $\{a_1 b + a_2 \colon a_2 \in \mathbb{R}\}$ and $\{a_1 c + a_3 \colon a_3 \in \mathbb{R}\}$ are both equal to $\mathbb{R}$.

Let's talk a bit about what we just did. We started with the span of three vectors, and we were able to reduce to a set of two vectors that has the same span as the original set. This is something we can do in general.

Theorem 1.3.2 *If S is a linearly dependent set of vectors in some vector space V, then there is some vector $\vec{v}$ in S such that* $\operatorname{Span}\{S\} = \operatorname{Span}\{S \backslash \{\vec{v}\}\}$.

The proof of this theorem is very similar to the discussion preceding it, so we'll leave the details to the exercises for now.

Exploration 26 Consider the following vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

- ▶ The set $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent. Find scalars $a, b, c, d \in \mathbb{R}$, not all 0, so that $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 + d\vec{v}_4 = \vec{0}$. Hint: There isn't a unique answer. You'll need to make a choice for one of the variables.

- ▶ From the equation above, it should be possible to identity a vector in S that could be removed without changing the span of the set of vectors. Actually, there are three vectors that could be chosen as the one removed! The only one that cannot be removed is $\vec{v}_4$. Why is this?

Example 1.3.8 Let's see how this looks in $\mathbb{P}_2$. Consider the vectors

$$\vec{p}_1 = 1 + x^2, \quad \vec{p}_2 = x, \quad \vec{p}_3 = 1 + 3x + x^2.$$

The set $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ is linearly dependent. To see this, we will find scalars $a, b, c \in \mathbb{R}$, not all zero, so that $a\vec{p}_1 + b\vec{p}_2 + c\vec{p}_3 = 0$. That is,

$$a(1 + x^2) + bx + c(1 + 3x + x^2) = 0.$$

Rearranging, we have

$$(a + c) + (b + 3c)x + (a + c)x^2 = 0.$$

Therefore $a + c = 0$ and $b + 3c = 0$. This gives us $a = -c$ and $b = -3c$. If we choose $c = 1$, we can see that $\vec{p}_3 = \vec{p}_1 + 3\vec{p}_2$. Now, we have realized $\vec{p}_3$ as a linear combination of $\vec{p}_1$ and $\vec{p}_2$, but we could rearrange this equation to realize $\vec{p}_1$ as a linear combination of $\vec{p}_2$ and $\vec{p}_3$ or $\vec{p}_2$ as a linear combination of $\vec{p}_1$ and $\vec{p}_3$. All of this tells us that $\text{Span}\,\{\vec{p}_1, \vec{p}_2, \vec{p}_3\} = \text{Span}\,\{\vec{p}_1, \vec{p}_2\} = \text{Span}\,\{\vec{p}_2, \vec{p}_3\} = \text{Span}\,\{\vec{p}_1, \vec{p}_3\}$.

Let's talk about what Theorem 1.3.1 tells us about the span of a linearly independent set of vectors. Suppose S is a linearly independent set of vectors and $\vec{v} \in S$. By Theorem 1.3.1, we know $\vec{v}$ is *not* a linear combination of other vectors in S, since if it were, the set would be linearly dependent. Thus, if we were to compare $\text{Span}\,\{S\}$ with $\text{Span}\,\{S \backslash \{\vec{v}\}\}$, these would be different! In particular, $\vec{v} \in \text{Span}\,\{S\}$, but $\vec{v} \notin \text{Span}\,\{S \backslash \{\vec{v}\}\}$.

Section Highlights

- A *linear combination* of a set of vectors is a sum of scalar multiplied vectors from the set. See Definition 1.3.1.
- The set of all possible linear combinations of a set of vectors is the *span* of that set of vectors. See Definition 1.3.2.
- A system of equations can be set up and solved to determine whether a set of vectors is linearly independent. See Example 1.3.6 and Explorations 23 and 24.
- A set of vectors can be shown to be linearly dependent by finding one vector as a linear combination of the others. See Example 1.3.6 and Exploration 25.
- A system of equations can be set up and solved to determine whether the vector $\vec{v}$ in $\text{Span}\,\{\vec{v}_1, \ldots, \vec{v}_n\}$. See Exploration 22.
- A linearly dependent set of vectors contains vectors that can be removed without altering the span of that set of vectors. The ones that can be removed are determined by dependence relations. See Exploration 26.

Exercises for Section 1.3

1.3.1. Some linear combinations are given, but they are missing some information. Fill in the missing information.

(a) In $\mathbb{R}^2$,
$$3\begin{bmatrix} \underline{\qquad} \\ -1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}.$$

(b) In $\mathbb{R}^2$,
$$3\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \underline{\qquad}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}.$$

(c) In $\mathbb{R}^3$,
$$2\begin{bmatrix} 1 \\ \underline{\qquad} \\ -2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 2 \\ \underline{\qquad} \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix}.$$

(d) In $\mathbb{P}_2$,
$$3(1+x) + 2(1-x-x^2) - (\underline{\qquad}) = 5 - 2x - 3x^2.$$

1.3.2. Find the linear combination $5\vec{x} + 3\vec{y} - 2\vec{z}$ for the vectors $\vec{x}$,$\vec{y}$, and $\vec{z}$ given below.

(a) $\vec{x} = 3x + 2x^2$, $\vec{y} = 1 + x^2$, $\vec{z} = 3$ in $\mathbb{P}_2$

(b) $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ in $\mathbb{R}^2$

(c) $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^3$

1.3.3. Determine whether $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is in each span below:

(a) $\operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$

(b) $\operatorname{Span}\left\{\begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}\right\}$

(c) $\operatorname{Span}\left\{\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}\right\}$

(d) $\operatorname{Span}\left\{\begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right\}$

(e) $\operatorname{Span}\left\{\begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right\}$

(f) $\operatorname{Span}\left\{\begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

(g) $\operatorname{Span}\left\{\begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

(h) $\operatorname{Span}\left\{\begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

1.3.4. Consider the set $\mathrm{Span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}0\\0\\2\end{bmatrix}\right\}$. Which of the vectors below are in this set?

(a) $\begin{bmatrix}1\\-1\\2\end{bmatrix}$

(b) $\begin{bmatrix}1\\-1\\2\end{bmatrix} + \begin{bmatrix}0\\0\\4\end{bmatrix}$

(c) $\begin{bmatrix}3\\3\\0\end{bmatrix}$

(d) $\begin{bmatrix}3\\-3\\0\end{bmatrix}$

(e) $\begin{bmatrix}-2\\0\\2\end{bmatrix}$

(f) $\begin{bmatrix}-2\\2\\2\end{bmatrix}$

(g) $\begin{bmatrix}-2\\2\\a\end{bmatrix}$ for any $a \in \mathbb{R}$

(h) $\begin{bmatrix}b\\-b\\a\end{bmatrix}$ for any $a, b \in \mathbb{R}$

1.3.5. Let $\vec{u}$ and $\vec{v}$ be vectors in some vector space V. Explain why $\vec{u}$ and $\vec{v}$ are both vectors in $\mathrm{Span}\{\vec{u}, \vec{v}\}$.

1.3.6. Suppose $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a subset of vectors from a vector space V.

(a) Suppose $\vec{u} \in \mathrm{Span}\{S\}$. Explain how this implies $\mathrm{Span}\{\vec{u}\} \subseteq \mathrm{Span}\{S\}$.

(b) Suppose $\vec{u}, \vec{v} \in \mathrm{Span}\{S\}$. Explain how this implies $\mathrm{Span}\{\vec{u}, \vec{v}\} \subseteq \mathrm{Span}\{S\}$.

1.3.7. Let $\vec{u}$ and $\vec{v}$ be vectors in some vector space V.

(a) Explain why $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are in $\mathrm{Span}\{\vec{u}, \vec{v}\}$.

(b) Show that $\vec{u}$ and $\vec{v}$ are both vectors in $\mathrm{Span}\{\vec{u} + \vec{v}, \vec{u} - \vec{v}\}$.

(c) What can you conclude then about $\mathrm{Span}\{\vec{u}, \vec{v}\}$ and $\mathrm{Span}\{\vec{u} + \vec{v}, \vec{u} - \vec{v}\}$?

1.3.8. Let H be the set of all vectors in $\mathbb{R}^3$ of the form

$$\left\{\begin{bmatrix}-a_1 - 3a_2\\4a_1\\a_1 - 2a_2\end{bmatrix} : a_1, a_2 \in \mathbb{R}\right\}.$$

Rewrite this as a linear combination of two vectors with coefficients a_1 and a_2. Use this to find two vectors $\vec{v}_1$ and $\vec{v}_2$ such that $H = \mathrm{Span}\{\vec{v}_1, \vec{v}_2\}$.

1.3.9. Let K be the set of all vectors in $\mathbb{R}^3$ of the form

$$\left\{\begin{bmatrix}a_1 + a_2 + a_3\\4a_1\\a_1 - 2a_2 - 2a_3\end{bmatrix} : a_1, a_2, a_3 \in \mathbb{R}\right\}.$$

Rewrite this as a linear combination of three vectors with coefficients a_1, a_2 and a_3. Then find two vectors $\vec{v}_1$ and $\vec{v}_2$ such that $K = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

1.3.10. Let J be the set of all vectors in $\mathbb{R}^3$ of the form

$$\left\{ \begin{bmatrix} -a_1 - 3a_2 \\ 4a_3 \\ a_1 - 2a_2 \end{bmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

Find vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ such that $J = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

1.3.11. Determine whether the sets below are linearly independent or linearly dependent.

(a) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

(f) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

(g) $\{1 + x, x + x^2, x^2\}$

(h) $\{2 + 2x, 1 - x + x^2, 4 + 2x^2\}$

(i) $\{1 + x + x^2, x + x^2, x^2\}$

(j) $\{1 + x, 2x, 1 - x^2, 1 + x^2\}$

(k) $\{1 + x, 2x + x^2, x^2, 1\}$

1.3.12. Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set in a vector space V. Which of the following sets must also be linearly independent? Give a complete argument to support your conclusion.

(a) $\{\vec{v}_2, \vec{v}_3, \vec{v}_1\}$

(b) $\{\vec{v}_1, \vec{v}_3\}$

(c) $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_2\}$

(d) $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2\}$

(e) $\{\vec{v}_1 - \vec{v}_2, \vec{v}_2 - \vec{v}_3, \vec{v}_1 - \vec{v}_3\}$

(f) $\{\vec{v}_2 - \vec{v}_3, \vec{v}_1 - \vec{v}_2\}$

1.3.13. The set of vectors below is linearly dependent. However, it contains many linearly independent subsets. Find all nonempty linearly independent subsets. There should be 11.

$$\left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Circle the ones that have the same span as the original set. (Hint: They should all be the same size.)

1.3.14. For each set S below, reduce the set to a linearly independent one that has the same span.

(a) $S = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \right\}$

(b) $S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$

(c) $S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

(d) $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

In Problems 1.3.15, 1.3.16, and 1.3.17 below, determine what the span of the vectors looks like geometrically. Explicitly, find whether it is a point, a line, a plane, or all of $\mathbb{R}^3$.

1.3.15. Let $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Determine what $\operatorname{Span}\{\vec{v}, \vec{u}\}$ looks like geometrically.

1.3.16. Let $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{z} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$. Determine what $\operatorname{Span}\{\vec{w}, \vec{z}\}$ looks like geometrically.

1.3.17. Let $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$. Determine what $\operatorname{Span}\{\vec{u}, \vec{w}, \vec{z}\}$ looks like geometrically.

1.3.18. Consider the following vectors in $\mathbb{P}_2$:

$$\begin{aligned} \vec{p}_1 &= \pi + \pi^2 x + \pi^3 x^2 \\ \vec{p}_2 &= \pi - x^2 \\ \vec{p}_3 &= \pi^2 - \pi x^2. \end{aligned}$$

Show that $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ is a linearly dependent set.

1.3.19. Let

$$\begin{aligned} \vec{p}_1 &= x + x^3 \\ \vec{p}_2 &= 1 + x^2 \\ \vec{p}_3 &= 1 + x \end{aligned}$$

be vectors in $\mathbb{P}_4$. Describe $\operatorname{Span}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ algebraically (in set notation) and in words.

1.3.20. Let $\vec{w}$ be an arbitrary vector in $\mathbb{R}^2$. Then $\vec{w} = \begin{bmatrix} a \\ b \end{bmatrix}$ for some $a, b \in \mathbb{R}$. Find a way to write $\vec{w}$ as a linear combination of the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Explain why $\mathbb{R}^2 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

1.3.21. Show that $\mathbb{R}^2 = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \end{bmatrix} \right\}$. (Hint: Follow the technique of the previous exercise.)

1.3.22. Let $\vec{w}$ be an arbitrary vector in $\mathbb{R}^3$. Then $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ for some $a, b, c \in \mathbb{R}$. Find a way to write $\vec{w}$ as a linear combination of the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Conclude that $\mathbb{R}^3 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

1.3.23. Let $H = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ and $J = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. Show that $H = J$.

1.3.24. Let $H = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ and $J = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. Show that $H \neq J$.

1.4 Subspaces

Vector spaces are sometimes too big. Oftentimes, the collection of vectors you actually care about is only a small piece of the vector space with which you are stuck. In some cases, a collection of vectors in a vector space turns out to be a vector space itself, and this can be very convenient. Who needs extraneous vectors just hanging around everywhere? You know what's *not* convenient? Verifying all ten axioms to show that the smaller set of vectors is a vector space.[22]

22: I don't like verifying all ten axioms. It takes too long.

Verifying axioms is a wonderful exercise that helps build understanding.

Ready for some good news?

Definition 1.4.1 *A* ***subspace*** *of a vector space V is a subset H of V with the following three properties:*

- *The zero vector is in H.*
- *(Closure under vector addition) For any $\vec{v}$ and $\vec{u}$ in H, the vector $\vec{v}+\vec{u}$ is also in H.*
- *(Closure under scalar multiplication) For any $\vec{v}$ in H and any a in $\mathbb{R}$, the vector $a\vec{v}$ is also in H.*

Note that vector additional and scalar multiplication for H are the same as for V.

It only makes sense that a sub*space* is a vector *space*. Thus, if you want to show a set is a vector space and it's actually a subset of some vector space, then you can just use this definition instead to show it is a subspace, right? That is *definitely* better than verifying all those axioms in the definition of a vector space. It really seems too good to be true, though, right? Well, we were due for some good news; it is true:

Theorem 1.4.1 *A subspace of a vector space is itself a vector space.*

PROOF. The three axioms for a subspace take care of three of our vector space axioms. Then the addition and scalar multiplication are the same as for the ambient vector space, and all the other properties are inherited. □

"Inherited" is an interesting math word, and it works a lot like one might expect. If H is a subset of a vector space V with the same operations as V, then properties of V are oftentimes also passed on to H, like from parent to offspring. For example, if V has properties such as commutativity and associativity for some operations, then provided H is closed under the operations, they still hold for H because the operations on H are the same as those on V.

Example 1.4.1 The set

$$H = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 \colon v_1 + v_2 + v_3 = 0 \right\}.$$

It turns out H is a subspace of $\mathbb{R}^3$, so H is a vector space. You can check this later, but for now, just trust us that it's true. Then

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \in H \qquad \text{and} \qquad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \notin H.$$

This is because H is the set of vectors in $\mathbb{R}^3$ whose components sum to zero, and we have that $1 + (-1) + 0 = 0$ but $1 + (-1) + 1 \neq 0$. Thus, we can also check, more generally, that

$$H_0 = \left\{ \begin{bmatrix} a \\ -a \\ 0 \end{bmatrix} \in \mathbb{R}^3 \colon a \in \mathbb{R} \right\} \subseteq H, \quad \text{and}$$

$$\left\{ \begin{bmatrix} a \\ -a \\ 1 \end{bmatrix} \in \mathbb{R}^3 \colon a \in \mathbb{R} \right\} \not\subseteq H.$$

Since H_0 is a subset of the vector space H with the same operations as H, we would now only need to check the three axioms from the definition of subspace to verify that H_0 is also a vector space. It inherits the remaining vector space properties from H!

Now, before doing anything interesting, we should note that any vector space V has two uninteresting subspaces. It's not hard to check[23] that V is a subspace of itself. How does one check? Just verify each of the axioms[24] in Definition 1.4.1 for V. The other uninteresting subspace is the zero vector. Yep, this is the only finite set we'll get in this course that turns out to be a vector space. It's also a subspace of *any* vector space. You should check that, too, with Definition 1.4.1. A reasonable response to the question "How many subspaces does vector space V have?" is "At least two."[25] It turns out, however, that there are often[26] many more.

23: You should check.

24: You really should do this, too.

25: Unless $V = \{\vec{0}\}$.

26: When $V = \mathbb{R}$, there are exactly these two. Maybe we should show this, though. Exercise!

You mean like running laps?

No, at the end of the section. A homework exercise!

Example 1.4.2 Let's start off with the familiar vector space $\mathbb{P}_2$. Consider all polynomials of the form $a + bx$ for any $a, b \in \mathbb{R}$. Let's call this set H for convenience. H is a subset of $\mathbb{P}_2$ since $\mathbb{P}_2$ is all polynomials of the form $a + bx + cx^2$ for $a, b, c \in \mathbb{R}$, and we could let $c = 0$. Also, it is a subspace of $\mathbb{P}_2$:

- H contains the zero vector 0 since we could let $a = b = 0$.
- Let $a_1 + b_1x$ and $a_2 + b_2x$ be any polynomials in H. Then

  $$(a_1 + b_1x) + (a_2 + b_2x) = (a_1 + a_2) + (b_1 + b_2)x \in H$$

  since $(a_1 + a_2)$ and $(b_1 + b_2)$ are again in $\mathbb{R}$. Thus, H is closed under addition.
- Let $k \in \mathbb{R}$. Then $k(a + bx) = ka + kbx \in H$ since $ka, kb \in \mathbb{R}$, so H is closed under scalar multiplication.

Exploration 27 Now it's your turn! Let $J = \{a + cx^2 \colon a, c \in \mathbb{R}\}$. We see that these will all be either degree 2 or 0 polynomials, so $J \subseteq \mathbb{P}_2$. Show that all 3 of the axioms for a subspace are also satisfied.

- What should a and c be to see that J contains the zero vector?

- Let $a_1 + c_1x^2$ and $a_2 + c_2x^2$ be any polynomials in J. Show the sum of these two polynomials is still in J.

- Let $k \in \mathbb{R}$. Show that $k(a + cx^2) \in J$ for any $a + cx^2 \in J$.

Let's consider the two subspaces H and J of $\mathbb{P}_2$ for a bit. How many of you noticed that H is really $\mathbb{P}_1$? How often does something like this happen? Well, for $k \leq n$, we can always show that $\mathbb{P}_k$ is a subspace of $\mathbb{P}_n$.[27] However, these are not all of the subspaces of $\mathbb{P}_n$! The subspace J from the exploration above is not $\mathbb{P}_k$ for any integer k, and you'll see several other examples of subspaces of $\mathbb{P}_n$ in the exercises.

27: Expect to see this as an exercise!

We should really see an example of something that is *not* a subspace, too.

Example 1.4.3 Let's consider the set $K = \left\{ \begin{bmatrix} a \\ a+2 \end{bmatrix} : a \in \mathbb{R} \right\}$. Since for any $a \in \mathbb{R}$ we know $a + 2 \in \mathbb{R}$ as well, this is a subset of $\mathbb{R}^2$. However, it is not a subspace.

- First of all, this set does not contain the zero vector of $\mathbb{R}^2$. To see this, suppose for some $a \in \mathbb{R}^2$
$$\begin{bmatrix} a \\ a+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
Then $a = 0$ and $a + 2 = 0$. Both of these can't be true. At this point, we can definitively state it is not a subspace, but let's see what happens with the other axioms.[28]
- We can check that closure under vector addition also fails. To show this, it is enough to check that it fails for specific vectors rather than general ones. Let's consider
$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$
These are in K since one corresponds to $a = 0$ and the other to $a = 1$, but the sum
$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$
is not in K since we cannot have $a = 1$ and $a + 2 = 5$ both be true.
- Lastly, we can show that K is not closed under scalar multiplication. Again, to show a property fails, it is enough to give a specific case where it fails. We can choose
$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} \in K$$

28: We only need to fail one axiom?

Yep! Once you fail one axiom, you can't satisfy all axioms of a definition.

Neat.

and the scalar $5 \in \mathbb{R}$. Then

$$5\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

is not in K since $a = 0$ and $a + 2 = 10$ cannot both be true.

Exploration 28 The following subset L is not a subspace of $\mathbb{P}_1$. Which of the three axioms fail?[29]

$$L = \{mx + b \colon m, b > 0\}$$

29: Remember that we only need to show *one* axiom fails to show L is not a subspace!

Now that we've seen some examples, let's explore in detail what happens in $\mathbb{R}^n$.

Subspaces of $\mathbb{R}^n$

If you've been keeping up with the sidenotes, then we've already mentioned the subspaces of $\mathbb{R}$. We've claimed that the only subspaces of $\mathbb{R}$ are the "uninteresting" ones, $\{\vec{0}\}$ and $\mathbb{R}$ itself. However, we also saw in the exercises of Section 1.1 that the interval $(0, \infty)$ is a vector space when addition is given by $a \boxplus b = ab$ and scalar multiplication is $ka = a^k$ for $k \in \mathbb{R}$ and $a, b \in (0, \infty)$.[30] Why is this not a subspace of $\mathbb{R}$? Well, the operations of addition and scalar multiplication *must* be the exact same ones from the larger vector space in order for a subset to be a subspace. Thus, Theorem 1.4.1 lets us say "A subspace of a vector space V is a subset that is also a vector space" but only with the caveat that the operations are the same as those for V.

30: Unless you skipped that one like I did.

Now let's focus on subspaces of $\mathbb{R}^3$. This will give us some nice geometric intuition to go along with the algebraic computations.

Example 1.4.4 First we'll show that

$$H = \left\{ \begin{bmatrix} 3t \\ 0 \\ -5t \end{bmatrix} : t \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^3$. Note that the vectors here have three real components, so H is a subset of $\mathbb{R}^3$. Next we need to verify the three axioms in Definition 1.4.1. The first requires that $\vec{0} \in H$. Indeed, when $t = 0$, we see that $\vec{0} \in H$. That one was easy (this time). It only remains to show H is closed under vector addition and scalar multiplication. As we did in Section 1.1, we need to have general elements of the set H to satisfy Definition 1.4.1, which needs to hold "for any" vectors in H. Let $x, y \in \mathbb{R}$, so

$$\vec{x} = \begin{bmatrix} 3x \\ 0 \\ -5x \end{bmatrix}, \vec{y} = \begin{bmatrix} 3y \\ 0 \\ -5y \end{bmatrix} \in H$$

are general elements. Let's see what happens when we add them:

$$\vec{x}+\vec{y}=\begin{bmatrix} 3x \\ 0 \\ -5x \end{bmatrix}+\begin{bmatrix} 3y \\ 0 \\ -5y \end{bmatrix}=\begin{bmatrix} 3x+3y \\ 0+0 \\ -5x-5y \end{bmatrix}=\begin{bmatrix} 3(x+y) \\ 0 \\ -5(x+y) \end{bmatrix}.$$

This last vector is an element of H since $x+y \in \mathbb{R}$. Thus, the sum of any two vectors in H is still in H, so H is closed under vector addition. To check closure under scalar multiplication, let $a \in \mathbb{R}$. Then

$$a\vec{x}=a\begin{bmatrix} 3x \\ 0 \\ -5x \end{bmatrix}=\begin{bmatrix} a(3x) \\ a(0) \\ a(-5x) \end{bmatrix}=\begin{bmatrix} 3(ax) \\ 0 \\ -5(ax) \end{bmatrix}.$$

Since $ax \in \mathbb{R}$, this last vector is also in H. Thus, H is closed under scalar multiplication. It follows that H is a subspace of $\mathbb{R}^3$.

Before moving on, let's think a bit about the geometry of this subspace in $\mathbb{R}^3$. Sketch this set of vectors in $\mathbb{R}^3$. Observe that H is actually a line through the origin in the direction of the vector

$$\begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix}.$$

What if we considered a line that does not go through the origin? Can that be a subspace of $\mathbb{R}^3$? This would be a fun exploration, but let's do this one together.

Example 1.4.5 Here's another subset of $\mathbb{R}^3$.

$$H=\left\{\begin{bmatrix} 3t \\ 1 \\ -5t \end{bmatrix} : t \in \mathbb{R}\right\}$$

The only difference between this set and the set in Example 1.4.4 is that these vectors have 1 as their second component, rather than 0. Geometrically, this is equivalent to a line in $\mathbb{R}^3$ that does not go through the origin, like we were just speculating about. It turns out this is not a subspace since it does not contain the zero vector from $\mathbb{R}^3$.

Example 1.4.6 At the risk of developing a theme, here's another subset of $\mathbb{R}^3$.

$$H=\left\{\begin{bmatrix} 0 \\ y \\ z \end{bmatrix} : y, z \in \mathbb{R}\right\}$$

Spoiler: this one's a subspace. Checking is very similar to Example 1.4.4, but you should check it anyway. Let's consider

$$\left\{\begin{bmatrix} y \\ z \end{bmatrix} : y, z \in \mathbb{R}\right\}=\mathbb{R}^2.$$

Well, this seems like it's the same as H, in which case we'd have $\mathbb{R}^2$ as a subspace of $\mathbb{R}^3$ (just like $\mathbb{P}_2$ is a subspace of $\mathbb{P}_3$). Seems okay, right? . . . Right?

No! Absolutely not! While they have many things in common and "look alike," the definition of a subspace H of the vector space V requires first for H to be a *subset* of V. Vectors in $\mathbb{R}^2$ are not vectors in $\mathbb{R}^3$; they are two different mathematical objects. While H resembles $\mathbb{R}^2$ in many ways, $\mathbb{R}^2$ *is not a subspace of* $\mathbb{R}^3$ because it's not even a subset of $\mathbb{R}^3$. You may find this annoying. Indeed, the vectors

$$\begin{bmatrix} 0 \\ y \\ z \end{bmatrix} \text{ and } \begin{bmatrix} y \\ z \end{bmatrix}$$

carry the same information. They are alike in many ways, but they are, strictly speaking, different mathematical objects. While this may seem obnoxious, this degree of rigor in definitions is necessary for consistently functional (and understandable) mathematics. Later we shall investigate what we can gain by defining and understanding what it means for vectors or vector spaces to "look alike."

Exploration 29 True or false? $\mathbb{R}$ is a subspace of $\mathbb{R}^3$.[31]

31: FALSE!!

What's different about the definition of $\mathbb{P}_n$ that allows $\mathbb{P}_k$ to be a subspace of $\mathbb{P}_n$ for $k \leq n$?

We've now seen examples of several subspaces of $\mathbb{R}^3$ and some subsets that are not subspaces in $\mathbb{R}^3$. What do subspaces of $\mathbb{R}^3$ look like geometrically?

- Well, we have $\{\vec{0}\}$ as a subspace. This is just the point at the origin.
- From Examples 1.4.4 and 1.4.5 we saw a subspace that was a line through the origin and that lines not traveling through the origin are not subspaces. Really, we saw one example of this, but they all fail for the exact same reason. Any line not through the origin does not contain $\vec{0}$.
- Then in Example 1.4.6, we saw that a plane could be a subspace. Are all planes subspaces, though? Nope. Just like lines must go through the origin, planes that are subspaces must also go through the origin in order to contain the zero vector.
- Lastly, we know that $\mathbb{R}^3$ is a subspace of itself, and we all know what $\mathbb{R}^3$ looks like, right?[32]

32: If you said a giant purple gorilla, you are incorrect.

We've discussed subspaces of $\mathbb{R}^n$ for $n = 1$ and $n = 3$. Note that we skipped over $\mathbb{R}^2$. That's because it will make a glorious exercise for you! We also have not addressed subspaces for $n > 3$. That's because once we leave the comfortable 3-dimensional world we live in, we lose our geometric tools and must rely upon just algebra. With that in mind, we will now turn to more algebraic tools.

Spans as Subspaces

Suppose you were given a set and you suspected it was a vector space. Initially, there were roughly ten axioms you had to verify before you could declare your set a vector space. At the beginning of this section, we cut that list down to three things if your set was already contained in a vector space. What's better than checking three things? One. Checking just one thing is better.[33]

33:  Wait. What's the catch? There has to be a catch. This feels like a trap!

Theorem 1.4.2 *Let $\vec{v}_1, \ldots, \vec{v}_n$ be vectors in a vector space V. Then* $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ *is a subspace of V. Note that* $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ *can be referred to as* ***the subspace spanned by*** $\vec{v}_1, \ldots, \vec{v}_n$.

PROOF. Recall that

$$\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_n\} = \{a_1\vec{v}_1 + \cdots + a_n\vec{v}_n : a_i \in \mathbb{R} \text{ for } i = 1, \ldots, n\}.$$

Since the a_i's can be any real number, we observe that when $a_1 = \cdots = a_n = 0$, we have

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = 0\vec{v}_1 + \cdots + 0\vec{v}_n = \vec{0},$$

so $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ contains the zero vector. Now we need general vectors in $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$:

$$\begin{aligned} \vec{x} &= a_1\vec{v}_1 + \cdots + a_n\vec{v}_n \\ \vec{y} &= b_1\vec{v}_1 + \cdots + b_n\vec{v}_n. \end{aligned}$$

To see closure under vector addition, we add $\vec{x}$ and $\vec{y}$:

$$\begin{aligned} \vec{x} + \vec{y} &= (a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) + (b_1\vec{v}_1 + \cdots + b_n\vec{v}_n) \\ &= (a_1\vec{v}_1 + b_1\vec{v}_1) + \cdots + (a_n\vec{v}_n + b_n\vec{v}_n) \\ &= (a_1 + b_1)\vec{v}_1 + \cdots + (a_n + b_n)\vec{v}_n, \end{aligned}$$

noting that the only reason we can pull off all that algebraic manipulation (associativity and commutivity of vector addition and distributivity for scalar multiplication) is because these vectors are all part of a vector space V already. The last line of this equation is a vector in $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ since $a_i + b_i \in \mathbb{R}$, so $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ is closed under vector addition. To see closure under scalar multiplication, we multiply $\vec{x}$ by $c \in \mathbb{R}$:

$$\begin{aligned} c\vec{x} &= c(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) \\ &= ca_1\vec{v}_1 + \cdots + ca_n\vec{v}_n \\ &= (ca_1)\vec{v}_1 + \cdots + (ca_n)\vec{v}_n. \end{aligned}$$

Again, the last line of this equation is a vector in $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ since $ca_i \in \mathbb{R}$, so $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ is closed under scalar multiplication. Behold! $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a subspace of V! $\square$

How does one use this theorem? If you have a set of vectors in a vector space V and want to show they form a subspace, all you have to do is show your set is the span of some set of vectors. Just show that, and you're done. Pretty great, right?

Example 1.4.7 Let

$$H = \left\{ \begin{bmatrix} a-2b \\ b-a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\} \subset \mathbb{R}^4.$$

You could show this is a subspace using the definition of a subspace, or you could show it's a subspace by showing it's the span of some set of vectors. Let's do the latter. Note that for any $a, b \in \mathbb{R}$,

$$\begin{bmatrix} a-2b \\ b-a \\ a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} -2b \\ b \\ 0 \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Neat, eh? Here we've *undone* vector addition and scalar multiplication, but look what we can do now:

$$\begin{aligned} H &= \left\{ \begin{bmatrix} a-2b \\ b-a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\} \\ &= \left\{ a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\} \\ &= \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

This tells us H is the span of two vectors in $\mathbb{R}^4$. Now it follows from Theorem 1.4.2 that H is a subspace of $\mathbb{R}^4$.

Exploration 30 Let

$$J = \left\{ \begin{bmatrix} a+b \\ a+b+c \\ a+b \end{bmatrix} : a, b, c \in \mathbb{R} \right\} \subset \mathbb{R}^3.$$

Find a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ such that $J = \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$.

We now have a theorem that says that the span of a set of vectors in a vector space V must be a subspace of V, which is neat. What about the other way around? Is every subspace of V a span of some set of vectors in V? Actually, this turns out to be true! It's very exciting. Unfortunately, the proof of this fact requires a bit more than the scope of this text, so this is really all we'll say about it.

Intersections and Sums of Subspaces

Perhaps you have two subspaces of a particular vector space V that you are interested in. A natural question would perhaps be "How are they related to one another?" or maybe instead "How could you combine these subspaces?" The first of these two questions leads us to the idea of the intersection of two subspaces. First, let's be sure we know what an intersection is.

Definition 1.4.2 *The* intersection *of two sets A and B is all of the elements that are in both A and B. We denote this intersection as $A \cap B$.*

For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$, then the intersection of A and B is $\{2, 4\}$.

Now, what is the intersection of two subspaces of a vector space V? Why, a subspace of V!

Theorem 1.4.3 *Let V be a vector space over $\mathbb{R}$ with subspaces U and W. Then the intersection $U \cap W$ is also a subspace of V.*

Exploration 31 Let's go through this proof together.

- We need to argue that $\vec{0}$ is in $U \cap W$. Thus, we need $\vec{0}$ to be in *both* U and W. Why is this true?

- Now, let $\vec{x}$ and $\vec{y}$ be in $U \cap W$. Why is $\vec{x} + \vec{y}$ in U? Why is it in W?

  Since $\vec{x} + \vec{y}$ is in both U and W, it must be in $U \cap W$. Thus the intersection is closed under addition.

- Lastly, suppose $k \in \mathbb{R}$ and $\vec{x}$ is again in $U \cap W$. Follow the logic above to show $k\vec{x}$ must be in $U \cap W$.

Exploration 32 Let's look at an explicit example of this. The following are all subspaces of $\mathbb{R}^3$.

$$U = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad V = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Let's find the intersections of these subspaces.

- $U \cap V$: A vector that is in both U and V will satisfy the following equation

$$a\begin{bmatrix}1\\1\\0\end{bmatrix}+b\begin{bmatrix}0\\0\\1\end{bmatrix}=c\begin{bmatrix}0\\1\\0\end{bmatrix} \tag{1.12}$$

for some real numbers a, b, and c. The top row (first component) gives us the equation $a = 0$. Then, the middle row (second component) gives us the equation $a = c$. The last row (third component) gives us $b = 0$. Thus, $a = b = c = 0$. So the only vector satisfying Equation 1.12 is the zero vector and $U \cap V = \{\vec{0}\}$.

- $U \cap W$: A vector that is in both U and W will satisfy the following equation

$$a\begin{bmatrix}1\\1\\0\end{bmatrix}+b\begin{bmatrix}0\\0\\1\end{bmatrix}=c\begin{bmatrix}1\\0\\1\end{bmatrix}+d\begin{bmatrix}0\\1\\0\end{bmatrix} \tag{1.13}$$

for some real numbers a, b, c and d. Find the equations from each row and show that $a = b = c = d$.

Now, what does that tell us about the intersection? Well, any vector in the intersection must be of the form

$$a\begin{bmatrix}1\\1\\0\end{bmatrix}+a\begin{bmatrix}0\\0\\1\end{bmatrix} \quad \text{or equivalently} \quad a\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Thus, $U \cap W = \operatorname{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$.

- $V \cap W$: Follow the methods used above to compute this intersection.

The second question mentioned earlier dealt with combining subspaces. The correct notion for this is to take the *sum* of the subspaces.

Definition 1.4.3 *Let U and W be subspaces of a vector space V. The* ***sum*** *of these subspaces $U+W$ is defined as*

$$\{\vec{u}+\vec{w} \colon \vec{u} \in U, \vec{w} \in W\}.$$

Additionally, if U and W have the property that $U \cap W = \{\vec{0}\}$, then we call this a ***direct sum*** *and denote it $U \oplus W$.*

Theorem 1.4.4 *Let U and W be subspaces of a vector space V. Then the sum $U+W$ is a subspace of V.*

The proof of this follows a similar format as the previous one that said $U \cap V$ is a subspace, so we'll just leave that as an exercise.[34]

Example 1.4.8 Let's determine the sums of some of the vector spaces we considered in Exploration 31. Let's recall the definitions of the subspaces of $\mathbb{R}^3$ to start.

$$U = \text{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \qquad V = \text{Span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$$

$$W = \text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$$

Whenever our subspaces are given to us as the span of a set of vectors, finding the sum is fairly straightforward. We just combine the sets defining each of the two subspaces to get a new set, then take the span. For example,

$$\begin{aligned} U+W &= \text{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} \\ &= \text{Span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} = \mathbb{R}^3 \end{aligned}$$

Note here that the initial list of vectors we obtained was not linearly independent, despite the fact the original two lists were when considered separately. This will always be the case when your subspaces have an intersection other than $\{\vec{0}\}$.

Let's consider now the case of $U+V$. Since we know from Exploration 31 that $U \cap V = \{\vec{0}\}$, what we'll have instead is the direct sum $U \oplus V$. In particular,

$$U \oplus V = \text{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} = \mathbb{R}^3.$$

Note that here the new list of vectors was linearly independent.

Direct sums give us a way to decompose a vector space neatly and will be a topic that comes up again in the following chapters.

34: At this point, you should have a good idea of what the homework for this section will look like!

Section Highlights

- A *subspace* (Definition 1.4.1) is a subset of a vector space that is itself a vector space.
- For a vector space V, V and the set consisting of just the zero vector are both subspaces.
- The span of a set of vectors is always a subspace. See Theorem 1.4.2.
- The sum of two subspaces is a subspace (Definition 1.4.3 and Theorem 1.4.4), and the intersection of two subspaces is a subspace (Theorem 1.4.3).

Exercises for Section 1.4

1.4.1. The following subsets all fail to be closed under vector addition. Give an example that illustrates this failure.

(a) $\{a + bx\colon a, b \in \mathbb{R}, a \neq 0\} \subset \mathbb{P}_2$

(b) $\left\{\begin{bmatrix} 1 \\ a \end{bmatrix} : a \in \mathbb{R}\right\} \subset \mathbb{R}^2$

(c) $\left\{\begin{bmatrix} ab \\ b \\ a \end{bmatrix} : a, b \in \mathbb{R}\right\} \subset \mathbb{R}^3$

(d) $\left\{\begin{bmatrix} a+3 \\ a \end{bmatrix} : a \in \mathbb{R}\right\} \subset \mathbb{R}^2$

1.4.2. The following subsets all fail to be closed under scalar multiplication. Give an example that illustrates this failure.

(a) $\{a + ax : a \in \mathbb{R}, a \geq 0\} \subset \mathbb{P}_2$

(b) $\left\{\begin{bmatrix} 1 \\ a \end{bmatrix} : a \in \mathbb{R}\right\} \subset \mathbb{R}^2$

(c) $\left\{\begin{bmatrix} ab \\ b \\ a \end{bmatrix} : a, b \in \mathbb{R}\right\} \subset \mathbb{R}^3$

(d) $\left\{\begin{bmatrix} a+3 \\ a \end{bmatrix} : a \in \mathbb{R}\right\} \subset \mathbb{R}^2$

1.4.3. Show that the following subsets are subspaces of $\mathbb{P}_3$. Then, write each as the span of some set of vectors.

(a) $\{a + bx^3\colon a, b \in \mathbb{R}\}$

(b) $\{4(a+c) + bx + cx^2\colon a, b, c \in \mathbb{R}\}$

1.4.4. Show that $\{4 + ax + bx^2\colon a, b \in \mathbb{R}\}$ is *not* a subspace of $\mathbb{P}_3$. Identify which of the three properties fail.

1.4.5. Show that

$$H = \left\{\begin{bmatrix} t \\ 0 \\ 2t \end{bmatrix} : t \in \mathbb{R}\right\}$$

is a subspace of $\mathbb{R}^3$, and write it as the span of some set of vectors.

1.4.6. Show that

$$H = \left\{\begin{bmatrix} 0 \\ x \\ y \end{bmatrix} : x, y \in \mathbb{R}\right\}$$

is a subspace of $\mathbb{R}^3$, and write it as the span of some set of vectors.

1.4.7. Show that

$$H = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2t \end{bmatrix} : t \in \mathbb{R} \right\}$$

is *not* a subspace of $\mathbb{R}^3$. Identify which of the three properties fail.

1.4.8. Show that

$$H = \left\{ \begin{bmatrix} t \\ 0 \\ 2t \end{bmatrix} : t \in \mathbb{R}, t > 0 \right\}$$

is *not* a subspace of $\mathbb{R}^3$. Identify which of the three properties fail.

1.4.9. Show that

$$H = \left\{ \begin{bmatrix} t \\ 0 \\ 2t \end{bmatrix} : t \in \mathbb{R}, t \geq 0 \right\}$$

is *not* a subspace of $\mathbb{R}^3$. Identify which of the three properties fail.

1.4.10. Suppose V is a vector space. Show that $\{\vec{0}\}$ is a subspace of V.

1.4.11. Suppose V is a vector space. Show that V is a subspace of V.

1.4.12. Show that the only subspaces of $\mathbb{R}$ are $\mathbb{R}$ and $\{\vec{0}\}$. To do this, suppose there is some other subspace H of $\mathbb{R}$. If $H \neq \{\vec{0}\}$, then there must be some nonzero vector $v \in H$. Conclude that $H = \mathbb{R}$.

1.4.13. Suppose that H is a subset of a vector space V and you've shown that the second and third axioms from the definition of subspace hold (that is, that H is vector addition under and scalar multiplication). Did you know that if H is nonempty that this implies that the first axiom holds (that is, that H contains the zero vector) as well? It's true! Now prove it.

1.4.14. We will now investigate subspace in $\mathbb{R}^2$.

(a) Show $\{\vec{0}\}$ is a subspace of $\mathbb{R}^2$.

(b) Show $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

(c) Show the set $L(a, b)$ below is a subspace of $\mathbb{R}^2$ for any real numbers a and b.

$$L(a,b) = \left\{ \begin{bmatrix} ka \\ kb \end{bmatrix} : k \in \mathbb{R} \right\}.$$

1.4.15. Show that $\mathbb{P}_k = \{a_0 + a_1x + a_2x^2 + \cdots + a_kx^k : a_i \in \mathbb{R} \text{ for } 0 \leq i \leq k\}$ is a subspace of $\mathbb{P}_n = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_i \in \mathbb{R} \text{ for } 0 \leq i \leq n\}$ for any $0 \leq k \leq n$. Hint: This should look similar to Example 1.4.2.

1.4.16. The following are all subspaces of $\mathbb{R}^3$.

$$U = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad V = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad W = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(a) Is $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in U \cap V$?

(b) Is $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in U \cap W$?

(c) Find $U \cap V$.

(d) Find $U \cap W$.

(e) Find $V \cap W$.

(f) Find $U + W$. Is this a direct sum?

(g) Find $V + W$. Is this a direct sum?

1.4.17. The following are all subspaces of $\mathbb{R}^3$.

$$U = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \quad V = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \quad W = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(a) Is $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in U \cap V$?

(b) Is $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in U \cap W$?

(c) Is $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in V \cap W$?

(d) Find $U \cap V$.

(e) Find $U \cap W$.

(f) Find $V \cap W$.

(g) Find $U + W$. Is this a direct sum?

(h) Find $V + W$. Is this a direct sum?

1.4.18. Prove Theorem 1.4.4, which says the sum of two subspaces is a subspace.

1.4.19. Along with the concept of intersection, we often discuss the *union* of two sets. Let A and B be sets. The union of A and B, denoted $A \cup B$, is the set of all elements in either A or B. For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$, then the union of A and B is $\{1, 2, 3, 4, 6, 8\}$. Let U and W be subspaces of a vector space V. Show that $U \cup W$ is *not* a subspace of V in general. Under which conditions will it be a subspace?

1.5 A Menagerie of Vector Spaces

We've had quite a few different examples of real vectors spaces already. Here's a list with links to where you can go back and read about them again:

- $\mathbb{R}^n$: Equation 1.2, Section 1.1
- $\mathbb{P}_n$: Equation 1.7, Section 1.1
- arrows: beginning of Section 1.2
- $\mathbb{C}$: Exercise 1.1.3, Section 1.1
- $(0, \infty)$: Exercise 1.1.12, Section 1.1
- sums and intersections of subspaces: Theorem 1.4.4, Section 1.4

But wait! There's more![35]

35: I'm not paying for anything.

A Peek Into the Future

There are other examples of vector spaces that will appear for us naturally later. However, we can tell you a little about them now.[36]

36: Spoilers!

Example 1.5.1 A rectangular array of numbers with m rows and n columns is called an $m \times n$ matrix, and we call the set of all such matrices $\mathcal{M}_{m\times n}(\mathbb{R})$. When $m = 2$ and $n = 2$, we more specifically have

$$\mathcal{M}_{2\times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

If we define vector addition componentwise,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+d & b+e \\ c+g & d+h \end{bmatrix},$$

and for scalars k, we define scalar multiplication as

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}.$$

You might have noticed that this set and its operations look a lot like $\mathbb{R}^4$, just with the vector entries arranged in a different shape. If that's the case, it should not surprise you that $\mathcal{M}_{2\times 2}(\mathbb{R})$ is a vector space. Should you require a review of the ten axioms, though, you should check them for practice.

Exploration 33 Let

$$\begin{aligned} M_0 &= = \left\{ \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} : b, c, d \in \mathbb{R} \right\} \text{ and} \\ M_1 &= = \left\{ \begin{bmatrix} 1 & b \\ c & d \end{bmatrix} : b, c, d \in \mathbb{R} \right\}. \end{aligned}$$

Show that M_0 is a subspace of $\mathcal{M}_{2\times 2}(\mathbb{R})$ but M_1 is not.

Example 1.5.2 Consider the general form of a linear equation in n variables:

$$a_1x_1 + \cdots + a_nx_n = b.$$

Moving the constant term to the left side, we have

$$a_1x_1 + \cdots + a_nx_n - b = 0,$$

and this looks very much like the general form of a vector in $\mathbb{P}_n$. Leveraging what we know about $\mathbb{P}_n$, we can make a vector space of linear equations. Let

$$V = \{a_1x_1 + \cdots + a_nx_n = a_{n+1} \colon a_i \in \mathbb{R} \text{ for } i = 1, \ldots, n+1\},$$

and define vector addition by combining like terms and scalar multiplication by multiplication on both sides of the equation. With these operations, this set is a vector space.

Example 1.5.3 Suppose V and W are both vector spaces. Just like how we took Cartesian products of sets in Chapter 0, we can do the same with vector spaces. Consider the set

$$V \times W = \{(\vec{v}, \vec{w}) \colon \vec{v} \in V \text{ and } \vec{w} \in W\}.$$

If we use operations from V in the first component and operations from W in the second, then the set $V \times W$ is a vector space. Check!

Exploration 34 Do these computations in $\mathbb{R}^2 \times \mathbb{P}_2$.

- $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, 1 + x + x^2 \right) + \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 3 + x^2 \right) =$

- $5 \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, 1 + x + x^2 \right) =$

Function Spaces

While we're on the topic of additional examples of vector spaces, here's a nice class of them that you may find interesting.[37]

Let $I \subseteq \mathbb{R}$ be any interval, and define

$$\begin{aligned} \mathcal{C}(I) &= \{f \colon I \to \mathbb{R} \colon f \text{ is continuous}\} \\ \mathcal{D}(I) &= \{f \colon I \to \mathbb{R} \colon f \text{ is differentiable}\} \\ \mathcal{R}(I) &= \{f \colon I \to \mathbb{R} \colon f \text{ is integrable}\} \end{aligned}$$

Again, we're going to think of $\vec{f} \in \mathcal{D}(I)$ as a vector, knowing in our hearts that this vector is a real-valued function defined on I. For $\vec{f}, \vec{g} \in \mathcal{C}(I)$ (or $\mathcal{D}(I)$ or $\mathcal{R}(I)$[38]) and $a \in \mathbb{R}$, define

$$\begin{aligned} \vec{f} + \vec{g} &= f \vec{+} g \colon I \to \mathbb{R} \text{ defined by } (f \vec{+} g)(x) = f(x) + g(x) \text{ and} \\ a\vec{f} &= a\vec{f} \colon I \to \mathbb{R} \text{ defined by } a\vec{f}(x) = af(x). \end{aligned}$$

37: Ooh! Look at all the fancy calligraphy!

C'mon, Ricky. Don't get taken in by substance-free flash; seriously–Wait. That "C" is very cool.

38: Hang on. Why "R" for "integrable?" Why not "I?"

Do you want to write "$I(I)$?" The "R" is probably for Riemann.

Using these typical operations, function addition and scalar multiplication of functions, $\mathcal{C}(I)$, $\mathcal{D}(I)$, and $\mathcal{R}(I)$ are all vector spaces. Of course, you should really check this. Let's talk about what would go into doing that. There are some properties of continuous, differentiable, and integrable[39] functions that should be easily located in just about any calculus text that tell us these sets are closed under the operations of vector addition and scalar multiplication. Then the remaining properties actually hold more broadly for the set of functions from I to $\mathbb{R}$.

39: I didn't come here for calculus.

Settle down. No one's asking you to integrate anything.

Example 1.5.4 Note that $\vec{f} = \sin x$ and $\vec{g} = \cos x$ are both vectors in $\mathcal{C}([0,1])$ because both $\sin x$ and $\cos x$ are continuous functions of x on the interval $[0,1]$. Note that $\sin \pi/4 = \cos \pi/4$. Nevertheless, $\vec{f} \neq \vec{g}$; for these vectors to be equal, they would have to be the same *for all values of* x in $[0,1]$.

Moreover, $\vec{f}$ and $\vec{g}$ are linearly independent. If $\vec{f}$ and $\vec{g}$ were linearly dependent, we would have $\vec{g} = a\vec{f}$ for some nonzero scalar a, and this would have to be true *for all* x. However, note that for $x = 0 \in [0,1]$, we have

$$1 = \cos 0 = a \sin 0 = 0,$$

so there is no a such that $\vec{g} = a\vec{f}$ for all $x \in [0,1]$. Thus, $\vec{f}$ and $\vec{g}$ are linearly independent.

Exploration 35 Let $\vec{f} = e^x$ and $\vec{g} = x^2$. We know that $\operatorname{Span}\left\{\vec{f}, \vec{g}\right\}$ is a subspace of $\mathcal{D}(\mathbb{R})$. Show that $\operatorname{Span}\left\{\vec{f}, \vec{g}\right\} \neq \mathcal{D}(\mathbb{R})$.

Example 1.5.5 Here's a differential equation:

$$y''' + 3y'' + 2y' = 0. \tag{1.14}$$

We can check that $y_1 = e^{-x}$, $y_2 = e^{-2x}$, and $y_3 = 87$ are all solutions:

$$\begin{aligned}
y_1''' + 3y_1'' + 2y_1' &= -e^{-x} + 3e^{-x} - 2e^{-x} = 0 \\
y_2''' + 3y_2'' + 2y_2' &= -8e^{-x} + 12e^{-x} - 4e^{-x} = 0 \\
y_3''' + 3y_3'' + 2y_3' &= -0 + 3(0) - 2(0) = 0
\end{aligned}$$

Each function y_i for $i = 1, 2, 3$ makes Equation 1.14 true when substituted in for y, so all three are solutions for the differential equation.

This all may seem like a wild tangent, but note that y_1, y_2, and y_3 from Example 1.5.5 are all vectors in $\mathcal{D}(\mathbb{R})$. Obviously, $\operatorname{Span}\{y_1, y_2, y_3\}$ is a subspace of $\mathcal{D}(\mathbb{R})$.[40]

40: Wait, why is this obvious?

Exploration 36 Show that any vector in $\operatorname{Span}\{y_1, y_2, y_3\}$, that is, any linear combination of y_1, y_2, and y_3, is a solution to Equation 1.14.

Thus, $\mathrm{Span}\{y_1, y_2, y_3\}$ is an entire *subspace of solutions* for the given differential equation. This is an example of the Superposition Principle, and it actually holds for a large class of differential equations. We should definitely think about this more, but let's do it later.[41]

41: Later, like in a different book?

Exercises for Section 1.5

1.5.1.Let

$$H \quad = \quad = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \text{ and } a + d = 0 \right\}.$$

Show H is a subspace of $\mathcal{M}_{2\times 2}(\mathbb{R})$.

1.5.2.Determine whether $H = \left\{ \begin{bmatrix} x & x+y \\ x-y & y \end{bmatrix} : x, y \in \mathbb{R} \right\}$ is a subspace of $\mathcal{M}_{2\times 2}$.

1.5.3.Consider the vector space

$$V = \{a_1x_1 + a_2x_2 = a_3 \colon a_i \in \mathbb{R} \text{ for } i = 1, 2, 3\}.$$

Note that $x_1 + x_2 = 2$ is a vector in V and that $x_1 = x_2 = 1$ is a solution to this particular equation. Determine whether the set of vectors in V for which $x_1 = x_2 = 1$ is a solution is a subspace of V. Justify your determination.

1.5.4.Let

$$S = \{(s_1, s_2, \dots) \colon s_i \in \mathbb{R} \text{ for } i = 1, 2, \dots\}$$

be the set of infinite real-valued sequences. Using

$$\begin{aligned} (s_1, s_2, \dots) + (r_1, r_2, \dots) &= (s_1 + r_2, s_2 + r_2, \dots) \text{ and} \\ k(s_1, s_2, \dots) &= (ks_1, ks_2, \dots), \end{aligned}$$

determine whether S is a vector space.

1.5.5.Show that $\mathbb{R}^2 \times \mathbb{P}_2$ is a vector space.

1.5.6.Let

$$S_0 = \{(s_1, s_2, \dots) \colon s_i \in \{0, 1\} \text{ for } i = 1, 2, \dots\}$$

be the set of infinite binary sequences. Here's how addition will work on $\{0, 1\}$:

$$\begin{aligned} 1 + 0 &= 1, \\ 0 + 1 &= 1, \\ 0 + 0 &= 0, \text{ and} \\ 1 + 1 &= 0. \end{aligned}$$

For $(s_1, s_2, \dots), (r_1, r_2, \dots) \in S_0$, and $k \in \{0, 1\}$, define

$$\begin{aligned} (s_1, s_2, \dots) + (r_1, r_2, \dots) &= (s_1 + r_2, s_2 + r_2, \dots) \text{ and} \\ k(s_1, s_2, \dots) &= (ks_1, ks_2, \dots). \end{aligned}$$

Determine whether S_0 is a vector space. Note that k is only allowed to be 0 or 1; we're not looking for a *real* vector space here because we're not using real scalars.

1.5.7.For two functions $f \colon \mathbb{R} \to \mathbb{R}$ and $g \colon \mathbb{R} \to \mathbb{R}$, both differentiable, the Wronskian of f and g is defined as $W(x) = f(x)g'(x) - g(x)f'(x)$. Show that if $W(x) \neq 0$ for some x, then f and g are linearly independent.

2 Bases

In this chapter, we will begin by expanding upon the topics of span and linear independence from Section 1.3 to build the concept of a basis of a vector space. This will allow us to revisit our geometric tools from Section 1.2. We'll see that there are (usually) perfectly reasonable ways to think about and make use of the geometry of *any* vector space. First, we have a bit of organizing to do.

We'll need some basic definitions. [1]

1: We leave it to the reader to determine for themselves whether that qualifies as wordplay.

2.1 Introduction to Bases

The following definition builds off of our discussion of the span of a set of vectors in both Sections 1.3 and 1.4.

Definition 2.1.1 *Let V be a vector space and $\vec{v}_1, \ldots, \vec{v}_p \in V$. We say the vectors $\vec{v}_1, \ldots, \vec{v}_p$* ***span*** *$V$ if*

$$\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_p\} = V.$$

In this situation, we call $\{\vec{v}_1, \ldots, \vec{v}_p\}$ a ***spanning set*** *for V.*

Compare this with Definition 1.3.2.[2] The difference is subtle. Basically, all we've done here is make verb and adjective versions of the noun "span."

2: Did you notice citations like this are hyperlinked? You're welcome.

Example 2.1.1 Let's consider the vectors

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Is the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a spanning set for $\mathbb{R}^2$?

To answer this question, we should first try to better understand $\operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Because $\vec{v}_1 = \vec{v}_2 - \vec{v}_3$, we know $\operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{Span}\{\vec{v}_2, \vec{v}_3\} = \operatorname{Span}\{\vec{v}_1, \vec{v}_3\} = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$. While it is not necessary to reduce our set, it will simplify the algebra involved in the problem since two vectors are easier to work with than three.

Let's now show that $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$ is a spanning set for $\mathbb{R}^2$. Since $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$, we know automatically that $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} \subseteq \mathbb{R}^2$. So we only need to check that $\mathbb{R}^2 \subseteq \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$ to have the desired set equality. To do this, we choose a general vector in $\mathbb{R}^2$ and show it can be obtained as a linear combination of $\vec{v}_1$ and $\vec{v}_2$. For our general vector, we choose

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $x_1, x_2 \in \mathbb{R}$. Then we need to find $a, b \in \mathbb{R}$ such that $a\vec{v}_1 + b\vec{v}_2 = \vec{x}$. That is,

$$a \begin{bmatrix} -1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This gives us the equations $-a = x_1$ and $a + b = x_2$. Our goal is to find a and b in terms of x_1 and x_2. We quickly have $a = -x_1$, and we can substitute and rearrange to get $b = x_1 + x_2$. This gives us a way to write any vector in $\mathbb{R}^2$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, so $\mathbb{R}^2 \subseteq \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$. Since we started by arguing that $\operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$, we now know that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is also a spanning set.

Exploration 37 Use the equations solved for in the example above to write

$$\begin{bmatrix} 2 \\ 10 \end{bmatrix}$$

as a linear combination of

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Spanning sets may be a convenient way to describe a vector space, but this method doesn't preclude us from doing silly things like

$$\mathbb{R}^2 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 1,234 \end{bmatrix} \right\}.$$

It's probably a good time to decide what makes a spanning set a "good" spanning set, where by good, we mean avoiding inefficiencies like the preceding example. Surely we don't need more than one thousand vectors to span $\mathbb{R}^2$! What exactly do we need then? Well, in Example 2.1.1 we used the fact that the set was linearly dependent to reduce it to a more manageable size, and we can do this in general. That means linearly independent sets are more desirable, since they cannot be reduced down further without changing the span.

Definition 2.1.2 *Let V be a vector space. A finite set of vectors $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_p\}$ is a* ***basis*** *for V if*

(a) *$\mathcal{B}$ is linearly independent, and*

(b) *$\mathcal{B}$ spans V.*

Any set satisfying the above definition is a basis. However, when we are using a specific basis in this text, we will implicitly impose an ordering on the elements of the set as determined by the order they are listed. This is sometimes referred to as an "ordered basis," but we will not use this terminology. Just know that for us, if the set $\{\vec{v}_1, \vec{v}_2\}$ is a basis, then the set $\{\vec{v}_2, \vec{v}_1\}$ is also a basis. However, the orderings on these two bases[3] are different.

Now, we've claimed that a basis is better than just any old spanning set, and it is. However, before we tell you why, let's spend some time with the definition and some specific examples. How exactly do you show a set meets these conditions?

...So You Think Your Set's a Basis

Example 2.1.2 Let $\mathcal{B} = \{\vec{p}_1 = 1 + x, \vec{p}_2 = x + x^2, \vec{p}_3 = x\}$. This is a basis for the vector space $\mathbb{P}_2$. To show this using the definition of basis, we'd need to show that $\mathcal{B}$ is linearly independent and that it spans $\mathbb{P}_2$. Let's verify!

- *Linearly Independent:* Suppose $a\vec{p}_1 + b\vec{p}_2 + c\vec{p}_3 = \vec{0}$. Then we have $a(1+x)+b(x+x^2)+c(x) = 0$ which simplifies to $a+(a+b+c)x + bx^2 = 0$. For this equation to be true, the coefficients on each of the terms must be zero. This can only happen when $a = b = 0$. Thus, we also see that $c = 0$ since $a + b + c = 0$. The only solution is then $a = b = c = 0$, where all coefficients are zero.
- *Spans* $\mathbb{P}_2$. Let $a_0 + a_1x + a_2x^2$ be any polynomial in $\mathbb{P}_2$. Then we have

$$\begin{aligned} &a_0\vec{p}_1 + a_2\vec{p}_2 + (a_1 - a_2 - a_0)\vec{p}_3 \\ &\qquad = a_0(1 + x) + a_2(x + x^2) + (a_1 - a_2 - a_0)x \\ &\qquad = a_0 + a_1x + a_2x^2 \end{aligned}$$

  This gives us a recipe for writing any vector of $\mathbb{P}_2$ as a linear combination of the vectors in $\mathcal{B}$, so $\mathcal{B}$ spans $\mathbb{P}_2$!

Exploration 38 Let's do another one! Let

$$\mathcal{B} = \left\{ \vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Verify this is a basis for $\mathbb{R}^2$.

- Argue that $\mathcal{B}$ is linearly independent.

- Now we show $\mathcal{B}$ spans $\mathbb{R}^2$. Let

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

  be any vector in $\mathbb{R}^2$. Find a "recipe" for writing $\vec{x}$ as a linear combination of $\vec{b}_1$ and $\vec{b}_2$.

3: The plural of "basis" is "bases," pronounced with a long e.

Exploration 39 Let $\mathcal{B} = \{\vec{p}_1 = 1 + x, \vec{p}_2 = x + x^2\}$. Note that $\mathcal{D}$ is linearly independent.[4] Show that this is not a basis by finding a polynomial in $\mathbb{P}_2$ that is not a linear combination of $\vec{p}_1$ and $\vec{p}_2$. This will mean it fails to span all of $\mathbb{P}_2$.

4: This is actually seen in the example above since any subset of a linearly independent set will still be linearly independent.

Oh yeah, then why not prove it?

No, *you* prove it.

Exercise?

Deal.

Standard Basis Vectors

We've found bases now for $\mathbb{R}^2$ and $\mathbb{P}_2$, but neither of these bases is actually the most preferred basis for its space. (Of course, preferences are debatable...)[5] Instead, let's define the *standard basis* for each of these spaces, or really, for the more general vector spaces $\mathbb{R}^n$ and $\mathbb{P}_n$.

5: What if I have a mathematically rigorous way to quantify preferences?

Do you though?

Not yet, but I'm working on it.

- *Standard Basis for* $\mathbb{R}^n$ Let $\vec{e}_i$ be the vector with 0 in every entry except for the ith entry, which is 1. For example, in $\mathbb{R}^3$,

$$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then the standard basis for $\mathbb{R}^n$ is

$$\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$$

where each $\vec{e}_i \in \mathbb{R}^n$. To be thorough, we should say a bit about verifying this is a basis. Even though its name is *standard basis*, such things shouldn't be taken for granted. The key thing to note is that for any $a_1, \ldots, a_n \in \mathbb{R}$,

$$a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

Thus, the only way that $a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n = \vec{0}$ is if each $a_i = 0$. Also, any vector in $\mathbb{R}^n$ can be written as a linear combination of the vectors in $\mathcal{B}$ by using the ith entry of the vector for a_i, the coefficient of $\vec{e}_i$ in the sum. For example,

$$\begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 7\vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3.$$

- *Standard Basis for* $\mathbb{P}_n$ The standard basis for $\mathbb{P}_n$ is

$$\{1, x, x^2, ..., x^n\}.$$

Again, we should pause to convince ourselves this is a basis. Have you convinced yourself...? Good! Share your thoughts about this below:

Another Method to Show a Set Spans

Suppose you have a set S that you suspect is a basis for some vector space V. To verify it's a basis as we've illustrated, there are then two things to show. First, you need to argue that the set is linearly independent. We have a method for how to do this from Section 1.3, which is what we did in Example 2.1.2 as well. However, you also need to show it spans the vector space. Perhaps you noticed in the previous examples that this part can be a bit more difficult. Here are two strategies:

- *Method 1:* Argue that any general element of V can be obtained as a linear combination of the vectors in S.
- *Method 2:* Argue that a known basis for V is a subset of $\operatorname{Span}\{S\}$.

The first method is what we employed in Example 2.1.2 and Exploration 38. It has the advantage of giving a recipe for how to write any vector in V as a linear combination of the vectors in S. The disadvantage is that sometimes the number of variables involved can get a bit overwhelming. Let's talk a bit about the second method now. It relies on the same logic we used in Example 2.1.1 in which we reasoned that if a subset is a spanning set, then the whole set must also be a spanning set. Let's formalize that.

Theorem 2.1.1 *Suppose* $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ *and* $U = \{\vec{u}_1, \ldots, \vec{u}_k\}$ *are subsets of a vector space* V*.*
(a) *If* $U \subseteq \operatorname{Span}\{S\}$, *then* $\operatorname{Span}\{U\} \subseteq \operatorname{Span}\{S\}$.
(b) *If* $U \subseteq \operatorname{Span}\{S\}$ *and* $\operatorname{Span}\{U\} = V$, *then* $\operatorname{Span}\{S\} = V$.

Before proceeding to our proof, we must welcome a new notation. The small problem we have is that we will need a bunch of scalars for each of a bunch of different vectors, and there simply aren't enough letters... or subscripts. Thus, we must implement a second subscript. For example, if given an ith vector $\vec{v}_i$ that requires j scalars (for some reason), we could (that is, we will) use the notation $a_{i1}, \ldots, a_{ij}$ for these j scalars. This is nice[6] because it also indicates these scalars' association with this given ith vector $\vec{v}_i$. Now, on to the proof.

6: I object to the description of this as "nice." Having two subscripts is annoying.

Those subscripts are actually necessary, unlike, for example, having two unicorns.

You would be so bored without me.

PROOF. Suppose $U = \{\vec{u}_1, \ldots, \vec{u}_k\}$ is a subset of $\operatorname{Span}\{S\}$. Note that the span of a set of vectors is all linear combinations of the vectors in that set. Thus, every vector in U is a linear combination of the vectors in S. More specifically, we have

$$\begin{aligned}
\vec{u}_1 &= a_{11}\vec{v}_1 + a_{12}\vec{v}_2 + \cdots + a_{1n}\vec{v}_n \\
\vec{u}_2 &= a_{21}\vec{v}_1 + a_{22}\vec{v}_2 + \cdots + a_{2n}\vec{v}_n \\
&\ \vdots \\
\vec{u}_k &= a_{k1}\vec{v}_1 + a_{k2}\vec{v}_2 + \cdots + a_{kn}\vec{v}_n
\end{aligned}$$

Suppose now that $\vec{x} \in \text{Span}\{U\}$. Then

$$\begin{aligned}\vec{x} &= b_1\vec{u}_1 + \cdots b_k\vec{u}_k \\ &= b_1(a_{11}\vec{v}_1 + a_{12}\vec{v}_2 \cdots + a_{1n}\vec{v}_2) + \cdots + b_k(a_{k1}\vec{v}_1 + a_{k2}\vec{v}_2 + \cdots + a_{kn}\vec{v}_n) \\ &= (b_1a_{11} + b_2a_{21} \cdots + b_ka_{k1})\vec{v}_1 + \cdots + (b_1a_{1n} + b_2a_{2n} \cdots + b_ka_{kn})\vec{v}_n.\end{aligned}$$

Thus, $\vec{x} \in \text{Span}\{S\}$ which tells us $\text{Span}\{U\} \subseteq \text{Span}\{S\}$. Now, suppose U is a spanning set. That means $\text{Span}\{U\} = V$, and thus, $V \subseteq \text{Span}\{S\}$. Since we always have $\text{Span}\{S\} \subseteq V$, we know then that $\text{Span}\{S\} = V$ and S must also be a spanning set. □

This theorem gives us the tools for our second strategy. We know a basis is also a spanning set. Thus, if we want to check that $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a spanning set for V when we already know a basis for V, we can check that the basis is contained in $\text{Span}\{S\}$. If V is $\mathbb{R}^n$ or $\mathbb{P}_n$, a convenient basis to look for is the standard one.

Example 2.1.3 Let's show that $S = \{1, 1+x, 1+x+x^2\}$ is a spanning set for $\mathbb{P}_2$ by showing the standard basis $\{1, x, x^2\} \subseteq \text{Span}\{S\}$.

- First, we must argue that $1 \in \text{Span}\{S\}$. Well, wait. We know $1 \in S$, so there's nothing to check here. Great!

- Now, we need to argue that $x \in \text{Span}\{S\}$. This one requires a bit more thought, but it's still not bad. We need to find coefficients $a, b, c \in \mathbb{R}$ such that
$$a(1) + b(1+x) + c(1+x+x^2) = x.$$
This simplifies to the equations $a + b = 0$, $b + c = 1$, and $c = 0$. Thus, we have $a = -1$, $b = 1$, and $c = 0$. That is, $x = (-1)(1) + (1+x) \in \text{Span}\{S\}$.

- Finally, we can argue that $x^2 \in \text{Span}\{S\}$. Again, we need to find coefficients $a, b, c \in \mathbb{R}$ such that
$$a(1) + b(1+x) + c(1+x+x^2) = x^2.$$
This simplifies to the equations $a + b = 0$, $b + c = 0$, and $c = 1$. Thus, we have $a = 0$, $b = -1$, and $c = 1$. That is, $x^2 = (-1)(1+x) + (1+x+x^2) \in \text{Span}\{S\}$.

Now that we've checked that the standard basis is contained in $\text{Span}\{S\}$, we can conclude that S is a spanning set.

Now, if you are trying to use this method for say $\mathbb{R}^{10}$ or $\mathbb{P}_9$, it might take a while to show all 10 basis vectors are in the span of the set. This method is less efficient than Method 1 in many cases, but sometimes it is preferred just because of the fewer variables involved.

Exploration 40 Complete the argument that S from Example 2.1.3 is a basis by showing the vectors are linearly independent.

Exploration 41 Use Method 2 to argue

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

is a spanning set for $\mathbb{R}^2$.

Finding a Basis

Alright. We've defined a basis. We've seen some examples of bases, and we've discussed how to verify a set is a basis. What's left? Well, there are several facts we can state about bases and also things they tell us about their respective vector spaces. Much of the next section will be involved with stating and proving these facts.[7] For now, we will focus on a very useful theorem that helps us to *find* a basis for a given vector space.

7: Those facts better be theorems. Bubbles gets testy when you don't do proofs.

Part of the statement below is a repeat from Theorem 1.3.2

Theorem 2.1.2 *Let V be a vector space, let $S = \{\vec{v}_1, \dots, \vec{v}_p\} \subset V$, and let S span V.*

(a) *If one of the vectors $\vec{v}_k \in S$ is a linear combination of the other vectors in S, then the set formed from S by removing $\vec{v}_k$ still spans V.*

(b) *If $V \neq \{\vec{0}\}$, then some subset of S is a basis for V.*

PROOF. Let's prove the first statement. Suppose $\vec{v}_k \in S$ is a linear combination of the other vectors in S; that is

$$\vec{v}_k = a_1\vec{v}_1 + \cdots a_{k-1}\vec{v}_{k-1} + a_{k+1}\vec{v}_{k+1} + \cdots + a_p\vec{v}_p.$$

Since S spans V, for any $\vec{v} \in V$, there exist scalars $b_1, \dots, b_p \in \mathbb{R}$ such that

$$\begin{aligned} \vec{v} &= b_1\vec{v}_1 + \cdots + b_k\vec{v}_k + \cdots + b_p\vec{v}_p \\ &= b_1\vec{v}_1 + \cdots + b_k(a_1\vec{v}_1 + \cdots + a_{k-1}\vec{v}_{k-1} + a_{k+1}\vec{v}_{k+1} + \cdots + a_p\vec{v}_p) + \\ &\quad \cdots + b_p\vec{v}_p \\ &= (b_1 + b_k a_1)\vec{v}_1 + \cdots + (b_{k-1} + b_k a_{k-1})\vec{v}_{k-1} + (b_{k+1} + b_k a_{k+1})\vec{v}_{k+1} + \\ &\quad \cdots + (b_p + b_k a_p)\vec{v}_p. \end{aligned}$$

Thus, any vector in V can be written as a linear combination of the vectors in the set formed from S by removing $\vec{v}_k$. It follows that the set formed from S by removing $\vec{v}_k$ spans V.

Let's move now to the second statement. If the set formed from S by removing $\vec{v}_k$ is linearly independent, then we have a basis for V and we are done. If it is linearly dependent, then one of the vectors in the set is a linear combination of the others by Theorem 1.3.1. In this case, we start this whole procedure again; repeating this as many times as necessary, the set will eventually be linearly independent since the set is finite and a set with one nonzero vector is trivially linearly independent. Also, from part one of this theorem, we know the set still spans V and is therefore a basis of V. $\square$

This theorem gives us a nice recipe for finding a basis for any vector space when we have a spanning set. Just remove the linearly dependent vectors one at a time until there are none. Nice! Let's try it!

Example 2.1.4 Recall from the beginning of this section:

$$\mathbb{R}^2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 1,234 \end{bmatrix} \right\}.$$

We're already told the set

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 1,234 \end{bmatrix} \right\}$$

spans $\mathbb{R}^2$, but it would be easy to check that for any vector $\vec{x} \in \mathbb{R}^2$, we have

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \cdots + 0 \begin{bmatrix} 0 \\ 1,234 \end{bmatrix}.$$

By Theorem 2.1.2, all we have to do is throw out linearly dependent vectors from S until it is linearly independent, and we'll have a basis. Note that for any integer $2 \leq k \leq 1,234$, we have

$$\begin{bmatrix} 0 \\ k \end{bmatrix} = k \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

so the last $1,232$ vectors in S are linear combinations of the second vector. By Theorem 2.1.2, we can throw them all out of S and the resulting set still spans $\mathbb{R}^2$. Specifically,

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

still spans $\mathbb{R}^2$. These vectors are linearly independent (since there are two of them and they're not scalar multiples of each other), so this is a basis[8] for $\mathbb{R}^2$.

8: It's the standard basis for $\mathbb{R}^2$!

Bases of Subspaces

Theorem 2.1.2 is also very helpful[9] when finding the basis for a subspace. Let's see an example illustrating this.

9: Like me!
Debatable.

Example 2.1.5 First, let's define a few vectors to work with for this example. Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \text{ and } \vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 4 \\ 1 \end{bmatrix}.$$

Now, we can define $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, which is a subspace of $\mathbb{R}^4$. Because of the way H is described to us, we are starting with a spanning set. So we just need to reduce it down to a linearly independent set without changing the span. To do this, we need to find a vector that can be written as a linear combination of other vectors.

The vector $\vec{v}_1$ is a good place to start. Suppose the there exist scalars $a_1, a_2, a_3 \in \mathbb{R}$ such that

$$\vec{v}_1 = a_1\vec{v}_2 + a_2\vec{v}_3 + a_3\vec{v}_4.$$

That is,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 1 \\ 4 \\ 1 \end{bmatrix}.$$

Then we would have the equations $a_1 + a_2 + a_3 = 1$ from the top row of the vectors and also $a_1 + a_2 + a_3 = 2$ from the last row of the vectors. Both of these can't be true, so $\vec{v}_1$ is not a linear combination of the remaining vectors.

Instead, we can check whether $\vec{v}_2$ is a linear combination of $\vec{v}_3$ and $\vec{v}_4$. (We don't need to consider $\vec{v}_1$ here because we've already ruled it out.) Let $b_1, b_2 \in \mathbb{R}$ be such that

$$\vec{v}_2 = b_1\vec{v}_3 + b_2\vec{v}_4.$$

That is,

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ 1 \\ 4 \\ 1 \end{bmatrix}.$$

Then we have the equations $1 = b_1 + b_2$ and $2 = 3b_1 + 4b_2$, which have the mutual solution $b_1 = 2$ and $b_2 = -1$, so

$$\vec{v}_2 = 2\vec{v}_3 - \vec{v}_4.$$

Now by Theorem 2.1.2, we can remove one of these three vectors and still have a spanning set. Thus, $H = \text{Span}\{\vec{v}_1, \vec{v}_3, \vec{v}_4\}$. Now, is the set $\{\vec{v}_1, \vec{v}_3, \vec{v}_4\}$ linearly independent? Yes! We can conclude this fairly quickly by our previous work. We know $\vec{v}_1$ is not a linear combination of $\vec{v}_3$ and $\vec{v}_4$, and we also see from inspection that $\vec{v}_3$ and $\vec{v}_4$ are not scalar multiples of each other. We have found a basis!

Exploration 42 In Example 2.1.5 above, we found a basis for the subspace H of $\mathbb{R}^4$. Can you find a different basis for H?

What about when a subspace is described differently?[10] Well, in that case, we start with finding a spanning set and then proceed like Example 2.1.5.

10: There's nothing wrong with being different.

How is that a substantive contribution to the narrative?

Exploration 43 Let's consider the subspace K of $\mathbb{R}^4$ defined below.

$$K = \left\{ \begin{bmatrix} a+b+c+d \\ a+b+c+d \\ a+2b+3c+4d \\ 2a+b+c+d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

Find a set of vectors that span K. (Hint: These should look familiar!) Then use this spanning set to find a basis for K.

Section Highlights

- Determine when the span of a set of vectors is a *spanning set* for a vector space. See Definition 2.1.1.
- There are multiple methods for determining whether a set is a spanning set. See Examples 2.1.1 and 2.1.3.
- A *basis* for a vector space (Definition 2.1.2) is a linearly independent spanning set.
- Any spanning set can be reduced to a basis by carefully removing vectors that are linear combinations of other basis vectors. See Example 2.1.5.

Exercises for Section 2.1

2.1.1. The following sets are not bases for $\mathbb{R}^2$. Determine whether they fail to be a spanning set, fail to be linearly independent, or both.

(a) $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

(b) $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

2.1.2. The following sets are not a basis for $\mathbb{R}^3$. Determine whether they fail to be a spanning set, fail to be linearly independent, or both.

(a) $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

(b) $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

2.1.3. The following sets are not a basis for $\mathbb{P}_1$. Determine whether they fail to be a spanning set, fail to be linearly independent, or both.

(a) $\{1+x\}$

(b) $\{1+x, -1-x\}$

(c) $\{1+x, -1-x, x\}$

2.1.4. The set $\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$ is not a basis for $\mathbb{R}^3$, even though it spans $\mathbb{R}^3$. Use the procedure from Theorem 2.1.2 to find a basis for $\mathbb{R}^3$.

2.1.5. Let $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$.

(a) Use Method 1 to show this is a spanning set for $\mathbb{R}^2$.

(b) Use Method 2 to show this is a spanning set for $\mathbb{R}^2$.

2.1.6. Let $\mathcal{B} = \left\{1+x, 1+x^2, 1+x+x^2\right\}$.

(a) Use Method 1 to show this is a spanning set for $\mathbb{P}_2$.

(b) Use Method 2 to show this is a spanning set for $\mathbb{P}_2$.

2.1.7. Let $\mathcal{B}=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

(a) Show $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

(b) Find a way to write $\begin{bmatrix}4\\6\end{bmatrix}$ as a linear combination of the vectors in $\mathcal{B}$.

(c) Find a way to write $\begin{bmatrix}1\\0\end{bmatrix}$ as a linear combination of the vectors in $\mathcal{B}$.

2.1.8. Let $\mathcal{B}=\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$.

(a) Show $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

(b) Find a way to write $\begin{bmatrix}1\\1\\1\end{bmatrix}$ as a linear combination of the vectors in $\mathcal{B}$.

2.1.9. Show $\mathcal{B}=\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}-1\\1\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^3$. How do you write $\begin{bmatrix}0\\1\\1\end{bmatrix}$ as a linear combination of these basis vectors?

2.1.10. Show $\mathcal{B}=\{1+x,x\}$ is a basis for $\mathbb{P}_1$.

2.1.11. Show $\mathcal{B}=\left\{5+x,1+x+x^2,x^2\right\}$ is a basis for $\mathbb{P}_2$.

2.1.12. Let $\mathcal{B}=\left\{1+x,2x,x+x^2\right\}$.

(a) Show $\mathcal{B}$ is a basis for $\mathbb{P}_2$.

(b) Find 1 as a linear combination of these basis vectors.

(c) Find $1+x+x^2$ as a linear combination of these basis vectors.

2.1.13. If $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ is a basis for a vector space V, show that $\{\vec{v}_1,\vec{v}_1+\vec{v}_2,\vec{v}_1+\vec{v}_2+\vec{v}_3\}$ is also a basis. (Hint: Method 2 may be a helpful way to show this is a spanning set.)

2.1.14. Let $\vec{p}_1=x^2+1$, $\vec{p}_2=x^2-1$, and $\vec{p}_3=3$. Show that $\{\vec{p}_1,\vec{p}_2,\vec{p}_3\}$ is a linearly dependent set, and find a basis for $\operatorname{Span}\{\vec{p}_1,\vec{p}_2,\vec{p}_3\}$.

2.1.15. As mentioned in the section, any nonempty subset of a linearly independent set is itself linearly independent. Let's show this.

Let $S=\{\vec{v}_1,\ldots,\vec{v}_n\}$ be a subset of some vector space V and suppose $\widehat{S}=\{\vec{v}_1,\ldots,\vec{v}_k\}$ is a subset of S. Suppose S is a linearly independent set.

- ► Suppose $\widehat{S}$ is not linearly independent. Then there are scalars $a_1,\ldots,a_k\in\mathbb{R}$ not all zero such that

$$a_1\vec{v}_1+\cdots a_k\vec{v}_k=\vec{0}$$

Explain how this violates the definition of linear independence for the set S. This allows us to conclude $\widehat{S}$ is also linearly independent since otherwise it can't be true that S was linearly independent.

2.1.16. The subset H below is a subspace of $\mathbb{R}^4$. Find a basis for H.

$$H = \left\{ \begin{bmatrix} x \\ -y \\ x \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}.$$

2.1.17. The subset J below is a subspace of $\mathbb{R}^4$. Find a basis for J.

$$J = \left\{ \begin{bmatrix} 2x + 3y + z \\ -y \\ x \\ y + z \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$$

2.1.18. The subset K below is a subspace of $\mathbb{R}^4$. Find a basis for K.

$$K = \left\{ \begin{bmatrix} x + y + 2z \\ -y - 2z \\ x \\ y + 2z \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$$

2.1.19. Find a basis for each subspace below.

(a) $\text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) $\text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(c) $\text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

2.1.20. Let

$$\vec{v}_1 = \begin{bmatrix} 0 \\ -5 \\ 1 \\ 2 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 3 \\ 4 \\ -2 \\ 5 \end{bmatrix}, \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \end{bmatrix}.$$

and

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \qquad \vec{u}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 6 \end{bmatrix}$$

(a) Check that $\vec{v}_1 + 2\vec{v}_2 - 3\vec{v}_3 = \vec{0}$.

(b) Find a basis for $H_1 = \operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

(c) Let $H_2 = \operatorname{Span}\{\vec{u}_1, \vec{u}_2\}$. Find a basis for $H_1 + H_2$.

2.1.21. Show that $\{1, i\}$ is a basis for $\mathbb{C}$.

2.1.22. Show that $\{1 + i, i\}$ is also a basis for $\mathbb{C}$.

2.1.23. In the Section 1.1 Exercises, you showed $V = \{a \colon a \in \mathbb{R}, a \geq 0\} = (0, \infty)$ is a vector space with addition given by $a \boxplus b = ab$ and scalar multiplication given by $ka = a^k$ for any $k \in \mathbb{R}$, $a, b \in V$. Find a basis for this vector space.

2.2 More Fun with Bases

In the previous section, we introduced the concept of a basis for a vector space and we focused on finding bases or showing a suspected basis is a basis. Now, we will see several results related to a basis for a vector space. Some of these will help us in our search to find a basis or to tell whether a set is a basis. Overall, they will show us how finding a basis for a vector space reveals an intrinsic property of the vector space, called dimension.[11]

11: Cool! Will I be able to use this to visit parallel universes?

Yes.

Really?!

...

How Large Can An Independent Set Be?

We'll start with a motivating example.

Example 2.2.1 Let's consider the set

$$S = \left\{ \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

While we could show the set S is linearly dependent in the same manner as we have in the past, we'll take a different approach for this example. Recall that the standard basis for $\mathbb{R}^2$ is

$$\left\{ \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

We can write each of the vectors in S as a linear combination of $\vec{e}_1$ and $\vec{e}_2$. In particular,

$$\begin{aligned} \vec{u}_1 &= \vec{e}_1 + \vec{e}_2, \\ \vec{u}_2 &= \vec{e}_1 - \vec{e}_2, \text{ and} \\ \vec{u}_3 &= \vec{e}_1 + 2\vec{e}_2. \end{aligned}$$

Now, we can solve for $\vec{e}_1$ in our first equation to get

$$\vec{e}_1 = \vec{u}_1 - \vec{e}_2.$$

Then we have

$$\begin{aligned} \vec{u}_2 &= \vec{e}_1 - \vec{e}_2 \\ &= (\vec{u}_1 - \vec{e}_2) - \vec{e}_2 \\ &= -\vec{u}_1 - 2\vec{e}_2. \end{aligned}$$

Rearranging this we see that

$$\vec{e}_2 = \frac{\vec{u}_1 + \vec{u}_2}{2}.$$

Then

$$\begin{aligned}\vec{u}_3 &= \vec{u}_1 - \vec{e}_2 + 2\vec{e}_2 \\ &= \vec{u}_1 + \vec{e}_2 \\ &= \vec{u}_1 + \frac{\vec{u}_1 + \vec{u}_2}{2} \\ &= \frac{3\vec{u}_1 + \vec{u}_2}{2} \\ &= \frac{3}{2}\vec{u}_1 + \frac{1}{2}\vec{u}_2.\end{aligned}$$

Since $\vec{u}_3$ is a linear combination of $\vec{u}_1$ and $\vec{u}_2$, this shows the set S is linearly dependent by Theorem 1.3.1. This method relied only on the fact that the set of vectors was larger than the size of the standard basis. Below we will generalize this example to prove this is always enough to conclude that such a set is linearly dependent.

Theorem 2.2.1 *If a vector space V has a basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$, then any set in V containing more than p vectors is linearly dependent.*

PROOF. Let $S = \{\vec{u}_1, \ldots, \vec{u}_{p+1}\} \subseteq V$. If any of these vectors are the zero vector, we are done, so assume that $\vec{u}_i \neq \vec{0}$ for $1 \leq i \leq p+1$. We shall attempt to write one of the vectors $\vec{u}_i$ as a linear combination of the other vectors in S. Since $\mathcal{B}$ is a basis for V and $\vec{u}_1 \in V$, there are weights such that

$$\vec{u}_1 = a_{11}\vec{v}_1 + \cdots + a_{1p}\vec{v}_p. \tag{2.1}$$

Since $\vec{u}_1 \neq \vec{0}$, we know $a_{1j} \neq 0$ for some $1 \leq j \leq p$. Suppose without loss of generality that $a_{11} \neq 0$. Then we may solve for $\vec{v}_1$ in Equation 2.1:

$$\vec{v}_1 = \frac{1}{a_{11}}\vec{u}_1 - \frac{a_{12}}{a_{11}}\vec{v}_2 - \cdots - \frac{a_{1p}}{a_{11}}\vec{v}_p. \tag{2.2}$$

Thus, $\vec{v}_1 \in \text{Span}\{\vec{u}_1, \vec{v}_2, \ldots, \vec{v}_p\}$. Again, since $\mathcal{B}$ is a basis for V and $\vec{u}_2 \in V$, there are weights such that

$$\vec{u}_2 = a_{21}\vec{v}_1 + \cdots + a_{2p}\vec{v}_p. \tag{2.3}$$

Since $\vec{v}_1 \in \text{Span}\{\vec{u}_1, \vec{v}_2, \ldots, \vec{v}_p\}$, we can find weights such that $\vec{v}_1 = b_{21}\vec{u}_1 + b_{22}\vec{v}_2 + \cdots + b_{2p}\vec{v}_p$. In fact, we already found these weights in Equation 2.2! Then

$$\begin{aligned}\vec{u}_2 &= a_{21}\vec{v}_1 + \cdots + a_{2p}\vec{v}_p \\ &= a_{21}\left(b_{21}\vec{u}_1 + b_{22}\vec{v}_2 + \cdots + b_{2p}\vec{v}_p\right) + a_{22}\vec{v}_2 + \cdots + a_{2p}\vec{v}_p \\ &= a_{21}b_{21}\vec{u}_1 + (a_{21}b_{22} + a_{22})\vec{v}_2 + \cdots + (a_{21}b_{2p} + a_{2p})\vec{v}_p \\ &= c_{21}\vec{u}_1 + c_{22}\vec{v}_2 + \cdots + c_{2p}\vec{v}_p, \qquad (2.4)\end{aligned}$$

Where the weights, $c_{21}, \ldots, c_{2p}$, are defined by the line in Equation 2.4 that immediately precedes them. What have we done here?[12] We can write $\vec{u}_2$ as a linear combination of the vectors in $\mathcal{B}$, but we can find different weights to replace $\vec{v}_1$ in our linear combination with $\vec{u}_1$. This is one step on the way to writing one of the vectors in S as a linear combination of the others. If $c_{22} = \cdots = c_{2p} = 0$, then $\vec{u}_2$ is a linear combination of $\vec{u}_1$. In that case,

12: What have we done? Was it good?

Yes. Pay attention!

the set S is linearly dependent by Theorem 1.3.1, and we are done. Otherwise, $c_{2j} \neq 0$ for some $2 \leq j \leq p$, so suppose without loss of generality that $c_{22} \neq 0$. We'll have to do this procedure again: Solve for $\vec{v}_2$ in Equation 2.4 to show $\vec{v}_3 \in \operatorname{Span}\{\vec{u}_1, \vec{u}_2, \vec{v}_3, \ldots, \vec{v}_p\}$; then use that to show $\vec{u}_3 \in \operatorname{Span}\{\vec{u}_1, \vec{u}_2, \vec{v}_3, \ldots, \vec{v}_p\}$.

How many times could we do this? One of two things must happen; either $\vec{u}_i$, for some $2 \leq i \leq p$, will be written as a linear combination of $\vec{u}_1, \ldots, \vec{u}_{i-1}$ (in which case S would be linearly dependent by Theorem 1.3.1), or we will have to run this procedure p times, writing each $\vec{v}_j$, for $1 \leq j \leq p$, with linear combinations of vectors in S. That is, for each $1 \leq j \leq p$,

$$\vec{v}_j = c_{1j}\vec{u}_1 + \cdots + c_{pj}\vec{u}_p = \sum_{i=1}^{p} c_{ij}\vec{u}_i. \tag{2.5}$$

In this case, we still have one more vector $\vec{u}_{p+1} \in S$. Since $\vec{u}_{p+1} \in V$, we can write $\vec{u}_{p+1}$ as a linear combination of vectors in $\mathcal{B}$ and then substitute the linear combinations in Equation 2.5 for each $\vec{v}_j$:

$$\begin{aligned} \vec{u}_{p+1} &= a_1\vec{v}_1 + \cdots + a_p\vec{v}_p \\ &= a_1\left(\sum_{i=1}^{p} c_{i1}\vec{u}_i\right) + \cdots + a_p\left(\sum_{i=1}^{p} c_{ip}\vec{u}_i\right). \end{aligned}$$

Thus, $\vec{u}_{p+1}$ is a linear combination of the other vectors in S, so by Theorem 1.3.1, S is linearly dependent. □

Now, let's consider how this works for us in $\mathbb{R}^4$.[13] According to the theorem we just proved, any set of five vectors in $\mathbb{R}^4$ is linearly dependent (because the standard basis for $\mathbb{R}^4$ has four vectors). But what about four vectors?

13: No particular reason to choose 4 here, just feeling like a 4 apparently.

Example 2.2.2 Let's consider two different sets of four vectors in $\mathbb{R}^4$. Here's the first one:

$$S_1 = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

This set is linearly dependent. To see this, note that $\vec{v}_1 = \vec{v}_2 + \vec{v}_3$. Here's another set:

$$S_2 = \left\{ \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{u}_3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

This set is linearly independent. If $a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 + d\vec{u}_4 = \vec{0}$, then we get the equations $a = 0$, $b + d = 0$, $a + d = 0$, $c + d = 0$. The only real numbers satisfying all of these are $a = b = c = d = 0$.

From these examples, we see that while our theorem says every set of vectors with 5 vectors in $\mathbb{R}^4$ must be linearly dependent, sets with 4 vector in $\mathbb{R}^4$ can go either way, dependent or independent.

Dimension

Let's make an observation. From our explorations and examples in Section 2.1, we've seen that $\mathbb{R}^2$ has bases

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

From this, we see that a vector space does not have a unique basis.[14] Notice something else, though: each of these bases has two vectors. We also saw that $\mathbb{P}_2$ has bases $\{1+x, x+x^2, x\}$ and $\{1, x, x^2\}$. Again, these bases have the same number of vectors. Well, we were due for some good news; that's always true!

14: This seems very inconvenient, but we will eventually make this work to our advantage.

Theorem 2.2.2 *If a vector space V has a basis with p vectors, then every basis for V must contain exactly p vectors.*

PROOF. Let $\mathcal{B}$ be a basis for V with p vectors. Suppose $\mathcal{B}_0$ is another basis for V. We know that $\mathcal{B}_0$ cannot have more than p vectors since then, by Theorem 2.2.1, $\mathcal{B}_0$ would be linearly dependent. We also know that $\mathcal{B}_0$ cannot have fewer than p vectors since then, by Theorem 2.2.1, $\mathcal{B}$ would be linearly dependent. Thus, $\mathcal{B}_0$ also has exactly p vectors. $\square$

Note that we didn't really talk about spans in that proof. However, there's a nice corollary about spanning sets that comes from this result. Since any set that is a basis must have the same number of vectors, we know any linearly independent set that has too few vectors to be a basis must fail to span. Otherwise, we would contradict this result!

Corollary 2.2.3 *If a vector space V has a basis with p vectors, then every spanning set for V must contain at least p vectors.*

Every vector space has many different bases;[15] Theorem 2.2.2 guarantees that they all have the same number of vectors. Good. This makes the following well-defined:

15: Actually, most of the vector spaces over $\mathbb{R}$ we will see have infinitely many bases!

Definition 2.2.1 *Let V be a vector space. The **dimension of** V, denoted $\dim V$, is the number of vectors in a basis for V.*

The term "dimension" is likely not new to you. People regularly make use of three spacial dimensions, and movie-going often offers us a choice between two and three dimensional experiences. What's new here is giving it a formal definition that relies on a basis. We can now say that $\mathbb{R}^n$ has dimension n and also that $\mathbb{P}_n$ has dimension $n+1$. This tells us immediately how large a basis for these spaces should be. This also gives us a nice way to talk about the subspaces in these vector spaces. The only subspace of any vector space with dimension 0 is the trivial subspace $\{\vec{0}\}$. In $\mathbb{R}^n$, the subspaces with dimension 1 are the lines through the origin, and the subspaces with dimension 2 are the planes through the origin. While we're talking about subspaces, here's a formal statement about their dimensions.

Theorem 2.2.4 *If H is a subspace of a vector space V, then $\dim H \leq \dim V$. Moreover, the only subspace of V with dimension $\dim V$ is V itself.*

Well, it would be lovely to prove this theorem. We need something else though first.

Theorem 2.2.5 *Let V be a p-dimensional vector space, where $p \geq 1$. Then*

(a) *Any linearly independent set of exactly p vectors in V is a basis for V.*

(b) *Any set of p vectors that spans V is a basis for V.*

Exploration 44 Let's do something a bit different. Let's prove this one together!

PROOF.

- First, let's suppose $\mathcal{D}$ is a set of exactly p linearly independent vectors. Well, suppose this is not a basis for V. Then, it must not span all of V. So there must be some vector $v \in V$ that is not in the span of $\mathcal{D}$. Then v is specifically not a linear combination of any of the vectors in $\mathcal{D}$, and none of the vectors in $\mathcal{D}$ are a linear combination of each other. So adding v to $\mathcal{D}$ *must* still give a linearly independent set of vectors. On the other hand, some of the results in the previous section can be used to argue that the set that is $\mathcal{D}$ with v added *must* be linearly dependent. Explain why it must be linearly dependent. It can't be *both* linearly independent and linearly dependent. What's going on here? Well, this was all built on the supposition that $\mathcal{D}$ is not a basis. Since we have arrived at something absurd based on this assumption, it must be that $\mathcal{D}$ *is* a basis. (Which is what we wanted to show!)[16]

- Now suppose $S \subseteq V$ is a set of exactly p vectors that spans all of V. We want to show that S is a basis for V. Since we know it spans V, all that is left is that we argue it must be linearly independent. Okay, suppose it's not. Suppose instead that it is linearly dependent. Like before, our goal will be to arrive at something absurd using this supposition. What does Theorem 2.1.2 allow us to do since this is a spanning set that is linearly dependent? How does this contradict Theorem 2.2.2?

□

That was just what we needed to prove Theorem 2.2.4!

16: This technique of proof is called "proof by contradiction." As a rule of thumb, it is a lovely way to include words such as "absurd", "preposterous", "silly", or "ridiculous" in your formal mathematical writing.

Formal writing like this, right?

Well. . .

PROOF OF THEOREM 2.2.4. Suppose H is a subspace of a vector space V with $\dim V = p$. If H is also dimension p, then it must have a basis of size p. However, from Theorem 2.2.5, we know any linearly independent set of size p in V must be a basis for p. Thus, we must have $H = V$ since any basis of H is also a basis for V. Otherwise, H must have a basis that is smaller in size than p since we know any set with more than p vectors will be linearly dependent by Theorem 2.2.1. □

Theorem 2.2.5 has already been useful, but it gets better. To be a basis for a vector space V, a set must do two things: it must be linearly independent and it must span V. That is, unless you know the dimension of V, say p, and your set of vectors already has p vectors in it. Then you only need one of the two criterion for a basis to be true! That is sometimes very helpful.

Example 2.2.3 Hey, remember that set we showed was a spanning set in Example 2.1.1? Oh, well, maybe we showed several sets were spanning sets actually since we had $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{Span}\{\vec{v}_2, \vec{v}_3\} = \text{Span}\{\vec{v}_1, \vec{v}_3\} = \text{Span}\{\vec{v}_1, \vec{v}_2\}$. Now, we can conclude, by Theorem 2.2.5, that any of those spanning sets of size two are a basis for $\mathbb{R}^2$ because they each have two vectors and span $\mathbb{R}^2$! Yay!
Let's be slightly more specific. We can all agree that $\mathbb{R}^2$ has dimension two because the standard basis $\{\vec{e}_1, \vec{e}_2\}$ for $\mathbb{R}^2$ has two vectors. By Theorem 2.2.5, any spanning set of two vectors in $\mathbb{R}^2$ is a basis for $\mathbb{R}^2$. This will work the same for $\mathbb{R}^n$ in general. Also, any linearly independent set of n vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

Example 2.2.4 Note that $\mathbb{P}_2$ has standard basis $\{1, x, x^2\}$, so it has dimension three. While $\{1 + x, x^2\}$ is a linearly independent set in $\mathbb{P}_2$, by Theorem 2.2.5 it does not span. It does not have enough vectors to be linearly independent *and* span $\mathbb{P}_2$; a set needs exactly three vectors to be linearly independent and span (i.e. be a basis for) a three dimensional vector space.

Exploration 45 Let's show that $\{\vec{p}_1 = 1, \vec{p}_2 = 2x, \vec{p}_3 = -2 + 4x^2, \vec{p}_4 = -11x + 8x^3\}$ is a basis for $\mathbb{P}_3$! Well, first of all, it has 4 vectors, which is luckily the correct number. Now we need only show that it is either linearly independent or that it spans all of $\mathbb{P}_3$. But which one is easier? You decide!

- *Linearly Independent.* Let a, b, c, d be real numbers such that

  $$a\vec{p}_1 + b\vec{p}_2 + c\vec{p}_3 + d\vec{p}_4 = \vec{0}.$$

  Then

  $$a + 2bx - 2c + 4cx^2 - 11dx + 8dx^3 = 0.$$

  Why can we conclude that $a = b = c = d = 0$?

- *Spans* $\mathbb{P}_3$. Let $a + bx + cx^2 + dx^3$ be any vector in $\mathbb{P}_3$. Find coefficients of $\vec{p}_1$, $\vec{p}_2$, $\vec{p}_3$, and $\vec{p}_4$ so that

$$a + bx + cx^2 + dx^3 = \underline{\qquad}\vec{p}_1 + \underline{\qquad}\vec{p}_2 + \underline{\qquad}\vec{p}_3 + \underline{\qquad}\vec{p}_4.$$

  It will be easiest to first determine the coefficient of $\vec{p}_4$, then $\vec{p}_3$, then $\vec{p}_2$, and lastly $\vec{p}_1$. (This one seems harder...)

