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Algebra 1

HIGH SCHOOL TEXAS EDITION

Algebra 1

SENIOR CONTRIBUTING AUTHORS

Lynn Marecek, Santa Ana College

MaryAnne Anthony-Smith, Formerly of Santa Ana College

Andrea Honeycutt Mathis, Northeast Mississippi Community College

Jay Abramson, Arizona State University

Sharon North, St. Louis Community College

Amy Baldwin, University of Central Arkansas

Alyssa Howell, OpenStax, Rice University



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OpenStax
Rice University
6100 Main Street MS-375
Houston, Texas 77005

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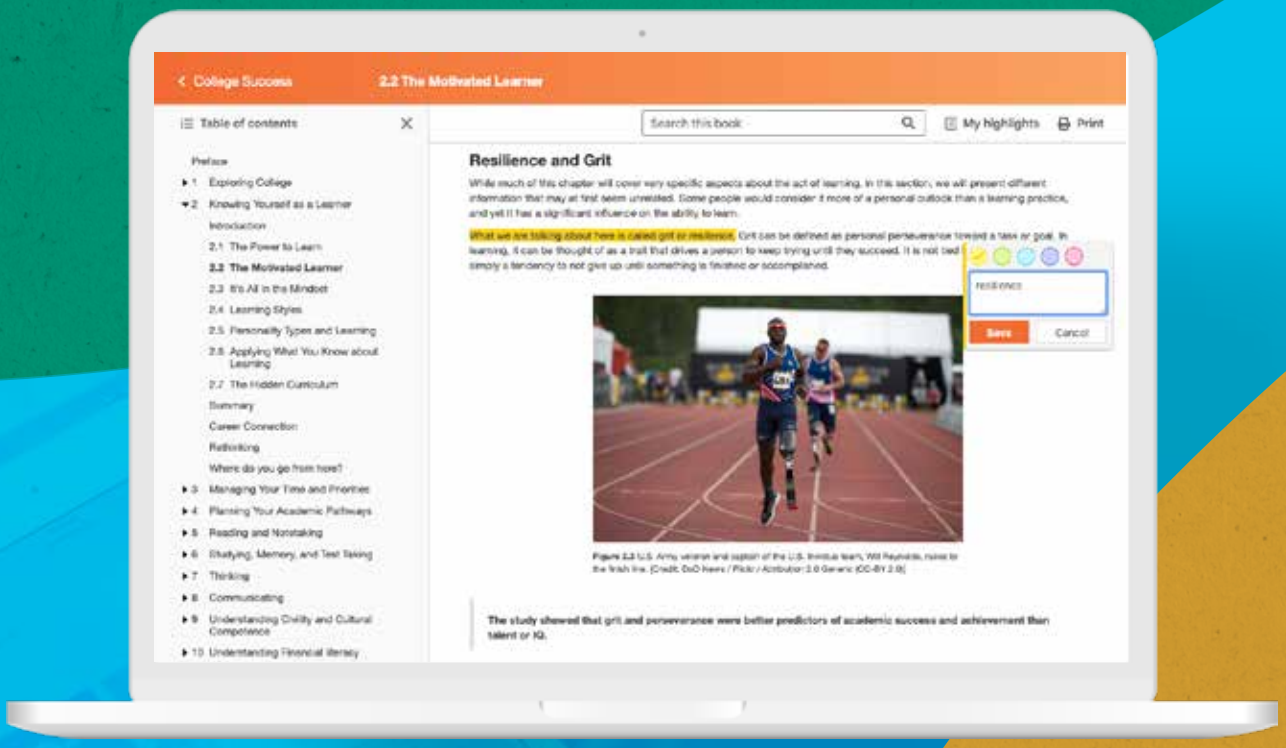
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Contents

GETTING STARTED

About this Course 1

Students Start Here 3

Quick Start 3
Using Desmos 5
Additional Resources 8

UNIT 1 LINEAR EQUATIONS

Unit 1 Overview and Readiness 9

Unit 1 Overview 9
Solve Linear Equations: Mini-Lesson Review 9
Classify Equations: Mini-Lesson Review 11
Find Coordinates: Mini-Lesson Review 13
Unit 1 Overview: Wrap Up 15