You may perhaps be wondering why you bothered at all to learn how to show a set is a spanning set.[17] The algebra of that is usually much trickier than showing a set is linearly independent. However, there is something extremely useful about being able to write a vector as a linear combination of basis vectors, as we'll see in just the next section.

17: I was definitely wondering.

Before concluding our initial discussion of dimension, it is important to not that our definition of basis specifies that a basis is a finite set of vectors. Hence, any vector space for which we use this definition (basis or dimension) must be *finite dimensional*. Indeed, one should assume that for remainder of this book, all vectors spaces have are finite dimensional. Infinite dimensional vectors spaces do exist and are a lot of fun, but their fun is simply too great to be contained in this book.

Section Highlights

- A basis $\mathcal{B}$ for vector space V is a minimal spanning set in the following sense: If any vector is removed from $\mathcal{B}$, the resulting set will fail to span V.
- Any set with fewer vectors than a basis will automatically fail to span the vector space. See Corollary 2.2.3.
- A basis $\mathcal{B}$ is a maximal linearly independent set in the vector space V in the following sense. If any vector is apended to $\mathcal{B}$, the resulting set will be linearly dependent. See Theorem 2.2.1.
- Every basis for a specific vector space V has the same number of vectors. This number of vectors is the *dimension* of the vector space. See Theorem 2.2.2 and Definition 2.2.1.
- If the dimension of a vector space is known, it can simplify the determination of whether a set is a basis. See Example 2.2.3.

Exercises for Section 2.2

2.2.1. Let $\vec{x}_1, \vec{x}_2, \vec{x}_3 \in \mathbb{R}^3$ be such that $\operatorname{Span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\} = \mathbb{R}^3$. Explain why $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is a basis for $\mathbb{R}^3$.

2.2.2. Let $\vec{v}_1, \ldots, \vec{v}_7 \in \mathbb{R}^7$. Suppose $H_6 = \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_6\}$ and $H_7 = \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_7\}$. Explain why $\mathbb{R}^7 \neq H_6$. Must it be true that $\mathbb{R}^7 = H_7$?

2.2.3. Recall that in Exercise 1.3.21 in Section 1.3 we showed that $\mathbb{R}^2 = \operatorname{Span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \end{bmatrix}\right\} = H$. Here's another fun way to do that using the fact that $\mathbb{R}^2 = \operatorname{Span}\{\vec{e}_1, \vec{e}_2\}$. I know

$$\begin{bmatrix} 7 \\ 0 \end{bmatrix} \in H,$$

so $\vec{e}_1 \in H$ as well. But that means

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} - 2\vec{e}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \in H,$$

so $\vec{e}_2 \in H$. Since $\dim H = 2$, we know $H = \operatorname{Span}\{\vec{e}_1, \vec{e}_2\} = \mathbb{R}^2$. Neat, eh? Adapt this procedure to show that

$$\operatorname{Span}\left\{\begin{bmatrix} 7 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}\right\} = \mathbb{R}^3.$$

2.2.4. The following sets are all too large to be linearly independent. Find a vector that is a linear combination of the others in the set to verify this.

(a) $\{1+x, 1-x, 1+x+x^2, 2x^2\}$ in $\mathbb{P}_2$.

(b) $\{1+x^2, 1-x, 1+2x+x^2, 1-2x^2, x\}$ in $\mathbb{P}_2$.

(c) $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \end{bmatrix}\right\}$ in $\mathbb{R}^2$

(d) $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ 1 \end{bmatrix}\right\}$ in $\mathbb{R}^3$

2.2.5. The following sets S are linearly independent but too small to be spanning sets. Find a vector that is not in $\operatorname{Span}\{S\}$ and add it to the set to form a basis for the vector space. (Hint: If it is not a spanning set, there must be at least one standard basis vector that is missing.)

(a) $\{1+x, 1-x^2\}$ in $\mathbb{P}_2$.

(b) $\{1+x^2, 1-x\}$ in $\mathbb{P}_2$.

(c) $\left\{\begin{bmatrix} 5 \\ 1 \end{bmatrix}\right\}$ in $\mathbb{R}^2$

(d) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^3$

(e) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^3$

(f) $\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^3$

2.2.6. If a set is the correct size to be a basis according to the dimension of the vector space, but it is not a basis, it must fail both conditions to be a basis. That is, it must be linearly dependent and fail to be a spanning set. Verify that each of these sets fail both conditions to be a basis.

(a) $\{1 + x, 1 + 2x + x^2, 2 + 3x + x^2\}$ in $\mathbb{P}_2$.

(b) $\{1 + x^2, 1 - x, x + x^2\}$ in $\mathbb{P}_2$.

(c) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^3$

(d) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix} \right\}$ in $\mathbb{R}^3$

(e) $\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 5 \end{bmatrix} \right\}$ in $\mathbb{R}^3$

2.2.7. Show the sets below are bases for the given vector space.

(a) $\left\{\vec{5}\right\}$ for $\mathbb{R}$

(b) $\{5, 1 + x\}$ for $\mathbb{P}_1$

(c) $\{x, 1 + x\}$ for $\mathbb{P}_1$

(d) $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \end{bmatrix} \right\}$ for $\mathbb{R}^2$

(e) $\left\{ \begin{bmatrix} -2 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \end{bmatrix} \right\}$ for $\mathbb{R}^2$

(f) $\left\{x, 1 + x, x^2\right\}$ for $\mathbb{P}_2$

(g) $\{1+x, x+x^2, x^2\}$ for $\mathbb{P}_2$

(h) $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^3$

(i) $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^3$

(j) $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^4$

2.2.8. Find a basis for each subspace below. State the dimension of the subspace.

(a) $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \right\}$

(b) $H = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -7 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \right\}$

(c) $H = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

2.2.9. Find a basis for the set of vectors in $\mathbb{R}^3$ that lie in the subspace

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x + 4y - z = 0 \right\}.$$

To do this, first identify a vector in $\mathbb{R}^3$ that is not in W. Then, find two linearly independent vectors that are in W. We know W cannot be 3-dimensional because you found a vector that is not in W. Since you were able to find two linearly independent vectors that are in W, we can conclude that $\dim W = 2$ and these vectors form a basis.

2.2.10. Consider the subspaces below for $\mathbb{R}^3$.

$$U = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad V = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(a) Show that $\dim U = 2$ and $\dim V = 2$.

(b) Since both U and V are 2-dimensional subspaces of a 3-dimensional space, they must have an intersection that is at least 1-dimensional. Why? Find $U \cap V$ to verify this.

2.3 Coordinates, Inner Products, and Orthogonality: Oh my!

At the start of this chapter, we promised to focus more on geometry. We are now ready to see how the concept of a basis for a vector space allows us to extend the geometry we already talked about in $\mathbb{R}^n$ to other vector spaces.[18]

18: What geometry?

Length and distance via an inner product.

Bases In Action: Coordinates

One of the best things about bases is that they provide a convenient way to organize vectors in a vector space. If every vector in the vector space V is a linear combination of the vectors in some set $\mathcal{B} \subseteq V$, then those linear combinations can be used as a description for the vectors. This is something you are actually very much accustomed to; consider the standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ for $\mathbb{R}^3$. Consider the following example:

$$\begin{aligned}
\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} &= \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix} \\
&= 7\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 8\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 9\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= 7\vec{e}_1 + 8\vec{e}_2 + 9\vec{e}_3.
\end{aligned}$$

Now we know that $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a basis for $\mathbb{R}^3$, so we shouldn't be surprised that we can write any vector in $\mathbb{R}^3$ as a linear combination of $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$. It also turns out that the weights on that linear combination, 7, 8, and 9 respectively, have meaning. What is the vector

$$\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}?$$

It's 7 $\vec{e}_1$'s, 8 $\vec{e}_2$'s, and 9 $\vec{e}_3$'s. What if we used a different basis? Those weights would probably have to change, right?

Exploration 46 Let

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ and } \vec{x} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Note that $\mathcal{B}$ is a basis for $\mathbb{R}^3$.[19] Write $\vec{x}$ as a linear combination using this basis.

19: Or instead of "note", maybe you should check this?

Or maybe get a group of friends together for a basis-checking party!

What if there was more than one linear combination for a particular vector and a particular basis? Well, fortunately that cannot happen:

Theorem 2.3.1 *Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be a basis for a vector space V. Then for each $\vec{v} \in V$, there exists a unique set of scalars $c_1, \ldots, c_p \in \mathbb{R}$ such that*

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p. \tag{2.6}$$

PROOF. The existence of an equation such as Equation 2.6 is guaranteed by the fact every basis must span the vector space. Thus, the content here is really in showing that such an equation is unique. Suppose there is another set of scalars $d_1, \ldots, d_p \in \mathbb{R}$ such that

$$\vec{v} = d_1\vec{v}_1 + \cdots + d_p\vec{v}_p. \tag{2.7}$$

Subtracting Equation 2.7 from Equation 2.6, we have

$$\vec{0} = \vec{x} - \vec{x} = (c_1 - d_1)\vec{v}_1 + \cdots + (c_p - d_p)\vec{v}_p.$$

Since $\mathcal{B}$ is a basis, we know it is linearly independent, so by Definition 1.3.3 we have that $c_i - d_i = 0$ for all $1 \leq i \leq p$. Thus, $c_i = d_i$ for all $1 \leq i \leq p$; our new batch of scalars has to be the same as the original ones. □

This is great. Given a basis for a vector space, we can write every vector in that vector space as a *unique* linear combination of the basis vectors. Then we only really need to know the weights, right? Each set of weights associated to each vector provides all the information you need for that vector.

Definition 2.3.1 *Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be a basis for a vector space V, and suppose $\vec{v} \in V$. The **coordinates for $\vec{v}$ relative to $\mathcal{B}$** (or the **$\mathcal{B}$-coordinates of $\vec{v}$**) are the weights $c_1, \ldots, c_p$ such that*

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p. \tag{2.8}$$

*The **coordinate vector of $\vec{v}$ relative to $\mathcal{B}$** is*

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}. \tag{2.9}$$

Yay! Now we can use coordinates relative[20] to a basis to write any vector in any p-dimensional vector space as a vector in $\mathbb{R}^p$! Note that the order of the coordinates is determined entirely by the order of the basis vectors.

20: Some other books say "with respect to $\mathcal{B}$" rather than "relative to $\mathcal{B}$."

What if $\mathcal{B}$ is not a respectable basis?

Then those other books look very silly.

Example 2.3.1 Let

$$\vec{x} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \in \mathbb{R}^2.$$

Using the standard basis for $\mathbb{R}^2$, $\{\vec{e}_1, \vec{e}_2\}$, we have that $\vec{x} = 4\vec{e}_1 + 0\vec{e}_2$, but this is neither interesting nor exciting. Perhaps we could use a different basis, $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 5 \\ -6 \end{bmatrix}.$$

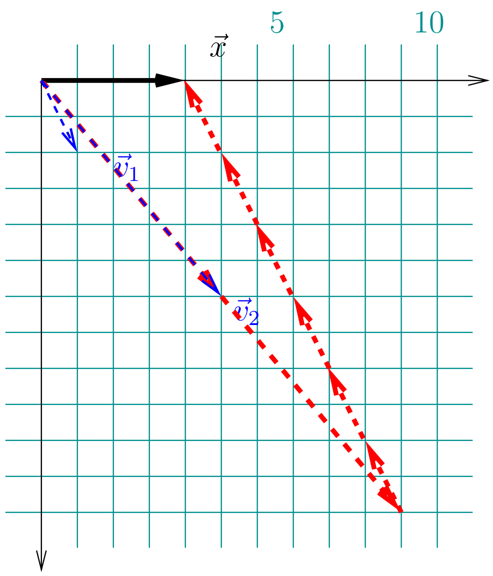


FIGURE 2.1. The vector $\vec{x} \in \mathbb{R}^2$ is shown with a solid arrow line. Using $\{\vec{v}_1, \vec{v}_2\}$ (with dashed arrow line) as a basis for $\mathbb{R}^2$, we see that $\vec{x} = -6\vec{v}_1 + 2\vec{v}_2$.

We can still write $\vec{x}$ as a linear combination of the vectors in this basis:

$$\vec{x} = -6\vec{v}_1 + 2\vec{v}_2,$$

and this can be seen in Figure 2.1. Thus, the coordinates for $\vec{x}$ relative to $\mathcal{B}$ are

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -6 \\ 2 \end{bmatrix},$$

indicating that to get the vector $\vec{x}$ with vectors from $\mathcal{B}$, you'll need -6 $\vec{v}_1$'s and 2 $\vec{v}_2$'s. You can think of these coordinates (the weights) as an address using "the directions" given in your basis.

Exploration 47 Let's apply what we just learned to $\mathbb{P}_2$. Let

$$\mathcal{B}_1 = \{1, x, x^2\} \text{ and } \mathcal{B}_2 = \{1+x, x+x^2, x\}.$$

- Let $\vec{v} = 2 + 3x + 4x^2$. Find $[\vec{v}]_{\mathcal{B}_1}$ and $[\vec{v}]_{\mathcal{B}_2}$.

- Let $\vec{u} = a + bx + cx^2$. Find $[\vec{u}]_{\mathcal{B}_1}$ and $[\vec{u}]_{\mathcal{B}_2}$.

Let's think about what we've just done. We've used two different bases for $\mathbb{P}_2$ to write a vector in $\mathbb{P}_2$ in a way that looks like a vector in $\mathbb{R}^3$. This seems quite useful. In fact, based on what we know from Chapter 0, it seems like there could be a function from $\mathbb{P}_2$ to $\mathbb{R}^3$ floating around here. However, the vectors are different when the bases are different, so there's not just a single way to do this.

Geometry in Vector Spaces

We need to tidy up our geometric intuition a bit with respect to more general vector spaces. In the previous chapter, we learned about an inner product on $\mathbb{R}^n$ as well as the concept of distance in $\mathbb{R}^n$; we just need to generalize these definitions so they can be used for any vector space.

If only there was a way to write vectors from a vectors space V as vectors in $\mathbb{R}^n$...[21]

21: I know this. Don't tell me.

Yes! Very good! Coordinate vectors![22]

22: I was just about to say that!

Definition 2.3.2 *Let V be a vector space with basis $\mathcal{B}$ and let $[\cdot]_{\mathcal{B}} : V \to \mathbb{R}^n$ be the function that relates vectors in V to their coordinate vector relative to $\mathcal{B}$ in $\mathbb{R}^n$. The* ***inner product on** V **relative to** $\mathcal{B}$ *is the function $\cdot_{\mathcal{B}} : V \times V \to \mathbb{R}$ defined as the composition of $[\cdot]_{\mathcal{B}} \times [\cdot]_{\mathcal{B}}$ on $V \times V$ with the standard inner product $\cdot$ on $\mathbb{R}^n \times \mathbb{R}^n$ (from Section 1.2). That is, for any vectors $\vec{v}, \vec{u} \in V$, we define*

$$\vec{v} \cdot_{\mathcal{B}} \vec{u} = [\vec{v}]_{\mathcal{B}} \cdot [\vec{u}]_{\mathcal{B}} . \tag{2.10}$$

There are a couple of annoying things about this. First, the inner product on a vector space depends on the basis you're using. Unfortunately, you're just going to have to deal with that; it's a fact of life. Second, the notation is a bit obnoxious; since the inner product depends on the basis, the dot for our function name has to carry around a $\mathcal{B}$ as a subscript. We propose this compromise. If we all agree to understand that the inner product on a vector space is dependent on the basis you're using, we can just suppress the subscript $\mathcal{B}$. Then Equation (2.10) becomes

$$\vec{v} \cdot \vec{u} = [\vec{v}]_{\mathcal{B}} \cdot [\vec{u}]_{\mathcal{B}} ,$$

which definitely looks better.

We also quickly glossed over that "composition" bit in Definition 2.3.2. We'll discuss composition in greater detail in Section 3.1, but for now, just think of function composition as using the outputs of the first functions as the inputs of the second function. A diagram like this

$$\begin{tikzcd}
\mathbb{R}^n \times \mathbb{R}^n \arrow[dr, "\bullet"] & \\
V \times V \arrow[u, "{[\cdot]_{\mathcal{B}} \times [\cdot]_{\mathcal{B}}}"] \arrow[r, "\bullet_{\mathcal{B}}"'] & \mathbb{R},
\end{tikzcd}$$

called a **commuting diagram**, is often used to describe this situation. It is read as follows: to get the inner product from $V \times V$ to $\mathbb{R}$ (the arrow from left to right), you take coordinate vectors (the up arrow) then the standard inner product on $\mathbb{R}^n$ (the diagonal arrow).

This all probably sounds much more complicated than it actually is. Let's try some examples.

Example 2.3.2 Let $\vec{p} = x^2 - 2x + 1$ and $\vec{q} = x^2 + 2x + 6$ be vectors in $\mathbb{P}_2$. What is the inner product of $\vec{p}$ and $\vec{q}$? How does one take the inner product of two polynomials? Right. This is exactly the issue we've been solving. Before you can take an inner product, you need the coordinate vectors for $\vec{p}$ and $\vec{q}$ relative to some basis. Let's use the standard basis for $\mathbb{P}_2$: $\mathcal{B} = \{1, x, x^2\}$. Then

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \text{ and } [\vec{q}]_{\mathcal{B}} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}.$$

Using the coordinate vectors, we have

$$\vec{p} \cdot \vec{q} = [\vec{p}]_{\mathcal{B}} \cdot [\vec{q}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} = 6 - 4 + 1 = 3.$$

Exploration 48 We've said that this inner product depends on the basis, since the coordinate vectors depend on the basis. Let's see this in action. Again, let $\vec{p} = x^2 - 2x + 1$ and $\vec{q} = x^2 + 2x + 6$. Back in Section 2.1, we showed that $\mathcal{B}_1 = \{1 + x, x + x^2, x\}$ also formed a basis of $\mathbb{P}_2$.

- Find the coordinate vectors $[\vec{p}]_{\mathcal{B}_1}$ and $[\vec{q}]_{\mathcal{B}_1}$. (You might find it helpful to look back at Example 2.1.2.)

- Now, use these coordinate vectors to compute $\vec{p} \cdot \vec{q}$.

The answer's different, right? So, the inner product can depend on the basis.

Exploration 49 In the exercises of Section 2.2, we saw that $\{\vec{1}, \vec{i}\}$ forms a basis for the complex numbers $\mathbb{C} = \{a + bi\colon a, b \in \mathbb{R}\}$ when viewed as a real vector space. Use this basis to compute $(1 \vec{+} i) \cdot (-1 \vec{+} 2i)$. Keep in mind that the dot here indicates inner product of vectors and *not* multiplication of complex numbers.

This whole "just use the coordinate vector" strategy is pretty great. Let's apply it to length, too!

Definition 2.3.3 *Let V be a vector space with basis $\mathcal{B}$ and let $[\cdot]_{\mathcal{B}} : V \to \mathbb{R}^n$ be the function that relates vectors in V to their coordinate vector relative to $\mathcal{B}$ in $\mathbb{R}^n$.* ***Length relative to*** *$\mathcal{B}$ (or the* ***norm relative to*** *$\mathcal{B}$) is the function $\|\cdot\|_{\mathcal{B}} : \mathbb{V} \to \mathbb{R}$ defined by relating vectors to their length by composing the function $[\cdot]_{\mathcal{B}} \times [\cdot]_{\mathcal{B}}$ on V with $\|\cdot\|$ on $\mathbb{R}^n$. That is, for any vector $\vec{v} \in V$, we define*

$$\|\vec{v}\|_{\mathcal{B}} = \|[\vec{v}]_{\mathcal{B}}\| . \tag{2.11}$$

Again, we'll all agree that length is also relative to basis and regularly omit the subscript $\mathcal{B}$ on the norm of vectors in general vector spaces. As before, the idea is that if you don't have a vector in $\mathbb{R}^n$, you'll need to make it a coordinate vector before you can take its length.

Example 2.3.3 What is the norm of $\vec{p} = x^3 + 4x - 2 \in \mathbb{P}_3$? The length of this vector depends on your choice of basis. For lack of imagination, we'll use the standard basis again, $\mathcal{B} = \{1, x, x^2, x^3\}$. Then

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 4 \\ 0 \\ 1 \end{bmatrix},$$

so

$$\|\vec{p}\| = \|[\vec{p}]_{\mathcal{B}}\| = \left\| \begin{bmatrix} -2 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\| = \sqrt{(-2)^2 + 4^2 + 0^2 + 1^2} = \sqrt{21}$$

Exploration 50 Now let's do something similar using $\mathbb{P}_2$. As before, let $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{B}_1 = \{1 + x, x + x^2, x\}$. Let $\vec{p} = x + 5$.

- Find $[\vec{p}]_{\mathcal{B}}$ and $[\vec{p}]_{\mathcal{B}_1}$.
- Find $\|\vec{p}\|_{\mathcal{B}}$.
- Find $\|\vec{p}\|_{\mathcal{B}_1}$.

Example 2.3.4 Now, we again have the concept of unit vectors. Let's find a unit vector in the direction of $\vec{q} = 5x^4 - 2x + 3 \in \mathbb{P}_4$ using the standard basis $\mathcal{B}$ for $\mathbb{P}_4$. First, the coordinate vector for $5x^4 - 2x + 3$ is

$$[\vec{q}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \\ 5 \end{bmatrix}.$$

Now, we find that the length of this vector is

$$\|\vec{q}\| = \sqrt{9+4+25} = \sqrt{38}.$$

The unit vector is then the vector with coordinate vector given by

$$\begin{bmatrix} \frac{3}{\sqrt{38}} \\ \frac{-2}{\sqrt{38}} \\ 0 \\ 0 \\ \frac{5}{\sqrt{38}} \end{bmatrix}.$$

To find the actual unit vector, we should then translate this back into a vector in $\mathbb{P}_4$. Our final answer is then

$$\frac{3}{\sqrt{38}} - \frac{2x}{\sqrt{38}} + \frac{5x^4}{\sqrt{38}}.$$

Moving forward, we'll often need to make use of inner products. Since one can choose from a lot of different bases and inner products on a vector space, we'll usually need to agree on a basis and inner product before proceeding with the use of any inner product.

Definition 2.3.4 *We call a vector space, V, together with inner product relative to basis $\mathcal{B}$ an **inner product space**.*

In practice, whenever you see the words "inner product space," know that we're just talking about a vectors space with a specifically chosen inner product and basis.

Now for something completely different. What does it mean for vectors to be perpendicular in $\mathbb{R}^2$? Right,[23] their directions differ by $90°$ or, better yet, $\pi/2$ radians. What about in $\mathbb{R}^3$ or, better yet, $\mathbb{R}^5$? Can you imagine what it means for "directions differ by $90°$" in $\mathbb{R}^5$? Right. The notion of perpendicular is too restricted by geometry in $\mathbb{R}^n$ (for small n!) to be useful for general vector spaces. We need something better.

23: Pun intended.

Definition 2.3.5 *Vectors $\vec{v}$ and $\vec{u}$ in vector space V with chosen basis $\mathcal{B}$ are said to be **orthogonal** if $\vec{v} \cdot \vec{u} = 0$.*

Great, but does it still work the same as perpendicular in $\mathbb{R}^n$ for small n? It does!

Theorem 2.3.2 *Two nonzero vectors $\vec{v}$ and $\vec{u}$ in $\mathbb{R}^2$ are orthogonal with respect to the standard inner product if and only if they are perpendicular.*

PROOF. We first need to better define what it means to be perpendicular in $\mathbb{R}^2$. Let α be the angle between $\vec{v}$ and $\vec{u}$. By the Law of Cosines and definition of distance,

$$\begin{aligned} \operatorname{dist}(\vec{v},\vec{u})^2 &= \|\vec{v}\|^2 + \|\vec{u}\|^2 - 2\|\vec{v}\|\|\vec{u}\|\cos\alpha \quad \text{and} \\ \operatorname{dist}(\vec{v},\vec{u})^2 &= \|\vec{v}-\vec{u}\|^2 \\ &= (\vec{v}-\vec{u})\cdot(\vec{v}-\vec{u}) \\ &= \|\vec{v}\|^2 + \|\vec{u}\|^2 - 2\vec{v}\cdot\vec{u}. \end{aligned}$$

This gives us two different expressions, each equal to $\operatorname{dist}(\vec{v}, \vec{u})^2$. Thus we have
$$\|\vec{v}\|^2 + \|\vec{u}\|^2 - 2\|\vec{v}\|\|\vec{u}\| \cos\alpha = \|\vec{v}\|^2 + \|\vec{u}\|^2 - 2\vec{v} \cdot \vec{u}.$$
Once we simplify this a bit, it follows that
$$\vec{v} \cdot \vec{u} = \|\vec{v}\|\|\vec{u}\| \cos\alpha.$$
Since $\vec{v}$ and $\vec{u}$ are nonzero, $\vec{v} \cdot \vec{u} = 0$ if and only if $\alpha = 90°$. □

Also, we do have a concept of the angle between vectors in $\mathbb{R}^3$. Since any two vectors in $\mathbb{R}^3$ lie in a plane together, the angle between the vectors can be determined by the angle between them specifically in that plane.[24] Our definition of orthogonality also agrees with the use of perpendicular to mean the angle between two vectors in $\mathbb{R}^3$ is $90°$.

24: The Law of Cosines also works in that plane as well!

Orthogonality is great. In $\mathbb{R}^n$ (for small n), it means the same thing as perpendicular, but it also works in any vector space. That said, for the sake of consistency, we shall only use the word "orthogonal" in this course.

Exploration 51 Let
$$\vec{v}_1 = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
Note that $\vec{v}_2 \cdot \vec{v}_3 = 0$, so $\vec{v}_2$ and $\vec{v}_3$ are orthogonal to each other. Find all pairs of orthogonal vectors.

Exploration 52 Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The vector $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is orthogonal to $\vec{v}$. Find a vector $\vec{w}$ orthogonal to $\vec{v}$ so that $\{\vec{u}, \vec{w}\}$ is a linearly independent set.

As was noted in Section 1.2, length in vector spaces is a generalization of absolute value in $\mathbb{R}$. It should come as little surprise then that many of your favorite facts about absolute value are true for length in vector spaces.

Theorem 2.3.3 (The Triangle Inequality) *For any two vectors $\vec{v}$ and $\vec{u}$ in vector space V,*
$$\|\vec{v} + \vec{u}\| \leq \|\vec{v}\| + \|\vec{u}\|.$$

The proof for this fact would take us a bit off track, so we've omitted it.

Theorem 2.3.4 (The Pythagorean Theorem) *For any two orthogonal vectors $\vec{v}$ and $\vec{u}$ in a vector space V,*
$$\|\vec{v} + \vec{u}\|^2 = \|\vec{v}\|^2 + \|\vec{u}\|^2$$

Note that in the theorem above, the length function is computed using the same basis for which the vectors are orthogonal.

PROOF. First, we need a fact about coordinate vectors. Let $\vec{v}$ and $\vec{u}$ be vectors in a vector space V with basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$. Then there are real numbers $v_1, \ldots, v_n$ and $u_1, \ldots, u_n$ so that

$$\begin{aligned}\vec{v} =& v_1\vec{b}_1 + \cdots + v_n\vec{b}_n \\ \vec{u} =& u_1\vec{b}_1 + \cdots + u_n\vec{b}_n\end{aligned}$$

Thus, $\vec{v} + \vec{u} = (v_1\vec{b}_1 + \cdots + v_n\vec{b}_n) + (u_1\vec{b}_1 + \cdots + u_n\vec{b}_n) = (v_1 + u_1)\vec{b}_1 + \cdots + (v_n + u_n)\vec{b}_n$. From this equation we see that

$$[\vec{v} + \vec{u}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{B}} + [\vec{u}]_{\mathcal{B}}.$$

Now, we will use this to prove our theorem. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then, by the definition of length in $\mathbb{R}^n$ we have

$$\begin{aligned}\|\vec{x} + \vec{y}\|^2 =& \left(\sqrt{(x_1 + y_1)^2 + \cdots + (x_n + y_n)^2}\right)^2 \\ =& (x_1 + y_1)^2 + \cdots + (x_n + y_n)^2 \\ =& (x_1^2 + 2x_1y_1 + y_1^2) + \cdots + (x_n^2 + 2x_ny_n + y_n^2) \\ =& (x_1^2 + \cdots + x_n^2) + 2(x_1y_1 + \cdots + x_ny_n) + (y_1^2 + \cdots + y_n^2) \\ =& \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2\end{aligned}$$

Then if we know in addition that $\vec{x}$ and $\vec{y}$ are orthogonal, then $2\vec{x} \cdot \vec{y} = 0$ and we have

$$\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2.$$

Thus, the theorem holds for vectors in $\mathbb{R}^n$.

Now let $\vec{v}$ and $\vec{u}$ be orthogonal vectors in a vector space V with basis $\mathcal{B}$. We then have

$$\begin{aligned}\|\vec{v} + \vec{u}\|^2 =& \| [\vec{v} + \vec{u}]_{\mathcal{B}} \|^2 \\ =& \| [\vec{v}]_{\mathcal{B}} + [\vec{u}]_{\mathcal{B}} \|^2 \\ =& \| [\vec{v}]_{\mathcal{B}} \|^2 + \| [\vec{u}]_{\mathcal{B}} \|^2 \\ =& \|\vec{v}\|^2 + \|\vec{u}\|^2\end{aligned}$$

□

Section Highlights

- There is a unique way to write any vector $\vec{v}$ in a vector space V as a linear combination of the vectors in a basis for V. The coefficients from this linear combination form the *coordinate vector* for $\vec{v}$. See Theorem 2.3.1 and Definition 2.3.1.
- Coordinate vectors can be used to define the inner product like in Section 1.2 for any vector space. See Definition 2.3.2.

- ► The concept of perpendicular vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ is extended to general vector spaces of dimension n as orthogonality. Two vectors are said to be orthogonal if their inner product is zero. See Definition 2.3.5 and Theorem 2.3.2.

Exercises for Section 2.3

2.3.1. Let $\mathcal{B} = \left\{1+x, 2x, x+x^2\right\}$. This is a basis for $\mathbb{P}_2$.

(a) Find $\left[1+x+x^2\right]_{\mathcal{B}}$ and $[1]_{\mathcal{B}}$.

(b) Suppose $\vec{p} \in \mathbb{P}_2$ has coordinate vector $[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. Find the polynomial $\vec{p} \in \mathbb{P}_2$.

2.3.2. Let $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \end{bmatrix} \right\}$. This is a basis for $\mathbb{R}^2$.

(a) Find $\begin{bmatrix} 4 \\ 6 \end{bmatrix}_{\mathcal{B}}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{B}}$.

(b) Suppose $\vec{x} \in \mathbb{R}^2$ has coordinate vector $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $\vec{x}$.

2.3.3. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. This is a basis for $\mathbb{R}^3$.

(a) Find $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{B}}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}}$.

(b) Suppose $\vec{x} \in \mathbb{R}^3$ has coordinate vector $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$. Find $\vec{x}$.

2.3.4. Below are two bases for $\mathbb{P}_2$:

$$\begin{aligned} \mathcal{B}_1 &= \left\{x, 1+x, x^2\right\} \text{ and} \\ \mathcal{B}_2 &= \left\{1+x, x+x^2, x^2\right\}. \end{aligned}$$

Find $\left[1+2x-3x^2\right]_{\mathcal{B}_1}$ and $\left[1+2x-3x^2\right]_{\mathcal{B}_2}$.

2.3.5. Below are two bases for $\mathbb{R}_3$:

$$\begin{aligned} \mathcal{B}_1 &= \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \right\} \text{ and} \\ \mathcal{B}_2 &= \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

Find $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{B}_1}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{B}_2}$.

2.3.6. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ be a basis for a vector space V. Show that for any constants k_1 and k_2,

$$\left[k_1\vec{b}_1\right]_{\mathcal{B}} = \begin{bmatrix} k_1 \\ 0 \end{bmatrix} \quad \text{and} \quad \left[k_2\vec{b}_2\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ k_2 \end{bmatrix}.$$

2.3.7. Find the distance between

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}.$$

2.3.8. Find the distance, using the standard basis $\{1, x, x^2, x^3\}$ for $\mathbb{P}_3$, between $1 + x + x^3$ and $-2x - 2x^3$.

2.3.9. Let

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \text{ and } \vec{v}_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

Find all pairs of orthogonal vectors among $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$.

2.3.10. Let

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \text{ and } \vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Find all pairs of orthogonal vectors among $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$.

2.3.11. Let $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. The vector $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is orthogonal to $\vec{v}$. Find a vector $\vec{w}$ orthogonal to $\vec{v}$ so that $\{\vec{u}, \vec{w}\}$ is a linearly independent set.

2.3.12. Let $\vec{v}, \vec{u}_1, \vec{u}_2$ be vectors in some vector space V.

(a) Suppose $\vec{v}$ is orthogonal to $\vec{u}_1$ and $\vec{u}_2$. Use properties of the dot product to argue that $\vec{v}$ is orthogonal to $\vec{u}_1 + \vec{u}_2$.

(b) Suppose $\vec{v}$ is orthogonal to $\vec{u}_1 + \vec{u}_2$. Must it be true that $\vec{v}$ is orthogonal to $\vec{u}_1$ and $\vec{u}_2$? Explain.

(c) Suppose $\vec{v}$ is orthogonal to $\vec{u}_1$ and $\vec{u}_2$. If $\vec{x} \in \text{Span}\,\{\vec{u}_1, \vec{u}_2\}$, must $\vec{v}$ be orthogonal to $\vec{x}$? Explain.

2.3.13. Let $\vec{v} \in \mathbb{R}^n$. Why is $\|\vec{v}\| \geq 0$? When is $\|\vec{v}\| = 0$? Explain.

2.3.14. Let $\mathcal{B}$ be the standard basis for $\mathbb{P}_3$ and $\mathcal{C} = \{1, 1 + x, x + x^2, x^2 + x^3\}$, which is also a basis of $\mathbb{P}^3$. Let $\vec{v} = 2 + x + x^3$ and $\vec{u} = x + 3x^2 - x^3$.

(a) Find $[\vec{v}]_{\mathcal{B}}$ and $[\vec{u}]_{\mathcal{B}}$.

(b) Find $\vec{v} \cdot \vec{u}$ with respect to $\mathcal{B}$.

(c) Are $\vec{v}$ and $\vec{u}$ orthogonal with respect to $\mathcal{B}$?

(d) Find $\|\vec{v}\|$ and $\|\vec{u}\|$ with respect to $\mathcal{B}$.

(e) Find $[\vec{v}]_{\mathcal{C}}$ and $[\vec{u}]_{\mathcal{C}}$.

(f) Find $\vec{v} \cdot \vec{u}$ with respect to $\mathcal{C}$.

(g) Are $\vec{v}$ and $\vec{u}$ orthogonal with respect to $\mathcal{C}$?

(h) Find $\|\vec{v}\|$ and $\|\vec{u}\|$ with respect to $\mathcal{C}$.

2.3.15. Let $\mathcal{B}$ be the standard basis for $\mathbb{P}_3$. Let $\vec{v} = x + 3x^2 + x^3$ and $\vec{u} = 2 + 3x - x^2$.

(a) Find $[\vec{v}]_{\mathcal{B}}$ and $[\vec{u}]_{\mathcal{B}}$.

(b) Find $\vec{v} \cdot \vec{u}$ with respect to $\mathcal{B}$.

(c) Are $\vec{v}$ and $\vec{u}$ orthogonal with respect to $\mathcal{B}$?

(d) Give a new vector $\vec{w}$ so that $\vec{w}$ is orthogonal to $\vec{v}$.

(e) Find $\|\vec{v}\|$ and $\|\vec{u}\|$ with respect to $\mathcal{B}$.

2.3.16. Note that $\mathcal{B}_1 = \{1, 1 + x, x + x^2, x^3\}$ is a basis for $\mathbb{P}_3$. Let $\vec{v} = x + 3x^2 + x^3$ and $\vec{u} = 2 + 3x - x^2$.

(a) Find $[\vec{v}]_{\mathcal{B}_1}$ and $[\vec{u}]_{\mathcal{B}_1}$.

(b) Find $\vec{v} \cdot \vec{u}$ with respect to $\mathcal{B}_1$.

(c) Are $\vec{v}$ and $\vec{u}$ orthogonal with respect to $\mathcal{B}_1$?

(d) Give a new vector $\vec{w}$ so that $\vec{w}$ is orthogonal to $\vec{v}$.

(e) Find $\|\vec{v}\|$ and $\|\vec{u}\|$ with respect to $\mathcal{B}_1$.

2.3.17. Let S be the unit cube in $\mathbb{R}^3$. That is, let S be the cube with corners $(0,0,0)$, $(0,1,0)$, $(1,0,0)$, $(1,1,0)$, $(0,0,1)$, $(0,1,1)$, $(1,0,1)$, and $(1,1,1)$. There are four different diagonals in S (segments that connect one corner of S to another and go through the center of S). By subtracting corners, we can write these diagonals as vectors in $\mathbb{R}^3$. Show that any two of these diagonals are *not* orthogonal.

2.3.18. Let V be a vector space with basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$. Define $T\colon V \to \mathbb{R}^n$ by $T(\vec{v}) = [\vec{v}]_{\mathcal{B}}$. Such a mapping T is often called a *coordinate mapping*. Show that for any $\vec{v}_1, \vec{v}_2 \in V$ and any $a \in \mathbb{R}$, we have

$$T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) \quad \text{and} \quad T(a\vec{v}_1) = aT(\vec{v}_1).$$

2.3.19. Let $\vec{v}, \vec{u} \in \mathbb{R}^n$. Following techniques similar to the proof of the Pythagorean Theorem, show $\|\vec{v} + \vec{u}\|^2 + \|\vec{v} - \vec{u}\|^2 = 2\|\vec{v}\|^2 + 2\|\vec{u}\|^2$. This is called the Parallelogram Law.

2.3.20. Let $\vec{v}, \vec{u} \in \mathbb{R}^n$. Use the Triangle Inequality to show $\big|\|\vec{v}\| - \|\vec{u}\|\big| \leq \|\vec{v} - \vec{u}\|$ using the fact that $\vec{v} = \vec{v} - \vec{u} + \vec{u}$. This is called the Reverse Triangle Inequality.

2.4 Orthogonal Sets

Here we will extend the notion of orthogonality to sets; it will end up being very useful. When we say useful here, we mean it will produce a formula that can save you a *lot* of time.

Definition 2.4.1 *Let V be an inner product space and W be a subspace of V. If a vector $\vec{v} \in V$ is orthogonal to every vector in W, then we say $\vec{v}$ is orthogonal to W. The set of all vectors $\vec{v} \in V$ that are orthogonal to W is called the* ***orthogonal complement of*** *W. The orthogonal complement of W is denoted $W^{\perp}$.*

Note that orthogonality is defined in terms of an inner product, which depends on the basis one uses. Thus, orthogonality always depends on the choice of basis in a vector space, so we'll almost always be working in inner product spaces (where we have a chosen basis and inner product) when orthogonality is relevant. Also for clarification, the notation $W^{\perp}$ is usually pronounced "W perp."[25]

Example 2.4.1 Let

$$W = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}.$$

This is a subspace in $\mathbb{R}^3$. In particular, this is a line through the origin in $\mathbb{R}^3$. Thus, the vectors orthogonal to this line form a plane. Specifically, they form a plane through the origin since the zero vector is orthogonal to every vector in $\mathbb{R}^3$. Now let us determine this plane.

Let $\vec{v} \in W$. Since $\vec{0}$ is orthogonal to every vector, we should specify that $\vec{v} \neq \vec{0}$. Then for some $c \in \mathbb{R}$ with $c \neq 0$,

$$\vec{v} = \begin{bmatrix} c \\ 2c \\ 3c \end{bmatrix}.$$

Let

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

be a vector orthogonal (using the standard basis for $\mathbb{R}^3$) to $\vec{v}$. We know

$$\begin{bmatrix} c \\ 2c \\ 3c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = cx + 2cy + 3cz = 0.$$

Thus,

$$W^{\perp} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + 2y + 3z = 0 \right\}.$$

Since $c \neq 0$, we were able to algebraically clean up our condition and require simply that $x + 2y + 3z = 0$.

25: As in "purple?"
No.
As in "perpetrator?"
No.
"Porpoise?"
It's not pronounced "porp!"
It is now.

However, you may perhaps feel unsatisfied with this description of $W^\perp$. We can also describe $W^\perp$ as

$$\text{Span}\left\{\begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}\right\},$$

where the appropriate vectors were determined by substituting into the equation $x + 2y + 3z = 0$. We know these two vectors are linearly independent since they are not scalar multiples of each other. Thus, they must span our 2-dimensional plane. This feels more like a description we would typically see in this book.

We saw in the example that a way to find the orthogonal complement of a subspace W is to consider a general element of W and determine what it means for some other vector to be orthogonal to that vector. However, we can actually streamline this a bit. Rather than taking a general element, we can instead separately consider each element in a spanning set of W. In the example above, this would mean it was enough to just find all vectors orthogonal to

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

which amounts to not including the scalar multiple c. Since we simplified out our c anyway, you can see that was exactly what we needed!

Theorem 2.4.1 *Let W be a subspace of an inner product space V, and let $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a set of vectors that spans W. Then $\vec{v} \in W^\perp$ if and only if $\vec{v}$ is orthogonal to every vector in S.*

PROOF. Suppose $\vec{v} \in W^\perp$. Then $\vec{v}$ is orthogonal to every vector in W, including those in S. Now suppose $\vec{v}$ is orthogonal to every vector in S. Since S spans W, any vector $\vec{w} \in W$ can be written as a linear combination of the vectors in S:

$$\vec{w} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n.$$

Since $\vec{v}$ is orthogonal to $\vec{v}_i$ for $1 \le i \le n$,

$$\begin{aligned}
\vec{v} \cdot \vec{w} &= \vec{v} \cdot (a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) \\
&= \vec{v} \cdot (a_1\vec{v}_1) + \cdots + \vec{v} \cdot (a_n\vec{v}_n) \\
&= a_1(\vec{v} \cdot \vec{v}_1) + \cdots + a_n(\vec{v} \cdot \vec{v}_n) \\
&= a_1(0) + \cdots + a_n(0) \\
&= 0.
\end{aligned}$$

Thus, $\vec{v} \in W^\perp$. It follows that $\vec{v} \in W^\perp$ if and only if $\vec{v}$ is orthogonal to every vector in S. □

Theorem 2.4.2 *If W is a subspace of and inner product space V, then $W^\perp$ is a subspace of V.*

Exploration 53 To show $W^\perp$ is a subspace, we will verify the three axioms of being a subspace.

- First, we know $\vec{0} \in W^\perp$ because the zero vector is orthogonal to every vector.
- Now, let $\vec{v}$ and $\vec{u}$ be in $W^\perp$. Let $\vec{w} \in W$. To show $W^\perp$ is closed under addition, we need to verify that $(\vec{v}+\vec{u})\cdot\vec{w} = 0$. Use a property of the inner product to argue that this must be true.
- Lastly, we need to show that $W^\perp$ is closed under scalar multiplication. Let $k \in \mathbb{R}$. Suppose as before that $\vec{v} \in W^\perp$ and $\vec{w} \in W$. Then we need to verify $(k\vec{v}) \cdot \vec{w} = 0$. Again, use a property of the inner product to show this.

Example 2.4.2 Let $\vec{v}$ and $\vec{u}$ be any two linearly independent vectors in $\mathbb{R}^3$ with basis $\mathcal{B}$. Then $W = \operatorname{Span}\{\vec{v}, \vec{u}\}$ is a two-dimensional subspace of $\mathbb{R}^3$ that can be visualized as a plane through the origin in $\mathbb{R}^3$. What do we know about $W^\perp$? It's a subspace of $\mathbb{R}^3$, it must contain the zero vector, and all of its vectors must be orthogonal to W, which is a plane through the origin. What does it mean to be orthogonal to all the vectors in a plane in $\mathbb{R}^3$? As an arrow vector, it would have to stick straight out of the plane. Imagine, for example, that a tabletop is the plane, and you set a marker or pen standing up on its end on the table; an arrow vector orthogonal to that plane would point in the same direction as the marker (or pen). It could also point directly down. It could also be any length! Our set of orthogonal vectors does, as we've noted, have to contain the zero vector, too. Thus, $W^\perp$ is the line through the origin orthogonal to W. Moreover, $\dim W^\perp = 1$.

Exploration 54 In Example 2.4.1, we saw the orthogonal complement of a 1-dimensional subspace in $\mathbb{R}^3$ had dimension 2. Then in Example 2.4.2, we used our geometric understanding to note that the orthogonal complement of a 2-dimensional subspace of $\mathbb{R}^3$ is 1-dimensional. Now let $\vec{v}$ and $\vec{u}$ be any two linearly independent vectors in $\mathbb{R}^4$. Again, $W = \operatorname{Span}\{\vec{v}, \vec{u}\}$ is a two-dimensional subspace of $\mathbb{R}^4$, but it's harder to visualize this time. What is $W^\perp$, and what is $\dim W^\perp$?

Well, if you are having trouble figuring this out, let's see if a more specific example will help. Use the standard basis for $\mathbb{R}^4$, and let

$$W_0 = \operatorname{Span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right\} \text{ and } \vec{v} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix} \in W_0^\perp.$$

Then we know

$$\begin{bmatrix}a\\b\\c\\d\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = a = 0, \quad \text{and} \quad \begin{bmatrix}a\\b\\c\\d\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\\0\end{bmatrix} = b = 0.$$

Using this information, can you determine $W_0^\perp$?

Well, these examples do seem to be following a pattern. However, examples are not enough to conclude a theorem. Let's instead make a "conjecture." This is really just a guess based on some evidence. Hopefully, we'll be able to prove this conjecture in a future section.[26]

26: Perhaps even the next section.

Conjecture 1 *Let W be a subspace of an inner product space V such that $\dim V = n$ and $\dim W = p$. Then $\dim W^\perp = n - p$.*

Orthogonal Sets and Bases

Definition 2.4.2 *If S is a set of vectors in an inner product space such that all pairs of vectors in S are orthogonal, then S is said to be an* ***orthogonal set****.*

Theorem 2.4.3 *Orthogonal sets of nonzero vectors are linearly independent.*

PROOF. We will use the definition of linear independence to show an orthogonal set $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ of nonzero vectors is linearly independent. Suppose there are scalars $a_1, \ldots, a_p$ such that $a_1\vec{v}_1 + \cdots + a_p\vec{v}_p = \vec{0}$. Taking the inner product of both sides of this equation with $\vec{v}_i$ for some $1 \leq i \leq p$, we have

$$\begin{aligned} 0 = \vec{v}_i \cdot \vec{0} &= \vec{v}_i \cdot (a_1\vec{v}_1 + \cdots + a_p\vec{v}_p) \\ &= a_1\vec{v}_i \cdot \vec{v}_1 + \cdots + a_i\vec{v}_i \cdot \vec{v}_i + \cdots + a_p\vec{v}_i \cdot \vec{v}_p \\ &= a_i\vec{v}_i \cdot \vec{v}_i, \end{aligned}$$

since $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$ (by definition of orthogonal set). Since $\vec{v}_i \neq \vec{0}$, we know $\vec{v}_i \cdot \vec{v}_i \neq 0$. Thus, $a_i = 0$. Since this argument holds for any $1 \leq i \leq p$, it follows that S is linearly independent. □

Sometimes a another useful result follows immediately from a theorem. We call such a result a corollary to the theorem. The following is a corollary to Theorem 2.4.3.

Corollary 2.4.4 *If S is an orthogonal set in an inner product space V, then S is a basis for the subspace* $\mathrm{Span}\,\{S\}$.

Orthogonality is a property enjoy by sets of vectors in a variety of contexts. In fact, it's such an enjoyable property, it gets its own definition with one of favorite sets of vectors:

Definition 2.4.3 *An* ***orthogonal basis*** *for a subspace W is a basis for W that is also an orthogonal set.*

What makes an orthogonal basis so enjoyable? We're glad you asked.

Theorem 2.4.5 *Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be an orthogonal basis for a subspace W of an inner product space V. Then for any $\vec{w} \in W$,*

$$\vec{w} = \left(\frac{\vec{w} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{w} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2 + \cdots + \left(\frac{\vec{w} \cdot \vec{v}_p}{\vec{v}_p \cdot \vec{v}_p}\right)\vec{v}_p,$$

where the inner product is taken with respect to $\mathcal{B}$.

It's no surprise that $\vec{w}$ can be written as a linear combination of the vectors in $\mathcal{B}$ because $\mathcal{B}$ is a basis. The innovation here is the weights; when $\mathcal{B}$ is an *orthogonal* basis, we get a formula for the weights. That's extremely enjoyable. Oh, we should prove it, though.

PROOF. This is a good one. Since $\mathcal{B}$ is a basis, we know there are weights c_i for $1 \leq i \leq p$ such that

$$\vec{w} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p.$$

To solve for c_i for $1 \leq i \leq p$, apply the dot product with $\vec{v}_i$ to both sides of this equation. We then have

$$\begin{aligned}
\vec{w} \cdot \vec{v}_i &= (c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) \cdot \vec{v}_i \\
&= c_1(\vec{v}_1 \cdot \vec{v}_i) + \cdots + c_i(\vec{v}_i \cdot \vec{v}_i) + \cdots + c_p(\vec{v}_p \cdot \vec{v}_i) \\
&= c_1(0) + \cdots + c_i(\vec{v}_i \cdot \vec{v}_i) + \cdots + c_p(0) \\
&= c_i\vec{v}_i \cdot \vec{v}_i,
\end{aligned}$$

where all but one of the dot products is zero because $\mathcal{B}$ is an orthogonal set. Then just solve for c_i; this can be done for each $1 \leq i \leq p$. □

Exploration 55 Let's do an example together to illustrate that theorem!

- ▶ Using the standard basis for $\mathbb{R}^3$, show that the set $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

  $$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and } \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

  is an orthogonal set. This is readily verified using the definition of inner product. Find $\vec{v}_1 \cdot \vec{v}_2$, $\vec{v}_1 \cdot \vec{v}_3$ and $\vec{v}_2 \cdot \vec{v}_3$. (Hopefully, you get 0 for each one!)

- ▶ Now, we want to write

  $$\vec{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

  as a linear combination of the vectors in S. According to Theorem 2.4.5, we may write

  $$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \text{ where } c_i = \frac{\vec{x} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i},$$

  where $1 \leq i \leq 3$. We compute each of the c_i below:

$$\begin{aligned}
c_1 &= \frac{\vec{x}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1} = \frac{4(1)+5(2)+6(-5)}{1(1)+2(2)+(-5)(-5)} = \frac{-16}{30},\\
c_2 &= \frac{\vec{x}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2} = \frac{4(-2)+5(1)+6(0)}{(-2)(-2)+1(1)+0(0)} = \frac{-3}{5},\\
c_3 &= \frac{\vec{x}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3} = \frac{4(1)+5(2)+6(1)}{1(1)+2(2)+1(1)} = \frac{20}{6}.
\end{aligned}$$

Thus,

$$\vec{x} = -\frac{8}{15}\vec{v}_1 - \frac{3}{5}\vec{v}_2 + \frac{10}{3}\vec{v}_3.$$

► Now that we've seen how it's done, let's write

$$\vec{y} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

as a linear combination of the vectors in S. Find each c_i below

$$\begin{aligned}
c_1 &= \frac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1} = \underline{\hspace{6cm}}\\
c_2 &= \frac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2} = \underline{\hspace{6cm}},\\
c_3 &= \frac{\vec{y}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3} = \underline{\hspace{6cm}}.
\end{aligned}$$

Now, use your answers for c_1, c_2, and c_3 to fill in the blanks below

$$\vec{y} = \underline{\hspace{2.5cm}}\vec{v}_1 + \underline{\hspace{2.5cm}}\vec{v}_2 + \underline{\hspace{2.5cm}}\vec{v}_3.$$

If you have a vector space V, having a basis is very nice. Having an orthogonal basis is even better. Shall we introduce a superlative?

Definition 2.4.4 *If S is an orthogonal set of vectors in an inner product space such that any vector in S is a unit vector, then S is said to be an* ***orthonormal set****. If S happens to be an orthogonal basis and any vector in S is a unit vector, then S is said to be an* ***orthonormal basis****.*

An orthonormal set is just an orthogonal set of unit vectors. Why should we care? In what way is this better?[27]

27: How is being normal better than being "-gonal?" You should never strive for normality. Always be exceptional.

What are you talking about?!

Theorem 2.4.6 *Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be an orthonormal basis for subspace W of inner product space V. Then for any $\vec{w} \in W$,*

$$\vec{w} = (\vec{w}\cdot\vec{v}_1)\,\vec{v}_1 + (\vec{w}\cdot\vec{v}_2)\,\vec{v}_2 + \cdots + (\vec{w}\cdot\vec{v}_n)\,\vec{v}_n,$$

where the inner product is taken with respect to $\mathcal{B}$.

This theorem follows immediately from the proof of Theorem 2.4.5 once you realize that $\vec{v}_i \cdot \vec{v}_i = 1$ for all $1 \le i \le p$ since each $\vec{v}_i$ is a unit vector (since $\mathcal{B}$ is an orthonormal basis).

Example 2.4.3 The set $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$\vec{v}_1 = \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}, \quad \text{and } \vec{v}_3 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

is an orthonormal set. One can see this by noting that

$$\vec{v}_1 = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and } \vec{v}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix};$$

then the fact that this is an orthogonal set follows from properties of inner product and Exploration 55. For example,

$$\begin{aligned} \vec{v}_1 \cdot \vec{v}_2 &= \left(\frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} \right) \cdot \left(\frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right) \\ &= \left(\frac{1}{\sqrt{30}} \frac{1}{\sqrt{5}} \right) \left(\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right) = \left(\frac{1}{\sqrt{30}} \frac{1}{\sqrt{5}} \right) 0 = 0. \end{aligned}$$

The remaining pairs of vectors yield similar calculations and could readily be checked. However, we must also verify that each of these vectors is a unit vector:

$$\begin{aligned} \|\vec{v}_1\| &= \left\| \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} \right\| = \frac{1}{\sqrt{30}} \left\| \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} \right\| = \frac{1}{\sqrt{30}} \sqrt{30} = 1, \\ \|\vec{v}_2\| &= \left\| \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\| = \frac{1}{\sqrt{5}} \left\| \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\| = \frac{1}{\sqrt{5}} \sqrt{5} = 1, \\ \|\vec{v}_3\| &= \left\| \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\| = \frac{1}{\sqrt{6}} \left\| \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\| = \frac{1}{\sqrt{6}} \sqrt{6} = 1. \end{aligned}$$

Then we can write

$$\vec{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

as a linear combination of the vectors in S. According to Theorem 2.4.6, we may write

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \text{ where } c_i = \vec{x} \cdot \vec{v}_i,$$

where $1 \leq i \leq 3$. Since

$$\begin{aligned} c_1 &= \vec{x} \cdot \vec{v}_1 = 4(1/\sqrt{30}) + 5(2/\sqrt{30}) + 6(-5/\sqrt{30}) = \frac{-16}{\sqrt{30}}, \\ c_2 &= \vec{x} \cdot \vec{v}_2 = 4(-2/\sqrt{5}) + 5(1/\sqrt{5}) + 6(0) = \frac{-3}{\sqrt{5}}, \\ c_3 &= \vec{x} \cdot \vec{v}_3 = 4(1/\sqrt{6}) + 5(2/\sqrt{6}) + 6(1/\sqrt{6}) = \frac{20}{\sqrt{6}}. \end{aligned}$$

Thus,
$$\vec{x} = -\frac{16}{\sqrt{30}}\vec{v}_1 - \frac{3}{\sqrt{5}}\vec{v}_2 + \frac{20}{\sqrt{6}}\vec{v}_3.$$
Note that this agrees with what we found in Exploration 55:
$$\begin{aligned}
\vec{x} &= -\frac{16}{\sqrt{30}}\frac{1}{\sqrt{30}}\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} - \frac{3}{\sqrt{5}}\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \frac{20}{\sqrt{6}}\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\
&= -\frac{8}{15}\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} - \frac{3}{5}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \frac{10}{3}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.
\end{aligned}$$

Exploration 56 Write
$$\vec{y} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$
as a linear combination of the orthonormal basis vectors from the example above.

Perhaps you're convinced at this point that orthogonal bases are pretty great, and of course, orthonormal bases are even better. However, if you're given a random basis for a vector space, how likely do you think it is that this basis is an orthogonal basis, much less an orthonormal one? Right;[28] it is very unlikely. Thus, it would be good to develop a method for making sets of vectors into orthogonal sets. (Then we could take any basis and make a new orthogonal one!)

28: Pun intended, again.

Orthogonal Projection

Before we take on the task of making a new basis in general, we should start with the very simple case of two vectors. That is, given two distinct, linearly independent vectors $\vec{v}$ and $\vec{u}$ in a vector space V, is there a way to write $\vec{v}$ as the sum of $\vec{u}$ and some other vector $\vec{w}$? Yes!

Lemma 2.4.7 *Let $\vec{v}$ and $\vec{u}$ be vectors in a vector space V, and $\vec{w} = \vec{v} - \vec{u}$. Then*

$$\text{Span}\{\vec{v}, \vec{u}\} = \text{Span}\{\vec{u} + \vec{w}, \vec{u}\} = \text{Span}\{\vec{w}, \vec{u}\}. \tag{2.12}$$

PROOF. Note that $\vec{u}+\vec{w} = \vec{u}+(\vec{v}-\vec{u}) = \vec{v}$. Using the definition of span, we have

$$\begin{aligned}
\mathrm{Span}\{\vec{v},\vec{u}\} &= \{a\vec{v}+b\vec{u}\colon a,b\in\mathbb{R}\} \\
&= \{a(\vec{u}+\vec{w})+b\vec{u}\colon a,b\in\mathbb{R}\} = \mathrm{Span}\{\vec{u}+\vec{w},\vec{u}\} \\
&= \{a\vec{u}+a\vec{w}+b\vec{u}\colon a,b\in\mathbb{R}\} \\
&= \{a\vec{w}+(a+b)\vec{u}\colon a,b\in\mathbb{R}\} \\
&= \{a\vec{w}+c\vec{u}\colon a,c\in\mathbb{R}\} = \mathrm{Span}\{\vec{w},\vec{u}\}.
\end{aligned}$$

□

Exploration 57 Let's be sure you believe this before moving on. Let

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{u} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

- ▶ Find the vector $\vec{w}$ such that $\vec{v} = \vec{u}+\vec{w}$.

- ▶ Let

$$\vec{x} \in \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}, \text{ so } \vec{x} = \begin{bmatrix} a+b \\ a \\ 2b \end{bmatrix}$$

for some $a,b\in\mathbb{R}$. Write $\vec{x}$ as a linear combination of $\vec{w}$ and $\vec{u}$.
Hint: The coefficients should be a and $(a+b)$.

What does Equation (2.12) do for us? Well, as mentioned, it gives us a way to interchange elements in a spanning set. However, our goal is to ultimately replace a basis with an orthogonal basis, so what we really would like is for $\vec{u}$ and $\vec{w}$ to be orthogonal and satisfy $\vec{v} = \vec{u}+\vec{w}$. However, as we attempt this, we see that $\vec{u}$ needs a little adjusting. Let's instead require that some scalar multiple of $\vec{u}$, say $\alpha\vec{u}$, and some vector $\vec{w}$ be orthogonal such that $\vec{v} = \alpha\vec{u}+\vec{w}$. Just like before, we need $\vec{w} = \vec{v}-\alpha\vec{u}$. If you're having trouble visualizing this, refer to Figure 2.2.

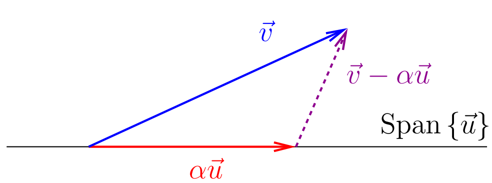


FIGURE 2.2. We need to find a scalar α such that $\vec{v}-\alpha\vec{u}$ is orthogonal to $\vec{u}$.

According to Figure 2.2, we need to "tune" α so that $\vec{v}-\alpha\vec{u}$ is orthogonal to $\vec{u}$, or

$$(\vec{v}-\alpha\vec{u})\cdot\vec{u} = 0.$$

Using a distributive property of our inner product, we see this is true if and only if $\alpha(\vec{u}\cdot\vec{u}) = \vec{v}\cdot\vec{u}$. Solving for α, we have

$$\alpha = \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}.$$

We've just constructed a vector $\vec{v} - \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$ that is orthogonal to $\vec{u}$ and satisfies $\vec{v} = \vec{u} + \vec{w}$. Mission accomplished! Well, the short-term mission at least. This takes care of a basis of size 2, but we'll actually wait until the next section for that whole orthogonal basis part in general because this needs a bit more discussion.

Didn't α look a tad familiar?[29] What we've actually done here in finding $\alpha\vec{u}$ is we have found a vector to represent the part of $\vec{v}$ that is traveling in the direction of $\vec{u}$. This actually has its own name and special notation.

29:  After making you compute it in those explorations, it really should!

Definition 2.4.5 *For any two vectors $\vec{v}$ and $\vec{u}$ in an inner product space, the* ***orthogonal projection of $\vec{v}$ onto $\vec{u}$*** *is*

$$\operatorname{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}.$$

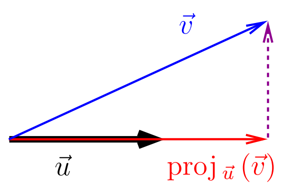


FIGURE 2.3. Here we project $\vec{v}$ onto $\vec{u}$.

Exploration 58 Let

$$\vec{v} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \quad \text{and } \vec{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

- Let's compute the orthogonal projection of $\vec{v}$ onto $\vec{u}$. First, we see $\vec{v}\cdot\vec{u} = -2-2+3 = -1$ and $u \cdot u = 1+4+1 = 6$. Then we get

$$\operatorname{proj}_{\vec{u}}(\vec{v}) = -\frac{1}{6}\vec{u} = \begin{bmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ -\frac{1}{6} \end{bmatrix}$$

- Now, let's compute the orthogonal projection of $\vec{u}$ onto $\vec{v}$. We already know $\vec{v}\cdot\vec{u} = \vec{u}\cdot\vec{v} = -1$ from above. Now, compute $\vec{v}\cdot\vec{v}$. Combine

  these to find $\operatorname{proj}_{\vec{v}}(\vec{u})$.

Now, why would this be useful? Let's consider a scenario. You have a cart loaded with gold on a train track. The cart is unfortunately not self-propelled, but luckily, you also own a sturdy plow-horse and a very thick rope. The horse cannot walk on the train tracks, so he cannot pull the cart directly forward along the track... Wait. From the picture below, your plow horse is actually a unicorn, and it flies above the track so that the rope makes a $45°$ angle with the train track. Either way, it's the same basic problem, so we'll go with the unicorn.[30] See the diagram below.

30: Is *that* unicorn drawing why we're here?

I guess so.

Yikes.

Yeah. Maybe we should rethink our standards?

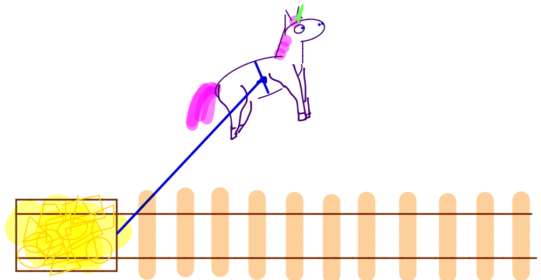


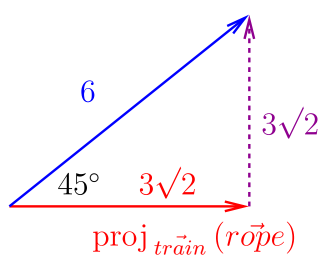
FIGURE 2.4. It's just as you pictured, right?

Suppose the forward force that must be exerted on the cart for it to move is 5 N (where N stands for Newtons, a metric unit for force). If the "horse" is able to pull with a force of 6 N, will the cart move? The way to answer this is by computing the orthogonal projection of the "rope vector" onto the "train track vector." The vector representing the rope should have length 6 to represent the force with which the unicorn can pull. Since we know the angle is $45°$, we see that the "rope vector" is the vector

$$\begin{bmatrix} 3\sqrt{2} \\ 3\sqrt{2} \end{bmatrix}$$

when we are assigning the "train track vector" to be the vector

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$



Because of our diagram, we can actually compute that

$$\operatorname{proj}_{\vec{train}}(\vec{rope}) = \begin{bmatrix} 3\sqrt{2} \\ 0 \end{bmatrix}$$

from the picture, but we also get this using the formula in our definition. Unfortunately, this vector has length $3\sqrt{2}$, which is less than 5. Therefore, our cart will not be moving anywhere unless we either unload some of the gold or we provide additional fairy dust to our flying unicorn. Let this be a lesson. You should never fill your cart before you compute the strength of your horse.[31]

31: ...or unicorn.

Section Highlights

- For any subspace W of an inner product space, there is an *orthogonal complement* $W^\perp$ that is also a subspace. See Theorem 2.4.2.
- Any vector in $W^\perp$ will be orthogonal to any vector in W. See Definition 2.4.1 and Theorem 2.4.1.
- Any set of vectors whose vectors are pairwise orthogonal is called an *orthogonal set*. See Definition 2.4.2.
- Any orthogonal set is linearly independent, and therefore, an orthogonal set of n vectors in an n-dimensional vector space will be an orthogonal basis. See Theorem 2.4.5 and Corollary 2.4.4.
- If a basis is orthogonal, then there is a formula that can be used to compute the coordinate vector with respect to that basis. See Theorem 2.4.5 and Exploration 55.
- An *orthogonal basis* can be turned into an orthonormal basis by scaling each vector to be a unit vector. See Definition 2.4.4 and Example 2.4.3.
- The projection of a vector $\vec{u}$ onto a vector $\vec{v}$ is the part of the vector $\vec{u}$ that is in the direction of $\vec{v}$. See Definition 2.4.5.

Exercises for Section 2.4

2.4.1.

Using the standard inner product in $\mathbb{R}^3$, determine all orthogonal subsets of this set of vectors:

$$\left\{\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \vec{v}_6 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right\}.$$

2.4.2. Let

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 8 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and } \vec{x} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$ with standard inner product. Then express $\vec{x}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

2.4.3. Let

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 8 \\ 5 \\ -4 \end{bmatrix}, \quad \text{and } \vec{x} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$ standard inner product. Then express $\vec{x}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

2.4.4. Let $\vec{p}_1 = 2 + x - 2x^2$, $\vec{p}_2 = -1 + 4x + x^2$, $\vec{p}_3 = 1 + x^2$, and $\vec{q} = 8 - 4x - 3x^2$. Show that $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ is an orthogonal basis for $\mathbb{P}_2$ with respect to the standard basis of $\mathbb{P}_2$. Then express $\vec{q}$ as a linear combination of $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$.

2.4.5. Using the standard inner product in $\mathbb{R}^3$, compute the projection from $\vec{v}$ onto $\vec{u}$, $\mathrm{proj}_{\vec{u}}(\vec{v})$, for each pair of vectors below. Note that the answer should always be a scalar multiple of $\vec{u}$.

(a) $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ onto $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

(b) $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ onto $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(c) $\vec{v} = \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}$ onto $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(d) $\vec{v} = \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}$ onto $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(e) $\vec{v} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$ onto $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

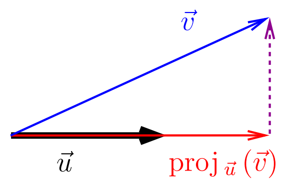
2.4.6. Using the standard basis for $\mathbb{P}_2$, compute the projection from $\vec{p}$ onto $\vec{q}$ below. Note that the answer should always be a scalar multiple of $\vec{q}$.

(a) $\vec{p} = 1 + x + x^2$ onto $\vec{q} = -1 - x$.

(b) $\vec{p} = 1 + 2x - x^2$ onto $\vec{q} = -1 - x + x^2$.

(c) $\vec{p} = x + x^2$ onto $\vec{q} = -1 - x$.

2.4.7. We consider the distance between a vector and a subspace to be the distance from the tip of the vector to the closest point in the subspace. When the subspace is just the span of a single vector, we've seen how to handle this.



As you can see from the diagram, the distance is the length of $\vec{v} - \text{proj}_{\vec{u}}(\vec{v})$, denoted $\|\vec{v} - \text{proj}_{\vec{u}}(\vec{v})\|$. Answer this question for each $\vec{v}$ and $\vec{u}$ below: How far is $\vec{v}$ from $\text{Span}\{\vec{u}\}$?

(a) $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(b) $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(c) $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(d) $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

2.4.8. Let $\vec{v} \in \mathbb{R}^n$. Using the standard inner product, show that the orthogonal projection of $\vec{v}$ onto $\vec{v}$ is $\vec{v}$.

2.4.9. Let $W = \text{Span}\left\{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$, and use the standard inner product in $\mathbb{R}^3$.

(a) Show that $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$ is an orthogonal basis for W. Note, you must confirm that $\mathcal{B} \subset W$.

(b) Find a vector in $W^\perp$ by solving for $a, b, c \in \mathbb{R}$ such that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$$

(c) What must e, f, and g be so that $\mathcal{B}_1 = \left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} e \\ f \\ g \end{bmatrix}\right\}$ is also an orthogonal basis for W? Note that you must choose e, f, and g so that $\mathcal{B}_1 \subset W$.

2.4.10. Let $W = \text{Span}\left\{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\}$, and use the standard inner product in $\mathbb{R}^3$.

(a) Find conditions to assure a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in $W^\perp$. Use these to write

$$W^\perp = \left\{\begin{bmatrix} \rule{2cm}{0.4pt} \\ \rule{2cm}{0.4pt} \\ \rule{2cm}{0.4pt} \end{bmatrix} : a, b, c \in \mathbb{R}\right\}$$

(b) Use the description above to find a basis for $W^\perp$.

(c) Verify $\vec{v} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \in W^\perp$ and write it as a linear combination of your basis vectors.

2.4.11. Let $W = \text{Span}\left\{1 + x^2\right\}$ be a subspace of $\mathbb{P}_2$. Use the method from Exercise 2.4.10 to find $W^\perp$ with respect to the standard basis for $\mathbb{P}_2$.

2.4.12. Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is an orthogonal set in inner product space V. Show that $\vec{v}_1 + \vec{v}_2$ is orthogonal to $\vec{v}_3 + \vec{v}_4$.

2.4.13. Let W be a subspace of an inner product space V. Show that $W \cap W^\perp = \{\vec{0}\}$.

2.4.14. Let V be an inner product space and $\vec{v}_0 \in V$. Define $T\colon V \to V$ by $T(\vec{v}) = \text{proj}_{\,\vec{v}_0}(\vec{v})$. Show that for any $\vec{v}_1, \vec{v}_2 \in V$ and any $a \in \mathbb{R}$, we have

$$T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) \quad \text{and} \quad T(a\vec{v}_1) = aT(\vec{v}_1).$$

2.5 The Gram-Schmidt Process

We're well on our way at this point to having a very efficient and geometrically intuitive way to describe vectors in a vector space. For reasons that will be obvious much later, it's often nice to be able to think about vectors relative to a given subspace. We actually just did this in a small way in the previous section![32] When we found the projection of a vector $\vec{v}$ onto a vector $\vec{u}$, we obtained a scalar multiple of our vector $\vec{u}$. Really, we related $\vec{v}$ to the subspace $\operatorname{Span}\{\vec{u}\}$.

32: We did?

We did! Pay attention!

Now, we'll want a way to do this in general, when our subspace is more than the span of a single vector. For example, if W is a subspace of a vector space V, then we could think of any vector in V as having two parts: the part in W and the part not in W. We're going to explore the right way to do this, where by "right" we mean orthogonal. You see, it's funny because "right" has two meanings; you may have thought we meant "the correct way," but we really meant it in both senses. Mediocre wordplay is really best when over-explained, don't you think?

Theorem 2.5.1 (The Orthogonal Decomposition Theorem) *Let W be a subspace of an inner product space V such that $\dim W = p < n = \dim V$, and suppose W has an orthogonal basis. Then any $\vec{v} \in V$ can be written uniquely as*

$$\vec{v} = \vec{w} + \vec{u},$$

where $\vec{w} \in W$ and $\vec{u} \in W^{\perp}$.

According to this theorem, we can take *any* vector in V and write it as the sum of a vector in W and some other vector orthogonal to W, and there is only one way to do this. That's actually pretty amazing; we should prove it.

PROOF. Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be the orthogonal basis for W. The choice for vectors $\vec{w}$ and $\vec{u}$ is key here. Inspired by the fact that we could write the part of $\vec{v}$ in the direction of $\vec{v}_1$ using a projection, let's try doing that for all p vectors in $\mathcal{B}$. Define

$$\begin{aligned}\vec{w} &= \frac{\vec{v}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \cdots + \frac{\vec{v}\cdot\vec{v}_p}{\vec{v}_p\cdot\vec{v}_p}\vec{v}_p. \text{ and}\\ \vec{u} &= \vec{v} - \vec{w};\end{aligned}$$

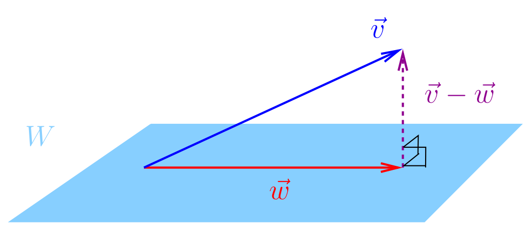


FIGURE 2.5. This is the most important picture in Section 2.5.

these are pictured in Figure 2.5. From these definitions, it is obvious that $\vec{v} = \vec{w} + \vec{u}$ and $\vec{w} \in W$. We only have to show that $\vec{u} \in W^{\perp}$. This is fun to check; note that for any $1 \leq i \leq p$, we have

$$\begin{aligned}
\vec{u} \cdot \vec{v}_i &= (\vec{v} - \vec{w}) \cdot \vec{v}_i \\
&= \vec{v} \cdot \vec{v}_i - \vec{w} \cdot \vec{v}_i \\
&= \vec{v} \cdot \vec{v}_i - \left(\frac{\vec{v} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \cdots + \frac{\vec{v} \cdot \vec{v}_p}{\vec{v}_p \cdot \vec{v}_p} \vec{v}_p \right) \cdot \vec{v}_i \\
&= \vec{v} \cdot \vec{v}_i - \left(\frac{\vec{v} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i} \vec{v}_i \cdot \vec{v}_i \right) \\
&= \vec{v} \cdot \vec{v}_i - \vec{v} \cdot \vec{v}_i \ = \ 0.
\end{aligned}$$

Since $\vec{u}$ is orthogonal to every vector in $\mathcal{B}$, a basis for W, we have from Theorem 2.4.1 that $\vec{u} \in W^{\perp}$.

The last thing to prove is that the decomposition $\vec{v} = \vec{w} + \vec{u}$ is unique. To do this, we will assume there is some other decomposition and show that it has to be the same as our original decomposition. Assume $\vec{v} = \vec{w}_0 + \vec{u}_0$ for some vectors $\vec{w}_0 \in W$ and $\vec{u}_0 \in W^{\perp}$. Then we have $\vec{w} + \vec{u} = \vec{w}_0 + \vec{u}_0$ (since both sides are equal to v), which implies

$$\vec{w} - \vec{w}_0 = \vec{u}_0 - \vec{u}.$$

Since W is a subspace of V and $\vec{w}, \vec{w}_0 \in W$, we have $\vec{w} - \vec{w}_0 \in W$. Moreover, since $W^{\perp}$ is a subspace of V and $\vec{u}, \vec{u}_0 \in W^{\perp}$, we have $\vec{w} - \vec{w}_0 = \vec{u} - \vec{u}_0 \in W^{\perp}$. Since $\vec{w} - \vec{w}_0 \in W$ and $\vec{w} - \vec{w}_0 \in W^{\perp}$, we have

$$\|\vec{w} - \vec{w}_0\|^2 = (\vec{w} - \vec{w}_0) \cdot (\vec{w} - \vec{w}_0) = 0,$$

and this can only happen if $\vec{w} - \vec{w}_0 = 0$. Thus, $\vec{w} = \vec{w}_0$. A similar argument shows that $\vec{u} = \vec{u}_0$. Behold! The decomposition is unique! $\square$

Corollary 2.5.2 *Let W be a subspace of a vector space V. Then $V = W \oplus W^{\perp}$.*

PROOF. By the Orthogonal Decomposition Theorem, every vector in V is the sum of a vector in W and a vector in $W^{\perp}$. Thus, $V = W + W^{\perp}$ by the definition of the sum of two subspaces. Also, suppose $\vec{v} \in W \cap W^{\perp}$. Then, $\vec{v}$ is orthogonal to itself, which means $\vec{v} \cdot \vec{v} = 0$. This only happens when $\vec{v} = \vec{0}$. Thus, the intersection is only the zero vector, and we have $V = W \oplus W^{\perp}$. $\square$

In Figure 2.6, we introduce a convenient way to picture V decomposed as $W \oplus W^{\perp}$. Note that $W \cup W^{\perp}$ is not equal to V; there are vectors in V that are in neither W nor $W^{\perp}$; those vectors are represented by the white part of the diagram. Every vector in $V = W \oplus W^{\perp}$ can be written as a sum of a vector in W and a vector in $W^{\perp}$, though, so we've represented the whole vector space V as W and $W^{\perp}$, bridged by the direct sum symbol.

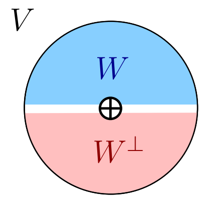


FIGURE 2.6. Here is a convenient way to picture the vector space, V, decomposed as $W \oplus W^{\perp}$ and its subspaces, W and $W^{\perp}$.

It turns out that for direct sums, like the one in Corollary 2.5.2, finding a basis for the direct sum is actually pretty easy, too.

Theorem 2.5.3 *Suppose V is an inner product space with subspace U and W such that $V = U \oplus W$. Let $\mathcal{B}_U$ be a basis for U and $\mathcal{B}_W$ be a basis for W. Then $\mathcal{B} = \mathcal{B}_U \cup \mathcal{B}_W$ is a basis for V.*

PROOF. Note that if $\mathcal{B}_U = \{\vec{u}_1, \ldots, \vec{u}_k\}$ and $\mathcal{B}_W = \{\vec{w}_1, \ldots, \vec{w}_\ell\}$, then $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_k, \vec{w}_1, \ldots, \vec{w}_\ell\}$. We know first of all that $\mathcal{B}$ spans V. Any vector can be written as a sum of a vector in U and a vector in W, and these vectors in turn can be written as linear combinations of vectors in $\mathcal{B}_U$ and $\mathcal{B}_W$, respectively. We also know that these vectors must be linearly independent separately. Thus, we need to consider the case where a vector in $\mathcal{B}_U$ is a linear combination of the vectors in $\mathcal{B}_W$, or vice versa. Really, the argument will be the same either way, so we'll focus on the first case. Suppose $\vec{v} \in \mathcal{B}_U$ is a linear combination of the vectors in $\mathcal{B}_W$. Then $\vec{v} \in \operatorname{Span}\{\mathcal{B}_W\} = W$. This would mean $\vec{v} \in U \cap W$; since $U \cap W = \{\vec{0}\}$, we must have $\vec{v} = \vec{0} \in \mathcal{B}_U$, but this is not possible because a basis cannot contain the zero vector. So, $\mathcal{B}$ must be linearly independent as well as span V. So, $\mathcal{B}$ is a basis for V. □

Perhaps you will recall we made a conjecture in the last section.[33] Now we can prove it!

33: …ummmm…

Ricky! Go back and look!

Corollary 2.5.4 *Let W be a subspace of an inner product space V such that $\dim V = n$ and $\dim W = p$. Then $\dim W^{\perp} = n - p$.*

PROOF. Let $\mathcal{B}_1$ be a basis for W and $\mathcal{B}_2$ be a basis for $W^{\perp}$. Now consider $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. From Theorem 2.5.3, we know that $\mathcal{B}$ is a basis for V since $V = W \oplus W^{\perp}$. We can conclude the statement since the dimensions are computed using the sizes of the bases. □

Corollary 2.5.5 *Let W be a subspace of an inner product space V. Suppose $\mathcal{B}_1$ is an orthogonal basis for W and $\mathcal{B}_2$ is an orthogonal basis for $W^{\perp}$. Then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is an orthogonal basis for V.*

Another fun byproduct of Theorem 2.5.1 is that we can project any vector onto a subspace,[34] and this projection is unique (for the given subspace). Thus, we have the following definition.

34: You need an *orthogonal* basis to use the formula, though.

Right.

It's not funny anymore, Ricky.

Definition 2.5.1 *Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be an orthogonal basis for a subspace W of an inner product space V. For any vector $\vec{v} \in V$, the* ***orthogonal projection of $\vec{v}$ onto W*** *is*

$$\operatorname{proj}_W(\vec{v}) = \frac{\vec{v}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \cdots + \frac{\vec{v}\cdot\vec{v}_p}{\vec{v}_p\cdot\vec{v}_p}\vec{v}_p.$$

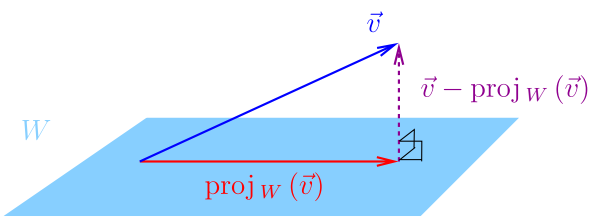


FIGURE 2.7. This is the same picture as in Figure 2.5, but now we can stop pretending that $\vec{w}$ wasn't the projection onto W all along.

Example 2.5.1 Let's see this Orthogonal Decomposition in action and use the standard inner product in $\mathbb{R}^3$. Define

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

Let $W = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$. Note that $\vec{v}_1 \cdot \vec{v}_2 = 0$, so $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for W. Let's find the $\operatorname{proj}_W(\vec{v}_3)$ and use this to find the orthogonal decomposition of $\vec{v}_3$. First, we compute the coefficients.

$$\frac{\vec{v}_3\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1} = \frac{1}{3} \quad \text{and} \quad \frac{\vec{v}_3\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2} = \frac{1}{2}$$

Now, we see

$$\operatorname{proj}_W(\vec{v}_3) = \frac{1}{3}\vec{v}_1 + \frac{1}{2}\vec{v}_2.$$

This will be our $\vec{w}$ in our orthogonal decomposition. Let $\vec{u} = \vec{v}_3 - \vec{w}$. Then we have

$$\vec{w} = \frac{1}{3}\vec{v}_1 + \frac{1}{2}\vec{v}_2 = \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ -\frac{1}{3} \\ -\frac{1}{6} \end{bmatrix}$$

$$\vec{u} = \vec{v}_3 - \vec{w} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{5}{6} \\ -\frac{1}{3} \\ -\frac{1}{6} \end{bmatrix} = \begin{bmatrix} \frac{7}{6} \\ \frac{7}{3} \\ \frac{7}{6} \end{bmatrix}$$

If we've done this all correctly, we should have $\vec{u} \in W^{\perp}$. Let's check!

$$\vec{u} \cdot \vec{v}_1 = \begin{bmatrix} \frac{7}{6} \\ \frac{7}{3} \\ \frac{7}{6} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{7}{6} - \frac{7}{3} + \frac{7}{6} = 0$$

$$\vec{u} \cdot \vec{v}_2 = \begin{bmatrix} \frac{7}{6} \\ \frac{7}{3} \\ \frac{7}{6} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{7}{6} - \frac{7}{6} = 0$$

Since $\vec{u}$ is orthogonal to the basis for W, it is orthogonal to all of W. So it's in $W^{\perp}$!

Example 2.5.2 Let $W = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Let's find a basis for $W^{\perp}$. First, we should note that $\vec{v}_1$ and $\vec{v}_2$ are orthogonal. So $\dim W = 2$ and $\{\vec{v}_1, \vec{v}_2\}$ forms a basis of W. According to Corollary 2.5.4, we now know $\dim W^{\perp} = 1$, so we only need to find one vector. It has to be orthogonal to both $\vec{v}_1$ and $\vec{v}_2$. Thus, if the vector we seek is of the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

then $\vec{v}_2 \cdot \vec{x} = \vec{v}_1 \cdot \vec{x} = 0$ yields

$$x_1 - x_3 = 0 \text{ and } x_1 + x_2 + x_3 = 0.$$

From the first equation, we see that our vector $\vec{x}$ must have $x_1 = x_3$. Applying this to the second equation, we have $x_2 = -2x_1$. There are a lot of vectors that satisfy these criteria; here's one:

$$\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

One can check that $\vec{v}_2 \cdot \vec{x} = \vec{v}_1 \cdot \vec{x} = 0$. Thus, $\vec{x} \in W^{\perp}$ and will serve nicely as a basis for $W^{\perp}$. Thus, a basis for $\mathbb{R}^3 = W \oplus W^{\perp}$ is given by $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{x}\}$.

Exploration 59 Now that you've seen some examples, let's try an exploration. Define

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 6 \\ 6 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{y} = \begin{bmatrix} 0 \\ -2 \\ 8 \\ 2 \end{bmatrix}.$$

Let $W = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$. Note that $\vec{v}_1 \cdot \vec{v}_2 = 0$, so this is an orthogonal basis of W. We would like to find a basis now for $W^{\perp}$. We'll use our Orthogonal Decomposition Theorem to do this!

- Compute $\operatorname{proj}_W(\vec{x})$. Use this to find a vector $\vec{u}_1 \in W^{\perp}$.

► Compute $\operatorname{proj}_W(\vec{y})$. Use this to find a vector $\vec{u}_2 \in W^{\perp}$.

You should now have two linearly independent vectors in $W^{\perp}$, thus a basis for $W^{\perp}$. (Caution: This may not always work. You are not guaranteed that the projections of distinct vectors will be linearly independent, even if those vectors are linearly independent or orthogonal. However, in the event this fails, it could be attempted again with different vectors.)

So, we found a basis of $W^{\perp}$. Wouldn't it be better to find an *orthogonal* basis for $W^{\perp}$?

Orthogonal Basis Through the Gram-Schmidt Process

This orthogonality business all seems pretty good, but you may have noticed at this point that we've not provided a way to actually generate an orthogonal basis.[35] The good news is there is a procedure to make an orthogonal basis from any given basis, and it should seem fairly intuitive after the two vector example of the previous section. The bad news is that it's not a lot of fun to actually do.

Theorem 2.5.6 (The Gram-Schmidt Process) *Let W be a subspace of vector space V with basis $\{\vec{v}_1, \ldots, \vec{v}_p\}$. Define*

$$\begin{aligned}
\vec{w}_1 &= \vec{v}_1, \\
\vec{w}_2 &= \vec{v}_2 - \operatorname{proj}_{\operatorname{Span}\{\vec{w}_1\}}(\vec{v}_2) \\
&= \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1, \\
\vec{w}_3 &= \vec{v}_3 - \operatorname{proj}_{\operatorname{Span}\{\vec{w}_1, \vec{w}_2\}}(\vec{v}_3) \\
&= \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 - \frac{\vec{v}_3 \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2}\vec{w}_2, \\
&\vdots \\
\vec{w}_p &= \vec{v}_p - \operatorname{proj}_{\operatorname{Span}\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_{p-1}\}}(\vec{v}_p) \\
&= \vec{v}_p - \frac{\vec{v}_p \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 - \frac{\vec{v}_p \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2}\vec{w}_2 - \cdots - \frac{\vec{v}_p \cdot \vec{w}_{p-1}}{\vec{w}_{p-1} \cdot \vec{w}_{p-1}}\vec{w}_{p-1}.
\end{aligned}$$

Then $\{\vec{w}_1, \ldots, \vec{w}_p\}$ is an orthogonal set.

There isn't much to prove here. We get orthogonality from repeated application of the Orthogonal Decomposition Theorem. We get the fact that $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_p\} = \operatorname{Span}\{\vec{w}_1, \ldots, \vec{w}_p\}$ by a similar repeated application of

35: We talked about this in the last section before we got sidetracked with the orthogonal projection and the unicorn example.

what we did in Section 2.4. Just replace each vector one at a time so that the span is unchanged.

Now we can take any set of linearly independent vectors and make it orthogonal, but you should probably be warned at this point. The coefficients from the projections used in the Gram-Schmidt process are usually pretty terrible. The process actually has you feed those terrible coefficients back into a projection formula, too. Perhaps this is better experienced than described...

Example 2.5.3 Let $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \text{and } \vec{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 0 \end{bmatrix}.$$

Find an orthogonal set $\mathcal{B}$ such that $\mathrm{Span}\{S\} = \mathrm{Span}\{\mathcal{B}\}$. According to Theorem 2.5.6, we're going to build an orthogonal set of vectors $\mathcal{B} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ from S, and we can start by just letting $\vec{w}_1 = \vec{v}_1$. Then we use a projection to find $\vec{w}_2$:

$$\begin{aligned} \vec{w}_2 &= \vec{v}_2 - \mathrm{proj}_{\,\mathrm{Span}\{\vec{w}_1\}}(\vec{v}_2) \\ &= \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{4+10+18}{1+4+9}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 12/7 \\ 3/7 \\ -6/7 \end{bmatrix}. \end{aligned}$$

Now that we have $\vec{w}_1$ and $\vec{w}_2$, we can use these with $\vec{v}_3$ to find $\vec{w}_3$:

$$\begin{aligned} \vec{w}_3 &= \vec{v}_3 - \mathrm{proj}_{\,\mathrm{Span}\{\vec{w}_1, \vec{w}_2\}}(\vec{v}_3) \;=\; \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 - \frac{\vec{v}_3 \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2}\vec{w}_2 \\ &= \begin{bmatrix} 7 \\ 8 \\ 0 \end{bmatrix} - \frac{7+16+0}{1+4+9}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{12 + \frac{24}{7} + 0}{\frac{144}{49} + \frac{9}{49} + \frac{36}{49}}\begin{bmatrix} 12/7 \\ 3/7 \\ -6/7 \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ 8 \\ 0 \end{bmatrix} - \frac{23}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 4\begin{bmatrix} 12/7 \\ 3/7 \\ -6/7 \end{bmatrix} \\ &= \begin{bmatrix} -3/2 \\ 3 \\ -3/2 \end{bmatrix}. \end{aligned}$$

Thus, $\mathcal{B} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$, where

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 12/7 \\ 3/7 \\ -6/7 \end{bmatrix}, \quad \text{and } \vec{w}_3 = \begin{bmatrix} -3/2 \\ 3 \\ -3/2 \end{bmatrix}.$$

You can check this computation by verifying that $\mathcal{B}$ is an orthogonal set and that $\mathrm{Span}\{S\} = \mathrm{Span}\{\mathcal{B}\}$. Also, let's note something here. Can you see a way to get a different orthogonal basis $\mathcal{B}_0$ using S and this procedure? What if the vector you started with was $\vec{v}_2$? or $\vec{v}_3$? Yes, this algorithm could be used to find multiple orthogonal bases for the same subspace.

Exploration 60 In Exploration 59, we found a basis for $W^{\perp}$, or really you found a basis. Now, make this an orthogonal basis using the Gram-Schmidt Process.

Section Highlights

- If W is any subspace of an inner product space V, then there is an orthogonal decomposition of V into $W \oplus W^{\perp}$. See Theorem 2.5.1.
- Any basis can be turned into an orthogonal basis using the Gram-Schmidt Process. See Theorem 2.5.6.
- The Gram-Schmidt Process is a recursively-defined procedure where at each step the basis vector $\vec{b}_i$ is replaced with $\vec{b}_i - \mathrm{proj}_W\left(\vec{b}_i\right)$, where W is the span of the previously constructed orthogonal vectors. See Theorem 2.5.6 and Example 2.5.3.

Exercises for Section 2.5

For fun, we will introduce a new verb for the exercise section by verbing* a noun. Henceforth, the noun "Gram-Schmidt" will also be the verb "to Gram-Schmidt" with the meaning "to perform the Gram-Schmidt process on."

2.5.1.

Below are several sets of linearly independent vectors. Use the standard inner product and Gram-Schmidt each one to produce an orthogonal set.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(f) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

(g) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$

(h) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(i) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

(j) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(k) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

(l) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

2.5.2. Let $\vec{p}_1 = 2+x$, $\vec{p}_2 = 2x+x^2$, and $W = \text{Span}\,\{\vec{p}_1, \vec{p}_2\}$. Use the standard basis for $\mathbb{P}_2$ and the Gram-Schmidt Process to find an orthogonal basis for W.

2.5.3. Here is a linearly dependent set of vectors:

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} \right\}$$

Use the standard inner product to Gram-Schmidt this set. Your result cannot be a basis for $\mathbb{R}^3$ since the original vectors did not form a basis. How is this reflected in your answer?

*Yes, we have used the noun "verb" to implicitly introduce the act of turning a noun into a verb. This is not upsetting.

2.5.4. Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set of vectors using the standard inner product. Write out and simplify the formulas from the Gram-Schmidt process. What can you conclude about the outcome of the Gram-Schmidt process when the initial set is orthogonal?

2.5.5. Consider the subspace

$$W = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \right\}.$$

(a) Use the standard inner product to construct an orthogonal basis for W.

(b) Note that $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not in W and compute $\vec{y} = \vec{x} - \operatorname{proj}_W(\vec{x})$.

2.5.6. Let

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 8 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and } \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

As you might recall from the previous section, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$ using the standard inner product. Feel free to also verify this.

(a) Write the vector $\vec{x}$ as the sum of two vectors, one in $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$ and one in $\operatorname{Span}\{\vec{v}_3\}$.

(b) Write the vector $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as the sum of two vectors, one in $\operatorname{Span}\{\vec{v}_1\}$ and one in $\operatorname{Span}\{\vec{v}_2, \vec{v}_3\}$.

(c) Normalize the vectors in the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ to get an orthonormal basis for $\mathbb{R}^3$.

2.5.7. Consider the subspace

$$H = \left\{ \begin{bmatrix} a+b \\ a-b \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

(a) Construct an orthogonal basis for H using the standard inner product.

(b) Find a vector $\vec{x}$ that is not in H and compute $\vec{y} = \vec{x} - \operatorname{proj}_H(\vec{x})$.

(c) Verify $\vec{y} \in H^\perp$.

2.5.8. Let W be a subspace of $\mathbb{R}^n$ with any inner product. Show that $\vec{v} \in W$ if and only if $\operatorname{proj}_W(\vec{v}) = \vec{v}$.

2.5.9. Let W be a subspace of $\mathbb{R}^n$ with any inner product. Show that $\vec{v} \in W^\perp$ if and only if $\operatorname{proj}_W(\vec{v}) = \vec{0}$.

2.6 Least Squares Applications

We've spent quite a lot of time learning to describe vector spaces in terms of bases. Then we spent quite a lot more time trying to make bases that are nice for various reasons. You probably won't be surprised to find this was all for a grand purpose. There are, in fact, many good uses for orthogonal bases, but we're going to introduce just one now. Of course by one, we mean two.

Theorem 2.6.1 *Let W be a subspace of an inner product space V, and let $\vec{v} \in V$. Then $\operatorname{proj}_W(\vec{v})$ is the closest vector in W to $\vec{v}$ in the sense that for any $\vec{w} \in W$,*

$$\|\vec{v} - \operatorname{proj}_W(\vec{v})\| \leq \|\vec{v} - \vec{w}\|. \tag{2.13}$$

PROOF. Let $\vec{v} \in V$ and $\vec{w} \in W$. Then

$$\begin{aligned}\vec{v} - \vec{w} &= \vec{v} + (-\operatorname{proj}_W(v) + \operatorname{proj}_W(v)) - \vec{w}\\ &= (\vec{v} - \operatorname{proj}_W(v)) + (\operatorname{proj}_W(v) - \vec{w})\end{aligned}$$

Thus, by the Pythagorean Theorem,

$$\begin{aligned}\|\vec{v} - \vec{w}\|^2 &= \|(\vec{v} - \operatorname{proj}_W(v)) + (\operatorname{proj}_W(v) - \vec{w})\|^2\\ &= \|\vec{v} - \operatorname{proj}_W(v)\|^2 + \|\operatorname{proj}_W(v) - \vec{w}\|^2\end{aligned}$$

Note now that $\vec{v} - \operatorname{proj}_W(v)$ is orthogonal to $\operatorname{proj}_W(v) - \vec{w}$ because $\operatorname{proj}_W(\vec{v}) - \vec{w} \in W$ (since W is closed under vector addition) and $\vec{v} - \operatorname{proj}_W(\vec{v})$ is orthogonal to everything in W. See Figure 2.8 for the geometric picture of this. We can rearrange now to get

$$\|\vec{v} - \vec{w}\|^2 - \|\operatorname{proj}_W(v) - \vec{w}\|^2 = \|\vec{v} - \operatorname{proj}_W(v)\|^2.$$

Since we know $\|\operatorname{proj}_W(v) - \vec{w}\|^2 \geq 0$, we have

$$\begin{aligned}\|\vec{v} - \vec{w}\|^2 &\geq \|\vec{v} - \vec{w}\|^2 - \|\operatorname{proj}_W(v) - \vec{w}\|^2\\ &= \|\vec{v} - \operatorname{proj}_W(v)\|^2\end{aligned}$$

Since both $\|\vec{v} - \vec{w}\|$ and $\|\vec{v} - \operatorname{proj}_W(v)\|$ are positive, this gives the desired inequality. $\square$

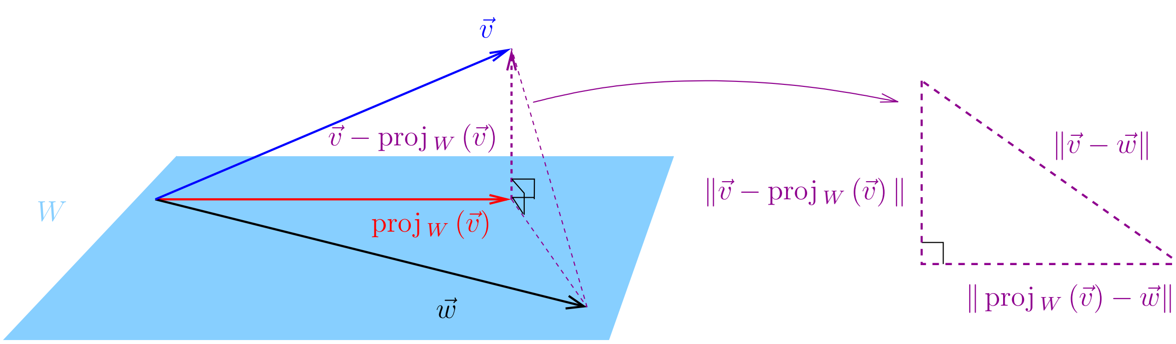


Figure 2.8: This is the most important picture in Section 2.6.

Example 2.6.1 Let

$$\vec{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix},$$

and $W = \text{Span}\{\vec{v}_1, \vec{v}_2\}$. How far is $\vec{y}$ from W? Let's see! According to Theorem 2.6.1, $\text{proj}_W(\vec{y})$ is the closest vector in W to $\vec{y}$, so we'll start by calculating $\text{proj}_W(\vec{y})$.

$$\begin{aligned} \text{proj}_W(\vec{y}) &= \frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{y} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 \\ &= \frac{15}{30}\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \frac{-21}{6}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix}. \end{aligned}$$

Then

$$\|\vec{y} - \text{proj}_W(\vec{y})\| = \left\| \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} \right\| = 3\sqrt{5}.$$

Exploration 61 Define

$$\vec{y} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

Let $W = \text{Span}\{\vec{v}_1, \vec{v}_2\}$. Find $\text{proj}_W(\vec{y})$.

Did you get $\text{proj}_W(\vec{y}) = \vec{y}$? Why would that happen? What does it mean that the "closest vector" to $\vec{y}$ in W is $\vec{y}$?

The idea of "the closest vector" to some subspace feels like something that could be useful in general scientific settings.[36] This concept allows us to specifically find a "line of best fit" for a set of data points in $\mathbb{R}^2$. Suppose we have an independent variable x, a dependent variable y, and have four observed data points (x, y),

$$(-2,-2), (-1,0), (1,2), \text{ and } (2,1),$$

as seen in Figure 2.9.

If all of these data points were on the same line, then we would be able to find scalars m and b such that each data point satisfied the equation $y = mx + b$. As seen in Figure 2.9, though, when graphed in $\mathbb{R}^2$, these points do not lie on the same line together, so we shouldn't expect to find one m and one b to make this equation true for all four data points. This is annoying; let's try it with vectors. We could interpret the set of x coordinates for these data points as a vector $\vec{x} \in \mathbb{R}^4$ and do likewise for the y coordinates:

36: Quite useful indeed!

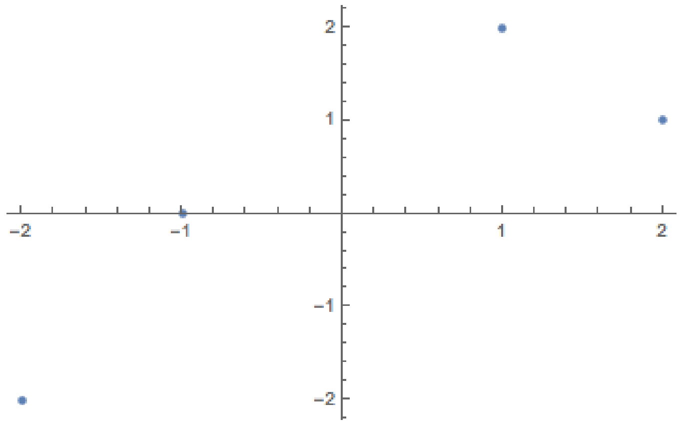


FIGURE 2.9. Here's some data.

$$\begin{matrix}(-2,-2)\\(-1,0)\\(1,2)\\(2,1)\end{matrix} \qquad \vec{x} = \begin{bmatrix}-2\\-1\\1\\2\end{bmatrix} \qquad \vec{y} = \begin{bmatrix}-2\\0\\2\\1\end{bmatrix}$$

The four data points satisfy the equation $y = mx + b$ if and only if

$$\begin{aligned}-2 &= -2m + b\\ 0 &= -1m + b\\ 2 &= 1m + b \quad \text{and}\\ 1 &= 2m + b.\end{aligned}$$

This translates into the vector equation

$$\begin{bmatrix}-2\\0\\2\\1\end{bmatrix} = m\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix} + b\begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$$

In other words, the four data points satisfy the equation $y = mx + b$ if and only if

$$\vec{y} \in \operatorname{Span}\left\{\vec{x}, \vec{1}\right\}, \text{ where } \vec{1} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$$

However, we know for a fact that $\vec{y} \notin \operatorname{Span}\left\{\vec{x}, \vec{1}\right\}$.[37] For simplicity of notation, let $W = \operatorname{Span}\left\{\vec{x}, \vec{1}\right\}$. It would be very convenient right now to have the vector in W that was closest to $\vec{y}$. According to Theorem 2.6.1, $\operatorname{proj}_W(\vec{y})$ is the vector in W that was closest to $\vec{y}$. Let us calculate $\operatorname{proj}_W(\vec{y})$!

37: Prove it!

Note first that $\vec{x} \cdot \vec{1} = 0$, so $\{\vec{x}, \vec{1}\}$ is an orthogonal set. If you think this is one of those scams where the example in the book contains a mathematical miracle that makes everything simple, you're half right. While it does make things simpler for us, keep in mind that x was our independent variable, so we could've chosen whatever values we wanted for x's. As long as you pick

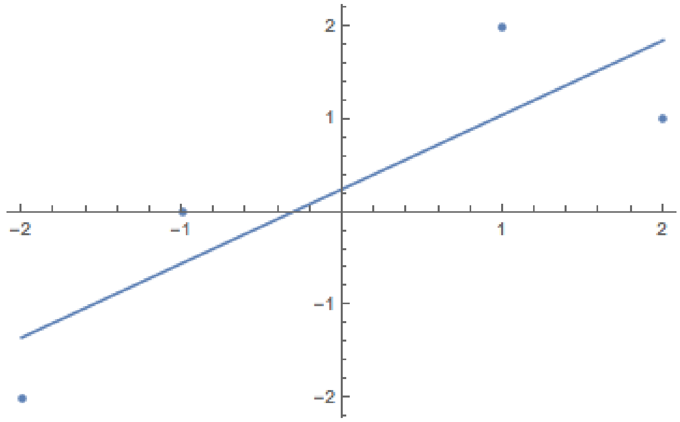


FIGURE 2.10. Here's some data with the line of best fit.

x's in pairs symmetric about 0, this procedure will yield an orthogonal set.[38] Thus, we may use Theorem 2.4.5 to find

$$\text{proj}_W(\vec{y}) = \frac{\vec{x}\cdot\vec{y}}{\vec{x}\cdot\vec{x}}\vec{x} + \frac{\vec{1}\cdot\vec{y}}{\vec{1}\cdot\vec{1}}\vec{1} = \frac{8}{10}\vec{x} + \frac{1}{4}\vec{1} = \frac{8}{10}\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix} + \frac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$$

While we couldn't find an m and b so that $\vec{y} = m\vec{x} + b\vec{1}$, it seems like we just did for $\text{proj}_W(\vec{y}) = m\vec{x} + b\vec{1}$. Using $m = 8/10$ and $b = 1/4$, we have an equation for a line that best approximates our four data points:

$$y = \frac{8}{10}x + \frac{1}{4}.$$

Figure 2.10 shows this line with the four data points. We call this line a **least squares approximation** for the given data.

When working with data, it's far more common that it was collected by someone else. What are we to do if we aren't fortunate enough to have an orthogonal set?

Example 2.6.2 Find the line of best fit for the following data:

$$(-2,-1), (-1,0), (0,2), \text{ and } (1,4).$$

One can quickly check that these points are not colinear, so there are no scalars m and b such that

$$\vec{y} = m\vec{x} + b\vec{1}, \quad \text{where} \quad \vec{y} = \begin{bmatrix}-1\\0\\2\\4\end{bmatrix}, \quad \vec{x} = \begin{bmatrix}-2\\-1\\0\\1\end{bmatrix}, \quad \vec{1} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$$

Thus, we should find the closest possible vector to $\vec{y}$ in $W = \text{Span}\left\{\vec{x}, \vec{1}\right\}$, which is $\text{proj}_W(\vec{y})$. Unfortunately, you probably noticed our basis $\{\vec{x}, \vec{1}\}$ for W is not orthogonal, so we cannot find $\text{proj}_W(\vec{y})$ as we did before. Fortunately, we have a way of making an orthogonal basis $\{\vec{w}_1, \vec{w}_2\}$ for W from $\{\vec{x}, \vec{1}\}$ by using the Gram-Schmidt process. By Theorem 2.5.6, we

38: Prove that, too!

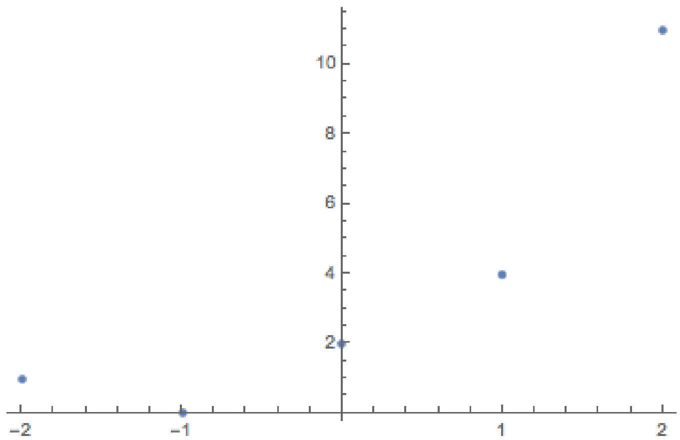


FIGURE 2.11. One last aggressively nonlinear data set.

have

$$\begin{aligned}\vec{w}_1 &= \vec{x} \\ \vec{w}_2 &= \vec{1} - \frac{\vec{1}\cdot\vec{w}_1}{\vec{w}_1\cdot\vec{w}_1}\vec{w}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix} = \begin{bmatrix}1/3\\2/3\\1\\4/3\end{bmatrix}.\end{aligned}$$

For good measure, one should check that $\vec{w}_1 \cdot \vec{w}_2 = 0$ so we have an orthogonal basis. We do? Excellent! Then

$$\operatorname{proj}_W(\vec{y}) = \frac{\vec{w}_1\cdot\vec{y}}{\vec{w}_1\cdot\vec{w}_1}\vec{w}_1 + \frac{\vec{w}_2\cdot\vec{y}}{\vec{w}_2\cdot\vec{w}_2}\vec{w}_2 = \vec{w}_1 + \frac{21}{10}\vec{w}_2$$

However, to get the line of best fit, we need $\operatorname{proj}_W(\vec{y})$ in terms of $\vec{x}$ and $\vec{1}$. Since $\vec{w}_1 = \vec{x}$ and $\vec{w}_2 = \vec{1} + \frac{1}{3}\vec{x}$, we have

$$\operatorname{proj}_W(\vec{y}) = \vec{w}_1 + \frac{21}{10}\vec{w}_2 = \vec{x} + \frac{21}{10}\left(\vec{1} + \frac{1}{3}\vec{x}\right) = \frac{17}{10}\vec{x} + \frac{21}{10}\vec{1}.$$

Thus, using $m = 17/10$ and $b = 21/10$, the line $y = \frac{17}{10}x + \frac{21}{10}$ is the line of best fit, or least squares approximation, for the data.

Let's do one more, but this time with a twist (or turn, depending on your point of view).

Example 2.6.3 Find the line of best fit for the following data:

$$(-2,1), (-1,0), (0,2), (1,4) \text{ and } (2,11).$$

One can quickly check that these points are not colinear. The first four data points are a single minus sign away from the four points in the last example. However, our new fifth point makes this set of point even farther from being on a line; see Figure 2.11.

In fact, these points look a lot more like they lie on a parabola than a line; shall we try to find the quadratic curve of best fit? Yes. Yes, we shall. We need to find scalars a, b, and c such that $y = ax^2 + bx + c$ for all five of our points. You can tell from looking at Figure 2.11 that these points don't *actually* all lie on the same parabola. Can you verify this algebraically?

Thus, we know that no such a, b, and c exist; that is there is no set of scalars a, b, and c such that $\vec{y} = a\vec{q} + b\vec{x} + c\vec{1}$, where

$$\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 11 \end{bmatrix}, \vec{q} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \vec{x} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \vec{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Wait. Where did that vector $\vec{q}$ come from? We got the vector $\vec{x}$ from the first coordinates of our data points because we had an x in the equation $y = ax^2 + bx + c$. Note also that we have an x^2 term. Thus, we need a second vector comprised of the square of the first coordinates of our data points.

Again, we should find the closest possible vector to $\vec{y}$ that is in $W = \text{Span}\left\{\vec{q}, \vec{x}, \vec{1}\right\}$, which is $\text{proj}_W(\vec{y})$. Fortunately for us, $\{\vec{q}, \vec{x}, \vec{1}\}$ is *almost* an orthogonal set; note that $\vec{x} \cdot \vec{1} = 0$ and $\vec{x} \cdot \vec{q} = 0$. Thus, to find a new orthogonal basis for W, $\{\vec{w}, \vec{x}, \vec{1}\}$, we only need to replace $\vec{q}$ with a new vector $\vec{w}$. Then

$$\vec{w} = \vec{q} - \frac{\vec{x} \cdot \vec{q}}{\vec{x} \cdot \vec{x}}\vec{x} - \frac{\vec{1} \cdot \vec{q}}{\vec{1} \cdot \vec{1}}\vec{1} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \\ -1 \\ 2 \end{bmatrix}.$$

Then we have

$$\text{proj}_W(\vec{y}) = \frac{\vec{w} \cdot \vec{y}}{\vec{w} \cdot \vec{w}}\vec{w} + \frac{\vec{x} \cdot \vec{y}}{\vec{x} \cdot \vec{x}}\vec{x} + \frac{\vec{1} \cdot \vec{y}}{\vec{1} \cdot \vec{1}}\vec{1} = \frac{8}{7}\vec{w} + \frac{12}{5}\vec{x} + \frac{18}{5}\vec{1}.$$

It's a good thing we didn't try to guess these coefficients. This isn't quite what we need, though; $\vec{w}$ is the wrong vector. For this to be the parabola of best fit, we need $\vec{q}$, the vector of squared x's. Fortunately, we know that $\vec{w} = \vec{q} - (2)\vec{1}$. Thus,

$$\begin{aligned} \text{proj}_W(\vec{y}) &= \frac{8}{7}\vec{w} + \frac{12}{5}\vec{x} + \frac{18}{5}\vec{1} \\ &= \frac{8}{7}(\vec{q} - (2)\vec{1}) + \frac{12}{5}\vec{x} + \frac{18}{5}\vec{1} \\ &= \frac{8}{7}\vec{q} + \frac{12}{5}\vec{x} + \left(\frac{18}{5} - \frac{16}{7}\right)\vec{1} = \frac{8}{7}\vec{q} + \frac{12}{5}\vec{x} + \frac{46}{35}\vec{1} \end{aligned}$$

The quadratic equation $y = \frac{8}{7}x^2 + \frac{12}{5}x + \frac{46}{35}$ is the best quadratic least squares approximation for the given data. See Figure 2.12.

It seems like we could generalize this procedure pretty easily.

Exploration 62 Let $a, b, c, d, e \in \mathbb{R}$, and consider the data points

$$(-3, a), (-1, b), (0, c), (1, d), \text{ and } (3, e).$$

Find the line of best fit for this data. Note that the equation for your line should depend on the scalars a, b, c, d, and e.

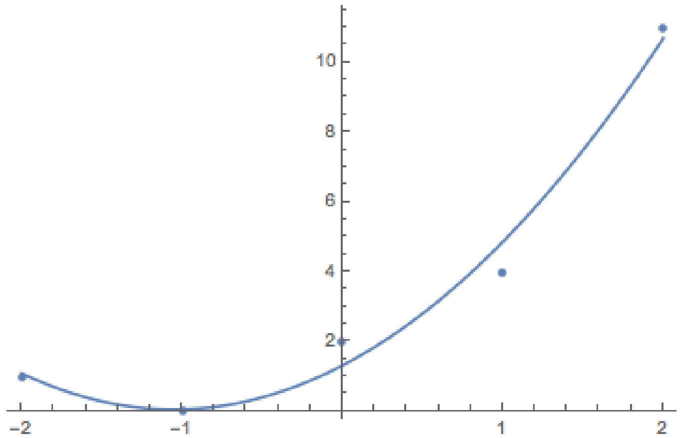


FIGURE 2.12. The last aggressively nonlinear data set with the parabola of best fit.

Exercises for Section 2.6

2.6.1. Let $W = \operatorname{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. Using the standard inner product, how far is $\vec{v}$ from W?

2.6.2. Let $W = \operatorname{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Using the standard inner product, how far is $\vec{v}$ from W?

2.6.3. Let $W = \operatorname{Span}\left\{x^2 - 2, 3x^3 + 1\right\} \subset \mathbb{P}_3$ and $\vec{p} = x + 1$. How far is $\vec{p}$ from W? (Hint: Use coordinate vectors relative to the standard basis for $\mathbb{P}_3$.)

3 Linear Transformations

In this chapter, we're going to finally begin exploring functions on vector spaces.[1] Linear algebra is the study of linear transformations on vector spaces. We did vector spaces, so once we do linear transformations, we should be done, right?[2] You will not be surprised that the implications for everything currently known that springs forth from the definition of a linear transformation *alone* can fill an entire chapter. Indeed, the entanglement and conceptual symbiosis of these two main characters in our story, vector spaces and linear transformations, could fill volumes.

Alas, we have but this one volume and this short time together to celebrate these two glorious concepts and their relationship with each other. Thus, we'll just have to hit the high points. In doing so, we'll find that linear transformations, while being an incredibly diverse and flexible type of function, can be understood *almost entirely* in a very systematic and concrete way.

As has become our custom at the beginning of chapters, though, we are getting a bit ahead of ourselves. Let's take a small step back to better prepare ourselves for the coming of linear transformations. Let's talk more about functions…

3.1 More Fun with Functions

There are a few more fun facts about functions for which a fresh look would be good.[3] You should be familiar, in some way, with each of the ideas we'll cover in this section, but you may not have previously seen this level of formality.[4] You are encouraged to go through this material very carefully and thoroughly; it will serve you well in future sections.

1: It feels like this should be the end, right?

2: Yes!

3: Favorable? Fine? Funicular? Well, at least two of those.

4: …and silliness. That should be expected by now, right?

Onto Functions

Definition 3.1.1 *For sets A and B and a function $f\colon A \to B$, the function f is* **onto** *if for every element $b \in B$, there is an element $a \in A$ such that f relates a to b, that is, $f(a) = b$.*

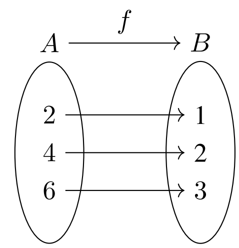
Besides the formal definition of onto, we also have a more geometric characterization that will be extremely useful from time to time.

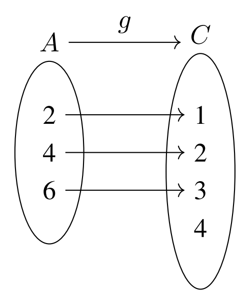
Theorem 3.1.1 *For sets A and B and a function $f\colon A \to B$, the function f is* ***onto*** *if and only if*

$$\operatorname{ran}(f) = \operatorname{codom}(f).$$

The proof of this theorem follows almost directly from Definition 3.1.1. Note that it *almost* follows.[5]

5:  Exercise!





Example 3.1.1 Let's see some examples!

- Let $A = \{2, 4, 6\}$ and $B = \{1, 2, 3\}$. Define the function $f\colon A \to B$ by the rule $f(a) = a/2$ for every $a \in A$. This function is onto because every element in B is half of one of the elements in A. If we had instead mapped $g\colon A \to C$ where $C = \{1, 2, 3, 4\}$ using the same rule as for f, then g would not be onto. This really illustrates how much the codomain has to do with this property.
- Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by the rule

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1.$$

  This map is onto. Let's convince ourselves.
  - First, let's look at a specific example. Consider 5. It's a real number, so it's in the codomain. What does f map to 5? Yep,

$$f\left(\begin{bmatrix} 5 \\ 0 \end{bmatrix}\right) = 5.$$

    Also,

$$f\left(\begin{bmatrix} 5 \\ 1 \end{bmatrix}\right) = 5.$$

    In fact,

$$f\left(\begin{bmatrix} 5 \\ x_2 \end{bmatrix}\right) = 5$$

    where x_2 is anything in $\mathbb{R}$. Let's be honest, there was nothing special about 5. Sometimes it's just nice to see how these things look with numbers, but to actually prove or verify that f is onto, we'll need to use a general element from the codomain.
  - Let $y \in \mathbb{R}$. Then

$$f\left(\begin{bmatrix} y \\ 0 \end{bmatrix}\right) = y.$$

    Since y is a general element of the codomain, we can conclude that f is onto. Note that we just needed to find *one* element in the domain that mapped to our general element y. As in our previous discussion, we could actually find infinitely many, but according to our definition of onto, that's overkill because we only need one.[6]

- Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 2$. Let's show this is onto! Let $y \in \mathbb{R}$ denote any element in the codomain. We need to find an x in the domain that f maps to y. That is, we need to solve $f(x) = y$ for a value of x. Then $f(x) = x + 2$, so we have $x + 2 = y$. Solving this for x we see $x = y - 2$. Well $f(y - 2) = (y - 2) + 2 = y$, so $y - 2$ is the correct x value to satisfy $f(x) = y$. Because we were able to do this for a general y in the codomain, we know f is onto.

6: There is still value in knowing precisely what in the domain maps to an element in the range. We'll see this again later.

Exploration 63 Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$. This function is not onto. Find a value in $\mathbb{R}$ that is not in $\operatorname{ran}(f)$.

Exploration 64 Show that the map $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is onto. First, let y represent a general element in the codomain $\mathbb{R}$. Now, what must x be so that $f(x) = y$?

Exploration 65 For each function below, determine whether it is onto. If you are having trouble deciding, ask yourself whether there's anything *not* in the range of this function.[7] If you can't think of anything, see if there's an input that you could use to give you any desired output.

7: How about an ice cream cone? I bet that's not in the range! Oh, maybe you should only consider things that are also in the codomain, too...

- Let $f: \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 x_2.$$

- Let $g: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 x_2 \\ x_1 x_2 \end{bmatrix}.$$

One-to-one Functions

The definition companion to onto is one-to-one. This is an often misunderstood concept; at the heart of these misunderstandings is the oversimplification of this concept to something having to do with horizontal and vertical lines. We shall require a thorough understanding of this particular concept in a very broad context, so let's get the formal definition and just say no more about that standard oversimplification.

Definition 3.1.2 *For sets A and B and a function $f\colon A \to B$, the function f is* ***one-to-one*** *if for any $b \in \operatorname{ran}(f)$, we have $a_1 = a_2 \in A$ if $f(a_1) = b$ and $f(a_2) = b$.*

This definition is just requiring that if f maps both a_1 and a_2 to b, then it must be that $a_1 = a_2$. That way, no more than one element from the domain can be mapped by f to an element in the range of f.

Theorem 3.1.2 *For sets A and B and a function $f\colon A \to B$, the function f is* ***one-to-one*** *if and only if for all $a_1, a_2 \in A$,*

$$f(a_1) = f(a_2) \text{ implies } a_1 = a_2.$$

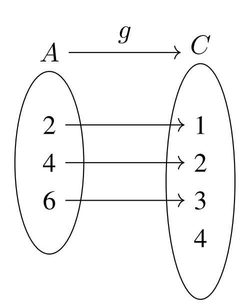
Note that in the definition, the universal quantifier (the "for any" bit) is on the elements of B, but in the equivalent definition of one-to-one given in Theorem 3.1.2, the universal quantifier is on the elements of A. This makes this theorem a bit trickier than Theorem 3.1.1.

PROOF. Let's suppose the function $f\colon A \to B$ is one-to-one by our definition. That is, for any $b \in \operatorname{ran}(f)$, we have $a_1 = a_2$ whenever $f(a_1) = b$ and $f(a_2) = b$. Now, let a_1 and a_2 be in A and suppose they have the property that $f(a_1) = f(a_2)$. Then, $f(a_1) = f(a_2) = b$ for some $b \in B$. Thus, $a_1 = a_2$ by the definition of one-to-one.

Now, suppose we know the function $f\colon A \to B$ has the property that for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$. Let $b \in \operatorname{ran}(f)$. Then we know there must be some $a_1 \in A$ such that $f(a_1) = b$. Suppose we also have $a_2 \in A$ such that $f(a_2) = b$. Then since $f(a_1) = b = f(a_2)$, we know $a_1 = a_2$. So this function is one-to-one by our definition. $\square$

Example 3.1.2 Let's revisit some familiar functions, but now, we can ask whether they are one-to-one!

- Let $A = \{2, 4, 6\}$ and $C = \{1, 2, 3, 4\}$. Define the function $g\colon A \to C$ by the rule $g(a) = a/2$ for every $a \in A$. This function is one-to-one! To see this, suppose $c, d \in A$ are such that $g(c) = g(d)$. Using the definition of g, we see $c/2 = d/2$, which can be simplified to see $c = d$.



- Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by the rule

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1.$$

This map was onto, but it is *not* one-to-one. Recall that we had options about what mapped to 5. We saw that

$$f\left(\begin{bmatrix} 5 \\ 0 \end{bmatrix}\right) = 5 = f\left(\begin{bmatrix} 5 \\ 1 \end{bmatrix}\right).$$

This example with specific numbers is actually enough to show that the function does not satisfy our definition!

- Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x+2$. This one was onto, and it is also one-to-one! Suppose $a, b \in \mathbb{R}$ are such that $f(a) = f(b)$. then we know $a + 2 = b + 2$. This says $a = b$.

Exploration 66 Determine whether the following functions are one-to-one. If you are having trouble deciding, pick something in the range of the function. Ask yourself whether there are multiple inputs to get that same output.

- Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1x_2.$$

- Let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 \end{bmatrix}.$$

As you probably expect, one-to-one and onto are both very nice properties for a function to have. Additionally, functions with both are really quite grand. We'll see why shortly.

Composition of Functions

We saw composition of functions appear briefly in Section 2.3 to define inner product on vector spaces. Let us now have the formal definition.

Definition 3.1.3 *Let A, B, and C be sets and $f\colon A \to B$ and $g\colon B \to C$ be functions. The* ***composition of the functions*** f ***and*** g *is the function $(g \circ f)\colon A \to C$ such that $(a, c) \in g \circ f$ if and only if there is a $b \in B$ such that $(a, b) \in f$ and $(b, c) \in g$. That is, for any $a \in A$,*

$$(g \circ f)(a) = g(f(a)).$$

As was mentioned in Section 2.3, what makes the composition of functions work is the fact that outputs of f are inputs of g. This can be seen in both of the commuting diagrams below:

$$
\begin{tikzcd}
B \arrow[dr, "g"] & \\
A \arrow[u, "f"] \arrow[r, "g \circ f"'] & C,
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C.
\end{tikzcd}
$$

Example 3.1.3 Define

$$S = \left\{ \frac{z}{2} \colon z \in \mathbb{Z} \right\}.$$

Let $f\colon \mathbb{Z} \to S$ be defined by $f(z) = \dfrac{z}{2}$ and $g\colon S \to \mathbb{Z}$ be defined by $g(x) = 2x$. Then $g \circ f\colon \mathbb{Z} \to \mathbb{Z}$ is the map sending an integer z to itself since $(g \circ f)(z) = g(f(z)) = g\left(\dfrac{z}{2}\right) = 2\left(\dfrac{z}{2}\right) = z$.

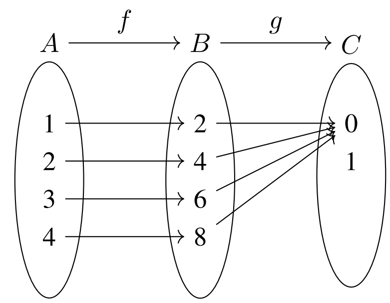
Exploration 67 Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$, $C = \{0, 1\}$. Let $f\colon A \to B$ be given by

$$f(a) = 2a \text{ for any } a \in A.$$

Let $g\colon B \to C$ be given by

$$g(b) = 0 \text{ for all } b \in B.$$

- Consider $g \circ f\colon A \to C$. We see $g(f(1)) = g(2) = 0$ and $g(f(2)) = g(4) = 0$. Since g maps everything to 0, this is the map from A to C which maps everything in A to 0.



Here's a picture for $g \circ f$. You should try drawing similar ones for the other functions mentioned.

Let $k\colon C \to B$ be defined by

$$k(0) = 2 \text{ and } k(1) = 6.$$

- Consider $g \circ k\colon C \to C$. We see $g(k(0)) = g(2) = 0$ and $g(k(1)) = g(6) = 0$. Then $g \circ k$ is the map from C to itself sending both elements of C to 0.
- Consider $k \circ g\colon B \to B$. What is $k(g(2))$? What about $k(g(4))$? Describe $k \circ g$.

Let $h\colon A \to C$ be given by

$$h(a) = 0 \text{ if } a \in A \text{ is even and } h(a) = 1 \text{ if } a \in A \text{ is odd.}$$

- Consider $k \circ h\colon A \to B$. Describe this map.

Example 3.1.4 Suppose $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are both one-to-one functions. We can show that the composition $f \circ g$ must also be one-to-one! Suppose we have $a_1, a_2 \in \mathbb{R}$ such that $(f \circ g)(a_1) = (f \circ g)(a_2)$. Then we know $f(g(a_1)) = f(g(a_2))$. Since we know f is one-to-one, we know it must be true that $g(a_1) = g(a_2)$. This is because $g(a_1)$ and $g(a_2)$ are both inputs to f with the same output from f. Now, we can use the fact that g

is one-to-one to say that $a_1 = a_2$ since $g(a_1) = g(a_2)$. This is what we needed! We started with two inputs to $f \circ g$ that had the same output from $f \circ g$ and were able to argue that the two inputs were really the same.

Exploration 68 Suppose $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are both onto. Must $f \circ g$ be onto?

Invertible Functions

A problem that comes up a *lot* in mathematics and science is whether or not a process (often modeled by a function) can be undone in a reasonable way. For example, if I were to replace every song file on the internet with a single song file, then I strongly suspect many people would want a way to undo that process. Could the "undoing" be done with a function? That's not immediately clear, but the forward process certainly could: I hereby relate every song file on the internet to Jimmy Buffet's "Pencil Thin Mustache." [8] You're welcome world. Oh? What's that? You want a different function that relates "Pencil Thin Mustache" back to every song that used to be on the internet? Ha! Good luck with that! While there is a "Buffeting" function, I'm afraid the "Un-Buffeting" is not functional.[9]

Suppose instead that we related every song on the internet to itself, played backwards.[10] Well, if we did it again, we'd be back where we started. This is an example of a process that can be modeled by what we call an *invertible* function.

Definition 3.1.4 *A function $f\colon A \to B$ is* ***invertible*** *if there is another function $g\colon B \to A$ such that*

- *for all $a \in A$, $(g \circ f)(a) = a$, and*
- *for all $b \in B$, $(f \circ g)(b) = b$.*

If such a function exists, we call it the ***inverse of*** *f, and denote it f^{-1}.*

Example 3.1.5 Let's recall some maps from an earlier example. Define

$$S = \left\{ \frac{z}{2} \colon z \in \mathbb{Z} \right\}.$$

Let $f\colon \mathbb{Z} \to S$ be defined by $f(z) = \dfrac{z}{2}$ and $g\colon S \to \mathbb{Z}$ be defined by $g(x) = 2x$. Then we saw $g \circ f\colon \mathbb{Z} \to \mathbb{Z}$ has the property that $g \circ f(z) = z$ for any $z \in \mathbb{Z}$. We can also construct $f \circ g\colon S \to S$, and we see that $f \circ g = f(g(a)) = f(2a) = \dfrac{2a}{2} = a$ for any $a \in S$. Thus, $g = f^{-1}$.

8: This is a perfectly reasonable function with domain and codomain being all song files on the internet, but range just the single specific song file for "Pencil Thin Mustache."

9: This is not actually intended as a pun. A relation that is a function is often called a "functional" relation. However, we gladly accept the dual meaning here.

10: Disclaimer: The authors do not endorse the playing of songs backwards or the following of any nefarious instructions heard when doing so.

Example 3.1.6 Let's see an example of a function that is not invertible. Let $h\colon \mathbb{R} \to \mathbb{R}^2$ be defined by

$$h(x) = \begin{bmatrix} x \\ x \end{bmatrix}.$$

Note that h is not onto because any vector of the form

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

with $x \neq y$ will not be in the range. Suppose h has an inverse that we will call $k\colon \mathbb{R}^2 \to \mathbb{R}$. Then $(h \circ k)(\vec{v}) = \vec{v}$ for any $\vec{v} \in \mathbb{R}^2$. Let's consider then the vector

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2.$$

If k exists, then we have

$$(h \circ k)\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = h\left(k\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

This says, however, that

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \operatorname{ran}(h),$$

which is not true. Thus, the inverse k does not exist and h is not invertible.

Example 3.1.7 Let's see another function that fails to be invertible. Consider the function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ that relates a vector $\vec{x} \in \mathbb{R}^2$ to the vector with the same first coordinate but 0 for the second coordinate. That is, for any vector in $\mathbb{R}^2$,

$$\varphi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

The domain and codomain of φ is $\mathbb{R}^2$, but the range of φ is the horizontal line

$$\left\{\begin{bmatrix} x_1 \\ 0 \end{bmatrix} : x_1 \in \mathbb{R}\right\}.$$

Thus, the function φ collapses all of $\mathbb{R}^2$ (a plane!) to a single line. That doesn't sound like one-to-one behavior, so let's prove this function is not one-to-one. Note that

$$\varphi\left(\begin{bmatrix} 42 \\ 10 \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} 42 \\ 11 \end{bmatrix}\right) = \begin{bmatrix} 42 \\ 0 \end{bmatrix}.$$

Since φ maps two vectors to the same vector in the codomain, it is not one-to-one. Why does this matter? Suppose g is a function that we'd like to be an inverse of φ. By Definition 3.1.4, for any vector $\vec{x} \in \mathbb{R}^2$, we must have $(g \circ \varphi)(\vec{x}) = \vec{x}$. However, that means

$$(g \circ \varphi)\left(\begin{bmatrix} 42 \\ 10 \end{bmatrix}\right) = \begin{bmatrix} 42 \\ 10 \end{bmatrix} \text{ and } (g \circ \varphi)\left(\begin{bmatrix} 42 \\ 11 \end{bmatrix}\right) = \begin{bmatrix} 42 \\ 11 \end{bmatrix}.$$

This first equation tells us

$$g\left(\varphi\left(\begin{bmatrix} 42 \\ 10 \end{bmatrix}\right)\right) = g\left(\begin{bmatrix} 42 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 42 \\ 10 \end{bmatrix},$$

and the second tells us that

$$g\left(\varphi\left(\left[\begin{array}{c}42\\11\end{array}\right]\right)\right)=g\left(\left[\begin{array}{c}42\\0\end{array}\right]\right)=\left[\begin{array}{c}42\\11\end{array}\right].$$

Hang on. Those two vectors aren't equal. In order for g to be an inverse for φ, it has to map one vector to two different vectors. That is not how any respectable function behaves![11]

11:  Here, you should take a respectable function to mean any function. Yes, we have just claimed all functions are respectable, even the Buffetting one from earlier.

The main idea here is that if you have a function that relates elements of A to elements of B, does there exist another function that unrelates them? More specifically, is there another function g that relates B to A so that if f relates $a_0 \in A$ to $b_0 \in B$, then g relates b_0 back to a_0. Sounds pretty easy, right? The part that makes this nontrivial is all that "for all" business. This property of "unrelating" has to hold for every element in A and every element in B. That is a very rigid condition! Fortunately, as with one-to-one and onto, there is an alternate characterization of invertibility.

In Example 3.1.6 we saw a function that failed to be invertible because it was not onto. In Example 3.1.7, we saw a function that failed to be invertible because it was not one-to-one. Perhaps then this theorem will not come as a surprise.[12]

12:  Oh, this makes me sad. I love a surprise...

Theorem 3.1.3 *A function $f\colon A \to B$ is invertible if and only if f is both one-to-one and onto.*

Exploration 69 Let's start this proof together!

PROOF. First, suppose $f\colon A \to B$ is invertible. Then f^{-1} exists. Let's use f^{-1} to show that f is both one-to-one and onto.

One-to-one: Let $a, b \in A$ be such that $f(a) = f(b)$. We need to show $a = b$. To do this, let's consider the expression $f^{-1}(f(a))$. Since $f(a) = f(b)$, we have $f^{-1}(f(a)) = f^{-1}(f(b))$. Now, why does this tell us $a = b$?

Onto: Let $b \in B$. Then we need to find an element $a \in A$ that maps to b under f. Use f^{-1} to find a.

Now, we need to prove the other direction. Suppose that f is both one-to-one and onto. We need to define a function $g\colon B \to A$ so that f has an inverse. Since we know that f is onto B, we can write any element of B as $f(a)$ for some $a \in A$. Moreover, since f is one-to-one, we know for any $b \in B$ that there is exactly one $a \in A$ such that $b = f(a)$. Thus, we can define $g\colon B \to A$ to be the relation sending $f(a)$ to a. Now, we need to verify that this g is actually a function. Suppose $g(b) = a_1$ and $g(b) = a_2$. By the definition of g, this says $f(a_1) = b = f(a_2)$. Since f is one-to-one, we know $a_1 = a_2$, so g is a valid function. Note that we needed f to be both onto and one-to-one in order for g to be a function with domain B. We now just need to verify that g is the inverse of f. Our first condition is that for all $a \in A$, $(g \circ f)(a) = a$. We see this quickly since $g(\circ f)(a) = g(f(a)) = a$ by definition. Next we have that second condition: for all $b \in B$, $(f \circ g)(b) = b$. Let $b \in B$, then there is some $a \in A$ such that $f(a) = b$ since f is onto. So $f(g(b)) = f(g(f(a))) = f(a) = b$. Thus, the second condition is also satisfied by this function g because f is onto. □

Exploration 70 Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 2$. We saw in our previous examples that this function is both one-to-one and onto. So it's invertible! Can you find the inverse?

Recall our Buffeting function from earlier? Well, we can very quickly convince ourselves that this function is not one-to-one.[13] Thus, we have the following fun corollary to Theorem 3.1.3.

13: We can also convince ourselves that this function was ridiculous.

Corollary 3.1.4 *There is no function to undo the Buffeting.*

Functions Between Finite Sets

Something a little special happens when talking about the properties of one-to-one and onto when the function is between two finite sets.

Theorem 3.1.5 *Let A and B be finite sets and suppose $f\colon A \to B$ is a function. Let n denote the number of elements in A and m denote the number of elements in B.*

(a) *If $n > m$, then f is not one-to-one.*
(b) *If $n < m$, then f is not onto.*
(c) *If $n = m$, then f is either both one-to-one and onto or f is neither one-to-one nor onto.*

PROOF. First, suppose $n > m$. Then there are more elements in the domain of f than in its codomain. If f is one-to-one, then we would see each element in A map to a distinct element of B. However, there just aren't enough elements of B for this to happen!

Next, suppose $n < m$. Then this function has more elements in its codomain than its domain. In order for f to be onto, we must have $\operatorname{ran}(f) = \operatorname{codom}(f)$. However, to be a function, f must map each element of A to only one element of B. This means the largest $\operatorname{ran}(f)$ can be is n, so f cannot be onto.

Lastly, let's suppose $n = m$. If f is onto, then every element in B is mapped to by an element in A. Because these sets have the same size, the map is forced to be one-to-one since it can only send each element of A to one distinct element in B. Suppose now that f is one-to-one. Then it will send each element in A to a different element in B. Since these sets have the same size, we must use every element in B in this mapping, so f is also onto. We've just argued that f is one-to-one if and only if it is onto if $n = m$. This gives our result. □

Since almost all of our vector spaces are infinite sets,[14] you may be wondering why we bothered to tell you about Theorem 3.1.5. You'll see in a few sections

14: All except the trivial one, $\{\vec{0}\}$, in fact.

Nicky! What are *you* doing here?!

It looked like you needed help, so here I am. Helping.

that there is actually a wonderful analog to this theorem in the case of functions between vector spaces.

Section Highlights

- A function is one-to-one if every element in the range is mapped to exactly once. In terms of function input from the domain and output from the range, this means each output has exactly one input that maps to it. See Definition 3.1.2 and Theorem 3.1.2
- A function is onto if its range is the entirety of the codomain. This means every possible element in the codomain is an output mapped to by some input in the domain. See Definition 3.1.1 and Theorem 3.1.1.
- Function composition is a way to combine functions. See Definition 3.1.3.
- If a function is both one-to-one and onto, then it is invertible. This means that there exists an inverse function with which it composes to form the identity map. See Definition 3.1.4 and Theorem 3.1.3.

Exercises for Section 3.1

3.1.1.Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 + x_2.$$

(a) Is f onto? Prove or give a counterexample.

(b) Is f one-to-one? Prove or give a counterexample.

3.1.2.Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 + 3.$$

(a) Is f onto? Prove or give a counterexample.

(b) Is f one-to-one? Prove or give a counterexample.

3.1.3.For each function below, determine whether it is one-to-one.

(a) $f\colon \{a, b, c\} \to \{1, 2, 3, 4\}$ defined by $\{(a, 1), (b, 2), (c, 1)\}$

(b) $f\colon \{a, b, c\} \to \{1, 2, 3, 4\}$ defined by $\{(a, 1), (b, 2), (c, 4)\}$

(c) $f\colon \{a, b, c\} \to \{1, 2, 3\}$ defined by $\{(a, 1), (b, 2), (c, 3)\}$

(d) $f\colon \{a, b, c, d\} \to \{1, 2, 3\}$ defined by $\{(a, 1), (b, 2), (c, 3), (d, 2)\}$

(e) $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \left(\dfrac{x+1}{4}\right)$

(f) $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1$

(g) $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 + 3x_2$

(h) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ x_1x_2 \end{bmatrix}$

(i) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ x_1 \end{bmatrix}$

(j) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_1 + 3 \end{bmatrix}$

(k) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 + 3 \end{bmatrix}$

(l) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 2x_1 + 2x_2 \end{bmatrix}$

(m) $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 2x_1 + 2x_2 \\ x_2 \end{bmatrix}$

3.1.4.For each function below, determine whether it is onto.

(a) $f\colon \{a, b, c\} \to \{1, 2, 3, 4\}$ defined by $\{(a, 1), (b, 2), (c, 1)\}$

(b) $f\colon \{a, b, c\} \to \{1, 2, 3, 4\}$ defined by $\{(a, 1), (b, 2), (c, 4)\}$

(c) $f\colon \{a, b, c\} \to \{1, 2, 3\}$ defined by $\{(a, 1), (b, 2), (c, 3)\}$

(d) $f\colon \{a, b, c, d\} \to \{1, 2, 3\}$ defined by $\{(a, 1), (b, 2), (c, 3), (d, 2)\}$

(e) $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \left(\dfrac{x+1}{4}\right)$

(f) $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1$

(g) $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 + 3x_2$

(h) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ x_1 x_2 \end{bmatrix}$

(i) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ x_1 \end{bmatrix}$

(j) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_1 + 3 \end{bmatrix}$

(k) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 + 3 \end{bmatrix}$

(l) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 2x_1 + 2x_2 \end{bmatrix}$

(m) $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 2x_1 + 2x_2 \\ x_2 \end{bmatrix}$

3.1.5.Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 3$. Show that f is one-to-one and onto. Find f^{-1}.

3.1.6.Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_3 \\ 2x_1 + x_2 \\ x_2 \end{bmatrix}$. Show that f is both one-to-one and onto. Find f^{-1}.

3.1.7.A constant function maps every element in the domain to the same element in the codomain. Give an example of

(a) a domain, codomain, and constant function that is onto and not invertible;

(b) a domain, codomain, and constant function that is onto and invertible.

3.1.8.Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 + x_2.$$

Does f have the property that $f(\vec{v} + \vec{u}) = f(\vec{v}) + f(\vec{u})$ for any $\vec{v}, \vec{u} \in \mathbb{R}^2$?

3.1.9.Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 + 3.$$

Does f have the property that $f(\vec{v} + \vec{u}) = f(\vec{v}) + f(\vec{u})$ for any $\vec{v}, \vec{u} \in \mathbb{R}^2$?

3.1.10.Suppose $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are both onto. Must $f + g$ be onto? Give an example to support your claim.

3.1.11.Let A, B, and C be sets and $f\colon A \to B$ and $g\colon B \to C$ be functions.

(a) Suppose $g \circ f$ is one-to-one. Then the function f is one-to-one. Let's see why. Suppose $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$. We want to show $a_1 = a_2$, and we know we need to involve $g \circ f$ to do this. Let's consider $g \circ f(a_1) = g(f(a_1))$ and $g \circ f(a_2) = g(f(a_2))$. How are these related?

Now use the fact that $g \circ f$ is one-to-one to conclude $a_1 = a_2$.

(b) Suppose g is onto. It's not necessarily true that $g \circ f$ is onto. Can you come up with an example where g is onto but $g \circ f$ is not?

3.1.12.Let A, B, and C be sets and $f\colon A \to B$ and $g\colon B \to C$ be functions.

(a) Suppose f is one-to-one and g is onto.

(i) Is $g \circ f$ one-to-one? Prove or give a counterexample.

(ii) Is $g \circ f$ onto? Prove or give a counterexample.

(b) Suppose $g \circ f$ is onto and g is one-to-one.

(i) Is g onto? Prove or give a counterexample.

(ii) Is $g \circ f$ one-to-one? Prove or give a counterexample.

3.2 Linear Transformations

We've now talked quite a bit about functions, but very little about functions in relation[15] to vector spaces.[16] It's time to fix that. It's time to celebrate. It's time to finally formally familiarize ourselves with the functions that best fit this narrative we've been following. We're all here to hear about linear transformations on vector spaces. Well, we may never actually be done with amateur word play, but the long wait is over. Get ready. It's linear transformation time.

15: Pun not intended, but it's there anyway.

16: Well, except for those inner product ones, but that was ages ago now.

Respect the Operations

Let's think about what it means for a function to map one vector space to another by starting with an example.

Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+2 \\ b+2 \\ c+2 \end{bmatrix}.$$

This seems like a perfectly good function between vector spaces. Commence exploration!

Exploration 71 To get a sense for what this function does, let us experiment and see what happens with a few specific vectors.

- ▶ Compute $f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$ and $f\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right)$.

- ▶ Compute $2f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$.

From these computations, we see that for a vector $\vec{v} \in \mathbb{R}^3$, this function has

$$f(\vec{v}+\vec{v}) \neq f(\vec{v}) + f(\vec{v}) \qquad \text{and} \quad f(2\vec{v}) \neq 2f(\vec{v}).$$

While f is a perfectly good function between the *set* of vectors in $\mathbb{R}^3$ and the *set* of vectors in $\mathbb{R}^3$, it is not a useful function between the *vector spaces*. We have $\vec{v} \in \mathbb{R}^3$ and $f(\vec{v}) \in \mathbb{R}^3$; since $\mathbb{R}^3$ is closed under vector addition, we also have that

$$\vec{v}+\vec{v} \in \mathbb{R}^3 \quad \text{and} \quad f(\vec{v}) + f(\vec{v}) \in \mathbb{R}^3.$$

Since our function related $\vec{v}$ to $f(\vec{v})$, our function would, ideally, relate $\vec{v}+\vec{v}$ to $f(\vec{v})+f(\vec{v})$; that is, it would be very nice if our function related our notion of vector addition in the domain to our notion of vector addition in the codomain. However, our function relates $\vec{v}+\vec{v}$ to $f(\vec{v}+\vec{v})$, and as we've already shown, $f(\vec{v}+\vec{v}) \neq f(\vec{v})+f(\vec{v})$. In order for our function to relate the vector space structure in the domain to the vector space structure in the codomain, it needs to behave well with respect to the vector space operations: vector addition and scalar multiplication.

There is a very useful class of functions for which this respect of operations is cleverly guaranteed.

Definition 3.2.1 *A function $f: V \to W$, where V and W are vector spaces, is called a* ***linear transformation*** *if for any vectors $\vec{v}, \vec{u} \in V$ and any scalar $a \in \mathbb{R}$,*

- $f(\vec{v}+\vec{u}) = f(\vec{v})+f(\vec{u})$ *and*
- $f(a\vec{v}) = af(\vec{v})$.

Linear transformations[17] resolve the issue of how vector addition and scalar multiplication are defined in the range of a function (function first or operation first) by saying it doesn't matter. For linear transformations, the two vectors are equal:

- $f(\vec{v}+\vec{u}) = f(\vec{v})+f(\vec{u})$; sum and then function, or function then sum; it's the same either way.
- $f(a\vec{v}) = af(\vec{v})$; scale and then function, or function then scale; it's the same either way.

Linear transformations are often said to "preserve" vector addition and scalar multiplication for this reason. If this seems overly restrictive, keep in mind that these are the only operations on V, and making sure those work correctly in the range is really the only restriction we've imposed. Speaking of the "range," you'll find this arrangement has its advantages:

Definition 3.2.2 *Let V and W be vector spaces and $f: V \to W$ be a linear transformation. The* ***image of*** *f is the set of vectors $\vec{w} \in W$ such that there is a vector $\vec{v} \in V$ with $\vec{w} = f(\vec{v})$. We shall use the notation*

$$\operatorname{Imag} f = \{\vec{w} \in W : \vec{w} = f(\vec{v}) \text{ for some } \vec{v} \in V\}.$$

You're probably asking yourself, "Isn't that just the range of f?" Yeah, it is. That's true. However, for any function $f: V \to W$ where V and W are just sets and $v \in V$, we often refer to $f(v)$ as the image of v. The idea is that the image of the *point* v is the *point* $f(v)$. Now, when V and W are vector spaces and f is a linear transformation, we extend this term to encompass all of the range. Thus, the image of a vector space V is the vector space $f(V)$ when f is a linear transformation. The word "image" is commonly used instead of "range" when a function is the type that preserves the algebraic structure of the domain.

Theorem 3.2.1 *Let V and W be vector spaces and $f: V \to W$ be a linear transformation. Then the image of f, $\operatorname{Imag}(f)$, is a subspace of W.*

Exploration 72 Let's prove this one together!

17:  Yay! That's on the cover of the book!

PROOF. We know that $\operatorname{Imag} f$ is a subset of W, so we just need to verify closure for addition and scalar multiplication and also that it contains $\vec{0}$.

- Let $\vec{v}$ and $\vec{u}$ be vectors in V. Use the properties of the linear transformation f to show $f(\vec{v}) + f(\vec{u})$ is in $\operatorname{Imag} f$. (To show a vector is in $\operatorname{Imag} f$, find a way to write it as $f(\text{a vector in } V)$.)

- Let $\vec{v} \in V$ and $a \in \mathbb{R}$. Show $af(\vec{v}) \in \operatorname{Imag} f$.

- The fact that $\vec{0} \in \operatorname{Imag} f$ is actually related to an exercise from all the way back in Chapter 0. Linear transformations have the property that $f(a\vec{v}) = af(\vec{v})$ for any $a \in \mathbb{R}$ and any $\vec{v} \in V$. Thus, if $\vec{v} = \vec{0}$ and $a = 0$, we have $f(\vec{0}) = f(0\vec{0}) = 0f(\vec{0}) = \vec{0}$. Thus, $\vec{0} \in \operatorname{Imag} f$ since any linear transformation maps $\vec{0}$ to $\vec{0}$.

□

Examples Abound

Example 3.2.1 Let's start with a fairly straightforward function. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 + x_2$$

where $x_1, x_2 \in \mathbb{R}$. Let's verify that this is a linear transformation. We need the axioms in the definition to hold for *all* vectors, so we need general vectors. Let $\vec{x}, \vec{y} \in \mathbb{R}^2$. Then

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

for some $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Then

$$\begin{aligned} f(\vec{x} + \vec{y}) = f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) &= f\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) \\ &= (x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) \\ &= f(\vec{x}) + f(\vec{y}). \end{aligned}$$

Thus, f preserves vector addition. Suppose $a \in \mathbb{R}$. We will show now that f preserves scalar multiplication. We will use the same vector $\vec{x}$ as above.[18]

$$\begin{aligned} f(a\vec{x}) &= f\left(a\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} ax_1 \\ ax_2 \end{bmatrix}\right) \\ &= ax_1 + ax_2 = a(x_1 + x_2) = af(\vec{x}). \end{aligned}$$

Thus, f is a linear transformation.

18: Now wait a minute. Weren't we using $\vec{v}$ and $\vec{u}$ for our vectors?

Yeah, we were! What's with all these $\vec{x}$'s and $\vec{y}$'s?

Settle down. Using x for domain elements is pretty standard. Now that we're dealing with functions, it makes sense to change our vector naming convention.

Example 3.2.2 Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(\vec{x}) = \alpha\vec{x}$ for some fixed real number $\alpha > 1$. This function rescales vectors in $\mathbb{R}^2$ by a factor of α. To see this in action, let S be the following square in $\mathbb{R}^2$:

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \colon 0 \leq x_1 \leq 1 \text{ and } 0 \leq x_2 \leq 1 \right\}.$$

Figure 3.1 shows the image of S under the function T.
Let's verify that T is a linear transformation; we already know T is a function, so we just need to verify the two properties of a linear transformation. Let $\vec{x}, \vec{y} \in \mathbb{R}^2$ and $k \in \mathbb{R}$. Note that

$$\begin{aligned} T(\vec{x}+\vec{y}) &= \alpha(\vec{x}+\vec{y}) = \alpha\vec{x} + \alpha\vec{y} = T(\vec{x}) + T(\vec{y}) \text{ and} \\ T(k\vec{x}) &= \alpha(k\vec{x}) = (\alpha k)\vec{x} = (k\alpha)\vec{x} = k(\alpha\vec{x}) = kT(\vec{x}). \end{aligned}$$

It follows that T is a linear transformation.

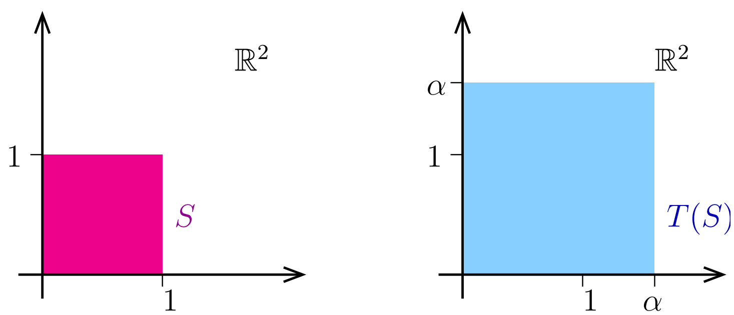


FIGURE 3.1. T is a linear transformation that rescales vectors by $\alpha > 1$.

Example 3.2.3 Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the function such that for any

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \quad \text{we define} \quad T(\vec{x}) = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

This perhaps appears more complicated, but all we're doing here is assigning vectors in $\mathbb{R}^3$ to a specific linear combination of vectors in $\mathbb{R}^2$ by using the components of our domain vectors as the weights of the linear combination. Not only does it turn out this is also a linear transformation, this also is an easy and convenient way to define a linear transformation. We should probably remember this example! Again, T is a function, so we just verify the two properties of a linear transformation. Let $\vec{x}, \vec{y} \in \mathbb{R}^2$, where

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

and $a \in \mathbb{R}$. This is gonna be a little gross. Note first that

$$\begin{aligned}
T(\vec{x}+\vec{y}) &= T\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}\right) \\
&= (x_1+y_1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (x_2+y_2)\begin{bmatrix} 3 \\ 1 \end{bmatrix} + (x_3+y_3)\begin{bmatrix} 5 \\ 4 \end{bmatrix} \\
&= \left(x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right) \\
&\quad + \left(y_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y_2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + y_3\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right) \\
&= T(\vec{x}) + T(\vec{y}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
T(a\vec{x}) = T\left(\begin{bmatrix} ax_1 \\ ax_2 \\ ax_3 \end{bmatrix}\right) &= ax_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + ax_2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + ax_3\begin{bmatrix} 5 \\ 4 \end{bmatrix} \\
&= a\left(x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right) \\
&= aT(\vec{x}).
\end{aligned}$$

Exploration 73 Recall Example 3.1.7 from Section 3.1. The function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ relates a vector $\vec{x} \in \mathbb{R}^2$ to the vector with the same first coordinate but 0 for the second coordinate. That is, for any vector in $\mathbb{R}^2$,

$$\varphi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

We showed this was a function that is not invertible already. Let's show that it's a linear transformation. We'll need some general vectors in $\mathbb{R}^2$ to start. Let $\vec{x}, \vec{y} \in \mathbb{R}^2$. Then

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

for some $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

- First, verify that $\varphi(\vec{x}+\vec{y}) = \varphi(\vec{x}) + \varphi(\vec{y})$.

- Second, verify that $\varphi(a\vec{x}) = a\varphi(\vec{x})$ for any $a \in \mathbb{R}$.

Example 3.2.4 We've seen several examples of functions that are linear transformations. Let's see another one that is *not* a linear transformation.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 x_2$$

where $x_1, x_2 \in \mathbb{R}$. We can show this fails to be a linear transformation in multiple ways. First, we can show it fails to preserve vector addition.

$$f\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = f\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right) = (3)(5) = 15,$$

but

$$f\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) + f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = (2)(3) + (1)(2) = 8.$$

Next, note that

$$f\left(5\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = f\left(\begin{bmatrix} 10 \\ 15 \end{bmatrix}\right) = (10)(15) = 150,$$

but

$$5f\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = 5(2)(3) = 30.$$

Thus, f also does not preserve scalar multiplication. Failing either condition is enough though to see that it is not a linear transformation.

Example 3.2.5 Let $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ be defined by

$$T(\vec{p}) = \begin{bmatrix} p(1) \\ p(0) \end{bmatrix}.$$

Let's show T is a linear transformation; this one is particularly interesting because T has a completely different domain and codomain. Let $\vec{p} = ax^2 + bx + c$ and $\vec{q} = dx^2 + ex + f$ be arbitrary vectors in $\mathbb{P}_2$. Note that

$$T(\vec{p}) = \begin{bmatrix} a+b+c \\ c \end{bmatrix} \quad \text{and} \quad T(\vec{q}) = \begin{bmatrix} d+e+f \\ f \end{bmatrix}.$$

Checking vector addition, we have

$$\begin{aligned}
T(\vec{p}+\vec{q}) &= T((ax^2+bx+c)+(dx^2+ex+f)) \\
&= T((a+d)x^2+(b+e)x+(c+f)) \\
&= \begin{bmatrix} (a+d)+(b+e)+(c+f) \\ c+f \end{bmatrix} \\
&= \begin{bmatrix} (a+b+c)+(d+e+f) \\ c+f \end{bmatrix} \\
&= \begin{bmatrix} a+b+c \\ c \end{bmatrix} + \begin{bmatrix} d+e+f \\ f \end{bmatrix} = T(\vec{p}) + T(\vec{q}).
\end{aligned}$$

All that remains is to check scalar multiplication; let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} T(\alpha\vec{p}) = T(\alpha(ax^2+bx+c)) &= T((\alpha a)x^2+(\alpha b)x+(\alpha c)) \\ &= \begin{bmatrix} \alpha a+\alpha b+\alpha c \\ \alpha c \end{bmatrix} \\ &= \begin{bmatrix} \alpha(a+b+c) \\ \alpha c \end{bmatrix} \\ &= \alpha \begin{bmatrix} a+b+c \\ c \end{bmatrix} = \alpha T(\vec{p}). \end{aligned}$$

Exploration 74 Let $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ be defined by

$$T(\vec{p}) = \begin{bmatrix} p(2) \\ p(0) \end{bmatrix}.$$

Is T a linear transformation? Justify your response.

Some Noteworthy Examples

Until now, we've focused on examples to illustrate how we verify a function is a linear transformation. Now, we'll see some examples of common linear transformations that will be important for us in future sections.

Theorem 3.2.2 *Let V be a vector space with basis $\mathcal{B}$ and dimension n. The function $\varphi_{\mathcal{B}}\colon V \to \mathbb{R}^n$ that relates vectors in V to their coordinate vector relative to $\mathcal{B}$ in $\mathbb{R}^n$ is a linear transformation; that is, the function given by*

$$\varphi_{\mathcal{B}}(\vec{v}) = [\vec{v}]_{\mathcal{B}}$$

*is a linear transformation. This function is sometimes called the **coordinate mapping**.*

PROOF. In the proof of the Pythagorean Theorem in Section 2.3 we established that $[\vec{v}+\vec{u}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{B}} + [\vec{u}]_{\mathcal{B}}$ for any $\vec{v}, \vec{u} \in V$. We just now need to see that $[\alpha\vec{v}]_{\mathcal{B}} = \alpha\,[\vec{v}]_{\mathcal{B}}$ for any scalar α and any $\vec{v} \in V$. However, this seems more like an exercise at this point. Expect to see this for homework. $\square$

It turns out, quite conveniently, that compositions of linear transformations are also linear transformations.

Theorem 3.2.3 *Let V, W, and U be vector spaces, $T\colon V \to W$ a linear transformation, and $S\colon W \to U$ a linear transformation. Then the composition $S \circ T\colon V \to U$ is a linear transformation.*

This can be seen in the commuting diagram below:

$$\begin{tikzcd} W \arrow[dr, "S"] & \\ V \arrow[u, "T"] \arrow[r, "S\circ T"'] & U \end{tikzcd}$$

PROOF. To show $S \circ T$ is a linear transformation, we need to show it respects vector addition and that it respects scalar multiplication. Let $\vec{v}_1$ and $\vec{v}_2$ be vectors in V, and let $a \in \mathbb{R}$. Then since S and T are both linear transformations,

$$S \circ T(\vec{v}_1 + \vec{v}_2) = S(T(\vec{v}_1 + \vec{v}_2)) = S(T(\vec{v}_1) + T(\vec{v}_2)) = S(T(\vec{v}_1)) + S(T(\vec{v}_2)).$$

Similarly, we have

$$S \circ T(a\vec{v}_1) = S(T(a\vec{v}_1)) = S(aT(\vec{v}_1)) = aS(T(\vec{v}_1)).$$

Thus, the composition of two linear transformations is a linear transformation. □

While not all linear transformations are invertible, something interesting happens when they are. Inverses of linear transformations are also linear transformations.

Theorem 3.2.4 *Suppose V and W are vector spaces and $T\colon V \to W$ is a linear transformation. If T is invertible with inverse T^{-1}, then T^{-1} is a linear transformation.*

PROOF. Recall from Theorem 3.1.3 that T is invertible if and only if it is both one-to-one and onto. Also, from the definition of invertible, we know for any $\vec{v} \in V$ that $(T^{-1} \circ T)(\vec{v}) = \vec{v}$. Suppose then that $\vec{u}, \vec{w}$ are in W. Since T is onto, there exist some $\vec{x}, \vec{y} \in V$ such that $\vec{u} = T(\vec{x})$ and $\vec{w} = T(\vec{y})$. Because T^{-1} is the inverse of T, we must also have that $T^{-1}(\vec{u}) = \vec{x}$ and $T^{-1}(\vec{w}) = \vec{y}$. Thus

$$\begin{aligned} T^{-1}(\vec{u} + \vec{w}) &= T^{-1}(T(\vec{x}) + T(\vec{w})) = T^{-1}(T(\vec{x} + \vec{y})) \\ &= (T^{-1} \circ T)(\vec{x} + \vec{y}) = \vec{x} + \vec{y} \\ &= T^{-1}(\vec{u}) + T^{-1}(\vec{w}), \end{aligned}$$

and we see T^{-1} preserves vector addition. Let $a \in \mathbb{R}$. Then we have

$$T^{-1}(a\vec{u}) = T^{-1}(aT(\vec{x})) = T^{-1}(T(a\vec{x})) = (T^{-1} \circ T)(a\vec{x}) = a\vec{x} = aT^{-1}(\vec{u}).$$

Thus, T^{-1} is a linear transformation because T is a linear transformation. □

We'll talk quite a bit more about these invertible linear transformations in the next section.

Linear Transformations and Bases

We spent quite a bit of time in Chapter 2 convincing you how great it is to have a basis, but we had to save one of the best things about them until now. Bases are extremely useful in helping us to understand linear transformations.

Theorem 3.2.5 *Suppose V is a vector space with a spanning set $\mathcal{P} = \{\vec{v}_1, \ldots, \vec{v}_k\}$. Any linear transformation $T\colon V \to W$ is determined by $T(\vec{v}_1), \ldots, T(\vec{v}_k)$. That is, for any $\vec{v} \in V$, we can realize $T(\vec{v})$ as a linear combination of the vectors $T(\vec{v}_1), \ldots, T(\vec{v}_k)$.*

PROOF. This theorem follows directly from the definitions of spanning set and linear transformation. Since $\mathcal{P}$ is a spanning set of V, we know there are coefficients $c_1, \ldots, c_k$ such that $\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ for any $\vec{v} \in V$ and

$$T(\vec{v}) = T(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) = c_1T(\vec{v}_1) + \cdots + c_kT(\vec{v}_k).$$

□

Corollary 3.2.6 *Suppose V is a vector space with a spanning set $\mathcal{P} = \{\vec{v}_1, \ldots, \vec{v}_k\}$. If $T\colon V \to W$ is a linear transformation, then* $\operatorname{Imag} T = \operatorname{Span}\{T(\vec{v}_1), \ldots, T(\vec{v}_k)\}$.

Corollary 3.2.7 *Suppose V is a vector space with a basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$. Any linear transformation $T\colon V \to W$ is determined by $T(\vec{v}_1), \ldots, T(\vec{v}_n)$.*

Example 3.2.6 First, let's consider the vectors

$$\vec{v} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \quad \text{and} \quad \vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Since these are two linearly independent vectors in $\mathbb{R}^2$, we see that $\{\vec{v}, \vec{u}\}$ is a basis for $\mathbb{R}^2$. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T(\vec{v}) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad T(\vec{u}) = \begin{bmatrix} -5 \\ 7 \end{bmatrix}.$$

First, let us find the images under T of $2\vec{v}$ and $4\vec{u}$. It seems pretty straightforward using our properties of a linear transformation.

$$\begin{aligned} T(2\vec{v}) &= 2T(\vec{v}) = 2\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \text{ and} \\ T(4\vec{u}) &= 4T(\vec{u}) = 4\begin{bmatrix} -5 \\ 7 \end{bmatrix} = \begin{bmatrix} -20 \\ 28 \end{bmatrix}. \end{aligned}$$

Similarly, we can also use linearity to find $2\vec{v} + 4\vec{u}$:

$$T(2\vec{v} + 4\vec{u}) = T(2\vec{v}) + T(4\vec{u}) = \begin{bmatrix} 6 \\ 2 \end{bmatrix} + \begin{bmatrix} -20 \\ 28 \end{bmatrix} = \begin{bmatrix} -14 \\ 30 \end{bmatrix}$$

We saw in Theorem 3.2.5 that the linear transformation is determined by what it does on a basis, so T is determined by $T(\vec{u})$ and $T(\vec{v})$.

Now, suppose we want to know what T maps $\vec{x} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$ to? We need to find the coordinate vector for $\vec{x}$! That is, we need to solve for $a, b \in \mathbb{R}$ such that $\vec{x} = a\vec{v} + b\vec{u}$. This is equivalent to

$$\begin{bmatrix} 3 \\ 10 \end{bmatrix} = a\begin{bmatrix} 1 \\ -4 \end{bmatrix} + b\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

which gives us the two equations $3 = a + 2b$ and $10 = -4a + 3b$. Solving these gives us $a = -1$ and $b = 2$. So $\vec{x} = -\vec{v} + 2\vec{u}$. Now we know how to find $T(\vec{x})$!

$$T(\vec{x}) = T(-\vec{v}+2\vec{u}) = -T(\vec{v})+2T(\vec{u}) = -\begin{bmatrix} 3 \\ 1 \end{bmatrix}+2\begin{bmatrix} -5 \\ 7 \end{bmatrix} = \begin{bmatrix} -13 \\ 13 \end{bmatrix}.$$

This is the procedure we could use to find where T sends any vector, but of course, don't expect the coordinate vector weights to always be quite so nice.

Respect the Kernel

We've seen that for a linear transformation $T\colon V \to W$, where V and W are vector spaces, $\mathrm{Imag}\ T$ is a subspace of the codomain, W. That's great for the codomain. Oh, to be guaranteed a subspace! How very nice, indeed. Perhaps we should try to do the same for the domain.

Exploration 75 Recall again the function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined in Example 3.1.7 by

$$\varphi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix},$$

which we saw was a linear transformation in Exploration 73. Give an example of a nonzero vector that maps to $\vec{0}$.
Can you describe all the vectors that map to $\vec{0}$?

Definition 3.2.3 *Let V and W be vector spaces and $T\colon V \to W$ be a linear transformation. The* ***kernel of*** *T is the set of vectors $\vec{v} \in V$ such that $T(\vec{v}) = \vec{0}$. We shall use the notation*

$$\mathrm{Ker}\ T = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}.$$

Theorem 3.2.8 *Let V and W be vector spaces and $T\colon V \to W$ be a linear transformation. Then the kernel of T is a subspace of V.*

Exploration 76 PROOF. Let V and W be vector spaces and $T\colon V \to W$ be a linear transformation. We know that $\mathrm{Ker}\ T$ is a subset of V, so we just need to verify closure for addition and scalar multiplication and also that it contains $\vec{0}$.

- Let $\vec{v}$ and $\vec{u}$ be vectors in $\mathrm{Ker}\ T$, so $T(\vec{v}) = \vec{0}$ and $T(\vec{u}) = \vec{0}$. Show $T(\vec{v}+\vec{u}) = \vec{0}$ so that $\vec{v}+\vec{u}$ is in $\mathrm{Ker}\ T$.

- Let $\vec{v} \in \mathrm{Ker}\ T$ and $a \in \mathbb{R}$. Then $T(\vec{v}) = \vec{0}$. Show $T(a\vec{v}) = \vec{0}$ so that $a\vec{v} \in \mathrm{Ker}\ T$.

- In our proof that $\operatorname{Imag} T$ is a subspace, we established that a linear transformation always maps $\vec{0}$ to $\vec{0}$. Thus, $\vec{0} \in \operatorname{Ker} T$.

□

Hooray! A linear transformations guarantees a subspace in the domain *and* another one in the codomain, each of which will be useful for us.

Example 3.2.7 Let's look at some of the linear transformations we've seen already and see what their kernels are.

- Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(\vec{x}) = \alpha\vec{x}$ for some fixed real number $\alpha > 1$. For this linear transformation, we only have $\alpha\vec{x} = \vec{0}$ if $\vec{x} = \vec{0}$ since $\alpha \neq 0$. The kernel is then just the zero vector space.
- Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the function such that for any

$$\begin{aligned} \vec{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \quad \text{we define} \\ T(\vec{x}) &= x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 4 \end{bmatrix}. \end{aligned}$$

Let's determine the kernel for this one as well. This means we need to solve

$$\vec{0} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 4 \end{bmatrix}. \tag{3.1}$$

Since we know any set of three vectors in $\mathbb{R}^2$ must be linearly dependent, we know there must be nontrivial solutions to this equation. Note that

$$-7 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Thus we can rewrite our Equation 3.1 as

$$\begin{aligned} \vec{0} &= x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_3 \left(-7 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right) \\ &= (x_1 - 7x_3) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (x_2 + 4x_3) \begin{bmatrix} 3 \\ 1 \end{bmatrix} \end{aligned}$$

The vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ are linearly independent, so the only solution to this is given by

$$x_1 - 7x_3 = 0 \quad \text{and} \quad x_2 + 4x_3 = 0 \tag{3.2}$$

We can rearrange these to get $x_1 = 7x_3$ and $x_2 = -4x_3$. Thus,

$$\operatorname{Ker} T = \left\{ \begin{bmatrix} 7x_3 \\ -4x_3 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R} \right\} = \operatorname{Span} \left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

Exploration 77 Define $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ by

$$T(\vec{p}) = \begin{bmatrix} p(1) \\ p(0) \end{bmatrix}.$$

What must $\operatorname{Ker} T$ be?

Now that we've spent some time talking about these special functions called linear transformations, we should spend some time talking about how they relate to the concepts of Section 3.1. We'll do that in the next section because this seems like enough for now.[19]

19: It is. It is enough.

Section Highlights

- A linear transformation is a function between vector spaces that preserves vector space structure. In particular, the range of a linear transformation, called its image, is a vector space. See the discussion before Definition 3.2.1 and Theorem 3.2.1.
- A function between vector spaces is a linear transformation if it preserves the operations of vector addition and scalar multiplication. This means that for $f: V \to W$ to be a linear transformation, it must satisfy $f(\vec{x}+\vec{y}) = f(\vec{x})+f(\vec{y})$ and $f(a\vec{x}) = af(\vec{x})$ where the operations on the left of the equals are in V and the operations on the right are in W for any vectors $\vec{x}$ and $\vec{y}$ in V. See Definition 3.2.1.
- The kernel of a linear transformation, T, is the subspace in the domain consisting of the vectors mapped to the zero vector by T. See Definition 3.2.3 and Theorem 3.2.8. See Example 3.2.7 on how to compute the kernel.

Exercises for Section 3.2

3.2.1. Consider the function $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

Show that T is a linear transformation.

3.2.2. Let $T\colon \mathbb{R}^5 \to \mathbb{R}^5$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} 5x_1 + 4 \\ x_2 + x_3 \\ x_3 - 1 \\ x_1 \\ x_2 \end{bmatrix}.$$

Show that T is *not* a linear transformation.

3.2.3. All of the functions between vector spaces below fail to be linear transformations. Give specific examples illustrating why they fail.

(a) $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 + 5$

(b) $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 x_2$

(c) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ x_1 x_2 \end{bmatrix}$

(d) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_1 + 3 \end{bmatrix}$

(e) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1^3 \end{bmatrix}$

3.2.4. Consider the map $T\colon \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + x_3 + x_4 \\ x_1 + x_2 - x_3 - x_4 \end{bmatrix}.$$

Show this is a linear transformation.

3.2.5. Let $T\colon \mathbb{R}^5 \to \mathbb{R}^5$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} 5x_1 \\ x_2 + x_3 \\ x_3 \\ x_1 \\ x_2 \end{bmatrix}.$$

(a) Show that T is a linear transformation.

(b) Find $\operatorname{Ker} T$.

3.2.6.Consider the function $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]\right) = \left[\begin{array}{c} x_1 \\ x_1 + x_2 \\ 0 \end{array}\right].$$

(a) Show that T is a linear transformation.

(b) Find $\operatorname{Ker} T$.

3.2.7.Consider the function $T\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$T\left(\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]\right) = x_1 + 3x_2.$$

(a) Show that T is a linear transformation.

(b) Find $\operatorname{Ker} T$.

3.2.8.Let $T\colon \mathbb{P}_2 \to \mathbb{R}^4$ be defined by

$$T\left(a_0 + a_1 x + a_2 x^2\right) = \left[\begin{array}{c} a_0 + a_1 \\ a_0 - a_1 \\ a_1 - a_2 \\ a_1 + a_2 \end{array}\right].$$

(a) Show T is a linear transformation.

(b) Find $\operatorname{Ker} T$.

3.2.9.Let $T\colon \mathbb{P}_2 \to \mathbb{R}^4$ be defined by

$$T\left(a_0 + a_1 x + a_2 x^2\right) = \left[\begin{array}{c} a_0 \\ a_1 \\ a_2 \\ (a_2)^2 \end{array}\right].$$

Show that T is *not* a linear transformation.

3.2.10.Let $\vec{v}_1 = \left[\begin{array}{c} 1 \\ -1 \end{array}\right]$ and $\vec{v}_2 = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$. Then $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbb{R}^2$. Suppose $T : \mathbb{R}^2 \to \mathbb{P}_2$ is the linear transformation determined by

$$T(\vec{v}_1) = 1 + x \quad \text{and} \quad T(\vec{v}_2) = x + x^2.$$

(a) Find $T(\vec{v}_1 + 3\vec{v}_2)$.

(b) Suppose $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Find $T(\vec{x})$.

(c) Find $T(\vec{e}_2)$.

(d) Find $T\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)$.

3.2.11. Suppose $T : \mathbb{P}_2 \to \mathbb{R}^2$ is the linear transformation determined by

$$T(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad T(1+x) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad T(1+x^2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) Find $T(1 + x + x^2)$.

(b) Find $T(x)$.

(c) Find $T(x^2)$.

3.2.12. Show that any nonzero constant function between vector spaces is not a linear transformation.

3.2.13. Let V be a vector space with basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$. Complete the proof from Theorem 3.2.2 that the coordinate mapping is a linear transformation by showing $[\alpha\vec{v}]_{\mathcal{B}} = \alpha\, [\vec{v}]_{\mathcal{B}}$ for any real number α and any $\vec{v} \in V$.

3.2.14. Let V be a vector space such that $\dim V = 4$; let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subset V$ be linearly independent and $W = \operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Show that the function $f : V \to V$ that relates $\vec{v} \in V$ to $\operatorname{proj}_W(\vec{v})$ is a linear transformation.

3.3 One-to-one and Onto Linear Transformations

In Section 3.1, we learned about when a function is one-to-one, onto, and invertible. Let's revisit all of these concepts in the more specific context of linear transformations. We'll begin with one-to-one.

Example 3.3.1 Consider the function $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

This is a linear transformation.[20] Moreover, it is one-to-one. To show this from the definition, suppose $T(\vec{v}) = T(\vec{u})$ for some $\vec{v}, \vec{u} \in \mathbb{R}^2$. Then, we know that if

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

then

$$T(\vec{v}) = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} \quad \text{and} \quad T(\vec{u}) = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}.$$

Thus $T(\vec{v}) = T(\vec{u})$ means

$$\begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}.$$

From this we see that $v_1 = u_1$ and $v_2 = u_2$, so $\vec{v} = \vec{u}$; this tells us, by definition, that T is one-to-one.
Now, let's consider $\text{Ker } T$. If $\vec{x} \in \text{Ker } T$, then $T(\vec{x}) = \vec{0}$. Then

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, the only possibility for $\vec{x}$ is the vector $\vec{0} \in \mathbb{R}^2$. In fact, this is always the kernel of a one-to-one function, and this can even be used to tell whether a function is one-to-one.

20:  Does this look familiar? Could this have been an exercise in the last section?

This example illustrates a convenient fact that is true in general.

Theorem 3.3.1 *Let V and W be vector spaces. A linear transformation $T\colon V \to W$ is one-to-one if and only if* $\text{Ker } T = \{\vec{0}\}$.

PROOF. Suppose first that the linear transformation $T\colon V \to W$ is one-to-one. Then there is a unique element of V which maps to $\vec{0}$ in W. Since we know $T(\vec{0}) = \vec{0}$ whenever T is a linear transformation, $\text{Ker } T = \{\vec{0}\}$.

Now suppose $\text{Ker } T = \{\vec{0}\}$. Let $\vec{v}_1$ and $\vec{v}_2$ be vectors in V such that $T(\vec{v}_1) = T(\vec{v}_2)$. We need to show that $\vec{v}_1 = \vec{v}_2$ in order for T to be one-to-one. Since

T is a linear transformation, we have

$$\begin{aligned} T(\vec{v}_1) &= T(\vec{v}_2) \\ T(\vec{v}_1) - T(\vec{v}_2) &= \vec{0} \\ T(\vec{v}_1 - \vec{v}_2) &= \vec{0}. \end{aligned}$$

Thus, $\vec{v}_1 - \vec{v}_2 \in \operatorname{Ker} T = \{\vec{0}\}$, so $\vec{v}_1 - \vec{v}_2 = \vec{0}$. We then see $\vec{v}_1 = \vec{v}_2$, as desired. □

Exploration 78 Consider our favorite linear transformation, $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$\varphi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

Is this function one-to-one?

Now we should consider what it means for a linear transformation to be onto. In particular, Theorem 3.1.1 from Section 3.1 can be restated for linear transformations.

Theorem 3.3.2 *Let V and W be vector spaces. A linear transformation $T\colon V \to W$ is onto if and only if* $\operatorname{Imag} T = W$.

Example 3.3.2 Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation such that for any

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \quad \text{we define} \quad T(\vec{x}) = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

We know this is not one-to-one since $\operatorname{Ker} T \neq \{\vec{0}\}$.[21] Let's see if it is onto, though. Based on the definition of the function, we know x_1, x_2, and x_3 can be any real numbers. Thus,

$$\operatorname{Imag} T = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \end{bmatrix}\right\} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}.$$

Since this is a subspace of dimension 2 in $\mathbb{R}^2$, we see that it is all of $\mathbb{R}^2$. Thus, T is onto.

21:  This was done in the previous section.

Exploration 79 Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 + x_2$$

where $x_1, x_2 \in \mathbb{R}$. We showed in the previous section that this was a linear transformation. Now, find $\operatorname{Ker} f$ and $\operatorname{Imag} f$ to determine whether it is one-to-one, onto, both, or neither.

Isomorphisms

Now that we have discussed one-to-one linear transformations and onto linear transformations, we should talk about when a linear transformation has *both* of these properties. In Section 3.1, we called such functions invertible. We have a special name for an invertible linear transformation.[22]

22:  No, the name is not "Ricky," but maybe it should be...

Definition 3.3.1 *Let V and W be vector spaces. A linear transformation $T\colon V \to W$ is called an* ***isomorphism*** *if it is both one-to-one and onto. When such a linear transformation exists, we say V and W are* ***isomorphic*** *vector spaces, and denote this by $V \cong W$.*

The notation here of $V \cong W$ suggests that there is a sense that V and W are "equal" if they are isomorphic. This is essentially true, at least as vector spaces. A one-to-one and onto map between sets means they are in many ways interchangeable, and the fact here that such a map preserves the vector space structure means that they have identical structure as vector spaces. However, to claim that isomorphic is a form of equality, there are some properties of equality that should be satisfied. [23]

23:  The properties outlined here are those needed to form an *equivalence relation* on a set. What we've actually saying here is that being isomorphic gives an equivalence relation on the set of all vector spaces.

Theorem 3.3.3 (a) *If V is any vector space, then $V \cong V$.*
(b) *If V and W are vector spaces such that $V \cong W$, then $W \cong V$.*
(c) *If V, W, and U are vector spaces such that $V \cong W$ and $W \cong U$, then $V \cong U$.*

These properties might just come in handy later. Rather than prove these all here, we'll do it in the Appendix.

You might recall that the concept of vector spaces being essentially the same has come up for us before. In Section 1.2, we equated the appropriate vector space of arrow vectors with each of the vector spaces $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$. In truth, these are *isomorphic* vector spaces. The elements in these sets and the ways that they are described are very different, except they are really the same *as vector spaces*; that is, they are *isomorphic*. We also talked in Section 1.4 about how any plane through the origin in $\mathbb{R}^3$ "looks like" $\mathbb{R}^2$. In truth, each of these is isomorphic to $\mathbb{R}^2$.

Here's an important one; the coordinate mapping is *always* an isomorphism.

Theorem 3.3.4 *Let V be a vector space of dimension n with basis $\mathcal{B}$. The coordinate mapping $\varphi_{\mathcal{B}}\colon V \to \mathbb{R}^n$ defined by $\varphi_{\mathcal{B}}(\vec{v}) = [\vec{v}]_{\mathcal{B}}$ is an isomorphism.*

PROOF. From Theorem 3.2.2, we know $\varphi_{\mathcal{B}}$ is a linear transformation. We need now to show that it is one-to-one and onto. Since Theorem 2.3.1 tells us that each element in V is represented uniquely on the basis $\mathcal{B}$, we know $\varphi_{\mathcal{B}}$ is one-to-one. By Theorem 3.2.1, $\varphi_{\mathcal{B}}$ is onto if and only if $\operatorname{Imag}\,\varphi_{\mathcal{B}} = \mathbb{R}^n$. By the definition of $\varphi_{\mathcal{B}}$, we already have that $\operatorname{Imag}\,\varphi_{\mathcal{B}} \subseteq \mathbb{R}^n$. Since $\operatorname{Span}\{\mathcal{B}\} = V$, every possible linear combination of the elements of $\mathcal{B}$ must be in V. Thus, every vector in $\mathbb{R}^n$ appears in $\operatorname{Imag}\,\varphi_{\mathcal{B}}$. It follows that $\operatorname{Imag}\,\varphi_{\mathcal{B}} = \mathbb{R}^n$. $\square$

Let's think about what this theorem tells us for a moment. Suppose V is a vector space of dimension n. Then the coordinate mapping gives us an isomorphism between V and $\mathbb{R}^n$. This is actually incredibly important; it tells us that every vector space of dimension n is in essence "the same" as $\mathbb{R}^n$. This shouldn't make you think other vector spaces are not important, but it means a thorough understanding of how things work in $\mathbb{R}^n$ can be useful for predicting how they work in other real vector spaces.

There's something else that should be apparent from this; the dimensions of isomorphic vector spaces match. Let's formalize this.

Theorem 3.3.5 *Let V and W be vector spaces, and suppose $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis of V. If $T\colon V \to W$ is an isomorphism, then $\widehat{\mathcal{B}} = \{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is a basis of W.*

We'll walk through the proof of this fact in the exercises. The above theorem tells us that isomorphic vector spaces have bases of the same size. Thus, we have this useful corollary.

Corollary 3.3.6 *Two real vector spaces are isomorphic if and only if they have the same dimension.*

PROOF. We see from Theorem 3.3.5 that two isomorphic vector spaces will have the same dimensions since they have the same size bases. The other direction, where we begin by assuming two vector spaces have the same dimension, is given to us from Theorem 3.3.4 and Theorem 3.3.3 since they would both be isomorphic to $\mathbb{R}^n$ for the same n. $\square$

We mentioned above that any plane through the origin in $\mathbb{R}^n$ is actually isomorphic to $\mathbb{R}^2$, but what's the isomorphism? Let's see an explicit example of this.

Example 3.3.3 Depending on your background, you may have seen planes in $\mathbb{R}^3$ described differently than the span of two vectors. For instance, the solutions to the equation $3x+2y-z=0$ in $\mathbb{R}^3$ form a plane. Let's translate that plane into more of our language.

$$3x + 2y - z = 0 \quad \text{says} \quad z = 3x + 2y.$$

Thus, the plane is all vectors in the set

$$\begin{aligned}\left\{\begin{bmatrix} x \\ y \\ 3x+2y \end{bmatrix} : x, y \in \mathbb{R}\right\} &= \left\{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} y \colon x, y \in \mathbb{R}\right\} \\ &= \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right\}.\end{aligned}$$

Since this is now a vector space of dimension 2, we know it must be isomorphic to $\mathbb{R}^2$ and a specific isomorphism here can be given by

$$T\colon \text{ Span}\left\{\begin{bmatrix}1\\0\\3\end{bmatrix},\begin{bmatrix}0\\1\\2\end{bmatrix}\right\}\to\mathbb{R}^2 \quad \text{defined by}$$

$$T\left(\begin{bmatrix}1\\0\\3\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}0\\1\\2\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}.$$

What if the plane we want to consider does not go through the origin? Well, it is then not a subspace of $\mathbb{R}^3$, but there is a way to define our operations of vector addition and scalar multiplication so that it is a vector space. For an example of such operations, see the Appendix. This vector space would again be isomorphic to $\mathbb{R}^2$.

Let's take note of something from that previous example. Once we had a basis for our plane in $\mathbb{R}^3$, we defined our isomorphism by mapping each basis vector in the space to one of the standard basis vectors in $\mathbb{R}^2$. This is really the easiest way to define an isomorphism.

Theorem 3.3.7 *Suppose V and W are vector spaces of the same dimension. Let $\{\vec{v}_1,\ldots,\vec{v}_n\}$ be any basis for V. A linear transformation $T\colon V\to W$ is an isomorphism if and only if T maps each basis vector in $\{\vec{v}_1,\ldots,\vec{v}_n\}$ to a distinct vector in a basis $\{\vec{w}_1,\ldots,\vec{w}_n\}$ of W.*

PROOF. This is just the combination of Theorem 3.3.5 and Theorem 3.2.5. □

Exploration 80 Let $V=\text{Span}\left\{1+x,x^2\right\}$ in $\mathbb{P}_2$. Just like we did in Example 3.3.3, define an isomorphism T from V to $\mathbb{R}^2$. Say explicitly what $T(1+x)$ and $T(x^2)$ are.

Exploration 81 Again let $V=\text{Span}\left\{1+x,x^2\right\}$ in $\mathbb{P}_2$. Also, let $W=\text{Span}\{1,x\}$ in $\mathbb{P}_2$. We know that these vector spaces both have dimension 2, so they should be isomorphic. Use the given bases to give an isomorphism T from V to W. Say explicitly what $T(1+x)$ and $T(x^2)$ are.

Exploration 82 Can you find an example of a function between vector spaces that is both one-to-one and onto but is *not* an isomorphism? Hint: There's one somewhere in the previous section.

Rank-Nullity Theorem

There is a theorem known as the *Rank-Nullity Theorem* that we should probably talk about now. We name it this because that's what all the cool textbooks call it. For now, you should just assume the theorem is due to amateur mathematicians named Ronnie Rank and Noether Nullity.[24] This is, of course, a lie,[25] but the mathematical definitions of the words rank and nullity will come later.

24:  They were also full-time mimes; this was the gig that paid the bills.

25:  The bit about who proved the Rank-Nullity Theorem was a lie. We, however, choose to believe that in an infinite universe, there exist both Ronnie Rank and Noether Nullity, full-time mimes. Oh yeah, they're unicorns, too.

Theorem 3.3.8 *Let V and W be inner product spaces and $T\colon V \to W$ be a linear transformation between them. Then*

$$(\text{Ker } T)^{\perp} \cong \text{Imag } T.$$

Corollary 3.3.9 *Let V and W be inner product spaces with $\dim V = \dim W$ and $T\colon V \to W$ be a linear transformation between them. Then*

$$\begin{aligned}(\text{Ker } T)^{\perp} &\cong \text{Imag } T \textit{ and}\\ \text{Ker } T &\cong (\text{Imag } T)^{\perp}.\end{aligned}$$

Again, the various subspaces to which we will refer in the proof can be seen in Figure 3.2.

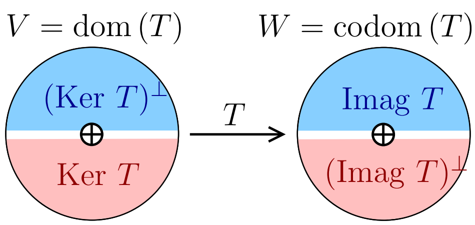


FIGURE 3.2. The linear transformation $T\colon V \to W$ generates four subspaces: $\text{Ker } T$ and $(\text{Ker } T)^{\perp}$ in the domain, V, and $\text{Imag } T$ and $(\text{Imag } T)^{\perp}$ in the codomain, W. The top two subspaces are isomorphic., and when $\dim V = \dim W$, the bottom subspaces are also isomorphic.

PROOF OF THEOREM 3.3.8. This proof has several elements. We'll use bullets to keep track.

- ▶ We know from Theorem 3.2.8 that $\text{Ker } T$ is a subspace of V. Thus, from Corollary 2.5.2, we know $V = \text{Ker } T \oplus (\text{Ker } T)^{\perp}$. So every element of V can be written as $\vec{n} + \vec{v}$ for some $\vec{n} \in \text{Ker } T$ and some $\vec{v} \in (\text{Ker } T)^{\perp}$.
- ▶ To show these vector spaces are isomorphic, we need an isomorphism. We'll use T restricted to $(\text{Ker } T)^{\perp}$, and we'll call this map $\overline{T}$ to keep it straight. To be clear,

  $$\overline{T}\colon (\text{Ker } T)^{\perp} \longrightarrow \text{Imag } T$$

  is defined by

  $$\overline{T}(\vec{x}) = T(\vec{x})$$

for any $\vec{x} \in (\text{Ker } T)^{\perp}$. This is a function since T is a function, and it is a linear transformation since T is a linear transformation. We need to show now that it is an isomorphism. More specifically, we need to show this map is one-to-one and onto.

- To show $\overline{T}$ is onto, let $\vec{y} \in \text{Imag } T$. Then there is some $\vec{u} \in V$ such that $T(\vec{u}) = \vec{y}$. We know from above that

$$\vec{u} = \vec{n}_u + \vec{v}_u$$

for some $\vec{n}_u \in \text{Ker } T$ and some $\vec{v}_u \in (\text{Ker } T)^{\perp}$. Thus,

$$\vec{y} = T(\vec{u}) = T(\vec{n}_u + \vec{v}_u) = T(\vec{n}_u) + T(\vec{v}_u) = \vec{0} + T(\vec{v}_u) = \overline{T}(\vec{v}_u).$$

This tells us that $\overline{T}$ maps onto $\text{Imag } T$.

- We need lastly to show that $\overline{T}$ is one-to-one. We could do this from the definition, but Theorem 3.3.1 says we need only establish that $\text{Ker } \overline{T} = \{\vec{0}\}$. Suppose $\vec{x} \in \text{Ker } \overline{T}$. Then $\overline{T}(\vec{x}) = T(\vec{x}) = \vec{0}$ and $\vec{x} \in \text{Ker } T$. Since $\text{Ker } T \cap (\text{Ker } T)^{\perp} = \{\vec{0}\}$, this says $\vec{x} = \vec{0}$ and therefore $\overline{T}$ is one-to-one.

□

The following is more of a corollary to the above result, but all the other textbooks give it a fancy-sounding name. So we'll make it a theorem. No, it's not just because all the other books are doing it. . . No, we wouldn't jump off a bridge if all the other books did. Look, it's just that we wouldn't want to miss an opportunity to sound really fancy! Whatever. Just call it Theorem 3.3.10 if you want. Can we just drop it now?[26]

26:  Seriously though, you should know the name; we expect you to want to talk about Linear Algebra with many people throughout your lifetime, and sadly, many have not been fortunate enough to learn from this text. Therefore, you need to know what everyone else means when they refer to the Rank-Nullity Theorem.

Theorem 3.3.10 (Rank-Nullity Theorem) *Let V and W be vector spaces and $T\colon V \to W$ be a linear transformation between them. Then* $\dim V = \dim \text{Ker } T + \dim \text{Imag } T$.

PROOF. We know from Corollary 3.3.6 that $(\text{Ker } T)^{\perp}$ and $\text{Imag } T$ have the same dimension. From Corollary 2.5.2 of the Orthogonal Decomposition theorem and Theorem 2.5.3, we have that

$$\dim V = \dim \text{Ker } T + \dim (\text{Ker } T)^{\perp} = \dim \text{Ker } T + \dim \text{Imag } T.$$

□

Let's talk about those words, *rank* and *nullity*.

Definition 3.3.2 *The **rank** of a linear transformation is the dimension of its image. The **nullity** of a linear transformation is the dimension of its kernel.*

Thus, the Rank-Nullity Theorem is aptly named.[27] Let's see how the theorem can be useful.

27:  It was originally named (gesticulate wildly with your front hooves) by Ronnie and Noether.

Example 3.3.4 Let $T\colon \mathbb{R}^5 \to \mathbb{R}^5$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} 5x_1 \\ x_2 + x_3 \\ x_3 \\ x_1 \\ x_2 \end{bmatrix}.$$

Since x_1, x_2, x_3 can be any real numbers, we get that

$$\begin{aligned}
\text{Imag } T &= \left\{\begin{bmatrix} 5x_1 \\ x_2 + x_3 \\ x_3 \\ x_1 \\ x_2 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R}\right\} \\
&= \text{Span}\left\{\begin{bmatrix} 5 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}
\end{aligned}$$

We can see then that $\dim \text{Imag } T = 3$. We can then conclude $\text{Ker } T$ has dimension 2. This helps us to find $\text{Ker } T$ because if we can find two linearly independent vectors in the kernel, we know they form a basis of the kernel.

$$\text{Ker } T = \text{Span}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Exploration 83 Consider the linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + x_3 + x_4 \\ x_1 + x_2 - x_3 - x_4 \end{bmatrix}.$$

Convince yourself that this map is onto by identifying vectors in $\mathbb{R}^4$ that map to the basis

$$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

for $\mathbb{R}^2$.

Now, the dimension of $\text{Ker } T$ must be 2 from the Rank-Nullity Theorem. Find two linearly independent vectors in the kernel.

Another Useful Theorem

At the end of Section 3.1, we proved Theorem 3.1.5 about the properties of one-to-one and onto when a function is defined between two finite sets. Then in Theorem 3.2.5, we saw that linear transformations are completely determined by what they do on a spanning set. This result implies linear transformations between finite dimensional vector spaces are similar in some way to functions between finite sets, and we can now prove a linear transformation version of Theorem 3.1.5.

Theorem 3.3.11 *Suppose $T\colon V \to W$ is a linear transformation between finite dimensional vector spaces V and W.*

(a) *If $\dim V > \dim W$, then T is not one-to-one.*
(b) *If $\dim V < \dim W$, then T is not onto.*
(c) *If $\dim V = \dim W$, then either T is both one-to-one and onto or T is neither one-to-one nor onto.*

PROOF. Suppose first that $\dim V > \dim W$. Since we know $\operatorname{Imag} T$ is a subspace of W, we know $\dim \operatorname{Imag} T \leq \dim W$ from Theorem 2.2.4. Together, this says $\dim \operatorname{Imag} T < \dim V$. Then rearranging the Rank-Nullity Theorem tells us that $\dim \operatorname{Ker} T = \dim V - \dim \operatorname{Imag} T$ and $\dim \operatorname{Ker} T$ must be nonzero since the inequality $\dim \operatorname{Imag} T < \dim V$ is strict. Theorem 3.3.1 says T then is not one-to-one.

Suppose next that $\dim V < \dim W$. Again, we turn to the Rank-Nullity Theorem to see that $\dim \operatorname{Imag} T = \dim V - \dim \operatorname{Ker} T$. From this we see that $\dim \operatorname{Imag} T \leq \dim V$ and we can combine the inequalities to see that $\dim \operatorname{Imag} T < \dim W$. We know from Theorem 3.3.2 that T is not onto then since $\operatorname{Imag} T = W$ would mean they have the same dimension.

Suppose lastly that $\dim V = \dim W$. If T is onto, then we know $\dim W = \dim \operatorname{Imag} T$. Then the Rank-Nullity Theorem says

$$\dim W = \dim V = \dim \operatorname{Imag} T + \dim \operatorname{Ker} T = \dim W + \dim \operatorname{Ker} T.$$

This can only be true if $\dim \operatorname{Ker} T = 0$. In that case, T is one-to-one by Theorem 3.3.1. Now, if we begin by assuming that T is one-to-one, we know $\dim \operatorname{Ker} T = 0$ and the Rank-Nullity Theorem says $\dim W = \dim V = \dim \operatorname{Imag} T$. This means $W = \operatorname{Imag} T$ and T is onto. We have argued that T is onto if and only if it is one-to-one. The result follows. □

Section Highlights

- A linear transformation is one-to-one if and only if its kernel is exactly the zero vector. See Theorem 3.3.1.
- For a linear transformation, T, the Rank-Nullity Theorem tells us that $\dim \operatorname{dom}(T) = \dim \operatorname{Ker} T + \dim \operatorname{Imag} T$. See Theorem 3.3.10.
- For any linear transformation T, $\dim \operatorname{Ker} T$ can be used to determine whether the function is one-to-one, onto, both, or neither. In particular:

- T is one-to-one if and only if $\dim \operatorname{Ker} T = 0$;
- T is onto if and only if $\dim \operatorname{Imag} T = \dim \operatorname{codom}(T)$, and by the Rank-Nullity Theorem, $\dim \operatorname{Imag} T = \dim \operatorname{dom}(T) - \dim \operatorname{Ker} T$.

See Theorem 3.3.10, Theorem 3.3.1, and Theorem 3.3.2.

- If there is a one-to-one and onto linear transformation between two vector spaces, we say the vector spaces are isomorphic, a form of equivalence for vector spaces. See Definition 3.3.1.
- Any n-dimensional vector space is isomorphic to $\mathbb{R}^n$ via the coordinate mapping. See Theorem 3.3.4.

Exercises for Section 3.3

3.3.1. Determine whether the linear transformation is one-to-one, onto, both or neither. If it is not onto, find $\mathrm{Imag}\ T$. If it is not one-to-one, find $\mathrm{Ker}\ T$.

(a) $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ defined by

$$T(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_0 + a_1 \\ a_2 \end{bmatrix}$$

(b) $T\colon \mathbb{P}_2 \to \mathbb{R}^3$ defined by

$$T(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

(c) $T\colon \mathbb{P}_2 \to \mathbb{R}^3$ defined by

$$T(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_0 + a_1 \\ 2a_1 \\ a_1 + a_2 \end{bmatrix}$$

(d) $T\colon \mathbb{P}_2 \to \mathbb{R}^4$ defined by

$$T(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_0 \\ a_1 + a_2 \\ a_1 + a_2 \\ a_2 \end{bmatrix}$$

(e) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_1 \\ x_2 \end{bmatrix}$$

(f) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 \\ x_2 \end{bmatrix}$$

(g) $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

3.3.2. Determine whether the linear transformation is one-to-one, onto, both or neither.

(a) $T\colon \mathbb{P}_2 \to \mathbb{R}^4$ defined by

$$T(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_0 - a_1 \\ a_1 - a_2 \\ a_2 \\ a_1 \end{bmatrix}$$

(b) $T\colon \mathbb{P}_2 \to \mathbb{R}^4$ defined by

$$T(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_1 \\ a_1 - a_2 \\ a_2 \\ a_1 \end{bmatrix}$$

3.3.3.Let $T\colon \mathbb{P}_2 \to \mathbb{R}^3$ defined by

$$T(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_0 - a_1 \\ a_1 - a_2 \\ a_2 - a_0 \end{bmatrix}.$$

(a) Which of the following vectors are in $\operatorname{Ker} T$?

$$1 - x, \qquad 1 + x + x^2, \qquad x - x^2, \qquad 1 - x - x^2$$

(b) Which of the following vectors are in $\operatorname{Imag} T$?