1.1 Exploring Expressions and Equations 17

1.1.0 Lesson Overview 17
1.1.1 Understanding Value 17
1.1.2 Creating Expressions to Estimate Cost, Part 1 17
1.1.3 Understanding Constraints 20
1.1.4 Creating Expressions to Estimate Cost, Part 2 22
1.1.5 Practice 23
1.1.6 Lesson Summary 25

1.2 Writing Equations to Model Relationships, Part 1 27

1.2.0 Lesson Overview 27
1.2.1 Finding the Percent of 200 27
1.2.2 Modeling with Equations to Find Edges in Platonic Solids 27
1.2.3 Writing Equations to Represent Relationships 30
1.2.4 Writing Equations to Represent Relationships with Percentages 32
1.2.5 Writing Equations to Represent T-Shirts 33
1.2.6 Practice 34
1.2.7 Lesson Summary 36

1.3 Writing Equations to Model Relationships, Part 2 37

1.3.0 Lesson Overview 37
1.3.1 Finding a Relationship between x and y 37
1.3.2 Describing Relationships Using Words and Equations 37
1.3.3 Identifying and Representing Relationships 40
1.3.4 Writing an Equation to Represent a Relationship 42
1.3.5 Practice 43
1.3.6 Lesson Summary 45

1.4 Equations and Their Solutions 47

1.4.0 Lesson Overview 47
1.4.1 Finding the Solution to an Equation in One Variable, Part 1 47
1.4.2 Writing Equations to Represent Constraints 47
1.4.3 Finding the Solution to an Equation in Two Variables 49
1.4.4 Finding the Solution to an Equation in One Variable, Part 2 50
1.4.5 Practice 52
1.4.6 Lesson Summary 54

1.5 Equations and Their Graphs 55

1.5.0 Lesson Overview 55
1.5.1 Analyzing Graphs of Linear Equations 55
1.5.2 Graphing Linear Functions in Two Variables 56
1.5.3 Examining an Equation in Two Variables and Its Graph, Part 1 57
1.5.4 Writing, Graphing, and Solving a Linear Equation 60
1.5.5 Examining an Equation in Two Variables and Its Graph, Part 2 63
1.5.6 Practice 64
1.5.7 Lesson Summary 66

1.6 Equivalent Equations 67

1.6.0 Lesson Overview 67
1.6.1 Exploring Equivalent Expressions 67
1.6.2 Expressing Relationships as Equations 67
1.6.3 Exploring Related Equations 69
1.6.4 Recognizing Related Equations 71
1.6.5 Practice 72
1.6.6 Lesson Summary 74

1.7 Explaining Steps for Rewriting Equations 75

1.7.0 Lesson Overview 75
1.7.1 Determining if Zero Is a Solution 75
1.7.2 Explaining Acceptable Moves to Solve an Equation 75
1.7.3 Understanding Equations with No Solution or Infinitely Many 78
1.7.4 Solving Equations by Dividing Values 81
1.7.5 Practice 82
1.7.6 Lesson Summary 83

1.8 Choosing the Correct Variable to Solve For, Part 1 85

1.8.0 Lesson Overview 85
1.8.1 Expressing Relationships between Two Quantities 85
1.8.2 Identifying the Most Useful Form of an Equation 86
1.8.3 Writing Equations and Isolating Variables 90
1.8.4 Writing Equations Given Dimensions of Shapes 92
1.8.5 Practice 93
1.8.6 Lesson Summary 95

1.9 Choosing the Correct Variable to Solve For, Part 2 97

1.9.0 Lesson Overview 97

1.9.1 Rearranging an Equation in Three Variables 97
1.9.2 Writing an Equation to Represent a Constraint 98
1.9.3 Writing and Rearranging Equations in Two Variables 100
1.9.4 Solving for a Specified Variable 104
1.9.5 Practice 105
1.9.6 Lesson Summary 106