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3.3.4.Find the dimension of $\operatorname{Ker} T$ using the Rank-Nullity Theorem.

- $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

- $T\colon \mathbb{R}^5 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} x_3 \\ x_1 \end{bmatrix}$$

- $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

3.3.5.Suppose $T\colon V \to W$ is an isomorphism of vector spaces and $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for V.

(a) Use the fact that T is onto to show that $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is a spanning set of W.

(b) Use the fact that T is one-to-one to argue that $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is a linearly independent set.

This completes the proof of Theorem 3.3.5.

3.3.6.Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 47x_2 \\ x_2 - 48x_3 \\ x_3 \end{bmatrix}.$$

Calculate $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$, and explain how this proves that T is an isomorphism.

3.3.7.Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ and $S\colon \mathbb{R}^n \to \mathbb{R}^n$.

(a) Is it possible that $\dim \operatorname{Ker}\ T < \dim \operatorname{Ker}(S \circ T)$? Explain.

(b) Is it possible that $\dim \operatorname{Ker}\ S < \dim \operatorname{Ker}(S \circ T)$? Explain.

(c) Is it possible that $\dim \operatorname{Ker}(S \circ T) < \dim \operatorname{Ker}\ T$? Explain.

(d) Is it possible that $\dim \operatorname{Ker}(S \circ T) < \dim \operatorname{Ker}\ S$? Explain.

3.3.8.Let $V = \operatorname{Span}\left\{x, 1 + x^2\right\}$ in $\mathbb{P}_2$. Find two distinct isomorphisms T and S from V to $\mathbb{R}^2$.

3.4 Matrices

At some point, you might have been given the impression[28] that Linear Algebra is all about matrices. Don't feel bad; matrix theory is often confused with linear algebra. We know now that Linear Algebra is the study of linear transformations on vector spaces.[29] In this section, we will begin the discussion of what a matrix is and also how it can be connected to a linear transformation.

28: Certainly not from us.

29: One of the three unicorns does, anyway.

What is. . . a Matrix?

Definition 3.4.1 *An $m \times n$ **matrix** A is a rectangular array of numbers with m rows and n columns:*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

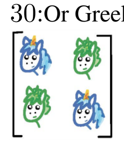
*The number a_{ij} in the ith row and jth column is called the ij**th entry**. Matrices are sometimes also written as*

$$A = [a_{ij}]_{\substack{1\le i\le m,\\ 1\le j\le n}}.$$

*An $n \times n$ matrix is often called a **square matrix**. For convenience, the set of all $m \times n$ matrices with real number entries will be denoted by $\mathcal{M}_{m\times n}$.*

The numbers in a matrix may be integers, real numbers, complex numbers, etc. The entries don't even have to be numbers! You could make a matrix of polynomials or even emojis[30] if you really wanted. In this course, we will use real numbers unless otherwise specified.[31]

30:Or Greek symbols. Or even. . .



31: I caught that! They're planning to specify non-real matrices at some point!

Probably in Chapter 5. I've heard everything gets a bit more complex there.

Do you think they'll change the meaning of $\mathcal{M}_{m\times n}$ then?

Quite possibly.

Another common notation (that will be particularly useful for us) is to think of an $m \times n$ matrix as n vectors from $\mathbb{R}^m$ all lined up next to each other, so

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ &= \left[\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \cdots \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \right] = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n], \end{aligned}$$

where

$$\vec{a}_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}, \quad \text{for} \quad 1 \le j \le n.$$

We call $\vec{a}_j$ a *column vector* for the matrix A.

Example 3.4.1 Here's a 2×3 matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = [\vec{a}_1 \; \vec{a}_2 \; \vec{a}_3],$$

where

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad \text{and} \quad \vec{a}_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Fun facts: $a_{21} = 4$ and $a_{12} = 2$.

Exploration 84 In the matrix A from Example 3.4.1, what are a_{13} and a_{22}?

Building a Linear Transformation from a Matrix

Before we can use a matrix to make a linear transformation, we will need to define a type of multiplication between a matrix and a vector.

Definition 3.4.2 *Let $A \in \mathcal{M}_{m\times n}$ with columns $\vec{a}_1, \ldots, \vec{a}_n$, and let $\vec{x} \in \mathbb{R}^n$. The product of a matrix and a vector, that is, the* ***product of*** A ***and*** $\vec{x}$*, is the linear combination of the columns of A with the entries of $\vec{x}$ as weights. That is,*

$$A\vec{x} = [\vec{a}_1 \; \vec{a}_2 \cdots \vec{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n.$$

Example 3.4.2 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \quad \text{and} \quad \vec{u} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}.$$

We can calculate $A\vec{v}$ and $B\vec{u}$; see Exploration 85.
Note, however, that $A\vec{u}$ and $B\vec{v}$ are not defined. For $A\vec{u}$, the matrix A has three columns, while $\vec{u}$ has only two components; thus, there is no way to form a linear combination. Similarly for $B\vec{v}$, the matrix B has only two columns, and $\vec{v}$ has three components.

Exploration 85 Let

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 2 & -2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & -1 \end{bmatrix},$$

$$\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Calculate $A\vec{v}$ and $B\vec{u}$.

$$\begin{aligned} A\vec{v} &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9\begin{bmatrix} 3 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ 28 \end{bmatrix} + \begin{bmatrix} 16 \\ 40 \end{bmatrix} + \begin{bmatrix} 27 \\ 54 \end{bmatrix} = \begin{bmatrix} 50 \\ 122 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} B\vec{u} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 8\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ 21 \\ 35 \end{bmatrix} + \begin{bmatrix} 16 \\ 32 \\ 48 \end{bmatrix} = \begin{bmatrix} 23 \\ 53 \\ 83 \end{bmatrix} \end{aligned}$$

With this product definition in hand, we are able to make any matrix into a linear transformation.[32]

32:  Read that last sentence again. It's a big deal.

Theorem 3.4.1 *Let $A \in \mathcal{M}_{m\times n}$, and define $T\colon \mathbb{R}^n \to \mathbb{R}^m$ by*

$$T(\vec{x}) = A\vec{x}.$$

Then T is a linear transformation.

When a linear transformation is defined as multiplication by a specific matrix, we often use the following terminology and notation.

Definition 3.4.3 *Let $A \in \mathcal{M}_{m\times n}$, and define $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ by*

$$T_A(\vec{x}) = A\vec{x}.$$

*We will call T_A the **linear transformation induced by** A.*

PROOF. We need to show that for any vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ and any scalar $\alpha \in \mathbb{R}$, we have

$$T_A(\vec{x}+\vec{y}) = T_A(\vec{x}) + T_A(\vec{y}) \quad \text{and} \quad T_A(\alpha\vec{x}) = \alpha T_A(\vec{x}).$$

We may write $A = [\vec{a}_1 \cdots \vec{a}_n]$. First note that

$$\begin{aligned} T_A(\vec{x}+\vec{y}) = A(\vec{x}+\vec{y}) &= [\vec{a}_1 \cdots \vec{a}_n] \begin{bmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{bmatrix} \\ &= (x_1+y_1)\vec{a}_1 + \cdots + (x_n+y_n)\vec{a}_n \\ &= (x_1\vec{a}_1 + \cdots + x_n\vec{a}_n) + (y_1\vec{a}_1 + \cdots + y_n\vec{a}_n) \\ &= [\vec{a}_1 \cdots \vec{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + [\vec{a}_1 \cdots \vec{a}_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \\ &= A\vec{x} + A\vec{y} = T_A(\vec{x}) + T_A(\vec{y}). \end{aligned}$$

Now, we need to show that for any scalar $\alpha \in \mathbb{R}$, we have $T_A(\alpha\vec{x}) = \alpha T_A(\vec{x})$. To see this,

$$\begin{aligned}
T_A(\alpha\vec{x}) = A(\alpha\vec{x}) &= [\vec{a}_1 \cdots \vec{a}_n]\left(\alpha \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) \\
&= [\vec{a}_1 \cdots \vec{a}_n]\begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix} \\
&= (\alpha x_1)\vec{a}_1 + \cdots + (\alpha x_n)\vec{a}_n \\
&= \alpha(x_1\vec{a}_1 + \cdots + x_n\vec{a}_n) \\
&= \alpha A\vec{x} = \alpha T_A(\vec{x}).
\end{aligned}$$

□

Example 3.4.3 Let

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

and define $T_A\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $T_A(\vec{x}) = A\vec{x}$. Find the images $T_A(\vec{u})$ and $T_A(\vec{v})$. Here our linear transformation is defined as multiplication by a matrix A. We can compute

$$T_A(\vec{u}) = A\vec{u} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = (1)\begin{bmatrix} 3 \\ 0 \end{bmatrix} + (-2)\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

and both $\vec{u}$ and its image under T_A can be seen in Figure 3.3.
For the arbitrary vector $\vec{v}$, we have

$$T_A(\vec{v}) = A\vec{v} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1\begin{bmatrix} 3 \\ 0 \end{bmatrix} + v_2\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3v_1 \\ -v_2 \end{bmatrix},$$

and now we have a nice formula for the image under T_A of any vector in $\mathbb{R}^2$.

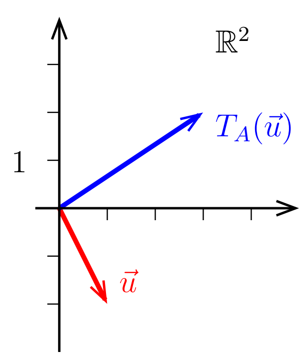


FIGURE 3.3. The vector $\vec{u} \in \mathbb{R}^2$ and its image $T_A(\vec{u}) \in \mathbb{R}^2$.

Exploration 86 Let's use our matrix B from earlier to define a linear transformation. Specifically, let

$$B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix},$$

and define $T_B \colon \mathbb{R}^2 \to \mathbb{R}^3$ by $T_B(\vec{x}) = B\vec{x}$.

- ► What is $T_B\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$?

- ► What is $T_B\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$?

Example 3.4.4 Suppose we wanted to define a map $G \colon \mathbb{R}^9 \to \mathbb{R}^{14}$ by $G(\vec{x}) = A\vec{x}$? How many rows and columns must A have for this linear transformation to be well-defined? Since

$$A\vec{x} = [\vec{a}_1 \cdots \vec{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n,$$

It follows that $\vec{x}$ needs as many components as A has columns. Since $\vec{x} \in \mathbb{R}^9$, we see that A must have nine columns. Moreover, $A\vec{x}$ is a linear combination of the column vectors $\vec{a}_j$. Since $A\vec{x} \in \mathbb{R}^{14}$, we must also have $\vec{a}_j \in \mathbb{R}^{14}$ for each $1 \leq j \leq 9$. It follows that A must have fourteen rows. Thus, $A \in \mathcal{M}_{14\times 9}$.

Revisiting Image and Kernel

Now that we've seen that a matrix can define a linear transformation, we can talk about the kernel and image of such a function. Let's start with an example.

Example 3.4.5 Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -2 & 4 \end{bmatrix},$$

and define $T_A \colon \mathbb{R}^4 \to \mathbb{R}^2$ by $T_A(\vec{x}) = A\vec{x}$. We shall find Ker T_A; that is, we would like to describe the set

$$\text{Ker } T_A = \left\{\vec{x} \in \mathbb{R}^4 \colon T_A(\vec{x}) = \vec{0}\right\} = \left\{\vec{x} \in \mathbb{R}^4 \colon A\vec{x} = \vec{0}\right\}.$$

Solving $A\vec{x} = \vec{0}$ means solving

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{where} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

After doing the matrix-vector multiplication, this becomes

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or

$$\begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ x_2 - 2x_3 + 4x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Now we just have a system of two equation in four variables:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ x_2 - 2x_3 + 4x_4 &= 0. \end{aligned}$$

Solving for x_2 in the second component, and substituting it into the first equation, we have

$$\begin{aligned} x_1 &= -7x_3 + 8x_4 \\ x_2 &= 2x_3 - 4x_4. \end{aligned}$$

These are the conditions on the vector $\vec{x}$ that must be satisfied in order for $A\vec{x} = \vec{0}$. Thus

$$\text{Ker } T_A = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \colon x_1 = -7x_3 + 8x_4 \text{ and } x_2 = 2x_3 - 4x_4 \right\}.$$

However, this is not very satisfying. We can, in fact, clean this up quite a bit. Note that

$$\begin{aligned} &\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \colon x_1 = -7x_3 + 8x_4 \text{ and } x_2 = 2x_3 - 4x_4 \right\} \\ &= \left\{ \begin{bmatrix} -7x_3 + 8x_4 \\ 2x_3 - 4x_4 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \colon x_3, x_4 \in \mathbb{R} \right\} \\ &= \left\{ x_3 \begin{bmatrix} -7 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -4 \\ 0 \\ 1 \end{bmatrix} \colon x_3, x_4 \in \mathbb{R} \right\}. \end{aligned}$$

It follows that

$$\text{Ker } T_A = \text{Span} \left\{ \begin{bmatrix} -7 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

It turns out we really only needed the matrix A to tell us what T_A does to a vector in $\mathbb{R}^4$. Since T_A is completely determined by the matrix A, the following definition makes sense.

Definition 3.4.4 *Suppose $A \in \mathcal{M}_{m \times n}$ and $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation induced by A. We then define **the kernel of the matrix** A,* *denoted* $\text{Ker } A$, *to be* $\text{Ker } T_A$. *That is,*

$$\text{Ker } A = \{\vec{x} \in \mathbb{R}^n \colon A\vec{x} = \vec{0}\} = \text{Ker } T_A.$$

Sometimes $\text{Ker } A$ *is called the* ***nullspace of the matrix*** A.

Example 3.4.6 Define $A = \begin{bmatrix} -1 & 0 & 4 \\ 0 & -1 & 2 \end{bmatrix}$. Then we can find $\text{Ker } A$ by solving for all vectors

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

such that $A\vec{x} = \vec{0}$. That is,

$$\begin{bmatrix} -1 & 0 & 4 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This gives us the equations

$$-x_1 + 4x_3 = 0$$
$$-x_2 + 2x_3 = 0$$

which simplify to $x_1 = 4x_3$ and $x_2 = 2x_3$. So

$$\text{Ker } A = \left\{ \begin{bmatrix} 4x_3 \\ 2x_3 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Exploration 87 Define $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Find $\text{Ker } A$.

Now that we've seen that we can extend our definition of the kernel of a linear transformation to a matrix, let's look at an example of the image of a linear transformation defined by a matrix. Let's start with a familiar example.

Example 3.4.7 Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -2 & 4 \end{bmatrix},$$

and define $T_A \colon \mathbb{R}^4 \to \mathbb{R}^2$ by $T_A(\vec{x}) = A\vec{x}$. This time, we will find $\text{Imag } T_A$. Let

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

We can use this general vector to see what a general element of $\operatorname{Imag} T_A$ must look like. That is,

$$T_A(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

From this equation, we conclude

$$\operatorname{Imag} T_A = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right\}.$$

Since this is definitely a spanning set for $\mathbb{R}^2$, we can say $\operatorname{Imag} T_A = \mathbb{R}^2$. Now, how does this relate to the matrix A? Well, it's the span of the column vectors of A! This is true in general.

Definition 3.4.5 *Let* $A = [\vec{a}_1 \cdots \vec{a}_n] \in \mathcal{M}_{m\times n}$. *The* ***column space of*** A, *denoted* $\operatorname{Col} A$, *is the span of the column vectors* $\vec{a}_j$ *for* $1 \le j \le n$. *That is,*

$$\operatorname{Col} A = \operatorname{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}.$$

Exploration 88 Find $\operatorname{Col} A$ where

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Theorem 3.4.2 *Let* $A \in \mathcal{M}_{m\times n}$ *and suppose* $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ *is the linear transformation induced by* A. *Then* $\operatorname{Imag} T_A = \operatorname{Col} A$.

PROOF. Let $\vec{x} \in \mathbb{R}^n$ and suppose $A = [\vec{a}_1 \cdots \vec{a}_n]$ where $a_j \in \mathbb{R}^m$ for all $1 \le j \le n$. We know

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

for some $x_i \in \mathbb{R}$ for each $1 \le i \le n$. Thus,

$$T_A(\vec{x}) = A\vec{x} = [\vec{a}_1 \cdots \vec{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n.$$

From this equation, we see that a vector $\vec{y} \in \mathbb{R}^m$ is in $\operatorname{Imag} T_A$ if and only if $\vec{y} \in \operatorname{Col} A$, and the sets are equal. $\square$

One-to-one and Onto for T_A

Recall that we say a linear transformation is onto if its codomain and image are equal. Thus, we have the following corollary.

Corollary 3.4.3 *Let $A \in \mathcal{M}_{m\times n}$ and suppose $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation induced by A. Then T_A is onto if and only if the columns of A span $\mathbb{R}^m$.*

Now that we've brought up the concept of T_A being onto, what about determining when it's one-to-one?

Theorem 3.4.4 *Let $A \in \mathcal{M}_{m\times n}$ and suppose $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation induced by A. Then T_A is one-to-one if and only if the columns of A are linearly independent.*

PROOF. The columns of A are linearly dependent if and only if there are scalars (not all 0) $c_1, \ldots, c_n$ such that $c_1\vec{a}_1 + \cdots c_n\vec{a}_n = \vec{0}$. This is true if and only if

$$A\vec{c} = [\vec{a}_1 \cdots \vec{a}_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}.$$

Then we have a nonzero vector $\vec{c} \in \mathrm{Ker}\ A$. By Theorem 3.3.1, this is true if and only if T_A is *not* one-to-one. □

Example 3.4.8 Define the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$, and

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Since the columns of A fail to span $\mathbb{R}^3$, we have from Corollary 3.6.3 that $\mathrm{Imag}\ T$ has dimension two. Since $\mathbb{R}^3$ has dimension three, it is not possible that $\mathrm{Imag}\ T = \mathbb{R}^3$, so by Theorem 3.1.1, T is not onto.
However, since we can see the two column vectors are linearly independent, we know that T is one-to-one.

Exploration 89 Define the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Is this linear transformation onto? Is it one-to-one?

Section Highlights

- A matrix is a rectangular array of numbers. See Definition 3.4.1.

- A matrix, A, with m rows and n columns, induces a linear transformation, $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$. See Theorem 3.4.1 and Definition 3.4.3.
- The span of the column vectors in a matrix A is called the column space of A and denoted $\mathrm{Col}\ A$, and $\mathrm{Imag}\ T_A = \mathrm{Col}\ A$. See Definition 3.6.1 and Theorem 3.4.2.
- The kernel of matrix A, denoted $\mathrm{Ker}\ A$, is defined as $\mathrm{Ker}\ T_A$. See Definition 3.4.4.
- The linear transformation T_A is one-to-one if and only if the columns of the matrix A are linearly independent. See Theorem 3.6.2.
- The linear transformation T_A is onto if and only if the columns of A span the codomain. See Corollary 3.6.3.

Exercises for Section 3.4

3.4.1. Let $\vec{x} \in \mathbb{R}^3$, $\vec{y} \in \mathbb{R}^4$, $\vec{z} \in \mathbb{R}^5$, and $A \in \mathcal{M}_{m\times n}$. What must m and n be...

(a) ...so that $A\vec{x} = \vec{y}$?

(b) ...so that $A\vec{x} = \vec{z}$?

(c) ...so that $A\vec{z} = \vec{y}$?

3.4.2. Consider the vectors below:

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \quad \vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

For each matrix A below, exactly one of $A\vec{x}$, $A\vec{y}$, $A\vec{u}$, and $A\vec{v}$ can be computed. Compute the one that is defined.

(a) $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & -2 \\ 1 & 5 \\ -2 & 3 \\ 1 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & 5 & 3 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 2 & 3 \\ -2 & 0 & 3 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ -2 & 5 & 0 & 2 \end{bmatrix}$

(f) $A = \begin{bmatrix} 1 & 3 & 2 & 3 & 1 \\ -2 & 1 & 3 & 2 & 0 \\ -1 & 1 & 2 & 0 & 0 \end{bmatrix}$

3.4.3. Let $T_A : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation induced by A where

$$A = \begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -1 & 4 & -6 \end{bmatrix}.$$

Find $\operatorname{Ker} A$.

3.4.4. Let

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 3 & -5 \end{bmatrix},$$

$$\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad \text{and} \quad \vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

(a) Compute $A\vec{u}$ and $B\vec{v}$.

(b) Let $T_A\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation induced by A. Note that means $T_A(\vec{x}) = A\vec{x}$ for any $\vec{x} \in \mathbb{R}^3$.

(i) Find $\text{Imag } T_A = \text{Col } A$.

(ii) Find $\text{Ker } T_A = \text{Ker } A$.

(iii) Is T_A one-to-one?

(iv) Is T_A onto?

(c) Define $T_B\colon \mathbb{R}^2 \to \mathbb{R}^3$ to be the linear transformation induced by B.

(i) Find $\text{Imag } T_B = \text{Col } B$.

(ii) Find $\text{Ker } T_B = \text{Ker } B$.

(iii) Is T_B one-to-one?

(iv) Is T_B onto?

3.4.5.Let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix},$$

$$\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad \text{and} \quad \vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

(a) Compute $A\vec{u}$ and $B\vec{v}$.

(b) Let $T_A\colon \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation induced by A. Note that means $T_A(\vec{x}) = A\vec{x}$ for any $\vec{x} \in \mathbb{R}^3$.

(i) Find $\text{Imag } T_A = \text{Col } A$.

(ii) Find $\text{Ker } T_A = \text{Ker } A$.

(iii) Is T_A one-to-one?

(iv) Is T_A onto?

(c) Define $T_B\colon \mathbb{R}^2 \to \mathbb{R}^2$ to be the linear transformation induced by B.

(i) Find $\text{Imag } T_B = \text{Col } B$.

(ii) Find $\text{Ker } T_B = \text{Ker } B$.

(iii) Is T_B one-to-one?

(iv) Is T_B onto?

3.4.6. For each matrix A below, do the following in order:

(a) Find the the domain and codomain of T_A.

(b) Find a basis for $\mathrm{Ker}\ A$ and compute $\dim \mathrm{Ker}\ A$.

(c) Use the Rank-Nullity Theorem to compute $\dim \mathrm{Col}\ A$.

(d) Find a basis for $\mathrm{Col}\ A$.

(e) Determine whether T_A is one-to-one, onto, both, or neither.

(f) Based on your previous answers, are the columns of A linearly independent?

(a) $A = \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 3 \end{bmatrix}$

(f) $A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 0 & -4 \\ 0 & 0 & 3 \end{bmatrix}$

(g) $A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 5 & 0 & 2 \end{bmatrix}$

(h) $A = \begin{bmatrix} 1 & 3 & 4 & 1 \\ 0 & 5 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix}$

(i) $A = \begin{bmatrix} 1 & 3 & 2 & 3 & 1 \\ -2 & 0 & 3 & 2 & 0 \\ -1 & 1 & 2 & 0 & 0 \end{bmatrix}$

3.4.7. Suppose we have the matrix

$$A = \begin{bmatrix} a & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

For what values of a is the induced linear transformation T_A an onto linear transformation? Why?

3.4.8. Determine by inspection of the columns whether these matrices correspond to transformations that are one-to-one. Explain your reasoning.

(a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

3.4.9. Determine by inspection whether these matrices correspond to transformations that are onto. Explain your reasoning.

(a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

3.4.10. Suppose $B \in \mathcal{M}_{n\times n}$ with induced linear transformation $T_B\colon \mathbb{R}^n \to \mathbb{R}^n$. Let $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^n$ be such that $\{\vec{x}_1, \vec{x}_2\}$ is a linearly independent set. If $B\vec{x}_1 = B\vec{x}_2 = \vec{0}$, what is the largest and smallest $\dim \operatorname{Ker} T_B$ and $\dim \operatorname{Imag} T_B$ can be? Explain your reasoning.

3.4.11. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(\vec{x}) = A\vec{x}$ where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

For any vector $\vec{x} \in \mathbb{R}^3$, describe its image $T(\vec{x})$ geometrically.

3.5 The Matrix of a Linear Transformation

In the last section, we learned about matrices and how a matrix can be used to define a linear transformation. But can we go the other way? If we start with a linear transformation, is there a matrix that can be associated to it? Yes![33]

Matrix Representation with the Standard Basis of $\mathbb{R}^n$

Let's start with an example.

Example 3.5.1 Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that for the standard basis vectors $\vec{e}_1, \vec{e}_2 \in \mathbb{R}^2$, we have

$$T(\vec{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\vec{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}.$$

Note that for any vector $\vec{x} \in \mathbb{R}^2$, we have

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2.$$

Then using the definition of linear transformation,

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + x_2\vec{e}_2) \\ &= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) \;=\; x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}. \end{aligned}$$

Then using the definition of the product of a matrix and a vector,

$$T(\vec{x}) \;=\; x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Thus, $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix}.$$

This strategy works in general!

Theorem 3.5.1 *Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Using the standard basis $\{\vec{e}_1, \dots, \vec{e}_n\}$ for $\mathbb{R}^n$, define*

$$A = [T(\vec{e}_1) \cdots T(\vec{e}_n)].$$

Then A is the unique matrix in $\mathcal{M}_{m\times n}$ such that $T = T_A$. That is, $T(\vec{x}) = A\vec{x}$ for any $\vec{x} \in \mathbb{R}^n$.

Because of its connection to the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, this matrix is sometimes referred to as the *standard matrix for the linear transformation T*. If you look carefully at the steps in Example 3.6.2, you will see how to prove this result. Let's walk through that together.

33: Yes?

YES! There are actually infinitely many.

That's too many. You can keep almost all of them.

Exploration 90 PROOF. Let $\vec{x} \in \mathbb{R}^n$. We would normally write $\vec{x}$ as a column vector, but let's write it as

$$\vec{x} = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n$$

instead. Note that this is the unique way to represent $\vec{x}$ in terms of the standard basis by Theorem 2.3.1. Using $\vec{x}$ in this fashion and the linearity of T, find a way to write $T(\vec{x})$ as $A\vec{x}$ for some matrix A.

Our theorem states that this matrix A is unique. How does this follows from the uniqueness result of Theorem 2.3.1?

□

Let's see an example of this theorem in action.

Example 3.5.2 Let $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ 0 \\ x_3 \\ 0 \end{bmatrix}.$$

Using Theorem 3.5.1, we can build A by finding the image by T of each standard basis vector for $\mathbb{R}^3$; these will be the columns of A. Specifically, we have $T(\vec{x}) = A\vec{x}$, where

$$A = [T(\vec{e}_1)\ T(\vec{e}_2)\ T(\vec{e}_3)] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that we weren't told at the start that this was a linear transformation. However, if we verify that $T(\vec{x}) = A\vec{x}$ for any $\vec{x} \in \mathbb{R}^3$, we'll know T is a linear transformation from Theorem 3.4.1.

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} x_1 - x_2 \\ 0 \\ x_3 \\ 0 \end{bmatrix} = T(\vec{x}) \end{aligned}$$

This tells us T is a linear transformation!

Example 3.5.3 First, let us take the frequent geometric convention to label the three coordinate directions in $\mathbb{R}^3$ as x, y, and z, respectively. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined as the linear transformation rotating all vectors

$\pi/4$ radians around the z-axis and increasing by a factor of 2 in the z direction. Let's also assume this rotation is counterclockwise in the plane[34]

$$\left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}.$$

Let's use the standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ for the domain and codomain. Note that

$$T(\vec{e}_1) = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix}, T(\vec{e}_2) = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix}, T(\vec{e}_3) = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

By Theorem 3.5.1, we have

$$A = [T(\vec{e}_1)\; T(\vec{e}_2)\; T(\vec{e}_3)] = \begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Exploration 91 Find a linear transformation $T \colon \mathbb{R}^4 \to \mathbb{R}^4$ that interchanges the x_1 axis with the x_3 axis, maps the x_2 axis to $\vec{0}$, and does nothing to the x_4 axis. Then find a matrix representation for T using the standard basis for $\mathbb{R}^4$ in the domain and the codomain.

General Version of a Matrix Representation

Using coordinate vectors, Theorem 3.5.1 generalizes nicely for arbitrary finite dimensional vector spaces.

Theorem 3.5.2 *Let V and W be vector spaces and $T \colon V \to W$ be a linear transformation. For each pair of fixed bases, $\mathcal{B}_V$ for V and $\mathcal{B}_W$ for W, there exists a unique matrix A such that for all vectors $\vec{v} \in V$,*

$$[T(\vec{v})]_{\mathcal{B}_W} = A\, [\vec{v}]_{\mathcal{B}_V}.$$

Moreover, if $\mathcal{B}_V = \{\vec{v}_1, \ldots, \vec{v}_n\}$, then

$$A = \left[[T(\vec{v}_1)]_{\mathcal{B}_W} \cdots [T(\vec{v}_n)]_{\mathcal{B}_W} \right].$$

For any linear transformation $T \colon V \to W$, we call the matrix A obtained from Theorem 3.5.2 the **matrix representation of T relative to the bases $\mathcal{B}_V$ and $\mathcal{B}_W$**. Note that a different choice of basis for either V or W would result in a different matrix representation. Before we delve deep into the proof of the theorem, let's see it in action with an example.

Example 3.5.4 Let $T \colon \mathbb{P}_3 \to \mathbb{P}_3$ be defined by

$$T(a + bx + cx^2 + dx^3) = (a - b) + (c - d)x^2.$$

34: This is the usual xy plane in $\mathbb{R}^3$.

The advantage to Theorem 3.5.2 over Theorem 3.5.1 is that with the more general theorem, we can build a matrix representation for a linear transformation between any vector spaces, even if their vectors aren't column vectors in $\mathbb{R}^n$ for some positive integer n. This is done, of course, by way of coordinate vectors, so the matrix representation, A, will depend on our choice of bases. Let's use $\mathcal{B} = \{1, x, x^2, x^3\}$ as a basis for $\mathbb{P}_3$ in both the domain and codomain. Recall that a general vector $\vec{p} = a + bx + cx^2 + dx^3$ has coordinate vector

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix},$$

so, for example,

$$\left[T(x - x^3)\right]_{\mathcal{B}} = \left[-1 + x^2\right]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

To use Theorem 3.5.2, we need to find the coordinate of the image by T of each basis vector in $\mathcal{B}$. That is,

$$\begin{aligned} A &= \left[[T(1)]_{\mathcal{B}}\ [T(x)]_{\mathcal{B}}\ \left[T(x^2)\right]_{\mathcal{B}}\ \left[T(x^3)\right]_{\mathcal{B}}\right] \\ &= \left[[1]_{\mathcal{B}}\ [-1]_{\mathcal{B}}\ \left[x^2\right]_{\mathcal{B}}\ \left[-x^2\right]_{\mathcal{B}}\right] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Great! We just built a matrix A that is a matrix representation for T. Let's think a bit about what all this was on the level of functions. We started with $T\colon \mathbb{P}_3 \to \mathbb{P}_3$. We know that $\mathbb{P}_3 \cong \mathbb{R}^4$ because the coordinate mapping $\varphi_{\mathcal{B}}\colon \mathbb{P}_3 \to \mathbb{R}^4$ is an isomorphism. You probably noticed that $A \in \mathcal{M}_{4\times 4}$, so we can define a linear transformation $T_A\colon \mathbb{R}^4 \to \mathbb{R}^4$ by $T_A(\vec{x}) = A\vec{x}$. This gives us the following commuting diagram of linear transformations:

$$\begin{tikzcd} \mathbb{P}_3 \arrow[r, "T"] \arrow[d, "\varphi_{\mathcal{B}}"'] & \mathbb{P}_3 \arrow[d, "\varphi_{\mathcal{B}}"] \\ \mathbb{R}^4 \arrow[r, "T_A"] & \mathbb{R}^4 \end{tikzcd}$$

The way you should think about this diagram is that the coordinate isomorphisms represented by the vertical arrows allow us to translate between T and T_A. Then, we can examine all the stuff with T and $\mathbb{P}_3$ *or* we can use the matrix representation T_A and $\mathbb{R}^4$. We'll talk a bit more about diagrams like this below.

We should be ready now to talk about the proof of Theorem 3.5.2, so let's get started.

PROOF. Let's suppose V and W are vector spaces with $\dim V = n$ and $\dim W = m$, and fix some bases $\mathcal{B}_V = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $\mathcal{B}_W = \{\vec{w}_1, \dots, \vec{w}_m\}$ for V and W, respectively. We know from Theorem 3.3.4 that the coordinate mapping gives an isomorphism between V and $\mathbb{R}^n$ and between W and $\mathbb{R}^m$.

This statement actually involves four different linear transformations, and they are all relevant here. Specifically, we have

- $\varphi_{\mathcal{B}_V} : V \to \mathbb{R}^n$ defined by $\varphi_{\mathcal{B}_V}(\vec{v}) = [\vec{v}]_{\mathcal{B}_V}$ for any $\vec{v} \in V$
- $\varphi_{\mathcal{B}_V}^{-1} : \mathbb{R}^n \to V$ defined by $\varphi_{\mathcal{B}_V}^{-1}\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$
- $\varphi_{\mathcal{B}_W} : W \to \mathbb{R}^m$ defined by $\varphi_{\mathcal{B}_W}(\vec{w}) = [\vec{w}]_{\mathcal{B}_W}$ for any $\vec{w} \in W$
- $\varphi_{\mathcal{B}_W}^{-1} : \mathbb{R}^m \to W$ defined by $\varphi_{\mathcal{B}_W}^{-1}\left(\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}\right) = x_1\vec{w}_1 + \cdots + x_m\vec{w}_m$.

Suppose now that we have a linear transformation $T\colon V \to W$. We know from Theorem 3.2.3 that we can compose linear transformations to form new linear transformations. Consider then the composition

$$\varphi_{\mathcal{B}_W} \circ T \circ \varphi_{\mathcal{B}_V}^{-1} : \mathbb{R}^n \to \mathbb{R}^m.$$

Let's call this new linear transformation $\widehat{T}$. Here's a useful commuting diagram for this situation.

(3.3)
$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\varphi_{\mathcal{B}_V}^{-1}"] \arrow[rrr, bend right, "\widehat{T}"'] & V \arrow[r, "T"] & W \arrow[r, "\varphi_{\mathcal{B}_W}"] & \mathbb{R}^m. \end{tikzcd}$$

This $\widehat{T}$ is a linear transformation which has input of coordinate vectors for $\mathcal{B}_V$ and whose output is in coordinate vectors for $\mathcal{B}_W$. Also, it is determined by our linear transformation T! Now, let's compute the matrix A from Theorem 3.5.1 for $\widehat{T}$.

$$\begin{aligned} A &= \left[\widehat{T}(\vec{e}_1) \ \cdots \ \widehat{T}(\vec{e}_n)\right] \\ &= \left[\varphi_{\mathcal{B}_W} \circ T \circ \varphi_{\mathcal{B}_V}^{-1}(\vec{e}_1) \ \cdots \ \varphi_{\mathcal{B}_W} \circ T \circ \varphi_{\mathcal{B}_V}^{-1}(\vec{e}_n)\right] \\ &= \left[\varphi_{\mathcal{B}_W} \circ T(\vec{v}_1) \ \cdots \ \varphi_{\mathcal{B}_W} \circ T(\vec{v}_n)\right] \\ &= \left[[T(\vec{v}_1)]_{\mathcal{B}_W} \ \cdots \ [T(\vec{v}_n)]_{\mathcal{B}_W}\right] \end{aligned}$$

So by Theorem 3.5.1, we know $\widehat{T} = T_A$ for this matrix A. Moreover, by examining our functions we see that this says exactly that

$$[T(\vec{v})]_{\mathcal{B}_W} = A\,[\vec{v}]_{\mathcal{B}_V}.$$

□

The commuting diagram in Figure 3.4 is a convenient way to organize all of the sets and functions that came up in Theorem 3.5.1 and its proof. As long as we follow the maps in the direction they point, composing functions as we go, all that matters are the starting and ending points. As a bonus, remember that the coordinate mappings are isomorphisms, so, while they are not pictured, there are inverses for each of the coordinate mappings with arrows that point in the opposite direction. Since these vertical arrows represent isomorphisms, we can think of T and T_A as "equivalent." They aren't completely the same since they map between different spaces, but they will share all the linear transformation properties. [35]

35:  This is the same thing that isomorphisms do for vector spaces!

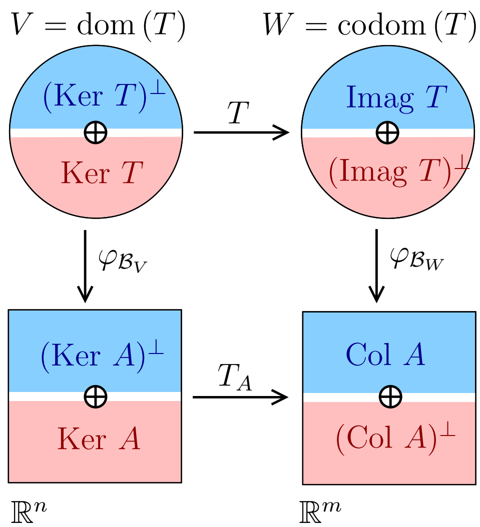


FIGURE 3.4. The linear transformation $T\colon V \to W$ generates four subspaces: $\mathrm{Ker}\ T$ and $(\mathrm{Ker}\ T)^{\perp}$ in the domain V and $\mathrm{Imag}\ T$ and $(\mathrm{Imag}\ T)^{\perp}$ in the codomain W. The matrix transformation T_A given by the matrix representation A of T also generates subspaces.

Figure 3.4 also strongly suggests that there is a natural decomposition of the domain and codomain of T_A that is analogous to the decompositions for T. This should not be terribly surprising, but we'll put off a formal statement and proof of that fact until the next chapter.

Kernel and Image of T from A

We have said that the advantage of this matrix representation is that we can work with the matrix rather than the original linear transformation. In Section 3.4, we saw this with $\mathrm{Ker}\ A$ and $\mathrm{Ker}\ T_A$. Now we can define the relationship more generally between $\mathrm{Ker}\ T$ and $\mathrm{Ker}\ A$, where A is any matrix representation for T. Like we said before, it's not quite true that these are equal. What *is* true is that these are isomorphic via the coordinate mapping, which is admittedly pretty close to being equal. Let's state this formally and prove it.

Theorem 3.5.3 *Let V and W be vector spaces with bases $\mathcal{B}_V$ and $\mathcal{B}_W$, respectively. Let $T\colon V \to W$ be a linear transformation whose matrix representation relative to $\mathcal{B}_V$ and $\mathcal{B}_W$ is $A \in \mathcal{M}_{m\times n}$. Then*

$$\mathrm{Ker}\ T = \{\vec{x} \in V \colon A\,[\vec{x}]_{\mathcal{B}_V} = \vec{0}\}.$$

That is, $\vec{x} \in \text{Ker } T$ if and only if $\varphi_{\mathcal{B}_V}(\vec{x}) \in \text{Ker } A$, and $\text{Ker } T \cong \text{Ker } A$ via the coordinate mapping.

PROOF. Suppose $\vec{x} \in \text{Ker } T$, so $T(\vec{x}) = \vec{0}$. Since $T = \varphi_{\mathcal{B}_W}^{-1} \circ T_A \circ \varphi_{\mathcal{B}_V}$, we know $T(\vec{x}) = \vec{0}$ if and only if

$$\begin{aligned} \left(\varphi_{\mathcal{B}_W}^{-1} \circ T_A \circ \varphi_{\mathcal{B}_V}\right)(\vec{x}) &= \vec{0} \text{ if and only if} \\ \left(T_A \circ \varphi_{\mathcal{B}_V}\right)(\vec{x}) &= \varphi_{\mathcal{B}_W}(\vec{0}) \text{ if and only if} \\ T_A\,[\vec{x}]_{\mathcal{B}_V} &= \vec{0}. \end{aligned}$$

Thus, $\vec{x} \in \text{Ker } T$ if and only if $[\vec{x}]_{\mathcal{B}_V} \in \text{Ker } T_A = \text{Ker } A$. $\square$

We can do something similar with $\text{Imag } T$ and $\text{Col } A$. These are also isomorphic by the coordinate mapping.

Theorem 3.5.4 *Let V and W be vector spaces with bases $\mathcal{B}_V$ and $\mathcal{B}_W$, respectively. Let $T\colon V \to W$ be a linear transformation whose matrix representation with respect to $\mathcal{B}_V$ and $\mathcal{B}_W$ is $A \in \mathcal{M}_{m\times n}$. Then*

$$\text{Imag } T = \{\vec{w} \in W\colon\ [\vec{w}]_{\mathcal{B}_W} \in \text{Col } A\}.$$

That is, $\vec{w} \in \text{Imag } T$ if and only if $\varphi_{\mathcal{B}_W}(\vec{w}) \in \text{Col } A$, and $\text{Imag } T \cong \text{Col } A$ via the coordinate mapping.

The proof of Theorem 3.6.1 is similar to the proof of Theorem 3.6.10.[36]

36: Exercise!

Now that we know $\text{Ker } T \cong \text{Ker } A$ and $\text{Imag } T \cong \text{Col } A$ whenever A is a matrix representation for the matrix T, we actually can tell quite a lot about T from any matrix representation, A. For instance, if we know either $\dim \text{Ker } T$ or $\dim \text{Imag } T$, we can say whether T is one-to-one or onto. Now, we can conclude these from properties of the matrix A!

Theorem 3.5.5 *Let V and W be vector spaces. Let $T\colon V \to W$ be a linear transformation with matrix representation $A \in \mathcal{M}_{m\times n}$. Then we know the following:*

(a) *T is one-to-one if and only if the columns of A are linearly independent.*
(b) *T is onto if and only if the columns of A span $\mathbb{R}^m$.*
(c) *T is an isomorphism if and only if the columns of A form a basis for $\mathbb{R}^m$.*

This doesn't really need a proof. For the first two statements, we've just restated Corollary 3.6.3 and Theorem 3.6.2 in light of Theorems 3.6.10 and 3.6.1. Then the last statement is a combination of the first two with the definitions of a basis and an isomorphism.

More Examples!

The big takeaway here is that anything you want to know about *any* linear transformation on *any* finite dimensional vector spaces can be found using matrices and column vectors. That's actually quite amazing. We should look into that more.

Example 3.5.5 Define $T\colon \mathbb{P}_1 \to \mathbb{P}_2$ by taking the indefinite integral of vectors in $\mathbb{P}_1$ and using zero for the constant of integration. That is, for $\vec{p} = a + bx$, we have

$$T(\vec{p}) = T(a + bx) = ax + \frac{1}{2}bx^2 \in \mathbb{P}_2.$$

Since a and b are arbitrary real numbers, we see that $\mathrm{Imag}\ T$ is the set of vectors in $\mathbb{P}_2$ with zero as a constant term.
Using the standard bases $\mathcal{B}_1 = \{1, x\}$ and $\mathcal{B}_2 = \{1, x, x^2\}$ for $\mathbb{P}_1$ and $\mathbb{P}_2$ respectively, we have from Theorem 3.5.2 that $[T(\vec{p})]_{\mathcal{B}_2} = A\,[\vec{p}]_{\mathcal{B}_1}$, where

$$A = \left[[T(1)]_{\mathcal{B}_2}\ [T(x)]_{\mathcal{B}_2}\right] = \left[[x]_{\mathcal{B}_2}\ \left[\frac{1}{2}x^2\right]_{\mathcal{B}_2}\right] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Then $\mathrm{Col}\ A$ is (by definition) the span of the vectors

$$\vec{a}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}.$$

Since $\mathrm{Imag}\ T$ is the set of all vectors in $\mathbb{P}_2$ with zero as a constant term, we have

$$\begin{aligned} \{[\vec{p}]_{\mathcal{B}_2} \in \mathbb{R}^3 \colon \vec{p} \in \mathrm{Imag}\ T\} &= \left\{ a\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\} \\ &= \mathrm{Span}\,\{\vec{e}_2, \vec{e}_3\} = \mathrm{Span}\,\{\vec{a}_1, \vec{a}_2\}, \end{aligned}$$

where $\vec{e}_i$ are the standard basis vectors in $\mathbb{R}^3$. This is precisely what we were told by Theorem 3.6.1:

$$\mathrm{Col}\ A = \{[\vec{p}]_{\mathcal{B}_2} \in \mathbb{R}^3 \colon \vec{p} \in \mathrm{Imag}\ T\}.$$

Exploration 92 Let $T\colon \mathbb{P}_2 \to \mathbb{P}_1$ be defined by $T(a+bx+cx^2) = (b-a)+ax$ for any $a+bx+cx^2 \in \mathbb{P}_2$. Find the matrix A that represents this transformation relative to the bases $\{x^2, x, 1\}$ and $\{x, 1\}$.

What is $\mathrm{Ker}\ A$? Use the coordinate mapping to translate your answer back to vectors in $\mathbb{P}_2$ to find $\mathrm{Ker}\ T$.

Example 3.5.6 Let $H_D = \text{Span}\left\{\vec{b}_1 = 1 + x, \vec{b}_2 = x + x^2, \vec{b}_3 = x^3\right\}$, so $\mathcal{B}_D = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is a basis for H_D. Similarly, let

$$H_C = \text{Span}\left\{\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\},$$

so $\mathcal{B}_C = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for H_C. Let $T\colon H_D \to H_C$ be the linear transformation such that $T(\vec{b}_i) = \vec{v}_i$ for each $i = 1, 2, 3$.
We know $H_D \cong \mathbb{R}^3$ and $H_C \cong \mathbb{R}^3$, so we'd like very much to build a linear transformation $T_A\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $T_A(\vec{x}) = A\left[\vec{b}\right]_{\mathcal{B}_D} = \left[T(\vec{b})\right]_{\mathcal{B}_C}$, where $A \in \mathcal{M}_{3\times3}$. This is why we have Theorem 3.5.2:

$$\begin{aligned} A &= \left[\left[T(\vec{b}_1)\right]_{\mathcal{B}_C} \ \left[T(\vec{b}_2)\right]_{\mathcal{B}_C} \ \left[T(\vec{b}_3)\right]_{\mathcal{B}_C}\right] \\ &= \left[[v_1]_{\mathcal{B}_C} \ [v_2]_{\mathcal{B}_C} \ [v_3]_{\mathcal{B}_C}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Wait, what? This actually checks out, but why is our matrix so simple this time? We started with isomorphic vector spaces; note that $H_C \cong H_D$. Then our linear transformation was built on the basis of our choosing; this flexibility to choose whatever basis you like can make your life much simpler.

In that previous example, we saw how nice our matrix can be given a careful choice of the bases for the vector spaces. What if we start with a linear transformation defined by a matrix, but we'd prefer a simpler-looking matrix. Well, we could *change the basis* we are using...

Example 3.5.7 Let $A = \begin{bmatrix} -1 & -2 & -1 \\ -1 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix}$ and define $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$ for any $\vec{x} \in \mathbb{R}^3$. Well, the form of A is not that bad, but let's see what it looks like if we were to replace the standard basis with the basis

$$\mathcal{B} = \left\{\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}\right\}.$$

Well, how do we do this? Actually, Theorem 3.5.2 tells us exactly what to do. We are still using coordinate vectors, even though we didn't actually change vector spaces. So, we first need to find $T(\vec{v}_1)$, $T(\vec{v}_2)$, and $T(\vec{v}_3)$.

$$T(\vec{v}_1) = \begin{bmatrix} -1 & -2 & -1 \\ -1 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$T(\vec{v}_2) = \begin{bmatrix} -1 & -2 & -1 \\ -1 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

$$T(\vec{v}_3) = \begin{bmatrix} -1 & -2 & -1 \\ -1 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, we need to convert all of these to coordinate vectors relative to $\mathcal{B}$. Normally, to do this, we need to find coefficients $a, b, c \in \mathbb{R}$ such that $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{x}$ where $\vec{x}$ is any vector in $\mathbb{R}^3$. However, we can do these quickly by inspection. Note that $T(\vec{v}_1) = -\vec{v}_1$, $T(\vec{v}_2) = \vec{v}_2$, and $T(\vec{v}_3) = \vec{0}$. Thus, we have the matrix B for the linear transformation T relative to the basis $\mathcal{B}$.

$$B = [[T(\vec{v}_1)]_{\mathcal{B}}\ [T(\vec{v}_2)]_{\mathcal{B}}\ [T(\vec{v}_3)]_{\mathcal{B}}\ ] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This new matrix B is also a matrix representation for T, and from this, we can see quickly that T is neither one-to-one nor onto. We can also identify that $\vec{v}_3 \in \operatorname{Ker} T$ and that $\dim \operatorname{Imag} T = 2$. But where did this new basis come from? You'll have to wait a bit to find out, but we'll get there.

Exploration 93 Let $A = \begin{bmatrix} 4 & 2 & 1 \\ 4 & 6 & 3 \\ -8 & -8 & -4 \end{bmatrix}$ and define $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$ for any $\vec{x} \in \mathbb{R}^3$. Just like in the example above, find a new matrix B that represents T with respect to the basis

$$\mathcal{B} = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

Section Highlights

- Every linear transformation on a finite dimensional vector space can be represented by a matrix. The matrix representation depends on the bases chosen for the domain and codomain. See Theorem 3.5.2
- If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then the matrix

  $$A = [T(\vec{e}_1) \cdots T(\vec{e}_n)]$$

  is called the standard matrix for T. This matrix is the matrix for which $T = T_A$. See Theorem 3.5.1.
- If $T\colon V \to V$ is a linear transformation and $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for the vector space V, then the matrix for T with respect to $\mathcal{B}$ is given by

  $$[[T(\vec{v}_1)]_{\mathcal{B}} \cdots [T(\vec{v}_n)]_{\mathcal{B}}].$$

  See Theorem 3.5.2.

Exercises for Section 3.5

3.5.1. Using the standard bases for $\mathbb{R}^n$ and $\mathbb{P}_n$, find a matrix A such that $T(\vec{x}) = A\vec{x}$ for each of the following linear transformations:

(a) $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_3 \\ x_2 + 4x_3 \end{bmatrix}$$

(b) $T\colon \mathbb{P}_2 \to \mathbb{R}^4$ defined by

$$T(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_0 + 3a_2 \\ a_1 - a_2 \\ a_2 \\ a_1 \end{bmatrix}$$

(c) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(d) $T\colon \mathbb{P}_2 \to \mathbb{R}^4$ defined by

$$T(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_1 \\ a_1 - a_2 \\ a_2 \\ a_1 \end{bmatrix}$$

(e) $T\colon \mathbb{R}^2 \to \mathbb{R}^5$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_2 \\ 0 \\ x_1 \\ 2x_1 + x_2 \\ 4x_1 - 5x_2 \end{bmatrix}$$

(f) $T\colon \mathbb{P}_3 \to \mathbb{R}^4$ defined by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = \begin{bmatrix} a_1 + a_3 \\ a_1 - a_2 + 4a_3 \\ 2a_2 + a_0 \\ a_1 + a_2 + a_3 \end{bmatrix}$$

(g) $T\colon \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_2 + x_4 \\ 4x_1 - 5x_2 + x_3 \end{bmatrix}$$

(h) $T\colon \mathbb{P}_3 \to \mathbb{R}^2$ defined by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = \begin{bmatrix} a_0 - a_1 + a_3 \\ 3a_1 - 2a_2 + 4a_3 \end{bmatrix}$$

3.5.2. Let $T\colon \mathbb{P}_3 \to \mathbb{P}_1$ be defined by $T(ax^3 + bx^2 + cx + d) = (b - a)x + (c - d)$.

(a) Using the standard bases for $\mathbb{P}_3$ and $\mathbb{P}_1$, find the matrix representation A of T.

(b) Find $\text{Ker } A$ and $\text{Col } A$.

(c) Use your answers to part $b)$ to find $\text{Ker } T$ and $\text{Imag } T$.

3.5.3. Let $T\colon \mathbb{P}_3 \to \mathbb{P}_2$ be defined by $T(ax^3 + bx^2 + cx + d) = (a - c)x^2 + (b - d)x + (c - a)$.

(a) Using the standard bases for $\mathbb{P}_3$ and $\mathbb{P}_1$, find the matrix representation A of T.

(b) Find $\text{Ker } A$ and $\text{Col } A$.

(c) Use your answers to part $b)$ to find $\text{Ker } T$ and $\text{Imag } T$.

3.5.4. Consider the subspace

$$H = \left\{ \begin{bmatrix} -2y-3z \\ y \\ z \end{bmatrix} : y, z \in \mathbb{R} \right\}$$

of $\mathbb{R}^3$. (This is the plane determined by the equation $x + 2y + 3z = 0$.)

(a) Find a basis for H.

(b) Define a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ that has $\operatorname{Imag} T = H$.

(c) Find the matrix representation A for T with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

3.5.5. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 2x_1 \end{bmatrix}.$$

(a) Using the standard basis for $\mathbb{R}^2$, find a matrix A such that $T(\vec{x}) = A\vec{x}$.

(b) Using the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$, find a matrix B such that $[T(\vec{x})]_{\mathcal{B}} = B\,[\vec{x}]_{\mathcal{B}}$. That is, find the matrix representation of T relative to the basis $\mathcal{B}$.

3.5.6. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear transformation defined by

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad T\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Find the matrix for T with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

3.5.7. Suppose $T : \mathbb{P}_2 \to \mathbb{P}_2$ is the linear transformation defined by

$$T(1+x) = 1 - x \quad T(x) = 1 - x + x^2 \quad T(1+x^2) = 1.$$

Find the matrix for T with respect to the standard basis of $\mathbb{P}_2$.

3.5.8. Ricky has affixed a picture of Bubbles onto the square in $\mathbb{R}^2$ whose vertices are $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. He decides this is much too small, and he prefers that Bubbles faces the other direction.

(a) Find the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that makes the picture of Bubbles ten times bigger and reflects the image across the vertical axis. In particular, determine $T(\vec{e}_1)$ and $T(\vec{e}_2)$.

(b) Find the matrix A that represents T with respect to the standard basis of $\mathbb{R}^2$.

3.5.9. Bubbles has now also affixed a picture of Ricky onto the square in $\mathbb{R}^2$ whose vertices are $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. He would like to find a way to animate his picture by rotating around the origin.

(a) Find the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates the picture 45° counterclockwise, keeping the bottom tip at $(0,0)$. In particular, determine $T(\vec{e}_1)$ and $T(\vec{e}_2)$.

(b) Find the matrix A that represents T with respect to the standard basis of $\mathbb{R}^2$.

(c) Let $\vec{x} = A\vec{e}_1$. Find $A\vec{x}$. (Repeatedly applying A would give the animation Bubbles was looking for.)

3.5.10. Fix an angle $\theta \in [0, 2\pi)$, and let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates vectors about the origin by θ radians (counterclockwise). Find the matrix A that represents T with respect to the standard basis of $\mathbb{R}^2$.

3.5.11. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 4x_2 \\ 2x_2 + x_3 \\ 0 \end{bmatrix}.$$

For each basis $\mathcal{B}_i$ below, find the matrix representation for T with respect to $\mathcal{B}_i$.

(a) $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

(b) $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

(c) $\mathcal{B}_3 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

Note that this basis is orthogonal. Use this to simplify the computations.

3.5.12. Suppose $T : \mathbb{P}_2 \to \mathbb{P}_2$ is the linear transformation defined by

$$T\left(a_0 + a_1 x + a_2 x^2\right) = (a_0 + a_1) + 3a_2 x^2.$$

For each basis $\mathcal{B}_i$ below, find the matrix representation for T with respect to $\mathcal{B}_i$.

(a) $\mathcal{B}_1 = \left\{1 + x, 1 - x, x^2\right\}$

(b) $\mathcal{B}_2 = \left\{1, 1 - x, x + x^2\right\}$

(c) $\mathcal{B}_3 = \left\{1 + x, 1 - x + x^2, x^2\right\}$

3.5.13. Prove Theorem 3.6.1.

3.6 More Fun with Linear Transformations

Now that we have a fun and convenient way to represent our linear transformations with matrices, we should go back and think about all the things we've learned about linear transformations. Some things become a bit easier in the context of matrices, so we're devoting this section to understanding what we can learn about a linear transformation from its matrix representation.

Kernel and Image of a Matrix

Let V and W be vector spaces (with $\dim V = n$ and $\dim W = m$), and let $T\colon V \to W$ be a linear transformation. Fix bases $\mathcal{B}_V$ and $\mathcal{B}_W$ for V and W respectively, and using Theorem 3.5.2, let $A \in \mathcal{M}_{m\times n}$ be the matrix that represents T with respect to these bases. To be clear,

$$A = [[T(\vec{v}_1)]_{\mathcal{B}_W} \dots [T(\vec{v}_n)]_{\mathcal{B}_W}] \quad \text{and} \quad T(\vec{v}) = A[\vec{v}]_{\mathcal{B}_V}$$

where $\mathcal{B}_V = \{\vec{v}_1, \dots, \vec{v}_n\}$.

Recall from Example 3.5.4 that the matrix A determines a linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T_A(\vec{v}) = A\vec{v}$ for any $\vec{v} \in \mathbb{R}^n$. This transformation differs from T since we are not assuming V is $\mathbb{R}^n$, only that it is *isomorphic* to $\mathbb{R}^n$ under the coordinate mapping. Using the isomorphisms we get from the coordinate mappings $\varphi_{\mathcal{B}_V}$ and $\varphi_{\mathcal{B}_W}$, we have the following commuting diagram:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\varphi_{\mathcal{B}_V}"'] & W \arrow[d, "\varphi_{\mathcal{B}_W}"] \\ \mathbb{R}^n \arrow[r, "T_A"] & \mathbb{R}^m \end{tikzcd}$$

With all of these maps in hand, consider

$$\text{Ker } A = \{\vec{v} \in \mathbb{R}^n \colon A\vec{v} = \vec{0}\}.$$

This is something we can compute from the matrix A. By the definition of T_A, this subspace of $\mathbb{R}^n$ is Ker T_A. Its image under the inverse of the coordinate mapping $\varphi_{\mathcal{B}_V}$ is then Ker T. Thus, the representation A can identify a set in $\mathbb{R}^n$ that is isomorphic to Ker T in V, but it also actually identifies the *exact* kernel of T in V when the subspace is tracked back through the coordinate mapping. This is best illustrated by an example.

Example 3.6.1 Let $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ be defined by

$$T(\vec{p}) = \begin{bmatrix} p(0) \\ 0 \end{bmatrix}.$$

We showed a similar function was linear in Example 3.2.5, and T here is also a linear transformation. It is also not hard to check that Ker T is the set of polynomials in $\mathbb{P}_2$ with 0 for the constant term. We want to show here that using the basis $\mathcal{B} = \{x^2, x, 1\}$ for V, we have

$$\text{Ker } T = \{\vec{p} \in \mathbb{P}_2 \colon T(\vec{p}) = \vec{0}\} = \{\vec{p} \in \mathbb{P}_2 \colon A\,[\vec{p}]_{\mathcal{B}} = \vec{0}\}.$$

According to Theorem 3.5.2,

$$A = [T([\vec{e}_1]_{\mathcal{B}})\, T([\vec{e}_2]_{\mathcal{B}})\, T([\vec{e}_3]_{\mathcal{B}})] = \left[T(x^2)\, T(x)\, T(1)\right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

For any polynomial $\vec{p} = ax^2 + bx + c \in \mathbb{P}_2$, we have

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

so we must solve the equation

$$A\,[\vec{p}]_{\mathcal{B}} = \vec{0}, \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.4}$$

From computing the matrix multiplication, we have

$$\begin{bmatrix} c \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, $c = 0$ while a and b are free to be any real number; that is, the solution to Equation 3.4 is

$$\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : c = 0 \right\}.$$

This set of coordinate vectors corresponds to the set of vectors in Ker $T \subset \mathbb{P}_2$.

While we can use different bases to get different matrix representations for T, they all identify the same set of vectors in V that T maps to $\vec{0}$. Because of this, we choose not to feel guilty about the following slightly abusive notation:

$$\text{Ker } T = \text{Ker } A.$$

Indeed, for future reference, Ker A will refer to the kernel of the linear transformation T which A represents.

Exploration 94 Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(\vec{x}) = A\vec{x}$ for any $\vec{x} \in \mathbb{R}^3$ where

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Find Ker T by finding all vectors $\vec{x}$ such that

$$A\vec{x} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Exploration 95 Let $T\colon \mathbb{P}_2 \to \mathbb{P}_1$ be defined by $T(a+bx+cx^2) = (b-a)+ax$ for any $a+bx+cx^2 \in \mathbb{P}_2$. Find the matrix A that represents this transformation relative to the bases $\{x^2, x, 1\}$ and $\{x, 1\}$.

You should have gotten an A that looks very familiar. What is $\text{Ker } A$? How does this translate into finding $\text{Ker } T$?

Definition 3.6.1 *Let $A = [\vec{a}_1 \cdots \vec{a}_n] \in \mathcal{M}_{m\times n}$. The* ***column space of*** *A, denoted $\text{Col } A$, is the span of the column vectors $\vec{a}_j$ for $1 \leq j \leq n$. That is,*

$$\text{Col } A = \text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}.$$

Exploration 96 Find $\text{Col } A$ where

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Theorem 3.6.1 *Let V and W be vector spaces with bases $\mathcal{B}_V = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\mathcal{B}_W = \{\vec{w}_1, \ldots, \vec{w}_m\}$, respectively. Let $T\colon V \to W$ be a linear transformation with matrix representation $A \in \mathcal{M}_{m\times n}$. Then*

$$\text{Col } A = \{[\vec{w}]_{\mathcal{B}_W} \in \mathbb{R}^m \colon \vec{w} \in \text{Imag } T\}.$$

Simply put, this theorem says that $\text{Col } A$ **is** $\text{Imag } T$ **as coordinate vectors relative to** $\mathcal{B}_W$. This will provide us with an easy way to describe $\text{Imag } T$. The proof of this theorem is similar to that of Theorem 3.6.10 at the end of this section, so we do not include it here.

Again, because of this, we choose not to feel guilty about this other slightly abusive notation:

$$\text{Imag } T = \text{Col } A.$$

For future reference, $\text{Col } A$ can be thought of as the image of the linear transformation T which A represents.

Example 3.6.2 Define $T\colon \mathbb{P}_1 \to \mathbb{P}_2$ by taking the indefinite integral of vectors in $\mathbb{P}_1$ and using zero for the constant of integration. That is, for $\vec{p} = a + bx$, we have

$$T(\vec{p}) = T(a + bx) = ax + \frac{1}{2}bx^2 \in \mathbb{P}_2.$$

Since a and b are arbitrary real numbers, we see that $\text{Imag } T$ is the set of vectors in $\mathbb{P}_2$ with zero as a constant term.

Using the standard bases $\mathcal{B}_1 = \{1, x\}$ and $\mathcal{B}_2 = \{1, x, x^2\}$ for $\mathbb{P}_1$ and $\mathbb{P}_2$ respectively, we have from Theorem 3.5.2 that $[T(\vec{p})]_{\mathcal{B}_2} = A\,[\vec{p}]_{\mathcal{B}_1}$, where

$$A = \left[[T(1)]_{\mathcal{B}_2}\ [T(x)]_{\mathcal{B}_2}\right] = \left[[x]_{\mathcal{B}_2}\ \left[\frac{1}{2}x^2\right]_{\mathcal{B}_2}\right] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Then $\mathrm{Col}\ A$ is (by definition) the span of the vectors

$$\vec{a}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}.$$

Since $\mathrm{Imag}\ T$ is the set of all vectors in $\mathbb{P}_2$ with zero as a constant term, we have

$$\begin{aligned} \{[\vec{p}]_{\mathcal{B}_2} \in \mathbb{R}^3 \colon \vec{p} \in \mathrm{Imag}\ T\} &= \left\{ a\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\} \\ &= \mathrm{Span}\,\{\vec{e}_2, \vec{e}_3\} = \mathrm{Span}\,\{\vec{a}_1, \vec{a}_2\}, \end{aligned}$$

where $\vec{e}_i$ are the standard basis vectors in $\mathbb{R}^3$. This is precisely what we were told by Theorem 3.6.1:

$$\mathrm{Col}\ A = \{[\vec{p}]_{\mathcal{B}_2} \in \mathbb{R}^3 \colon \vec{p} \in \mathrm{Imag}\ T\}.$$

We get some very useful results for A immediately. We'll state them for linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ for simplicity of notation, but be aware that these results hold for general vector spaces as well.

Theorem 3.6.2 *Let A be the matrix such that the linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^m$ is given by $T(\vec{x}) = A\vec{x}$. Then T is one-to-one if and only if the columns of A are linearly independent.*

PROOF. The columns of A are linearly dependent if and only if there are scalars (not all 0) $c_1, \ldots, c_n$ such that $c_1\vec{a}_1 + \cdots c_n\vec{a}_n = \vec{0}$. This is true if and only if

$$A\vec{c} = [\vec{a}_1 \cdots \vec{a}_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}.$$

Then we have a nonzero vector $\vec{c} \in \mathrm{Ker}\ A$. By Theorem 3.3.1, this is true if and only if T is *not* one-to-one. $\square$

Example 3.6.3 As we saw in Example 3.6.2, the linear transformation $T \colon \mathbb{P}_1 \to \mathbb{P}_2$ given by taking the indefinite integral of vectors in $\mathbb{P}_1$ and using zero for the constant of integration has matrix representation (with respect to the standard basis)

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Since the columns of A are linearly independent, we have from Theorem 3.6.2 that T is one-to-one.

Theorem 3.6.3 *Let A be the matrix such that the linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is given by $T(\vec{x}) = A\vec{x}$. The linearly independent columns of A form a basis for* $\operatorname{Imag} T$.

PROOF. Let $\vec{x} \in \mathbb{R}^n$, so $T(\vec{x})$ is an arbitrary vector in $\operatorname{Imag} T$. Since

$$T(\vec{x}) = A\vec{x} = x_1\vec{a}_1 + \cdots x_n\vec{a}_n,$$

it follows that $\vec{x} \in \operatorname{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$, so $\operatorname{Imag} T \subseteq \operatorname{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$. Now pick $\vec{y} \in \operatorname{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$, so $\vec{y} = c_1\vec{a}_1 + \cdots + c_n\vec{a}_n$ for some scalars $c_1, \ldots, c_n$. It follows that

$$\vec{y} = [\vec{a}_1 \cdots \vec{a}_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = A\vec{c},$$

so $\vec{y} \in \operatorname{Imag} T$. Thus, $\operatorname{Span}\{\vec{a}_1, \ldots, \vec{a}_n\} \subseteq \operatorname{Imag} T$, so $\operatorname{Span}\{\vec{a}_1, \ldots, \vec{a}_n\} = \operatorname{Imag} T$. The result then follows from Theorem 2.1.2, the fact that a subset of a spanning set is a basis. $\square$

Corollary 3.6.4 *Let A be the matrix such that the linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is given by $T(\vec{x}) = A\vec{x}$. Then T is onto if and only if the columns of A span $\mathbb{R}^m$.*

Example 3.6.4 Define the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$, and

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Since the columns of A fail to span $\mathbb{R}^3$, we have from Theorem 3.6.3 that $\operatorname{Imag} T$ has dimension two. Since $\mathbb{R}^3$ has dimension three, it is not possible that $\operatorname{Imag} T = \mathbb{R}^3$, so by Theorem 3.1.1, T is not onto.

Exploration 97 Define the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$, and

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Is this linear transformation onto?

Exploration 98 Suppose V and W are vector spaces with the property that $\dim W > \dim V$. Let $T\colon V \to W$ be a linear transformation. Use Theorem 3.6.3 and an argument similar to the one in the example above to explain why T is not onto.

Subspaces Induced by Matrix Representations

Not only do matrices allow us to discover more about our favorite subspaces generated by linear transformations, they naturally generate some new subspaces we've not yet discussed. Before we get to those, though, there's another tool we'll need.

Definition 3.6.2 *Let $A \in \mathcal{M}_{m\times n}$. The* ***transpose of*** *A, denoted A^T, is the matrix in $\mathcal{M}_{n\times m}$ derived from A by making the jth column of A into the jth row for each $1 \leq j \leq n$.*

Example 3.6.5 Let

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$$

Then

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}.$$

Theorem 3.6.5 *Let $A, B \in \mathcal{M}_{m\times n}$ and $\alpha \in \mathbb{R}$. Then*

- $(A+B)^T = A^T + B^T$,
- $(\alpha A)^T = \alpha A^T$, *and*
- $(A^T)^T = A$.

PROOF. These are obvious.[37] □

37: Prove it!

Corollary 3.6.6 *Let $T_T : \mathcal{M}_{m\times n} \to \mathcal{M}_{n\times m}$ be the function that relates any matrix $A \in \mathcal{M}_{m\times n}$ to $A^T \in \mathcal{M}_{n\times m}$. Then T_T is a linear transformation.*

Exploration 99 Let $T_T : \mathcal{M}_{m\times n} \to \mathcal{M}_{n\times m}$ be the linear transformation that relates any matrix $A \in \mathcal{M}_{m\times n}$ to $A^T \in \mathcal{M}_{n\times m}$. Find a matrix representation for T_T when $m = 3$ and $n = 2$ using the bases in Figure 3.5.

Hint: The answer is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\mathcal{B}_{3\times 2} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\mathcal{B}_{2\times 3} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Figure 3.5: These are basis for $\mathcal{M}_{3\times 2}$ and $\mathcal{M}_{2\times 3}$

Speaking of rows of a matrix,

Definition 3.6.3 *For a matrix $A \in \mathcal{M}_{m\times n}$, let $\vec{r}_i$ be the vector formed from the ith row of A for each $1 \leq i \leq m$. The* ***row space of*** *A, denoted $\operatorname{Row} A$, is the span of these row vectors. That is,*

$$\operatorname{Row} A = \operatorname{Span}\{\vec{r}_1, \ldots, \vec{r}_m\}.$$

Theorem 3.6.7 *For a matrix $A \in \mathcal{M}_{m\times n}$, $\operatorname{Row} A$ is a subspace of $\mathbb{R}^m$.*

PROOF. This follows from Theorem 3.6.3 by taking the transpose of your matrix. □

Now for something really cool. We have fun subspaces of domains and codomains for linear transformations (the kernel and image, respectively), but what about the rest of the domain and codomain? You would not be shocked to find that the orthogonal complement of the kernel is a subspace of the domain[38], and the orthogonal complement of the image is a subspace of the codomain (Theorem 2.4.2). What is surprising is that these orthogonal complements are also given by the matrix representation for the linear transformation. Behold!

38: This is even isomorphic to the image from a theorem in Section 3.3.

Theorem 3.6.8 *Let A be the matrix such that the linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is given by $T(\vec{x}) = A\vec{x}$. Then*

$$\operatorname{Ker} A = (\operatorname{Row} A)^{\perp} \quad \textit{and} \quad \operatorname{Imag} A = \left(\operatorname{Ker} A^T\right)^{\perp}.$$

See Figure 4.3 for some geometric intuition.

PROOF. Let $A = [a_{ij}]$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, let $\vec{r}_i$ for $1 \leq i \leq m$ be the row vectors of A, and let $\vec{a}_j$ for $1 \leq j \leq n$ be the column vectors of A.

Then

$$\begin{aligned}
& \vec{x} \in \text{Ker } A \\
\Leftrightarrow \quad & A\vec{x} = \vec{0} \\
\Leftrightarrow \quad & x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{0} \\
\Leftrightarrow \quad & x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \\
\Leftrightarrow \quad & x_1 a_{i1} + \cdots + x_n a_{in} = 0 \text{ for } 1 \le i \le n \\
\Leftrightarrow \quad & \vec{x} \cdot \vec{r}_i = 0 \text{ for } 1 \le i \le n.
\end{aligned}$$

Thus, $\vec{x} \in \text{Ker } A$ if and only if it is orthogonal to every row of A. Since $\text{Row } A$ is the span of the rows of A, the result follows. A similar argument proves $(\text{Ker } A^T)^\perp = \text{Col } A$. □

Exploration 100 Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

We spent some time earlier considering $\text{Col } A$ and $\text{Ker } A$. Find $\text{Row } A$ and $\text{Ker } A^T$.

Check that the claims of the theorem line up as expected.

Corollary 3.6.9 *Let A be the matrix such that the linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^m$ is given by $T(\vec{x}) = A\vec{x}$. Then*

$$\text{dom}\,(T) = \text{Ker } A \oplus \text{Row } A \quad \textit{and} \quad \text{codom}\,(T) = \text{Imag } A \oplus \text{Ker } A^T.$$

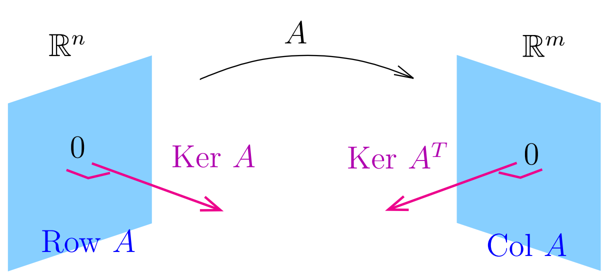


FIGURE 3.6. Some people refer to this as “the splits” of $\text{dom}\,(T)$ and $\text{codom}\,(T)$.

At the beginning of this section, we spent some time relating $\mathrm{Ker}\ A$ to $\mathrm{Ker}\ T$. Let's state this formally and prove it.

Theorem 3.6.10 *Let V and W be vector spaces with bases $\mathcal{B}_V = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\mathcal{B}_W = \{\vec{w}_1, \ldots, \vec{w}_m\}$, respectively. Let $T\colon V \to W$ be a linear transformation with matrix representation $A \in \mathcal{M}_{m\times n}$. Then*

$$\mathrm{Ker}\ T = \{\vec{v} \in V \colon A\,[\vec{v}]_{\mathcal{B}_V} = \vec{0}\}.$$

PROOF. Suppose $\vec{v} \in \mathrm{Ker}\ T$, so $T(\vec{x}) = \vec{0}$, and there are weights $a_1, \ldots, a_n$ such that $\vec{x} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$, that is,

$$[\vec{x}]_{\mathcal{B}_V} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

By Theorem 3.2.2, the coordinate mapping is a linear transformation, so it maps the zero vector to the zero vector. Thus $T(\vec{x}) = \vec{0}$ if and only if $[T(\vec{x})]_{\mathcal{B}_W} = \vec{0}$. Moreover, since T and the coordinate mapping are both linear,

$$\begin{aligned} & [T(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n)]_{\mathcal{B}_W} = \vec{0} \\ \Leftrightarrow\quad & a_1\,[T(\vec{v}_1)]_{\mathcal{B}_W} + \cdots + a_n\,[T(\vec{v}_n)]_{\mathcal{B}_W} = \vec{0} \\ \Leftrightarrow\quad & \left[[T(\vec{v}_1)]_{\mathcal{B}_W} \cdots [T(\vec{v}_n)]_{\mathcal{B}_W}\right] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \vec{0} \\ \Leftrightarrow\quad & A\,[\vec{x}]_{\mathcal{B}_V} = \vec{0}. \end{aligned}$$

□

Exercises for Section 3.6

3.6.1.n Define the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$, and

$$A = \begin{bmatrix} a & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

For what values of a is this an onto linear transformation?

3.6.2. Prove Theorem 4.4.8. That is, let $A, B \in \mathcal{M}_{m\times n}$ and $\alpha \in \mathbb{R}$. Prove

(a) $(A+B)^T = A^T + B^T$,

(b) $(\alpha A)^T = \alpha A^T$, and

(c) $(A^T)^T = A$.

3.6.3. Let $T\colon \mathbb{P}_3 \to \mathbb{P}_1$ by $T(ax^3 + bx^2 + cx + d) = (b-a)x + (c-d)$. Using the standard bases for $\mathbb{P}_3$ and $\mathbb{P}_1$, find the matrix representation of T and use this to find $\mathrm{Ker}\ A$ and $\mathrm{Col}\ A$. What are $\dim \mathrm{Ker}\ A^T$ and $\dim \mathrm{Row}\ A$?

3.6.4. Let $T\colon \mathbb{P}_3 \to \mathbb{P}_2$ by $T(ax^3 + bx^2 + cx + d) = (a-c)x^2 + (b-d)x + (c-a)$. Using the standard bases for $\mathbb{P}_3$ and $\mathbb{P}_2$, find the matrix representation of T and use this to find $\mathrm{Ker}\ A$ and $\mathrm{Col}\ A$. What are $\dim \mathrm{Ker}\ A^T$ and $\dim \mathrm{Row}\ A$?

3.6.5. Determine by inspection of the columns whether these matrices correspond to transformations that are one-to-one.

(a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

3.6.6. Determine by inspection whether these matrices correspond to transformations that are onto.

(a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

3.6.7.Let A be any $n \times m$ matrix. Prove $(\mathrm{Ker}\ A^T)^\perp = \mathrm{Col}\ A$.

3.6.8.Find $\mathrm{Ker}\ A$, $\mathrm{Row}\ A$, $\mathrm{Col}\ A$, and $\mathrm{Ker}\ A^T$ when A is the matrix below.

(a) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

3.7 Applications of Linear Tranformations

Computer Graphics and Animation

See the picture of Ricky's beautiful unicorn hooves in Figure 3.7. Yes, Ricky likes to wear zebra leg warmers.[39] A single vector would be a reasonable model[40] of Ricky's leg if unicorn legs did not bend. However, since Ricky has both knees and ankles, we should use three vectors; again, see Figure 3.7.

39: Doesn't everyone?

40: As we'll see, it's actually a very reasonable model.

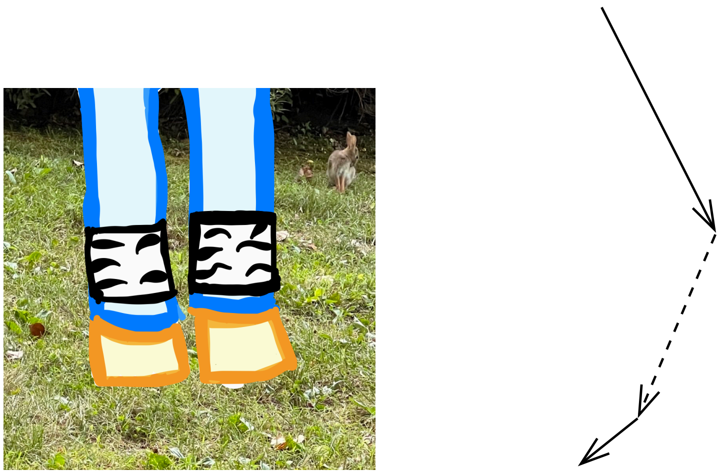


FIGURE 3.7. There are unicorn feet to the left and a vector impression of a unicorn leg to the right. The middle vector is dashed to simulate Ricky's beautiful zebra leg warmers; they're arranged to suggest a very horse-like galloping motion.

Our goal here is to animate our picture of Ricky's leg; more specifically, we will animate the vector representation of Ricky's leg. What should this entail? We should be able to translate, rotate, and scale the image. In this context, scaling might be weird. Legs usually stay the same size, but perhaps unicorns have some strange, little-known, femur-stretching powers. All kidding aside, scaling is actually extremely useful in images to give the impression that something is getting nearer or farther away. That's three things we'll need to do then: translation, rotation, and scaling.

When animating an image using a computer, sometimes it's better to use a vector (to represent a unicorn femur, for example), and sometimes it makes more sense to use a point (to represent a hoof, for example). For the entirety of this book, we've used different types of brackets or fonts to indicate whether something is a point or a vector. Since it's relatively difficult to explain this distinction to a computer, an additional component is often added to the vectors; we put a 1 in that component if our object is a vector and a 0 if it's a point. For example,

$$\text{The vector} \quad \vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{is} \quad \vec{v} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}.$$

$$\text{The point} \quad v = (x, y, z) \quad \text{is} \quad v = \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}.$$

Since vector convention has vectors rooted at the origin and leg convention has bones connected to other bones, we need to translate our second and third leg vectors so they are appropriately attached to the preceding leg vector. This is another handy[41] feature of the point/vector distinction; they allow us to *translate* our vectors in space. If you are concerned that translation is not a linear transformation, your concern is very well placed. Any translation by a nonzero vector is *not* a linear transformation. Oh, hey. We should prove that.

41:  No pun intended. Seriously! We promise!

Exploration 101 Let $A \in \mathcal{M}_{m\times n}$, and let $\vec{v} \in \mathbb{R}^m$ be nonzero. Define $T\colon \mathbb{R}^n \to \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x} + \vec{v}$. Show that T is not a linear transformation.

Looks like we're gonna need a new definition.

Definition 3.7.1 *An* ***affine transformation*** *is a linear transformation composed with a translation.*

To avoid ambiguity, when doing an affine transformation, we will use the point notation for the translation part of the transformation.

Example 3.7.1 Let $M \in \mathcal{M}_{m\times n}$, and let b be a point in $\mathbb{R}^m$. Then $T\colon \mathbb{R}^n \to \mathbb{R}^m$ by $T(\vec{x}) = M\vec{x} + b$ is an affine transformation. Moreover, T is a linear transformation if and only if b is the point at the origin. Lastly, it's important to note that the "+" in the equation is *not* vector addition; b isn't even a vector! The duplicitous plus, in this very specific context, indicates simply to "do translation by b when you're done with your matrix multiplication by M."

Let's get back to vector unicorn legs. See Figure 3.8. Note that the vector $\vec{t}+\vec{f}$ is not attached to the end of $\vec{f}$, it's way too long, *and* it's pointing in the wrong direction. However, the *translated* vector $\vec{t}+f$ has none of the aforementioned problems. Hooray for affine transformations!

Now we can do translations; that was the hard part. We've actually done rotation already; see Exercise 3.5.10 in Section 3.5. Indeed, to rotate a vector

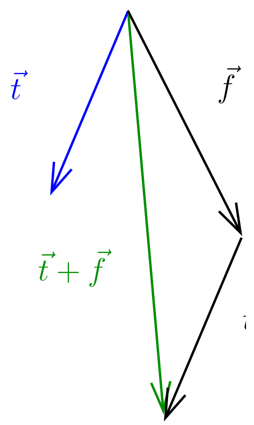


FIGURE 3.8. The shin bone $\vec{t}$ is connected to the thigh bone $\vec{f}$. The thigh bone $\vec{f}$ is connected to the hip bone (not pictured). We have chosen the letters "f" and "t" for *femur* and *tibia* because we don't know anything about biology, no one knows anything about unicorn biology, and couldn't be bothered to do a quick internet search.

$\vec{v} \in \mathbb{R}^3$ counterclockwise around the z-axis by an angle θ, we multiply

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad \text{by} \quad \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The last row and column are there to accommodate our vector/point notational convention. More generally, if we'd like to rotate a vector $\vec{v} \in \mathbb{R}^3$ counterclockwise around an arbitrary unit vector $\vec{u}$ by an angle θ, we multiply

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

by

$$\begin{bmatrix} \cos\theta + u_1^2(1-\cos\theta) & u_1u_2(1-\cos\theta) - u_3\sin\theta & u_1u_3(1-\cos\theta) + u_2\sin\theta & 0 \\ u_1u_2(1-\cos\theta) + u_3\sin\theta & \cos\theta + u_2^2(1-\cos\theta) & u_2u_3(1-\cos\theta) - u_1\sin\theta & 0 \\ u_1u_3(1-\cos\theta) - u_2\sin\theta & u_2u_3(1-\cos\theta) + u_1\sin\theta & \cos\theta + u_3^2(1-\cos\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

That's not hard to verify, but it is pretty gross and annoying.

The only thing remaining to do is scaling, and this, like rotation, is a linear transformation, so it can be represented as a matrix transformation.

Exploration 102 Let $a, b, c \in \mathbb{R}$ and $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ by scaling the first coordinate of vectors by a, the second coordinate by b, and the third coordinate by c. Find a matrix representation for T.

Alright! We now have a way to do translation, rotation, and scaling of our vector model of Ricky's leg. This is gonna be great. As we decided earlier, our vector model of Ricky's leg has three vectors, but now we label them carefully using affine transformations so that they are appropriately connected to each other. For example, since we want $\vec{t}$ to connect to the end of vector $\vec{f}$, we use an affine transformation to translate it to the end of $\vec{f}$, giving us $\vec{t} + f$. Similarly, to appropriately place $\vec{a}$, we use $\vec{a} + t + f$; see Figure 3.9.

Let A, B, and C be rotation matrices that rotate vectors counterclockwise about the origin by θ_1, θ_2, and θ_3 radians respectively. If we use A to rotate $\vec{f}$ by θ_1 radians, we must according rotate the translation parts of each affine transformation we used that involved $\vec{f}$. However, we might also want $\vec{t}$ to move, so we could apply B to $\vec{t}$ and the translation parts of each affine transformation we used that involved $\vec{t}$. Lastly, we can apply C to the little ankle vector $\vec{a}$. Again, see Figure 3.9. By appropriately moving the angles θ_1, θ_2, and θ_3, we can make this vector leg gallop (or spin in wildly unnatural ways if so desired).

In practice, we would then construct three more vector unicorn legs and use translations to place them on a vector unicorn body. At this point, we leave it to the professionals, but we know, by the power of linear algebra, we could do it were we so inclined. In the late 1970's and early 1980's many video games used just vectors to create "graphics" of tanks and asteroids and the like. More recently, carefully rendered, realistic images are associated to each vector. Indeed, we could overlay Ricky's zebra leg-warmer covered tibia over the vector $\vec{t}$, and that would be a significant improvement for realism's sake over the blue vector. Again, we leave this to the professionals.

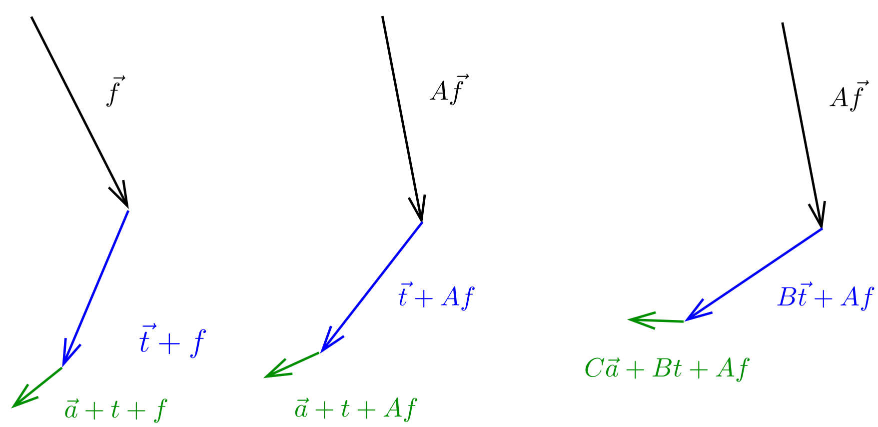


Figure 3.9: Various transformations of various leg vectors

4 More Fun with Matrices

4.1 Systems of Equations and Matrices

In the previous chapter, we introduced the concept of a matrix and explained the connections they enjoy with linear transformations. Many texts actually begin with matrices because they are a rich and convenient computational tool, especially useful for solving systems of equations. You'll note that many of the questions we asked previously boiled down to solving a system of equations. It shouldn't be a surprise then to learn that we can revisit topics, such as linear independence and coordinate vectors, using matrices. In this chapter, we will do all this. . . and more.

Systems of Equations Algebraically

As mentioned above, matrices are an important tool for solving systems of equations. Let's start there.

Consider this situation. A new dorm is being built on campus and the rooms come in two types. The first type is a "pod" which holds 6 students, and the second type is a standard room that holds 2 students. The university would like the dorm to hold 952 students in a total of 200 rooms.[1] We can set up a *system of equations* to represent this scenario and determine how many of each rooms should be included in the new dorm.

1: We're not sure why these specific numbers, but we know not to question such decisions.

Let x represent the number of pods and y represent the number of standard rooms. Then we have

$$6x + 2y = 952 \quad \text{and} \quad x + y = 200.$$

There are multiple ways you learned in previous algebra courses to solve this system of equations. You could graph these lines and see where they intersect.[2] You could solve one equation for x and then substitute the result into the other. Then there's *elimination*. We're going to use something pretty close to elimination to solve this by manipulating our system using two operations.

2: Do *not* do this. This is the worst way to solve this problem!

- We can replace either equation in the system by a scalar multiple of that equation.
- We can replace either equation in the system by the addition of that equation and a scalar multiple of the other equation.

Let's begin.

$$6x + 2y = 952 \tag{4.1}$$
$$x + y = 200 \tag{4.2}$$

We replace Equation 4.1 with Equation 4.1 scaled by $1/2$.

$$3x + y = 476 \tag{4.3}$$
$$x + y = 200 \tag{4.4}$$

We replace Equation 4.3 with Equation 4.3 plus Equation 4.4 scaled by (-1).

$$2x \quad = 276 \tag{4.5}$$
$$x + y = 200 \tag{4.6}$$

We scale Equation 4.5 by $1/2$.

$$x \quad = 138 \tag{4.7}$$
$$x + y = 200 \tag{4.8}$$

We replace Equation 4.8 by Equation 4.8 plus Equation 4.7 scaled by (-1).

$$x \quad = 138 \tag{4.9}$$
$$y = 62 \tag{4.10}$$

Solved! We need 138 pods and 62 standard rooms. Ready for some good news? We can use matrices as a shorthand for those operations. We'll use a matrix for the coefficients on the left hand side of our linear equations, a vertical bar to separate the left hand side from the right hand side, and a column for the constants on the right hand side. For example, given a system of m equations in n variables, such as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array},$$

we can form the matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

This matrix is called an **augmented matrix**, and there's definitely more to it than we're presenting here. For now, though, it certainly provides a more convenient notational convention. You're welcome.[3]

3:  Hey! I'll handle the snarkiness around here!

Here's the previous example as an augmented matrix and the sequence of operations we did to it.

$$\left[\begin{array}{cc|c} 6 & 2 & 952 \\ 1 & 1 & 200 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 1 & 476 \\ 1 & 1 & 200 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 0 & 276 \\ 1 & 1 & 200 \end{array}\right] \to$$

$$\left[\begin{array}{cc|c} 1 & 0 & 138 \\ 1 & 1 & 200 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 138 \\ 0 & 1 & 62 \end{array}\right]$$

That's so much less to write! Matrices might be another way to solve systems of equations that is worth further exploration, right?[4]

4: You know what that means. . .

Exploration 103 Let's translate the rules we followed to solve the system of equations into the language of matrices.

- Any row may be replaced by a scalar multiple of itself.
- Any row may be replaced by the sum of that row and a scalar multiple of another row.

Suppose the total number of rooms in the above example were changed to 240. Solve this new system using the matrix notation.

Our primary concern in this section will be identifying solutions for systems of equations, so we should probably define *solution* formally.

Definition 4.1.1 *A* ***solution for a system*** *of linear equations*

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

is an n-tuple $(x_1, \ldots, x_n)$ that makes all the linear equations in the system true.

System Representations

The matrix notation we used at the beginning of the section was sold as a notational convenience, but it's actually a lot more natural than it may seem. Given a system of m equations in n variables,

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m, \end{array} \tag{4.11}$$

we could use the definition of the product of a scalar and a vector to write this as the **vector equation**

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}, \tag{4.12}$$

where for $1 \leq j \leq n$,

$$\vec{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Using the definition of the product of a matrix and a vector, we find that the vector equation (4.12) is equivalent to

$$[\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n]\,\vec{x} = \vec{b},$$

where

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

This is a **matrix equation**

(4.13) $$A\vec{x} = \vec{b},$$

where

$$A = [\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

is called the **coefficient matrix**. We summarize this in the following theorem.

Theorem 4.1.1 *Using the matrix and vectors*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn}. \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \textit{ and } \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

the following are all equivalent representations

(a) *System of equations:*

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m, \end{array}$$

(b) *Vector equation:*

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b},$$

(c) *Matrix equation:*

$$A\vec{x} = \vec{b},$$

(d) *Augmented matrix:*

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

This theorem gives us the flexibility to view a system of equations in four different contexts.

Exploration 104 Consider the system below.

$$\begin{aligned} 6x_1 + 2x_2 - x_3 &= 111 \\ x_1 + 7x_2 - 2x_3 &= 56 \\ -5x_1 + x_2 + 4x_3 &= 12 \end{aligned}$$

Write it as a vector equation, matrix equation, and an augmented matrix. When you're done, do it again, but make it really, really small.

Gauss-Jordan Elimination

The goal here is to change our system of equations *without altering the solution(s)* to make it easier to identify the solution(s). Thinking back to the example at the beginning of this section (or other methods you've used to solve systems), this is precisely what one typically does. The strategy we are about to show you is called Gauss-Jordan elimination, named for Carl Friedrich Gauss and Wilhelm Jordan, despite the fact that Gauss apparently had nothing to do with it. First, we'll outline the operations we can do in this elimination procedure.[5] Next, we'll formalize a goal for the procedure so that we know when we can stop. Lastly, we'll see the procedure described as an algorithm. Got all that? Good. Let's get started.

5:  Two of these already made an appearance in that opening example.

Operations: Three Things You Can Do

What are things you can *do* to a system that don't change the solution for the system? Some things are so simple, they're not obvious. We could change the order of our equations; that certainly wouldn't alter any of the solutions for any of the equations, so the system solution would remain unchanged as well. We could also multiply both sides of an equation by a nonzero scalar; again, the solution set for that equation wouldn't change, so the system solution remains unchanged.

We keep "doing stuff" to equations; let's get a little more specific. Let V be the set of all linear equations in n variables with real coefficients. That is,

$$V = \{a_1x_1 + a_2x_2 + \cdots + a_nx_n = b\colon\ a_i, b \in \mathbb{R} \text{ for } 1 \le i \le n\}. \tag{4.14}$$

For any nonzero scalar α and any equation $v \in V$, define the nonzero scalar multiple of v by α as the equation one gets by multiplying both sides of v by α. For any $v, u \in V$, define the sum of the equations v and u to be the equation one gets by equating the sum of the left sides of v and u to the sum of the right sides of v and u. [6]

There is a third, less obvious, thing one could do to a system without changing the solution. Let's pile it onto the other two and call it a theorem.[7]

Theorem 4.1.2 *The following three operations on a system of m equations in n variables do not change the system's solution:*

(a) *interchanging the position of any two equations in the system;*

(b) *multiplying an equation by a nonzero scalar; and*

(c) *replacing the ith equation with the sum of the ith equation and any nonzero scalar multiple of any of the other equations.*

PROOF. We've already covered both a and b. To prove c, the following observation will simplify things: we could write any equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ as

$$\vec{a} \cdot \vec{x} = b,$$

where

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Suppose we have two equations from a system in this notation:

$$\vec{a} \cdot \vec{x} = b \tag{4.15}$$

$$\vec{c} \cdot \vec{x} = d. \tag{4.16}$$

Suppose $\vec{x}$ is a solution to both of these equations. Then for any nonzero scalar α,

$$\begin{aligned} \vec{c} \cdot \vec{x} + \alpha b &= d + \alpha b && \text{since } \vec{c} \cdot \vec{x} = d, \\ \vec{c} \cdot \vec{x} + \alpha(\vec{a} \cdot \vec{x}) &= d + \alpha b && \text{since } \vec{a} \cdot \vec{x} = b, \text{ and} \\ (\vec{c} + \alpha\vec{a}) \cdot \vec{x} &= d + \alpha b && \text{by distributive properties of inner product.} \end{aligned}$$

Thus, $\vec{x}$ is also a solution to the sum of equations (4.15) and (4.16). Now suppose $\vec{x}$ is a solution to the system

$$\vec{a} \cdot \vec{x} = b \tag{4.17}$$

$$(\vec{c} + \alpha\vec{a}) \cdot \vec{x} = \alpha b + d. \tag{4.18}$$

Then for any nonzero scalar α,

$$\begin{aligned} (\vec{c} + \alpha\vec{a}) \cdot \vec{x} - \alpha b &= d && \text{by Equation (4.18)}, \\ (\vec{c} + \alpha\vec{a}) \cdot \vec{x} - \alpha(\vec{a} \cdot \vec{x}) &= d && \text{since } \vec{a} \cdot \vec{x} = b, \text{ and} \\ \vec{c} \cdot \vec{x} &= d && \text{by distributive properties of inner product.} \end{aligned}$$

Thus, $\vec{x}$ is also a solution to $\vec{c} \cdot \vec{x} = d$. Since $\vec{x}$ is a solution to $(\vec{c} + \alpha\vec{a}) \cdot \vec{x} = d + \alpha b$ if and only if it is a solution to both $\vec{a} \cdot \vec{x} = b$ and $\vec{c} \cdot \vec{x} = d$, we have

6: These operations should remind you of something if you have not forgotten everything we talked about in Section 1.1.

Or if you haven't skipped ahead to Chapter 4 because matrices are your favorite topic...

What!? That's not allowed!

Actually, it is... There's a note in the introduction and everything.

7: You may note that Gauss-Jordan elimination fans on the internet often neglect proving this all-important fact.

No!

It's true.

that the following systems have the same set of solutions:

$$\begin{array}{rcl} \vec{a}\cdot\vec{x} & = & b \\ \vec{c}\cdot\vec{x} & = & d \end{array} \qquad \begin{array}{rcl} \vec{a}\cdot\vec{x} & = & b \\ (\vec{c}+\alpha\vec{a})\cdot\vec{x} & = & d+\alpha b \end{array}$$

□

Corollary 4.1.3 *One may apply any number of the operations in Theorem 4.1.2 in any order to a system of equations without changing the set of solutions for that system.*

Since a system of equations can also be represented as an augmented matrix, we can do the analogous operations to a matrix.

Definition 4.1.2 *Let $A \in \mathcal{M}_{m\times n}$. The following manipulations of A are called* ***row operations****:*

(a) *interchanging any two rows in A;*
(b) *multiplying any row by a nonzero scalar; and*
(c) *replacing the ith row with the sum of the ith row and any nonzero scalar multiple of any of the other rows.*

Any matrix resulting from any row operation on A is called ***row equivalent*** *to A.*[8]

Corollary 4.1.4 *Let $[A|\vec{b}]$ be the augmented matrix for a given system of m equations in n variables. Row operations on $[A|\vec{b}]$ do not change the set of solutions for the associated system.*

PROOF. This follows immediately from Theorem 4.1.2 and Theorem 4.1.1. □

Corollary 4.1.4 and augmented matrices are the primary means by which systems of equations are solved in most linear algebra courses, so we should probably work on that a bit. Keep in mind, though, that Theorem 4.1.1 provides multiple ways to represent a system of equations, so there may be situations where other methods are preferred.

For organizational purposes, we need to pick a notational convention for row operations. Here's one:

(a) $A \xrightarrow{\vec{r}_i \leftrightarrow \vec{r}_j} B$ means interchanging row i and row j in A results in the matrix B;

(b) $A \xrightarrow{a\vec{r}_i \to \vec{r}_i} B$ means multiplying row i by $a \neq 0$ results in the matrix B; and

(c) $A \xrightarrow{a\vec{r}_i + \vec{r}_j \to \vec{r}_j} B$ means replacing the jth row with the sum of the jth row and $a \neq 0$ times the ith row results in the matrix B.

This gives us a way to keep track of our operations, but most of us omit these once we've mastered row reduction.[9]

8: *"Row equivalent" is really nothing like "equivalent." Maybe there should be another word.*

What about "row similar"?

Noooo! Don't say that. "Similar" will mean something better than "row equivalent" in the next chapter, and now you've confused everyone!

It'll be fine. You're assuming they're reading these remarks. I'm sure no one saw that one... Also, who picks these words anyway?

9: You'll likely find yourself doing the steps of row reduction "in your head" sooner than you'd expect. At that point, though, you should use this notation to punish yourself when you make an error.

Example 4.1.1 Solve the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 5x_1 + 6x_2 + 7x_3 &= 8 \\ 10x_2 + 11x_3 &= 12. \end{aligned}$$

By Theorem 4.1.1, this system is equivalent to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 10 & 11 & 12 \end{array}\right].$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 10 & 11 & 12 \end{array}\right] \xrightarrow{-5\vec{r}_1+\vec{r}_2\to\vec{r}_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 10 & 11 & 12 \end{array}\right]$$

$$\xrightarrow{-1/4\vec{r}_2\to\vec{r}_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 10 & 11 & 12 \end{array}\right]$$

$$\xrightarrow{-10\vec{r}_2+\vec{r}_3\to\vec{r}_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -9 & -18 \end{array}\right]$$

$$\xrightarrow{-1/9\vec{r}_3\to\vec{r}_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

We could be done here; we see from the last row of the last matrix that $x_3 = 2$, and we could substitute this into the other two equations to find x_1 and x_2 as well. Alternatively, we can just keep cooking with the row reduction.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-2\vec{r}_3+\vec{r}_2\to\vec{r}_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{-3\vec{r}_3+\vec{r}_1\to\vec{r}_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{-2\vec{r}_2+\vec{r}_1\to\vec{r}_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Well, *that's* easy to read. We have $x_1 = 0$, $x_2 = -1$, and $x_3 = 2$, which is the unique solution to the original system.

Target Format: Reduced Row-Echelon Form

Note that Corollary 4.1.3 implies that one has a lot of options when applying row operations to an augmented matrix. This can be a bit overwhelming. It is entirely possible to apply one thousand row operations (correctly) to an augmented matrix only to get the augmented matrix you started with. This can be a bit frustrating. We should avoid feeling overwhelmed and frustrated. It will help us to have a target in mind for our row operations.

Definition 4.1.3 *Let $A \in \mathcal{M}_{m\times n}$. We say the matrix A is in **row-echelon form** if*

- *the first nonzero number from the left, also called the **pivot**, of any nonzero row is always strictly to the right of the pivot of the row above, and*
- *any row with nonzero entries is above any row of all zeros.*

*We say A is in **reduced row-echelon form** if*

- *it is in row-echelon form,*
- *every pivot is a 1, and*
- *every pivot is the only nonzero entry in its column.*

This probably seems like a really weird definition; it definitely reads more like a tax form than most definitions. Nevertheless, it will have its uses. Let us explore the echelon-iness of some matrices.[10]

Example 4.1.2 Here are some matrices. Let's determine whether they are in row-echelon form, reduced row-echelon form, or neither.

- $\begin{bmatrix} 0 & 2 & 4 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

  This matrix is in neither form. In fact, it does not appear to be row reduced whatsoever.

- $\begin{bmatrix} 4 & 2 & 4 & 8 \\ 0 & 10 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & -2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix},$

  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

  These matrices are all in row-echelon form. The pivots have been colored red.[11]

- $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix},$

  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

  These matrices are all in reduced row-echelon form. Almost all the pivots have been colored red; one rogue pivot[12] has been colored blue for no good reason at all.

Exploration 105 The matrices in each bullet list below are row reductions of the same matrix. Circle the row-echelon form matrices. Put a square around the matrix in reduced row-echelon form. Highlight your favorite column vector with a pink sparkle pen.

- $\begin{bmatrix} 0 & 2 & 4 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 3/4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/4 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix},$

10: What does "echelon" mean?
Model name of a Ford sedan from the 1970's?
No.
I think there's a sci-fi series-
No.
Michael Jordan's middle name?
No! I'm quite sure it's not a proper noun!

11: Do I have to pronounce the t in "pivot"?
Yes. Can I pronounce it with a long i?
Sure. What could go wrong?

12: Rogue Pivot! Dibs on band name!
Only with a long i and a silent t.
I hate to admit it, but that is a solid name.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/4 \end{bmatrix}$$

▶ $$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, consider if these had been augmented matrices with the bar before the final column missing. Which form is the easiest to use when determining the solution of your system of linear equations?

Indeed, reduced row-echelon form is the form in which it is arguably[13] easiest to find the solution(s) to the associated system, and there are other benefits we shall uncover later.

13: I will fight you.

Now that we've talked about reduced row-echelon form and implied that this is the target format from the subsection title, you might be wondering whether this target is actually always achievable. Well, it is. Here's a theorem that says so, so it must be true.[14]

14: This is a classic example of "proof by intimidation." The idea here is that if you argue something with strong enough language, even without sound logic, the reader might believe it.

Theorem 4.1.5 *Suppose $A \in \mathcal{M}_{m\times n}$. Then there exists a unique matrix $B \in \mathcal{M}_{m\times n}$ in reduced row-echelon form that can be obtained from A by performing row operations.*

The proof of this statement relies on the technique of mathematical induction and Gauss-Jordan Elimination, so we will leave it to the Appendix.[15] Note here that it is *reduced* row-echelon form that is unique. In fact, scaling any row in a row-echelon form matrix produces a different row-echelon matrix associated to the same system of equations. Perhaps you are wondering why we even bother defining row-echelon form then? Well, the columns which contain pivots can be identified once the matrix is placed in row-echelon form, and there are some situations where the only information needed to answer a question is this. We'll see examples of these types of questions in the next two sections in particular. For now, though, it really will be *reduced* row-echelon form that is our target.

15: Oh, good. I was worried we were just going to declare it as fact without proof.

Procedure for Gauss-Jordan Elimination

We mentioned Gauss-Jordan elimination before, but never precisely defined it. That's because we needed a bit of terminology. **Gauss-Jordan elimination** is an algorithm that uses only the row operations to put any matrix into reduced row-echelon form. Thus, by Corollary 4.1.4, the resulting reduced row-echelon matrix has the same solution as the original matrix. It is definitely a thing you (or a computer!) could do. Here are the steps:

(1) Move any row of all zeros to the bottom of the matrix.

(2) Start at the leftmost column that has a nonzero entry in a row not already containing a reduced pivot. Designate an entry in this column to be a pivot and scale that row so that the pivot is a 1.

(3) Move the row of the pivot you are currently working with to be the highest row without a pivot already in reduced position.

(4) Use this pivot to produce zeros in every entry below it in the same column (by adding appropriate multiples of the pivot's row to the rows below).

(5) Repeat Steps 1–4.

(6) Continue this process until there are no more columns satisfying Step 2.

(7) Now begin at the bottom nonzero row and use the pivot to produce zeros (as in Step 3) in every entry *above* it in the same column.

(8) Move up one row and repeat Step 7.

(9) Repeat Step 8 until there are no more rows.

Example 4.1.3 Example 4.1.1 was this exact procedure. Go look at it again. Did you look? Seriously, you have to go look at it.

Example 4.1.4 Fine. We'll do another one. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 6 & -1 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix}.$$

(a) Move any row of all zeros at the bottom of the matrix. Alright::

$$A \xrightarrow{\vec{r}_1 \leftrightarrow \vec{r}_4} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 2 & 4 & 6 & -1 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Start at the leftmost column that has a nonzero entry in a row not already containing a reduced pivot and designate a pivot. Scale this pivot's row so that the pivot is 1 (by multiplying the row by the appropriate nonzero scalar). Already done! Take that, Gauss! You're not the boss of me!

(c) Move the row of the pivot to be the highest row without a pivot already in reduced position. Hmmm... That seems to already be done as well. We are rocking this!

(d) Use this pivot to produce zeros in every entry below it in the same column (by adding appropriate multiples of the pivot's row to the rows below). So ordered:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 2 & 4 & 6 & -1 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2\vec{r}_1+\vec{r}_2 \to \vec{r}_2} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 0 & 2 & 4 & -3 & 8 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(e) Repeat steps 1–4. Ugh, that's a lot to ask.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 0 & 2 & 4 & -3 & 8 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{1/2\vec{r}_2 \to \vec{r}_2} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & -3/2 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Well, I guess that wasn't so bad.

(f) Continue this process until there are no more columns satisfying Step 2. Hey, all those zeros and ones were pretty convenient!

(g) Now begin at the bottom nonzero row and use the pivot to produce zeros (as in step 4) in every entry *above* it in the same column.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & -3/2 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{3/2\vec{r}_3 + \vec{r}_2 \to \vec{r}_2} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-\vec{r}_3 + \vec{r}_1 \to \vec{r}_1} \begin{bmatrix} 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(h) Move up one row and repeat step 6.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\vec{r}_2 + \vec{r}_1 \to \vec{r}_1} \begin{bmatrix} 1 & 0 & -1 & 0 & -5 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Victory!

(i) Repeat step 8 until there are no more columns satisfying Step 2. Thanks for the advice, but we already declared victory.

Exploration 106 Let's play a bit of fill in the blanks. We'll row reduce a matrix using Gauss-Jordan Elimination, but leave out the names of the row operations. You fill in the operation that was done at each step. Ready? Go!

$$\begin{bmatrix} 2 & 4 & 0 \\ 3 & 6 & 3 \\ 0 & 0 & 0 \\ 2 & 2 & 2 \end{bmatrix} \text{——} \to \text{——} \begin{bmatrix} 2 & 4 & 0 \\ 3 & 6 & 3 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{——} \to \text{——} \begin{bmatrix} 1 & 2 & 0 \\ 3 & 6 & 3 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{——} \to \text{——} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{——} \to \text{——} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{——} \to \text{——}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \text{——} \to \text{——} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \text{——} \to \text{——} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{——} \to \text{——} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{——} \to \text{——} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Section Highlights

- A system of equations can be represented equivalently as a vector equation, a matrix equation, or an augmented matrix. See Theorem 4.1.1.

- There are three types of row operations that can be done to a matrix as part of Gauss-Jordan Elimination:

  - Rows can be scaled by any nonzero real number.

  - Rows can be swapped.

  - Any row can be replaced by a linear combination of other rows.

  See Definition 4.1.2.

- Gauss-Jordan Elimination is a process that uses row operations to transform a matrix into reduced row-echelon form. See Definition 4.1.3 and Example 4.1.4.

- If row operations from Gauss-Jordan Elimination are used on an augmented matrix, then the system of equations corresponding to the resulting matrix has the same set of solutions as the system corresponding to the original augmented matrix. This gives a convenient way to solve a system of equations since reduced row-echelon form makes the solution easier to identify. See Corollary 4.1.4 and Example 4.1.1.

Exercises for Section 4.1

4.1.1. Write the system as a vector equation, a matrix equation, and an augmented matrix.

(a)
$$\begin{aligned} x_1 + 3x_2 &= 5 \\ 3x_1 &= 2 \\ x_1 - x_2 &= 1 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 - x_3 &= 5 \\ 3x_1 + x_3 &= 2 \\ x_2 - x_3 &= 0 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + x_2 - x_3 &= 5 \\ 3x_1 + 2x_3 + x_5 &= 2 \\ x_2 - x_3 + x_4 &= 1 \end{aligned}$$

(d)
$$\begin{aligned} x_1 + x_2 - x_3 - x_4 &= 1 \\ 3x_1 + x_3 + x_4 &= 1 \\ x_2 - x_3 &= 1 \end{aligned}$$

4.1.2. Determine whether the matrices below are in row-echelon form, reduced row-echelon form, or neither.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4.1.3. Write down every possible reduced row-echelon matrix in $\mathcal{M}_{2\times 4}$. You may use any symbol you like to represent real numbers that are neither 1 nor 0.

4.1.4. Write down every possible reduced row-echelon matrix in $\mathcal{M}_{3\times 3}$. You may use any symbol you like to represent real numbers that are neither 1 nor 0.

4.1.5. Determine whether or not the pair of matrices is row equivalent. If they are list the row operations that transform the former into the latter.

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

4.1.6. Row reduce the matrices to obtain reduced row-echelon form. (That is, perform Gauss-Jordan Elimination.)

(a) $\begin{bmatrix} 1 & -2 \\ 1 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -2 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & -4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -4 \\ 3 & 0 & -6 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 6 & 6 \\ 0 & 0 & 6 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 3 & 0 & 6 \end{bmatrix}$

(h) $\begin{bmatrix} 1 & 1 & -1 & 1 & 2 \\ 2 & 2 & -2 & 1 & 0 \\ 0 & 0 & 6 & 1 & 4 \end{bmatrix}$

(i) $\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 6 & 6 \\ 0 & 3 & 0 & 6 \end{bmatrix}$

(j) $\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 6 & 6 \\ 1 & 2 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

(k) $\begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 6 & 2 & 6 \\ 0 & 3 & 0 & 6 & 0 \end{bmatrix}$

4.1.7.Is V from Equation 4.14, the set of all linear equations in n variables with real coefficients, a vector space? Justify your argument.

4.2 More Systems of Equations and Matrices

Now that we have a new method to solve a system of equations, we should spend some time talking about what a solution might look like and say a bit more about how matrices help us find them.

Parametric Solutions to Systems

Augmented matrices in reduced row-echelon form have the wonderful property that one can "read" the solution to the associated system right off the matrix. Just look back at Example 4.1.1. There is a small subtlety, though.

Example 4.2.1 Let's solve the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 5x_1 + 6x_2 + 7x_3 &= 8 \\ 9x_1 + 10x_2 + 11x_3 &= 12. \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right] \xrightarrow[]{\substack{-5\vec{r}_1 + \vec{r}_2 \to \vec{r}_2 \\ -9\vec{r}_1 + \vec{r}_3 \to \vec{r}_3}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{array}\right]$$

$$\xrightarrow{-2\vec{r}_2+\vec{r}_3\to\vec{r}_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{(-1/4)\vec{r}_2\to\vec{r}_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-2\vec{r}_2+\vec{r}_1\to\vec{r}_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The final matrix above is in reduced row-echelon form. Now we can solve the system. The top row says $x_1 - x_3 = -2$ and the second row says $x_2 + 2x_3 = 3$. Solving for x_1 and x_2 gives us

$$x_1 = x_3 - 2$$

$$x_2 = -2x_3 + 3.$$

In terms of vectors, this would be all vectors of the form

$$\begin{bmatrix} x_3 - 2 \\ -2x_3 + 3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$$

where $x_3 \in \mathbb{R}$.

The augmented matrix in reduced row-echelon form in the previous example did not have a pivot in every variable column. That seems somewhat annoying, but it happens a lot. How did we deal with it in the example? We were able to solve for all the variables associated to a column containing a pivot, and some of those variables were in terms of the variable not associated to a pivot. This is not a coincidence.

Definition 4.2.1 *A* ***pivot variable*** *is a variable in a system of equations whose column in the associated augmented matrix in reduced echelon form contains a pivot. A* ***free variable*** *is a variable in a system of equations that is not a pivot variable. That is, a free variable in a system of equations is one whose column in the associated augmented matrix in reduced echelon form does not contain a pivot.*

Exploration 107 Solve for each of the pivot variables in terms of free variables in the augmented matrices below.

- $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

- $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

- $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 2 \\ 0 & 1 & 5 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Definition 4.2.2 *A* ***parametric solution*** *for a system of m equations in n variables that has an infinite number of solutions is a representation of the solutions in which the pivot variables are given in terms of the free variables (often called* ***parameters****).*

Your answers from the exploration above were all parametric solutions to the systems represented by the matrices. We will often also make note of which variables are free variables in solutions of this form.

Example 4.2.2 Consider the augmented matrix below.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 5 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solving for the pivot variables in terms of the free variables and taking note of which variables are free gives us a solution that looks like

$$\begin{array}{c} x_1 = 2 - x_4, \\ x_2 = -1 - x_4 - 5x_3, \\ x_3 \text{ free}, \\ x_4 \text{ free}. \end{array}.$$

We could write this in set notation with vectors as

$$\left\{ x_3 \begin{bmatrix} 0 \\ -5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} : x_3, x_4 \in \mathbb{R} \right\}.$$

Zero, One, or Many

The Gauss-Jordan elimination procedure and row-echelon form make determining whether a system has a solution pretty simple. First of all, when an augmented matrix is in even just row-echelon form, we can tell quickly that a solution does not exist.

Theorem 4.2.1 *A system of m equations in n variables has no solutions if the associated augmented matrix in row-echelon form has a pivot in the last column.*

PROOF. Let $[A|\vec{b}]$ be the associated augmented matrix in row-echelon form. Suppose $[A|\vec{b}]$ has a pivot in the last row. Then this row corresponds to an equation in the associated system of the form

$$0x_1 + 0x_2 + \cdots + 0x_n = b_i,$$

for some $b_i \neq 0$. This equation has no solutions, so any system of which it is a part also has no solution. □

Now, if we do not have a pivot in the final column, there are actually two possibilities.

Theorem 4.2.2 *A system of m equations in n variables has a unique solution if the associated augmented matrix in reduced row-echelon form has a pivot in every column except the last.*

PROOF. Suppose $[A|\vec{b}]$ is the augmented matrix in reduced row-echelon form. We note that any nonzero row of this matrix has a 1 in a single pivot location and a 0 in all remaining locations except the final one. Thus, any associated equation will be of the form

$$x_i = b_i$$

where x_i is a variable and b_i is the i^{th} entry in the final column $\vec{b}$. Thus, each variable is determined by the entries of $\vec{b}$, and there is a unique solution. □

Theorem 4.2.3 *A system of m equations in n variables has infinitely many solutions if the associated augmented matrix in reduced row-echelon form has no pivot in the last column and at least one other column. That is, the system has infinitely many solutions if it has a free variable and no pivot in the final column.*

PROOF. Suppose $[A|\vec{b}]$ is the augmented matrix in reduced row-echelon form. Any nonzero row of this matrix has a 1 in a single pivot location and a 0 in all remaining pivot locations. Thus, we can solve for the pivot variable in terms of the free variables and the entry in our solution column. That is, if x is a pivot variable, there is some row of $[A|\vec{b}]$ whose associated equation is

$$x + a_1y_1 + \cdots + a_ky_k = b$$

where $a_1, \ldots, a_k \in \mathbb{R}$, $y_1, \ldots y_k$ are free variables, and b is the corresponding entry in the solution column. This allows us to say

$$x = b - a_1 y_1 - \cdots - a_k y_k.$$

Since we assumed there is no pivot in the final column, each row containing a pivot can be treated thusly, and doing so gives us a full parametric solution for the system of equations. All rows not containing a pivot are rows of zero since we are in reduced row-echelon form. Then, we can choose a real number for each free variable. Any choice gives a valid solution, and there are infinitely many possible choices for these real numbers. Furthermore, each distinct choice gives rise to a distinct solution since the free variables themselves would then be distinct. Thus, there are infinitely many solutions. □

Combining these three theorems, we can state the following:

Corollary 4.2.4 *A system of m equations in n variables has either zero solutions, one solution, or infinitely many solutions.*

Corollary 4.2.5 *A system of m equations in n variables has no solutions if and only if the associated augmented matrix in row-echelon form has a pivot in the last column.*

Note also that Theorem 4.2.2 and Theorem 4.2.3 stated that the matrix was in *reduced* row-echelon form. This is really just to make the proofs clearer. Since the locations of pivots can be identified once the matrix is in row-echelon form, we could have stated these theorems using the weaker condition instead. Therefore, if you are trying to answer the question of how many solutions rather than actually finding them, you could actually save yourself some time by only reducing the matrix to row-echelon form.

Exploration 108 The matrices below are the examples of row-echelon form from the previous section. Cross out the ones that correspond to systems with no solution. Circle the ones with a unique solution. Do the ones left have infinitely many solutions?

$$\left[\begin{array}{ccc|c} 4 & 2 & 4 & 8 \\ 0 & 10 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & -2 & 2 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \text{ and } \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Exploration 109 The augmented matrices below are in row-echelon form. Determine whether the corresponding systems of equations have no solution, a single solution, or infinitely many solutions. If there are infinitely many, write the parametric solution.

- $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

- $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

- $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$

Geometry of Solutions

There are a lot of subtle misconceptions about the relationship between linear equations and their graphical representations. For example, it's very common to hear the equation $y = mx + b$ referred to as "a line." It's not that this is completely *wrong*, but it's definitely inaccurate. An equation and a line are two different things. Before we proceed, we should establish some concrete lingo.

Definition 4.2.3 *A* ***solution*** *for a linear equation $a_1x_1 + \cdots + a_nx_n = b$ is an n-tuple $(x_1, \ldots, x_n)$ that makes the linear equation true. The* ***graph*** *of a linear equation $a_1x_1 + \cdots + a_nx_n = b$ is a visual representation of the set of all n-tuples $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ that make the linear equation true.*

There is a not-so-subtle distinction between the linear equation $a_1x_1 + \cdots + a_nx_n = b$, its set of solutions, and its graph. These three things are respectively an algebraic equation (that may or may not be true), a set of points, and a visual representation of that set of points.

Let us explore the consequences of this definition for systems. A solution for a system must be a solution for each equation in the system. We now have multiple ways of thinking about this:

- The solution for a system is the intersection of the sets of solutions for each equation in the system.

- The graph of the solution for a system is a visual representation of this intersection. That is, the graph of the solution for a system is where the graphs of each equation in the system intersect.

We need at least two variables for this to be the slightest bit interesting, so we'll start there.

Two Variable Cases:

- Two variables, one equation:
  Classic. If we have $a_1x_1 + a_2x_2 = b$, we could write
  $$x_2 = -\frac{a_1}{a_2}x_1 + \frac{b}{a_2},$$
  the standard "slope-intercept" form. Note that since we're defining x_2 as a function of x_1, we need $a_2 \neq 0$ to have an x_2, which works out really well since we also have to divide by a_2. In this case, there's always an infinite number of solutions in $\mathbb{R}^2$ for one linear equation in two variables. The graph of this equation, the set of all solutions to the equation, forms a line in the $\mathbb{R}^2$ plane.

- Two variables, two equations:
  This is significantly more interesting. With two equations, we have an infinite number of solutions for each equation. However, since a solution for a system must be a solution for *all* equations in the system, there are three possibilities. The set of solutions for each equation is a line in the plane, and these lines may be parallel, overlap completely, or intersect at one point. These three cases correspond to the system having infinite solutions, no solutions, or exactly one solution, respectively. See Figure 4.1.

- Two variables, three or more equations:
  This is not all that different from the previous case; the graph of the third equation simply provides a third line in the $\mathbb{R}^2$ plane. Any additional equations do the same. Convince yourself that we still only have three options for this system: infinite solutions, no solutions, or exactly one solution.

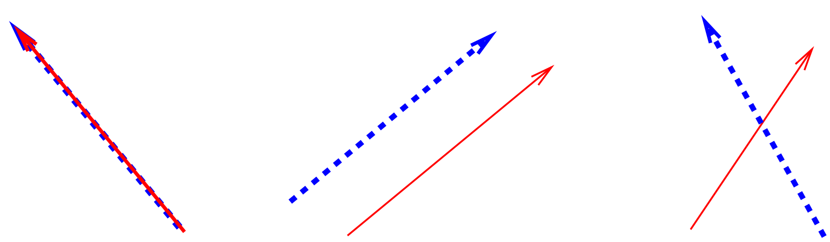


Figure 4.1: There are only three possible ways for two lines in $\mathbb{R}^2$ to intersect.

Three Variable Cases:

- Three variables, one equation:
  This is an equation of the form $a_1x_1 + a_2x_2 + a_3x_3 = b$, and the graph of this equation is a plane in $\mathbb{R}^3$. We can easily find y_1, y_2, y_3 such that $b = -(a_1y_1 + a_2y_2 + a_3y_3)$. Then $a_1x_1 + a_2x_2 + a_3x_3 = b$ if and only if
  $$a_1(x_1 - y_1) + a_2(x_2 - y_2) + a_3(x_3 - y_3) = 0,$$
  or $\vec{a} \cdot (\vec{x} - \vec{y}) = 0$, where
  $$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Since $\vec{a} \cdot (\vec{x} - \vec{y}) = 0$ if and only if $\vec{a}$ and $\vec{x} - \vec{y}$ are orthogonal, we have that $(x_1, x_2, x_3) \in \mathbb{R}^3$ is a solution for $a_1x_1 + a_2x_2 + a_3x_3 = b$ if and only if $\vec{x} - \vec{y}$ is orthogonal to $\vec{a}$. Thus, the graph of $a_1x_1 + a_2x_2 + a_3x_3 = b$ is the plane of points orthogonal to $\vec{a}$, containing the point (y_1, y_2, y_3).

- Three variables, two equations:
  Now we have two planes, each the graph of one of the equations, in $\mathbb{R}^3$. How can two planes intersect in $\mathbb{R}^3$? They can be the same plane, parallel planes, or intersect in a line. Thus, we have infinite solutions for the system (a plane of solutions or a line of solutions) or no solutions.

- Three variables, three equations:
  This one's nice. There are a lot of ways three planes can intersect in $\mathbb{R}^3$. See Figure 4.2 for a couple of them. The three planes could coincide, yielding a plane of solutions for the system. All three planes could intersect in a line, and all three planes could intersect in a point. Lastly, if any two of the planes are parallel, then there are no solutions for the system.

- Three variables, four or more equations:
  Well, now there are planes shooting all over the place. Good luck drawing this one. It'll probably be harder to make them all intersect, but it's not hard to arrange cases where we have a plane of solutions for the system, a line of solutions for the system, a single point for the solution, or no solution for the system.

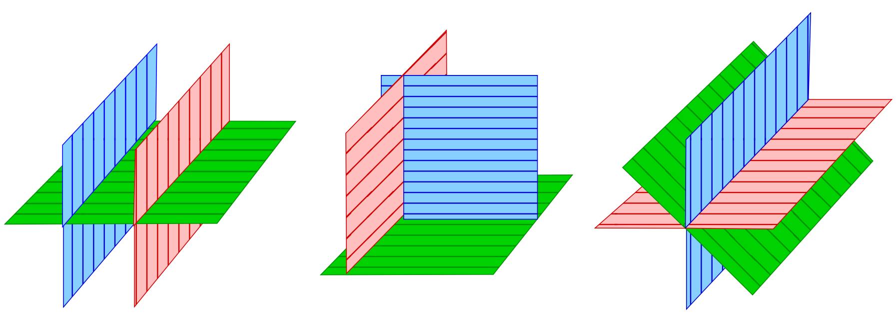


Figure 4.2: There are many ways for three planes in $\mathbb{R}^3$ to intersect.

The Many Dimensions of "Infinitely Many"

We saw before that when a system has a solution and a free variable, there are infinitely many solutions. Yet, in the examples we just looked at, we see that perhaps there is an infinite "line-worth" of solutions, or perhaps there is an infinite"plane-worth" of solutions. Actually, there's a bit more we can say in the case where there are infinitely many solutions.

Theorem 4.2.6 *If a system of m linear equations in n variables has a solution, then the set of solutions is in one-to-one correspondence onto a $k-$dimensional subspace of $\mathbb{R}^n$, where k is the number of free variables in the reduced row-echelon form of the coefficient matrix associated to the system.*

Now, in the case that there are no free variables, this statement reduces naturally to the statement that there is a unique solution. What's really interesting here is that we get a concept of size based on the number of free variable in the situation of infinitely many solutions. The proof of this statement gets a bit off track, so we have moved it to the Appendix. However, we can show you an example that will help illustrate the idea of the proof.

Example 4.2.3 Let's start with a system of equations.

$$\begin{aligned} x_1 + x_2 + 2x_3 + 2x_4 &= 3 \\ 2x_1 + 2x_2 + x_3 + 4x_4 &= 3 \\ 4x_1 + 4x_2 + 2x_3 + 8x_4 &= 6 \end{aligned}$$

Great! Now, let's make that into an augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 1 & 4 & 3 \\ 4 & 4 & 2 & 8 & 6 \end{array}\right]$$

Wonderful! Now let's row reduce to reduced row-echelon form. We'll skip the steps, just to keep this example short.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Stupendous! We can see from this reduced row-echelon form that x_2 and x_4 are free variables and x_1 and x_3 are pivot variables. Moreover, we can write out a parametric solution for this system.

$$\begin{aligned} x_1 &= 1 - x_2 - 2x_4 \\ x_3 &= 1 \\ x_2 &\text{ free} \\ x_4 &\text{ free} \end{aligned}$$

This says the set of solutions to this system is

$$U_{\text{soln}} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4 \colon x_2, x_4 \in \mathbb{R} \right\}.$$

By Theorem 4.2.6, we expect then that there is a 2-dimensional subspace in one-to-one correspondence with this set U_{soln}. The 2-dimensional subspace is actually the kernel of a matrix. Define

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is the matrix obtained by just omitting the last column of the reduced row-echelon augmented matrix. To find $\text{Ker } A$, we need to solve $A\vec{x} = \vec{0}$. Since A is in reduced row-echelon form and the augmented column will be all 0's, we can go straight to a parametric solution.

$$\begin{aligned} x_1 &= -x_2 - 2x_4 \\ x_3 &= 0 \\ x_2 &\text{ free} \\ x_4 &\text{ free} \end{aligned}$$

That is,

$$\text{Ker } A = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4 \colon x_2, x_4 \in \mathbb{R} \right\}.$$

By comparing U_{soln} to Ker A we see that U_{soln} is actually a *shifted* version of Ker A. The one-to-one and onto correspondence is then just the function that shifts Ker A appropriately; specifically, the map $S\colon$ Ker $A \to U_{\text{soln}}$ given by

$$S(\vec{x}) = \vec{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

is the invertible shift function.[16] It follows that geometrically, the solution space, U_{soln}, is a plane that does not go through the origin in $\mathbb{R}^4$.

16: Is there a joke to be made about S being "shifty"?

Ugh. Let's just assume you made it. Well done. Moving on...

Did you two catch that S is not an isomorphism since U_{soln} is not a subspace of $\mathbb{R}^4$?

Exploration 110 Describe the solution space for the system geometrically if possible.

$$\begin{aligned} x_1 + x_2 + 2x_3 + 2x_4 &= 3 \\ 2x_1 + 2x_2 + 4x_4 &= 3 \\ 3x_1 + 3x_2 + 2x_3 + 4x_4 &= 0 \end{aligned}$$

Let's think for a minute about what Theorem 4.2.6 tells us about the kernel of a matrix. Suppose $A \in \mathcal{M}_{m\times n}$. Then finding Ker A is equivalent to solving the matrix equation $A\vec{x} = \vec{0}$. Thus, there is a system of m equations and n unknowns involved here, and Ker A is the solution space for this system.

Corollary 4.2.7 *If $A \in \mathcal{M}_{m\times n}$ row reduces to a matrix with k free variables, then* $\dim \text{Ker } A = k$.

Additionally, we know that if A is a matrix representation for some linear transformation $T\colon V \to W$, then Ker T is isomorphic to Ker A. We also know $\dim \text{Imag } T$ can be determined once $\dim \text{Ker } T$ is known because of the Rank-Nullity Theorem.

Corollary 4.2.8 *Suppose $T\colon V \to W$ is a linear transformation with matrix representation $A \in \mathcal{M}_{m\times n}$. Then $\dim \operatorname{Ker} T$ is the number of free variables identified once A is row reduced and $\dim \operatorname{Imag} T$ is the number of pivot variables.*

Section Highlights

- A system of equations has either one solution, infinitely many solutions, or no solutions. See Corollary 4.2.4.
- The number of solutions for a system of equations can be quickly determined by examining the pivots in the row-echelon form of the corresponding augmented matrix. In particular:
  - If there is a pivot in the augmented column, then the system has no solutions.
  - If there is a pivot in all columns except the final augmented column, then the system has exactly one solution.
  - If neither of the above two situations occur, then the system has infinitely many solutions.

  See Corollary 4.2.5, Theorem 4.2.2, and Theorem 4.2.3.
- If a system has infinitely many solutions, then they can be described parametrically in terms of the free variables. See Definition 4.2.2 and Example 4.2.2.

Exercises for Section 4.2

4.2.1. Determine by inspection whether the system represented by the augmented matrix has no solution, one solution, or infinitely many solutions.

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

(d) $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 2 \\ 0 & 1 & 5 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

(e) $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 2 \\ 0 & 1 & 5 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(f) $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 2 \\ 0 & 1 & 5 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

4.2.2. For each augmented matrix in reduced row-echelon form below, give either the unique solution or the parametric solution.

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(c) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(d) $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 5 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

(e) $\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 1 & 4 \\ 0 & 1 & 5 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(f) $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$

(g) $\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 2 & 0 & 5 & 2 \\ 0 & 1 & 0 & 3 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

4.2.3. Determine the value (or values) of k for which the system will have zero, exactly one, or infinitely many solutions, if possible.

(a) $\begin{array}{rcrcr} 2x & + & ky & = & 6 \\ 3x & + & 6y & = & 0 \end{array}$

(b) $\begin{array}{rcrcr} 2x & + & ky & = & 0 \\ x & + & 4y & = & 0 \end{array}$

(c)
$$\begin{aligned} x - ky + z &= k \\ x + 2y - 2z &= k \\ 2kx + 2y - 2z &= 2k. \end{aligned}$$

4.2.4. Show that for any $k_1, k_2 \in \mathbb{R}$, if $ad - bc \neq 0$, then
$$\begin{aligned} ax + by &= k_1 \\ cx + dy &= k_2 \end{aligned}$$
has a unique solution and state the solution.

4.2.5. Find the line of intersection for the planes given by the equations in the system.

(a)
$$\begin{aligned} -2x - 7y + z &= 12 \\ x + 2y + 4z &= 0. \end{aligned}$$

(b)
$$\begin{aligned} x - y + z &= 3 \\ 3x - 3y &= 6. \end{aligned}$$

4.2.6. Set up an augmented matrix and row reduce to solve the system of equations.

(a)
$$\begin{aligned} x_1 + x_2 &= 4 \\ -x_1 + 2x_2 &= 2 \\ x_1 + x_2 &= 3 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 &= 4 \\ -x_1 + 2x_2 &= 2 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + x_2 &= 4 \\ -x_1 - x_2 &= -4 \end{aligned}$$

(d)
$$\begin{aligned} x_1 + x_3 &= 4 \\ -x_1 + 2x_2 &= 2 \\ 3x_2 + x_3 &= 3 \end{aligned}$$

(e)
$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ -x_1 + 2x_2 + 3x_3 &= 2 \\ x_1 + 3x_2 + x_3 &= 3 \end{aligned}$$

(f)
$$\begin{aligned} x_1 + 5x_2 + 5x_3 &= 5 \\ -x_1 + 2x_2 + 3x_3 &= 2 \\ 2x_1 + 3x_2 + 2x_3 &= 3 \end{aligned}$$

(g)
$$\begin{aligned} x_1 + 5x_2 + 5x_3 + x_4 &= 5 \\ -x_1 + 2x_2 + 3x_3 + x_4 &= 2 \\ 2x_1 + 3x_2 + 2x_3 &= 3 \end{aligned}$$

(h)
$$\begin{aligned} x_1 + 5x_2 + 6x_3 &= 5 \\ -x_1 + 2x_2 + 3x_3 &= 2 \\ 2x_1 + 3x_2 + 3x_3 &= 3 \end{aligned}$$

(i)
$$\begin{aligned} x_1 + 5x_2 + 4x_3 &= 1 \\ -x_1 + 2x_2 + 1x_3 &= 2 \\ 2x_1 + 3x_2 + 3x_3 &= 3 \end{aligned}$$

(j)
$$\begin{aligned} x_1 - 2x_2 + 5x_3 + x_4 + x_5 &= 2 \\ -x_1 + 2x_2 - x_3 + x_4 &= 2 \\ x_1 + 3x_2 + 2x_3 + x_5 &= 3 \end{aligned}$$

(k)
$$\begin{aligned} x_1 + 5x_3 + x_4 + x_5 &= 2 \\ -x_1 + 2x_2 - x_4 &= 2 \\ x_1 + 3x_2 + 2x_3 + x_5 &= 3 \\ x_1 + 2x_4 + x_5 &= 0 \end{aligned}$$

4.3 Matrix Techniques

Gauss-Jordan elimination was a useful tool for finding solutions for a system of equations in the last section. Now, think back to all those times we've needed to solve a system of equations so far. All the work to learn this technique is about to pay off in a huge way.

It should also be noted that technology is often quite useful and helpful when implementing Gauss-Jordan elimination.[17] As with all technology, there are advantages (efficiency!) and disadvantages (potential for inaccuracy!), but it is not difficult to find reasonably simple-to-use matrix calculators.[18]

Linear Independence

If we'd like to determine whether or not a set is linearly independent, there is a matrix we can construct and row reduce to answer this question.

Theorem 4.3.1 *A set of vectors $S = \{\vec{v}_1, \ldots, \vec{v}_n\} \subset \mathbb{R}^n$ is linearly independent if and only if the matrix $A = [\vec{v}_1 \cdots \vec{v}_n]$ row reduces to a matrix with a pivot in every column.*

PROOF. We know that the set S is linearly independent if and only if the only solution to

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0} \tag{4.19}$$

is the trivial solution with $x_1 = x_2 = \cdots = x_n = 0$. Let $A = [\vec{v}_1 \cdots \vec{v}_n]$. We can then rewrite Equation 4.19 as $A\vec{x} = \vec{0}$ where

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

We know this equation has the trivial solution, so whether or not that is the only solution comes down to whether there are free variables. Now, we can row reduce A. By Theorem 4.2.2, we know this will be the only solution if and only if A has a pivot in every column. □

Example 4.3.1 Let us discover if the following set of vectors are linearly independent:

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 2 \end{bmatrix}, \text{ and } \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

Using these vectors as columns in a matrix, we have

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 0 & -1 & 1 \\ -1 & 2 & -2 & 1 \\ 1 & -2 & 2 & -1 \end{bmatrix},$$

17: You mean with robots?
No, not specifically robots.

18: Robots!
No, Ricky! Not robots!

which in reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By Theorem 4.3.1, these vectors are not linearly independent since the last column does not contain a pivot.

Exploration 111 Recall a set of three vectors in $\mathbb{R}^3$ will be a basis if it is linearly independent since $\dim \mathbb{R}^3 = 3$. Use Theorem 4.3.1 to determine whether this set is a basis for $\mathbb{R}^3$. Note that row-echelon form is enough to identify the locations of pivots.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Finding a Basis

In that last exploration, we used row reduction to identify a basis since we already knew the dimension of our space. What about when we don't know the dimension? Well, row reduction can do this for us too. However, we first need to define some terminology to help us out.

Definition 4.3.1 *The ith column in a matrix A is called* ***pivot column*** *if the ith column of the row-echelon form of A contains a pivot. Since the rows and columns of a matrix are each vectors, we will often refer to a pivot column as a vector.*

Note that a pivot column is the original column in the matrix *before* you put the matrix in row-echelon form. Doing Gauss-Jordan elimination will identify, *not yield*, which columns are pivot columns. For example, if you row reduce a matrix A and find that the resulting matrix B has a pivot in the third column. The third column of B is *not* a pivot column of A. However, since B has a pivot in the third column, we know that the third column of A is a pivot column.

Pivot columns, it turns out, are very important.[19]

Theorem 4.3.2 *Let $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a set vectors in $\mathbb{R}^m$. The pivot columns of the matrix $[\vec{v}_1 \cdots \vec{v}_n]$ form a basis for* $\operatorname{Span}\{S\}$.

19: Yes, but what about pea-voh columns?

I think you're supposed to accent the second syllable in pea-*voh*.

PROOF. Let $A = [\vec{v}_1 \cdots \vec{v}_n]$. We can row reduce A to reduced row-echelon form and identify the pivot columns. For reference, we will denote this reduced row echelon form of A by B. From Theorem 4.3.1, we see that in the case where all the columns of A are pivot columns, the set S is linearly independent and forms a basis for $\mathrm{Span}\{S\}$. Suppose now that there are free variables, so that $A\vec{x} = \vec{0}$ or equivalently

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}. \tag{4.20}$$

has infinitely many solutions. Let us re-label the variables in Equation 4.20 so that we sort between pivot and free variables. We will use $y_1, \ldots, y_k$ to denote the pivot variables and $z_1, \ldots, z_l$ to denote the free variables. Then Equation 4.20 can be rearranged as

$$y_1\vec{v}_{y_1} + \cdots + y_k\vec{v}_{y_k} + z_1\vec{v}_{z_1} + \cdots + z_l\vec{v}_{z_l} = \vec{0}. \tag{4.21}$$

Recall that our matrix B is the reduced row-echelon form of A. Thus, each row is either a row of all zeros or it contains a pivot. Since we are trying to determine linear independence, we can augment our matrix A by a column of zeros, and the result for B will remain a column of zeros. We can then use our technique of parametric solutions to write each pivot variable as a linear combination of the free variables. That is, we can write

$$\begin{aligned} y_1 &= c_{11}z_1 + \cdots + c_{1l}z_l \\ &\vdots \\ y_k &= c_{k1}z_1 + \cdots + c_{kl}z_l \end{aligned}$$

for some real numbers c_{ij} determined by the entries of B. Now, any choice for the free variables gives one specific solution in our solution space for $A\vec{x} = \vec{0}$. If we choose $z_1 = -1$ and $z_2 = \cdots = z_l = 0$, then $y_1, \ldots, y_k$ are determined and give a solution to

$$y_1\vec{v}_{y_1} + \cdots + y_k\vec{v}_{y_k} - \vec{v}_{z_1} = \vec{0}.$$

This rearranges to give us

$$y_1\vec{v}_{y_1} + \cdots + y_k\vec{v}_{y_k} = \vec{v}_{z_1}.$$

So the free variable vector $\vec{v}_{z_1}$ can be written as a linear combination of the pivot column vectors $\vec{v}_{y_1}, \cdots, \vec{v}_{y_k}$. The same can be done with a similar choice for the free variables to solve for each of the vectors $\vec{v}_{z_i}$. Thus, the pivot column vectors $\vec{v}_{y_1}, \cdots, \vec{v}_{y_k}$ are a spanning set for $\mathrm{Span}\{S\}$.

To be a basis, we must also show that they are linearly independent. The easiest way to confirm this is to form the new matrix $C = [\vec{y}_1 \cdots \vec{y}_k]$ and perform the same row operations on C as those used to reduce A to B. Because of the definition of pivot columns, we will have a matrix with a pivot in every column, so the vectors $\{\vec{y}_1, \ldots, \vec{y}_k\}$ are linearly independent by Theorem 4.3.1. Therefore, the pivot columns are a linearly independent spanning set, a.k.a. a basis for $\mathrm{Span}\{S\}$. $\square$

Example 4.3.2 Let's use this theorem to find a basis for a subspace. Let

$$S = \left\{ \begin{bmatrix} 1 \\ 4 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Now, we can use a matrix and row reduction to find a basis for $\operatorname{Span}\{S\}$.

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 0 \\ 4 & 0 & 4 & 1 & 5 & 0 \\ 3 & -1 & 2 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 2 & 1 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Wow, we row reduced that one fast! Okay, fine. We just omitted all of the steps. If you're looking for some additional row reduction practice, this would be a good one to try since you already have the answer. Now, let's talk about our basis for $\operatorname{Span}\{S\}$. We see that columns 1, 2, 4, and 6 are the pivot columns, so

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 4 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis for $\operatorname{Span}\{S\}$.

Exploration 112 Use the technique from the previous example to find a basis for

$$\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -6 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

One place we have needed to find a basis for a subspace in the past is when computing the column space of a matrix. The concept of pivot columns helps here naturally.

Example 4.3.3 Consider the matrix A defined below.

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 & 1 \\ 3 & 1 & 2 & 1 & 2 \\ 3 & 1 & 2 & 1 & 2 \\ 1 & 0 & 2 & 0 & 1 \end{bmatrix}$$

Then we can row reduce to get

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -4 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Based on pivot columns, we see then that

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

forms a basis for $\text{Col } A$. Recall that $\text{Col } A = \text{Imag } T_A$, where T_A is the linear transformation induced by the matrix A. Thus, this gives us a way to find the image of a linear transformation using matrix representations as well.

We've seen now that row reducing a matrix can help us find a basis for the span of a set of vectors. However, what would that look like when the set of vectors is a spanning set for the vector space? Well, to be a spanning set, the expected number of basis vectors identified would be at least the dimension of the vector space. Since the dimension of the vector space is the same as the number of rows in the matrix, we would see a pivot in every row of the corresponding matrix, since the number of pivot columns must match the dimension of the vector space.

Corollary 4.3.3 *Suppose $S = \{\vec{v}_1, \ldots, \vec{v}_m\}$ is a subset of $\mathbb{R}^n$. If the matrix $A = [\vec{v}_1 \ \ldots \ \vec{v}_m]$ has a pivot in every row when row reduced, then S is a spanning set for $\mathbb{R}^n$.*

Linear Combinations

We can actually do better than just determining whether vectors are linearly dependent or not. If they are in fact linearly dependent, this matrix method provides a nice way to write one of the vectors as a linear combination of the others. To see this, let's look at an example.

Example 4.3.4 Let's just use the matrix from Example 4.3.2 for this one.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 0 \\ 4 & 0 & 4 & 1 & 5 & 0 \\ 3 & -1 & 2 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 2 & 1 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In our proof for Theorem 4.3.2, we saw that there was a way to solve for any free variable vector in terms of the pivot columns by choosing -1 for one free variable and 0 for the rest. Let's see what happens when we do this here. Using the convention of y's for pivot variables and z's for free variables as in the proof, our parametric solution for $A\vec{x} = \vec{0}$ becomes

$$y_1 = -z_1 - z_2 \tag{4.22}$$
$$y_2 = -z_1 \tag{4.23}$$
$$y_3 = -z_2 \tag{4.24}$$
$$y_4 = 0 \tag{4.25}$$

where y_1, y_2, y_3, and y_4 correspond to the pivot columns $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_4$, and $\vec{v}_6$ and z_1 and z_2 correspond to the free variable columns $\vec{v}_3$ and $\vec{v}_5$. Choosing $z_1 = -1$ and $z_2 = 0$ gives us a way to write $\vec{v}_3$ as a linear combination of the pivot columns, and choosing $z_1 = 0$ and $z_2 = -1$ gives us a way to write $\vec{v}_5$ as a linear combination of the pivot columns. Specifically, we have

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2$$
$$\vec{v}_5 = \vec{v}_1 + \vec{v}_4.$$

Because of how nice reduced row-echelon form is, we can actually see the outcomes of these directly from the reduced free variable column.

Exploration 113 Form the same matrix as from Example 4.3.3 and then use the entries in the reduced free variable columns to write the free variable column vectors as linear combinations of the pivot columns. Check your answer by computing the linear combinations with the vectors.

Exploration 114 The following set of vectors are clearly linearly dependent. Set up a matrix and row reduce to find a way to write one of the vectors as a linear combination of the others.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Coordinate Vectors

Now that we are talking about how matrices allow us to write vectors as linear combinations of basis vectors, we should really say something about coordinate vectors, too.

Recall that if we have a basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ for a vector space V, then for any vector $\vec{v} \in V$, we can write $\vec{v}$ as a unique linear combination of the vectors in $\mathcal{B}$; that is, $\vec{v} = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n$ for some scalars $a_1, \ldots, a_n$. These scalars are the coordinates for $\vec{v}$ relative to $\mathcal{B}$:

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

We now have a very convenient method for finding these coordinates. We want to solve the vector equation

$$a_1\vec{b}_1 + \cdots + a_n\vec{b}_n = \vec{v} \tag{4.26}$$

for the scalars $a_1, \ldots, a_n$. If $V = \mathbb{R}^n$, then this is equivalent to the matrix equation and augmented matrix,

$$\begin{bmatrix} \vec{b}_1 \cdots \vec{b}_n \end{bmatrix} [\vec{v}]_{\mathcal{B}} = \vec{v} \quad \text{and} \quad \begin{bmatrix} \vec{b}_1 \cdots \vec{b}_n \mid \vec{v} \end{bmatrix},$$

respectively. This is also equivalent to a system of equations, but at this point, we suspect solving by way of the augmented matrix is the preferred method.

Example 4.3.5 Let

$$H = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 2 \end{bmatrix} \right\}.$$

From the row reduction in Example 4.3.1 and Theorem 4.3.2, we know these three vectors

$$\mathcal{B} = \left\{ \vec{b}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 0 \\ -1 \\ -2 \\ 2 \end{bmatrix} \right\}.$$

form a basis for H. Let's find the coordinates for

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

relative to $\mathcal{B}$. That is, we want to find scalars a_1, a_2, a_3 where

$$\vec{v} = a_1\vec{b}_1 + a_2\vec{b}_2 + a_3\vec{b}_3, \text{ so that } [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

This vector equation can be written as an augmented matrix and its reduced row-echelon form

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & 2 \\ 2 & 0 & -1 & 1 \\ -1 & 2 & -2 & 1 \\ 1 & -2 & 2 & -1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

From this we see that $\vec{v} = -\vec{b}_1 - 3\vec{b}_2 - 3\vec{b}_3$, so $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -3 \\ -3 \end{bmatrix}$.

Exploration 115 Consider the basis $\mathcal{B}_0 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^3$.

Write the vector $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ as a linear combination of these basis vectors.

What about if we want to find multiple coordinate vectors with respect to the same basis? This was what we had to do when finding the matrix representations for a linear transformation with respect to a non-standard basis. Well, we can augment with multiple vectors at once!

Example 4.3.6 Again consider the basis

$$\mathcal{B}_0 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ of } \mathbb{R}^3.$$

Suppose we have a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \\ T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \text{ and} \\ T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \end{aligned}$$

In order to find the matrix representation for T with respect to the basis $\mathcal{B}_0$, we need now to convert the outputs given here into coordinate vectors for $\mathcal{B}_0$. Instead of row-reducing 3 separate times, we can do this all at once.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & -1 & 0 \\ 1 & 0 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -1 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 1 \end{array}\right]$$

Then

$$A = \begin{bmatrix} -4 & -1 & 0 \\ 3 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

is the matrix representation for T with respect to the basis $\mathcal{B}_0$.

Lastly, can row reduction help us when our vector space is not $\mathbb{R}^n$ for some positive integer n? Of course it can. The key is finding coordinate vectors, and it's best to choose the standard basis since it's easier to work with. From there, we could apply any technique we want from the $\mathbb{R}^n$ setting and then translate our results back.

Example 4.3.7 Let's determine the coordinates for $\vec{p} = 2x^3 - 4x + 5$ relative to the vectors in the basis

$$\mathcal{B} = \left\{1, x, x^2 - 1, x^3 - 3x\right\}$$

for $\mathbb{P}_3$. Righto. We just need to find scalars $a_1, \ldots, a_4$ such that

$$a_1(1) + a_2(x) + a_3(x^2 - 1) + a_4(x^3 - 3x) = 2x^3 - 4x + 5.$$

Maybe that sounds like fun to you. If it does, great! There is, however, another way...

Recall that the standard basis for $\mathbb{P}_3$ is

$$\mathcal{E} = \left\{1, x, x^2, x^3\right\},$$

so we can write

$$[1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [x]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\left[x^2 - 1\right]_{\mathcal{E}} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \left[x^3 - 3x\right]_{\mathcal{E}} = \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

Similarly,

$$[\vec{p}]_{\mathcal{E}} = \left[2x^3 - 4x + 5\right]_{\mathcal{E}} = \begin{bmatrix} 5 \\ -4 \\ 0 \\ 2 \end{bmatrix}.$$

Now we can turn the problem of finding scalars such that

$$a_1\,[1]_{\mathcal{E}} + a_2\,[x]_{\mathcal{E}} + a_3\left[x^2 - 1\right]_{\mathcal{E}} + a_4\left[x^3 - 3x\right]_{\mathcal{E}} = \left[2x^3 - 4x + 5\right]_{\mathcal{E}},$$

or

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 0 \\ 2 \end{bmatrix}.$$

As we've seen, we can write this as the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 5 \\ 0 & 1 & 0 & -3 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right],$$

which, in reduced row-echelon form, is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right].$$

Thus, $a_1 = 5, a_2 = 2, a_3 = 0$, and $a_4 = 2$. Note that along the way we also verified that $\mathcal{B}$ is a basis for $\mathbb{P}_3$ since the columns in the reduced matrix corresponding to $\mathcal{B}$ all contain pivots and the set's size matches the dimension of $\mathbb{P}_3$.

Kernel of a Matrix

We've already discussed this in some examples in the previous section, but it's important enough to revisit. By definition,

$$\text{Ker } A = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$$

for any $A \in \mathcal{M}_{m\times n}$. Now, solving $A\vec{x} = \vec{0}$ for $\vec{x}$ can be done using the augmented matrix representation of the matrix equation.

Example 4.3.8 Let's find $\text{Ker } A$ where

$$A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 0 & -1 & 1 \\ -1 & 2 & -2 & 1 \\ 1 & -2 & 2 & -1 \end{bmatrix}.$$

Then we row reduce this to get

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since we are solving $A\vec{x} = \vec{0}$, the column we would augment would only be a column of zeros, and such a column is never changed by row operations. Thus, we can omit the augmentation when solving $A\vec{x} = \vec{0}$. To find $\text{Ker } A$, we write the parametric solution.

$$\begin{aligned} x_1 &= x_4 \\ x_2 &= 3x_4 \\ x_3 &= 3x_4 \\ x_4 &\text{ free} \end{aligned}$$

Now, we can convert this to a set of solutions. In this case, we know the set of solutions will actually be a subspace since we know $\text{Ker } A$ is always a subspace. Thus,

$$\text{Ker } A = \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} x_4 \colon x_4 \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

Recall that for any linear transformation $T\colon V \to W$, we can find a matrix representation A. Then $\text{Ker } A$ can be translated by a coordinate mapping into $\text{Ker } T$, and thus, computing $\text{Ker } A$ can be an essential step for finding $\text{Ker } T$.

Example 4.3.9 Suppose $T\colon \mathbb{P}_3 \to \mathbb{R}^4$ is defined by

$$T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a - b + 2d \\ 2a - c + d \\ -a + 2b - 2c + d \\ a - 2b + 2c - d \end{bmatrix}.$$

Then the matrix representation with respect to the standard bases for $\mathbb{P}_3$ and $\mathbb{R}^4$ is the familiar

$$A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 0 & -1 & 1 \\ -1 & 2 & -2 & 1 \\ 1 & -2 & 2 & -1 \end{bmatrix}.$$

From our previous example, we know

$$\text{Ker } A = \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} x_4 \colon x_4 \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

Now, to translate this to $\text{Ker } T$, we need to view these as coordinate vectors with respect to our standard basis for $\mathbb{P}_3$. Thus, $\text{Ker } T = \text{Span}\left\{1 + 3x + 3x^2 + x^3\right\}$.

Exploration 116 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_3 \\ x_3 \end{bmatrix}.$$

Find the matrix representation A for the linear transformation T on the standard basis vectors for $\mathbb{R}^3$. Use this matrix to find a basis for both $\text{Imag } T$ and $\text{Ker } T$.

Section Highlights

- Any column of a matrix, A, that has a pivot when row reduced is called a pivot column. Note that the pivot column is a column in the original matrix A, not its row reduced form. See Definition 4.3.1.
- A set of vectors in $\mathbb{R}^n$ is linearly independent if and only if the matrix with those vectors as columns has a pivot in every column. See Theorem 4.3.1.
- A set of vectors in $\mathbb{R}^n$ is a spanning set for $\mathbb{R}^n$ if and only if the matrix formed using those vectors as columns has a pivot in every row when row reduced. See Corollary 4.3.3.
- Suppose $H = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a subspace of $\mathbb{R}^m$. The set of pivot columns of $A = [\vec{v}_1 \ \ldots \ \vec{v}_n]$ is a basis for H. See Theorem 4.3.2.
- The set of pivot columns of a matrix, A, is a basis for $\text{Col } A$, which is the image of the linear transformation T_A. See Example 4.3.3.
- The coordinate vector for a vector $\vec{x}$ with respect to a basis $\mathcal{B}$ can be computed in the following manner:
  - Use the vectors in $\mathcal{B}$ as the columns of a matrix and augment it with $\vec{x}$.
  - Row reduce to reduced row-echelon form.
  - Read the coordinate vector for $\vec{x}$ from the augmented column.

  See Example 4.3.5 and Example 4.3.7
- Row operations do not change the kernel of a matrix. Thus, to find the kernel, row reduce to reduced row-echelon form; the parametric solution to the row reduce matrix augmented with the zero vector describes the kernel. See Example 4.3.8.

Exercises for Section 4.3

4.3.1. Determine whether the set is linearly dependent or linearly independent. If they are linearly dependent, set up a matrix and row reduce to find a way to write one of the vectors as a linear combination of the others.

(a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

(f) $\left\{ \begin{bmatrix} 2 \\ 0 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 6 \\ 0 \end{bmatrix} \right\}$

(g) $\left\{ \begin{bmatrix} 2 \\ 0 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

4.3.2. Use the concept of pivot columns to determine whether the set is a basis for $\mathbb{R}^3$.

(a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

4.3.3. Use the concept of pivot columns to determine a basis for each subspace H.

(a) $H = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(b) $H = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) $H = \operatorname{Span}\left\{1 + x, 2 - x + x^4, x^2 - 3x^3, 1 - 2x + x^4\right\}$

(d) $H = \operatorname{Span}\left\{1 + x + x^4, 1 - 2x, 1 + x^2 - 3x^3, 2 - x + x^4\right\}$

(e) $H = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

4.3.4. Consider the basis $\mathcal{B}_0 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^3$. Write the vector $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of these basis vectors.

4.3.5. Consider the basis $\mathcal{B}_1 = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$ of $\mathbb{R}^3$. Write the vector $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ as a linear combination of these basis vectors.

4.3.6. Consider the basis $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^3$. Find the coordinate vectors below.

(a) $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}_{\mathcal{B}_2}$

(b) $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}_{\mathcal{B}_2}$

(c) $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}_{\mathcal{B}_2}$

(d) $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}_{\mathcal{B}_2}$

4.3.7. Use the method from Example 4.3.6 to find the matrix for T with respect to the given basis $\mathcal{B}$.

(a) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_3 \\ 3x_1 + 3x_2 \end{bmatrix} \text{ with respect to the basis } \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_3 \\ 3x_1 + 3x_2 + x_3 \end{bmatrix} \text{ with respect to the basis } \mathcal{B} = \left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(c) $T\colon \mathbb{P}_2 \to \mathbb{P}_2$ be defined by

$T\left(a_0 + a_1x + a_2x^2\right) = (a_0 + a_1) + (a_2)x^2$ with respect to the basis $\mathcal{B} = \left\{1, 1 + x, 1 + x + x^2\right\}$

4.3.8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 - x_3 \\ x_2 + x_3 \\ x_3 \end{bmatrix}.$$

Find the matrix A that represents T on the standard basis vectors for $\mathbb{R}^3$. Use this matrix to compute $\operatorname{Imag} T$ and $\operatorname{Ker} T$.

4.3.9. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_3 \\ x_3 \end{bmatrix}.$$

Find the matrix A that represents T on the standard basis vectors for $\mathbb{R}^3$. Use this matrix to compute $\operatorname{Imag} T$ and $\operatorname{Ker} T$.

4.3.10. Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be defined by

$$T\left(a_0 + a_1x + a_2x^2\right) = a_0x + 3a_0x^2 + (a_1 + a_2)x^3.$$

Find the matrix A that represents T on the standard basis vectors for $\mathbb{P}_2$ and $\mathbb{P}_3$. Use this matrix to compute $\operatorname{Imag} T$ and $\operatorname{Ker} T$.

4.3.11. Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be defined by

$$T\left(a_0 + a_1x + a_2x^2\right) = \begin{bmatrix} a_0 + a_1 + a_2 \\ 2a_0 - a_2 \end{bmatrix}.$$

Find the matrix A that represents T on the standard basis vectors for $\mathbb{P}_2$ and $\mathbb{R}^2$. Use this matrix to compute $\operatorname{Imag} T$ and $\operatorname{Ker} T$.

4.4 Matrix Operations

Where are we now?[20] We've introduced matrices as a convenient way to represent linear transformations. We've also developed a systematic way to simplify them, row operations and Gauss-Jordan Elimination, and seen that since this helps us solve systems of equations, it is essentially an alternative method for doing everything we've learned so far. Well, except showing a set's a vector space. Row operations are no help there. That does remind us though that we should talk about vector space-style operations on matrices. That's where we are.

20: As a civilization?

I think I have a map here somewhere. . .

Oh! The margin! We are in the margin.

Addition, Scalar Multiplication, and Matrix/Vector Multiplication

We've made a big deal so far about how a matrix can be used to represent a linear transformation. Now, a linear transformation is a function, and an operation we have for two functions with the same domain and codomain is function addition.

Example 4.4.1 Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 3$ and $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x - 4$. There is then a function

$$f + g\colon \mathbb{R} \to \mathbb{R}$$

defined by

$$(f + g)(x) = f(x) + g(x) = (x + 3) + (x - 4) = 2x - 1.$$

This is likely something you've seen before. Now, we haven't discussed whether adding two linear transformations with the same domain and codomain should give us a new linear transformation. It does; we just haven't mentioned it before. We can also scale a linear transformation to obtain a new linear transformation. Maybe we should say this using official-sounding words.

Theorem 4.4.1 *Let $T\colon V \to W$ and $S\colon V \to W$ be linear transformations.*

(a) *Then $T + S\colon V \to W$ defined using the usual function addition is also a linear transformation.*

(b) *Let $k \in \mathbb{R}$. Then $kT\colon V \to W$ defined by scaling each output of the linear transformation T by k is also a linear transformation.*

The proof of this gets us a bit off-track, so we've put it in the exercises. You do remember how to show a function is a linear transformation, right?

Let's get back to talking about matrices. We've just defined two operations on linear transformations. Since we've made it quite clear that matrices and linear transformations are deeply linked, there should be analogous operations for matrices. We call these componentwise addition and scalar multiplication.

First, we can illustrate this with 2×2 matrices.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \quad \text{and}$$

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix},$$

for some scalar $k \in \mathbb{R}$. Generally, we have the following definition.

Definition 4.4.1 ► *The sum of two $m \times n$ matrices*[21] *$[a_{ij}]$ and $[b_{ij}]$ is the $m \times n$ matrix $[c_{ij}]$ whose entries satisfy $c_{ij} = a_{ij} + b_{ij}$.*

► *The scalar multiple of a matrix $[a_{ij}]$ by a scalar $k \in \mathbb{R}$ is the $m \times n$ matrix $[d_{ij}]$ whose entries satisfy $d_{ij} = ka_{ij}$.*

21:  *They must have the same dimensions!*

Exploration 117 Use the definitions above to compute these:

► $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & -2 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 1 \\ -1 & -2 & 4 \end{bmatrix} =$

► $3 \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & -2 \end{bmatrix} =$

We've saved matrices for the latter half of the book, but we could have discussed them back in Section 1.1 because we can also think of them as vectors. For example, observe that

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \in \mathcal{M}_{2\times 3} \quad \text{and} \quad \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} \in \mathbb{R}^6$$

contain the same information. Our operations of vector addition and scalar multiplication in this case would even match this new componentwise addition and scalar multiplication for matrices. Thus, the following theorem shouldn't be a surprise.

Theorem 4.4.2 *The set of $m \times n$ matrices, denoted $\mathcal{M}_{m\times n}$, is a vector space with componentwise addition and scalar multiplication.*

We could further explore the identification between matrices in $\mathcal{M}_{m\times n}$ and vectors in $\mathbb{R}^{mn}$ to actually form an isomorphism and prove this theorem. Moveover, once we've agreed $\mathcal{M}_{m\times n}$ is a vector space, we can get the following corollary.

Corollary 4.4.3 *The set of linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ is a vector space with function addition and scalar multiplication.*

The statement follows because we could map any matrix A to its induced linear transformation T_A and get an isomorphism. This game of declaring something is a vector space because we can define an isomorphism seems like a lot of fun,[22] but let's get back to matrices.

While we didn't previously talk about how matrices form a vector space, we did define a way to multiply a matrix $A = [\vec{a}_1 \cdots \vec{a}_n]$ and a vector $\vec{x} \in \mathbb{R}^n$:

$$A\vec{x} = [\vec{a}_1 \cdots \vec{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n.$$

We've gotten a lot of mileage out of this definition; most notably, it allowed us to think of matrices as functions, which paved the way for using matrices as representations for linear transformations. We should "jump start" something to complete the lap of car metaphors.[23]

Matrix Multiplication

The next logical operation on matrices would be multiplication of two matrices.[24] As with vectors, there are a lot of ways to think about a product of matrices, but a particularly useful and instructive one comes from function composition, an essential tool in function theory.[25] We saw in Theorem 3.2.3 that the composition of two linear transformations is again a linear transformation.

Just as we just did with addition and scalar multiplication, we'd like to translate this operation to matrices. Suppose $A = [\vec{a}_1 \cdots \vec{a}_n] \in \mathcal{M}_{m\times n}$ and $B = [\vec{b}_1 \cdots \vec{b}_p] \in \mathcal{M}_{n\times p}$; we have induced linear transformations $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ and $T_B\colon \mathbb{R}^p \to \mathbb{R}^n$. From Theorem 3.2.3, we know the composition $T_A \circ T_B\colon \mathbb{R}^p \to \mathbb{R}^m$ forms a linear transformation. How can the matrices A and B to reflect this?

Let $\vec{x} \in \mathbb{R}^p$. Then we have

$$\begin{aligned} (T_A \circ T_B)(\vec{x}) &= T_A\left(T_B(\vec{x})\right) && \text{by definition of composition} \\ &= T_A(B\vec{x}) && \text{since } T_B(\vec{x}) = B\vec{x} \\ &= A(B\vec{x}) && \text{since } B\vec{x} \in \mathbb{R}^n \\ & && \text{and } T_A(\vec{u}) = A\vec{u} \text{ for all } \vec{u} \in \mathbb{R}^n. \end{aligned}$$

Thus, for composition to agree with matrix multiplication, we need to have our matrix product AB satisfy $(AB)\vec{x} = A(B\vec{x})$. Calculating the expression on the right hand side of this, we have

$$\begin{aligned} A(B\vec{x}) &= A\left(x_1\vec{b}_1 + \cdots x_n\vec{b}_n\right) \\ &= x_1A\vec{b}_1 + \cdots + x_nA\vec{b}_n \;=\; [A\vec{b}_1 \cdots A\vec{b}_n]\vec{x}. \end{aligned}$$

This then dictates precisely what our matrix product should be.

Definition 4.4.2 *Let $A \in \mathcal{M}_{m\times n}$ and $B = [\vec{b}_1 \cdots \vec{b}_p] \in \mathcal{M}_{n\times p}$. Then we define the* ***product of matrices*** *A* ***and*** *B to be the matrix $AB \in \mathcal{M}_{m\times p}$*

22: Because it is!

23: or perhaps. . . we should tap on the brakes?

Oh, well done, Bubbles!

24: Recall from Chapter 1 that this was complicated for vectors.

25: Recall from Chapter 3 that we needed function composition to make inverses.

given by

$$AB = A[\vec{b}_1 \cdots \vec{b}_p] = [A\vec{b}_1 \cdots A\vec{b}_p].$$

Defining the product AB in this fashion gives us a natural way to have a matrix representation for a composition of linear transformations.

Theorem 4.4.4 *Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{n\times p}$, and let $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ and $T_B\colon \mathbb{R}^p \to \mathbb{R}^n$ be the induced linear transformations. Then the matrix $AB \in \mathcal{M}_{m\times p}$ has induced linear transformation $T_{AB} = T_A \circ T_B$.*

PROOF. Note first that the product AB makes sense based on the dimensions of the matrices A and B. This is equivalent to checking that the composition $T_A \circ T_B$ also maps between the appropriate spaces.[26] Let $\vec{x} \in \mathbb{R}^p$. Using the definition of matrix multiplication and our definition for the linear transformation induced by a matrix, we have $T_{AB}(\vec{x}) = (AB)\vec{x} = A(B\vec{x}) = A(T_B(\vec{x})) = T_A(T_B(\vec{x})) = T_A \circ T_B(\vec{x})$.[27] □

26: Exercise!

27: You will also justify each of these equalities as an exercise.

Example 4.4.2 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}.$$

Let's calculate AB. To make it clear what's happening, let's also shade the second column of B.

$$\begin{aligned}
AB = A & \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} \\
= & \begin{bmatrix} A\begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} & A\begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} \end{bmatrix} \\
= & \begin{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} \end{bmatrix} \\
= & \begin{bmatrix} 7\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 9\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 11\begin{bmatrix} 3 \\ 6 \end{bmatrix} & 8\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 10\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 12\begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{bmatrix} \\
= & \begin{bmatrix} 7+18+33 & 8+20+36 \\ 28+45+66 & 32+50+72 \end{bmatrix} \\
= & \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}.
\end{aligned}$$

Let's calculate BA. Hey, three columns in A? We'll shade the middle column to help keep things clear.

$$BA = B\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} B\begin{bmatrix} 1 \\ 4 \end{bmatrix} & B\begin{bmatrix} 2 \\ 5 \end{bmatrix} & B\begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} & \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} & \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1\begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} + 4\begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} & 2\begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} + 5\begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} & 3\begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} + 6\begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 7+32 & 14+40 & 21+48 \\ 9+40 & 18+50 & 27+60 \\ 11+48 & 22+60 & 33+72 \end{bmatrix}$$

$$= \begin{bmatrix} 39 & 54 & 69 \\ 49 & 68 & 87 \\ 59 & 82 & 105 \end{bmatrix}.$$

You probably noticed that $AB \neq BA$. Lots of types of multiplication are commutative, but evidently, matrix multiplication is not. Matrix multiplication is so badly not commutative, the product AB and the product BA are not even the same dimension!

Exploration 118 Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

Calculate AB and BA and verify that matrix multiplication does not commute.

Matrix Multiplication via Transpose

There is another way to think about this matrix multiplication that is often used. We will first need a new definition.

Definition 4.4.3 *Let $A \in \mathcal{M}_{m\times n}$. The **transpose of** A, denoted A^T, is the matrix in $\mathcal{M}_{n\times m}$ derived from A by making the jth column of A into the jth row for each $1 \leq j \leq n$.*

Example 4.4.3 Let

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$$

Then

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}.$$

Now that we have that, we make the following observation:

$$\mathcal{M}_{n\times 1} = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R} \right\} = \mathbb{R}^n,$$

and

$$\mathcal{M}_{1\times n} = \{[x_1 \cdots x_n] : x_i \in \mathbb{R}\} = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^T : x_i \in \mathbb{R} \right\} \cong \mathbb{R}^n.$$

[28] Clearly, the concept of a matrix transpose applies on a single row or column vector because they are matrices. This gives us the following fun interpretation of matrix multiplication.

28:  I always write my vectors horizontally with $\langle , \rangle$ brackets, so it is *your* weird set that is isomorphic to $\mathbb{R}^n$.

Theorem 4.4.5 *Let $A \in \mathcal{M}_{m\times p}$ and $B \in \mathcal{M}_{p\times n}$. Then the product $AB \in \mathcal{M}_{m\times n}$ is the matrix whose entry in the ith row and jth column is the inner product of the transpose of the ith row of A with the jth column of B for all $1 \le i \le m$ and $1 \le j \le n$. That is,*

$$AB = [(ab)_{ij}], \quad \text{where} \quad (ab)_{ij} = \vec{r}_i^{\,T} \cdot \vec{b}_j,$$

where $\vec{r}_i$ is the ith row of A and $\vec{b}_j$ is the jth column of B.

Exploration 119 We will prove Theorem 4.4.5 together! First, let $\vec{x} \in \mathbb{R}^p$ and write out $A\vec{x}$ as a *single* vector

Now convince yourself that for any vector $\vec{x} \in \mathbb{R}^p$, we have

$$A\vec{x} = \begin{bmatrix} \vec{r}_1^{\,T} \cdot \vec{x} \\ \vdots \\ \vec{r}_m^{\,T} \cdot \vec{x} \end{bmatrix},$$

where $\vec{r}_i$ are the row vectors of A for $1 \le i \le m$.

Now use the fact that $AB = [A\vec{b}_1 \cdots A\vec{b}_n]$ to complete the proof.

Example 4.4.4 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}.$$

Let's calculate AB using this new method. Note first that we have

$$\vec{r}_1^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{r}_2^T = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \vec{b}_1 = \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix}, \quad \text{and} \quad \vec{b}_2 = \begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix}$$

Then

$$\begin{aligned} AB &= \begin{bmatrix} \vec{r}_1^T \cdot \vec{b}_1 & \vec{r}_1^T \cdot \vec{b}_2 \\ \vec{r}_2^T \cdot \vec{b}_1 & \vec{r}_2^T \cdot \vec{b}_2 \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} \\ \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} & \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}. \end{aligned}$$

Properties of Matrix Multiplication

Like any other respectable mathematical operation, matrix multiplication has some nice properties. We've already discovered the deplorable fact that matrix multiplication is not commutative, and that makes everybody very sad.[29] Before we celebrate some things that matrix multiplication actually does well, we need a definition:

29:  Except the algebraists specializing in non-commutative algebras.

Definition 4.4.4 *The* ***identity matrix*** *is the square matrix $I_n \in \mathcal{M}_{n\times n}$ whose columns are the standard basis for $\mathbb{R}^n$ in order. That is,*

$$I_n = [\vec{e}_1\ \vec{e}_2\ \cdots\ \vec{e}_n] = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Note that the identity matrix corresponds to the identity map for a vector space, the linear transformation that sends each vector to itself. Formally, $T_I = T$, where $T\colon V \to V$ is given by $T(\vec{x}) = \vec{x}$ for all $\vec{x} \in V$. Now, commence celebration!

Theorem 4.4.6 *Let $k \in \mathbb{R}$, and $A \in \mathcal{M}_{m\times n}$, and let B and C be matrices of the size necessary for each of the following operations to be well-defined. Then*

(a) $(AB)C = A(BC)$ *(associativity of matrix multiplication)*
(b) $A(B+C) = AB + AC$ *(left distribution of matrix multiplication)*
(c) $(B+C)A = BA+CA$ *(right distribution of matrix multiplication)*
(d) $k(AB) = (kA)B = A(kB)$
(e) $I_m A = A = AI_n$ *(identity for matrix multiplication)*

Those look wonderful! Unfortunately, their proofs do not. They are not difficult, but the amount of notation and page space required makes them an eyesore we would rather avoid. Let's verify one of these for 2×2 matrices just so we don't feel quite so bad about cheating you out of the glorious satisfaction of a thorough proof.[30]

30: I always knew you authors were dirty cheaters! How dare you not include the proof?! I know Nicky, for one, will never forgive you.

Umm. . . I know the proof here, and I'm actually good with the fact that they chose not to show it.

Really? I'm not. I demand satisfaction!

Example 4.4.5 Let's start with some 2×2 matrices.

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \quad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \quad C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$$

Now, we will verify $A(B+C) = AB + AC$. This one should give us a nice flavor for what all of the proofs would have looked like.

$$\begin{aligned}
A(B+C) &= \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \left(\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} + \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \right) \\
&= \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} b_1 + c_1 & b_2 + c_2 \\ b_3 + c_3 & b_4 + c_4 \end{bmatrix} \\
&= \begin{bmatrix} a_1(b_1+c_1) + a_2(b_3+c_3) & a_1(b_2+c_2) + a_2(b_4+c_4) \\ a_3(b_1+c_1) + a_4(b_3+c_3) & a_3(b_2+c_2) + a_4(b_4+c_4) \end{bmatrix} \\
&= \begin{bmatrix} a_1b_1 + a_1c_1 + a_2b_3 + a_2c_3 & a_1b_2 + a_1c_2 + a_2b_4 + a_2c_4 \\ a_3b_1 + a_3c_1 + a_4b_3 + a_4c_3 & a_3b_2 + a_3c_2 + a_4b_4 + a_4c_4 \end{bmatrix} \\
&= \begin{bmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{bmatrix} + \begin{bmatrix} a_1c_1 + a_2c_3 & a_1c_2 + a_2c_4 \\ a_3c_1 + a_4c_3 & a_3c_2 + a_4c_4 \end{bmatrix} \\
&= AB + AC
\end{aligned}$$

Now that you've seen Example 4.4.5, you can imagine how the general proofs for these statements go.[31]

31: I take back that dirty cheater comment. Thank you for omitting these.

There's something cool we get as a consequence of these properties.

Corollary 4.4.7 *Let $A \in \mathcal{M}_{n\times m}$. There is a linear transformation*

$$M_A\colon \mathcal{M}_{m\times p} \to \mathcal{M}_{n\times p}$$

defined by $M_A(B) = AB$ for any $B \in \mathcal{M}_{m\times p}$.

The proof of this is a wonderful application of Theorem 4.4.6 that you should expect to see as an exercise.

Properties of the Transpose

As part of our second definition of matrix multiplication, we introduced the concept of the transpose of a matrix. This concept will actually have some nice connections to applications, so we should say a bit more about it. Recall that computing the transpose of a matrix is done by taking a row vector and making it a column vector.[32] There are several nice properties of this operation.

32:  Or vice-versa, right?

Theorem 4.4.8 *Let $A, B \in \mathcal{M}_{m\times n}$ and $\alpha \in \mathbb{R}$. Then*

(a) $(A+B)^T = A^T + B^T$,

(b) $(\alpha A)^T = \alpha A^T$, *and*

(c) $(A^T)^T = A$.

PROOF. Some of these should be done as an exercise, but we'll get the ball rolling by doing a together. Suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ for $1 \le i \le n$ and $1 \le j \le m$. Then, $A + B = [a_{ij} + b_{ij}]$ by the definition of addition of matrices. Note that we defined the transpose as swapping rows and columns, but we can also say what happens to each entry. If a_{ij} is in the ith row and jth column of A, then it is in the jth row and ith column of A^T. Let's denote this entry in A^T as $\overline{a}_{ji}$ and note that it is equal to a_{ij}. Keeping this notational convention, we can denote the entry in the jth row and ith column of $(A+B)^T$ as $\overline{a+b}_{ji}$ and it will be equal to $a_{ij}+b_{ij}$. This is then equal to $\overline{a}_{ji}+\overline{b}_{ji}$. Thus, $(A+B)^T = A^T + B^T$. □

This next one is a fun property.

Theorem 4.4.9 *Let $A \in \mathcal{M}_{m\times p}$ and $B \in \mathcal{M}_{p\times n}$. Then*

$$(AB)^T = B^T A^T.$$

PROOF. It is easy to check that both $(AB)^T, B^TA^T \in \mathcal{M}_{n\times n}$. Let c_{ij} be the entry in B^TA^T in the ith row and jth column. By Theorem 4.4.5, c_{ij} is the inner product of the ith row of B^T and the jth column of A^T. This is the same as the inner product of the j row of A and the ith column of B. It follows that c_{ij} is also the entry in $(AB)^T$ in the ith row and jth column. □

Row Operations as Matrix Multiplication

We've now discussed several matrix operations and seen how they are tied closely to analogous operations on linear transformations. One we haven't yet connected back to linear transformations is row operations, so let's do that now. Let $A \in \mathcal{M}_{n\times m}$. Since each of our row operations on A treat an entire row of A the same, there is actually a linear transformation $T_r\colon \mathbb{R}^n \to \mathbb{R}^n$ for each of the row operations r. For example, the row operation that swaps the first two rows of A corresponds to the linear transformation $T_{\vec{r}_1\leftrightarrow\vec{r}_2}\colon \mathbb{R}^n \to$

$\mathbb{R}^n$ defined as

$$T_{\vec{r}_1 \leftrightarrow \vec{r}_2}\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Swapping any other rows would be similar. The row operation that scales the first row of A by a nonzero scalar $k \in \mathbb{R}$ corresponds to the linear transformation $T_{k\vec{r}_1 \to \vec{r}_1} : \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$T_{k\vec{r}_1 \to \vec{r}_1}\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} kx_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Scaling any other rows would be similar. The row operation that replaces the first row with the sum of the first row and the second row scaled by $k \in \mathbb{R}$ corresponds to the linear transformation $T_{\vec{r}_1 + k\vec{r}_2 \to \vec{r}_1} : \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$T_{\vec{r}_1 + k\vec{r}_2 \to \vec{r}_1}\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} x_1 + kx_2 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

As before, doing this with other rows would be similar. Each of these linear transformations then has a matrix representation E_r with respect to the standard basis.

$$E_{\vec{r}_1 \leftrightarrow \vec{r}_2} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \qquad E_{k\vec{r}_1 \to \vec{r}_1} = \begin{bmatrix} k & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$E_{\vec{r}_1 + k\vec{r}_2 \to \vec{r}_1} = \begin{bmatrix} 1 & k & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Now, we didn't actually prove that these were linear transformations or that these were the matrix representations.[33] Since this section is already getting a bit long, we'll relegate these formalities to the exercises.

We defined these to correspond to row operations, but each linear transformation, $E_r : \mathbb{R}^n \to \mathbb{R}^n$, will really only convert a column at a time. However, using our definition of matrix multiplication and Theorem 4.4.7, we can define $E_r : \mathcal{M}_{n \times m} \to \mathcal{M}_{n \times m}$ by

$$E_r A = E_r[\vec{a}_1\ \vec{a}_2\ \cdots \vec{a}_m] = [E_r\vec{a}_1\ E_r\vec{a}_2\ \cdots E_r\vec{a}_m].$$

This means we can row reduce A using matrix multiplication! These matrices that correspond to row operations are important enough to have their own name.

33:  I believe that these are linear transformations. Seems pretty obvious.

 Didn't you demand satisfaction for matrix addition just a couple of pages ago?

Definition 4.4.5 *We call $E \in \mathcal{M}_{n\times n}$ an* ***elementary matrix*** *if for any $A \in \mathcal{M}_{n\times m}$, the matrix EA is the matrix A after performing a row operation on A.*

If we look at the specific elementary matrices we've seen so far, they are all one row operation away from the identity matrix I_n.

Theorem 4.4.10 *If $B \in \mathcal{M}_{n\times m}$ is the result of performing a row operation on a matrix $A \in \mathcal{M}_{n\times m}$ and E is the result of performing that same row operation on I_n, then $B = EA$.*

This is very convenient. If you want to do a row operation to A, then you could just do it to I_n and multiply that with A, and you get the same result as if you did the row operation to A. The proof of this theorem has a very similar flavor to how we found the E_r above, so we will omit it.[34]

34:  CHEATERS!

Example 4.4.6 Let's see it in action though. Let us define

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & -2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Suppose we would like to swap the first and third rows of A. Then the appropriate elementary matrix E would be

$$E = E_{\vec{r}_1 \leftrightarrow \vec{r}_3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

This is exactly the identity matrix with the first and third rows swapped. Now,

$$EA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & -2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 3 & -2 \\ 2 & 1 & 0 \end{bmatrix}.$$

Exploration 120 Consider the matrices A and B below.

$$A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 4 & 3 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & -9 & -8 & 0 \end{bmatrix}$$

Find the elementary matrix E such that $B = EA$.

Section Highlights

- Matrices of the same size can be added. Combining this with scalar multiplication of matrices gives us the fact that the space of matrices of a given size forms a vector space. See Theorem 4.4.2.
- A matrix of size $m \times n$ and a matrix of size $n \times k$ can be multiplied to form a matrix of size $m \times k$. This multiplication is the matrix

equivalent to function composition for linear transformations. See Definition 4.4.2 and Theorem 4.4.4.

- Since function composition is not commutative, matrix multiplication is not commutative. See Example 4.4.2.
- The $n \times n$ matrix, I_n, that has 1's on the diagonal and 0's everywhere else is called the identity matrix. Multiplying a matrix by I_n does not change the matrix. See Definition 4.4.4.
- The transpose of an $m \times n$ matrix is the $n \times m$ matrix formed by turning rows into columns (or vice versa). See Definition 4.4.3.
- Elementary matrices are matrix representations of row operations, providing a way to do row operations as matrix multiplication. See Definition 4.4.5 and Theorem 4.4.10

Exercises for Section 4.4

4.4.1. Perform the indicated operations.

(a) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}$

(b) $5\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}$

(d) $3\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix} - 2\begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}$

(e) $3\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix}^T - 2\begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}^T$

4.4.2. Multiply these matrices.

(a) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 3 \\ 2 & 2 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 2 & 2 \\ 1 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 2 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 2 & 2 \\ 1 & -1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 2 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 & 1 \\ 2 & 2 & -1 \\ 1 & -1 & -1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix}^T\begin{bmatrix} 0 & 4 \\ 2 & 1 \\ 1 & -2 \end{bmatrix}^T$

(g) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 2 \end{bmatrix}^T\begin{bmatrix} 0 & 3 \\ 2 & 2 \\ 1 & -1 \end{bmatrix}^T$

4.4.3. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}, \qquad C = \begin{bmatrix} -7 & 8 \\ 9 & -10 \\ -11 & 12 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 \\ -6 \end{bmatrix}, \qquad E = \begin{bmatrix} 7 \\ 9 \\ -10 \end{bmatrix}, \qquad F = \begin{bmatrix} -8 & 10 & 12 \end{bmatrix}$$

Compute the following matrices if it is possible. If it is not possible, draw your best dragon.

(a) AB

(b) BA

(c) $C(BD)$

(d) $B(CD)$

(e) EF

(f) FE

(g) $B(AE)$

(h) CE

(i) $(FC)D$

(j) $(DF)C$

4.4.4. Row reduce the matrices to reduced row-echelon form and find the elementary matrices for each row operation.

(a) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

4.4.5. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ and $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 5x_2 + x_3 \\ x_3 \end{bmatrix} \quad \text{and} \quad S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 \\ 3x_1 \end{bmatrix}.$$

(a) Find the matrix representation A for T and the matrix representation B for S with respect to the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$.

(b) Compute AB and BA.

(c) Verify that the matrix for $T \circ S$ with respect to the standard basis for $\mathbb{R}^2$ is AB.

(d) Verify that the matrix for $S \circ T$ with respect to the standard basis for $\mathbb{R}^3$ is BA.

4.4.6. Verify the associative property for matrix multiplication for 2×2 matrices. That is, show

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)\begin{bmatrix} x & y \\ u & w \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} x & y \\ u & w \end{bmatrix}\right)$$

4.4.7. Verify the left distribution of matrix multiplication for 2×2 matrices.

4.4.8. Verify Theorem 4.4.9 for 2×2 matrices.

4.4.9. Prove Theorem 4.4.1. That is, verify these maps are linear transformations.

4.4.10. Prove M_A from Corollary 4.4.7 is a linear transformation.

4.4.11. Complete the proof of Theorem 4.4.8. That is, let $A \in \mathcal{M}_{m\times n}$ and $\alpha \in \mathbb{R}$.

(a) Prove $(\alpha A)^T = \alpha A^T$.

(b) Prove $(A^T)^T = A$.

4.4.12. Prove the following are linear transformations and verify the matrix representations with respect to the standard basis as stated in the text.

(a) $T_{\vec{r}_1 \leftrightarrow \vec{r}_2}$

(b) $T_{k\vec{r}_1 \to \vec{r}_1}$

(c) $T_{\vec{r}_1+k\vec{r}_2\to\vec{r}_1}$

4.5 Invertible Matrices

In the previous section, we learned about matrix multiplication and how it strengthens the connection between matrices and linear transformations. Well, when you hear multiplication, often you think about division as well. Really though, you are thinking of a way to "undo" the multiplication.[35] While we do not have a concept of division for matrices, we do have an "inverse" of a matrix. This will match what you would expect from our discussion of inverses of functions from Section 3.1 because of the connection between matrix multiplication and composition of linear transformations, and our row operations will give us a nice computational algorithm to compute these inverses when they exist. We've given away enough of what's coming; we should just get started.

35: I was told that subtraction and division are a lie and that fields only have two operations, usually thought of as addition and multiplication.

Wha? Ricky?

That's right. Subtraction and division are just the inverse operations of addition and multiplication, respectively.

Oh. You.

Matrix Inverses from Linear Transformations

Matrix representations have already proven to be a very convenient tool for understanding pretty much everything about linear transformations, so it should come as little surprise that they also simplify determining invertibility and finding inverses for linear transformations as well.[36] Recall Definition 3.1.4; for vector spaces V and W and the linear transformation $T\colon V \to W$, we call $S\colon W \to V$ the inverse of T if

- for any $\vec{v} \in V$, $(S \circ T)(\vec{v}) = \vec{v}$ and
- for any $\vec{w} \in W$, $(T \circ S)(\vec{w}) = \vec{w}$.

36: Is there an isomorphism between matrix stuff and linear transformation stuff?

No. Maybe a one-to-one and onto function, though. You'd still probably have to be quite finicky about how you define your domain and codomain.

Exercise!

Moreover, we know from Theorem 3.1.3 that T is invertible if and only if it is both one-to-one and onto; thus, a linear transformation is invertible if and only if it is an isomorphism. Since T is an isomorphism, we also know that V and W are isomorphic, so by Corollary 3.3.6, they must have the same dimension.

If A and B are matrix representations of T and S respectively, then we know $A, B \in \mathcal{M}_{n\times n}$ for some n since V and W are the same dimension.[37] To keep things from being too crazy, let's restrict these matrix representations to be relative to a specific chosen basis for V and also a specific chosen basis for W. In this context, we see that the criteria for invertibility above becomes the following. For the linear transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^n$, we call $S_B\colon \mathbb{R}^n \to \mathbb{R}^n$ the inverse of T_A if for any $\vec{x} \in \mathbb{R}^n$,

$$(S_B \circ T_A)(\vec{x}) = BA\vec{x} = \vec{x} = AB\vec{x} = (T_A \circ S_B)(\vec{x}).$$

37: Did you catch that these are square matrices?

Wait, did we define a square matrix anywhere?

You're asking me? In case we didn't, it's just a matrix with the same number of rows and columns.

From Definition 4.4.4, we already know the identity matrix I_n is a matrix such that $I_n\vec{x} = \vec{x}$ for any $\vec{x} \in \mathbb{R}^n$. That's not all... well, actually, that is all.

Theorem 4.5.1 *The identity matrix $I_n \in \mathcal{M}_{n\times n}$ is the unique matrix such that for all $\vec{x} \in \mathbb{R}^n$, $I_n\vec{x} = \vec{x}$.*

PROOF. Suppose there is some matrix $A = [\vec{a}_1 \cdots \vec{a}_n] \in \mathcal{M}_{n\times n}$ such that for all $\vec{x} \in \mathbb{R}^n$, $A\vec{x} = \vec{x}$. We also know that for all $\vec{x} \in \mathbb{R}^n$, $I_n\vec{x} = \vec{x}$, so for all $\vec{x} \in \mathbb{R}^n$, we have

$$I_n\vec{x} = A\vec{x}.$$

In particular, for any $1 \leq j \leq n$, we have $\vec{e}_i = I_n\vec{e}_i = A\vec{e}_i = \vec{a}_i$. Thus, the columns of A are the vectors $\vec{e}_1, \ldots, \vec{e}_n$, so $A = I_n$. $\square$

Thus, the linear transformation T_A above is invertible if and only if there is another linear transformation S_B such that $AB = I_n = BA$. This motivates the following definition.

Definition 4.5.1 *A matrix $A \in \mathcal{M}_{n\times n}$ is* invertible *if there is another matrix $B \in \mathcal{M}_{n\times n}$ such that*

$$AB = I_n = BA.$$

We call the matrix B the inverse *of the matrix A and denote it as A^{-1}.*

Example 4.5.1 Here are a few matrices. Let's see if they are inverses of each other.

$$A = \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} -5 & -2 \\ 3 & 1 \end{bmatrix}.$$

It's easy to check that $AC = CA = I_2$, but $AB \neq I_2$ and $BC \neq I_2$. Thus, C is the inverse of A (and vice versa). That is, $C = A^{-1}$ and $A = C^{-1}$.

Note that this definition is stated just in terms of matrices and matrix multiplication. The concept of an inverse here is really "undoing" matrix multiplication, so it's our form of "division." Note that it only works for a square matrix, and even then, there will be many square matrices that are not invertible. Just like we can't divide by 0, there will be some square matrices with no inverses.[38] By the discussion prior to the definition and Theorem 4.4.4, we then have the following theorem.

38: Do you hear Jimmy Buffett music? I hear Jimmy Buffet suddenly for some reason.

What's he singing?

. . . something about boats?

Nope. It's just you.

Theorem 4.5.2 *A matrix $A \in \mathcal{M}_{n\times n}$ is invertible if and only if the induced linear transformation T_A is invertible.*

That's great! To tell whether a matrix is invertible, we just need to check whether the induced linear transformation is both one-to-one and onto. Maybe it'd be better, though, if we could tell this from the matrix itself. Let's revisit these concepts now that we know more about matrices.

One-to-one and Onto Using Pivots

In Chapter 3, we learned that a function is one-to-one if every element in the domain is mapped to a distinct output in the codomain. For linear transformations, Theorem 3.3.1 told us that a linear transformation will be one-to-one if and only if its kernel is just the zero vector. We can restate this now using a matrix equation to better align with our recent discussions.

Theorem 4.5.3 *A linear transformation $T\colon V \to W$ with matrix representation $A \in \mathcal{M}_{m\times n}$ is one-to-one if and only if the matrix equation $A\vec{x} = \vec{0}$ has only the trivial solution, $\vec{x} = \vec{0}$.*

PROOF. This follows immediately from Theorem 3.3.1 and the definition of $\mathrm{Ker}\ A$. $\square$

We also saw in Theorem 3.5.5 that a linear transformation will be one-to-one if and only if the columns of the matrix representation are linearly independent. We can now use row reduction to determine this, so we can make the following statement.

Corollary 4.5.4 *A linear transformation $T\colon V \to W$ with matrix representation $A \in \mathcal{M}_{m\times n}$ is one-to-one if and only if every column of A is a pivot column.*

PROOF. This is a restatement of Theorem 3.5.5 using Theorem 4.3.1. □

Exploration 121 Suppose the matrices below are matrices corresponding to linear transformations. Which ones correspond to a linear transformation that is one-to-one? Circle them.

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

That was so effective! It turns out detecting whether a linear transformation is onto can be similarly efficient. We learned in Chapter 3 that a function is onto when everything in the codomain appears as an output for some input from the domain. For linear transformations, we saw that the desired version is that the image is equal to the codomain. We shall restate the definition of onto in the context of matrix equations first.

Corollary 4.5.5 *A linear transformation $T\colon V \to W$ with matrix representation $A \in \mathcal{M}_{m\times n}$ is onto if and only if for every $\vec{b} \in \mathbb{R}^m$, the matrix equation $A\vec{x} = \vec{b}$ has a solution.*

PROOF. This follows immediately from the definition of onto, Theorem 3.5.2, and Theorem 4.1.1. □

We also saw in Theorem 3.5.5 that a linear transformation will be onto if the columns of the matrix representation span the codomain. From Theorem 4.2.8, we know the dimension of the image will be equal to the number of pivot columns. The dimension of the codomain is equal to the number of rows, so for a linear transformation to be onto, we need the number of rows to be equal to the number of pivots. Since each row can have only one pivot, we can state the following theorem.

Theorem 4.5.6 *A linear transformation $T\colon V \to W$ with matrix representation $A \in \mathcal{M}_{m\times n}$ is onto if and only if A can be row reduced to have a pivot in every row.*

PROOF. Read the paragraph before the theorem. That's the proof. □

Exploration 122 Suppose the matrices below are matrices corresponding to linear transformations. Which ones correspond to a linear transformation that is onto? Circle them.

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Now that we've talked about both one-to-one and onto, what about matrices for linear transformations that are both? We know for this we need to only consider square matrices. In this case, a pivot in every column is equivalent to a pivot in every row. This agrees with our expectations from Theorem 3.3.11 that a linear transformation between spaces of equal dimension will be either both one-to-one and onto or neither. This also gives us a nice condition for when a linear transformation is invertible.

Corollary 4.5.7 *A linear transformation $T: V \to V$ with matrix representation $A \in \mathcal{M}_{n\times n}$ is invertible if and only if A has n pivots.*

Computing the Inverse

When dealing with 2×2 matrices, there is a convenient formula for the inverse when it exists. This formula can be computed using a bit of algebra, and we've included just such a computation for you in the exercises. For now, we'll just tell you the answer though.

Theorem 4.5.8 *Given the matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in $\mathcal{M}_{2\times 2}$, the inverse is given by the formula

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

if $ad - bc \neq 0$.

Exploration 123 Use Theorem 4.5.8 to find A^{-1} for

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

Now we know how to find the inverse of a 2×2 matrix, but what about other for larger matrices? Well, from Theorem 4.5.2 and the discussion preceding it, the inverse of a matrix A is the matrix for the inverse linear transformation T_A^{-1}. Now, how do we find this matrix? Coordinate vectors!

Example 4.5.2 Let's use a 2×2 matrix so that we can check our work with the formula from Theorem 4.5.8. Let

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}.$$

Now, with the goal of understanding T_A^{-1}, let's begin by finding where T_A maps the standard basis vectors $\vec{e}_1$ and $\vec{e}_2$.

$$T_A(\vec{e}_1) = \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T_A(\vec{e}_2) = \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

Note that we just recovered the column vectors from A here! This is always what happens when we input the standard basis vectors to our linear transformation. Also, we can see that

$$\mathcal{B} = \left\{ \vec{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}^2$, so T_A is an isomorphism and A is invertible. Now, we know $T_A^{-1}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation such that

$$T_A^{-1}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.27}$$

$$T_A^{-1}\left(\begin{bmatrix} -1 \\ -3 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{4.28}$$

Now, to find the matrix for T_A^{-1}, we need to determine $T_A^{-1}(\vec{e}_1)$ and $T_A^{-1}(\vec{e}_2)$, but we only know what $T_A^{-1}(\vec{b}_1)$ and $T_A^{-1}(\vec{b}_2)$ are from Equations 4.27 and 4.28. If only there was a way to write $\vec{e}_1$ and $\vec{e}_2$ as linear combinations of $\vec{b}_1$ and $\vec{b}_2$... wait! There is! For this, we must find the coordinate vectors for $\vec{e}_1$ and $\vec{e}_2$ with respect to the basis $\mathcal{B}$. We can do this with row reduction!

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

This says that $\vec{e}_1 = 3\vec{b}_1 + 2\vec{b}_2$ and $\vec{e}_2 = -\vec{b}_1 - \vec{b}_2$. Most important for our purposes though, we have that $T_A^{-1}(\vec{e}_1) = 3T_A^{-1}(\vec{b}_1) + 2T_A^{-1}(\vec{b}_2) = 3\vec{e}_1 + 2\vec{e}_2$ and $T_A^{-1}(\vec{e}_2) = -T_A^{-1}(\vec{b}_1) - T_A^{-1}(\vec{b}_2) = -\vec{e}_1 - \vec{e}_2$. That is,

$$T_A^{-1}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$T_A^{-1}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Thus, the matrix representation for T_A^{-1} with respect to the standard basis of $\mathbb{R}^2$ is

$$B = \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}.$$

Note that this was exactly the right side of the augmented matrix we used for our row reduction. Because we are using the standard basis here and the matrix we started with was invertible, it will always work out like this. If the matrix we started with had not been invertible, we would have still gotten

column vectors as our outputs when we input the standard basis vectors to T_A, but these would not have formed a basis.
This method using row reduction gives us an algorithm for both determining invertibility and computing the inverse of the matrix when it exists.

Exploration 124 Use the formula for the inverse of a 2×2 matrix from Theorem 4.5.8 to check our answer from Example 4.5.2.

Theorem 4.5.9 *Suppose $A \in \mathcal{M}_{n\times n}$ is an invertible matrix. Then the augmented matrix $[A|I_n]$ row reduces to $[I_n|A^{-1}]$.*

The proof for this is essentially replacing the specific 2×2 matrix in Example 4.5.2 with a general $n \times n$ matrix. We'll save that for the Appendix.[39] There is also an alternative way to think about computing the inverse, but the outcome is this same algorithm. Note that we row reduced A to the identity matrix as part of Theorem4.5.9. There's a corollary there.

39: You can keep your row reduction proof.

Corollary 4.5.10 *A matrix A is invertible if and only if it is row equivalent to an identity matrix I_n for some positive integer n.*

Therefore, if a matrix $A \in \mathcal{M}_{n\times n}$ is invertible, there is a sequence of row operations $\{r_1, r_2, \ldots, r_k\}$ sending A to I_n. Converting this into elementary matrices gives us that

$$I_n = E_{r_k}(\cdots(E_{r_2}(E_{r_1}A) = (E_{r_k}\cdots E_{r_2}E_{r_1})A.$$

Thus, another way to find A^{-1} is to compute $(E_{r_k}\cdots E_{r_2}E_{r_1})$. Via Theorem 4.4.10, each of these matrices is obtained by applying the corresponding row operation to the identity matrix I_n. Thus, to compute A^{-1} with this method, we start with the identity matrix and then perform the same row operations as we used to row reduce A, in the same order. Thus, we can view Theorem 4.5.9 as keeping track of our elementary matrices with the augmented portion of the matrix.