1.10 **Connecting Equations to Graphs, Part 1 107**

1.10.0 Lesson Overview 107
1.10.1 Interpreting Equations in Context 107
1.10.2 Graphing Equations in Context 108
1.10.3 Writing and Graphing Equations in Standard Form 114
1.10.4 Matching Equations and Graphs 115
1.10.5 Practice 116
1.10.6 Lesson Summary 118

1.11 **Connecting Equations to Graphs, Part 2 119**

1.11.0 Lesson Overview 119
1.11.1 Applying the Distributive Property to Rewrite Expressions 119
1.11.2 Relating Two-Variable Equations, Their Graphs, and Situations 119
1.11.3 Reasoning Symbolically and Abstractly about Linear Equations 123
1.11.4 Identifying the Features of a Graph 125
1.11.5 Practice 126
1.11.6 Lesson Summary 128

1.12 **Writing the Equation of a Line 129**

1.12.0 Lesson Overview 129
1.12.1 Using the Slope Formula 129
1.12.2 Writing an Equation Given the Slope and *y*-intercept 129
1.12.3 Writing an Equation Given the Slope and a Point 132
1.12.4 Writing Equations in Different Forms 134
1.12.5 Writing an Equation Given Two Points 136
1.12.6 Choosing a Method to Write an Equation 138
1.12.7 Practice 140
1.12.8 Lesson Summary 142

1.13 **Lines from Tables and Graphs 143**

1.13.0 Lesson Overview 143
1.13.1 Writing Linear Equations 143
1.13.2 Creating Tables from Verbal Descriptions 143
1.13.3 Matching Tables, Equations, and Graphs 146
1.13.4 Creating a Table from an Equation 148
1.13.5 Practice 149
1.13.6 Lesson Summary 151

1.14 **Writing Equations of Parallel and Perpendicular Lines 153**

1.14.0 Lesson Overview 153
1.14.1 Using Point-Slope Form to Write the Equation of a Line 153

1.14.2 Writing Equations for Point-Slope and Slope-Intercept 154
1.14.3 Writing an Equation of a Line Parallel to a Given Line 155
1.14.4 Writing an Equation of a Line Perpendicular to a Given Line 158
1.14.5 Writing an Equation of a Line Parallel or Perpendicular to an Axis 160
1.14.6 Drawing Parallel and Perpendicular Lines 164
1.14.7 Practice 165
1.14.8 Lesson Summary 167

1.15 Direct Variation 169

1.15.0 Lesson Overview 169
1.15.1 Solving Proportions 169
1.15.2 Modeling Equations Using Direct Variation 169
1.15.3 Using Direct Variation to Solve Application Problems, Part 1 171
1.15.4 Using Direct Variation to Solve Application Problems, Part 2 173
1.15.5 Practice 174
1.15.6 Lesson Summary 175

Project 1: Slopes and Intercepts 177

Project 1 Overview 177
Notice and Wonder Graph Features 177
Matching Graph and Equations 177

Unit 1 Wrap Up 180

UNIT 2 LINEAR INEQUALITIES AND SYSTEMS

Unit 2 Overview and Readiness 181

Unit 2 Overview 181
Determine the Ordered Pair: Mini-Lesson Review 181
Find the Slope and y-intercept: Mini-Lesson Review 184
Find the x- and y-intercepts: Mini-Lesson Review 186
Unit 2 Overview: Wrap Up 188

2.1 Writing and Graphing Systems of Equations 189

2.1.0 Lesson Overview 189
2.1.1 Understanding Constraints and Values 189
2.1.2 Writing and Graphing Equations 190
2.1.3 Meeting Constraints 194
2.1.4 Exploring a System of Equations 196
2.1.5 Practice 198
2.1.6 Lesson Summary 200

2.2 Writing Systems of Equations 201

2.2.0 Lesson Overview 201
2.2.1 Interpreting Graphs of Systems of Equations 201
2.2.2 Writing Systems of Equations from Tables 202
2.2.3 Graphs of Systems of Equations 204
2.2.4 Systems of Equations in the Real-World 209

2.2.5 Practice 210
2.2.6 Lesson Summary 213

2.3 **Solving Systems by Substitution 215**

2.3.0 Lesson Overview 215
2.3.1 Finding Connections between Graphs and Equations 215
2.3.2 Checking Solutions in Systems 216
2.3.3 Solving Equations Using Substitution 218
2.3.4 Solving More Equations 220
2.3.5 Practice 222
2.3.6 Lesson Summary 224

2.4 **Solving Systems by Elimination, Part 1 225**

2.4.0 Lesson Overview 225
2.4.1 Combining Two True Equations 225
2.4.2 Adding Equations 226
2.4.3 Adding and Subtracting Systems of Equations and Using Their Graphs 228
2.4.4 Determining the Best Method for Solving a System of Equations 232
2.4.5 Practice 233
2.4.6 Lesson Summary 235

2.5 **Solving Systems by Elimination, Part 2 237**

2.5.0 Lesson Overview 237
2.5.1 Writing and Evaluating New Equations 237
2.5.2 Adding Two Equations in a System 238
2.5.3 Solving Systems of Linear Equations in Two Variables 240
2.5.4 Solving Systems of Equations Using Real-World Examples 242
2.5.5 Practice 243
2.5.6 Lesson Summary 245

2.6 **Solving Systems by Elimination, Part 3 247**

2.6.0 Lesson Overview 247
2.6.1 Multiplying Equations by a Number 247
2.6.2 Writing a New System to Solve a Given System 248
2.6.3 Finding Solutions to Unordered Sets of Equivalent Systems 250
2.6.4 Building Equivalent Systems 252
2.6.5 Practice 253
2.6.6 Lesson Summary 255

2.7 **Systems of Linear Equations and Their Solutions 257**

2.7.0 Lesson Overview 257
2.7.1 Systems of Linear Equations with Infinitely Many Solutions 257
2.7.2 Systems of Linear Equations with No Solution 257
2.7.3 Sorting Systems of Equations Based on Number of Solutions 259
2.7.4 Writing Consistent and Inconsistent Systems of Equations 260
2.7.5 Practice 261
2.7.6 Lesson Summary 263

2.8 Representing Situations with Inequalities 265

2.8.0 Lesson Overview 265
2.8.1 Understanding Inequality Symbols 265
2.8.2 Identifying Constraints in Inequalities 265
2.8.3 Writing Inequalities to Represent Constraints 267
2.8.4 Writing More Inequalities 268
2.8.5 Practice 269
2.8.6 Lesson Summary 271

2.9 Solutions to Inequalities 273

2.9.0 Lesson Overview 273
2.9.1 Writing Solutions to Inequalities 273
2.9.2 Graphing Inequalities 273
2.9.3 Understanding the Meaning of an Inequality 276
2.9.4 Making Sense of Inequalities and Their Solutions 278
2.9.5 Comparing Equality and Inequality 278
2.9.6 Using Technology to Visualize Inequality Solutions 281
2.9.7 Identifying Inequality Solutions on a Graph 282
2.9.8 Practice 283
2.9.9 Lesson Summary 285

2.10 Writing and Solving Inequalities in One Variable 287

2.10.0 Lesson Overview 287
2.10.1 Writing an Inequality to Represent a Constraint 287
2.10.2 Using Inequalities to Solve a Problem 287
2.10.3 Different Ways of Solving an Inequality 289
2.10.4 Matching Inequalities and Solutions 291
2.10.5 Solving Inequalities 292
2.10.6 Writing Inequalities to Represent a Situation 295
2.10.7 Practice 297
2.10.8 Lesson Summary 299

2.11 Graphing Linear Inequalities in Two Variables 301

2.11.0 Lesson Overview 301
2.11.1 Using Computation and Reasoning 301
2.11.2 Finding Solutions to Inequalities on the Coordinate Plane 301
2.11.3 Sketching Solutions to Inequalities 304
2.11.4 Shading Boundary Line Regions 310
2.11.5 Practice 311
2.11.6 Lesson Summary 313