Now that we've explained why our algorithm for computing the inverse works in two different ways, let's actually see it in action.

Example 4.5.3 Here's a matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Let's see if we can find an inverse. Well, after row reducing, we have

$$A \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

so A is not invertible. Alright. Let's try

$$B = \begin{bmatrix} 1 & -2 & 6 \\ 1 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Good. B is invertible. Oh, drat! We forgot to augment B with the identity matrix. Well, at least it was easy to row reduce. Let's augment B with I_3 and try again.

$$[B \,|\, I_3] = \left[\begin{array}{ccc|ccc} 1 & -2 & 6 & 1 & 0 & 0 \\ 1 & -3 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 2 & -14 \\ 0 & 1 & 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Thus, $B^{-1} = \begin{bmatrix} 3 & 2 & -14 \\ 1 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$.

Exploration 125 Now it's your turn! Find the inverse of the matrix below using the method of augmentation by I_3.

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Nice job! Now, check your answer by multiplying the original matrix by your new suspected inverse. Did you get I_3? If not, you might need to check your row reduction.

Matrix Inverses and Equations

Maybe now is a good time to think about why we should care about the inverse of a matrix. The inverse of a matrix will eventually turn out to be useful in a lot of contexts, but the definition alone immediately provides a great tool. To solve the equation $2x = 4$ for x, we know we should "divide both sides by 2." However, what we're really doing is *multiplying both sides of the equation by the multiplicative inverse of* 2.[40] Now suppose we have a matrix equation $A\vec{x} = \vec{b}$. If A is invertible, we can employ the exact same strategy. If B is the

40: See! I told you!

inverse of A, then we have $AB = BA = I_2$, so multiplying both sides of our matrix equation by B, we have

$$BA\vec{x} = B\vec{b},$$

and

$$\vec{x} = I_2\vec{x} = BA\vec{x} = B\vec{b}.$$

We can solve for $\vec{x}$ by multiplying by the inverse of A. This can be very convenient.

Exploration 126 Suppose the inverse of $A \in \mathcal{M}_{2\times 2}$ is

$$B = \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Solve the matrix equation

$$A\vec{x} = \begin{bmatrix} 42 \\ -12 \end{bmatrix}.$$

Section Highlights

- A matrix A is invertible if and only if its induced linear transformation T_A is invertible. This also means there is an inverse matrix A^{-1} such that
$$AA^{-1} = I_n = A^{-1}A.$$
See Definition 4.5.1 and Theorem 4.5.2.

- The inverse of an invertible matrix A is computed by augmenting A with an appropriately sized identity matrix I_n and row reducing to reduced row-echelon form. The inverse is the resulting augmented side. See Theorem 4.5.9.

- A linear transformation is onto if and only if the row reduced form of any matrix representing it has a pivot in every row. See Theorem 4.5.6.

- A linear transformation is one-to-one if and only if the row reduced form of any matrix representing it has a pivot in every column. See Corollary 4.5.4.

Exercises for Section 4.5

4.5.1. For each matrix, row reduce to determine whether the corresponding linear transformation is one-to-one, onto, both or neither.

(a) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 1 \\ 3 & 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 1 & 1 \\ -2 & 1 & -2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

4.5.2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ x_3 \\ x_3 \end{bmatrix}.$$

Find the matrix A that represents T on the standard basis vectors for $\mathbb{R}^3$. Use this matrix to determine whether T is one-to-one, onto, both or neither.

4.5.3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ x_1 + x_3 \\ x_2 \end{bmatrix}.$$

Find the matrix A that represents T on the standard basis vectors for $\mathbb{R}^3$. Use this matrix to determine whether T is one-to-one, onto, both or neither.

4.5.4. Determine whether the following matrices are invertible. If the matrix is invertible, find the inverse.

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & -4 \\ 1 & 1 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 2 \\ -2 & 0 & -4 \\ 0 & 1 & 2 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(h) $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

(i) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 2 & 1 & 1 \end{bmatrix}$

4.5.5. Let $A = \begin{bmatrix} 3 & 2 & 6 \\ -2 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ and

$$\vec{b}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \vec{b}_3 = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}.$$

Find the inverse of A and use it to solve $A\vec{x} = \vec{b}_1$, $A\vec{x} = \vec{b}_2$, and $A\vec{x} = \vec{b}_3$.

4.5.6. Find a matrix representation for the linear transformation to determine whether it is invertible.

(a) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 5x_2 \\ x_1 + x_2 \end{bmatrix}$$

(b) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix}$$

(c) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \\ x_1 + x_2 + x_3 \end{bmatrix}$$

(d) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \\ x_1 - x_2 + x_3 \end{bmatrix}$$

4.5.7. Let's see if we can find the general formula for the inverse of a 2×2 matrix using just matrices. Given a matrix $A \in \mathcal{M}_{2\times 2}$, we want to find a formula for a matrix B such that $AB = BA = I_2$. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

We need to solve for $e, f, g,$ and h in terms of a, b, c and d so that $AB = BA = I_2$.

(a) First, calculate AB.

(b) We want this result to be I_2, so use the fact that we should have 0 in two of the four components to solve for f and g in terms of the other constants.

(c) Substitute these expressions for f and g into the two expressions that should be equal to 1 and solve for e and h.

(d) Now you can substitute your expressions for e and h back into your expressions for f and g. You should now have e, f, g, and h in terms of a, b, c and d. A little more algebra should yield a nice formula for you.

4.6 Matrix Theorems

In this section, we've combined several nice results about matrices. The main work for these results has already occurred, so the goal here is primarily organize them and to remind you of them.

Subspaces Induced by Matrix Representations

First, let's introduce a new subspace related to a matrix. It shouldn't be surprising though since we know that each row is a vector.

Definition 4.6.1 *For a matrix $A \in \mathcal{M}_{m\times n}$, let $\vec{r}_i$ be the vector formed from the ith row of A for each $1 \leq i \leq m$. The* ***row space of*** *A, denoted $\text{Row } A$, is the span of these row vectors. That is,*

$$\text{Row } A = \text{Span}\{\vec{r}_1, \ldots, \vec{r}_m\}.$$

Theorem 4.6.1 *For a matrix $A \in \mathcal{M}_{m\times n}$, $\text{Row } A$ is a subspace of $\mathbb{R}^n$.*

PROOF. This follows from Theorem 3.6.3 by taking the transpose of your matrix. □

Now for something really cool. Suppose we have a linear transformation between two inner product spaces. We have fun subspaces of domains and codomains for linear transformations (the kernel and image, respectively), but what about the rest of the domain and codomain? You would not be shocked to find that the orthogonal complement of the kernel is a subspace of the domain,[41] and the orthogonal complement of the image is a subspace of the codomain (Theorem 2.4.2). What is surprising is that these orthogonal complements are also given by the matrix representation for the linear transformation. Behold!

41: The orthogonal complement of the kernel is even isomorphic to the image from a theorem in Section 3.3.

Theorem 4.6.2 *Suppose V and W are inner product spaces. Let $T : V \to W$ be a linear transformation represented by the $n \times m$ matrix A. Then*

$$\text{Ker } A = (\text{Row } A)^{\perp} \quad \textit{and} \quad \text{Ker } A^T = (\text{Col } A)^{\perp}.$$

PROOF. Let $A = [a_{ij}]$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, let $\vec{r}_i$ for $1 \leq i \leq m$ be the row vectors of A, and let $\vec{a}_j$ for $1 \leq j \leq n$ be the column vectors of A. Then

$$\begin{aligned}
& \vec{x} \in \text{Ker } A \\
\Leftrightarrow\ & A\vec{x} = \vec{0} \\
\Leftrightarrow\ & x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{0} \\
\Leftrightarrow\ & x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \\
\Leftrightarrow\ & x_1 a_{i1} + \cdots + x_n a_{in} = 0 \text{ for } 1 \leq i \leq n \\
\Leftrightarrow\ & \vec{x} \cdot \vec{r}_i = 0 \text{ for } 1 \leq i \leq n.
\end{aligned}$$

Thus, $\vec{x} \in \text{Ker } A$ if and only if it is orthogonal to every row of A. Since $\text{Row } A$ is the span of the rows of A, the result follows. The other equality is then achieved by noting $\text{Row } A^T = \text{Col } A$. □

Exploration 127 Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}.$$

Find $\text{Ker } A$ and check that it is orthogonal to each row vector.

Find $\text{Ker } A^T$ and check that it is orthogonal to each column vector.

Corollary 4.6.3 *Let A be an $n \times m$ matrix with induced linear transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$. Then*

$$\text{dom}\,(T_A) = \text{Ker } A \oplus \text{Row } A \quad \textit{and} \quad \text{codom}\,(T_A) = \text{Col } A \oplus \text{Ker } A^T.$$

We are now able to complete the vector spaces in our big commuting diagram; see Figure 4.3.

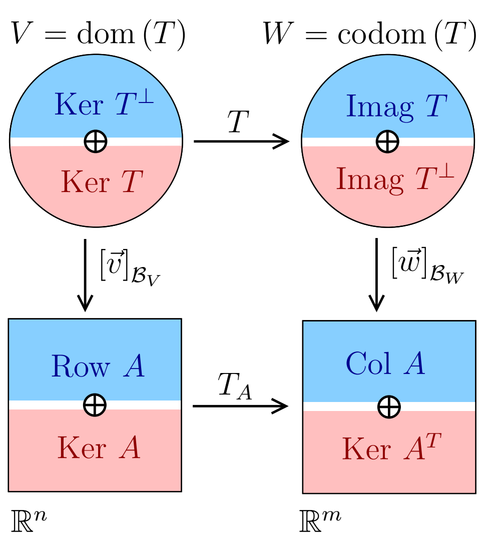


FIGURE 4.3. Some people refer to this as "the splits" of $\text{dom}\,(T_A)$ and $\text{codom}\,(T_A)$.

Theorem 4.6.4 (Invertible Matrix Theorem) *Let $A \in \mathcal{M}_{n\times n}$, and let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(\vec{x}) = A\vec{x}$. Note that the matrix A is a square matrix, one where the rows and columns have the same length. The following are equivalent statements.*

(a) *A is invertible.*
(b) *A can be row reduced to I_n.*
(c) *T is invertible.*
(d) *T is one-to-one.*
(e) *T is onto.*
(f) *A has a pivot in every column.*
(g) *A has a pivot in every row.*
(h) *The columns of A are linearly independent.*
(i) *The rows of A are linearly independent.*
(j) $\operatorname{Ker} A = \{\vec{0}\}$.
(k) *A^T is invertible.*

PROOF. Let's actually take it from the top and bottom here. We know from Theorem 4.4.9 that $(AB)^T = B^T A^T$. Now, A is invertible if and only if there is some matrix B such that $AB = I_n$. Combining this with the fact that the identity matrix is its own transpose gives us

$$I_n = I_n^T = (AB)^T = B^T A^T.$$

Thus, B^T is the inverse of A^T if B is the inverse of A. Since $(A^T)^T = A$, we see that A is invertible if and only if A^T is invertible. From this point, the equivalence of all of these statements follows from previous theorems, mostly contained or mentioned in the previous section. □

You should definitely believe every one of the statements in this theorem is equivalent to all the others at this point. We've definitely proved all of these results independently. If you are the slightest bit suspicious[42] about any of these connections, you should find the theorem and definitions in sections past that prove it.[43]

42: I am!

43: Fine! I will!

Example 4.6.1 Let's check whether the following matrices are invertible. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 2 & 3 \end{bmatrix}$$

A is invertible for any of the eleven reasons on Theorem 4.6.4. B and C are not, again for any of the ten reasons.

Exploration 128 Determine whether each matrix below is invertible without performing any row operations. Note which part of the Invertible Matrix Theorem you use.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 4 & 1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 6 \end{bmatrix}$$

Change of Basis Matrices

Our next topic is one we've seen in some examples, but we're finally ready to give it a proper discussion. We'll start, though, with an example.

Example 4.6.2 Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\},$$

which is a basis for $\mathbb{R}^3$. We can consider the coordinate map

$$\varphi_{\mathcal{B}} \colon \mathbb{R}^3 \to \mathbb{R}^3 \text{ defined as } \varphi_{\mathcal{B}}(\vec{v}) = [\vec{v}]_{\mathcal{B}}.$$

Then the matrix representation for $\varphi_{\mathcal{B}}$ with respect to the standard basis is

$$B = [[\vec{e}_1]_{\mathcal{B}} \ \dots \ [\vec{e}_2]_{\mathcal{B}}].$$

To find this, we need to compute all the coordinate vectors for the standard basis vectors with respect to $\mathcal{B}$. We can do this by augmenting the vectors of $\mathcal{B}$ with the standard basis vectors and row reducing. Does this sound familiar though? It should. Theorem 4.5.9 gave this as the algorithm for finding the inverse of a matrix. Let's see what this gives us.

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & -2/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & -2/3 \end{array}\right]$$

Then

$$C = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \end{bmatrix}.$$

It turns out that the matrix we were finding the inverse for is actually the matrix representation for $\varphi_{\mathcal{B}}^{-1}$, the linear transformation that converts a coordinate vector for $\mathcal{B}$ into a vector on the standard basis. Let's call our matrix representation of $\varphi_{\mathcal{B}}^{-1}$ here P since it's the one that's easier to find. Then we have

$$P = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad P^{-1} = C.$$

With these two matrices, we can efficiently travel back and forth between the standard basis and our basis $\mathcal{B}$! We can even use these two matrices to convert any matrix representation for a linear transformation T from the standard basis to this new basis $\mathcal{B}$. We did this for a different basis back in Example 3.5.7, but now we can use matrix multiplication instead of those methods.

Example 4.6.3 Let's use P and P^{-1} from Example 4.6.2 above to compute the matrix representation for $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ x_1 + x_3 \\ x_2 \end{bmatrix}$$

with respect to the basis $\mathcal{B}$. First, we need the matrix with respect to the standard basis.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Now, we should think about how to construct the desired matrix. When we think about this matrix as a function, it will have input of coordinate vectors for $\mathcal{B}$ and output of coordinate vectors for $\mathcal{B}$. Consider then the product $P^{-1}AP$.

- Since we read function composition right to left, we see that this will first apply P, which converts a coordinate vector for $\mathcal{B}$ to a vector in the standard basis.
- Then A will map this according to the linear transformation T.
- Finally, by applying P^{-1}, we convert the output of A to be a coordinate vector for $\mathcal{B}$.

This then will produce the matrix representation for T with respect to the basis $\mathcal{B}$.

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 4/3 & -2/3 & -2/3 \\ 1/3 & 4/3 & 4/3 \\ 1/3 & -5/3 & -5/3 \end{bmatrix}. \end{aligned}$$

Since we have an alternative method for finding this matrix representation, we can check that this method produces the same matrix representation. By "we" there, we mean you. You should check this.[44]

44: Explore!

Exploration 129 Use the method from Example 3.5.7 to find the matrix representation of T above with respect to the basis $\mathcal{B}$.

The matrices P and P^{-1} were examples of *change of basis matrices*. Specifically, P^{-1} is the change of basis matrix from the standard basis to the basis $\mathcal{B}$ and P is the change of basis matrix from the basis $\mathcal{B}$ to the standard basis. We can do this, though, with any two bases.

Definition 4.6.2 *Let V be an n-dimensional vector space with bases*

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\} \text{ and } \mathcal{C} = \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}.$$

Define the isomorphism

$$\varphi_{\mathcal{B}\triangleright\mathcal{C}} \colon V \to V \text{ by } \varphi_{\mathcal{B}\triangleright\mathcal{C}}(\vec{c}_i) = \vec{b}_i \text{ for each } 1 \le i \le n.$$

Then the ***change of basis matrix from*** $\mathcal{B}$ ***to*** $\mathcal{C}$ *is the matrix for $\varphi_{\mathcal{B}\triangleright\mathcal{C}}$ with respect to the basis $\mathcal{C}$. In particular, it is the matrix $P_{\mathcal{B}\triangleright\mathcal{C}} \in \mathcal{M}_{n\times n}$ defined* *by*

$$P_{\mathcal{B}\triangleright\mathcal{C}} = \left[\left[\vec{b}_1\right]_{\mathcal{C}} \cdots \left[\vec{b}_n\right]_{\mathcal{C}}\right].$$

Note that the inverse of a change of basis matrix is again a change of basis matrix. Perhaps that should be stated as a theorem.

Theorem 4.6.5 *For any two bases $\mathcal{B}$ and $\mathcal{C}$ of a vector space V, we have*

$$P^{-1}_{\mathcal{B}\triangleright\mathcal{C}} = P_{\mathcal{C}\triangleright\mathcal{B}}.$$

PROOF. This follows directly from the definition since we can quickly see $\varphi^{-1}_{\mathcal{B}\triangleright\mathcal{C}} = \varphi_{\mathcal{C}\triangleright\mathcal{B}}$. Thus, the matrix representations will also be inverses of one another. □

When we combine the definition of our change of basis matrix with our method from Section 4.3 for computing the coordinate vectors, we see there is a handy algorithm like Theorem 4.5.9 that we can use to compute these matrices.

Theorem 4.6.6 *Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ be bases for $\mathbb{R}^n$. Then the matrix*

$$[\vec{c}_1\ \vec{c}_2\ \cdots\ \vec{c}_n | \vec{b}_1\ \vec{b}_2\ \cdots\ \vec{b}_n] \text{ row reduces to } [I_n | P_{\mathcal{B}\triangleright\mathcal{C}}].$$

Similarly,

$$[\vec{b}_1\ \vec{b}_2\ \cdots\ \vec{b}_n | \vec{c}_1\ \vec{c}_2\ \cdots\ \vec{c}_n] \text{ row reduces to } [I_n | P_{\mathcal{C}\triangleright\mathcal{B}}].$$

In the case that one of the two bases involved is the standard basis for $\mathbb{R}^n$, we see that one direction will require no work and the other will only require us to compute the inverse of a matrix. This was what happened in Example 4.6.3. Let's see an example between two non-standard bases.

Example 4.6.4 Let's just do two different bases of $\mathbb{R}^2$. Define

$$\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right\} \text{ and } \mathcal{C} = \left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}.$$

We can see quickly that these are both bases of $\mathbb{R}^2$. Now, we can construct $P_{\mathcal{B}\triangleright\mathcal{C}}$ and $P_{\mathcal{C}\triangleright\mathcal{B}}$ using Theorem 4.6.6.

$$\left[\begin{array}{rr|rr} 2 & -1 & 1 & -1 \\ -1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rr|rr} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

and

$$\left[\begin{array}{rr|rr} 1 & -1 & 2 & -1 \\ 1 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{rr|rr} 1 & 0 & -1 & 1 \\ 0 & 1 & -3 & 2 \end{array}\right].$$

This tells us

$$P_{\mathcal{B}\triangleright\mathcal{C}} = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} \text{ and } P_{\mathcal{C}\triangleright\mathcal{B}} = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix}.$$

Note that these two matrices are inverses of each other, as they should be!

Now that we've defined the change of basis matrix and talked about how to find it, let's see what it does for us. Hopefully, it changes a basis.

Theorem 4.6.7 *Let $\mathcal{B}$ and $\mathcal{C}$ be bases for the vector space V. For any $\vec{x} \in V$,*

$$P_{\mathcal{B} \triangleright \mathcal{C}} [\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}} .$$

PROOF. Let $\vec{x} \in V$ and suppose $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$. Then

$$\begin{aligned} P_{\mathcal{B} \triangleright \mathcal{C}} [\vec{x}]_{\mathcal{B}} &= \left[\left[\vec{b}_1\right]_{\mathcal{C}} \cdots \left[\vec{b}_n\right]_{\mathcal{C}} \right] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \\ &= a_1 \left[\vec{b}_1\right]_{\mathcal{C}} + \cdots + a_n \left[\vec{b}_n\right]_{\mathcal{C}} \\ &= \left[a_1\vec{b}_1 + \cdots a_n\vec{b}_n\right]_{\mathcal{C}} = [\vec{x}]_{\mathcal{C}} . \end{aligned}$$

□

Oh, good. It is named appropriately! Let's see an example of this theorem in action. Note that we restricted Theorem 4.6.6 for simplicity to bases of $\mathbb{R}^n$, but this algorithm works for any vector space once coordinate vectors are computed.

Example 4.6.5 Here, let's consider two bases of $\mathbb{P}_2$. Let

$$\mathcal{B} = \{1, 1+x, x+x^2\} \text{ and } \mathcal{C} = \{x, 1+x^2, 1\}.$$

Then under the coordinate mapping using the standard basis, $\{1, x, x^2\}$, these become

$$\overline{\mathcal{B}} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \overline{\mathcal{C}} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

We can then apply Theorem 4.6.6 to these bases of coordinate vectors to find $P_{\mathcal{B} \triangleright \mathcal{C}}$.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

Thus, we have

$$P_{\mathcal{B} \triangleright \mathcal{C}} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

We can now check that this works as expected. Consider $\vec{p} = 2 + x + x^2$. Then from inspection we can see

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \text{ and } [\vec{p}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

To see for ourselves that $P_{\mathcal{B} \triangleright \mathcal{C}}$ converts coordinate vectors for $\mathcal{B}$ into ones for $\mathcal{C}$, we compute

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

as expected.

Now that we know these matrices convert between coordinate vectors for different bases, we can expand them to matrix representations of a linear transformation.

Corollary 4.6.8 *Let vector space V have bases $\mathcal{B}$ and $\mathcal{C}$ and $T\colon V \to V$ have matrix representation B with respect to $\mathcal{B}$ and C with respect to $\mathcal{C}$. Define the matrix $P \in \mathcal{M}_{n\times n}$ by*

$$P = P_{\mathcal{B} \triangleright \mathcal{C}} = \left[\left[\vec{b}_1\right]_{\mathcal{C}} \cdots \left[\vec{b}_n\right]_{\mathcal{C}} \right].$$

Then we know

$$C = PBP^{-1} \qquad \text{and} \qquad B = P^{-1}CP.$$

This was what we did in Example 4.6.3, but it's worth seeing again.

Example 4.6.6 Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ x_1 + x_3 \\ x_2 \end{bmatrix}.$$

Let's find the matrix for this linear transformation with respect to two different bases of $\mathbb{R}^3$. First, the standard basis $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. We have

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \quad T(\vec{e}_2) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}; \quad T(\vec{e}_3) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Thus, the matrix for the linear transformation on the basis $\mathcal{E}$ is

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let's write the matrix for T now with respect to the basis

$$\mathcal{B} = \left\{ \vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

We have

$$T(\vec{b}_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad T(\vec{b}_2) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}; \quad T(\vec{b}_3) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

but these are vectors in coordinates relative to the standard basis $\mathcal{E}$. We can augment a matrix and row reduce to convert these to coordinate vectors for

$\mathcal{B}$.

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

Thus, the matrix for the linear transformation on the basis $\mathcal{B}$ is

$$B = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

For notational simplicity, let's denote

$$P = P_{\mathcal{B} \triangleright \mathcal{E}} = \left[\left[\vec{b}_1\right]_{\mathcal{E}} \left[\vec{b}_2\right]_{\mathcal{E}} \left[\vec{b}_3\right]_{\mathcal{E}} \right] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

After verifying that

$$P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

one can check that

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = B. \end{aligned}$$

Exploration 130 Let

$$\mathcal{B}_0 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

This is a basis of $\mathbb{R}^3$. What is $P_{\mathcal{B}_0 \triangleright \mathcal{E}}$, where $\mathcal{E}$ is again the standard basis?

Now[45], use the matrix $P^{-1}_{\mathcal{B}_0 \triangleright \mathcal{E}}$ to find $[\vec{x}]_{\mathcal{B}_0}$ when $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

45: Hey, have you seen Ricky?

No.

It's not like Ricky to not just be hanging around.

[gasping] Hey guys. I've been all over this chapter. The proofs are all there.

You know they're all hyperlinked, right?

Section Highlights

- For an $m \times n$ matrix, A, the span of the rows of A forms a subspace of $\mathbb{R}^n$ called $\mathrm{Row}\ A$. See Definition 4.6.1 and Theorem 4.6.1.
- For any $m \times n$ matrix, A, there is an orthogonal decomposition of $\mathbb{R}^n$ into $\mathrm{Row}\ A \oplus \mathrm{Ker}\ A$ and an orthogonal decomposition of $\mathbb{R}^m$ into $\mathrm{Col}\ A \oplus \mathrm{Ker}\ A^T$. See Corollary 4.6.3 and Figure 4.3.
- There are many conditions equivalent to a matrix being invertible. See Theorem 4.6.4.
- If $\mathcal{B}$ and $\mathcal{C}$ are both bases for a vector space V, then the change of basis matrix $P_{\mathcal{B} \triangleright C}$ converts coordinate vectors for $\mathcal{B}$ into coordinate vectors for $\mathcal{C}$. See Definition 4.6.2.
- If A is the matrix representation with respect to a basis $\mathcal{B}$ for a linear transformation, then the matrix representation with respect to the basis $\mathcal{C}$ is $P_{\mathcal{B} \triangleright C} A P_{\mathcal{B} \triangleright C}^{-1}$. See Corollary 4.6.8 and Example 4.6.6.

Exercises for Section 4.6

4.6.1. Find $\text{Ker } A$, $\text{Row } A$, $\text{Col } A$, and $\text{Ker } A^T$ when A is the matrix below.

(a) $\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & -1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

4.6.2. Do each of these for the bases and vectors below.

- Find the change of basis matrix P that converts from the basis $\mathcal{B}$ to the standard basis.
- Find P^{-1}.
- Use P^{-1} to find $[\vec{x}]_{\mathcal{B}}$.

(a) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}; \quad \vec{x} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$

(b) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}; \quad \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(c) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}; \quad \vec{x} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

4.6.3. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Let $\mathcal{C}$ be the standard basis for $\mathbb{R}^3$.

(a) Find the matrix $P_{\mathcal{B} \triangleright \mathcal{C}}$ and the matrix $P_{\mathcal{C} \triangleright \mathcal{B}}$.

(b) Use $P_{\mathcal{B} \triangleright \mathcal{C}}$ and $P_{\mathcal{C} \triangleright \mathcal{B}}$ to convert each of these matrices to the basis $\mathcal{B}$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 3 \\ 0 & -1 & 1 \end{bmatrix}$$

4.6.4. Let V be a vector space with bases $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$. If

$$\begin{array}{rcrcrcr} \vec{c}_1 & = & \vec{b}_1 & + & 2\vec{b}_2 & + & \vec{b}_3 \\ \vec{c}_2 & = & -\vec{b}_1 & - & \vec{b}_2 & + & \vec{b}_3 \\ \vec{c}_2 & = & & & \vec{b}_2 & + & \vec{b}_3 \end{array}$$

For $\vec{x} \in V$, find a matrix P such that $P\,[\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}}$. Find P^{-1} and verify that $P^{-1}\,[\vec{x}]_{\mathcal{C}} = [\vec{x}]_{\mathcal{B}}$.

4.7 More Fun with Least Squares

Here's a matrix equation:

$$A\vec{x} = \vec{b}.$$

Depending on A and $\vec{b}$, sometimes this equation has exactly one solution or an infinite number of solutions. That's very nice; everyone likes solutions. However, from Corollary 4.2.4 we know it's also possible that this matrix equation has no solutions. That must be very frustrating for everyone involved. Surely there's something one can do in this case! We can't just *pretend* that solutions exist; Corollary 4.2.4 is pretty clear about solutions existing or *not* existing, as is the binary nature of existence. However, if you find you're stuck in that latter case, what's the next best thing to an honest solution?[46] Perhaps, maybe, just possibly... an *approximate* one will do.

46: A dishonest one! Some would argue honesty is also binary in nature.

Let's assume $A \in \mathcal{M}_{m\times n}$, so $\vec{x} \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^m$. Let's also assume some fool picked a rotten matrix A and a lonely vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ has no solutions.

Definition 4.7.1 *A* ***least squares solution*** *for the matrix equation $A\vec{x} = \vec{b}$ is a vector $\hat{\vec{x}} \in \mathbb{R}^n$ such that for all $\vec{x} \in \mathbb{R}^n$,*

$$\|A\hat{\vec{x}} - \vec{b}\| \leq \|A\vec{x} - \vec{b}\|.$$

The ***least squares error*** *of a least squares solution is $\|A\hat{\vec{x}} - \vec{b}\|$.*

These are the vectors in $\mathbb{R}^n$ whose images are *as close to* $\vec{b}$ as any other image vector. Good. In the absence of a solution to $A\vec{x} = \vec{b}$, these least-squares solutions are the closest you can get to a solution, and the least squares error quantifies how close. This inequality should look familiar; we've already encountered a situation like this. According to Theorem 2.6.1, if a vector $\vec{b}$ is not in a subspace W, then the vector in W closest to $\vec{b}$ is $\operatorname{proj}_W\left(\vec{b}\right)$. See Figure 4.4. Using Col A as the subspace W, we have the following corollary.

Corollary 4.7.1 *The least squares solutions for $A\vec{x} = \vec{b}$ are the solutions to the equation*

$$A\vec{x} = \operatorname{proj}_{\text{Col } A}\left(\vec{b}\right).$$

PROOF. By Theorem 2.6.1,

$$\|\operatorname{proj}_{\text{Col } A}\left(\vec{b}\right) - \vec{b}\| \leq \|\vec{y} - \vec{b}\|$$

for all $\vec{y} \in \text{Col } A$. By definition of column space, for every vector $\vec{y} \in \text{Col } A$, there is a set of vectors $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{y}$, and $\operatorname{proj}_{\text{Col } A}\left(\vec{b}\right) \in \text{Col } A$. The result follows from defining $\hat{\vec{x}}$ to be the set of vectors such that $A\hat{\vec{x}} = \operatorname{proj}_{\text{Col } A}\left(\vec{b}\right)$. □

Corollary 4.7.1 suggests a strategy for finding the least squares solutions; one only need to solve the equation $A\vec{x} = \operatorname{proj}_{\text{Col } A}\left(\vec{b}\right)$. Of course, this means first finding, $\operatorname{proj}_{\text{Col } A}\left(\vec{b}\right)$, and to do this efficiently, you need an orthogonal basis for Col A. Suddenly, this feels like a lot of work.

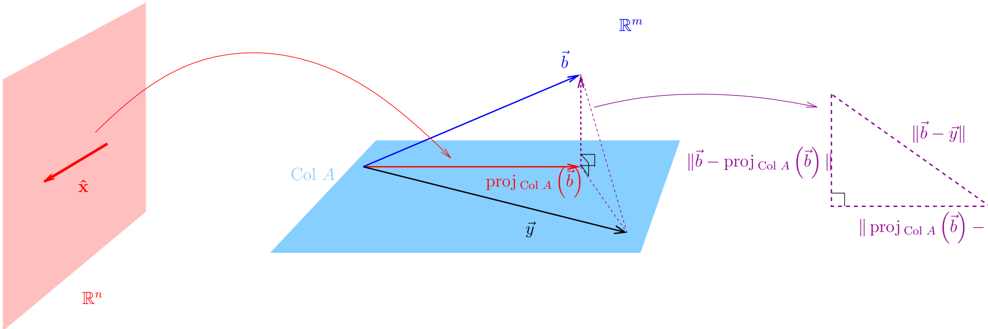


Figure 4.4: Here's a picture we've boosted from Section 2.6 and relabelled.

Example 4.7.1 Let's find the least squares solutions for the equation $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

One should first check that $\vec{b} \notin \text{Col } A$. It's not? Good. Thank you for checking. If we're going to solve the equation $A\vec{x} = \text{proj}_{\text{Col }A}\left(\vec{b}\right)$, we'd better first calculate $\text{proj}_{\text{Col }A}\left(\vec{b}\right)$, so we need an orthogonal basis for Col A. After using Gram-Schmit, we have

$$\left\{ \vec{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

as an orthogonal basis for Col A. Then

$$\text{proj}_{\text{Col }A}\left(\vec{b}\right) = \frac{\vec{w}_1 \cdot \vec{b}}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 + \frac{\vec{w}_2 \cdot \vec{b}}{\vec{w}_2 \cdot \vec{w}_2}\vec{w}_2 = \frac{2}{5}\vec{w}_1 + \vec{w}_2 = \begin{bmatrix} 2/5 \\ 4/5 \\ 1 \end{bmatrix}.$$

We know $\text{proj}_{\text{Col }A}\left(\vec{b}\right) \in \text{Col } A$, so the matrix equation $A\vec{x} = \text{proj}_{\text{Col }A}\left(\vec{b}\right)$ definitely has at least one solution. We can just row reduce the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2/5 \\ 2 & 4 & 6 & 4/5 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|r} 1 & 2 & 0 & -13/5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the least squares solutions for $A\vec{x} = \vec{b}$ are the vectors

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -13/5 \\ 0 \\ 1 \end{bmatrix},$$

for any real number x_2. These least squares solutions have least squares error

$$\|A\hat{\vec{x}} - \vec{b}\| = \|\text{proj}_{\text{Col }A}\left(\vec{b}\right) - \vec{b}\| = \left\| \begin{bmatrix} 2/5 \\ 4/5 \\ 0 \end{bmatrix} \right\| = \frac{2\sqrt{5}}{5}.$$

That didn't feel like a great way to find least squares solutions. Perhaps you're beginning to despair because you thought this way was our last hope. No, there is another...

We're all tired of writing $\operatorname{proj}_{\text{Col } A}\left(\vec{b}\right)$ over and over again, right? Right. Let's define

$$\hat{\vec{b}} = \operatorname{proj}_{\text{Col } A}\left(\vec{b}\right).$$

Then our least squares solutions are the vectors $\hat{\vec{x}} \in \mathbb{R}^n$ such that

$$A\hat{\vec{x}} = \hat{\vec{b}},$$

which looks like a totally respectable matrix equation.[47] By Theorem 2.5.1 (the orthogonal decomposition theorem!), we know that $\vec{b} - \hat{\vec{b}}$ is orthogonal to $\text{Col } A$. That is,

$$\vec{b} - \hat{\vec{b}} \in (\text{Col } A)^{\perp} = \left(\text{Span}\left\{\vec{a}_1, \ldots, \vec{a}_n\right\}\right)^{\perp},$$

where $a_1, \ldots a_n$ are the columns of A. See Figure 4.4. It follows that $\vec{b} - \hat{\vec{b}}$ is orthogonal to every column of A; that is, for any column $\vec{a}_i$ of A, we have

$$0 = \vec{a}_i \cdot (\vec{b} - \hat{\vec{b}}) = (\vec{a}_i)^T(\vec{b} - \hat{\vec{b}}).$$

Since we have $(\vec{a}_i)^T(\vec{b} - \hat{\vec{b}}) = 0$ for every row vector $(\vec{a}_i)^T$, it follows that

$$A^T(\vec{b} - \hat{\vec{b}}) = \vec{0},$$

or $A^T\hat{\vec{b}} = A^T\vec{b}$. Recall that $A\hat{\vec{x}} = \hat{\vec{b}}$, so we have

$$A^TA\hat{\vec{x}} = A^T\vec{b}$$

for any $\hat{\vec{x}} \in \mathbb{R}^n$. Wow. That seems like a very conventional equation that we got by exploiting orthogonality. I wonder what we should call it.

Definition 4.7.2 *The* ***normal equation*** *for a matrix $A \in \mathcal{M}_{m\times n}$ and a vector $\vec{b} \in \mathbb{R}^m$ is*

$$A^TA\hat{\vec{x}} = A^T\vec{b}.$$

We just proved the following theorem.

Theorem 4.7.2 *For a matrix $A \in \mathcal{M}_{m\times n}$ and a vector $\vec{b} \in \mathbb{R}^m$, a vector $\hat{\vec{x}} \in \mathbb{R}^n$ is a least squares solution for $A\vec{x} = \vec{b}$ if and only if $\hat{\vec{x}}$ is a solution to the normal equation*

$$A^TA\hat{\vec{x}} = A^T\vec{b}.$$

Since Corollary 4.7.1 indicates that least squares solutions to $A\vec{x} = \vec{b}$ are solutions to $A\vec{x} = \operatorname{proj}_{\text{Col } A}\left(\vec{b}\right)$, and $\operatorname{proj}_{\text{Col } A}\left(\vec{b}\right) \in \text{Col } A$, we know that least squares solutions always exist. Thus, we have another fun corollary:

Corollary 4.7.3 *For a matrix $A \in \mathcal{M}_{m\times n}$, the normal equation $A^TA\hat{\vec{x}} = A^T\vec{b}$ always has at least one solution.*

47: Especially with all the fancy hats!

Example 4.7.2 Let's find the least squares solutions (again!) for the equation $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

By Theorem 4.7.2, we want to find the solutions to $A^T A\hat{\vec{x}} = A^T\vec{b}$. Note that

$$A^T A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 15 \\ 10 & 20 & 30 \\ 15 & 30 & 46 \end{bmatrix} \text{ and}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}.$$

Thus, we need to solve the matrix equation

$$\begin{bmatrix} 5 & 10 & 15 \\ 10 & 20 & 30 \\ 15 & 30 & 46 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix},$$

whose augmented matrix in reduced row-echelon form is

$$\left[\begin{array}{ccc|r} 1 & 2 & 0 & -13/5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which definitely looks familiar. It follows that the least squares solutions to $A\vec{x} = \vec{b}$ are still the vectors

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -13/5 \\ 0 \\ 1 \end{bmatrix},$$

for any real number x_2.

Exploration 131 Let

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad \vec{b}_1 = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \text{and} \quad \vec{b}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 5 \end{bmatrix}.$$

Find the least squares solution(s) and least squares error for $A\vec{x} = \vec{b}_1$ and $A\vec{x} = \vec{b}_2$.

This particular method for finding least squares solutions is really handy when finding curves of best fit. Let's try!

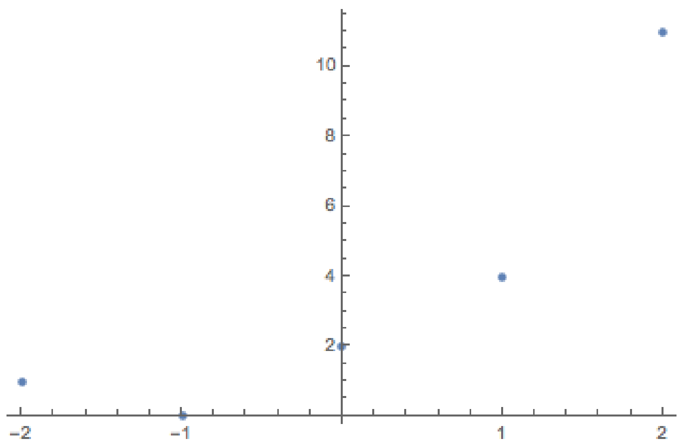


FIGURE 4.5. That old aggressively nonlinear data set again.

Example 4.7.3 Find (again!) the curve of best fit for the following data:

$$(-2,1),(-1,0),(0,2),(1,4) \text{ and } (2,11).$$

We checked before that these points are not colinear; see Figure 4.5.
Recall the example with the parabola of best fit? Sure you do. We need to find scalars a, b, and c such that $y = ax^2 + bx + c$ for all five of our points, but we know that no such a, b, and c exist; that is, there is no set of scalars a, b, and c such that

$$\vec{y} = a\vec{q} + b\vec{x} + c\vec{1}, \quad \text{where}$$

$$\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 11 \end{bmatrix}, \quad \vec{q} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \vec{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Here are two other ways of putting it:

$$\vec{y} \notin \operatorname{Span}\left\{\vec{q}, \vec{x}, \vec{1}\right\} = \operatorname{Col}[\vec{q}, \vec{x}, \vec{1}],$$

or the matrix equation

$$[\vec{q}, \vec{x}, \vec{1}] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{y}$$

has no solutions. Hey! No solutions? Let's find the least squares solutions! Let $A = [\vec{q}, \vec{x}, \vec{1}]$. Then

$$\begin{aligned} A^T A &= \begin{bmatrix} 4 & 1 & 0 & 1 & 4 \\ -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 34 & 0 & 10 \\ 0 & -10 & 0 \\ 10 & 0 & 5 \end{bmatrix}, \text{ and} \end{aligned}$$

$$A^T \vec{y} = \begin{bmatrix} 4 & 1 & 0 & 1 & 4 \\ -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 11 \end{bmatrix} = \begin{bmatrix} 52 \\ 24 \\ 18 \end{bmatrix}.$$

Since we're looking for least squares solutions, we can solve the normal equation

$$\begin{bmatrix} 34 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 52 \\ 24 \\ 18 \end{bmatrix},$$

whose augmented matrix in reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8/7 \\ 0 & 1 & 0 & 12/5 \\ 0 & 0 & 1 & 46/35 \end{array}\right].$$

Hey! No Gram-Schmit required. Nice!
The quadratic equation $y = \frac{8}{7}x^2 + \frac{12}{5}x + \frac{46}{35}$ is the best quadratic least squares approximation for the given data. See Figure 4.6.

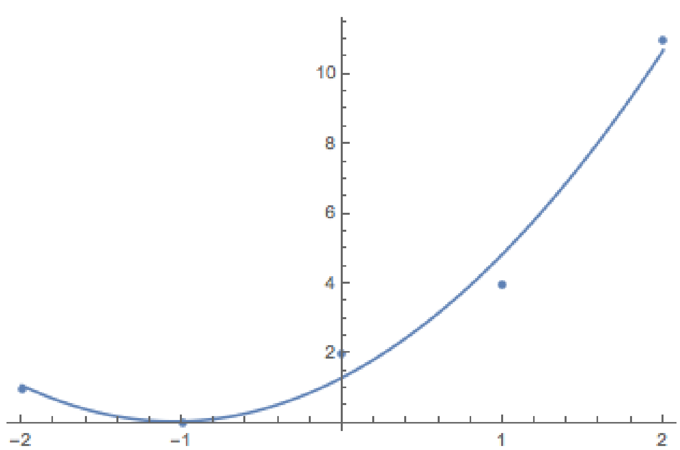


FIGURE 4.6. The last aggressively nonlinear data set with the parabola of best fit.

Section Highlights

- ▶ When a matrix equation $A\vec{x} = \vec{b}$ does not have a solution, an approximate solution can be found using the least squares solution. This is a solution to the matrix equation $A^T A\vec{x} = A^T\vec{b}$. See Definition 4.7.1 and Theorem 4.7.2.
- ▶ By converting data points into appropriate matrix equations, the technique of least squares can be used to find curves of best fit for data. See Example 4.7.3.

Exercises for Section 4.7

4.7.1. For any matrix $A \in \mathcal{M}_{m\times n}$, show that $A^T A$ is square and symmetric. That is, $A^T A \in \mathcal{M}_{n\times n}$ and $(A^T A)^T = A^T A$.

4.7.2. Suppose $A \in \mathcal{M}_{m\times n}$ and $\vec{b} \in \text{Col } A$. Prove that the solutions to $A\vec{x} = \vec{b}$ are exactly the least squares solutions to $A\vec{x} = \vec{b}$.

4.7.3. Make a matrix $A \in \mathcal{M}_{4\times 3}$ with twelve reasonably nice integers. Find the least squares solutions and least squares error for $A\vec{x} = \vec{e}_1$.

4.7.4. Repeat the previous exercise.

4.7.5. Find the cubic curve of best fit for the following data:

$$(-2, 1), (-1, 6), (0, 2), (1, 4) \text{ and } (2, 11).$$

4.8 Another Graphics Application

Convolution and Edge Detection

Definition 4.8.1 *Let* $A = [a_{ij}]$, $B = [b_{ij}] \in \mathcal{M}_{m\times n}$. *The* ***convolution*** *of* A *and* B*, denoted* $A * B$*, is*

$$A * B = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}b_{ij}|.$$

Example 4.8.1 Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 3 \\ 4 & -5 \end{bmatrix}.$$

Then

$$A*B = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}b_{ij}| = |-2a|+|3b|+|4c|+|-5d| = 2a+3b+4c+5d.$$

This probably seems like a weird thing to do with matrices. It is. Let's see just how weird. Fix a matrix $B \in \mathcal{M}_{m\times n}$ and define $T_B\colon \mathcal{M}_{m\times n} \to \mathbb{R}$ by $T_B(A) = A * B$. It turns out that T_B is *not* a linear transformation. Indeed, let

$$A = B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}.$$

One can readily check that

$$T_B(A + C) = 0 \neq 8 = T_B(A) + T_B(C).$$

Exploration 132 Is it true for any $k \in \mathbb{R}$ that $T_B(kA) = kT_B(A)$?

It's too bad for convolution that being "half linear" isn't a thing. It's worth noting that convolution is in part componentwise multiplication, and while this might seem like a reasonable way to define matrix multiplication, we see here that it definitely fails to preserve vector space properties in any reasonable way.

Definition 4.8.2 *Let*

$$G_x = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad G_y = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}.$$

*The **Sobel operator** is the function $S\colon \mathbb{M}_{3\times 3} \to \mathbb{R}$ defined by*

$$S(A) = (A * G_x) + (A * G_y).$$

"Operator" is just another word for a function. It's most commonly used in conjunction with "linear;" the term "linear operator" usually refers to a linear transformation with a vector space of functions as its domain. In the case of Sobel operators, though, we know well that it just means "function" because the Sobel operator, S, is not a linear transformation.[48]

48:  Exercise!

Example 4.8.2 Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then

$$\begin{aligned} S(A) &= (A * G_x) + (A * G_y) \\ &= |(a_{13} + 2a_{23} + a_{33}) - (a_{11} + 2a_{21} + a_{31})| \\ &\quad + |(a_{11} + 2a_{12} + a_{13}) - (a_{31} + 2a_{32} + a_{33})|. \end{aligned}$$

That's a pretty convenient way to calculate $S(A)$, but what does it mean? Why would anyone do something like this?

Let us experiment a bit with S. Perhaps we can find a use for it.

Example 4.8.3 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}, \text{and } C = \begin{bmatrix} 7 & 7 & 7 \\ 7 & 7 & 7 \\ 7 & 7 & 7 \end{bmatrix}.$$

One can check that

$$\begin{array}{lllllllll} G_x * A & = & 8 & G_x * B & = & 0 & G_x * C & = & 0 \\ G_y * A & = & 0 & G_y * B & = & 8 & G_y * C & = & 0 \end{array}$$

The last example illustrates the fact that $G_x * A$ quantifies how much the entries in a matrix A change *horizontally*. This is why $G_x * A$ is positive, and $G_y * B$ is zero; the entries of B don't change horizontally. Similarly, G_y quantifies how much the entries of a matrix A change *vertically*. This is something like a discrete version of a gradient of a function of two variables; we're quantifying the rate of change in the vertical and horizontal direction. Thus, the Sobel operator, S, quantifies the total change, both vertical and horizontal, in the entries of a 3×3 matrix.

Exploration 133 Find matrices A_1, A_2, and A_3 such that $S(A_i)$ is zero, small, and large, respectively.

The Sobel operator is a surprisingly effective detector of edges in images. Given a pixelated image (say of size 100×100), we can assign a value to each pixel to represent its color to create a matrix $A \in \mathcal{M}_{100\times 100}$. Then we apply the Sobel operator to every 3×3 section of A. Sufficiently large values for S would suggest the values in that section of A (that is, the colors in that section of the picture) were changing quickly either vertically or horizontally. This indicates the likely existence of an edge in the picture. Fun! Let's try it.

Example 4.8.4 Suppose we had a nice picture of a beautiful orange right triangle on a glorious brown plane. Using zeros for brown and ones for orange, we could construct a matrix $A \in \mathcal{M}_{12\times 12}$ to represent a badly pixilated version of such a picture. It might look something like

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&0&1&1&1&0\\ 0&0&0&0&0&0&0&1&1&1&1&0\\ 0&0&0&0&0&0&1&1&1&1&1&0\\ 0&0&0&0&0&1&1&1&1&1&1&0\\ 0&0&0&0&1&1&1&1&1&1&1&0\\ 0&0&0&1&1&1&1&1&1&1&1&0\\ 0&0&1&1&1&1&1&1&1&1&1&0\\ 0&1&1&1&1&1&1&1&1&1&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

See the triangle made of 1's? It's still pretty great; matrices make everything better. For $2 \le i, j \le 11$, let $A_{i,j}$ be the 3×3 submatrix of A centered at the entry in the ith row and jth column. For example, $A_{2,2}$ is the submatrix of entries in the shaded region in the upper left corner of the matrix, and $A_{6,6}$ is the submatrix of entries in the shaded region near the center of the matrix:

$$A_{2,2} = \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&0 \end{bmatrix} \quad \text{and} \quad A_{6,6} = \begin{bmatrix} 0&0&0\\ 0&0&1\\ 0&1&1 \end{bmatrix}.$$

Now let's construct a new matrix by applying the Sobel operator to each $A_{i,j}$:

$$\begin{bmatrix} S(A_{2,2}) & \cdots & S(A_{2,11})\\ \vdots & \ddots & \vdots\\ S(A_{11,2}) & \cdots & S(A_{11,11}) \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&0&0&2&6&4\\ 0&0&0&0&0&0&2&6&6&4\\ 0&0&0&0&0&2&6&6&2&4\\ 0&0&0&0&2&6&6&2&0&4\\ 0&0&0&2&6&6&2&0&0&4\\ 0&0&2&6&6&2&0&0&0&4\\ 0&2&6&6&2&0&0&0&0&4\\ 2&6&6&2&0&0&0&0&0&4\\ 6&6&2&0&0&0&0&0&0&4\\ 4&4&4&4&4&4&4&4&4&6 \end{bmatrix}$$

Returning to the original matrix A, we have colored white all the entries on the edge of the matrix (the Sobel operator was not defined for these entries), and we have also colored white all entries for which $S(A_{i,j}) < 4$. To be clear, we are just now seeing the entries that produce large (4 or greater) values with the Sobel operator, and we have nicely identified the boundary of the triangle.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is obviously a very simplified example for the purposes of being able to do the calculations by hand, but with minimal programing, one can implement this procedure very quickly with very high resolution images. See Figure 4.7.

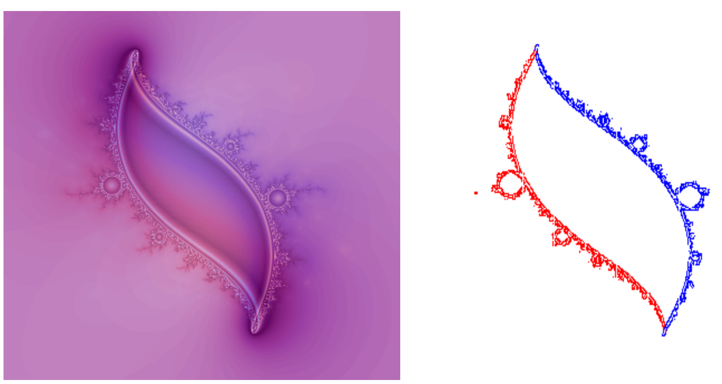


FIGURE 4.7. On the left is a fractal image. On the right is the boundary in the image, as identified using Sobel matrices.

Some optimized code and slightly more sophisticated linear algebra techniques (soon!) make applications like this very efficient. Again, this just scratches the surface of the power of linear algebra in computer graphics.

5 Square Matrices and Invariant Subspaces

It is extremely common in applications for a linear transformation to have the same vector space for its domain and codomain. In such a case, the standard matrix for this linear transformation will be a square matrix. In this chapter, we focus our attention on this very case. While most of our attention will be paid to matrix transformations, keep in mind this can all be applied to any linear transformation from a vector space to itself once we find a matrix representation for the linear transformation.

5.1 Eigenvalues and How to Find Them

In general, when we multiply a matrix A by a vector $\vec{x}$, it's not easy to know what the result will be before doing the computation. Will $A\vec{x}$ have a different magnitude than $\vec{x}$? Probably. Will it have a different direction? Yeah, probably.[1] When can we expect $A\vec{x}$ to maintain some of the properties of $\vec{x}$? This is actually a pretty common question in mathematics; many mathematical questions boil down to some kind of *invariance* issue like this.

1: Recall that quantities with magnitude and direction were a way to think about vectors from back in Chapter 1. Magnitude correlates to length and direction can be represented with a unit vector.

Eigenvalues and Eigenvectors

As we so often do, let us begin with an example.

Example 5.1.1 Consider the matrix and vector below.

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Okay, can you guess what happens when we multiply $\vec{x}$ by the matrix B? Did you do the computation in your head to figure it out? That's cheating.

Let's compute $B\vec{x}$.

$$B\vec{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

That's pretty cool. Multiplying by the matrix really just scales the vector by 3, so B preserves the direction of $\vec{x}$. Is B special, or is $\vec{x}$? What happens with other vectors? Let's check a few others.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Well, that's interesting. We found another one it scales by 3, one it scales by 1, and two that it seems to just change completely. There seems to be something special about some vectors relative to B. We should explore this more.

We state the following definition with the general term *scalar*, but for us, since our vector spaces are over $\mathbb{R}$, this means a real number.

Definition 5.1.1 *An* ***eigenvector*** *of a matrix $A \in \mathcal{M}_{n\times n}$ is a nonzero vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is called an* ***eigenvalue*** *of A if there is a nontrivial solution $\vec{x} \in \mathbb{R}^n$ of $A\vec{x} = \lambda\vec{x}$, and we call such an $\vec{x}$ an* ***eigenvector corresponding to*** *λ.*

Think about that $A\vec{x} = \lambda\vec{x}$ equation for a second. This is saying that when you multiply $\vec{x}$ by the matrix A, it's the same as just rescaling $\vec{x}$ by λ. That's dynamite! Think of all the things A could do to $\vec{x}$, and yet, it just rescales $\vec{x}$.

Example 5.1.2 Let

$$A = \begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Is $\vec{x}$ an eigenvector of A? Behold,

$$A\vec{x} = \begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = -2\vec{x}.$$

The matrix A rescales the vector $\vec{x}$ by -2, so yes, $\vec{x}$ is an eigenvector of A with eigenvalue $\lambda = -2$.

Example 5.1.3 Let

$$A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}.$$

Is $\lambda = 2$ an eigenvalue of A? This is a slightly more difficult question. We need to know if there are *nontrivial* solutions to $A\vec{x} = 2\vec{x}$. We could just solve the associated system of equations. Alternatively, note that

$$\begin{aligned} A\vec{x} &= \lambda\vec{x} && \text{if and only if} \\ A\vec{x} - \lambda\vec{x} &= \vec{0} && \text{if and only if} \\ (A - \lambda I)\vec{x} &= \vec{0}. \end{aligned}$$

where I here is the 2×2 identity matrix. This is a more familiar problem. We're looking for the kernel of the new matrix $A-2I$. That is, $\vec{x} \in \operatorname{Ker}(A-2I)$ if and only if $A\vec{x} = 2\vec{x}$. Observe that

$$A - \lambda I = A - 2I = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Thus,

$$\operatorname{Ker}(A-2I) = \operatorname{Span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}.$$

One can check that

$$\begin{aligned} A\begin{bmatrix} -2 \\ 1 \end{bmatrix} &= 2\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \\ A\begin{bmatrix} -4 \\ 2 \end{bmatrix} &= 2\begin{bmatrix} -4 \\ 2 \end{bmatrix}, \text{ and} \\ A\begin{bmatrix} 200 \\ -100 \end{bmatrix} &= 2\begin{bmatrix} 200 \\ -100 \end{bmatrix}. \end{aligned}$$

In fact, for any scalar $k \in \mathbb{R}$ and any $\vec{x} \in \mathbb{R}^2$, we know that $A(k\vec{x}) = kA\vec{x}$ so

$$A\left(k\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = 2\left(k\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right).$$

That example illustrates that if you have an eigenvector corresponding to λ, then you definitely have more than one. Specifically, if $\operatorname{Ker}(A-\lambda I)$ contains a nonzero vector, then $\operatorname{Ker}(A-\lambda I)$ is a nontrivial subspace.

Definition 5.1.2 *The set of all solutions of*

$$(A-\lambda I)\vec{x} = \vec{0}$$

is a subspace of $\mathbb{R}^n$ called the ***eigenspace corresponding to*** *λ relative to the matrix A.*

Theorem 5.1.1 *For any $n \times n$ matrix A, the eigenspace corresponding to λ is a subspace of $\mathbb{R}^n$. Note that it is a nontrivial subspace if and only if λ is an eigenvalue for A.*

PROOF. This follows from the fact that the eigenspace corresponding to λ is $\operatorname{Ker}(A-\lambda I)$, which is a subspace. We call λ an eigenvalue of A exactly when there are nontrivial solutions to $A\vec{x} = \lambda\vec{x}$. Thus, this will be a subspace of at least dimension one (i.e. nontrivial) exactly when λ is an eigenvalue. □

Example 5.1.4 Let

$$A = \begin{bmatrix} 4 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix}.$$

Let's find a basis for the eigenspace of A corresponding to the eigenvalue $\lambda = 3$.

$$A - 3I = \begin{bmatrix} 1 & 0 & -1 \\ 3 & -3 & 3 \\ 2 & -2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the eigenspace corresponding to $\lambda = 3$ is

$$\mathrm{Ker}\,(A - 3I) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\},$$

so $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis.

Example 5.1.5 Let's try that again for the same matrix A, but suppose that $\lambda = 1$.

$$A - 1I = \begin{bmatrix} 3 & 0 & -1 \\ 3 & -1 & 3 \\ 2 & -2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & -1 \\ 0 & -2 & 14/3 \\ 0 & 0 & 5/3 \end{bmatrix}$$

This matrix has a pivot in every column, so its kernel is trivial. What this calculation shows us is that $\lambda = 1$ is not a valid eigenvalue for the matrix A since $A - 1I$ does not have nontrivial solutions.

Exploration 134 Without calculation, find one eigenvalue and two linearly independent eigenvectors of

$$A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}.$$

Hint: If $\mathrm{Ker}\ A$ is nontrivial, then 0 is an eigenvalue with eigenspace $\mathrm{Ker}\ A$.

We've seen now that eigenspaces are subspaces. Let $A \in \mathcal{M}_{n\times n}$, and suppose E_1 is the eigenspace corresponding to the eigenvalue λ_1 of A and E_2 is the eigenspace corresponding to the eigenvalue λ_2 of A, where $\lambda_1 \neq \lambda_2$. What could the intersection $E_1 \cap E_2$ look like? Well, suppose $\vec{x} \in E_1 \cap E_2$. That means $\vec{x} \in E_1$, so $A\vec{x} = \lambda_1\vec{x}$. Also, $\vec{x} \in E_2$, so $A\vec{x} = \lambda_2\vec{x}$. The only way both of these equations can be true is if $\vec{x} = \vec{0}$. Thus, the intersection of eigenspaces corresponding to distinct eigenvalues must be trivial. This idea motivates the following extremely useful[2] theorem.

2:  We'll get to the useful bit in the next section. Be patient.

Theorem 5.1.2 *If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$, of an $n \times n$ matrix A, then the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ is linearly independent.*

PROOF. Suppose $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ is linearly dependent, so one of these vectors must be a linear combination of some of the others, which we may choose to be linearly independent. We can assume without loss of generality that $\vec{v}_1 = a_2\vec{v}_2 + \cdots + a_r\vec{v}_p$ for some scalars $a_2, \ldots, a_p$ (not all zero), where the vectors $\vec{v}_2, \ldots, \vec{v}_p$ are linearly independent. Then multiplying by A and using linearity of A and the fact that each $\vec{v}_i$ is an eigenvector corresponding to λ_i, we have

$$\begin{aligned} A\vec{v}_1 &= a_2 A\vec{v}_2 + \cdots + a_r A\vec{v}_p, \text{ so} \\ \lambda_1 \vec{v}_1 &= a_2 \lambda_2 \vec{v}_2 + \cdots + a_p \lambda_r \vec{v}_p. \end{aligned} \tag{5.1}$$

Multiplying $\vec{v}_1 = a_2\vec{v}_2 + \cdots + a_r\vec{v}_p$ by λ_1 and subtracting this from equation (5.1), we have

$$\vec{0} = a_2(\lambda_2 - \lambda_1)\vec{v}_2 + \cdots + a_p(\lambda_p - \lambda_1)\vec{v}_p.$$

Since the set $\{\vec{v}_2, \ldots, \vec{v}_p\}$ is linearly independent, we know that

$$a_2(\lambda_2 - \lambda_1) = \cdots = a_p(\lambda_p - \lambda_1) = 0.$$

Since the scalars $a_2, \ldots, a_p$ are not all zero, we must have $\lambda_1 = \lambda_i$ for some $2 \le i \le p$. This contradicts the fact that the eigenvalues are all distinct. Thus, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ is linearly independent. □

Corollary 5.1.3 *Suppose A is an $n \times n$ matrix with n distinct eigenvalues. Then the set of vectors formed by taking one eigenvector for each eigenvalue is a basis for $\mathbb{R}^n$. That is, there exists a basis of $\mathbb{R}^n$ consisting entirely of eigenvectors for A.*

Eigenvalue Finding Algorithms

If you're working with small enough matrices, say in $\mathcal{M}_{2\times 2}$ or $\mathcal{M}_{3\times 3}$, there are some algebraic methods for finding both eigenvalues and eigenvectors. We're going to tell you about these methods…in the next section. Unless you're working with a matrix in $\mathcal{M}_{2\times 2}$ or maybe $\mathcal{M}_{3\times 3}$, you're almost certainly going to use technology to find eigenvalues and eigenvectors, so let's explore how that works a little bit. There are *many* algorithms for finding eigenvalues and eigenvectors that make various compromises in accuracy, complication, and speed. We'll just look at the most common and simple ones here, but know that the interested reader is free to fall down that rabbit hole if they wish, but we're not gonna push you.

The Power Method

This method will be used to find just a single eigenvalue and eigenvector, and as the name implies, we will use powers of matrices to help us in our computation. Before we start though, let's take note of a property of eigenvalues and eigenvectors related to matrix powers.[3]

Theorem 5.1.4 *If $A \in \mathcal{M}_{n\times n}$ has eigenvalue λ with eigenvector $\vec{v}$, then*

$$A^k\vec{v} = \lambda^k\vec{v}$$

for any integer $k > 0$.

Rather than a formal proof, let's consider the case where $k = 2$. Then we have

$$A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda(A\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}.$$

3: I shall smite thee with my Matrix Powers!

No! Not the Matrix Powers! Ahh!

The proof for the general statement works the same, just with bulkier notation. Now, using this theorem as motivation, let's see an example of a numerical approximation procedure for finding an eigenvector.

Example 5.1.6 We'll start with a 2×2 matrix and a randomly selected vector,

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

That seems simple enough. Is $\vec{x}$ an eigenvector for A?

$$A\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Well, it seems not. Certainly $A\vec{x}$ is not a scalar multiple of $\vec{x}$. Let's see what happens when we compute $A^k\vec{x}$ for large values of k even though we know $\vec{x}$ is not an eigenvector. Since A preserves direction for eigenvectors, we'll also include here the unit vector for each result, so we can compare how the direction is changing.

	Unit Vector
$A^2\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 11 \end{bmatrix}$	$\begin{bmatrix} 0.763386 \\ 0.645942 \end{bmatrix}$
$A^8\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}^8 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 22,297 \\ 18,209 \end{bmatrix}$	$\begin{bmatrix} 0.774536 \\ 0.63253 \end{bmatrix}$
$A^{14}\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}^{14} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 37,567,993 \\ 30,674,171 \end{bmatrix}$	$\begin{bmatrix} 0.774596 \\ 0.632456 \end{bmatrix}$
$A^{20}\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}^{20} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 63,291,789,517 \\ 51,677,530,049 \end{bmatrix}$	$\begin{bmatrix} 0.774597 \\ 0.632456 \end{bmatrix}$

Well, that unit vector certainly seems to be converging to something. Could it be an eigenvector? How could we check? Let's see what A does to it.

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0.774597 \\ 0.632456 \end{bmatrix} = \begin{bmatrix} 2.67197 \\ 2.18165 \end{bmatrix} \approx 3.4495 \begin{bmatrix} 0.774597 \\ 0.632456 \end{bmatrix}$$

So, not quite an eigenvector, but approximately one.

In the following theorem, we need to refer to the magnitude of an eigenvalue. If $\lambda \in \mathbb{R}$, then the *magnitude* is the absolute value, $|\lambda|$. This corresponds to how we referred to the magnitude of a vector in $\mathbb{R}$ back in Chapter 1.

Theorem 5.1.5 (Power Method) *Suppose $A \in \mathcal{M}_{n\times n}$ has n distinct real eigenvalues $\lambda_1, \ldots, \lambda_n$ with corresponding eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ that can be arranged such that*

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|.$$

Let λ_1 be the eigenvalue with the largest magnitude, then for any $\vec{x} \notin$ Span $\{\vec{v}_2, \ldots, \vec{v}_n\}$, *$A^k\vec{x}$ approaches* Span $\{\vec{v}_1\}$ *as k increases; that is, as k goes to infinity,*

$$\left\| A^k\vec{x} - \operatorname{proj}_{\operatorname{Span}\{\vec{v}_1\}} \left(A^k\vec{x}\right) \right\| \to 0.$$

What a cool name for a theorem. I'm impressed. This says that for almost any $\vec{x}$ you pick, $A^k\vec{x}$ will approach, as k increases, an eigenvector whose eigenvalue has the largest magnitude. This must be exactly what happened in Example 5.1.6. Let's prove it.

PROOF. We can order and number all of the eigenvalues by magnitude:

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|.$$

Let $\vec{v}_1, \ldots, \vec{v}_n$ be the corresponding eigenvectors. Note that from Corollary 5.1.3 we know these eigenvectors form a basis for $\mathbb{R}^n$. Then for $\vec{x}$, we have $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ with $c_1 \neq 0$,[4] and

$$\begin{aligned} A^k\vec{x} &= A^k(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) \\ &= c_1A^k\vec{v}_1 + \cdots + c_nA^k\vec{v}_n \\ &= c_1\lambda_1^k\vec{v}_1 + \cdots + c_n\lambda_n^k\vec{v}_n. \end{aligned}$$

Factoring out λ_1^k, we have

$$\begin{aligned} A^k\vec{x} &= \lambda_1^k\left(c_1\vec{v}_1 + c_2\frac{\lambda_2^k}{\lambda_1^k}\vec{v}_2 + \cdots + c_n\frac{\lambda_n^k}{\lambda_1^k}\vec{v}_n\right) \\ &= \lambda_1^k\left(c_1\vec{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\vec{v}_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\vec{v}_n\right). \end{aligned}$$

Since $|\lambda_i/\lambda_1| < 1$ for every $i = 2, \ldots, n$, we have that $(\lambda_i/\lambda_1)^k$ is very small. Thus, for very large k, $A^k\vec{x}$ is very close to $\lambda_1^kc_1\vec{v}_1 \in \text{Span}\{\vec{v}_1\}$. More specifically,

$$\begin{aligned} \|A^k\vec{x} - \text{proj}_{\text{Span}\{\vec{v}_1\}}\left(A^k\vec{x}\right)\| &= \|A^k\vec{x} - \lambda^kc_1\vec{v}_1\| \\ &= \left\|c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\vec{v}_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\vec{v}_n\right\|, \end{aligned}$$

which goes to zero as k goes to infinity. □

More precise statements can be made and proved using limits. This is not calculus class, so we don't have to do that.[5] Nevertheless, Theorem 5.1.5 actually allows us to make an algorithm that rescales our vector at each step, and this rescaling allows it to converge to a specific vector as $k \to \infty$.

Corollary 5.1.6 *Suppose $A \in \mathcal{M}_{n\times n}$. Let λ_1 be the eigenvalue with the largest magnitude and unit eigenvector $\vec{v}_1$, and let $\{\vec{v}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a basis for $\mathbb{R}^n$. For any $\vec{x}_0 \notin \text{Span}\left\{\vec{b}_2, \ldots, \vec{b}_n\right\}$, and any integer $k \geq 0$, define $\vec{x}_{k+1} = A\vec{x}_k/\|A\vec{x}_k\|$. Then*

$$\lim_{k\to\infty}\|\vec{x}_k - \vec{v}_1\| = 0.$$

Note that here the unit vectors are incorporated into the algorithm itself to make a nice convergence statement.[6] Once we have our eigenvector, $\vec{v}_1$, finding the eigenvalue is not so hard because $A\vec{v}_1 = \lambda_1\vec{v}_1$. Let's see another example.

4: We know $c_1 \neq 0$ because we assumed $\vec{x} \notin \text{Span}\{\vec{v}_2, \ldots, \vec{v}_n\}$.

5: Note that Theorem 5.1.5 also works with complex eigenvalues by replacing absolute value with the complex number version of magnitude. However, we have simplified it here since we have so far only talked about vector spaces over $\mathbb{R}$.

So far?! Wait, are we going to talk about complex vector spaces?

Shhh! Not here.

6: It should also be noted here that convergence in norm like this only implies the convergence of the vectors here because our vector space is finite dimensional.

Wait. Things break when you need a basis with infinite vectors?

Yep. We're not going to talk about any of those things here, though.

Example 5.1.7 Now that we have the theorem, we can be a bit more efficient here. Let

$$B = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}.$$

Now, if our theorem applies, we should see that applying a large enough power of B to some vector $\vec{x}$ will approximate an eigenvector. Let's try $k = 20$ and

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then

$$B^{20}\vec{x} = \begin{bmatrix} 6,625,109 \\ -9,369,319 \end{bmatrix} \text{ with unit vector } \vec{u} = \begin{bmatrix} 0.57735 \\ -0.816497 \end{bmatrix}.$$

Now, we can check whether $\vec{u}$ is an approximate eigenvector.

$$\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0.57735 \\ -0.816497 \end{bmatrix} = \begin{bmatrix} 1.39385 \\ -1.9712 \end{bmatrix} \approx 2.414\begin{bmatrix} 0.57735 \\ -0.816497 \end{bmatrix}$$

Yay! We found an approximate eigenvalue and eigenvector.

Exploration 135 Use technology capable of computing powers of matrices to recreate the previous example with the matrix

$$C = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}.$$

While this method works almost all the time, you'll note that there were a few conditions on the theorem for when this exact technique applies and also on that initial vector choice. Also, be aware that we have worked examples of this with 2×2 matrices just for illustrative purposes. There are other techniques that are sometimes preferable for smaller matrices.

Perhaps you are sad that the Power Method only gives you one eigenvector and one eigenvalue. That is sad. It definitely makes me sad. Think of how left out all those other eigenvalues must feel. Fortunately, there are algorithms that find more eigenvalues. Let's see a different method.

The QR Method

For this method, we will need to *decompose* our matrix. This means we will write our matrix as the product of other matrices. We've already seen this in Chapter 4 when we discussed the change of basis matrices though we did not use this term, and we will discuss it quite a bit more in the upcoming sections. First, we need some handy definitions and theorems.

Definition 5.1.3 *The* ***main diagonal*** *of a square matrix $A \in \mathcal{M}_{n\times n}$ are the entries $a_{11}, a_{22}, \ldots a_{nn}$ starting at the upper left corner of the matrix and going diagonally to the lower right entry. A matrix is called* ***upper (lower) triangular*** *if all the entries below (above) the main diagonal are zero.*

Remember orthogonality from Chapter 2? What's better than orthogonality? Orthonormality!

Definition 5.1.4 *A matrix $A \in \mathcal{M}_{n\times n}$ is an* ***orthogonal matrix*** *if it has orthonormal columns.*

An "orthogonal" matrix has "ortho*normal*" columns? We concede this is confusing terminology but happily pass the buck to our mathematical ancestors. We'll gladly let this one slide because they also coined the terminology "spectral theory."[7] We'll talk about spectral theory soon. Anyway, orthogonal matrices, you will not be surprised, have very cool properties.

7: A term cool enough for superheroes to misuse so they sound smart.

Lemma 5.1.7 *If $A \in \mathcal{M}_{n\times n}$ is orthogonal, then $A^{-1} = A^T$.* [8]

8: Easiest inverse to calculate, ever!

What about I_n?

That's an orthogonal matrix, so it's inverse calculation is a subset of these calculations.

What calculation? It's its own inverse.

:...

PROOF. Let $A = [\vec{a}_1 \cdots \vec{a}_n]$. Using the definition of matrix multiplication from Theorem 4.4.5 gives that the a_{ij} entry of A^TA is $\vec{a}_i \cdot \vec{a}_j$. The result follows immediately then from the fact that $\vec{a}_i \cdot \vec{a}_j = 0$ if and only if $i \neq j$, and $\vec{a}_i \cdot \vec{a}_j = 1$ if and only if $i = j$ since the columns are orthonormal. □

Theorem 5.1.8 (QR Decomposition) *If $A \in \mathcal{M}_{n\times n}$, then there is an orthogonal matrix $Q \in \mathcal{M}_{n\times n}$ and an upper triangular matrix $R \in \mathcal{M}_{n\times n}$ such that $A = QR$.*

PROOF. All we have to do is perform the Gram-Schmidt process to the columns of $A = [\vec{a}_1 \ \cdots \ \vec{a}_n]$.[9] Let $\vec{w}_1, \ldots, \vec{w}_n$ be the resulting vectors. Then normalize them (divide each of them by their own norm so the resulting vector has norm one); for $i = 1, \ldots, n$, define $\vec{u}_i = \vec{w}_i/\|\vec{w}_i\|$. It follows that $Q = [\vec{u}_1 \ \cdots \ \vec{u}_n] \in \mathcal{M}_{n\times n}$ is an orthogonal matrix. Now define the upper triangular matrix

9: Yeah! Gram-Schmidt those columns!

$$R = [\vec{r}_1 \ \cdots \ \vec{r}_n] = \begin{bmatrix} \vec{a}_1\cdot\vec{u}_1 & \vec{a}_2\cdot\vec{u}_1 & \cdots & \vec{a}_n\cdot\vec{u}_1 \\ 0 & \vec{a}_2\cdot\vec{u}_2 & \cdots & \vec{a}_n\cdot\vec{u}_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \vec{a}_n\cdot\vec{u}_n \end{bmatrix} \in \mathcal{M}_{n\times n}.$$

One can check that $A = QR$; behold that for any $i = 1, \ldots, n$, we have

$$\begin{aligned} Q\vec{r}_i &= [\vec{u}_1 \ \cdots \ \vec{u}_n] \begin{bmatrix} \vec{a}_i\cdot\vec{u}_1 \\ \vec{a}_i\cdot\vec{u}_2 \\ \vdots \\ \vec{a}_i\cdot\vec{u}_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= (\vec{a}_i\cdot\vec{u}_1)\vec{u}_1 + (\vec{a}_i\cdot\vec{u}_2)\vec{u}_2 + \cdots + (\vec{a}_i\cdot\vec{u}_i)\vec{u}_i = \vec{a}_i. \end{aligned}$$

□

That last equality is some sneaky business with Gram-Schmidt. Expect to work out the details as an exercise. Also, fun fact: there is a version of this theorem for rectangular matrices as well.[10] Note that the proof of this theorem, quite importantly, gives us a way *to find* the QR decomposition for a matrix, not just tells us that it exists. That's so helpful! Also, if you perhaps skipped over the proof, you should definitely go back and look at that. Let's see an example.

10: Non-square matrices are dead to me!

That's mean, Bubbles! Guess who's writing the proof for the general statement now?

...Um...the readers?

Oh, I was thinking *you*, but sure. That's fine, too.

Example 5.1.8 Let's find this decomposition for the matrix from Example 5.1.6,

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}.$$

We'll refer to the columns here as $\vec{a}_1$ and $\vec{a}_2$, respectively. Now, the Gram-Schmidt process gives us

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and}$$

$$\vec{w}_2 = \vec{a}_2 - \operatorname{proj}_{\vec{w}_1}(\vec{a}_2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \frac{5}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Normalizing these vectors gives us

$$\vec{u}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \text{ and } \vec{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}.$$

Thus,

$$Q = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}.$$

Exploration 136 Verify that $A = QR$ with the matrices from the example above.

Now that we've learned about our decomposition, we can use it to find eigenvalues and eigenvectors. The following theorem gives us our algorithm.

Theorem 5.1.9 (QR Algorithm) *Suppose $A_0 \in \mathcal{M}_{n\times n}$ has n distinct eigenvalues such that*

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| > 0.$$

For any integer $k \geq 0$, let $Q_k R_k$ be the QR decomposition for A_k, and define

$$A_{k+1} = R_k Q_k.$$

Then the diagonal entries of A_k converge to the eigenvalues of A_0 as $k \to \infty$.

We're not going to prove this one; we'd need some machinery that's scattered throughout the next few sections. Thus, we will settle for seeing it in action. First, let's take note of a simplification. By using Lemma 5.1.7, the application of this theorem becomes a little easier in practice. Since $A_k = R_k Q_k$, we know $R_k = Q_k^{-1}A_k = Q_k^T A_k$ by Lemma 5.1.7. Then we have

$$A_{k+1} = Q_k^T A_k Q_k,$$

so we don't even need to compute R. That's neat! Now, it's time for that example.

Example 5.1.9 Here's a matrix,

$$A_0 = \begin{bmatrix} 5 & 6 & -6 \\ -7 & -12 & 15 \\ -4 & -8 & 11 \end{bmatrix},$$

with eigenvalues 3, 2, and -1. A quick[11] computation shows that, with entries rounded to the nearest thousandth,

$$\begin{aligned} A_1 &= \begin{bmatrix} 3.205 & 1.291 & 26.28 \\ 0.391 & 2.419 & 2.18 \\ -0.101 & -0.108 & -1.624 \end{bmatrix} \\ A_3 &= \begin{bmatrix} 2.92 & 0.604 & 26.172 \\ 0.215 & 2.141 & -4.046 \\ -0.011 & -0.007 & -1.061 \end{bmatrix} \\ A_{17} &= \begin{bmatrix} 3 & 0.334 & 24.288 \\ 0.001 & 2 & -10.582 \\ 0 & 0 & -1 \end{bmatrix}. \end{aligned}$$

Note that in A_1 the diagonal entries are already getting near the eigenvalues. Did we cheat by starting with a matrix whose diagonal entries were pretty close to the eigenvalues already? Yes. Yes, we did. It's not uncommon to need hundreds of steps for this algorithm for reasonable precision.

11: Ha!

This (or, technically, a more procedurally complicated, more efficient version[12] of this) is a super common way to find eigenvalues. However, you probably noticed that hypothesis about needing n distinct nonzero eigenvalues all with different magnitudes. Yeah. That's a strong one. It makes proving the convergence in this theorem possible, though. The good news is, *this algorithm works a lot of the time in general*, even without knowing anything about the eigenvalues.

12: How can something more complicated be more efficient?

There are more steps in the algorithm, but it converges *much* faster.

Section Highlights

- Suppose A is an $n \times n$ matrix and $\vec{x} \in \mathbb{R}^n$. We say nonzero $\vec{x}$ is an eigenvector for A if $A\vec{x} = \lambda\vec{x}$ for some $\lambda \in \mathbb{R}$. In this situation, we call λ an eigenvalue for A. See Definition 5.1.1.
- A real number λ is an eigenvalue for A if and only if $\dim \operatorname{Ker}(A - \lambda I) \geq 1$. Moreover, any nonzero vector in $\operatorname{Ker}(A - \lambda I)$ is an eigenvector with eigenvalue λ. See Theorem 5.1.1.
- Any collection of eigenvectors corresponding to distinct eigenvalues form a linearly independent set. See Theorem 5.1.2.
- The power method and QR method are used to approximate eigenvalues of matrices. See Example 5.1.6 and Example 5.1.9.

Exercises for Section 5.1

5.1.1. Let $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$. Multiply each of the following vectors by A to determine whether or not they are eigenvectors for A. If they are, state the eigenvalue.

(a) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, (e) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}$, (g) $\begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$,

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, (d) $\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, (f) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, (h) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

5.1.2. Ricky has affixed a picture of Bubbles onto the square in $\mathbb{R}^2$ whose vertices are $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. (He's done this at least once before.) He decides again that this is much too small, and he prefers that Bubbles faces the other direction. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that makes the picture of Bubbles twenty times bigger and reflects the image across the vertical axis. Find the eigenvalues for T. Justify. *Hint: No algebra required!*

5.1.3. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation that interchanges the x_1 axis with the x_3 axis, maps the x_2 axis to $\vec{0}$, and does nothing to the x_4 axis. Find all eigenvalues for T. Justify. *Hint: No algebra required!*

5.1.4. Determine whether or not 2 is an eigenvalue for each of the following matrices.

(a) $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 1 & \pi & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

5.1.5. The following matrices all have eigenvalues 1 and 2. Find a basis for the eigenspace for each eigenvalue.

(a) $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

5.1.6. Let $B = \begin{bmatrix} -1 & 4 & -1 \\ 1 & 2 & -1 \\ 3 & -1 & 0 \end{bmatrix}$. The eigenvalues for B are -2, 2, and 1. Find an eigenvector corresponding to each eigenvalue.

5.1.7. Let $B = \begin{bmatrix} -1 & 4 & 0 \\ 1 & 2 & 0 \\ 3 & -1 & 0 \end{bmatrix}$. The eigenvalues for B are -2, 3, and 0. Find an eigenvector corresponding to each eigenvalue.

5.1.8. The matrix

$$A = \begin{bmatrix} 1 & 0 & -2 & 2 \\ -3 & 7 & -6 & 3 \\ -3 & 5 & -4 & 3 \\ -3 & 4 & -6 & 6 \end{bmatrix}$$

has eigenvalues 1, 2, 3, and 4. According to Theorem 5.1.2, the associated eigenvectors should form a linearly independent set. Find an eigenvector for each eigenvalue, and verify that this set of four eigenvalues is linearly independent.

5.1.9. Explain why $A \in \mathcal{M}_{n\times n}$ can have at most n distinct eigenvalues.

5.1.10. Suppose $A^2 = I$. Show that the only possible eigenvalues are -1 and 1.

5.1.11. Determine whether or not the following matrices are orthogonal matrices.

(a) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & \sqrt{2/3} & -1/\sqrt{6} \end{bmatrix}$

(c) $C = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 11/\sqrt{3} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{23} \\ -1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$

5.1.12. Let $A = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$. Use the Power Method with $\vec{e}_1$ to approximate an eigenvector for the largest magnitude eigenvalue for A.

5.1.13. Let $A = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$. Find the QR decomposition for A.

5.1.14. Let $\vec{a}_1, \ldots, \vec{a}_n \in \mathbb{R}^n$, and let $\{\vec{u}_1, \ldots, \vec{u}_n\}$ be the orthonormal set of vectors resulting from performing Gram-Schmidt on $\vec{a}_1, \ldots, \vec{a}_n$ and normalizing. Show that for $i = 1, \ldots, n$,

$$(\vec{a}_i \cdot \vec{u}_1)\vec{u}_1 + (\vec{a}_i \cdot \vec{u}_2)\vec{u}_2 + \cdots + (\vec{a}_i \cdot \vec{u}_i)\vec{u}_i = \vec{a}_i.$$

5.1.15. Adapt the statement and proof of Theorem 5.1.8 to hold for $A \in \mathcal{M}_{m\times n}$, where $m \neq n$.

5.2 Determinants and More Fun with Eigenvalues

It's been a while since we dealt with real-valued functions, so this will be a nice change.[13] The first part of this section is devoted to defining a function from $\mathcal{M}_{n\times n}$ to $\mathbb{R}$ that has some very convenient properties. It's called the *determinant*. Then, we will use this function as a tool to compute eigenvalues, as we promised in the previous section. Since our ultimate goal here will be our hunt for eigenvalues, we have omitted many of the proofs related to the determinant function.

13: Really? Somehow I'm skeptical.

Determinants

Here's a neat fact. The determinant is the *unique* function from $\mathcal{M}_{n\times n}$ to $\mathbb{R}$ that is n-linear, alternating, and maps the identity matrix I_n to 1. If we had a couple dozen extra pages we could figure out what all of that means and use it to define the determinant function. That would be really great. However, we're going to take a more…um…*equestrian* approach. Yeah, that's the word I wanted. This way has more horses. Tons more horses. [14]

14: How dare you betray us after all we've done for you! Horses?!

Oh? Did you think I meant *pedestrian*? Nope. Not at all. There is nothing the slightest bit pedestrian about determinants. Ultimately, we're just building a function here (pedestrian), but this function is neat and hairy (equestrian). However, rather than build the spike-wheeled chariot of alternating, multilinear functionals, we're going to build a more equestrian/pedestrian saddle of a function. Just know in your heart that if you wanted to turn in your determinant saddle for a determinant chariot, all you have to do is watch *Ben Hur* and look up "determinant" on Wikipedia.

Commence construction! Let $\det\colon \mathcal{M}_{2\times 2} \to \mathbb{R}$, and let's make it so that for any $A \in \mathcal{M}_{2\times 2}$, we have $\det(A) \neq 0$ if and only if A is invertible.[15] From this, we can build a formula for $\det$. In Section 4.5, we saw that a matrix

15: Well, that's a handy property!

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathcal{M}_{2\times 2}$$

has an inverse

$$\frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} -a_{11} & a_{21} \\ a_{12} & -a_{22} \end{bmatrix}$$

if and only if $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Great! Then we define $\det\colon \mathcal{M}_{2\times 2} \to \mathbb{R}$ by

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

That was easy, like riding a bike (being pulled by a horse).[16] Here's a definition of a function $\det\colon \mathcal{M}_{n\times n} \to R$ that uses this $\det$ function we just defined on $\mathcal{M}_{2\times 2}$.

16: Strike that. The authors have it on good authority that you don't want to ride a bike being pulled by a horse. Ever.

Definition 5.2.1 *For $n \geq 2$, the* ***determinant*** *of a matrix $A = [a_{ij}] \in \mathcal{M}_{n\times n}$ is the sum*

$$\begin{aligned}\det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{n+1}a_{1n}\det A_{1n}\\ &= \sum_{j=1}^{n}(-1)^{j+1}a_{1j}\det A_{1j},\end{aligned}$$

where $A_{ij} \in \mathcal{M}_{(n-1)\times(n-1)}$ is the submatrix of A resulting from removing the ith row and jth column.

Defining the determinant this way allows us to maintain the property that $\det(A) \neq 0$ if and only if A is invertible; we'll prove that later in this section. It also turns out that this determinant function is unique (in a very specific sense). However, perhaps you are asking yourself the following question: What is this grossly complicated thing? That is a fair question. Hopefully an example will clarify.

Example 5.2.1 Let's calculate a determinant. Let $A \in \mathcal{M}_{3\times 3}$, so our formula from Definition 5.2.1 is

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}.$$

If

$$A = \begin{bmatrix} 1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix},$$

then

$$\begin{aligned} a_{11} = 1, \quad A_{11} &= \begin{bmatrix} 5 & 6\\ 8 & 9\end{bmatrix};\\ a_{12} = 2, \quad A_{12} &= \begin{bmatrix} 4 & 6\\ 7 & 9\end{bmatrix}; \text{ and}\\ a_{13} = 3, \quad A_{13} &= \begin{bmatrix} 4 & 5\\ 7 & 8\end{bmatrix}.\end{aligned}$$

It follows that

$$\begin{aligned}\det\begin{bmatrix} 1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix} &= \det\begin{bmatrix} 5 & 6\\ 8 & 9\end{bmatrix} - 2\det\begin{bmatrix} 4 & 6\\ 7 & 9\end{bmatrix} + 3\det\begin{bmatrix} 4 & 5\\ 7 & 8\end{bmatrix}\\ &= (-3) - 2(-6) + 3(-3) = 0\end{aligned}$$

That was anticlimactic. Neglect ye not the minus signs that alternate from submatrix to submatrix!

Exploration 137 Calculate the determinants of the matrices below.

$$A = \begin{bmatrix} 1 & 2 & 3\\ 4 & 5 & 6\\ 0 & 8 & 9\end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 3\\ 4 & 5 & 6\\ 0 & 8 & 9\end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0\\ 4 & 5 & 6\\ 400 & 8 & 9\end{bmatrix}$$

Did you appreciate the zeroes across the top row in that matrix C? Yes, we thought you would. Suppose those zeroes had been in the second row. It would have still simplified the computation, but not quite as much. It'd be great if we could just pick whichever row or column we wanted and use that to compute the determinant. We could always pick the one with the most zeroes. Well, guess what? We can do exactly that. Before stating this as a theorem, though, we need a definition.

Definition 5.2.2 *Let $A \in \mathcal{M}_{n\times n}$ and suppose A_{ij} is the submatrix formed by deleting the i^{th} row and j^{th} column from A. Then the number*

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

*is called the $(\mathbf{i}, \mathbf{j})-$**cofactor** of A.*

With this terminology and notation, we can write Definition 5.2.1 as

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

We call this a **cofactor expansion** of the determinant. Now, for that lovely theorem about picking your favorite row or column.

Theorem 5.2.1 *The determinant of a matrix $A \in \mathcal{M}_{n\times n}$ can be computed by a cofactor expansion across any row or down any column. In particular, the expansion across the i^{th} row is*

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

and the expansion down the j^{th} column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Example 5.2.2 Let's start with a matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

Now, we can compute the determinant across any row or column. Let's pick row 2!

$$\begin{aligned} \det A =& -0\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} - 0\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \\ =& 2(0-3) = -6 \end{aligned}$$

Let's do it again! This time, we'll use column 3!

$$\begin{aligned} \det A =& 1\begin{bmatrix} 0 & 2 \\ 3 & 2 \end{bmatrix} - 0\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} + 0\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \\ =& 1(0-6) = -6 \end{aligned}$$

Is 0 now your favorite number? Maybe it should be.

Now, let's talk about those negative signs. Did you catch in Example 5.2.2 that they alternated in the pattern $-$, $+$, $-$ when going across the second row but $+$, $-$, $+$ when going down the third column? Well, there's a nice way to remember where the negative signs go. Whatever size your matrix, you can build a checkerboard pattern to remind you how these work. Here are the checkerboards for 3×3 and 4×4 for reference.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \quad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Exploration 138 Compute the determinant by cofactor expansion along the row or column that involves the least amount of computation.

$$A = \begin{bmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{bmatrix}$$

That last theorem simplified our computations when we have a few zeroes. This next one will simplify it even more depending on where the zeroes are located.

Theorem 5.2.2 *The determinant of an $n \times n$ triangular matrix A is the product of the diagonal entries.*

PROOF. If $A \in \mathcal{M}_{n\times n}$ is upper triangular, the result follows from doing each cofactor expansion down the first column. If A is lower triangular, do each cofactor expansion down the last column. $\square$

Example 5.2.3 Let's calculate some more determinants!

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & \pi^2 \\ 0 & 1 & 4 & 6 & 42 \\ 0 & 0 & 4 & 8 & e \\ 0 & 0 & 0 & 1 & 1,234 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

By Theorem 5.2.2, $\det A = 4$.

This method was great, but it relied on having a very specific form for the matrix. Are we allowed to alter the form of the matrix? Well, yes and no. We have those handy row operations, but we'd need to know how those interact with the determinant. It seems we are ready to explore how this function behaves with some of the other things we've encountered.

Properties of Determinants

By carefully breaking down the definition of the determinant, we can say precisely how it is affected by each row operation.

Theorem 5.2.3 *(Row Operations and Determinants) Let A be a square matrix.*

(a) *If a multiple of one row of A is added to another to produce matrix B, then* $\det B = \det A$.

(b) *If two rows of A are interchanged to produce B, then* $\det B = -\det A$.

(c) *If one row of A is multiplied by k to produce B, then* $\det B = k \cdot \det A$.

Thus, we are allowed to modify the format of a matrix to help in calculating the determinant, but we'll need to carefully keep track of the row operations to do so.

Example 5.2.4 Let's use properties of determinants to simplify the computation of determinants.

$$A = \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{bmatrix}$$

Using only the row operation of adding a multiple of one row to another, one can show that

$$A \to \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 10 \end{bmatrix} = B$$

By Theorem 5.2.3, $\det A = \det B$, and by using the cofactor expansion across the third row of B, we see that $\det B = 0$. Thus, $\det A = 0$.

Recall that elementary matrices allow us to convert row operations to matrix multiplication. Thus, we have the following corollary.

Corollary 5.2.4 *Suppose $A \in \mathcal{M}_{n\times n}$ and $E \in \mathcal{M}_{n\times n}$ is an elementary matrix. Then* $\det EA = \det E \det A$.

This proof here can be done by computing $\det E$ for each type of elementary matrix and comparing to Theorem 5.2.3, but we will leave this for the exercises. This corollary might have you questioning whether we can always break up the determinant across matrix multiplication. Well, yes, actually we can. Before we can prove that though, we need our promised connection between the determinant and invertibility.

Theorem 5.2.5 *A square matrix A is invertible if and only if* $\det A \neq 0$.

PROOF. This follows from Theorem 5.2.3. Let B be the row-echelon form for the matrix A. We can always obtain row-echelon form without rescaling the rows since we are not required to make the pivots 1s, and since adding a scalar multiple of a row does not change the determinant, we only need to keep track of the row swaps to compute the relationship between $\det A$ and $\det B$. Suppose r row swaps were needed. Then

$$\det A = (-1)^r \det B.$$

Now, A is invertible if and only if B has a pivot in every column, which means A is invertible if and only if B has no zeroes on its diagonal. Thus, the result follows from Theorem 5.2.2. $\square$

Example 5.2.5 Use determinants to decide if the matrix is invertible.

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix}$$

Since $\det A = -1 \neq 0$, A is invertible by Theorem 5.2.5.

Because of the Invertible Matrix Theorem, knowing whether or not a matrix is invertible can also tell us other interesting things.

Exploration 139 Use determinants to decide if the set of vectors is linearly independent.

$$\begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} -8 \\ 5 \\ 7 \end{bmatrix}, \quad \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$$

Now, we have everything needed to expand Corollary 5.2.4 into a more general statement.

Theorem 5.2.6 *(Multiplicative Property) If $A, B \in \mathcal{M}_{n\times n}$, then*

$$\det AB = (\det A)(\det B)$$

PROOF. Let $A, B \in \mathcal{M}_{n\times n}$. Suppose first that A is not invertible. Then $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is not onto and $\operatorname{Imag} T_A \neq \mathbb{R}^n$. This means $T_{AB} = T_A \circ T_B$ cannot be onto since $\operatorname{Imag} T_{AB}$ must be a subset of $\operatorname{Imag} T_A$. Thus, AB is also not invertible. By Theorem 5.2.5, $\det A = 0$ and $\det AB = 0$. We can then conclude $\det AB = 0 = \det A \det B$.

Suppose now that A is invertible. Then A is row equivalent to the identity matrix and there is some sequence of row operations $\{r_1, \ldots, r_k\}$ such that $A = (E_{r_k} \cdots E_{r_1})I_n$ where E_{r_i} are each elementary matrices. We saw in Corollary 5.2.4 that the determinant breaks up across the product of an elementary matrix and a regular matrix, so we have

$$\begin{aligned} \det AB &= \det((E_{r_k} \cdots E_{r_1})I_n B) \\ &= (\det E_{r_k}) \cdots (\det E_{r_1})(\det I_n)(\det B) \\ &= (\det A)(\det B). \end{aligned}$$

$\square$

Beware! There is no analogue for sums of matrices. That is, $\det(A+B) \neq \det A + \det B$ in general. We'll verify this in an exercise.

Exploration 140 Let's use Theorem 5.2.6 to find the relationship between $\det A$ and $\det A^{-1}$ when A is invertible.

Suppose we know $\det A = k$ for some $k \in \mathbb{R}$. If A is invertible, we know $k \neq 0$ from Theorem 5.2.5. Then there exists some A^{-1} such that $AA^{-1} = I_n$. From Theorem 5.2.2, what must $\det I_n$ be?

Finally, what must $\det A^{-1}$ be?

Exploration 141 Compute $\det A^5$ if $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

Let's see one last property before moving on to eigenvalues. From Theorem 5.2.1, we know $\det A$ can be computed using a cofactor expansion along any row or column. Thus, since the transpose operation just swaps rows and columns, the following theorem should not be surprising.

Theorem 5.2.7 *If $A \in \mathcal{M}_{n\times n}$, then $\det A^T = \det A$.*

Finding Eigenvalues with Algebra

You've probably noticed we haven't explicitly outlined a method for finding eigenvalues by hand. That must be very annoying. We should fix that. But first, more terminology!

Definition 5.2.3 *For $A \in \mathcal{M}_{n\times n}$, the degree n polynomial $\det(A - \lambda I)$ is called the* ***characteristic polynomial*** *for A.*

Yes, take the determinant. What does the determinant tell us? Well, if it's nonzero, our matrix is invertible. But do we want $A - \lambda I$ to be invertible? No! We want a nontrivial kernel for $A - \lambda I$. We very specifically want to figure out which λ give us a non-invertible matrix $A - \lambda I$. Thus, we need to know when this characteristic polynomial is zero. Let's bundle this up into a theorem.

Theorem 5.2.8 *For $A \in \mathcal{M}_{n\times n}$, λ is an eigenvalue of A if and only if it is a zero of the characteristic polynomial for A.*

Yay! We've turned this into the algebra problem of finding zeros for a polynomial. That's easy, right?[17]

17:  Right... This is actually an extremely deep problem. For a polynomial of degree two, we have the quadratic equation, and there are also formulas that exist for degree 3 and degree 4. One of the most groundbreaking results of the early 1800s is that no similar formula exists for polynomials of degrees 5 and higher. Mathematicians had searched for a solution for *hundreds* of years before it was proven none existed.

 So... maybe not easy?

 *Hundreds* of years? Is that supposed to be a long time?

 ...

Example 5.2.6 Let's find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 0 \\ 5 & 1 & 0 \end{bmatrix}.$$

We have

$$\begin{aligned}
\det(A - \lambda I) &= \det\left(\begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 0 \\ 5 & 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 0 \\ 5 & 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) \\
&= \det\begin{bmatrix} -\lambda & 1 & 5 \\ 0 & 1-\lambda & 0 \\ 5 & 1 & -\lambda \end{bmatrix} \\
&= -\lambda[(1-\lambda)(-\lambda)] + 5[-5(1-\lambda)] \\
&= -\lambda^3 + \lambda^2 + 25\lambda - 25 = -(\lambda - 1)(\lambda + 5)(\lambda - 5).
\end{aligned}$$

Thus, we know that $\det(A - \lambda I) = 0$ when $\lambda = -5$, 1, and 5; thus, the eigenvalues of A are -5, 1, and 5.

Exploration 142 Find the eigenvalues of

$$A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

using Theorem 5.2.8.

Great! This gives us a systematic approach to finding eigenvalues by hand.[18] But sometimes, our task is even easier. For example, if it is quickly apparent that the matrix is itself not invertible, then 0 is one of its eigenvalues. Also, if the matrix is triangular, either upper or lower triangular, we have the following result.

18:  Hem, hem. When $n \leq 4$. Otherwise you need some machine to do factoring approximation for you.

Theorem 5.2.9 *The eigenvalues of a triangular matrix are the entries on its main diagonal.*

PROOF. Let $A \in \mathcal{M}_{n\times n}$ be triangular. Then the matrix $A - \lambda I$ is also triangular. For a triangular matrix, we can compute the determinant by multiplying the entries on the diagonal. Thus, if $a_1, a_2, \dots, a_n$ are the entries on the diagonal of A, the diagonal of $A - \lambda I$ is $(a_1 - \lambda), (a_2 - \lambda), \dots, (a_n - \lambda)$. So $\det(A - \lambda I) = (a_1 - \lambda)(a_2 - \lambda)\cdots(a_n - \lambda)$, and its zeros are exactly the entries on the diagonal of A. $\square$

Exploration 143 We know from Theorem 5.2.9 that the eigenvalues of

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

should be a, d, and f. Verify this by finding $\det(A - \lambda I)$, where $\lambda = a, d, f$.

At this point, we've defined all the eigenstuff[19] and developed a variety of methods for finding both eigenvalues and eigenvectors. We've proven our supposedly useful theorem about eigenvectors for different eigenvalues being linearly independent. What's left? More terminology, you say? Why, of course, if you insist.

19: What a fun word!
It's not a real word.

Definition 5.2.4 *For an eigenvalue λ of a matrix $A \in \mathcal{M}_{n\times n}$, the* ***algebraic multiplicity*** *of λ is the multiplicity of λ as a root of the characteristic polynomial for A. The* ***geometric multiplicity*** *of λ is the dimension of the eigenspace corresponding to λ.*

The multiplicity of λ as a root of a polynomial, $p(x)$, is then number of times $x - \lambda$ appears in the factorization of $p(x)$. For example, the multiplicity of 3 as a roof of $x^2 - 6x + 9$ is two because $x^2 - 6x + 9 = (x-3)^2$.

Algebraic and geometric multiplicity give us a language to help organize the information about a matrix's eigenvalues and eigenvectors.

Example 5.2.7 Let

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}.$$

Since A is triangular, we know its only eigenvalue is 5. Moreover, the characteristic polynomial is $(5-\lambda)^3$, so the eigenvalue 5 has algebraic multiplicity of 3. Computing the geometric multiplicity takes a bit more work. We need to find the dimension of $\mathrm{Ker}\,(A - 5I)$.

$$A - 5I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Oh, well maybe it's not much work in this case. This matrix has exactly two pivots, so the dimension of the kernel is one by the Rank-Nullity Theorem.[20] Thus, the geometric multiplicity of the eigenvalue 5 is 1.

20: Thank you, Ronnie Rank and Noether Nullity!

It would be really great if algebraic and geometric multiplicities were the same all the time.[21] It would also be great if the world was all candy canes and choo-choo trains, where the children of tomorrow dream away.[22]

21: Clearly this isn't the case. Besides the evidence of the previous example, why would there be two types of multiplicities with distinct names if they always agreed?

22: It's not.

Example 5.2.8 Let

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

By Theorem 5.2.9, 3 is the only eigenvalue of both A and B. For both matrices, the characteristic polynomial is $(\lambda - 3)^2$, so the algebraic multiplicity of the eigenvalue 3 is 2. However, it's easy to check that $\dim \operatorname{Ker}(A-3I) = 1$, and $\dim \operatorname{Ker}(B-3I) = 2$. Thus, the geometric multiplicity of the eigenvalue 3 for A is 1, and the geometric multiplicity of the eigenvalue 3 for B is 2.

You probably noticed that A and B are row equivalent. Yeah, that's right, they're row equivalent but have different eigenspaces. *Row operations mess up your eigenspace*. Fun fact: both A and B are also row equivalent to the identity, which doesn't even have the same eigenvalues, let alone eigenspaces. How messed up is that?

Exploration 144 Compare the eigenspaces of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -2 & -3 & 8 \\ 1 & 2 & -3 \\ -5 & -5 & 9 \end{bmatrix}.$$

Hint: They have the same eigenvalues.

Maybe you noticed that A and B are both invertible, so they're row equivalent matrices since both are row equivalent to I_3. What are we to conclude from this? Sometimes matrices with the same eigenvalues have the same geometric multiplicity and sometimes they don't? That is very dissatisfying. If only there were a way to classify when this happens? Clearly, row equivalence is not the right condition, but what could this condition be?[23]

23: Stay tuned for next week's episode, when our heroes Ronnie and Noether unravel the mystery of the unmatched algebraic and geometric multiplicities!

Section Highlights

- The determinant of a square matrix is a real number associated to that matrix that is computed through a recursive algorithm. See Definition 5.2.1, Example 5.2.1 and Theorem 5.2.1.
- A matrix is invertible if and only if its determinant is nonzero. See Theorem 5.2.5.
- If a matrix is diagonal, upper triangular, or lower triangular, then the determinant is the product of the diagonal entries, and the entries on

the diagonal are the eigenvalues. See Theorem 5.2.2 and Theorem 5.2.9.

- The characteristic polynomial of a matrix A is $\det(A - xI)$. Its zeros are the eigenvalues of A. See Definition 5.2.3 and Theorem 5.2.8.
- Every eigenvalue has an algebraic multiplicity (number of times it is the zero of the characteristic polynomial) and geometric multiplicity (dimension of the kernel of the associated eigenspace). See Definition 5.2.4.

Exercises for Section 5.2

5.2.1. Find the determinants of the matrices below.

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & -4 \\ 1 & 1 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 2 \\ -2 & 0 & -4 \\ 0 & 1 & 2 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(h) $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

(i) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 2 & 1 & 1 \end{bmatrix}$

5.2.2. Use determinants to determine whether each of the following sets are linearly independent.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

5.2.3. Let $A \in \mathcal{M}_{n\times n}$ and $k \in \mathbb{R}$. Find a formula for $\det(kA)$.

5.2.4. Let $A, B \in \mathcal{M}_{n\times n}$. Show that while AB may or may not be equal to BA, it is always the case that $\det(AB) = \det(BA)$.

5.2.5. Let $A \in \mathcal{M}_{n\times n}$ be such that $A^T A = I_n$. Show that either $\det A = 1$ or $\det A = -1$.

5.2.6. Let $A, B \in \mathcal{M}_{4\times 4}$ be such that $\det A = -1$ and $\det B = 2$. Calculate any of the following that can be calculated with the given information:

(a) $\det(4A^T)$

(b) $\det(A + B)$

(c) $\det(ABA^{-1})$

(d) $\det B^3$

5.2.7. Find the eigenvalues and corresponding eigenspaces for the following matrices. Also state the algebraic and geometric multiplicity of each eigenvalue.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

5.2.8. Bubbles was taking the train down to Gorky Park. A fellow passenger named Klaus claimed that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T. Decide if this is nonsense; then prove it or provide a counterexample.

5.2.9. Show that 0 is an eigenvalue of $A \in \mathcal{M}_{n\times n}$ if and only if A is not invertible.

5.2.10. Let

$$A = \begin{bmatrix} 13 & 113/2 & 113 & 25/2 \\ -50 & -251 & -478 & -25 \\ 25 & 235/2 & 241 & 25/2 \\ -25 & -253/2 & -253 & -49/2 \end{bmatrix}.$$

Check that $\vec{x} = \begin{bmatrix} -2 \\ 4 \\ -2 \\ 2 \end{bmatrix}$ is an eigenvector for A. Use this fact to calculate $A^{10}\vec{x}$. What is $\lim_{n\to\infty} A^n\vec{x}$?

5.2.11. If $\vec{v}$ is an eigenvector of both A and B, show it is an eigenvector of AB and $A + B$.

5.3 Diagonalization

In the last section we talked about this eigenstuff[24] for matrices. However, we all know by now that matrices are inextricably linked to linear transformations between vector spaces. Let's talk a bit about eigenstuff in the context of linear transformations.

24:  Still not a word.

Linear Transformations and Invariant Subspaces

Suppose $T\colon V \to V$ is a linear transformation from an n-dimensional vector space V to itself, and let $A \in \mathcal{M}_{n\times n}$ be the matrix representing this linear transformation with respect to some basis $\mathcal{B}$ for V. Suppose λ is an eigenvalue for A with eigenvector $\vec{w}_\lambda \in \mathbb{R}^n$. Then there is a vector $\vec{v}_\lambda \in V$ for which $\vec{w}_\lambda$ is the coordinate vector with respect to $\mathcal{B}$. Thus, we know $T(\vec{v}_\lambda) = \lambda\vec{v}_\lambda$. Since T is a linear transformation, we also know

$$T(a\vec{v}_\lambda) = aT(\vec{v}_\lambda) = a\lambda(\vec{v}_\lambda) = \lambda(a\vec{v}_\lambda)$$

for any $a \in \mathbb{R}$. This says $\operatorname{Span}\{\vec{v}_\lambda\}$ is *preserved* by the linear transformation T. That is, $T(\vec{x}) \in \operatorname{Span}\{\vec{v}_\lambda\}$ for any $\vec{x} \in \operatorname{Span}\{\vec{v}_\lambda\}$. In this situation, we call $\operatorname{Span}\{\vec{v}_\lambda\}$ an *invariant subspace* of V for T. Perhaps that should be a definition.

Definition 5.3.1 *Let $T\colon V \to V$ be a linear transformation from a vector space V to itself, and suppose W is a subspace of V. We say W is an **invariant subspace of** V **for** T if for any $\vec{x} \in W$, the vector $T(\vec{x})$ is also in W.*

In the discussion preceding the definition, we motivated this topic with its connection to eigenvectors, but that is not the only way these occur. It is quite possible to have invariant subspaces unrelated to eigenvectors.

Example 5.3.1 Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 \\ x_3 + x_4 \\ x_3 + x_4 \end{bmatrix}.$$

Then $W = \operatorname{Span}\{\vec{e}_1, \vec{e}_2\}$ and $U = \operatorname{Span}\{\vec{e}_3, \vec{e}_4\}$ are invariant subspaces of $\mathbb{R}^4$ for T. Let's verify this for W together. Note first that $T(\vec{e}_1) = \vec{e}_2 \in W$ and $T(\vec{e}_2) = \vec{e}_1 \in W$. Since these form a basis for W, this is actually enough to show the entire subspace is invariant. To see this, note that any element of W is of the form $a\vec{e}_1 + b\vec{e}_2$ for some $a, b \in \mathbb{R}$, and

$$T(a\vec{e}_1 + b\vec{e}_2) = a\vec{e}_2 + b\vec{e}_1 \in W.$$

Exploration 145 Check that U is an invariant subspace as claimed.

There are very practical reasons to be aware of invariant subspaces in general. For instance, we mentioned in Chapter 1 that we can use subspaces containing all of our relevant data to reduce the size of the vector space we consider. Now, we can use invariant subspaces to reduce the "size" of linear transformations, while continuing to work in the current square matrix setting.

Definition 5.3.2 *Let $T: V \to V$ be a linear transformation from a vector space V to itself, and suppose W is a subspace of V. The map*

$$T|_W : W \to V \text{ given by } T|_W(\vec{w}) = T(\vec{w})$$

for all $\vec{w} \in W$ is a map called the ***restriction of T to*** *W.*

Theorem 5.3.1 *Let $T: V \to V$ be a linear transformation from a vector space V to itself, and suppose W is a subspace of V. The restriction of T to W is a linear transformation. Moreover, if W is an invariant subspace of V for T, then $T|_W : W \to W$.*

This fact can quickly be proven since the linearity properties for T imply them for $T|_W$, so we will leave this for the exercises. Note that while we can always restrict ourselves to a subspace of the domain and define such a linear transformation, this is particularly nice when the subspace is invariant since this again gives us a linear transformation that can be represented by a square matrix.

Example 5.3.2 Suppose $T: V \to V$ is a linear transformation, V with basis $\mathcal{B}_V = \{\vec{b}_1, \ldots, \vec{b}_{7,000}\}$, and W is an invariant subspace of V for T with basis $\mathcal{B}_W = \{\vec{b}_1, \vec{b}_2\}$. Note first that the matrix representation for T will be $7,000 \times 7,000$, which isn't actually that large in practice (using computers), but it's certainly too large to write in this book. If $T(\vec{b}_1) = \vec{b}_1$, and $T(\vec{b}_2) = \vec{b}_1 + \vec{b}_2$, then the matrix representation with respect to $\mathcal{B}_W$ for $T|_W$ is

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

which doesn't even spill into the margin.[25]

25: Hey! You stay out of the margin!

Now, our goal was to discuss linear transformations and eigenstuff, so how does this fit in? Well, eigenspaces give us examples of invariant subspaces.

Theorem 5.3.2 *Suppose $T: V \to V$ is a linear transformation with matrix representation $A \in \mathcal{M}_{n\times n}$ with respect to the basis $\mathcal{B}$ of V for T. Suppose W is a subspace of V with a basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ such that $[\vec{v}_i]_{\mathcal{B}}$ is an eigenvector for A for each $1 \le i \le k$. Then W is an invariant subspace of T.*

PROOF. Suppose $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is the described basis of W, and suppose they have eigenvalues $\{\lambda_1, \ldots, \lambda_k\}$, respectively. If $\vec{w} \in W$, then $\vec{w} = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k$ for some $a_1, \ldots, a_k \in \mathbb{R}$. We know that $[T(\vec{x})]_{\mathcal{B}} = A[\vec{x}]_{\mathcal{B}}$, and thus, if $[\vec{x}]_{\mathcal{B}}$ is an eigenvector for A with eigenvalue λ, then $T(\vec{x}) = \lambda\vec{x}$. Therefore,

$$T(\vec{w}) = T(a_1\vec{v}_1 + \cdots + a_k\vec{v}_k) = a_1\lambda_1\vec{v}_1 + \cdots + a_k\lambda_k\vec{v}_k \in W$$

and W is preserved by T. □

Since any eigenspace is made up entirely of eigenvectors, this implies any eigenspace for the matrix representation corresponds to an invariant subspace for the linear transformation. Thus, eigenvalues and eigenvectors provide an incredibly convenient way to find invariant subspaces.

Example 5.3.3 Define $T\colon \mathbb{P}_1 \to \mathbb{P}_1$ by

$$T\left(a+bx\right) = 2b + 2ax.$$

Then the matrix A for T using the basis $\{1, x\}$ for $\mathbb{P}_1$ is

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

Let's find an eigenvalue and an eigenvector for this matrix.

$$\det(A - \lambda I) = \lambda^2 - 4$$
$$\lambda = -2, 2$$

For $\lambda = 2$, we have

$$A - 2I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}.$$

By computing the kernel of $A - 2I$ from this row reduction, we see that

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector for A corresponding to the eigenvalue 2.[26] Now, let's go back to our linear transformation T. The eigenvector we found corresponds to the polynomial $1 + x$. By our defining description of T, we have

$$T(1 + x) = 2 + 2x = 2(1 + x).$$

Aha! The "specialness" of 2 and $\vec{v}_2$ is present in the linear transformation T, not just it's matrix representation A. This says $\mathrm{Span}\,\{1 + x\}$ is an invariant subspace of $\mathbb{P}_1$ for the linear transformation T.

What happens if we have a different matrix representation for T? If we consider a different basis for $\mathbb{P}_1$, will we still have this property? Let's see! Let $\mathcal{B} = \{1 - x, 1 + x\}$. This is a new basis[27] for $\mathbb{P}_1$. Since

$$T\left(1-x\right) = -2+2x = -2(1-x) \quad \text{and} \quad T\left(1+x\right) = 2+2x = 2(1+x)$$

we see the matrix B which represents T on this basis is

$$B = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Because this matrix is diagonal, we know it's eigenvalues are the entries on the diagonal 2 and -2, the same as those for A.[28] Now, let's consider the eigenvectors for B. Repeating our procedure of computing the kernel of $(B - \lambda I)$ from above we see that

$$\text{for } \lambda = -2: \quad \vec{v}_{-2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and for } \lambda = 2: \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

26: Did you catch what just happened? Did we just review all the procedures from the previous section? Why, yes, now that you mention it, we did. Isn't reviewing fun?

27: In the spirit of review, could you show this?

28: Wow, the review topics just keep on coming!

Okay, these aren't the same eigenvectors as we had for A, even though these two matrices did have the same eigenvalues. However, what we should really be writing is this:

$$\text{for } \lambda = -2 : \quad [\vec{v}_{-2}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and for } \lambda = 2 : \quad [\vec{v}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since B is with respect to the basis $\mathcal{B}$, the eigenvectors for it are also with respect to the basis $\mathcal{B}$. Note that

$$[\vec{v}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ means } \vec{v}_2 = 0(1-x) + 1(1+x) = 1 + x.$$

This agrees with the invariant subspace of $\mathbb{P}_1$ we found using the matrix A!

The example above illustrates that the eigenspaces of a matrix can be used to determine an invariant subspace for the linear transformation that corresponds to that matrix. Also, note how nice that diagonal matrix B looked. We'll talk more about this later. Lastly, these two matrices representing the same linear transformation had the same eigenvalues. Well, that's true in general. However, it's a bit easier to see if we momentarily forget about our linear transformation and focus specifically on matrices.

Similar Matrices

The matrices in the example above have the property that they both represent the same linear transformation, just with respect to different bases. Well, considering what we know from Section 4.6 about changing the basis, we have the following definition.

Definition 5.3.3 *Matrices $A, B \in \mathcal{M}_{n\times n}$ are **similar** if there is an invertible matrix $P \in \mathcal{M}_{n\times n}$ such that $A = PBP^{-1}$, or equivalently, $B = P^{-1}AP$.*

As mentioned above, two square matrices are similar if and only if they are matrix representations for the same linear transformations, just with different bases. Now, we can prove all we observed in the previous example using this concept of similarity.

Theorem 5.3.3 *Similar matrices have the same characteristic polynomial.*

Exploration 146 Proof by exploration! Suppose A and B are similar, so there is an invertible matrix P such that $A = PBP^{-1}$. Convince yourself that $A - \lambda I = PBP^{-1} - \lambda PP^{-1}$. Use this to show that $A - \lambda I = P(B - \lambda I)P^{-1}$.

Use the last equation to show that $\det(A - \lambda I) = \det(B - \lambda I)$.

Corollary 5.3.4 *If A has an eigenvalue λ with algebraic multiplicity k and B is similar to A, then B has the same λ as an eigenvalue with algebraic multiplicity k as well.*

Example 5.3.4 Recall Exploration 144 from Section 5.1. The matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -2 & -3 & 8 \\ 1 & 2 & -3 \\ -5 & -5 & 9 \end{bmatrix}$$

have the same eigenvalues, each with the same multiplicities, both geometric and algebraic. This is not a coincidence; these matrices are similar by way of the matrix

$$P = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}.$$

This is readily verified:

$$\begin{aligned} PBP^{-1} &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & -3 & 8 \\ 1 & 2 & -3 \\ -5 & -5 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{bmatrix} = A. \end{aligned}$$

Theorem 5.3.3, however, is a one-way street. Having the same characteristic polynomial does not guarantee similarity. For example,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

have the same characteristic polynomial, but they are not similar.

This next theorem seems like maybe it should have been stated last section, when the concepts of algebraic and geometric multiplicities were introduced. Our proof, however, relies on Corollary 5.3.4.

Theorem 5.3.5 *For any eigenvalue, the algebraic multiplicity is greater than or equal to the geometric multiplicity.*

Proof. Suppose $A \in \mathcal{M}_{n\times n}$ has geometric multiplicity k for eigenvalue λ, and $\{\vec{x}_1, \dots, \vec{x}_k\}$ is a basis for the eigenspace E_λ associated to λ. Let $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ be a basis for $E_\lambda^\perp$. Then $\{\vec{x}_1, \dots, \vec{x}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ forms a basis of $\mathbb{R}^n$ since we know $\mathbb{R}^n = E_\lambda \oplus E_\lambda^\perp$. Consider the matrix $S = [\vec{x}_1 \cdots \vec{x}_k \, \vec{v}_{k+1} \cdots \vec{v}_n]$. The columns of this matrix are a basis and thus linearly independent, so the matrix is invertible. Then the first k columns of $B = S^{-1}AS$ are the vectors $\lambda\vec{e}_1, \dots \lambda\vec{e}_k$ since the first k basis vectors are eigenvectors for the eigenvalue λ.[29] Because λ is our specific eigenvalue, let us consider the characteristic polynomial as a polynomial in the variable x instead. Then $(\lambda - x)^k$ divides the characteristic polynomial of B. This can be seen by realizing the first k columns of B are all zero except with λ on the diagonal, so computing the determinant of $(B - xI)$ will be straightforward using the cofactor method down the first k columns. Thus, λ has algebraic

29:  If this seems unbelievable, you should do an example to convince yourself.

multiplicity at least k for B. By Corollary 5.3.4, it follows that λ has algebraic multiplicity at least k for A as well. □

Remember from Theorem 5.3.2 that the eigenspace of the matrix representation for a linear transformation is an invariant subspace for the linear transformation. Moreover, two matrices that both represent the linear transformation T with respect to different bases will have the same eigenvalues and their eigenspaces will identify the same invariant subspaces for T. In other words, similar matrices must have the same eigenvalues with the same geometric multiplicities. We could prove this result using Theorem 5.3.2, but we will give a proof here that is in line with our matrix results instead.[30]

30: Oh, I'm sad. I prefer the linear transformations.

You're in luck! The authors accidentally proved it both ways, so that proof is in the Appendix.

How do you accidentally do twice the amount of work?

(shrugs)

Theorem 5.3.6 *If $A \in \mathcal{M}_{n\times n}$ has an eigenvalue λ with geometric multiplicity k and $B \in \mathcal{M}_{n\times n}$ is similar to A, then B has λ as an eigenvalue with geometric multiplicity k as well.*

PROOF. Suppose $\{\vec{x}_1, \ldots, \vec{x}_k\}$ is a basis for the eigenspace corresponding to the eigenvalue λ of A. Since $A = PBP^{-1}$, we have $B = P^{-1}AP$, and as we've seen before, $B - \lambda I = P^{-1}(A - \lambda I)P$. Then for any $1 \le i \le k$, we have

$$\begin{aligned}
(B-\lambda I)P^{-1}\vec{x}_i &= [P^{-1}(A-\lambda I)P]P^{-1}\vec{x}_i \\
&= P^{-1}(A-\lambda I)\vec{x}_i && \text{since } PP^{-1} = I_n \\
&= P^{-1}\vec{0} && \text{since } \vec{x}_i \in \operatorname{Ker}(A-\lambda I) \\
&= \vec{0},
\end{aligned}$$

so $\mathcal{B} = \{P^{-1}\vec{x}_1, \ldots, P^{-1}\vec{x}_k\} \subset \operatorname{Ker}(B - \lambda I)$. Moreover, since P^{-1} is invertible, we know that $\mathcal{B}$ is a linearly independent set. Thus, the geometric multiplicity of λ for B is at least k. To see that it is exactly k, we could reverse this argument, starting with B rather than A, and conclude the geometric multiplicity for A must be at least that of B. Thus, they must be equal. □

Diagonalization

Let's discuss an application of all this. Well, it will eventually be an application. Suppose T is a linear transformation with matrix representation A, and suppose you have a valid, nay,[31] an *important* reason to apply T to a vector many, many times. That is, suppose you would like to *iterate* T. You would like to find $T(\vec{x})$ for some vector $\vec{x}$ and then you would also like to find $T(T(\vec{x}))$. Then maybe you would like to find $T(T(T(\vec{x})))$. Well, since composition of linear transformations corresponds to matrix multiplication, this means you need to know how to compute A^k for $k \ge 1$. Now, let's connect this to the concept of similarity we've been spending so much time on in this section.

31: Neigh? Is one of the authors trolling us?

It says "nay."

I prefer trolls to horses.

...

Example 5.3.5 Suppose A is similar to B, so there is an invertible matrix P such that $A = PBP^{-1}$. Then for any integer $k \ge 1$, A^k is similar to B^k.

Behold:

$$\begin{aligned} A^k &= \left(PBP^{-1}\right)^k \\ &= \left(PBP^{-1}\right)\left(PBP^{-1}\right)\cdots\left(PBP^{-1}\right) \\ &= PB(P^{-1}P)B\cdots(P^{-1}P)BP^{-1} \;=\; PB^kP^{-1}. \end{aligned}$$

Great! This seems to be leading somewhere. If only we had some particularly useful format of a matrix that we could use when raising it to a power...

Definition 5.3.4 *A matrix of the form*

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} \in \mathcal{M}_{n\times n}$$

is called a ***diagonal matrix***.

Diagonal matrices have the convenient property that

$$D^k = \begin{bmatrix} a_{11}^k & 0 & \cdots & 0 \\ 0 & a_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^k \end{bmatrix}$$

Similarity allows us to exploit this further; if P is invertible, D is diagonal, and $A = P^{-1}DP$, then as we saw in Example 5.3.5,

$$A^k = (P^{-1}DP)^k = P^{-1}D^kP.$$

Definition 5.3.5 *A matrix is called* ***diagonalizable*** *if it is similar to a diagonal matrix.*

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$$\begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix},$$

find

$$\begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}^k.$$

Where does one find such a matrix P? Eigenvectors! It's literally a change of basis matrix formed by eigenvectors.

Theorem 5.3.7 *A matrix $A \in \mathcal{M}_{n\times n}$ is diagonalizable if and only if A has n linearly independent eigenvectors.*

PROOF. Suppose A is diagonalizable, so there is an invertible matrix $P = [\vec{v}_1 \cdots \vec{v}_n]$ and a diagonal matrix D such that $A = PDP^{-1}$, or $AP = PD$. Note that

$$\begin{aligned} AP &= [A\vec{v}_1 \cdots A\vec{v}_n] \text{ and} \\ PD &= [d_{11}\vec{v}_1 \cdots d_{nn}\vec{v}_n]; \end{aligned}$$

since $AP = PD$, we have for all $1 \le j \le n$,

$$A\vec{v}_j = d_{jj}\vec{v}_j.$$

Thus, the jth column of P is an eigenvector corresponding to the jth entry on the diagonal of D. That is, every column of P is an eigenvector of A. Moreover, since P is invertible, its columns are linearly independent, so these eigenvectors must be linearly independent.

On the other hand, suppose A has n linearly independent eigenvectors. Then one can construct P and D (as above) and readily verify that $AP = PD$. □

Corollary 5.3.8 *The matrix $A \in \mathcal{M}_{n\times n}$ is diagonalizable if and only if for every eigenvalue of A, its geometric multiplicity is equal to its algebraic multiplicity.*

Corollary 5.3.9 *A matrix $A \in \mathcal{M}_{n\times n}$ with n distinct eigenvalues is diagonalizable.*

Here's a procedure for diagonalizing a matrix. Given a matrix $A \in \mathcal{M}_{n\times n}$,

(a) Find the eigenvalues of A.

(b) Find n linearly independent eigenvectors $\mathbf{v}_1, \ldots \mathbf{v}_n$. If there are not n of them, then be very sad; A is not diagonalizable.

(c) Construct P from these eigenvectors: $P = [\mathbf{v}_1 \cdots \mathbf{v}_n]$.

(d) Construct D from the eigenvalues with the eigenvalues along the diagonal in the same order as the eigenvectors in P.

(e) Check that $AP = PD$ (same as $A = PDP^{-1}$).[32]

Example 5.3.6 Let

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}.$$

Is A diagonalizable? Indeed. One can check that

$$\left\{ \begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

32:  Is checking really part of the procedure? Or is it just good advice?  I am glad that you are not a civil engineer.

is a basis for the eigenspace corresponding to the eigenvalue 5, and

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for the eigenspace corresponding to the eigenvalue -3. Then

$$P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

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$$A = \begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Is A diagonalizable?

Section Highlights

- Matrices A and B are *similar* if there is some change of basis matrix P such that $A = PBP^{-1}$. See Definition 5.3.3 and the discussion surrounding it. Since they only differ by the basis they are with respect to, A and B are both matrix representations for the same linear transformation.

- We say a matrix is *diagonalizable* if it is similar to a diagonal matrix. See Definition 5.3.5 and Definition 5.3.3. If matrices A and B are both matrix representations for the same linear transformation, we say A and B are similar.

- An $n \times n$ matrix A is diagonalizable if and only if there is a basis for $\mathbb{R}^n$ made of eigenvectors for A. See Theorem 5.3.7.

- To diagonalize a matrix, we use the change of basis matrix to convert from the standard basis to a basis of eigenvectors. See Example 5.3.6.

- If λ is an eigenvalue for A, the *algebraic multiplicity* of λ is the multiplicity of $(x - \lambda)$ in the characteristic polynomial for A, and the *geometric multiplicity* of λ is the dimension of $\operatorname{Ker}(A - \lambda I)$. It's always true that the geometric multiplicity is less than or equal to the algebraic multiplicity. See Definition 5.2.4 and Theorem 5.3.5.

- A matrix A is diagonalizable if and only if the geometric and algebraic multiplicities agree for every eigenvalue. See Corollary 5.3.8.

Exercises for Section 5.3

5.3.1. Let $P = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, and $A = PDP^{-1}$. Calculate A^9.

5.3.2. Here's a pretty big matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & 1 & 3 & 1 \\ 0 & 0 & 1 & 4 & 1 & 4 \\ 0 & 0 & 0 & 2 & 5 & 2 \\ 0 & 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Hey, at least it's triangular.

(a) Without calculation of any kind, what are the eigenvalues and their algebraic multiplicities?

(b) Without calculation of any kind, what are the upper and lower bounds on the geometric multiplicity of each eigenvalue?

(c) Calculate the geometric multiplicities of each eigenvalue. Is A diagonalizable?

5.3.3. With minimal calculation, determine whether each of the following matrices is diagonalizable.

(a) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

5.3.4.Let

$$A = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}.$$

Find an invertible matrix P and a diagonal matrix D such that that $A = PDP^{-1}$.

5.3.5.Let

$$A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 1 & 2 & -3 \\ -1 & 0 & 1 & -2 \\ -2 & -1 & 0 & -1 \end{bmatrix}.$$

Find an invertible matrix P and a diagonal matrix D such that that $A = PDP^{-1}$.

5.3.6.For each of the following pairs of matrices, determine whether or not the matrices are similar. If they are similar, find a matrix P such that $A = P^{-1}BP$.

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

5.3.7.Show that every eigenspace is an invariant subspace. That is, if $A \in \mathcal{M}_{n\times n}$ has an eigenvalue λ with eigenspace E, show that E is an invariant subspace of $\mathbb{R}^n$ for T_A.

5.3.8.Let

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 1 & -1 \\ -1 & 3 & 5 \end{bmatrix}.$$

It turns out that $\lambda = 2$ is an eigenvalue for A. Let E be the eigenspace associated to 2, and find a matrix representation for $A|_E$.

5.4 Jordan Canonical Form

Perhaps you're sold at this point on the greatness of diagonalizability. Whether or not that's the case, just how special are these diagonalizable matrices? What about matrices that aren't diagonalizable? How miserably "not diagaonalizable" can a matrix be? We'll see now, that they're not too miserable at all!

A New Form

Up to this point, we've only used vector spaces over $\mathbb{R}$, meaning all the scalars we use in scalar multiplication have been real numbers. See Definition 1.1.2. Even when we used the complex numbers as a vector space, it was as a *real* vector space. See Exercise 1.1.3. That ends now!

WARNING ALERT CAUTION ALERT WARNING

For the rest of the book, we're going to be using $\mathbb{C}$ for our scalars. That's what the garish warning was all about. Just know that it's happening, and don't panic. Vector spaces over $\mathbb{C}$ have their charm. Also, we won't explicitly be using them until we get to the proofs later in this section, and we'll keep the examples and explorations as real as possible.[33]

We have a good idea at this point how diagonalizable matrices work and which square matrices are diagonalizable. Now it is time to deal with the rest of them. That's right, *the rest of them*. It turns out that even square matrices that aren't diagonalizable are *almost diagonalizable* in a very specific, consistent sense. Let's see an example first, an incredibly long but deeply inspiring example.

Example 5.4.1 Here's a square matrix with its eigenvalue and eigenvector:

$$A = \begin{bmatrix} 2 & -5 & -6 \\ -2 & 2 & -4 \\ 5 & 7 & 14 \end{bmatrix}, \qquad \lambda = 6, \qquad \vec{v}_0 = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}.$$

That's right. There's only one eigenvalue with algebraic multiplicity 3 and geometric multiplicity 1. If we wanted to diagonalize A, we would need three linearly independent eigenvectors to use as the columns of P so that $AP = PD$, where D is a diagonal matrix. Alas, we only have one eigenvector. This matrix is not even a little bit diagonalizable. Well…
maybe a little bit…
We're missing two eigenvectors to build the matrix P used to diagonalize A, but perhaps we can find two stand-ins. What's so great about $\vec{v}_0$, anyway? Since it's an eigenvector for $\lambda = 6$, we know $(A - 6I)\vec{v}_0 = \vec{0}$. What if we found a vector that *maps into* $\operatorname{Ker}(A - 6I)$? In particular, we'd like to find a vector, $\vec{v}_1$ such that

$$(A - 6I)\vec{v}_1 = \vec{v}_0.$$

Then $(A - 6I)\vec{v}_1 = \vec{v}_0 \neq \vec{0}$ but $(A - 6I)^2\vec{v}_1 = \vec{0}$. This new vector, $\vec{v}_1$, isn't in $\operatorname{Ker}(A - 6I)$ because $A - 6I$ doesn't map it to $\vec{0}$, but if you apply $A - 6I$ to $\vec{v}_1$ *twice*, you do get $\vec{0}$. Thus, $\vec{v}_1$ isn't an eigenvector for $\lambda = 6$, but it's "the next best thing." It *eventually* gets mapped to $\vec{0}$ by repeated applications of $A - 6I$.

33: Really?
Really.
But aren't we all imaginary?
…!
I don't know about you two, but I'm totally real.

How do you find $\vec{v}_1$? Just solve $(A-6I)\vec{x} = \vec{v}_0$ for $\vec{x}$. Neat, right? Oh, why should a solution to $(A-6I)\vec{x} = \vec{v}_0$ exist? Yeah. That's a really good question. For now, let's just enjoy that is does, and we'll state and prove a theorem about that later. While we're at it, $(A-6I)\vec{x} = \vec{v}_1$ also has solutions; we'll call the one we're gonna use $\vec{v}_2$. Specifically, here are the solutions we chose:

$$\vec{v}_2 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}.$$

Alright. What exactly do these vectors, $\vec{v}_1$ and $\vec{v}_2$, do? They both eventually map to $\vec{0}$ after repeated application of $A-6I$. In fact,

$$\vec{v}_2 \xrightarrow{A-6I} \vec{v}_1 \xrightarrow{A-6I} \vec{v}_0 \xrightarrow{A-6I} \vec{0}$$

This is neat, but you're probably wondering what good this little sequence (sometimes called a *Jordan chain*) is. We have the following two convenient implications:

$$\begin{aligned} (A-6I)\vec{v}_1 = \vec{v}_0 \quad &\Rightarrow \quad A\vec{v}_1 = \vec{v}_0 + 6\vec{v}_1 \\ (A-6I)\vec{v}_2 = \vec{v}_1 \quad &\Rightarrow \quad A\vec{v}_2 = \vec{v}_1 + 6\vec{v}_2. \end{aligned}$$

Now, let's make a matrix $P = [\vec{v}_0\ \vec{v}_1\ \vec{v}_2]$. Then

$$\begin{aligned} AP &= A[\vec{v}_0 \quad \vec{v}_1 \quad \vec{v}_2] \\ &= [A\vec{v}_0 \quad A\vec{v}_1 \quad A\vec{v}_2] \\ &= [6\vec{v}_0 \quad \vec{v}_0 + 6\vec{v}_1 \quad \vec{v}_1 + 6\vec{v}_2] \\ &= [\vec{v}_0 \quad \vec{v}_1 \quad \vec{v}_2]\begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 6 \end{bmatrix} = P\begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 6 \end{bmatrix}, \end{aligned}$$

so

$$A = PJP^{-1}, \quad \text{where} \quad J = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 6 \end{bmatrix}.$$

You may have noticed that J is not diagonal, but it's *almost* diagonal (except for those 1's). Well, $\vec{v}_1$ and $\vec{v}_2$ are not eigenvectors, but we chose them because they were the next best thing.

We mentioned a term in that example that deserves its own definition box:

Definition 5.4.1 *If $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ has eigenvalue λ with eigenvector $\vec{v}_0$, then a* ***Jordan chain*** *for λ is a set of vectors $S = \{\vec{v}_1, \ldots \vec{v}_k\}$ for some $k < n$ such that*

$$\vec{v}_k \xrightarrow{A-\lambda I} \vec{v}_{k-1} \xrightarrow{A-\lambda I} \cdots \xrightarrow{A-\lambda I} \vec{v}_1 \xrightarrow{A-\lambda I} \vec{v}_0 \xrightarrow{A-\lambda I} \vec{0}$$

This almost-diagonal matrix with all the rogue 1's just above the diagonal is also going to come up a lot.

Definition 5.4.2 *A* ***Jordan block*** *is a square matrix whose entries are the same constant, $\lambda \in \mathbb{C}$, on the diagonal, 1 on each entry immediately above the diagonal, and zero elsewhere.*

Example 5.4.2 Here are some Jordan blocks:

$$J_1 = \begin{bmatrix} 4 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 5 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \qquad J_4 = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 6 \end{bmatrix}.$$

Using appropriately sized matrices $[\vec{0}]$, we can combine Jordan blocks into one big matrix, too:

$$\begin{bmatrix} J_1 & \vec{0} & \vec{0} & \vec{0} \\ \vec{0} & J_2 & \vec{0} & \vec{0} \\ \vec{0} & \vec{0} & J_3 & \vec{0} \\ \vec{0} & \vec{0} & \vec{0} & J_4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix},$$

Pretty close to diagonal, right?

Theorem 5.4.1 (Jordan Canonical Form) *Every square matrix, $A \in \mathcal{M}_{n\times n}(\mathbb{C})$, is similar by $P \in \mathcal{M}_{n\times n}(\mathbb{C})$ to a matrix, $J \in \mathcal{M}_{n\times n}(\mathbb{C})$, whose only nonzero entries are Jordan blocks on the diagonal. Moreover, J is unique (allowing for the reordering of the Jordan blocks). The matrix J is called the Jordan Canonical Form of the matrix A.*

Perhaps at this point, you find this very believable. There are some things to prove, but let's wait a minute and play with this fun new theorem first.

Example 5.4.3

$$A = \begin{bmatrix} 11 & 5 & 2 & 0 & -2 & -7 & -3 \\ -5 & 3 & 1 & 0 & 1 & 4 & 3 \\ 14 & 13 & 11 & 0 & -5 & -19 & -7 \\ 2 & 0 & -3 & 4 & 0 & 1 & -2 \\ 1 & 3 & 3 & 0 & 4 & -4 & 0 \\ 8 & 8 & 4 & 0 & -3 & -6 & -4 \\ -9 & -8 & -3 & 0 & 3 & 12 & 10 \end{bmatrix},$$

$$\begin{aligned} P &= [\vec{p}_1\ \vec{p}_2\ \vec{p}_3\ \vec{p}_4\ \vec{p}_5\ \vec{p}_6\ \vec{p}_7] \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 4 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 1 \end{bmatrix}, \text{ and} \end{aligned}$$

$$J = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}.$$

You can check that $A = PJP^{-1}$ (or you can trust us). J is the Jordan Canonical Form for A. Here are some fun facts about eigenvalues and eigenvectors for A that you can check:

eigenvalue	4	5	6
alg. mult.	1	3	3
eigenvectors	$\vec{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\vec{p}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \vec{p}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$	$\vec{p}_5 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

This begs the question, where did $\vec{p}_4$, $\vec{p}_6$, and $\vec{p}_7$ come from? As in the example from the beginning of this section, we found the vectors $\vec{p}_4$, $\vec{p}_6$, and $\vec{p}_7$ so that

$$\begin{aligned} (A-5I)\vec{p}_4 &= \vec{p}_3, \\ (A-6I)\vec{p}_6 &= \vec{p}_5, \text{ and} \\ (A-6I)\vec{p}_7 &= \vec{p}_6. \end{aligned}$$

Finding these vectors is as easy as solving these matrix equations. The remarkable thing about Theorem 5.4.1 is that is guarantees these vectors exist. We'll get into how this is done procedurally at the end of the section (after we prove the theorem).

Recall our heuristic for diagonalization. If an $n \times n$ matrix is diagonalizable, it's similar to a diagonal matrix, so the matrix is just scalar multiplication in n linearly independent directions. What are we to make of Jordan Canonical form? We need a couple more definitions.

Definition 5.4.3 *The* ***left shift*** *on $\mathbb{C}^n$ is the linear transformation $L\colon \mathbb{C}^n \to \mathbb{C}^n$ given by $L(x_1, \ldots, x_n) = (x_2, \ldots, x_n, 0)$.*

Or is it an *up* shift? Lots of people write their vectors vertically, so this linear transformation, even when written that way, is called the left shift. You should verify that the left shift is a linear transformation! Oh, hey; find its kernel and image, too![34]

34: Exercise!

Theorem 5.4.2 *As a linear transformation, any Jordan block is the sum of a scalar multiple of the identity map and the left shift.*

PROOF. Let $J \in \mathcal{M}_{n\times n}(\mathbb{C})$ be a Jordan block, so for $\vec{x} \in \mathbb{C}^n$,

$$\begin{aligned} J\vec{x} &= \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda x_1 + x_2 \\ \lambda x_2 + x_3 \\ \lambda x_3 + x_4 \\ \vdots \\ \lambda x_{n-1} + x_n \\ \lambda x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \\ \vdots \\ \lambda x_{n-1} \\ \lambda x_n \end{bmatrix} + \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \\ 0 \end{bmatrix} = \lambda\vec{x} + L(\vec{x}), \end{aligned}$$

where L is the left shift. How very gaudy. We probably should've just done this with a 3×3 and asked you to use your imagination. $\square$

Definition 5.4.4 *Let $T\colon V \to V$ be a linear transformation with invariant subspaces V_1 and V_2 such that $V = V_1 \oplus V_2$. Then every element of $\vec{v} \in V$ can be written as $\vec{v} = \vec{v}_1 + \vec{v}_2$ for some $\vec{v}_1 \in V_1$ and $\vec{v}_2 \in V_2$. Thus, we have*

$$T(\vec{v}) = T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = T|_{V_1}(\vec{v}_1) + T|_{V_2}(\vec{v}_2).$$

In this situation, we say T decomposes into the ***direct sum of linear transformations*** *$T|_{V_1}$ and $T|_{V_2}$ and denote this as $T = T|_{V_1} \oplus T|_{V_2}$.*

Note that we only defined this here for two invariant subspaces V_1 and V_2, but it's possible that the vector space $V = V_1 \oplus \cdots \oplus V_k$ for some $k \geq 2$, where each V_i is an invariant subspace. The definition makes sense in this setting as well.[35]

35: Really? Are you sure?

Exercise!

Now, we previously talked quite a bit about how two matrices are similar if they are each the matrix representation for the same linear transformation. Well, we can flip this as well, to talk about when two different linear transformations have a relationship given by their matrix representations.

Definition 5.4.5 *Two linear transformations are* ***similar linear transformations*** *if they have similar matrix representations.*

The following corollary nicely describes what Jordan Canonical Form actually does and is due to Terry Tao.

Corollary 5.4.3 *Every linear transformation is similar to a direct sum of linear transformations, where each summand is itself the sum of a scalar multiple of the identity map and the left shift.*

It's not *quite* as good as diagonalization, where a matrix is similar to a direct sum of scalar multiplication maps, but it's not too far off. It's just a little... *shiftier*.

Example 5.4.4 Let $T\colon \mathbb{P}_2 \to \mathbb{P}_2$ by $T(c+bx+ax^2) = b+2ax$. The matrix representation for T using the basis $\{1, x, x^2\}$ is a matrix A, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can see that A is similar to a matrix

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

since

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

Note that $B = 0I_3 + L$, so T is similar to T_1, where $T_1(c+bx+ax^2) = b+ax = 0(c+bx+ax^2)+L(c+bx+ax^2)$, where the first term is a scalar multiple of the identity and the second is a left shift.

In Example 5.4.4, we identified two matrices were similar by finding the matrix P such that $B = P^{-1}AP$. Well, it can sometimes be difficult to find that matrix. However, Jordan Canonical Form gives us a systematic approach to identifying whether two matrices are similar.

Corollary 5.4.4 *Suppose $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ and $B \in \mathcal{M}_{n\times n}(\mathbb{C})$. Then A and B are similar if and only if they are both similar to the same matrix in Jordan Canonical Form.*

This follows from the fact that the Jordan Canonical Form is unique up to rearranging placements of the Jordan blocks along the diagonal and one can use similarity to permute blocks along the diagonal.

Proof for Jordan Canonical Form

We still have some proving to do. We should also talk about how one actually finds a Jordan Canonical Form in practice. Proofs first!

Remember the beginning of this section when we announced the arrival of complex numbers? Right. Allowing complex numbers gives us the following super-handy theorem.

Theorem 5.4.5 (The Fundamental Theorem of Algebra) *Every degree $n \geq 1$ polynomial in one variable with complex coefficients has exactly n complex roots (counting the multiplicity of the repeated roots).*

Proofs of the Fundamental Theorem of Algebra are readily available. We provide none of them here.

Maybe we should have mentioned this before, but there is a small cost associated to the switch to complex numbers. Almost everything we've done up to this point remains the same, but with complex numbers, we need to make a small change to the definition of inner product (and hence, norm).

Complex numbers are like sand; once you have some of them in a vector space, they get all over everything. Even just using complex scalars, we end up having to deal with complex vectors as well because of linear combinations with complex scalars. For example, if $\vec{x}, \vec{y} \in \mathbb{R}^n$, then the linear combination

$$(1+i)\vec{x} + (2+4i)\vec{y}$$

is definitely *not* in $\mathbb{R}^n$. It is in $\mathbb{C}^n$, though. How is this a problem? Well, for $\vec{x} \in \mathbb{R}^n$, we defined $\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$. What happens with this definition for a complex vector?

$$\begin{aligned} \left\| \begin{bmatrix} 1+i \\ 1 \end{bmatrix} \right\| = \sqrt{\begin{bmatrix} 1+i \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1+i \\ 1 \end{bmatrix}} &= \sqrt{(1+i)(1+i)+1} \\ &= \sqrt{1+2i}. \end{aligned}$$

This is weird. You probably wouldn't be surprised to find that the square root of complex numbers is a complex number that is almost always not a real number; that is the case with $\sqrt{1+2i}$. If the norm of a vector is supposed to represent the *magnitude* of a vector, what are we to make of a complex magnitude? Right. We should fix that. The fix, it turns out, is pretty easy.

Definition 5.4.6 *For $z = x + iy \in \mathbb{C}$, the* ***conjugate of z*** *is the complex number $\bar{z} = x - iy$.*

Definition 5.4.7 *The* ***Hermitian Inner product*** *is the function $\cdot : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by*

$$\vec{v} \cdot \vec{u} = v_1\bar{u}_1 + \cdots + v_n\bar{u}_n = \sum_{i=1}^{n} v_i\bar{u}_i.$$

The notation is the same as the real inner product because the Hermitian inner product on real vectors is just the usual inner product.[36] The key is simply to remember that when you have complex vectors to conjugate the entries of the second vector when doing an inner product. How does this help? It turns out that for any $z \in \mathbb{C}$, we have $z\bar{z} \in \mathbb{R}$.[37]

36: Check this!

37: You check this!

Definition 5.4.8 *The* ***Hermitian norm*** *is the function $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}$ defined for any $\vec{v} \in \mathbb{C}^n$ as*

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1\bar{v}_1 + \cdots + v_n\bar{v}_n}.$$

Using the Hermitian inner product and norm is the only alteration we will need to make in this book when working with complex numbers. In the interest of full disclosure, here's a potentially upsetting fact; there are a lot of different ways we could've defined inner products. We chose some specific ones for simplicity's sake here and in earlier chapters.

Now, back to Jordan Canonical Form. We get to use the Fundamental Theorem of Algebra right away!

Lemma 5.4.6 *Suppose W_1 and W_2 are subspaces of vector space V with bases $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, and that $W_1 \oplus W_2 = V$. Let A be the matrix representation of $T|_{W_1}$ with respect to $\mathcal{B}_1$, and B be the matrix representation of $T|_{W_2}$ with respect to $\mathcal{B}_2$. If W_1 and W_2 are both invariant for T, then*

$$\begin{bmatrix} A & \vec{0} \\ \vec{0} & B \end{bmatrix}$$

is the matrix representation of T with respect to the basis $\mathcal{B}_1 \cup \mathcal{B}_2$. If just W_1 is invariant for T, then

$$\begin{bmatrix} A & R \\ \vec{0} & B \end{bmatrix}$$

is the matrix representation of T with respect to the basis $\mathcal{B}_1 \cup \mathcal{B}_2$.

Note that if $\mathcal{B}_1 = \{\vec{x}_1, \ldots, x_k\}$, $\mathcal{B}_2 = \{\vec{y}_{k+1}, \ldots, \vec{y}_n\}$ and both W_1 and W_2 are invariant for T, then for any $\vec{x}_i \in W_1$ and $\vec{y}_i \in W_2$, we have

$$\begin{array}{rcccccccc} T(\vec{x}_i) & = & a_{i1}\vec{x}_1 & +\cdots+ & a_{ik}\vec{x}_k & + & 0\vec{y}_{k+1} & +\cdots+ & 0\vec{y}_n, \\ T(\vec{y}_i) & = & 0\vec{x}_1 & +\cdots+ & 0\vec{x}_k & + & a_{i(k+1)}\vec{y}_{k+1} & +\cdots+ & a_{ik}\vec{y}_n. \end{array}$$

The proof follows then from computing the matrix for T with respect to this basis $\mathcal{B}_1 \cup \mathcal{B}_2$. We leave this proof then as an exercise, but we will revisit a familiar example illustrating this.

Exploration 149 Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 \\ x_3 + x_4 \\ x_3 + x_4 \end{bmatrix}.$$

In Example 5.3.1, we saw that T has two invariant subspaces $W = \text{Span}\,\{\vec{e}_1, \vec{e}_2\}$ and $U = \text{Span}\,\{\vec{e}_3, \vec{e}_4\}$. Compute the matrix for T with respect to the standard basis. Compare this to the form from Lemma 5.4.6.

Theorem 5.4.7 *If $A \in \mathcal{M}_{n\times n}(\mathbb{C})$, then there is an invertible matrix Q and an upper triangular matrix U such that $A = QUQ^{-1}$ and U has the eigenvalues of A on its diagonal.*

PROOF. By the Fundamental Theorem of Algebra, $\det(A - \lambda I) = 0$ has a solution, call it λ_1. By Theorem 5.2.8, λ_1 is an eigenvalue, so it has at least one eigenvector. Let $\{\vec{x}_1, \ldots, \vec{x}_j\}$ be a basis for E_1, the eigenspace for λ_1. Using this with a basis for $E_1^\perp$, $\{\vec{v}_{j+1}, \ldots, \vec{v}_n\}$, define the change of basis matrix, $Q_1 = [\vec{x}_1 \cdots \vec{x}_j \; \vec{v}_{j+1} \; \cdots \; \vec{v}_n]$. While E_1 is an invariant subspace, $E_1^\perp$ might not be,[38] so by Lemma 5.4.6

$$Q_1^{-1}AQ_1 = \begin{bmatrix} \lambda_1 I_j & R_1 \\ \vec{0} & C_1 \end{bmatrix},$$

where $R_1 \in \mathcal{M}_{1,n-j}(\mathbb{C})$, $\vec{0} \in \mathcal{M}_{n-j,1}(\mathbb{C})$, and $C_1 \in \mathcal{M}_{n-j,n-j}(\mathbb{C})$. This was a very nice and repeatable procedure.

38:  Make an example in which $E^\perp$ is not invariant!

Again by the Fundamental Theorem of Algebra, C_1 has an eigenvalue, λ_2. Let $\{\vec{y}_1, \ldots, \vec{y}_k\}$ be a basis for E_2, the eigenspace for λ_2. Using this with a basis for $E_2^{\perp}$, $\{\vec{u}_{n-j-k}, \ldots, \vec{u}_n\}$, define the change of basis matrix $Q_2 = [\vec{e}_1, \ldots, \vec{e}_j \vec{y}_1 \cdots \vec{y}_k\, \vec{u}_{n-k-j} \cdots \vec{u}_n]$, where each $\vec{e}_i$ is a standard basis vector. Then

$$Q_2^{-1}Q_1^{-1}AQ_1Q_2 = Q_2^{-1}\begin{bmatrix} \lambda_1 I_j & R_1 \\ \vec{0} & C_1 \end{bmatrix} Q_2 = \begin{bmatrix} \lambda_1 I_j & R_2 & R_3 \\ \vec{0} & \lambda_2 I_k & R_4 \\ \vec{0} & \vec{0} & C_2 \end{bmatrix},$$

where $C_3 \in \mathcal{M}_{n-j-k\times n-j-k}(\mathbb{C})$. Do this $n-2$ more times, and define $Q = Q_1Q_2\cdots Q_{n-1}$.

By the Fundamental Theorem of Algebra, $\det(A-\lambda I) = 0$ has n solutions, counting multiplicity, which are all eigenvalues; call them $\lambda_1, \ldots, \lambda_m$, and $\det(U-\lambda I) = \det(Q^{-1}AQ - \lambda I) = 0$ has the same n solutions, which are the eigenvalues of A. By construction, these eigenvalues are on the diagonal of U. □

Theorem 5.4.8 *Let $U \in \mathcal{M}_{n\times n}(\mathbb{C})$ be upper triangular with diagonal entries d_i, for $i = 1, \ldots, n$. Then for any integer $k \geq 0$, U^k is upper triangular with diagonal entries d_i^k, for $i = 1, \ldots, n$. If the diagonal entries of U are all zeros, then for any $0 \leq k \leq n$, U^k will have zero for all of its diagonal entries and for all k entries above each diagonal entry.*

Proof for this theorem can be found in the Appendix.

Lemma 5.4.9 *If $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ has eigenvalue, λ, with algebraic multiplicity k and geometric multiplicity j, then there is a positive integer $k_0 \leq k$ such that*

$$\dim \operatorname{Ker}(A-\lambda I)^{k_0} = k,$$

and for each $i = 1, \ldots, k_0 - 1$, there is an m_{i+1} such that $1 \leq m_{i+1} \leq k - j$,

$$\dim \operatorname{Ker}(A-\lambda I)^{i} + m_{i+1} = \dim \operatorname{Ker}(A-\lambda I)^{i+1},$$

and $j + m_2 + \cdots + m_{k_0} = k$.

PROOF. Using Theorem 5.4.7, we have that A is similar to an upper triangular matrix, U, such that the diagonal contains the eigenvalues of A, counting multiplicity. Let's cleverly arrange that the λ's are the first k entries on the diagonal. If E is the eigenspace for λ, then

$$U|_E = \begin{bmatrix} \lambda & * & \cdots & \cdots & * \\ \vec{0} & \lambda & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vec{0} & \cdots & \vec{0} & \lambda & * \\ \vec{0} & \cdots & \cdots & \vec{0} & \lambda \end{bmatrix}, (U-\lambda I)|_E = \begin{bmatrix} \vec{0} & * & \cdots & \cdots & * \\ \vec{0} & \vec{0} & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vec{0} & \cdots & \vec{0} & \vec{0} & * \\ \vec{0} & \cdots & \cdots & \vec{0} & \vec{0} \end{bmatrix}.$$

By Theorem 5.4.8, we know that $(U-\lambda I)|_E^{k_0} = \vec{0}$, the zero matrix. Since the last $n-k$ diagonal entries of U are different from λ, the last $n-k$ entries of $U-\lambda I$ must be nonzero. Thus, $\dim \operatorname{Ker}(U-\lambda I)^{k_0} = k$. Since $(U-\lambda I)^{k_0}$ is similar[39] to $(A-\lambda I)^{k_0}$, we know from Theorem 5.3.6 that $\dim \operatorname{Ker}(A-\lambda I)^{k_0} = k$ as well. Moreover, at each step from $(U-\lambda I)^i$ to $(U-\lambda I)^{i+1}$

39:  Do you remember why? If not, can you verify the calculation?

for $I = 1, \ldots, k_0 - 1$, at least one more free variable is produced, so we also have that $\dim \operatorname{Ker} (A - \lambda I)^i + 1 \leq \dim \operatorname{Ker} (A - \lambda I)^{i+1}$. $\square$

That "at least one more free variable" bit is annoying. Look at this matrix and its square:

$$B - 2I = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (B - 2I)^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that $B - 2I$ has two free variables, but $(B - 2I)^2$ has four; $\dim \operatorname{Ker} (B - 2I)^2$ has increased by two. What is happening here? It turns out that the eigenvalue 2 has two distinct Jordan chains corresponding to two distinct eigenvectors, $\vec{e}_1$ and $\vec{e}_4$:

$$\begin{array}{ccccccc} \vec{e}_3 & \xrightarrow{B-2I} & \vec{e}_2 & \xrightarrow{B-2I} & \vec{e}_1 & \xrightarrow{B-2I} & \vec{0} \\ & & \vec{e}_5 & \xrightarrow{B-2I} & \vec{e}_4 & \xrightarrow{B-2I} & \vec{0}. \end{array}$$

Here $\{\vec{e}_1, \vec{e}_4\}$ is a basis for $\operatorname{Ker} (B - 2I)$, and $\{\vec{e}_1, \ldots, \vec{e}_5\}$ is a basis for $\operatorname{Ker} (B - 2I)^3$.

Here, finally, is the formal statement of the fact that we can find nice substitutes for eigenvectors, like we did in Example 5.4.1. If a matrix has an eigenvalue with algebraic multiplicity k and geometric multiplicity j, then you can always find j Jordan chains with a combined total length of k. Here's the more formal statement:

Lemma 5.4.10 *Let $A \in \mathcal{M}_{n \times n}(\mathbb{C})$ have an eigenvalue, λ, with algebraic multiplicity k and geometric multiplicity j. Then there are j Jordan chains, $S_1, \ldots, S_j$, such that $S_1 \cup \cdots \cup S_j$ is a basis for $\operatorname{Ker} (A - \lambda I)^k$, and in particular,*

$$\operatorname{Ker} (A - \lambda I)^k = \operatorname{Span} \{S_1\} \oplus \cdots \oplus \operatorname{Span} \{S_j\}.$$

It's not hard to believe there are j Jordan chains because there are j distinct eigenvectors that form a linearly independent set. It is definitely worth checking, though, that the associated *Jordan chains* are also formed of vectors that collectively form a basis for $\operatorname{Ker} (A - \lambda I)^k$.

There is a lot of bookkeeping and paperwork in the proof for this lemma. The process is neither elegant nor enlightening, so we relegate most of the fine details to the Appendix. For now, we'll provide a sketch of how the proof works.

Our goal is to first argue that these j Jordan chains exist and are distinct, and then argue that they form a basis for $\operatorname{Ker} (A - \lambda I)^k$. Once they form a basis, we know we can rearrange these basis vectors as the Jordan chains, $S_1, \ldots, S_j$, to get

$$\operatorname{Ker} (A - \lambda I)^k = \operatorname{Span} \{S_1\} \oplus \cdots \oplus \operatorname{Span} \{S_j\}.$$

As was stated earlier, though, we begin with the argument that these distinct chains exist. We'll use the same notation as Lemma 5.4.9, noting that for some

$k_0 \le k$, we have $\text{Ker}\,(A - \lambda I)^k = \text{Ker}\,(A - \lambda I)^{k_0}$. One can also check that

$$\text{Ker}\,(A - \lambda I) \subset \text{Ker}\,(A - \lambda I)^2 \subset \cdots \subset \text{Ker}\,(A - \lambda I)^{k_0}.$$

Using the Orthogonal Decomposition Theorem several times,[40] we get a nice break down of $\text{Ker}\,(A - \lambda I)^{k_0}$ into orthogonal parts. Specifically, we can write

$$\begin{array}{rcll} E_1 & = & \text{Ker}\,(A - \lambda I), & \dim E_1 = j \\ E_2 & = & (E_1)^{\perp} \cap \text{Ker}\,(A - \lambda I)^2, & \dim E_2 = m_2 \\ E_3 & = & (E_1 \oplus E_2)^{\perp} \cap \text{Ker}\,(A - \lambda I)^3, & \dim E_3 = m_3 \\ & \vdots & & \\ E_{k_0} & = & (E_1 \oplus \cdots \oplus E_{k_0 - 1})^{\perp}, & \dim E_{k_0} = m_{k_0}. \end{array}$$

From this construction, we have

$$\begin{array}{rcl} \text{Ker}\,(A - \lambda I) & = & E_1 \\ \text{Ker}\,(A - \lambda I)^2 & = & E_1 \oplus E_2 \\ \text{Ker}\,(A - \lambda I)^3 & = & E_1 \oplus E_2 \oplus E_3 \\ & \vdots & \\ \text{Ker}\,(A - \lambda I)^{k_0} & = & E_1 \oplus \cdots \oplus E_{k_0} \end{array}$$

Now we can build all of our chains, all at once. Starting with a basis for E_{k_0}, we multiply each of these basis vectors by $(A - \lambda I)$ and extend the resulting set of vectors to make a basis for $E_1 \oplus \cdots \oplus E_{k_0 - 1}$. Next, we multiply some of the vectors in *that* basis by $(A - \lambda I)$ and again extend the resulting set of vectors to be a basis for $E_1 \oplus \cdots \oplus E_{k_0 - 2}$. We keep doing this until finally arriving in E_1. Not so bad, right? At each step, however, we have to argue that our sets of vectors are linearly independent so that we know the chains remain distinct; this is aided by the use of orthogonal complements to define each of the sets, E_i. The result will be a basis for E_1 built out of the final vectors from each of the Jordan chains (some of which will be of length one). This construction gives us our j distinct Jordan chains. Then we only need to argue that any two Jordan chains are linearly independent, so that the collection of all the chains forms a basis for $\text{Ker}\,(A - \lambda I)^k$ as desired.

Since we will use these Jordan chains to form a basis of our domain $\mathbb{C}^n$, we need to know that they are also linearly independent when they correspond to different eigenvalues.

Lemma 5.4.11 *If S_1 and S_2 are Jordan chains for $A \in \mathcal{M}_{n \times n}(\mathbb{C})$ for different eigenvalues, then $S_1 \cup S_2$ is linearly independent.*

The proof of this is similar to the one for Theorem 5.1.2, so we've put it as an exercise. Now, for one final lemma before we prove the theorem.

Lemma 5.4.12 *Let $A \in \mathcal{M}_{n \times n}(\mathbb{C})$ have an eigenvalue, λ, with Jordan chain S of length k, the algebraic multiplicity of λ. Then the matrix representation with respect to S for A is a Jordan block.*

This proof is very similar to the argument in Example 5.4.1, so this is also an exercise.

40: Is it exactly $k_0 - 1$ times?

It's cheating if you read ahead.

PROOF OF THEOREM 5.4.1. Applying Lemma 5.4.10 to all the eigenvalues of A, we have a set of Jordan chains with n vectors, and by Lemma 5.4.11, they are all linearly independent; thus, this set is a basis for $\mathbb{C}^n$, call it $\mathcal{B}$. Once we show the span of each chain, $\mathrm{Span}\,\{S_i\}$, from Lemma 5.4.10 is invariant for A, we can apply Lemmas 5.4.6 and 5.4.12 to each $\mathrm{Span}\,\{S_i\}$, and we're done.

Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a Jordan chain for eigenvalue λ and $\vec{v} \in \mathrm{Span}\,\{S\}$. Then $(A-\lambda I)\vec{v}_i = \vec{v}_{i-1}$ for each $i = 2, \ldots, k-1$, and $(A-\lambda I)\vec{v}_1 = \vec{0}$; it follows that $A\vec{v}_i = \lambda\vec{v}_i + \vec{v}_{i-1}$, and $A\vec{v}_1 = \lambda\vec{v}_1$. Moreover, $\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$ for some scalars $a_1, \ldots, a_n$. Observe that

$$\begin{aligned}
A\vec{v} &= A(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) \\
&= a_1(\lambda\vec{v}_1) + a_2(\lambda\vec{v}_2 + \vec{v}_1) + \cdots + a_n(\lambda\vec{v}_n + \vec{v}_{n+1}) \\
&= (a_1\lambda + a_2)\vec{v}_1 + (a_2\lambda + a_3)\vec{v}_2 + \cdots + (a_{n-1}\lambda + a_n)\vec{v}_{n-1} + (a_n\lambda)\vec{v}_n,
\end{aligned}$$

which is a vector in $\mathrm{Span}\,\{S\}$, so $\mathrm{Span}\,\{S\}$ is invariant.

To see that the Jordan form is unique up to rearranging the blocks on the diagonal, note that the sizes of the blocks correspond to the lengths of the Jordan chains. These in turn were determined by the dimensions of each relevant $\mathrm{Ker}\,(A-\lambda I)^l$ for the eigenvalues λ. Since these are intrinsic to the original matrix and did not rely upon choices, the form is unique. $\square$

Alright! Victory!

Computing Jordan Canonical Form

The procedure for finding Jordan Canonical Form is actually pretty thoroughly described in Example 5.4.1. We'll just go through one more example in detail and then call it a day.

Example 5.4.5 Consider the matrix

$$A = \begin{bmatrix} 5 & 2 & -1 & 0 & 2 \\ -2 & 1 & 2 & 1 & -3 \\ -1 & -1 & 2 & -1 & 0 \\ 1 & 1 & 0 & 3 & 1 \\ 0 & 0 & -1 & -1 & 4 \end{bmatrix}.$$

We can use techniques from the previous section to find the characteristic polynomial. Of course, then it's a degree 5 polynomial that needs factoring, so we'll just go ahead and tell you the characteristic polynomial here is $p(\lambda) = (3-\lambda)^5$. Thus, the only eigenvalue is 3, and it has an algebraic multiplicity of 5. Let's compute $\mathrm{Ker}\ A-3I$ to see the geometric multiplicity.

$$A - 3I = \begin{bmatrix} 2 & 2 & -1 & 0 & 2 \\ -2 & -2 & 2 & 1 & -3 \\ -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus,

$$\operatorname{Ker}(A-3I) = \operatorname{Span}\left\{ \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Since the geometric multiplicity is 2, we need 2 Jordan chains. To get the next step, we need to solve $(A-3I)\vec{v}_3 = \vec{v}_1$ and also $(A-3I)\vec{v}_4 = \vec{v}_2$. We can do this more efficiently by augmenting with two vectors and performing the same row reduction steps as we used above.

$$\left[\begin{array}{rrrrr|rr} 2 & 2 & -1 & 0 & 2 & -1 & -1 \\ -2 & -2 & 2 & 1 & -3 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{rrrrr|rr} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & -1 & -1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This tells us that

$$\vec{v}_3 \in \left\{ \begin{bmatrix} -x_2 - x_5 \\ x_2 \\ 1 \\ -1 + x_5 \\ x_5 \end{bmatrix} : x_2, x_5 \in \mathbb{R} \right\} \text{ and}$$
$$\vec{v}_4 \in \left\{ \begin{bmatrix} 1 - x_2 - x_5 \\ x_2 \\ 3 \\ -4 + x_5 \\ x_5 \end{bmatrix} : x_2, x_5 \in \mathbb{R} \right\}.$$

We have infinitely many choices here for $\vec{v}_3$ and $\vec{v}_4$, but we can simply choose that $x_2 = x_5 = 0$ in both sets to get

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -4 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

Now, since we need just one more vector to form a basis, we know only one of these chains continues. However, it's not obvious which one it is. We can again augment by two vectors and solve simultaneously for $(A-3I)\vec{v}_5 = \vec{v}_3$

and $(A-3I)\vec{v}_6 = \vec{v}_4$, knowing that only one of these has a solution.

$$\left[\begin{array}{rrrrr|rr} 2 & 2 & -1 & 0 & 2 & 1 & 0 \\ -2 & -2 & 2 & 1 & -3 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 3 & 1 \\ 1 & 1 & 0 & 0 & 1 & -4 & -1 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rrrrr|rr} 1 & 1 & 0 & 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 & 0 & -9 & -2 \\ 0 & 0 & 0 & 1 & -1 & 10 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

This says $\vec{v}_5$ does not exist and

$$\vec{v}_6 \in \left\{ \begin{bmatrix} -1-x_2-x_5 \\ x_2 \\ -2 \\ 2+x_5 \\ x_5 \end{bmatrix} : x_2, x_5 \in \mathbb{R} \right\}.$$

Again, we can choose that $x_2 = x_5 = 0$, so that

$$\vec{v}_6 = \begin{bmatrix} -1 \\ 0 \\ -2 \\ 2 \\ 0 \end{bmatrix}.$$

Now, our Jordan chains are

$$\begin{array}{ccccccc} \vec{v}_6 & \xrightarrow{A-3I} & \vec{v}_4 & \xrightarrow{A-3I} & \vec{v}_2 & \xrightarrow{A-3I} & \vec{0} \\ & & \vec{v}_3 & \xrightarrow{A-3I} & \vec{v}_1 & \xrightarrow{A-3I} & \vec{0}, \end{array}$$

and our basis for Jordan Canonical Form is $\{\vec{v}_1, \vec{v}_3, \vec{v}_2, \vec{v}_4, \vec{v}_6\}$.

Exploration 150 Verify that when A in Example 5.4.5 above is changed to the basis given for Jordan Canonical Form that the outcome is

$$J = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Note that in Example 5.4.5, we performed the same row reduction multiple times. This will always be the case if we compute our chains in this manner. If you'd prefer to avoid so many repetitive row reductions, you could instead find the product of elementary matrices that represents these row operations the first time you are doing them by augmenting with the identity matrix to keep track. Remember, this was one way we described how to find the inverse of a matrix back in Section 4.5. Then, you can do a simple matrix multiplication to see what the outcome would have been from the row reduction.

Example 5.4.6 Let's revisit Example 5.4.5 using the product of elementary matrices to see how those computations work. First, we augment $A-3I$ by the identity matrix and row reduce.

$$\left[\begin{array}{rrrrr|rrrrr} 2 & 2 & -1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ -2 & -2 & 2 & 1 & -3 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rrrrr|rrrrr} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

Notice that here we have lined up our pivots on the left hand side to be on the diagonal. This is then not in reduced row-echelon form, but this form will actually allow us to more quickly see our answers since we are always making the choice that the free variables are 0. Let us define now the matrix

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 2 & 0 \\ 2 & 1 & 0 & -2 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

We find $\text{Ker } A-3I$ in the same way as before to get the same $\vec{v}_1$ and $\vec{v}_2$. Now, we can use B to find $\vec{v}_3$ and $\vec{v}_4$ since $B\vec{v}_1 = \vec{v}_3$ and $B\vec{v}_2 = \vec{v}_4$. Once we have $\vec{v}_3$ and $\vec{v}_4$, we can again use B to find $\vec{v}_6$ since $B\vec{v}_4 = \vec{v}_6$. To see that $\vec{v}_5$ does not exist, note that $B\vec{v}_3$ would have a nonzero entry for x_5, which does not align with our choice that $x_2 = x_5 = 0$.

You should all be prepared now for the exercises that await.

Section Highlights

- A matrix in *Jordan Canonical Form* has eigenvalues on the diagonal, 1's or 0's immediately above the diagonal, and 0's everywhere else. See Definition 5.4.2 and Theorem 5.4.1.

- While not all matrices in $\mathcal{M}_{n\times n}$ are diagonalizable, it is always possible to find a basis of $\mathbb{C}^n$ that puts a matrix into Jordan Canonical Form. See Theorem 5.4.1.

- Every matrix in $\mathcal{M}_{n\times n}$ has a unique Jordan Canonical Form (allowing for rearrangement of blocks), and two matrices are similar if and only if they have the same Jordan Canonical Forms (again, allowing for rearrangement of blocks). See Theorem 5.4.1 and Corollary 5.4.4.

- ► The procedure for computing Jordan Canonical Form is illustrated in Example 5.4.1 and Example 5.4.5.

Exercises for Section 5.4

5.4.1. Now you will create examples.

(a) Make a matrix in $\mathcal{M}_{4\times 4}$ with 4 Jordan blocks.

(b) Make a matrix in $\mathcal{M}_{4\times 4}$ with 3 Jordan blocks. Use those same Jordan blocks to make a different matrix in $\mathcal{M}_{4\times 4}$.

(c) Make a matrix in $\mathcal{M}_{4\times 4}$ with 2 Jordan blocks. Use those same Jordan blocks to make a different matrix in $\mathcal{M}_{4\times 4}$.

(d) Make a matrix in $\mathcal{M}_{4\times 4}$ with 1 Jordan block.

5.4.2. Find the Jordan Canonical Form for the following matrices.

(a) $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix}$

(c) $C = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

5.4.3. Here are some matrices:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 1 \\ 2 & -3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 & 0 & -1 \\ 2 & 1 & 0 & -1 & -1 \\ -1 & 1 & 2 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

(a) A has eigenvalue 1 with algebraic multiplicity 3 and geometric multiplicity 1. Find a Jordan chain for 1 of length 3.

(b) B has eigenvalue 1 with algebraic multiplicity 5 and geometric multiplicity 1. Find a Jordan chain for 1 of length 5.

5.4.4. Let's explore the left shift! Define $L\colon \mathbb{R}^n \to \mathbb{R}^n$ by $L(\vec{x}_1, \ldots, \vec{x}_n) = (\vec{x}_2, \ldots, \vec{x}_n, 0)$.

(a) Prove that the left shift is a linear transformation.

(b) Find a matrix representation for L using the standard basis on $\mathbb{R}^n$.

(c) Find the kernel and image of L.

(d) Show that L is *nilpotent*; that is, show that there is some positive integer k such that $L^k = 0$, where 0 is the linear transformation that maps all vectors to $\vec{0}$.

5.4.5. Let $T\colon V \to V$ be a linear transformation with invariant subspaces $V_1, \ldots, V_n$ such that $V = V_1 \oplus \cdots \oplus V_n$. Show that T decomposes into the direct sum of linear transformations $T|_{V_1}, \ldots, T|_{V_n}$.

5.4.6. Complete the proof of Lemma 5.4.6.

5.4.7. Prove Lemma 5.4.11.

5.4.8. Prove Lemma 5.4.12.

5.5 Spectral Theory

As we saw in Section 5.3, some square matrices are diagonalizable, which is neat, but as we just saw in Section 5.4, *all* matrices are very nearly diagonalizable. Great, but you probably still wonder what precisely it is about a matrix that makes it diagonalizable. Sure, the diagonalizability of an $n \times n$ matrix is characterized by having n linearly independent eigenvectors, but what's that all about? Why should some matrices have a maximal number of invariant subspaces and others not? That is a good question, and we shall see that it has something like a good answer.

Symmetric Matrices

Recall that A^T, the transpose of the matrix A, is a matrix whose columns are the rows of A.[41]

41: This is true for all kinds of matrices, real or complex.

What about matrices of abstract shapes?

The definition still works, but what are you doing with abstract shapes?

You don't understand my art!

Definition 5.5.1 *A* ***symmetric*** *matrix is a matrix A such that $A^T = A$.*

Symmetric matrices must be square, and entries "mirror each other across the main diagonal."

Example 5.5.1 Here are some matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Note that $A^T = A$ (check it!), so A is symmetric. You could also note that $a_{12} = a_{21} = 2$, $a_{13} = a_{31} = 3$, and $a_{23} = a_{32} = 5$. However, B is not symmetric because $b_{12} = 2 \neq 3 = b_{21}$.

These neatly organized matrices we call "symmetric" have extremely nice properties. You'll recall from Theorem 5.1.2 that we already know eigenvectors from different eigenspaces are always going to be linearly independent. The additional hypothesis of symmetry improves this linear independence to orthogonality.

Theorem 5.5.1 *If a matrix A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.*

PROOF OF THEOREM 5.5.1. First of all, recall from Theorem 4.4.9 that for any two appropriately sized matrices B and C, we know $(BC)^T = C^T B^T$. Suppose $A\vec{v}_1 = \lambda_1 \vec{v}_2$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$, where $\lambda_1 \neq \lambda_2$. Since A is symmetric, we know $A^T = A$. Then

$$\begin{aligned} \lambda_1 \vec{v}_1 \cdot \vec{v}_2 &= A\vec{v}_1 \cdot \vec{v}_2 \\ &= (A\vec{v}_1)^T \vec{v}_2 \\ &= \vec{v}_1^T A^T \vec{v}_2 \\ &= \vec{v}_1^T A \vec{v}_2 \\ &= \vec{v}_1^T \lambda_2 \vec{v}_2 = \lambda_2 \vec{v}_1 \cdot \vec{v}_2. \end{aligned}$$

Then $(\lambda_1-\lambda_2)(\vec{v}_1\cdot\vec{v}_2)=0$. Since $\lambda_1-\lambda_2\neq 0$, we must have $\vec{v}_1\cdot\vec{v}_2=0$. $\square$

Example 5.5.2 Let's try Theorem 5.5.1 on the symmetric matrix

$$A=\begin{bmatrix} 1 & -6 & 4 \\ -6 & 2 & -2 \\ 4 & -2 & -3 \end{bmatrix}$$

One can check that the eigenvalues and eigenvectors are as follows:

$$\lambda_1=9,\qquad \vec{v}_1=\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix},$$

$$\lambda_2=-6,\qquad \vec{v}_2=\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix},\quad \text{and}$$

$$\lambda_3=-3,\qquad \vec{v}_3=\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

One can also verify that $\vec{v}_1\cdot\vec{v}_2=\vec{v}_1\cdot\vec{v}_3=\vec{v}_2\cdot\vec{v}_3=0$.

Exploration 151 Find the eigenvalues and eigenvectors for the symmetric matrix

$$A=\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}.$$

Verify that the eigenvectors are orthogonal.

The Spectral Theorem

Let's start with the most tongue-twistery terminology.

Definition 5.5.2 *A matrix A is* ***orthogonally diagonalizable*** *if there is an orthogonal matrix P and a diagonal matrix D such that*

$$A=PDP^T=PDP^{-1}.$$

Example 5.5.3 Orthogonally diagonalize

$$A=\begin{bmatrix} 1 & -6 & 4 \\ -6 & 2 & -2 \\ 4 & -2 & -3 \end{bmatrix}.$$

We know from Example 5.5.2 that

$$\left\{\begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix},\ \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix},\ \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}\right\}$$

is an orthogonal set of eigenvectors. Observe that we have also normalized each of them, so this is actually an ortho*normal* set of eigenvectors. If we define,

$$P = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ -2/3 & -1/3 & 2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -3 \end{bmatrix},$$

then we can then check that $A = PDP^T$.

Theorem 5.5.2 *If a matrix A is orthogonally diagonalizable, then A is symmetric.*

PROOF. Suppose $A = PDP^T$, where P is an orthogonal matrix and D is a diagonal matrix. Then

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A.$$

□

Theorem 5.5.2 is only half the story. It's actually true that A is orthogonally diagonalizable *if and only if* A is symmetric. The following theorem, known as the Spectral Theorem,[42] makes it so; we leave the proof of this theorem for the end of the section.

42:  The *spectrum* of a matrix A is the set of all eigenvalues, and the *spectral radius* is the maximum magnitude of all the eigenvalues of A.

 Yeah, that's not nearly as cool as you probably thought it would be. Sorry.

Theorem 5.5.3 (The Real Spectral Theorem) *A symmetric matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ has the following properties:*

(a) *A has n real eigenvalues, counting multiplicities.*
(b) *For each eigenvalue λ, the geometric multiplicity of λ equals the algebraic multiplicity of λ.*
(c) *The eigenspaces of A are mutually orthogonal.*
(d) *A is orthogonally diagonalizable.*

Besides being amazing, the Spectral Theorem is also quite useful. Behold!

Definition 5.5.3 *The **spectral decomposition** of an orthogonally diagonalizable matrix A is*

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \lambda_2 \vec{u}_2 \vec{u}_2^T + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T,$$

where for $1 \leq i \leq n$, λ_i are eigenvalues of A, and $\vec{u}_i$ are corresponding orthonormal eigenvectors of A.

Note that each $\vec{u}_i \vec{u}_i^T$ is a matrix that projects onto the subspace spanned by $\vec{u}_i$. In fact, we can make[43] a more general statement about such projection matrices.

43:  And prove!

Theorem 5.5.4 *Let $\mathcal{B} = \{\vec{w}_1, \ldots, \vec{w}_k\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$, and let $A = [\vec{w}_1 \cdots \vec{w}_k]$. For all $\vec{v} \in \mathbb{R}^n$,*

$$\operatorname{proj}_W(\vec{x}) = AA^T\vec{x}.$$

PROOF. Recall from way back in Chapter 2 that

$$\operatorname{proj}_W(\vec{x}) = \frac{\vec{x}\cdot\vec{w}_1}{\vec{w}_1\cdot\vec{w}_1}\vec{w}_1 + \cdots + \frac{\vec{x}\cdot\vec{w}_k}{\vec{w}_k\cdot\vec{w}_k}\vec{w}_k.$$

Using the fact that vectors in $\mathcal{B}$ are unit vectors, the fact that the inner product is commutative, and rewriting our inner products as matrix products, we have

$$\begin{aligned} \operatorname{proj}_W(\vec{x}) &= (\vec{x}\cdot\vec{w}_1)\vec{w}_1 + \cdots + (\vec{x}\cdot\vec{w}_k)\vec{w}_k \\ &= (\vec{w}_1\cdot\vec{x})\vec{w}_1 + \cdots + (\vec{w}_k\cdot\vec{x})\vec{w}_k \\ &= (\vec{w}_1^T\vec{x})\vec{w}_1 + \cdots + (\vec{w}_k^T\vec{x})\vec{w}_k \end{aligned}$$

Then we have

$$\operatorname{proj}_W(\vec{x}) = [\vec{w}_1 \cdots \vec{w}_k]\begin{bmatrix} \vec{w}_1^T\vec{x} \\ \vdots \\ \vec{w}_k^T\vec{x} \end{bmatrix} = AA^T\vec{x}.$$

□

Example 5.5.4 Let's find the spectral decomposition of

$$A = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}.$$

Note first that A is symmetric, so we know from the Spectral Theorem that the spectral decomposition exists. Here are the eigenvalues and eigenvectors of A:

$$\lambda_1 = 10, \vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \lambda_2 = 5, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

We need unit vectors, so let's use

$$\vec{u}_1 = \begin{bmatrix} -\sqrt{5}/5 \\ 2\sqrt{5}/5 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 2\sqrt{5}/5 \\ \sqrt{5}/5 \end{bmatrix}.$$

Then

$$\begin{aligned} \lambda_1\vec{u}_1\vec{u}_1^T + \lambda_2\vec{u}_2\vec{u}_2^T &= 10\begin{bmatrix} -\sqrt{5}/5 \\ 2\sqrt{5}/5 \end{bmatrix}\begin{bmatrix} -\sqrt{5}/5 & 2\sqrt{5}/5 \end{bmatrix} \\ &\quad +5\begin{bmatrix} 2\sqrt{5}/5 \\ \sqrt{5}/5 \end{bmatrix}\begin{bmatrix} 2\sqrt{5}/5 & \sqrt{5}/5 \end{bmatrix} \\ &= 10\begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix} + 5\begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix} \end{aligned}$$

This is the spectral decomposition of A. When written in this fashion, A is a linear combination of projections, which is computationally efficient. It is easy to verify from this point that this linear combination is equal to A:

$$\begin{aligned} 10\begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix} + 5\begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix} &= \begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix} \end{aligned}$$

The Complex Spectral Theorem

Definition 5.5.4 *For $z = x + iy \in \mathbb{C}$, the* ***conjugate of*** z *is the complex number $\bar{z} = x - iy$. For $A \in \mathcal{M}_{m\times n}(\mathbb{C})$, the* ***conjugate of*** A *is the matrix $\overline{A} = [\bar{a}_{ij}] \in \mathcal{M}_{m\times n}(\mathbb{C})$ obtained by conjugating every entry A. The*

conjugate transpose of A, *denoted by* A^H, *is obtained by conjugating the transpose of* A; *that is,* $A^H = \overline{A^T}$.

Definition 5.5.5 *A matrix* $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ *is called* ***Hermitian*** *if* $A = A^H$.

Example 5.5.5 Both of these matrices are Hermitian, but only one is symmetric.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 7 \\ 4 & 7 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2+3i & 4 \\ 2-3i & 5 & 7-8i \\ 4 & 7+8i & 9 \end{bmatrix}$$

Exploration 152 Prove that all symmetric matrices in $\mathcal{M}_{n\times n}(\mathbb{R})$ are Hermitian. Also, find an example to show that there are matrices in $\mathcal{M}_{n\times n}(\mathbb{C})$ that are symmetric but not Hermitian.

Exploration 153 Prove that all Hermitian matrices have real entries on their main diagonal.

There is a complex analog of Theorem 5.5.1 for Hermitian matrices. Its proof is nearly identical to the proof of Theorem 5.5.1. However, just as in Section 5.4, we need to use the Hermitian inner product and norm (Definitions 5.4.7 and 5.4.8) when dealing with matrices in $\mathcal{M}_{n\times n}(\mathbb{C})$. Other than *that*, the proof is nearly identical.

Corollary 5.5.5 *If* A *is Hermitian, then any two eigenvectors from different eigenspaces are orthogonal.*

Just like how Hermitian matrices are the complex generalization of symmetric matrices, we have a complex generalization of orthogonal matrices.

Definition 5.5.6 *A matrix* $U \in \mathcal{M}_{n\times n}(\mathbb{C})$ *is called* ***unitary*** *if* $UU^H = U^HU = I_n$.

Unitary matrices enjoy many properties analogous to the nice properties enjoyed by real symmetric matrices:

Theorem 5.5.6 *Let* $U \in \mathcal{M}_{n\times n}(\mathbb{C})$. *The following conditions are equivalent:*

(a) U *is unitary.*

(b) U *is invertible with* $U^{-1} = U^H$.

(c) *The columns of U are an orthonormal basis for $\mathbb{C}^n$.*

We leave the proof of Theorem 5.5.6 as an exercise.

Here's a surprising fact. Well, it's surprising if you're not at all familiar with the Spectral Theorem.

Theorem 5.5.7 *If $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ is Hermitian, then all of the eigenvalues of A are real.*

PROOF. We know from the Fundamental Theorem of Algebra that $\det(A - \lambda I) = 0$ has a solution, which by definition is an eigenvalue. Suppose $\lambda = x + iy$ for some $x, y \in \mathbb{R}$ and that $\vec{v}$ is an associated eigenvector, so $A\vec{v} = \lambda\vec{v}$. We will make extensive use of our Hermitian inner product and use the fact that for any $A, B \in \mathcal{M}_{n\times n}(\mathbb{C})$ we have $(AB)^H = B^H A^H$ and $A = (A^H)^H$.[44] Note first that

$$\begin{aligned}
(A\vec{v})\cdot\vec{v} &= \vec{v}^H(A\vec{v}) && \text{by conversion to matrix multiplication,}\\
&= (\vec{v}^H A)\vec{v} && \text{by associativity of matrix multiplication,}\\
&= (A^H\vec{v})^H\vec{v} && \text{since } (AB)^H = B^H A^H \text{ and } A = (A^H)^H,\\
&= \vec{v}\cdot(A^H\vec{v}) && \text{by conversion back to inner product,}\\
&= \vec{v}\cdot(A\vec{v}) && \text{since } A \text{ is Hermitian,}\\
&= \vec{v}\cdot(\lambda\vec{v}) &&\\
&= \bar{\lambda}(\vec{v}\cdot\vec{v}) && \text{by definition of Hermitian inner product.}
\end{aligned}$$

Also, we have

$$\begin{aligned}
(A\vec{v})\cdot\vec{v} &= (\lambda\vec{v})\cdot\vec{v} &&\\
&= \lambda(\vec{v}\cdot\vec{v}) && \text{by definition of Hermitian inner product.}
\end{aligned}$$

It follows that $\bar{\lambda}(\vec{v}\cdot\vec{v}) = \lambda(\vec{v}\cdot\vec{v})$, but since $\vec{v}$ is an eigenvector, we know that $\vec{v}\cdot\vec{v} > 0$. Thus, $\bar{\lambda} = \lambda$, or $x + iy = x - iy$. This implies that $y = -y$, so $y = 0$. Then $\lambda = x \in \mathbb{R}$. □

44: Exercise!

Theorem 5.5.8 (The Spectral Theorem) *If $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ is Hermitian, then there is a unitary matrix U and a real diagonal matrix D such that $A = U^H D U$.*

The proof is similar in flavor to the proof of Theorem 5.4.7, but the fact that A is Hermitian will make our calculations much nicer. Let's have a lemma first.

Lemma 5.5.9 *For square matrices A and B, $(AB)^H = B^H A^H$.*

PROOF. We will use the fact that for $z, w \in \mathbb{C}$, $\overline{zw} = \bar{z}\bar{w}$, and you will prove that as an exercise.

$$(AB)^H = \overline{(AB)^T} = \overline{B^T A^T} = B^H A^H.$$

□

PROOF OF THEOREM 5.5.8. Let λ_1 be an eigenvalue for A; we know λ_1 exists from the Fundamental Theorem of Algebra, and we know that λ_1 is real from Theorem 5.5.7. Let $\vec{x}_1$ be one of its associated eigenvectors such that $\|\vec{x}_1\| = 1$ and $E_1 = \mathrm{Span}\,\{\vec{x}_1\}$. Then $\dim E_1^\perp = n - 1$. Make a basis for

$\mathbb{C}^n$ using $\vec{x}_1$ and $n-1$ orthonormal vectors, $\vec{v}_2, \ldots, \vec{v}_n \in E_1^{\perp}$, and define $Q_1 = [\vec{x}_1\ \vec{v}_2\ \cdots\ \vec{v}_n]$. By Theorem 5.5.6, Q_1 is a unitary matrix, and

$$Q_1^H A Q_1 \quad = \quad \begin{bmatrix} \vec{x}_1^H \\ \vec{v}_2^H \\ \vdots \\ \vec{v}_n^H \end{bmatrix} [\lambda_1 \vec{x}_1\ A\vec{v}_2 \cdots A\vec{v}_n].$$

Observe that $\vec{x}_1^H \vec{x}_1 = \|\vec{x}_1\| = 1$ and $\vec{v}_i^H \vec{x}_1 = 0$ for $i = 2, \ldots, n$, so the first column of $Q_1^H A Q_1$ is $\lambda_1 \vec{e}_1$. Moreover, since

$$(Q_1^H A Q_1)^H = Q_1^H A^H (Q_1^H)^H = Q_1^H A Q_1,$$

$Q_1^H A Q_1$ is Hermitian. Thus,

$$Q_1^H A_1 Q_1 \quad = \quad \begin{bmatrix} \lambda_1 & \vec{0}^T \\ \vec{0} & C_1 \end{bmatrix},$$

where $\vec{0} \in \mathcal{M}_{n-1,1}$ and $C_1 \in \mathcal{M}_{n-1,n-1}(\mathbb{C})$ is also Hermitian. Since C_1 is Hermitian, this is a very nice and repeatable procedure.

Again, C_1 has a real eigenvalue, call it λ_2, with an eigenvector, call it $\vec{x}_2$, such that $\|\vec{x}_2\| = 1$ and $E_2 = \operatorname{Span}\{\vec{x}_2\}$. We can build a unitary matrix using $\vec{e}_1$, $\vec{x}_2$, and $n-2$ more orthonormal vectors, $\vec{u}_2, \ldots, \vec{u}_n \in E_2^{\perp}$ and define $Q_2 = [\vec{e}_1\ \vec{x}_2\ \vec{u}_2\ \cdots\ \vec{u}_n]$. Then

$$Q_2^H Q_1^H A Q_1 Q_2 = Q_2^H \begin{bmatrix} \lambda_1 & \vec{0}^T \\ \vec{0} & C_1 \end{bmatrix} Q_2 = \begin{bmatrix} \lambda_1 & 0 & \vec{0}^T \\ 0 & \lambda_2 & \vec{0}^T \\ \vec{0} & \vec{0} & C_2 \end{bmatrix},$$

where $\vec{0} \in \mathcal{M}_{n-2,1}$ and $C_2 \in \mathcal{M}_{n-2,n-2}(\mathbb{C})$ is also Hermitian.

Do this $n-2$ more times, and define $U = Q_1 Q_2 \ldots Q_{n-1}$. It remains only to verify that Q is unitary.[45] □

45:  As an exercise!

The proof of Theorem 5.5.3 now follows quickly, but for reference, we provide the details.

Proof of Theorem 5.5.3. Part a follows from Theorem 5.5.7, part b follows from part a, and part c follows from Theorem 5.5.1]. Since $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ is symmetric, it is Hermitian, so part d follows from Theorem 5.5.8, noting that since A is had only real entries, U is a symmetric matrix in $\mathcal{M}_{n\times n}(\mathbb{R})$ by construction. □

Section Highlights

- ▶ A matrix is orthogonally diagonalizable if there exists an orthogonal basis with respect to which it is diagonal. See Definition 5.5.2.
- ▶ A matrix is called symmetric if it is equal to its own transpose. See Definition 5.5.1.
- ▶ A real-valued matrix will be orthogonally diagonalizable if and only if it is symmetric. Thus, a symmetric real-valued matrix is always

diagonalizable and always has all real eigenvalues. This is part of the Real Spectral Theorem. See Theorem 5.5.2 and Theorem 5.5.3.

- For any real-valued symmetric matrix A, there is a spectral decomposition that decomposes A into a sum of symmetric matrices scaled by the eigenvalues of A. See Definition 5.5.3 and Example 5.5.4.
- For matrices with complex entries, the concept of Hermitian matrices replaces that of symmetric matrices, and there is a more-general version of the Spectral Theorem. See Definition 5.5.5, Theorem 5.5.7, Definition 5.5.6, and Theorem 5.5.8.

Exercises for Section 5.5

5.5.1. Here's a symmetric matrix:

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

According to Theorem 5.5.1, any two eigenvectors from different eigenspaces are orthogonal. Verify this is true for A.

5.5.2. Let

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

(a) Show that PDP^{-1} is not symmetric.

(b) Use the Gram-Schmidt process on the columns of P, normalize the resulting three vectors, and use them to make an orthogonal matrix Q.

(c) Show that QDQ^{-1} *is* symmetric.

5.5.3. Find the spectral decomposition for each of the following matrices.

(a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $C = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

5.5.4. Prove that for any $A, B \in \mathcal{M}_{n\times n}(\mathbb{C})$, $(AB)^H = B^H A^H$. *Hint: See Theorem 4.4.9 and make sure to keep track of the conjugates from the Hermitian inner product.*

5.5.5. Prove Theorem 5.5.6.

5.5.6. Recall that for $z = x + iy \in \mathbb{C}$, $\bar{z} = x - iy$. Prove that for $z, w \in \mathbb{C}$, $\overline{zw} = \bar{z}\bar{w}$.

5.5.7. Prove that if $A_1, \ldots, A_n \in \mathcal{M}_{n\times n}(\mathbb{C})$ are unitary, then $A_1 \cdots A_n$ is unitary.

5.6 Singular Value Decomposition

We've spent the entire chapter to this point dealing only with square matrices. Now we'll take what we learned and apply it to the more general setting of rectangular matrices. No mucking about. Let's have the theorem.

Theorem 5.6.1 (Real Singular Value Decomposition) *Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$. Then there is an orthogonal matrix $U \in \mathcal{M}_{m\times m}(\mathbb{R})$, an orthogonal matrix $V \in \mathcal{M}_{n\times n}(\mathbb{R})$, and a rectangular diagonal matrix $D \in \mathcal{M}_{m\times n}(\mathbb{R})$ such that*

$$A = UDV^T.$$

It doesn't take a lot of imagination to correctly guess what a *rectangular diagonal* matrix is. We're not giving the formal definition until you guess.

. . . ok. Ready?

Definition 5.6.1 *A matrix $D \in \mathcal{M}_{m\times n}$ with entries $d_{i,j}$ is called a **rectangular diagonal matrix** if $d_{i,j} = 0$ whenever $i \neq j$.*

For example, matrices of the form

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{mm} & 0 & \cdots & 0 \end{bmatrix}$$

or

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

are rectangular diagonal matrices.

Example 5.6.1 Here's a non-square matrix:

$$A = \begin{bmatrix} -20 & 8 & -2 \\ -10 & 19 & 14 \end{bmatrix}$$

You can check that $A = UDV^T$, $UU^T = U^TU = I_3$, and $VV^T = V^TV = I_4$, where

$$U = \frac{1}{5}\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix},$$

$$D = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 30 & 0 \end{bmatrix}, \text{ and}$$

$$V^T = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}.$$

How do you actually find such a decomposition? We need one more definition.

Definition 5.6.2 *A nonnegative $\sigma \in \mathbb{R}$ is a* ***singular value*** *for $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ if there are unit vectors $\vec{u} \in \mathbb{R}^m$ and $\vec{v}\mathbb{R}^n$ such that*

$$A\vec{v} = \sigma\vec{u} \qquad \text{and} \qquad A^T\vec{u} = \sigma\vec{v}.$$

In this case, $\vec{u}$ and $\vec{v}$ are called ***left-singular*** *and* ***right-singular*** *vectors, respectively.*

The left-singular vectors will be the columns of U, the right-singular vectors will be the columns of V, and the singular values will be the diagonal entries on D. If you're willing to believe that the singular value decomposition exists (and you should; we'll prove it soon), then finding the singular values and vectors is not terribly difficult. Note first that since U and V are orthogonal and D is rectangular diagonal, we have

$$\begin{aligned}
AA^T &= (UDV^T)(UDV^T)^T = (UDV^T)(VD^TU^T) = U(DD^T)U^T \\
A^TA &= (UDV^T)^T(UDV^T) = (VD^TU^T)(UDV^T) = V(D^TD)V^T
\end{aligned}$$

Here are some handy facts that we'll prove later:

Theorem 5.6.2 *If $A \in \mathcal{M}_{m\times n}(\mathbb{R})$, then AA^T and A^TA are both symmetric.*

Corollary 5.6.3 *If $D \in \mathcal{M}_{m\times n}(\mathbb{R})$ is rectangular diagonal, then DD^T and D^TD are both diagonal and have the same nonzero entries.*

Since AA^T and A^TA are both symmetric, we know from the Spectral Theorem that the columns of U and V should normalized eigenvectors of AA^T and A^TA, respectively. It's a little less obvious, but the eigenvalues of A, which become the entries on the diagonal of D, are *the square roots* of the eigenvalues of AA^T (or A^TA).

The only catch is that normalized eigenvectors aren't necessarily left-singular or right-singular vectors, but they do give us a starting point. If $\vec{u}$ is an eigenvector for AA^T, then calculate $A^T\vec{u}$. It should be one of the eigenvectors for A^TA. One may need to rescale some of these vectors by -1, but in this fashion, one can find and properly order the left-singular and right-singular vectors.

Example 5.6.2 Here's a matrix A with AA^T and A^TA:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^TA = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

As indicated by Theorem 5.6.2, both AA^T and A^TA are symmetric. Here is some additional handy information:

Eigenvalues/eigenvectors for AA^T

eigenvalue	eigenvectors
$\sqrt{3}$	$\vec{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$
1	$\vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

Eigenvalues/eigenvectors for A^TA

eigenvalue	eigenvectors
$\sqrt{3}$	$\vec{v}_1 = \begin{bmatrix} 2/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$
1	$\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
0	$\vec{v}_4 = \begin{bmatrix} -1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

Using the eigenvalues for AA^T, we have singular values 1, 1, and $\sqrt{3}$, so

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}.$$

You can check that $A\vec{v}_1 = \sqrt{3}\vec{u}_1$. However, $A\vec{v}_2 = -\vec{u}_3$ and $A^T\vec{u}_2 = -\vec{v}_3$. Thus, $\vec{u}_1$ and $\vec{v}_1$ are left-singular and right-singular vectors, but we need to use $-\vec{u}_3$ and $-\vec{u}_2$ (in that order) to have left-singular vectors if we use $\vec{v}_2$ and $\vec{v}_3$ as left-singular vectors. Defining

$$U = [\vec{u}_1 \ -\vec{u}_3 \ -\vec{u}_2] \qquad \text{and} \qquad V = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4],$$

we have the singular value decomposition $A = UDV^T$.

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$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The proof of Theorem 5.6.1 uses The Spectral Theorem and follows the roughly same procedure from Example 5.6.2, so we'll just take care of it in the exercises.

Pseudoinverses

We come immediately to a convenient application of the singular value decomposition. We also get a fun new term.

Definition 5.6.3 *Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ have singular value decomposition UDV^T, r be $\min\{m,n\}$, k be the number of nonzero singular values, and $\sigma_1,\ldots,\sigma_k,0_{k_1},\ldots,0_r$ be the diagonal entries of D. The **pseudoinverse** of A, denoted A^+, is the matrix VD^+U^T, where $D^+ \in \mathcal{M}_{n\times m}$ is rectangular diagonal with diagonal entries $1/\sigma_1,\ldots,1/\sigma_k,0,\ldots,0$.*

Example 5.6.3 Here's a matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Its singular value decomposition is UDV^T, where

$$\begin{aligned} U &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ D &= \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and} \\ V &= \begin{bmatrix} 2/\sqrt{6} & 0 & 0 & -1\sqrt{3} \\ 0 & 0 & 1 & 0 \\ 1/\sqrt{6} & -1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 0 & 1/\sqrt{3} \end{bmatrix}. \end{aligned}$$

The pseudoinverse of A is $A^+ = VD^+U^T$, where

$$D^+ = \begin{bmatrix} 1/\sqrt{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, we have

$$A^+ = VD^+U^T = \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix}.$$

That's a neat thing to do. What can we do this this fun new matrix, though? Recall that if A is invertible, A^{-1} can be used to find solutions to the matrix equation $A\vec{x} = \vec{b}$ by multiplying both sides of the equation by A^{-1}, so we have $\vec{x} = A^{-1}\vec{b}$. The compelling name, *pseudoinverse*, strongly suggests we can find vectors that are *almost* solutions; no invertibility required! That sure sounds familiar…

Theorem 5.6.4 *Let $A \in \mathcal{M}_{m\times n}(\mathbb{R}^n)$. Then $AA^+\vec{b} = \operatorname{proj}_{\text{Col } A}\left(\vec{b}\right)$. In particular, $\hat{x}$ is a least squares solution to $A\vec{x} = \vec{b}$ if and only if $\hat{A}x = AA^+\vec{b}$.*

PROOF. Recall that $\hat{x}$ is a least squares solution if and only if it holds that $A\hat{x} = \text{proj}_{\text{Col } A}\left(\vec{b}\right)$. Note that $\hat{x} = A^+\vec{b}$ if and only if $A\hat{x} = AA^+\vec{b}$, or

$$A\hat{x} = (UDV^T)(VD^+U^T)\vec{b} = UU^T\vec{b}.$$

Recall from Theorem 5.5.4 that $UU^T\vec{b} = \text{proj}_{\text{Col } U}\left(\vec{b}\right)$. Since you will prove Col $A =$ Col U as an exercise, we are done. $\square$

There is a subtle advantage here. To find the least squares solutions to $A\vec{x} = \vec{b}$, where $A \in \mathcal{M}_{m\times n}$, we can solve either

$$A^TA\vec{x} = A^T\vec{b}, \text{ or} \tag{5.2}$$

$$A\vec{x} = AA^+\vec{b}. \tag{5.3}$$

The augmented matrix for Equation 5.2 is n by $n+1$. The augmented matrix for Equation 5.6.3 is still m by $n+1$, just like the original augmented matrix..

Example 5.6.4 Using

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we have

$$A^+ = \frac{1}{8}\begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & -1 & 0 \end{bmatrix}.$$

One can quickly check that $\vec{e}_1 + \vec{e}_3 \notin$ Col A. To find the least squares solutions for $A\vec{x} = \vec{e}_1 + \vec{e}_3$ using the normal equation $A^TA\vec{x} = A^T(\vec{e}_1 + \vec{e}_3)$, we have a 6×7 auxiliary matrix. However, note that the auxiliary matrix for $A\vec{x} = AA^+\vec{b}$ is 3×7, which is substantially smaller and easier to solve. One can check that $AA^+\vec{b} = \vec{e}_1$ and that the augmented matrix for $A\vec{x} = AA^+(\vec{e}_1 + \vec{e}_3)$ is

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

so the least squares solutions for $A\vec{x} = \vec{e}_1 + \vec{e}_3$ are $\hat{x} =$

$$x_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_7\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where $x_2, x_4, x_5, x_6, x_7 \in \mathbb{R}$.

Great. The pseudoinverse does nice things for us, but it's also the *unique* matrix that does so... in the following very specific sense. This nicely mirrors the fact that actual inverses of invertible matrices are unique.

Theorem 5.6.5 *For $A \in \mathcal{M}_{m\times n}(\mathbb{R}^n)$ singular value decomposition UDV^T, the pseudoinverse $A^+ = VD^+U^T$ is the unique matrix satisfying all of the following properties:*

(a) $AA^+A = A$
(AA^+ maps the columns of A to the columns of A),

(b) $A^+AA^+ = A^+$
(A^+A maps the columns of A^+ to the columns of A^+),

(c) $(AA^+)^T = AA^+$
(AA^+ is symmetric), and

(d) $(A^+A)^T = A^+A$
(A^+A is symmetric).

Proof of Theorem 5.6.5 can be obtained by direct calculation, so... exercise!

The Complex Singular Value Decomposition

Corollary 5.6.6 *If $A \in \mathcal{M}_{n\times m}(\mathbb{C})$, then $AA^H \in \mathcal{M}_{n\times n}(\mathbb{R})$, $A^HA \in \mathcal{M}_{m\times m}(\mathbb{R})$, and both AA^H and A^HA are Hermitian.*

This corollary comes from clever conjugation. Exercise! Proof of the following theorem then follows from the (complex) Spectral Theorem.

Theorem 5.6.7 (Singular Value Decomposition) *Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$. Then there is a unitary matrix $U \in \mathcal{M}_{m\times m}(\mathbb{C})$, a unitary matrix $V \in \mathcal{M}_{n\times n}(\mathbb{C})$, and a rectangular diagonal matrix $D \in \mathcal{M}_{m\times n}(\mathbb{R})$ such that*

$$A = UDV^H.$$

It's worth noting that this works even if A has complex entries, and in that case, D still has only real entries.

Exploration 155 Find the singular value decomposition of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ i & i & i & i & i \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Section Highlights

- ► While diagonalization was a topic for square matrices, there is a way to decompose any matrix using the *singular value decomposition*. See Theorem 5.6.1 and Example 5.6.2.
- ► One application of the singular value decomposition is the existence of a pseudoinverse, which can be used to compute least squares solutions. See Definition 5.6.3 and Theorem 5.6.4.

Exercises for Section 5.6

5.6.1. We will prove Theorem 5.6.1 in a few steps. Suppose $n > m$. Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$. Use the Spectral Theorem on $A^T A \in \mathcal{M}_{n\times n}(\mathbb{R})$ to get orthonormal vectors $\vec{v}_1, \ldots, \vec{v}_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$ (counting multiplicity). For each $i = 1, \ldots, n$, define $\sigma_i = \sqrt{\lambda_i} > 0$.

(a) Use the σ_is to define D.

(b) For each $i = 1, \ldots, n$, define
$$\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i,$$
and verify that $U = [\vec{u}_1 \cdots \vec{u}_n]$ is orthogonal.

(c) Verify that $AV = UD$ and use that to show $A = UDV^T$.

(d) Now suppose $m > n$, and verify that a similar proof works after using the Spectral Theorem on AA^T.

(e) Complete the proof by verifying the $m = n$ case.

5.6.2. Prove Corollary 5.6.6.

5.6.3. Let $A \in \mathcal{M}_{m\times n}(\mathbb{R}^n)$ have singular value decomposition UDV^T. Verify that $\mathrm{Col}\ A = \mathrm{Col}\ U$.

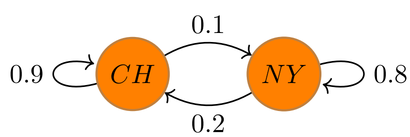
5.6.4. Let $A \in \mathcal{M}_{m\times n}(\mathbb{R}^n)$ have singular value decomposition UDV^T. Prove Theorem 5.6.5.

5.7 Applications of Invariant Subspaces

We've constructed an impressive catalog of incredible theorems, and there are dozens of practical uses for each of them. We will show you two.

Discrete Dynamics and Markov Chains

The fictitious city of Narwhal Springs has exactly two restaurants, and they both serve pizza exclusively. All is not well in Narwhal Springs, though. One restaurant serves Chicago deep-dish pizza, the other serves New York thin-crust pizza, and it turns out some people have very strong feelings about these two types of pizza. A weekly survey of the townspeople is conducted in which they are forced to choose which type of pizza they prefer. People, being very entrenched in their pizza preferences, tend to stick with the same type of pizza they currently prefer. Specifically, next week 90% of people that prefer Chicago pizza will still prefer Chicago pizza, and 80% of people that prefer New York pizza will still prefer New York pizza. That means 10% of Chicago pizza people convert to New York pizza people, and 20% of New York pizza people convert to Chicago pizza. Here is a convenient diagram, called a *transition diagram*, that summarizes all of this information:



Here are some important features of this situation:

- The probabilities predicting future preference only depend on the current preference.
- There are only a finite number of choices (two in this case).

We can put this situation into action using vectors and matrices. First let's start with a vector representing the distribution of people's pizza preference:

$$\vec{x} = \begin{bmatrix} c \\ n \end{bmatrix} \begin{array}{l} \longleftarrow \text{proportion of people preferring Chicago pizza} \\ \longleftarrow \text{proportion of people preferring New York pizza} \end{array}$$

Note that the entries of $\vec{x}$ must sum to 1; that is $c + n = 1$. This is because these are the proportions,[46] and we have proportions associated to all possible outcomes; in this case, there are only two possible restaurant choices, and we made people pick one. Using our transition diagram, we can predict the next week's proportions of pizza preference,

$$\begin{bmatrix} 0.9c + 0.2n \\ 0.1c + 0.8n \end{bmatrix} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} c \\ n \end{bmatrix} = A\vec{x},$$

which we can also write as a matrix, A, multiplied by $\vec{x}$. Such a matrix is sometimes called a *transition matrix*, and we can use $A\vec{x}$, $A^2\vec{x}$, $A^3\vec{x}$, and so on to predict how weekly changes in pizza preference progress.

Let's get more formal:

46:  We could also think about them as probabilities.

Definition 5.7.1 *Given a finite set of* ***states****, $\{1, 2, \ldots, n\}$, in which the probability of transition from the current state to another depends only on the current state, a* ***Markov chain*** *is a sequence describing how a distribution amongst the states evolves as a result of these probabilities.*

A common way to represent Markov chains is with vectors and matrices:

Definition 5.7.2 *A vector whose entries are all nonnegative and sum to 1 is called a* ***probability vector****. A square matrix whose columns are all probability vectors is called a* ***transition matrix****.*

Theorem 5.7.1 *The product of a transition matrix and a probability vector is a probability vector.*

Exploration 156 Let's prove Theorem 5.7.1 for a 2×2 transition matrix and a probability vector in $\mathbb{R}^2$. Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a transition matrix. Then $a + c = 1$ and $b + d = 1$. Suppose

$$\vec{x} = \begin{bmatrix} e \\ f \end{bmatrix}$$

is a probability vector. Then $e + f = 1$. Compute $A\vec{x}$ and show it is also a probability vector.

If A is a transition matrix, and $\vec{x}$ is a probability vector, then $\vec{x}, A\vec{x}, A^2\vec{x}, A^3\vec{x}, \ldots$ is a Markov chain. Transition matrices also have another convenient property.

Theorem 5.7.2 *If $A \in \mathcal{M}_{n\times n}$ is a transition matrix with eigenvalue λ, then $|\lambda| \leq 1$. Moreover, A has 1 as an eigenvalue.*

PROOF. First, suppose λ is an eigenvalue for A with $|\lambda| > 1$ with eigenvector $\vec{x}$. For large enough k, we have that $\|A^k\vec{x}\| = \|\lambda^k\vec{x}\| > (|x_1| + \cdots + |x_n|)$. By Theorem 5.7.1 and the definition of matrix multiplication, we know that A^k is a transition matrix, so each column of A^k, $\vec{a}_i$ for $i = 1, \ldots, n$, has $\|\vec{a}_i\| \leq 1$. Observe that

$$\begin{aligned} \|A^k\vec{x}\| &= \|x_1\vec{a}_1 + \cdots + x_n\vec{a}_n\| \\ &\leq |x_1|\|\vec{a}_1\| + \cdots + |x_n|\|\vec{a}_n\| \\ &\leq |x_1| + \cdots + |x_n|. \end{aligned}$$

However, we already have that $\|A^k\vec{x}\| > (|x_1| + \cdots + |x_n|)$, so it must be that there are no eigenvalues λ with $|\lambda| > 1$.

To see that A has an eigenvalue of 1, note first that since the entries in every column of A sum to 1, we have

$$A^T \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

so 1 is an eigenvalue for A^T. By Theorem 5.3.3, A and A^T have the same characteristic polynomial, so 1 is an eigenvalue for A as well. $\square$

Example 5.7.1 Recall the preceding pizza problem had transition matrix

$$A = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$

Then starting with two different probability vectors, $\vec{x}$ and $\vec{y}$,

$$\begin{array}{rclrcl} \vec{x} & = & \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} & \vec{y} & = & \begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix} \\ A\vec{x} & = & \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix} & A\vec{y} & = & \begin{bmatrix} 0.27 \\ 0.73 \end{bmatrix} \\ A^2\vec{x} & = & \begin{bmatrix} 0.585 \\ 0.415 \end{bmatrix} & A^2\vec{y} & = & \begin{bmatrix} 0.389 \\ 0.611 \end{bmatrix} \\ & \vdots & & & \vdots & \\ A^{100}\vec{x} & \approx & \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} & A^{100}\vec{y} & \approx & \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \end{array}$$

are both Markov chains. It's not surprising, based on the transition diagram, that the proportion of Chicago pizza preferences increases and New York pizza preferences decreases each week. Perhaps what is surprising is that it seems to settle, in both cases, to a specific set of proportions. There is actually an eigenvalue-based reason that this is happening; you can check that the eigenvalues for A are 1 and 0.7, with eigenvectors,

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

respectively. Since $\{\vec{v}_1, \vec{v}_2\}$ is a basis, any vector can be written as $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$ for some scalars c_1 and c_2. Then

$$\begin{aligned} A^k\vec{x} &= A\left(c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) \\ &= c_1 A^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 A^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= c_1 (1)^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (0.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \end{aligned}$$

Thus,

$$\lim_{k\to\infty} A^k\vec{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

From Theorem 5.7.1, we know that

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

must be a probability vector, so $c_1 = 1/3$. Thus, for any probability vector $\vec{x}$, we have proved the surprising fact that $A^k\vec{x}$ converges to $1/3\vec{v}_1$ as $k \to \infty$.

Or maybe you already know the following definitions and theorem and are not-at-all surprised. That is also possible.

Definition 5.7.3 *For a transition matrix, A, a* ***steady-state vector*** *is a probability vector, $\vec{x}$, such that $A\vec{x} = \vec{x}$.*

Example 5.7.2 Again, recall the preceding pizza problem had transition matrix

$$A = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$

The probability vector

$$\vec{x} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

is a steady-state vector because $A\vec{x} = \vec{x}$.

Definition 5.7.4 *A transition matrix, A, is called* ***regular*** *if A^k has no zero entries for some positive integer k.*

A transition matrix being regular is equivalent to being able to get from any state in a transition diagram to any other state by following paths in the diagram. A Markov chain from a regular transition matrices has very predictable long-term behavior:

Theorem 5.7.3 *If $A \in \mathcal{M}_{n\times n}$ is a regular transition matrix, then there is a unique steady-state vector, $\vec{x}_0$, such that for any probability vector, $\vec{x} \in \mathbb{R}^n$, we have*

$$\lim_{k\to\infty} A^k\vec{x} = \vec{x}_0.$$

There's a lot going on in Theorem 5.7.3. For regular transition matrices, Markov chains starting at *any* probability vector all converge to the same unique steady-state vector. That is exactly what happened in Example 5.7.1; note that the transition matrix, A, is regular (because it already has no zero entries).

The proof of Theorem 5.7.3 is hard, and we don't want to do it. It's the end of the book. Give us a break!

Exploration 157 The Office of Bureaucracy and Mismanagement has three queues for filing paperwork. Properly filed paperwork requires the queues to be done consecutively; that is, when a hapless citizen delivers their paperwork at the front of Queue A, they must then wait in Queue B to deliver paperwork there; then they do the same in Queue C; then, and only then, can they escape. Here are some more fun facts about the OBM:

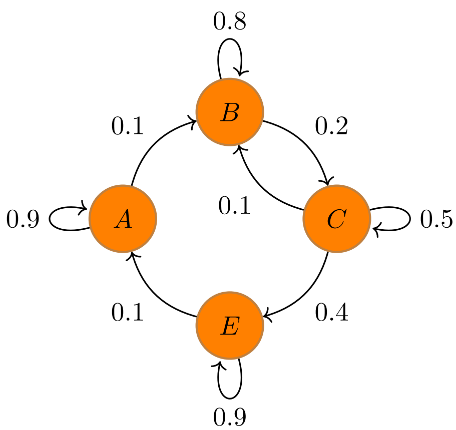
- 90% of citizens attempting to file paperwork in Queue A are asked to go to the back of the line and wait in Queue A again; the remaining 10% move on to Queue B.

- 80% filing in Queue B are told to go to the back of their line, and 20% move on to Queue C.

- The bureaucrats running Queue C are particularly cruel; while 50% of filers must go to the back of Queue C, 10% must go to the back of *Queue B*; the rest escape.

- 90% of citizens that escape the Office of Bureaucracy and Mismanagement never return; the remaining 10% go back to Queue A.

Here are the transition diagram and matrix:



$$A = \begin{bmatrix} 0.9 & 0 & 0 & 0.1 \\ 0.1 & 0.8 & 0.1 & 0 \\ 0 & 0.2 & 0.5 & 0 \\ 0 & 0 & 0.4 & 0.9 \end{bmatrix}$$

What percent of people eventually escape the Office of Bureaucracy and Mismanagement?

Exploration 158 The overlords at the Office of Bureaucracy and Mismanagement were dissatisfied with the insufficient level of misery they created, so some changes were implemented.[47] As a result, now 100% of people that escape never return. Nothing else changed. Show that the resulting transition matrix is not regular but that everyone eventually escapes.

47: That's right! Get those citizens in some queues!

As much fun as we've had with pizza and bureaucracy, it should definitely be noted that Markov processes have an incredible variety of uses, from flight scheduling to internet search engines. The possibilities are only limited by one's imagination. For more information, we refer you to said internet search engines.

Rank k Approximation

Here comes a handy technique. First, let's ruin your day with a big, awkward matrix:

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

You can check that A has singular values $\sqrt{10}$, $\sqrt{6}$, and 1. Thinking of the singular value decomposition, $A = UDV^T$, where $U = [\vec{u}_1 \cdots \vec{u}_4]$ and $V = [\vec{v}_1 \cdots \vec{v}_7]$ are orthogonal and

$$D = \begin{bmatrix} \sqrt{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that since $A = UDV^T$, we could, inspired by the spectral decomposition for square matrices, write a singular value decomposition similarly.

Theorem 5.7.4 *If $A \in \mathcal{M}_{m\times n}$ has singular value decomposition $A = UDV^T$ with $U = [\vec{u}_1 \cdots \vec{u}_m]$, $V = [\vec{v}_1 \cdots \vec{v}_n]$, and singular values $\sigma_1, \ldots, \sigma_m$, then*

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_k \vec{u}_m \vec{v}_m^T.$$

Proof of this theorem comes from direct calculation, so let's skip that[48] and just look at it in the context of our matrix A.

48: Aaarrgg!

$$A = \sqrt{10}\vec{u}_1\vec{v}_1^T + \sqrt{6}\vec{u}_2\vec{v}_2^T + \vec{u}_3\vec{v}_3^T + 0\vec{u}_4\vec{v}_4^T.$$

How much does that last term contribute to what the matrix A does to vectors? Literally nothing at all, but the third term doesn't contribute much either. Since $\sqrt{10}$ and $\sqrt{6}$ are much larger than the other singular values, 1 and 0, much more scaling is done by the first two terms of this decomposition. In this case, $\vec{v}_1$ and $\vec{v}_2$ are right-singular vectors for $\sqrt{10}$ and $\sqrt{6}$, respectively, so most of the linear transforming done by A is in the directions of $\vec{v}_1$ and $\vec{v}_2$. Working this way, we can feasibly restrict our attention to a linear transformation on a dimension two subspace that behaves a lot like A on a dimension seven space. In fact, all we have to do is cut off that last term and define

$$A_0 = \sqrt{10}\vec{u}_1\vec{v}_1^T + \sqrt{6}\vec{u}_2\vec{v}_2^T.$$

We can calculate $\vec{v}_1 = \vec{e}_1$ and $\vec{v}_2 = (-2\vec{e}_2 + \vec{e}_3\vec{e}_3)/\sqrt{6}$ and note that

$$\begin{aligned} A\vec{v}_1 &= \sqrt{10}\vec{u}_1 \text{ and} \\ A\vec{v}_2 &= \sqrt{6}\vec{u}_2 \end{aligned}$$

to see

$$A_0 = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Indeed, one can *see* that A and A_0 are very much alike. Moreover, one can test $A\vec{x}$ and $A_0\vec{x}$ with a variety of vectors, $\vec{x}$, to see how similarly they behave as linear transformations.[49]

49:  Do it!

While A_0 is a reasonable approximation for A, A_0 is still quite large; it's the exact same size as A! Let's see what we can do about that. Note that

$$A_0 = \sqrt{10}\vec{u}_1\vec{v}_1^T + \sqrt{6}\vec{u}_2\vec{v}_2^T = [\vec{u}_1\ \vec{u}_2] \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{6} \end{bmatrix} [\vec{v}_1\ \vec{v}_2]^T,$$

and we can rewrite this as

$$A_0[\vec{v}_1\ \vec{v}_2] = [\vec{u}_1\ \vec{u}_2] \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{6} \end{bmatrix}.$$

Would you believe that

$$A[\vec{v}_1\ \vec{v}_2] = [\vec{u}_1\ \vec{u}_2] \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{6} \end{bmatrix}$$

as well? Let's check.

$$\begin{aligned} A[\vec{v}_1\ \vec{v}_2] &= \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2/\sqrt{6} \\ 0 & 1/\sqrt{6} \\ 0 & 1/\sqrt{6} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & -\sqrt{6} \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = [\vec{u}_1\ \vec{u}_2] \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{6} \end{bmatrix}. \end{aligned}$$

When we define A_0 by cutting off that last term in the singular value decomposition of A, what's actually happening is that we're restricting A to the projection of the domain of A onto the span of $\vec{v}_1$ and $\vec{v}_2$. This sounds a lot like a theorem. First, we'll have a definition to formalize this "cutting off" procedure.

Definition 5.7.5 *If $A \in \mathcal{M}_{\times n}$ has singular value decomposition $A = UDV^T$ with $U = [\vec{u}_1 \cdots \vec{u}_m]$, $V = [\vec{v}_1 \cdots \vec{v}_n]$, and singular values $\sigma_1, \ldots, \sigma_m$, then for any positive integer $k \leq m$, a **rank** k **approximation of** A is*

$$A_k = \sigma_1\vec{u}_1\vec{v}_1^T + \cdots + \sigma_k\vec{u}_k\vec{v}_k^T.$$

Theorem 5.7.5 *If $A \in \mathcal{M}_{m\times n}$ has rank k approximation, $A_k = \sigma_1\vec{u}_1\vec{v}_1^T + \cdots + \sigma_k\vec{u}_k\vec{v}_k^T$, then for any positive integer $k \leq m$, A_k is the composition of A with the projection onto* $\mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$. *In particular,*

$$A_k = [\vec{u}_1 \cdots \vec{u}_k] \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_k \end{bmatrix} [\vec{v}_1 \cdots \vec{v}_k]^T = A[\vec{v}_1 \cdots \vec{v}_k][\vec{v}_1 \cdots \vec{v}_k]^T.$$

Moreover,

$$A[\vec{v}_1 \cdots \vec{v}_k] = [\sigma_1\vec{u}_1 \cdots \sigma_k\vec{u}_k].$$

PROOF. The first equality comes from matrix multiplication. The second comes from the singular value decomposition for A and Theorem 5.5.4, which states that $[\vec{v}_1 \cdots \vec{v}_k][\vec{v}_1 \cdots \vec{v}_k]^T$ is the projection onto $\mathrm{Span}\,\{\vec{v}_1, \ldots, \vec{v}_k\}$. The last inequality comes from the second after multiplication by $[\vec{v}_1 \cdots \vec{v}_k]$ and noting that $[\vec{v}_1 \cdots \vec{v}_k]^T[\vec{v}_1 \cdots \vec{v}_k] = I_k$. $\square$

Theorem 5.7.5 is particularly useful because

$$\begin{aligned} A &\in \mathcal{M}_{m\times n} \text{ and} \\ A[\vec{v}_1 \cdots \vec{v}_k] &\in \mathcal{M}_{m\times k}, \end{aligned}$$

and we could potentially choose k to be substantially smaller than n. [50]

50: Wait. Is that the end?

Looks like it is. I guess we should go home?

I like it here. I think I'll stay.

But what'll we do?

...

Appendix

Additional Proofs

As we were writing this book, some statements needed proofs for completeness, but we felt including those proofs detracted from the interactive experience of the readers. Thus, we created an appendix to include these results. Actually, first we lied about creating an appendix, and then we had to make it for real once we realized this was turning into something other instructors might one day use. We have ordered these based on where the results appear in the text.

Chapter 3

Theorem (3.3.3) (a) *If V is any vector space, then $V \cong V$.*
(b) *If V and W are vector spaces such that $V \cong W$, then $W \cong V$.*
(c) *If V, W, and U are all vector spaces such that $V \cong W$ and $W \cong U$, then $V \cong U$.*

PROOF. First, we should establish that if V is any vector space, then $V \cong V$. In this situation, the identity map $\mathrm{id}\colon V \to V$ that maps each vector to itself is the isomorphism. Second, suppose V and W are vector spaces such that $V \cong W$. We proved in Section 3.2 that if T is an invertible linear transformation, then T^{-1} is as well. Thus, we also have $W \cong V$ with the inverse function giving the isomorphism.

We also know from that section that the composition of two linear transformations will again be a linear transformation, and from Section 3.1 that the compositions of two maps that are one to one and onto will also be one to one and onto. Suppose $V \cong W$ and $W \cong U$. Then there are isomorphisms $T : V \to W$ and $S : W \to U$. The composition $S \circ T : V \to U$ is then an isomorphism as well, and $V \cong U$. □

We mentioned there is a way to define our operations of scalar multiplication and vector addition so that a plane in $\mathbb{R}^3$ that does not go through the origin is still a vector space. To do this, we need to shift the zero vector away from

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Suppose our goal is to impose a vector space structure on

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 6; x, y, z \in \mathbb{R} \right\}.$$

We need to first identify a vector in V. Well, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in V$ since $1+2+3=6$. Let's make this our new zero vector. For this, we need to shift our vector addition so that adding $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ does nothing.

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \boxplus \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 - 2 \\ z_1 + z_2 - 3 \end{bmatrix}$$

Now, we need to define a version of scalar multiplication that does not move our new zero vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

$$k \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} kx_1 - k + 1 \\ ky_1 - 2k + 2 \\ kz_1 - 3k + 3 \end{bmatrix}$$

Now, checking the vector space axioms for $\mathbb{R}^3$ with these operations would be a wonderful exercise, so we will leave it as that. If we assume this is a valid vector space, we can then just check the subspace axioms to see that V is a subspace.

Chapter 4

Theorem (4.1.5) *Suppose $A \in \mathcal{M}_{m\times n}$. Then there exists a unique matrix $B \in \mathcal{M}_{m\times n}$ in reduced row echelon form that can be obtained from A by performing row operations.*

PROOF. The statement we are claiming is that a reduced row echelon form matrix both exists and is unique. Suppose first of all that $A \in \mathcal{M}_{m\times 1}$, so that A is just a single column. Then, if this column has no nonzero entries, then the matrix is in reduced row echelon form already, and this is the unique reduced row echelon form since any row operation will preserve a column of all zeros. Suppose instead that A has a nonzero entry. Then we can select any nonzero entry in the column and scale that row so that it is a 1. We can then move that pivot to be in the top row if it is not currently. Then, we can use this pivot to reduce all the entries below to 0. This is the unique single column reduced row echelon form matrix with a pivot. Now, suppose $A \in \mathcal{M}_{m\times n}$ and any matrix in $\mathcal{M}_{m\times n-1}$ can be row reduced to a unique reduced row echelon form. So the first $n-1$ columns of A can be row reduced uniquely to reduced row echelon form. If the final column has only nonzero entries in rows already containing pivots, then the A is in reduced row echelon form. This form is unique since any row operation would change not just the final column but all the ones previously, creating a contradiction to our assumption that the first $n-1$ columns were arranged uniquely in reduced row echelon form. If the final column contains a nonzero entry in a row not already containing a pivot, we should scale that row so that the entry is a 1. We know this row must contain only 0 in all the previous entries since it is in a row that did not previously contain a pivot, so we can now clear out all the other entries in the final column and move the row with a pivot up to be the final nonzero row. This matrix is now in reduced row echelon form. None of the row operations affected the previous columns, and the final column is forced to be all zeros except for a 1 in the pivot location. Thus, this is the unique matrix in reduced row echelon form obtained from A by row operations, and the result holds by induction. □

Theorem (4.2.3) *If a system of m linear equations in n variables has a solution, then the set of solutions is in one to one correspondence onto a $k-$dimensional subspace of $\mathbb{R}^n$, where k is the number of free variables in the reduced row echelon form of the coefficient matrix associated to the system.*

PROOF. Given our system of m linear equations in n variable, we can form an augmented matrix $[C|\vec{d}]$. Suppose $[C|\vec{d}]$ row reduces to $[A|\vec{b}]$ in reduced row echelon form, and further suppose A has i columns containing pivots. Consider the equation $A\vec{x} = \vec{0}$. We can define the induced linear transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ by $T_A(\vec{x}) = A\vec{x}$. Then, solving the equation $A\vec{x} = \vec{0}$ is equivalent to finding $\text{Ker } A = \text{Ker } T_A$. We know from the Rank-Nullity Theorem that $\dim \text{Ker } T_A + \dim \text{Imag } T_A = n$. Also, we know $\text{Imag } T_A = \text{Col } A$. Each column containing a pivot in reduced row echelon form is a different standard basis vector for $\mathbb{R}^n$, so we know they are linearly independent. Also, any column not containing a pivot will only have nonzero entries corresponding to the locations of the pivots. Thus, the columns containing pivots are a basis for $\text{Imag } T_A$, so $\dim \text{Imag } T_A = i$, where i is the number of columns containing pivots. We must then have that $\dim \text{Ker } T_A = k$ since $i + k = n$.

We have shown that for the equation $A\vec{x} = \vec{0}$, the set of solutions is a subspace of dimension k, where k is the number of free variables in A. We have assumed that the original system of equations had a solution, or equivalently, the matrix equation $A\vec{x} = \vec{b}$ has a solution. Thus, there is some $\vec{z} \in \mathbb{R}^n$ such that $A\vec{z} = \vec{b}$. Now, let us show that there exists a *one to one* map from the set of solutions for $A\vec{x} = \vec{0}$ *onto* the set of solutions for $A\vec{x} = \vec{b}$. Let

$$U_{\vec{b}} = \{\vec{y} \in \mathbb{R}^n | A\vec{y} = \vec{b}\}.$$

We know $\vec{z} \in U_{\vec{b}}$. Let us define a map $f : \text{Ker } A \to U_{\vec{b}}$ by $f(\vec{k}) = \vec{k} + \vec{z}$. Let's first see that this map is well-defined. To do this, we need to establish that $\vec{k} + \vec{z} \in U_{\vec{b}}$ for any $\vec{k} \in \text{Ker } A$. We know $A(\vec{k} + \vec{z}) = A\vec{k} + A\vec{z}$ since this is a property of linear transformations. Then, since $\vec{k} \in \text{Ker } A$, we know $A\vec{k} + A\vec{z} = \vec{0} + A\vec{z} = \vec{b}$. Thus, $\vec{k} + \vec{z} \in U_{\vec{b}}$ for each $\vec{k} \in \text{Ker } A$. Now, we need to show this is one to one and onto. Let's start with one to one. Suppose that

$\vec{k}_1, \vec{k}_2 \in \text{Ker } A$ and that $f(\vec{k}_1) = f(\vec{k}_2)$. This means $\vec{k}_1 + \vec{z} = \vec{k}_2 + \vec{z}$, which says $\vec{k}_1 = \vec{k}_2$. Thus, f is one to one. Now, to see that it is onto. Suppose $\vec{y}_0 \in \mathbb{U}_{\vec{b}}$. We need to find some $\vec{k}_0 \in \text{Ker } A$ such that $\vec{y}_0 = f(\vec{k}_0) = \vec{k}_0 + \vec{z}$. Well, solving for $\vec{k}_0$ here gives us $\vec{k}_0 = \vec{y}_0 - \vec{z}_0$. This satisfies the desired equation and also is in Ker A since $A(\vec{y}_0 - \vec{z}) = A\vec{y}_0 - A\vec{z} = \vec{b} - \vec{b} = \vec{0}$. Thus, f is a one to one and onto map between $U_{\vec{b}}$ and Ker A. □

Theorem (4.5.9) *Suppose $A \in \mathcal{M}_{n\times n}$ is an invertible matrix. Then the augmented matrix $[A|I_n]$ row reduces to $[I_n|A^{-1}]$.*

PROOF. Since we know A is a invertible, the columns of A form a basis for $\mathbb{R}^n$. Let's name this basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$. Then A is the matrix for the coordinate mapping that sends each standard basis vector to the corresponding vector in $\mathcal{B}$. That is, $A\vec{e}_i = \vec{v}_i$. Thus, the inverse linear transformation will map $\vec{v}_i$ to $\vec{e}_i$. Now, from Theorem 3.5.2, we have

$$A^{-1} = [[T(\vec{v}_1)]_{\mathcal{B}} \cdots [T(\vec{v}_n)]_{\mathcal{B}}] = [[\vec{e}_1]_{\mathcal{B}} \cdots [\vec{e}_n]_{\mathcal{B}}] .$$

In order to compute coordinate vectors, we solve the equation

$$a_{i1}\vec{v}_1 + \cdots a_{in}\vec{v}_n = \vec{e}_i$$

for each standard basis vector $\vec{e}_i$. As seen in Section 4.3, we can do this all at once by augmenting the matrix A with each of the vectors that need to be converted to coordinate vectors and then row reducing. Specifically, we have

$$[\vec{v}_1 \cdots \vec{v}_n | \vec{e}_1 \cdots \vec{e}_n] \to [\vec{e}_1 \cdots \vec{e}_n | \vec{w}_1 \cdots \vec{w}_n]$$

where $A^{-1} = [\vec{w}_1 \cdots \vec{w}_n]$. □

Chapter 5

Theorem (5.3.6) *If $A \in \mathcal{M}_{n\times n}$ has an eigenvalue λ with geometric multiplicity k and $B \in \mathcal{M}_{n\times n}$ is similar to A, then B has λ as an eigenvalue with geometric multiplicity k as well.*

PROOF. Suppose $A \in \mathcal{M}_{n\times n}$ has an eigenvalue λ with geometric multiplicity k and A is the matrix representation for a linear transformation $T\colon V \to V$ with respect to some basis $\mathcal{B}$. Let $E_A \subseteq \mathbb{R}^n$ denote the eigenspace for A with respect to λ. By Theorem 5.3.2, we know there is a corresponding invariant subspace $W \subseteq V$ with $E_A \cong W$ under the coordinate mapping. Thus, $\dim W = \dim E_A = k$. Let $\{\vec{v}_1, \ldots, \vec{v}_k\}$ denote a basis of W. Suppose $B \in \mathcal{M}_{n\times n}$ is similar to A. Then there is some basis $\mathcal{P}$ of V such that B is the matrix representation for T with respect to $\mathcal{P}$. Then for each $1 \le i \le k$, we know $[\vec{v}_i]_{\mathcal{P}}$ must be an eigenvector for B with the eigenvalue λ since $T(\vec{v}_i) = \lambda\vec{v}_i$. Since the coordinate mapping is an isomorphism, we know $\{[\vec{v}_1]_{\mathcal{P}}, \ldots, [\vec{v}_k]_{\mathcal{P}}\}$ will be a linearly independent set of eigenvectors for B with eigenvalue λ. Thus, the geometric multiplicity of λ for B is at least k. To see that it is exactly k, we can suppose the geometric multiplicity of λ with respect to B is $j \ge k$ and repeat this argument starting with B instead of

A. We would then conclude the geometric multiplicity of λ for A is at least j, giving us $k \geq j$. We then conclude $j = k$. $\square$

Lemma (5.4.10) *Let $A \in \mathcal{M}_{n\times n}$ have an eigenvalue, λ, with algebraic multiplicity k and geometric multiplicity j. Then there are j Jordan chains, $S_1, \ldots, S_j$, such that $S_1 \cup \cdots \cup S_j$ is a basis for* $\operatorname{Ker}(A - \lambda I)^k$*, and in particular,*

$$\operatorname{Ker}(A - \lambda I)^k = \operatorname{Span}\{S_1\} \oplus \cdots \oplus \operatorname{Span}\{S_j\}.$$

PROOF. **Part 1: The Jordan chains exist and are distinct.** By Lemma 5.4.9, the dimension of $\operatorname{Ker}(A - \lambda I)^i$ increases by a amount, m_i, each time i increases. In particular, one can check that

$$\operatorname{Ker}(A - \lambda I) \subset \cdots \subset \operatorname{Ker}(A - \lambda I)^{k-1} \subset \operatorname{Ker}(A - \lambda I)^k.$$

We will use these increases in $\dim \operatorname{Ker}(A - \lambda I)^i$ to acquire extra "links" in our Jordan chains. To simplify all the notation, let's assume that $k_0 = 4$; that is, $\dim \operatorname{Ker}(A - \lambda I)^4 = k$, and note the general argument is similar. First, we will use the Orthogonal Decomposition Theorem several times to get a nice break down of $\operatorname{Ker}(A - \lambda I)^4$. Specifically, we can write

$$\begin{aligned}
&E_1 = \operatorname{Ker}(A - \lambda I)\\
&E_2 = E_1^{\perp} \cap \operatorname{Ker}(A - \lambda I)^2 \quad \Rightarrow \quad \operatorname{Ker}(A - \lambda I)^2 = E_2 \oplus E_1\\
&E_3 = (E_2 \oplus E_1)^{\perp} \cap \operatorname{Ker}(A - \lambda I)^3 \quad \Rightarrow \quad \operatorname{Ker}(A - \lambda I)^3 = E_3 \oplus E_2 \oplus E_1\\
&E_4 = (E_3 \oplus E_2 \oplus E_1)^{\perp} \quad \Rightarrow \quad \operatorname{Ker}(A - \lambda I)^4 = E_4 \oplus E_3 \oplus E_2 \oplus E_1
\end{aligned}$$

where $\dim E_1 = j$, $\dim E_2 = m_2$, $\dim E_3 = m_3$, and $\dim E_4 = m_4$.

Choose a basis $\{\vec{z}_1, \cdots, \vec{z}_{m_4}\}$ for E_4. It's not difficult to show that

$$\{(A - \lambda I)\vec{z}_1, \ldots, (A - \lambda I)\vec{z}_{m_4}\}$$

is a linearly independent set. In fact, let's do that.

Claim 1: If $\{\vec{u}_1, \ldots, \vec{u}_l\}$ is a linearly independent set such that

$$\operatorname{Span}\{\vec{u}_1, \ldots, \vec{u}_l\} \cap \operatorname{Ker}(A - \lambda I) = \{\vec{0}\},$$

then $\{(A - \lambda I)\vec{u}_1, \ldots, (A - \lambda I)\vec{u}_l\}$ is linearly independent.

Proof of Claim 1. To see this, without loss of generality, we can suppose

$$(A - \lambda I)\vec{u}_1 = a_2(A - \lambda I)\vec{u}_2 + \cdots a_l(A - \lambda I)\vec{u}_l.$$

Thus, we have

$$A(\vec{u}_1 - a_2\vec{u}_2 - \cdots - a_l\vec{u}_l) = \lambda(\vec{u}_1 - a_2\vec{u}_2 - \cdots - a_l\vec{u}_l).$$

This means

$$(\vec{u}_1 - a_2\vec{u}_2 - \cdots - a_l\vec{u}_l) \in \operatorname{Span}\{\vec{u}_1, \ldots, \vec{u}_l\} \cap \operatorname{Ker}(A - \lambda I) = \{\vec{0}\}.$$

However, this cannot be possible since $\{\vec{u}_1, \cdots, \vec{u}_l\}$ was linearly independent. Thus, we know $\{(A - \lambda I)\vec{u}_1, \ldots, (A - \lambda I)\vec{u}_l\}$ is also a linearly independent set. $\blacksquare$

Now, back to the main proof. We know each $(A-\lambda I)\vec{z}_i$ is in $\text{Ker}\,(A-\lambda I)^3 = E_3 \oplus E_2 \oplus E_1$ since

$$(A-\lambda I)^4\vec{z}_i = (A-\lambda I)^3(A-\lambda I)\vec{z}_i = \vec{0}.$$

Moreover, we know $(A-\lambda I)^3\vec{z}_i \neq \vec{0}$ since $\vec{z}_i \in E_4$. Thus, for each $i = 1,\ldots,m_4$, we have

$$(A-\lambda I)\vec{z}_i = \vec{y}_i + \vec{x}_i + \vec{v}_i$$

for some $\vec{y}_i \in E_3$, $\vec{x}_i \in E_2$, $\vec{v}_i \in E_1$ with $\vec{y}_i \neq \vec{0}$.

One can argue that $\{\vec{y}_1,\ldots,\vec{y}_{m_4}\}$ must be a linearly independent set in E_3. Again, let's do this.

Claim 2: The vectors $\{\vec{y}_1,\ldots,\vec{y}_{m_4}\}$ must be linearly independent.

Proof of Claim 2. To see this, suppose instead, without loss of generality, that $\vec{y}_1 = a_2\vec{y}_2 + \cdots + a_{m_4}\vec{y}_{m_4}$. Note that $\vec{y}_i = (A-\lambda I)\vec{z}_i - \vec{x}_i - \vec{v}_i$ for each i. Then we have

$$\begin{aligned}&(A-\lambda I)\vec{z}_1 - \vec{x}_1 - \vec{v}_1 = \\ &\quad a_2((A-\lambda I)\vec{z}_2 - \vec{x}_2 - \vec{v}_2) + \cdots + a_{m_4}((A-\lambda I)\vec{z}_{m_4} - \vec{x}_{m_4} - \vec{v}_{m_4})\end{aligned}$$

which means

$$\begin{aligned}&(A-\lambda I)(\vec{z}_1 - a_2\vec{z}_2 - \cdots a_{m_4}\vec{z}_{m_4}) = \\ &\qquad\qquad (\vec{x}_1 + \vec{v}_1) - a_2(\vec{x}_2 - \vec{v}_2)\cdots - a_{m_4}(\vec{x}_{m_4} - \vec{v}_{m_4})\end{aligned}$$

This would put $(\vec{z}_1 - a_2\vec{z}_2 - \cdots z_{m_4}\vec{z}_{m_4}) \in \text{Ker}\,(A-\lambda I)^3 = E_3 \oplus E_2 \oplus E_1$, which is not possible since $\vec{z}_1,\cdots,\vec{z}_{m_4} \in E_4$. ■

Since it's linearly independent and in E_3, we know $\{\vec{y}_1,\ldots,\vec{y}_{m_4}\}$ can be extended to a basis for E_3 with vectors $\{\vec{y}_{m_4+1},\ldots,\vec{y}_{m_3}\}$. This also tells us $\{\vec{y}_1+\vec{x}_1,\ldots,\vec{y}_{m_4}+\vec{x}_{m_4},\vec{y}_{m_4+1},\ldots,\vec{y}_{m_3}\}$ is a linearly independent set in $E_3 \oplus E_2$ since adding vectors from the orthogonal complement will not change the independence. Now, consider the set

$$\begin{aligned}&\{(A-\lambda I)(\vec{y}_1+\vec{x}_1),\ldots,(A-\lambda I)(\vec{y}_{m_4}+\vec{x}_{m_4}),(A-\lambda I)(\vec{y}_{m_4+1}),\ldots \\ &\qquad\qquad\qquad\qquad\ldots,(A-\lambda I)(\vec{y}_{m_3})\}.\end{aligned}$$

From Claim 1, we know this set is linearly independent, and it must be in $\text{Ker}\,(A-\lambda I)^2 = E_2 \oplus E_1$. Also, since $\vec{v}_i \in \text{Ker}\,(A-\lambda I)$, we know

$$(A-\lambda I)(\vec{y}_i+\vec{x}_i) = (A-\lambda I)(\vec{y}_i+\vec{x}_i) + (A-\lambda I)\vec{v}_i = (A-\lambda I)^2\vec{z}_i$$

for each $1 \le i \le m_4$. Thus, our set is really

$$\{(A-\lambda I)^2\vec{z}_1,\ldots,(A-\lambda I)^2\vec{z}_{m_4},(A-\lambda I)(\vec{y}_{m_4+1}),\ldots,(A-\lambda I)(\vec{y}_{m_3})\}.$$

Each of these must be equal to $\overline{\vec{x}}_i + \overline{\vec{v}}_i$ for some $\overline{\vec{x}}_i \in E_2$ and $\overline{\vec{v}}_i \in E_1$ with $\overline{\vec{x}}_i \neq 0$ by reasons similar to those before. Also, in an argument that mirrors the one for Claim 2, we can show $\{\overline{\vec{x}}_1,\ldots,\overline{\vec{x}}_{m_3}\}$ is a linearly independent set in E_2. We can now extend this to a basis for E_2 using $\{\overline{\vec{x}}_{m_3+1},\ldots,\overline{\vec{x}}_{m_2}\}$. This gives us a set

$$\{(A-\lambda I)\overline{\vec{x}}_1,\ldots,(A-\lambda I)\overline{\vec{x}}_{m_2}\}$$

which is linearly independent in E_1 by Claim 1. Additionally, we have

$$(A-\lambda I)\overline{\overline{x}}_i = (A-\lambda I)\overline{\overline{x}}_i + (A-\lambda I)\overline{\overline{v}}_i = \begin{cases} (A-\lambda I)^3 \vec{z}_i & \text{if } 1 \le i \le m_4 \\ (A-\lambda I)^2 \vec{y}_i & \text{if } m_4+1 \le i \le m_3. \end{cases}$$

Thus, the set is really

$$\{(A-\lambda I)^3\vec{z}_1, \dots, (A-\lambda I)^3\vec{z}_{m_4}, (A-\lambda I)^2(\vec{y}_{m_4+1}), \dots$$
$$\dots, (A-\lambda I)^2(\vec{y}_{m_3}), (A-\lambda I)\overline{\overline{x}}_{m_3+1}, \dots, (A-\lambda I)\overline{\overline{x}}_{m_2}\}.$$

This set can be extended to a basis of E_1 with the vectors $\{\widetilde{\vec{v}}_{m_2+1}, \dots \widetilde{\vec{v}}_j\}$, and each element in this basis corresponds to the end of a distinct Jordan chain, with the $\widetilde{\vec{v}}_i$'s being chains of length 1.

It remains only to show that any two of these chains must be linearly independent. This would then means we could form a basis for $\text{Ker}\,(A-\lambda I)^3$ out of the Jordan chains, giving us the direct sum decomposition claimed in the proof.

Part 2: The union of any two Jordan chains is linearly independent. Now suppose S_1 and S_2 are both Jordan chains for the same eigenvalue λ, so that

$$\begin{aligned} S_1 &= \{(A-\lambda I)^{n_1}\vec{x}, \dots, (A-\lambda I)^2\vec{x}, (A-\lambda I)\vec{x}, \vec{x}\}, \\ S_2 &= \{(A-\lambda I)^{n_2}\vec{y}, \dots, (A-\lambda I)^2\vec{y}, (A-\lambda I)\vec{y}, \vec{y}\}, \end{aligned}$$

and $n_1 \ge n_2$. Assume for some scalars, $c_{n_1}, \dots, c_0, d_{n_2}, \dots, d_0$, that

$$\begin{aligned} \vec{0} &= c_{n_1}(A-\lambda I)^{n_1}\vec{x} + \cdots + c_2(A-\lambda I)^2\vec{x} + c_1(A-\lambda I)\vec{x}, + c_0\vec{x} \\ &\quad + d_{n_2}(A-\lambda I)^{n_2}\vec{y} + \cdots + d_2(A-\lambda I)^2\vec{y} + d_1(A-\lambda I)\vec{y} + d_0\vec{y}. \end{aligned}$$

Then multiplying by $(A-\lambda I)^{n_1}$, we have

$$\vec{0} = c_0(A-\lambda I)^{n_1}\vec{x},$$

which implies $(A-\lambda I)^{n_1}\vec{x} = \vec{0}$ or $c_0 = 0$. Since we know $(A-\lambda I)^{n_1}\vec{x} = \vec{0}$ is not possible, we must have $c_0 = 0$. Now, we can multiply both sides by $(A-\lambda I)^{n_1-1}$ to get

$$\vec{0} = c_1(A-\lambda I)^{n_1}\vec{x}.$$

Again, this will allow us to conclude $c_1 = 0$. This repeats until we have $c_i = 0$ for each $0 \le i \le n_1 - n_2 - 1$. Then, we can multiply by $(A-\lambda I)^{n_2}$ to get

$$\vec{0} = c_{n_1-n_2}(A-\lambda I)^{n_1}\vec{x} + d_0(A-\lambda I)^{n_2}\vec{y}.$$

Since these are in the basis we built for $\text{Ker}\,(A-\lambda I)$, we know they must be linearly independent. Thus, $c_{m_1-m_2} = 0$ and $d_0 = 0$. The argument continues in this fashion until all coefficients are forced to be 0, thus the set $S_1 \cup S_2$ is linearly independent. Moreover, this tells us any set built as the union of these distinct Jordan chains will be linearly independent, so $S_1 \cup \cdots \cup S_j$ is a linearly independent set the size of a basis of $\text{Ker}\,(A-\lambda I)^k$ and must be a basis for $\text{Ker}\,(A-\lambda I)^k$. □

Answers to Selected Parts of Selected Explorations

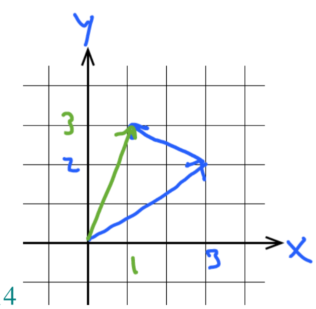
Chapter 0

3 $\{(1,2),(3,2),(5,2),(7,2)\}$; 5 Just r_2; 6 No. $(-1,1),(1,1)\in r$; 7 $\operatorname{dom}(f)=\mathbb{Q}$, $\operatorname{ran}(f)=\mathbb{Z}$, $\operatorname{codom}(f)=\mathbb{Q}$

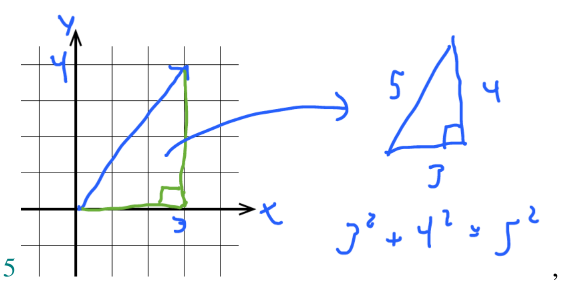
Section 1.1

10 $\mathbb{Q}$, $\mathbb{C}$; 11 commutativity of addition, associativity of addition, additive identity, additive inverse; 13 $\vec{p}=0$, $-5x+3x^3-4x^7$, $-a_0-a_1x-\cdots-a_nx^n$, both are equal to $60x-36x^3+48x^7$, $(ab)(a_0+a_1x+\cdots+a_nx^n)=a(b(a_0+a_1x+\cdots+a_nx^n))=a(ba_0+ba_1x+\cdots+ba_nx^n)$

Section 1.2

14 

;

15 

,
$\sqrt{6^2+8^2}=10$, twice the length of $\vec{v}$;
16 $\vec{v}\cdot\vec{w}=0+1+8=9$; 17 Vectors have different number of components; 18 $\vec{w}\cdot\vec{v}=0+1+8=9=\vec{v}\cdot\vec{w}$; 20 $\|\vec{v}\|=3$, $\frac{\vec{v}}{\|\vec{v}\|}=\begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}$, $5\frac{\vec{v}}{\|\vec{v}\|}=\begin{bmatrix}5/3\\10/3\\10/3\end{bmatrix}$

Section 1.3

24 Dependent, independent, dependent, yes, no ;

Section 1.4

28 Since $b>0$, there cannot be a zero vector, and the set is not closed under scalar multiplication by any negative scalar; 29 False; neither is a subset of the other; 30 $\begin{bmatrix}a+b\\a+b+c\\a+b\end{bmatrix}=(a+b)\begin{bmatrix}1\\1\\1\end{bmatrix}+c\begin{bmatrix}0\\1\\0\end{bmatrix}$; 32 $\begin{bmatrix}a\\a\\0\end{bmatrix}+\begin{bmatrix}0\\0\\b\end{bmatrix}=\begin{bmatrix}c\\0\\c\end{bmatrix}+\begin{bmatrix}0\\d\\0\end{bmatrix}$, so $a=c$, $a=d$, and $b=c$

Section 2.1

38 $\vec{b}_2$ is not a scalar multiple of $\vec{b}_1$, $\begin{bmatrix}x_1\\x_2\end{bmatrix}=(x_1-x_2)\begin{bmatrix}1\\0\end{bmatrix}+x_2\begin{bmatrix}1\\1\end{bmatrix}$;

Section 2.2

45 $a+\frac{c}{2}$, $\frac{8b+11d}{16}$, $\frac{c}{4}$, $\frac{d}{8}$

Section 2.3

46 $\vec{b}_1+8\vec{b}_2+16\vec{b}_3$; 47 $[\vec{v}]_{\mathcal{B}_1}=\begin{bmatrix}2\\3\\4\end{bmatrix}$, $[\vec{v}]_{\mathcal{B}_2}=\begin{bmatrix}2\\4\\-3\end{bmatrix}$, $[\vec{u}]_{\mathcal{B}_1}=\begin{bmatrix}a\\b\\c\end{bmatrix}$, $[\vec{u}]_{\mathcal{B}_2}=\begin{bmatrix}a\\c\\b-a-c\end{bmatrix}$; 48 $[\vec{p}]_{\mathcal{B}_1}=\begin{bmatrix}1\\1\\-4\end{bmatrix}$, $[\vec{q}]_{\mathcal{B}_1}=\begin{bmatrix}6\\1\\-5\end{bmatrix}$, $\vec{p}\cdot\vec{q}=27$; 49 $(1\vec{+}i)\cdot(-1\vec{+}2i)=1$; 50 $[\vec{p}]_{\mathcal{B}}=\begin{bmatrix}5\\1\\0\end{bmatrix}$, $[\vec{p}]_{\mathcal{B}_1}=\begin{bmatrix}5\\0\\-4\end{bmatrix}$, $\|[\vec{p}]_{\mathcal{B}}\|=\sqrt{26}$, $\|[\vec{p}]_{\mathcal{B}_1}\|=\sqrt{41}$; 51 $\vec{v}_1\cdot\vec{v}_3=\vec{v}_1\cdot\vec{v}_4=\vec{v}_2\cdot\vec{v}_4=\vec{v}_3\cdot\vec{v}_4=0$; 52 $\vec{w}=\begin{bmatrix}1\\-1\\0\end{bmatrix}$

Section 2.4

54 $\dim W^{\perp}=2$, so $W^{\perp}$ is a plane in $\mathbb{R}^4$, $W_0^{\perp}=$

$\text{Span}\left\{\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$; 55 $\vec{y} = -\frac{1}{30}\vec{v}_1 - \frac{3}{5}\vec{v}_2 - \frac{1}{6}\vec{v}_3$; 56 $\vec{y} = -\frac{1}{\sqrt{30}}\vec{v}_1 - \frac{3}{\sqrt{5}}\vec{v}_2 - \frac{1}{\sqrt{6}}\vec{v}_3$; ; 57 $\vec{w} = \begin{bmatrix}0\\1\\-2\end{bmatrix}$, $\vec{x} = a\vec{w} + (a+b)\vec{u}$; 58 $\vec{v}\cdot\vec{v} = 14$, $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}1/7\\-1/14\\-3/14\end{bmatrix}$

Section 2.5

59 $\text{proj}_W(\vec{x}) = \begin{bmatrix}5\\5\\3\\-1\end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix}1\\1\\-3\\1\end{bmatrix}$, $\text{proj}_W(\vec{y}) = \begin{bmatrix}1\\1\\2\\4\end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix}-1\\-3\\6\\-2\end{bmatrix}$; 60 $\left\{\begin{bmatrix}1\\1\\-3\\1\end{bmatrix},\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\right\}$

Section 2.6

61 $\text{proj}_W(\vec{y}) = \begin{bmatrix}3\\1\\1\end{bmatrix}$

Section 3.1

63 Any number less than 1; 64 $x = \sqrt[3]{y}$; 67 $k(g(2)) = 2$, $k\circ g$ maps everything in B to 2, $k\circ h$ maps evens in A to 2 and odds in A to 6; 68 Yes. There are many examples;

Section 3.2

71 $f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\\2\end{bmatrix}$, $f\left(\begin{bmatrix}2\\0\\0\end{bmatrix}\right) = \begin{bmatrix}4\\2\\2\end{bmatrix}$, $2f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}6\\4\\4\end{bmatrix}$; 74 Yes! Verify both axioms from the definition; 75 $\begin{bmatrix}0\\42\end{bmatrix}$, $\left\{\begin{bmatrix}0\\x_2\end{bmatrix} : x_2\in\mathbb{R}\right\}$; 76 $\vec{0} = T(\vec{v}) + T(\vec{u}) = T(\vec{v}+\vec{u})$, $\vec{0} = aT(\vec{v}) = T(a\vec{v})$; 77 $\text{Ker}\,T = \{ax^2 - ax\colon a\in\mathbb{R}\}$

Section 3.3

78 No, $f\left(\begin{bmatrix}7\\x_2\end{bmatrix}\right) = \begin{bmatrix}7\\0\end{bmatrix}$ for any $x_2\in\mathbb{R}$; 80 Note that $\left\{\begin{bmatrix}45\\46\end{bmatrix},\begin{bmatrix}47\\48\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$, so let $T(1+x) = \begin{bmatrix}45\\46\end{bmatrix}$ and $T(x^2) = \begin{bmatrix}47\\48\end{bmatrix}$ - answers may vary; 81 $T(1+x) = 1$ and $T(x^2) = x$; 82 $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = x_1 + x_2 + 3$ is not linear;

Section 3.4

85 $A\vec{v} = \begin{bmatrix}76\\100\end{bmatrix}$, $B\vec{u} = \begin{bmatrix}23\\53\\83\end{bmatrix}$; 87 $\text{Ker}\,A = \text{Span}\left\{\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right\}$

Section 3.5

96 $\text{Col}\,A = \text{Span}\left\{\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\}$; 97 No to both, $\text{Col}\,A = \text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\} \neq \mathbb{R}^3$; 90 $T(\vec{x}) = [T(\vec{e}_1)\cdots T(\vec{e}_n)]\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}$; 91 $A = \begin{bmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&1\end{bmatrix}$; 95 $A = \begin{bmatrix}0&0&1\\0&1&-1\end{bmatrix}$, $\text{Ker}\,A = \text{Span}\{\vec{e}_1\}$, $\text{Ker}\,T = \text{Span}\{x^2\}$;

Section 4.1

103 $x = 118$, $y = 122$; 105 The second, fourth, fifth, seventh, and eighth are in row echelon form. The second and eighth are in reduced row echelon form;

Section 4.2

107 $x_1 = -3x_3 + 2$, $x_2 = -5x_3 + 1$; $x_1 = -2x_2 + 2$, $x_3 = 1/5$; $x_1 = -3x_3 - 2x_4 + 2$, $x_2 = -5x_3 - 3x_4 + 1$ 108 The forth and fifth correspond to systems with no solution;

Section 4.3

111 The vectors are linearly dependent because $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so it's not a basis; 114 $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$; 115 $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{array}\right]$, so $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$; 116 Imag T = Span $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$, Ker T = Span $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$;

Section 4.4

118 $AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$, $BA = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$; 119 $A\vec{x} = \begin{bmatrix} x_1a_{11} + \cdots + x_pa_{1p} \\ \vdots \\ x_1a_{m1} + \cdots + x_pa_{mp} \end{bmatrix}$; if $\vec{r}_i$ is the ith row of A, then $\vec{r}_i^T = \begin{bmatrix} a_{i1} \\ \vdots \\ a_{ip} \end{bmatrix}$, so $\vec{r}_i^T \cdot \vec{x} = a_{i1}x_1 + \cdots + a_{ip}x_p$; $AB = \begin{bmatrix} A\vec{b}_1 \cdots A\vec{b}_n \end{bmatrix} = \begin{bmatrix} \vec{r}_1^T \cdot \vec{b}_1 + & \cdots & + \vec{r}_1^T \cdot \vec{b}_n \\ \vdots & \ddots & \vdots \\ \vec{r}_m^T \cdot \vec{b}_1 + & \cdots & + \vec{r}_m^T \cdot \vec{b}_n \end{bmatrix}$

Section 4.5

121 The third and fourth correspond to one-to-one linear transformations; 122 The second and third correspond to onto linear transformations; 125 $B^{-1} = \begin{bmatrix} 4 & -2 & -1 \\ 1 & -1 & 0 \\ -3 & 2 & 1 \end{bmatrix}$; 126 $\vec{x} = \begin{bmatrix} -24 \\ 9 \end{bmatrix}$;

Section 4.6

127 Row A = Span $\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$, Ker A = Span $\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\}$ 128 Not invertible, invertible, not invertible; 130 $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}$, $[\vec{x}]_{\mathcal{B}_0} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

Section 4.7

131 $\hat{x}_1 = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, 0$; $\hat{x}_1 = \begin{bmatrix} 57/13 \\ 25/26 \\ 61/26 \end{bmatrix}, 2\sqrt{13}$

Section 5.1

134 $\lambda = 9$ has eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\lambda = 0$ has linearly independent eigenvectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$;

Section 5.2

137 $\det A = 21$, $\det B = 93$, $\det C = -3$; 138 $\det A = 10$; 139 linearly independent; 140 $\det A^{-1} = (\det A)^{-1}$; 141 $\det A^5 = -32$ 142 $\det(A - \lambda I) = 5\lambda(15 - \lambda)$; 144 A has eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, respectively. B has eigenvectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$, respectively.

Section 5.3

147 $A^k = \begin{bmatrix} -3(2^k) + 4 & 12(2^k) - 12 \\ -(2^k) + 1 & 4(2^k) - 3 \end{bmatrix}$; 148 The eigenvalue $\lambda = 2$ has geometric multiplicity two, so A is diagonalizable.

Answers to Selected Exercises

These are at least as reliable as answers you'd find on the internet. In many cases, only answers without the appropriate explanation are provided.

Section 0.1

0.2.2 (a) function, (b) relation, (c) neither, (d) function; 0.2.3 no, check $1/2$ and $2/4$; 0.2.4 yes; 0.2.5 no; 0.2.8 (a) 0; 0.2.10 (a) $(1, 2)$; 0.2.11 (a) -1, (b) no

Section 1.1

Section 1.2

1.2.6 (a) $\|\vec{u}_1\| = \sqrt{10}$, (b) $\frac{\sqrt{10}}{10}\vec{u}_1$, (c) $\frac{7\sqrt{10}}{10}\vec{u}_1$ 1.2.8 $\vec{v}\cdot\vec{u} = 3$ 1.2.9 $\vec{v}\cdot\vec{u} = -1$

Section 1.3

1.3.3 (a) no, (b), no, (c) yes, (d) no, (e) yes, (f) yes; 1.3.11 (a) linearly independent, (b) linearly dependent, (c) linearly dependent, (d) linearly independent; 1.3.15 plane; 1.3.8 $\vec{v}_1 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -3 \\ 0 \\ -2 \end{bmatrix}$

Section 1.4

1.4.4 $\vec{0} \notin \{4 + ax + bx^2 \colon a, b \in \mathbb{R}\}$; 1.4.8 $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \in H$, but $-7\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \notin H$; 1.4.10 $\vec{0} \in \{\vec{0}\}$, $\vec{0}+\vec{0} = \vec{0} \in \{\vec{0}\}$, and $k\vec{0} = \vec{0} \in \{\vec{0}\}$; 1.4.16 (c) V, (d) $\operatorname{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$

Section 2.1

2.1.2 (a) doesn't span, (b) linearly dependent, (c) linearly dependent; 2.1.7 (b) $\begin{bmatrix} 4 \\ 6 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$; 2.1.16 $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$; 2.1.9 $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; 2.1.23 answers may vary, $\{2\}$ works.

Section 2.2

2.2.2 $\dim H_6 \le 6 < 7 = \dim \mathbb{R}^7$, it is possible that $\mathbb{R}^7 \neq H_7$;

Section 2.3

2.3.6 $k_1\vec{b}_1 = k_1\vec{b}_1 + 0\vec{b}_2$, similar for $k_2\vec{b}_2$; 2.3.1 (b) $\left[1 + x + x^2\right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}$, $[1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix}$;

Section 2.4

Section 2.5

Section 2.6

Section 3.1

3.1.1 Onto: $f\left(\begin{bmatrix} y \\ 0 \end{bmatrix}\right) = y$, not one to one: $f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = 0 = f\left(\begin{bmatrix} 7 \\ -7 \end{bmatrix}\right)$;

Section 3.2

3.2.6 Calculate $T(\vec{x}+\vec{y})$ and $T(\vec{x}) + T(\vec{y})$ to see they are the same. Then do the same for $T(a\vec{x})$ and $aT(\vec{x})$. Ker $T = \{\vec{0}\}$ 3.2.12 Since $T(\vec{u}+\vec{v}) = c$ and $T(\vec{v}) + T(\vec{v}) = 2c$, we would need $c = 0$ for T to be linear;

Section 3.3

3.3.1 (a) just onto, (b) both, (c) both, (d) just one to one; 3.3.4 1, 3, 2

Section 3.4

3.4.1 a) $m = 4$ and $n = 3$, b) $m = 5$ and $n = 3$, c) $m = 4$ and $n = 5$; 3.4.3

$\operatorname{Span}\left\{\begin{bmatrix}-1\\3\\1\\0\end{bmatrix},\begin{bmatrix}3/2\\9/2\\0\\1\end{bmatrix}\right\}$; 3.4.4 a) $\begin{bmatrix}-1\\1\end{bmatrix}$, $\begin{bmatrix}7\\-2\\19\end{bmatrix}$; 3.4.10 $2 \le \dim \operatorname{Ker} T_B \le n$ and $0 \le \dim \operatorname{Imag} T_B \le n-2$ since $\vec{x}_1, \vec{x}_2 \subseteq \operatorname{Ker} T_B$ and the Rank-Nullity Theorem tells us $\dim \operatorname{Imag} T_B = n - \dim \operatorname{Ker} T_B$.

Section 3.5

3.5.1 (a) $\begin{bmatrix}1&0&3\\0&1&4\end{bmatrix}$; 3.5.5 (a) $\begin{bmatrix}1&1\\2&0\end{bmatrix}$;

Section 3.6

3.6.1 It is never onto; 3.6.5 (a) yes, (b) no, (c) yes, (d) yes; 3.6.6 (a) no, (b) yes, (c) yes, (d) no

Section 4.1

4.1.5 (a) yes, (b) no, (c) yes; 4.1.6

Section 4.2

4.2.1 (a) no solution, (b) infinitely many, (c) one solution; 4.2.3 (a) There is no solution when $k = 4$, otherwise there is a unique solution 4.2.5 (a) $y = -\frac{3}{10}x - \frac{8}{5}$

Section 4.3

4.3.2 (a) yes, (b) no, (c) yes, operations left to reader; 4.3.1 (a) no; 4.3.8 $A = \begin{bmatrix}0&1&-1\\0&1&1\\0&0&1\end{bmatrix}$; 4.3.9 $A = \begin{bmatrix}1&0&-1\\0&1&-1\\0&0&1\end{bmatrix}$; 4.3.4 $\vec{x} = \frac{5}{2}\vec{b}_1 + \frac{1}{2}\vec{b}_2 - 2\vec{b}_3$

Section 4.4

4.4.2 (a) $\begin{bmatrix}4&5\\6&5\end{bmatrix}$, (b) $\begin{bmatrix}4&5&7\\6&5&3\end{bmatrix}$, (c) $\begin{bmatrix}5&6\\8&1\end{bmatrix}$, (d) $\begin{bmatrix}5&6\\8&1\\2&5\end{bmatrix}$, (e) $\begin{bmatrix}5&6&-2\\8&1&-6\\2&5&0\end{bmatrix}$, 4.4.3 (a) dragon, (b) $\begin{bmatrix}39&54&69\\49&68&87\end{bmatrix}$, (c) $\begin{bmatrix}-121\\-879\\-161\end{bmatrix}$, (d) dragon, (e) $\begin{bmatrix}-56&70&84\\-72&90&108\\80&-100&-120\end{bmatrix}$, (f) $\begin{bmatrix}-86\end{bmatrix}$, (g) $\begin{bmatrix}69\\85\end{bmatrix}$, (h) dragon, (i) $\begin{bmatrix}-1066\end{bmatrix}$, (j) $\begin{bmatrix}14&180\\-84&-1080\end{bmatrix}$;

Section 4.5

4.5.4 (a) $\begin{bmatrix}-1&1\\1&-\frac{1}{2}\end{bmatrix}$, (b) nope, (h) $\begin{bmatrix}\frac{1}{6}&-\frac{1}{6}&\frac{1}{2}\\-\frac{1}{3}&0&\frac{1}{3}\\\frac{1}{6}&\frac{1}{6}&\frac{1}{6}\end{bmatrix}$; 4.5.6 (a) yes, (b) no;

Section 4.6

4.6.2 (a) $P^{-1} = \begin{bmatrix}-1&2\\1&-1\end{bmatrix}$ and $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}17\\-7\end{bmatrix}$, (b) $P^{-1} = \begin{bmatrix}-1&2&-2\\1&-1&1\\0&0&1\end{bmatrix}$ and $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}-1\\1\\1\end{bmatrix}$

Section 4.7

4.7.3 Using $A = \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\\10&11&12\end{bmatrix}$, $\hat{x} = x_3\begin{bmatrix}1\\-2\\1\end{bmatrix} + \begin{bmatrix}2/5\\-3/10\\0\end{bmatrix}$; 4.7.5 $y = \frac{7}{6}x^3 + \frac{5}{7}x^2 - \frac{13}{6}x + \frac{118}{35}$

Section 5.1

5.1.1 (a) no, (b) 5, (c) -1, (d) 1, (e) 1, (f) no, (g) 1, (h) no;

Section 5.2

5.2.3 $\det(kA) = k^n \det A$; 5.2.6 (a) -256, (b) no, (c) 2, (d) 8 5.2.8 Nonsense! However, if A is square, then it it true. Prove it; 5.2.10 $\frac{1}{2^{10}}\vec{x}$, $\vec{0}$

Section 5.3

5.3.1 $A = P\begin{bmatrix}-1&0\\0&512\end{bmatrix}P^{-1}$, but you should finish the calculation; ; 5.3.4

$$P = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Glossary

Addition: Addition is the function $(+)\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by relating two real numbers to their sum. *Multiplication* is the function $(\cdot)\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by relating two real numbers to their product.. 14

affine transformation: An *affine transformation* is a linear transformation composed with a translation.. 229

algebraic multiplicity: For an eigenvalue λ of a matrix $A \in \mathcal{M}_{n\times n}$, the *algebraic multiplicity* of λ is the multiplicity of λ as a root of the characteristic polynomial for A.. 345

basis: Let V be a vector space. A finite set of vectors $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_p\}$ is a *basis* for V if

(a) $\mathcal{B}$ is linearly independent, and

(b) $\mathcal{B}$ spans V.

. 77

binary operation: Let A be a set. A *binary operation* on a set A is a function $f : A \times A \to A$ where the domain is $A \times A$.. 10

Cartesian product: Let A and B be sets. The *Cartesian product of A and B*, denoted $A \times B$, is the set

$$\{(a, b)\colon a \in A \text{ and } a \in B\}.$$

. 4

change of basis matrix: Let V be an n-dimensional vector space with bases

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\} \text{ and } \mathcal{C} = \{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}.$$

Define the isomorphism

$$\varphi_{\mathcal{B}\triangleright\mathcal{C}}\colon V \to V \text{ by } \varphi_{\mathcal{B}\triangleright\mathcal{C}}(\vec{c}_i) = \vec{b}_i \text{ for each } 1 \le i \le n.$$

Then the *change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$* is the matrix for $\varphi_{\mathcal{B}\triangleright\mathcal{C}}$ with respect to the basis $\mathcal{C}$. In particular, it is the matrix $P_{\mathcal{B}\triangleright\mathcal{C}} \in \mathcal{M}_{n\times n}$ defined by

$$P_{\mathcal{B}\triangleright\mathcal{C}} = \left[\left[\vec{b}_1\right]_{\mathcal{C}} \cdots \left[\vec{b}_n\right]_{\mathcal{C}}\right].$$

. 304

characteristic polynomial: For $A \in \mathcal{M}_{n\times n}$, the degree n polynomial $\det(A - \lambda I)$ is called the *characteristic polynomial* for A.. 343

closed under the operation: If $a * b \in A$ for any $a, b \in A$, we say the set A is *closed under the operation* $*$.. 10

codomain: Let $f\colon A \to B$ be a function. The *codomain of* f, written $\operatorname{codom}(f)$, is the set B.. 8

coefficient matrix: The matrix A in the matrix equation $A\vec{x} = \vec{b}$ or the augmented matrix $[A|\vec{b}]$ is called a *coefficient matrix*. . 235

column space: Let $A = [\vec{a}_1 \cdots \vec{a}_n] \in \mathcal{M}_{m\times n}$. The *column space of* A, denoted Col A, is the span of the column vectors $\vec{a}_j$ for $1 \leq j \leq n$. That is,

$$\text{Col } A = \text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}.$$

. 197, 219

composition: Let A, B, and C be sets and $f\colon A \to B$ and $g\colon B \to C$ be functions. The *composition of the functions* f *and* g is the function $(g \circ f)\colon A \to C$ such that $(a, c) \in g \circ f$ if and only if there is a $b \in B$ such that $(a, b) \in f$ and $(b, c) \in g$. That is, for any $a \in A$,

$$(g \circ f)(a) = g(f(a)).$$

. 151

conjugate of A: For $A \in \mathcal{M}_{m\times n}(\mathbb{C})$, the *conjugate of* A is the matrix $\overline{A} = [\bar{a}_{ij}] \in \mathcal{M}_{m\times n}(\mathbb{C})$ obtained by conjugating every entry A. . 382

conjugate of z: For $z = x + iy \in \mathbb{C}$, the *conjugate of* z is the complex number $\bar{z} = x - iy$. . 367, 382

conjugate transpose of A: The *conjugate transpose of* A, denoted by A^H, is obtained by conjugating the transpose of A; that is, $A^H = \overline{A^T}$. . 383

convolution: Let $A = [a_{ij}]$, $B = [b_{ij}] \in \mathcal{M}_{m\times n}$. The *convolution* of A and B, denoted $A * B$, is

$$A * B = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}b_{ij}|.$$

. 320

coordinate vector: The *coordinate vector of* $\vec{v}$ *relative to* $\mathcal{B}$ is

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}.$$

. 101

coordinates: Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be a basis for vector space V, and suppose $\vec{v} \in V$. The *coordinates for* $\vec{v}$ *relative to* $\mathcal{B}$ (or the $\mathcal{B}$*-coordinates of* $\vec{v}$) are the weights $c_1, \ldots, c_p$ such that

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p.$$

. 101

determinant: For $n \geq 2$, the *determinant* of a matrix $A = [a_{ij}] \in \mathcal{M}_{n\times n}$ is the sum

$$\begin{aligned} \det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{n+1}a_{1n}\det A_{1n} \\ &= \sum_{j=1}^{n}(-1)^{j+1}a_{1j}\det A_{1j}, \end{aligned}$$

where A_{ij} is the submatrix of A resulting from removing the ith row and jth column.. 338

diagonal matrix: A matrix of the form

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} \in \mathcal{M}_{n\times n}$$

is called a *diagonal matrix*.. 356

diagonalizable: A matrix is called *diagonalizable* if it is similar to a diagonal matrix.. 356

dimension: Let V be a vector space. The *dimension of* V, denoted $\dim V$, is the number of vectors in a basis for V.. 93

direct sum of linear transformations: Let V_1 and V_2 be vector spaces and $T_1\colon V_1 \to V_1$ and $T_2\colon V_2 \to V_2$ be linear transformations. The *direct sum of linear transformations* of a square matrix T_1 and T_2 is the linear transformation $T_1 \oplus T_2\colon V_1 \oplus V_2 \to V_1 \oplus V_2$ given by

$$(T_1 \oplus T_2)(\vec{v}_1, \vec{v}_2) = (T\vec{v}_1, T\vec{v}_2)$$

. 365

Distance: is the function $dist\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by relating two vectors to the length of their difference. That is, given $\vec{v}, \vec{u} \in \mathbb{R}^n$, we denote the distance between $\vec{v}$ and $\vec{u}$ as $\operatorname{dist}(\vec{v}, \vec{u})$ given by

$$\operatorname{dist}(\vec{v}, \vec{u}) = \|\vec{v} - \vec{u}\|.$$

. 34

domain: Let $f\colon A \to B$ be a function. The *domain of* f is the set

$$\begin{aligned} \operatorname{dom}(f) &= \{a \in A \colon \text{there exists } b \in B \text{ such that } (a, b) \in f\} \\ &= \{a \in A \colon \text{there exists } b \in B \text{ such that } f(a) = b\} \end{aligned}$$

. 8

eigenspace: The set of all solutions of

$$(A - \lambda I)\vec{x} = \vec{0}$$

is a subspace of R^n called the *eigenspace corresponding to* λ relative to the matrix A.. 326

eigenvalue: A scalar λ is called an *eigenvalue* of A if there is a nontrivial solution $\vec{x}\mathbb{R}^n$ of $A\vec{x} = \lambda\vec{x}$, and we call such an $\vec{x}$ an *eigenvector corresponding to* λ.. 325

eigenvector: An *eigenvector* of a matrix $A \in \mathcal{M}_{n\times n}$ is a nonzero vector $\vec{x}\mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. . 325

elementary matrix: We call $E \in \mathcal{M}_{n\times n}$ an *elementary matrix* if for any $A \in \mathcal{M}_{n\times n}$, the matrix EA is the matrix A after performing a row operation on A.. 284, 418

equal as a set: A is equal to B as a set, written as $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.. 2

free variable: A *free variable* is a variable in a system of equations that is not a pivot variable. That is, a free variable in a system of equations is one whose column in the associated augmented matrix in reduced echelon form does not contain a pivot. . 248

function: Let A and B be sets. A *function from A to B*, often written $f\colon A \to B$, is a relation f from A to B such that

$$\text{if } (a, b_1) \in f \text{ and } (a, b_2) \in f, \text{ then } b_1 = b_2.$$

For $(a, b) \in f$, it is often written $f(a) = b$.. 7

geometric multiplicity: The *geometric multiplicity* of λ is the dimension of the eigenspace corresponding to λ. . 345

Hermitian: A matrix $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ is called *Hermitian* if $A = A^H$. . 383

Hermitian Inner product: *Hermitian Inner product* is the function $\cdot\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by

$$\vec{v} \cdot \vec{v} = v_1\bar{u}_1 + \cdots + v_n\bar{u}_n = \sum_{i=1}^{n} v_i\bar{u}_i.$$

. 367

Hermitian norm: The *Hermitian norm* is the function $\|\cdot\|\colon \mathbb{C}^n \to \mathbb{R}$ defined for any $\vec{v} \in \mathbb{C}^n$ as

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1\bar{v}_1 + \cdots + v_n\bar{v}_n}.$$

. 367

identity matrix: The *identity matrix* is the square matrix $I_n \in \mathcal{M}_{n\times n}$ whose columns are the standard basis for $\mathbb{R}^n$ in order. That is,

$$I_n = [\vec{e}_1\ \vec{e}_2\ \cdots\ \vec{e}_n] = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

. 280

image: Let V and W be vector spaces and $f\colon V \to W$ be a linear transformation. The *image of f* is the set of vectors $\vec{w} \in W$ such that there is a vector $\vec{v} \in V$ and $\vec{w} = f(\vec{v})$. We shall use the notation

$$\operatorname{Imag} f = \{\vec{w} \in W\colon \vec{w} = f(\vec{v}) \text{ for some } \vec{v} \in V\}.$$

. 163

inner product: The inner product is the function $\cdot\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by relating two vectors to the real number given by summing the products of like components of the two vectors. That is, given $\vec{v}, \vec{u} \in \mathbb{R}^n$, we denote the inner product of $\vec{v}$ and $\vec{u}$ as $\vec{v}\cdot\vec{u}$, given by

$$\vec{v}\cdot\vec{u} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = v_1u_1 + \cdots + v_nu_n = \sum_{i=1}^{n} v_iu_i.$$

Let V be a vector space with basis $\mathcal{B}$ and let $[\cdot]_{\mathcal{B}} : V \to \mathbb{R}^n$ be the function that relates vectors in V to their coordinate vector relative to $\mathcal{B}$ in $\mathbb{R}^n$. The *inner product on V relative to $\mathcal{B}$* is the function $\cdot_{\mathcal{B}} \colon V \times V \to \mathbb{R}$ defined as the composition of $[\cdot]_{\mathcal{B}} \times [\cdot]_{\mathcal{B}}$ on $V \times V$ with the standard inner product $\cdot$ on $\mathbb{R}^n \times \mathbb{R}^n$. That is, for any vectors $\vec{v}, \vec{u} \in V$, we define

$$\vec{v} \cdot_{\mathcal{B}} \vec{u} = [\vec{v}]_{\mathcal{B}} \cdot [\vec{u}]_{\mathcal{B}} .$$

. 31, 103

inner product space: We call a vector space, V, together with inner product relative to basis $\mathcal{B}$ an *inner product space*.. 106

integers: The set of integers, $\mathbb{Z}$, is the set of counting numbers, negative counting numbers, and 0. That is,

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

. 1

intersection: The *intersection* of two sets A and B is all of the elements that are in both A and B. We denote this intersection as $A \cap B$.. 64

invariant subspace: Let $T \colon V \to V$ be a linear transformation from an n-dimensional vector space V to itself and suppose W is a subspace of V. We say W is an *invariant subspace of V for T* if for any $\vec{x} \in W$, the vector $T(\vec{x})$ is also in W. . 350

invertible: A function $f \colon A \to B$ is *invertible* if there is another function $g \colon B \to A$ such that

- for all $a \in A$, $(g \circ f)(a) = a$, and
- for all $b \in B$, $(f \circ g)(b) = b$.

If such a function exists, we call it the *inverse of f*, and denote it f^{-1}.
A matrix $A \in \mathcal{M}_{m\times n}$ is *invertible* if there is another matrix $B \in \mathcal{M}_{n\times n}$ such that

$$AB = I_n = BA.$$

We call the matrix B the *inverse* of the matrix A.. 153, 290

isomorphism: Let V and W be vector spaces. A linear transformation $T \colon V \to W$ is called an *isomorphism* if it is both one-to-one and onto. In this case, we say V and W are *isomorphic* vector spaces, and denote this by $V \cong W$.. 179

Jordan block: A *Jordan block* is a square matrix whose entries are the same constant, $\lambda \in \mathbb{C}$, on the diagonal, 1 on each entry immediately above the diagonal, and zero elsewhere.. 362

Jordan chain: If $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ has eigenvalue λ with eigenvector $\vec{v}_0$, then a *Jordan chain* for λ is a set of vectors $S = \{\vec{v}_1, \dots \vec{v}_k\}$ for some $k < n$ such that

$$\vec{v}_k \xrightarrow{A-\lambda I} \vec{v}_{k-1} \xrightarrow{A-\lambda I} \cdots \xrightarrow{A-\lambda I} \vec{v}_1 \xrightarrow{A-\lambda I} \vec{v}_0 \xrightarrow{A-\lambda I} \vec{0}$$

. 362

kernel: Let V and W be vector spaces and $f \colon V \to W$ be a linear transformation. The *kernel of f* is the set of vectors $\vec{v} \in V$ such that $f(\vec{v}) = \vec{0}$. We shall use the notation

$$\text{Ker } f = \{\vec{v} \in V \colon f(\vec{v}) = \vec{0}\}.$$

. 171

least squares solution: A *least squares solution* for the matrix equation $A\vec{x} = \vec{b}$ is a vector $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that for all $\vec{x} \in \mathbb{R}^n$,

$$\|A\hat{\mathbf{x}} - \vec{b}\| \leq \|A\vec{x} - \vec{b}\|.$$

The *least squares error* of a least squares solution is $\|A\hat{\mathbf{x}} - \vec{b}\|$.. 312

left shift: The *left shift* on $\mathbb{C}^n$ is the linear transformation $L\colon \mathbb{C}^n \to \mathbb{C}^n$ given by

$$L\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ 0 \end{bmatrix}.$$

. 364

left-singular: If there are unit vectors $\vec{u}$ and $\vec{v}$ such that

$$A\vec{v} = \sigma\vec{u} \qquad \text{and} \qquad A^T\vec{u} = \sigma\vec{v}$$

for some nonnegative scalar σ, then $\vec{u}$ and $\vec{v}$ are called *left-singular* and *right-singular* vectors, respectively. . 389

Length: is the function $\|\cdot\|\colon \mathbb{R}^n \to \mathbb{R}$ defined by relating vectors to their length. That is, given $\vec{v} \in \mathbb{R}^n$, we denote the length of $\vec{v}$ as $\|\vec{v}\|$, given by

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Let V be a vector space with basis $\mathcal{B}$ and let $[\cdot]_{\mathcal{B}} : V \to \mathbb{R}^n$ be the function that relates vectors in V to their coordinate vector relative to $\mathcal{B}$ in $\mathbb{R}^n$. *Length relative to* $\mathcal{B}$ is the function $\|\cdot\|_{\mathcal{B}}\colon \mathbb{V} \to \mathbb{R}$ defined by relating vectors to their length by composing the function $[\cdot]_{\mathcal{B}} \times [\cdot]_{\mathcal{B}}$ on V with $\|\cdot\|$ on $\mathbb{R}^n$. That is, for any vector $\vec{v} \in V$, we define

$$\|\vec{v}\|_{\mathcal{B}} = \|[\vec{v}]_{\mathcal{B}}\|.$$

. 33, 105

linear combination: The vector in V

$$a_1\vec{v}_1 + \cdots + a_p\vec{v}_p$$

is called a *linear combination* of the vectors $\vec{v}_1, \ldots, \vec{v}_p$ with *weights* $a_1, \ldots, a_p$.. 40

linear transformation: A function $f\colon V \to W$, where V and W are vector spaces, is called a *linear transformation* if for any vectors $\vec{v}, \vec{u} \in V$ and any scalar $a \in \mathbb{R}$,

- $f(\vec{v} + \vec{u}) = f(\vec{v}) + f(\vec{u})$ and
- $f(a\vec{v}) = af(\vec{v})$.

For $\vec{v} \in V$, the vector $f(\vec{v}) \in W$ is often called the *image of* $\vec{v}$.. 163

linearly dependent: The set $\{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$ is said to be *linearly dependent* if there are scalars $a_1, \ldots, a_n \in \mathbb{R}$, not all 0, such that

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}.$$

. 45

linearly independent: A set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\} \subseteq V$ is said to be *linearly independent* if

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$$

only when $a_1 = \cdots = a_n = 0$.. 45

main diagonal: The *main diagonal* of a square matrix $A \in \mathcal{M}_{n\times n}$ are the entries $a_{11}, a_{22}, \dots a_{nn}$ starting at the upper left corner of the matrix and going diagonally to the lower right entry. A matrix is called *upper (lower) triangular* if all the entries below (above) the main diagonal are zero.. 332

Markov chain: Given a finite set of *states*, $\{1, 2, \dots, n\}$, in which the probability of transition from the current state to another depends only on the current state, a *Markov chain* is a sequence describing how a distribution amongst the states evolves as a result of these probabilities. . 396

matrix: An $m \times n$ *matrix* A is a rectangular array of numbers with m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The number a_{ij} in the ith row and jth column is called the *ijth entry*. Matrices are sometimes also written as

$$A = [a_{ij}]_{\substack{1\le i\le m,\\ 1\le j\le n}}.$$

An $n \times n$ matrix is often called a *square matrix*.. 190

matrix representation: For any linear transformation $T\colon V \to W$, we call the matrix A obtained from Theorem 3.5.2 a *matrix representation of T*. . 206

norm: is the function $\|\cdot\|\colon \mathbb{R}^n \to \mathbb{R}$ defined by relating vectors to their length. That is, given $\vec{v} \in \mathbb{R}^n$, we denote the length of $\vec{v}$ as $\|\vec{v}\|$, given by

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Let V be a vector space with basis $\mathcal{B}$ and let $[\cdot]_{\mathcal{B}} : V \to \mathbb{R}^n$ be the function that relates vectors in V to their coordinate vector relative to $\mathcal{B}$ in $\mathbb{R}^n$. *Norm relative to* $\mathcal{B}$ is the function $\|\cdot\|_{\mathcal{B}}\colon \mathbb{V} \to \mathbb{R}$ defined by relating vectors to their length by composing the function $[\cdot]_{\mathcal{B}} \times [\cdot]_{\mathcal{B}}$ on V with $\|\cdot\|$ on $\mathbb{R}^n$. That is, for any vector $\vec{v} \in V$, we define

$$\|\vec{v}\|_{\mathcal{B}} = \|[\vec{v}]_{\mathcal{B}}\|.$$

. 33, 105

normal equation: The *normal equation* for a matrix $A \in \mathcal{M}_{m\times n}$ and a vector $\vec{b} \in \mathbb{R}^m$ is

$$A^T A\hat{\mathbf{x}} = A^T\vec{b}.$$

. 314

one-to-one: For sets A and B and a function $f\colon A \to B$, the function f is *one-to-one* if for any $b \in \operatorname{ran}(f)$, we have $a_1 = a_2$ if $f(a_1) = b$ and $f(a_2) = b$.. 150

onto: For sets A and B and a function $f\colon A \to B$, the function f is *onto* if for every element $b \in B$, there is an element $a \in A$ such that f relates a to b, that is, $f(a) = b$.. 148

orthogonal: Vectors $\vec{v}$ and $\vec{u}$ in vector space V with chosen basis $\mathcal{B}$ are said to be *orthogonal* if $\vec{v} \cdot \vec{u} = 0$. Let V be an inner product space and W be a subspace of V. If a vector $\vec{v} \in V$ is orthogonal to every vector in W, then we say $\vec{v}$ is *orthogonal to* W.. 106, 114

orthogonal basis: An *orthogonal basis* for a subspace W is a basis for W that is also an orthogonal set.. 117

orthogonal complement: The set of all vectors $\vec{v} \in V$ that are orthogonal to W is called the *orthogonal complement of* W. The orthogonal complement of W is denoted $W^{\perp}$.. 114

orthogonal projection: For any two vectors $\vec{v}$ and $\vec{u}$ in an inner product space, the *orthogonal projection of* $\vec{v}$ *onto* $\vec{u}$ is

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u}.$$

Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be an orthogonal basis for a subspace W of vector space V. For any vector $\vec{v} \in V$, the *orthogonal projection of* $\vec{v}$ *onto* W is

$$\text{proj}_W(\vec{v}) = \frac{\vec{v} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \cdots + \frac{\vec{v} \cdot \vec{v}_p}{\vec{v}_p \cdot \vec{v}_p}\vec{v}_p.$$

. 123, 132

orthogonal set: If S is a set of vectors in a vector space with inner product relative to basis $\mathcal{B}$ such that all pairs of vectors in S are orthogonal, then S is said to be an *orthogonal set*.. 117

orthonormal basis: If S is an orthogonal basis of vectors in a vector space such that any vector in S is a unit vector, then S is said to be an *orthonormal basis*.. 119

orthonormal set: If S is an orthogonal set of vectors in a vector space such that any vector in S is a unit vector, then S is said to be an *orthonormal set*.. 119

parametric solution: A *parametric solution* for a system of m equations in n variables that has an infinite number of solutions is a representation of the solutions in which the free variables serve as parameters.. 248

pivot column: A *pivot column* is a column in a matrix that would contain a pivot were the matrix put in row-echelon form. . 260

pivot variable: A *pivot variable* is a variable in a system of equations whose column in the associated augmented matrix in reduced echelon form contains a pivot. . 248

probability vector: A vector whose entries are all nonnegative and sum to 1 is called a *probability vector*. . 396

product of a matrix and a vector: Let $A \in \mathcal{M}_{m\times n}$ with columns $\vec{a}_1, \ldots, \vec{a}_n$, and let $\vec{x} \in \mathbb{R}^n$. The product of a matrix and a vector, that is, the *product of* A *and* $\vec{x}$, is the linear combination of the columns of A with the entries of $\vec{x}$ as weights. That is,

$$A\vec{x} = [\vec{a}_1 \; \vec{a}_2 \cdots \vec{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n.$$

. 191

product of matrices: Let $A \in \mathcal{M}_{m\times n}$ and $B = [\vec{b}_1 \cdots \vec{b}_p] \in \mathcal{M}_{n\times p}$. Then we define the *product of matrices A and B* to be the matrix $AB \in \mathcal{M}_{m\times p}$ given by

$$AB = A[\vec{b}_1 \cdots \vec{b}_p] = [A\vec{b}_1 \cdots A\vec{b}_p].$$

. 276

pseudoinverse: Let $A \in \mathcal{M}_{m\times n}$ have singular value decomposition UDV^T, r be $\min m, n$, k be the number of nonzero singular values, and $\sigma_1, \ldots, \sigma_k, 0_{k_1}, \ldots, 0_r$ be the diagonal entries of D. The *pseudoinverse* of A, denoted A^+, is the matrix VD^+U, where $D^+ \in \mathcal{M}_{n\times m}$ is rectangular diagonal with diagonal entries $1/\sigma_1, \ldots, 1/\sigma_k, 0, \ldots, 0$. . 391

range: Let $f\colon A \to B$ be a function. The *range of f* is the set

$$\begin{aligned} \operatorname{ran}(f) &= \{b \in B\colon \text{there exists } a \in A \text{ such that } (a,b) \in f\} \\ &= \{b \in B\colon \text{there exists } a \in A \text{ such that } f(a) = b\} \end{aligned}$$

. 8

rational numbers: The set of rational numbers, $\mathbb{Q}$, is the set of well-defined ratios of integers. That is,

$$\mathbb{Q} = \left\{ \frac{p}{q} \colon p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}.$$

. 2

regular: A transition matrix, A, is called *regular* if A^k has no zero entries for some positive integer k. . 398

relation: A *relation from A to B*, r, is a subset of $A \times B$; that is, $r \subseteq A \times B$.. 4

restriction of T: Let $T\colon V \to V$ be a linear transformation from a vector space V to itself, and suppose W is a invariant subspace of V for T. The linear transformation $T|_W\colon W \to W$ given by $T|_W(\vec{w}) = T(\vec{w})$ for all $\vec{w} \in W$ is called the *restriction of T to W*. . 351

right-singular: If there are unit vectors $\vec{u}$ and $\vec{v}$ such that

$$A\vec{v} = \sigma\vec{u} \qquad \text{and} \qquad A^T\vec{u} = \sigma\vec{v}$$

for some nonnegative scalar σ, then $\vec{u}$ and $\vec{v}$ are called *left-singular* and *right-singular* vectors, respectively. . 389

row operations: Let $A \in \mathcal{M}_{m\times n}$. The following manipulations of A are called *row operations*:

(a) interchanging any two rows in A;

(b) multiplying any row by a nonzero scalar; and

(c) replacing the ith row with the sum of the ith row and any nonzero scalar multiple of any of the other rows.

Any matrix resulting from any row operation on A is called *row equivalent* to A.. 238

row space: For a matrix $A \in \mathcal{M}_{m\times n}$, let $\vec{r}_i$ be the vector formed from the ith row of A for each $1 \leq i \leq m$. The *row space of A*, denoted Row A, is the span of these row vectors. That is,

$$\text{Row } A = \text{Span}\{\vec{r}_1, \ldots, \vec{r}_m\}.$$

. 223, 300

row-echelon form: Let $A \in \mathcal{M}_{m\times n}$. We say the matrix A is in *row-echelon form* if

- the first nonzero number from the left, also called the *pivot*, of any nonzero row is always strictly to the right of the pivot of the row above, and
- any row with nonzero entries is above any row of all zeros.

We say A is in *reduced row-echelon form* if

- it is in row-echelon form,
- every pivot is a 1, and
- every pivot is the only nonzero entry in its column.

. 240

set: A *set* is an unordered collection of objects we call *elements*.. 1

set difference: Let A and B be sets and $B \subseteq A$. The *set difference of A and B*, denoted $A\backslash B$, is the set of elements in A and not in B. Specifically,

$$A\backslash B = \{a\colon a \in A \text{ and } b \notin B\}.$$

. 3

similar: Matrices $A, B \in \mathcal{M}_{n\times n}$ are *similar* if there is an invertible matrix $P \in \mathcal{M}_{n\times n}$ such that $A = PBP^{-1}$, or equivalently, $B = P^{-1}AP$.. 353

similar linear transformations: Two linear transformations are *similar linear transformations* if they have similar matrix representations. . 365

singular value: A nonnegative $\sigma \in \mathbb{R}$ is a *singular value* for $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ if there are unit vectors $\vec{u} \in \mathbb{R}^m$ and $\vec{v} \in \mathbb{R}^n$ such that

$$A\vec{v} = \sigma\vec{u} \qquad \text{and} \qquad A^T\vec{u} = \sigma\vec{v}.$$

. 389

Sobel operator: Let

$$G_x = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad G_y = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}.$$

The *Sobel operator* is the function $S\colon \mathbb{M}_{3\times 3} \to \mathbb{R}$ defined by

$$S(A) = (A * G_x) + (A * G_y).$$

. 321

solution: A *solution* for a linear equation $a_1x_1 + \cdots + a_nx_n = b$ is an n-tuple $(x_1, \ldots, x_n)$ that makes the linear equation true. The *graph* of a linear equation $a_1x_1 + \cdots + a_nx_n = b$ is a visual representation of the set of all n-tuples $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ that make the linear equation true.. 251

solution for a system: A *solution for a system* of linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

is an n-tuple $(x_1, \ldots, x_n)$ that makes *all* the linear equations in the system true. . 234

span: Let V be a vector space and $\{\vec{v}_1, \ldots, \vec{v}_p\} \subseteq V$. The *span of* $\vec{v}_1, \ldots, \vec{v}_p$, denoted $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$, is the set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_p$. That is,

$$\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\} = \{a_1\vec{v}_1 + \cdots + a_p\vec{v}_p \colon a_i \in \mathbb{R} \text{ for } 1 \leq i \leq p\}.$$

. 42

steady-state vector: For a transition matrix, A, a *steady-state vector* is a probability vector, $\vec{x}$, such that $A\vec{x} = \vec{x}$. . 398

subset: A *subset* of a set A is a subcollection of the elements of A; that is, B is a subset of A, written as $B \subseteq A$, if and only if every element of B is an element of A.. 2

subspace: A *subspace* of a vector space V is a subset H of V with the following three properties:

- The zero vector is in H.
- (Closure under vector addition) For any $\vec{v}$ and $\vec{u}$ in H, the vector $\vec{v} + \vec{u}$ is also in H.
- (Closure under scalar multiplication) For any $\vec{v}$ in H and any a in $\mathbb{R}$, the vector $a\vec{v}$ is also in H.

. 56

sum: Let U and W be subspaces of a vector space V. The *sum* of these subspaces $U + W$ is defined as

$$\{\vec{u} + \vec{w} \colon \vec{u} \in U, \vec{w} \in W\}.$$

Additionally, if U and W have the property that $U \cap W = \{\vec{0}\}$, then we call this a *direct sum* and denote it $U \oplus W$.. 66

symmetric: A *symmetric* matrix is a matrix A such that $A^T = A$.. 379

transition matrix: A square matrix whose columns are all probability vectors is called a *transition matrix*. . 396

transpose: Let $A \in \mathcal{M}_{m\times n}$. The *transpose of* A, denoted A^T, is the matrix in $\mathcal{M}_{n\times m}$ derived from A by making the jth column of A into the jth row for each $1 \leq j \leq n$.. 222, 278

unit vector: A vector $\vec{v} \in \mathbb{R}^n$ is said to be a *unit vector* (or to have *unit length*) if $\|\vec{v}\| = 1$.. 33

unitary: A matrix $U \in \mathcal{M}_{n\times n}(\mathbb{C})$ is called *unitary* if $UU^H = U^HU = I_n$. . 383

vector space: A vector space is a set V together with two operations that satisfies all the ten vector space axioms.. 16