2.12 Using Linear Inequalities as Constraints 315

2.12.0 Lesson Overview 315
2.12.1 Sketching a Graph to Represent an Equation 315
2.12.2 Writing an Inequality to Represent a Constraint 316
2.12.3 Graphing Solutions and Interpreting Points 317
2.12.4 Recognizing Points Along Boundary Line 320
2.12.5 Recognizing Points for an Inequality 320

2.12.6 Practice 322
2.12.7 Lesson Summary 327

2.13 **Solving Problems with Inequalities in Two Variables 329**

2.13.0 Lesson Overview 329
2.13.1 Graphing Inequalities with Technology 329
2.13.2 Solving Problems with Inequalities in Two Variables 330
2.13.3 Matching Representations of Inequalities 334
2.13.4 Solving Inequality Problems Using Tables 336
2.13.5 Graphing Solutions to an Inequality 342
2.13.6 Practice 344
2.13.7 Lesson Summary 347

2.14 **Solutions to Systems of Linear Inequalities in Two Variables 349**

2.14.0 Lesson Overview 349
2.14.1 Modeling Equations to Solve a Riddle 349
2.14.2 Finding a Pair of Values that Satisfies Multiple Inequalities 349
2.14.3 Writing Systems of Inequalities that Represent Situations 352
2.14.4 Graphing Solutions of Systems of Inequalities 355
2.14.5 Writing Inequalities for Situations and Representing on a Graph 359
2.14.6 Practice 360
2.14.7 Lesson Summary 362

2.15 **Solving Problems with Systems of Linear Inequalities in Two Variables 363**

2.15.0 Lesson Overview 363
2.15.1 Analyzing Graphs that Represent Linear Equations and Inequalities 363
2.15.2 Determining if Points on Boundary Lines Are Solutions to a System 364
2.15.3 Solving Problems to Satisfy Multiple Constraints Simultaneously 368
2.15.4 Identifying Graph Regions Included in a System's Solution 371
2.15.5 Practice 372
2.15.6 Lesson Summary 375

Project 2: Modeling with Systems of Inequalities in Two Variables 377

Project 2 Overview 377
Determine a Solution to Inequalities 377
Interpret a Set of Mathematical Models 377
Perform Mathematical Modeling 378

Unit 2 Wrap Up 380

UNIT 3 TWO-VARIABLE STATISTICS

Unit 3 Overview and Readiness 381

Unit 3 Overview 381
Distinguish Patterns: Mini-Lesson Review 381
Identify Positive and Negative Trends: Mini-Lesson Review 384
Interpret Slope and y-intercept: Mini-Lesson Review 386
Unit 3 Overview: Wrap Up 388

3.1 Linear Models 391

3.1.0 Lesson Overview 391
3.1.1 Exploring Scatter Plots 391
3.1.2 Creating a Scatter Plot Using Data 392
3.1.3 Interpreting the Slope and Vertical Intercept of a Linear Model 396
3.1.4 Interpreting the Slope and Vertical Intercept of a Scatter Plot 398
3.1.5 Using an Equation for a Fit Line 403
3.1.6 Practice 405
3.1.7 Lesson Summary 409

3.2 Fitting Lines 411

3.2.0 Lesson Overview 411
3.2.1 Selecting the Best Line 411
3.2.2 Determining the Best Line That Fits the Data 412
3.2.3 Writing Linear Models without Technology 415
3.2.4 Assessing the Fit of a Linear Model 418
3.2.5 Working with Fitted Lines 423
3.2.6 Practice 424
3.2.7 Lesson Summary 426

3.3 Residuals 427

3.3.0 Lesson Overview 427
3.3.1 Subtracting an Estimated Value from an Actual Value 427
3.3.2 Plotting and Analyzing Residuals 428
3.3.3 Matching Residuals Graphs to Scatter Plots 431
3.3.4 Identifying Best Fit Line for Residuals on a Graph 435
3.3.5 Practice 437
3.3.6 Lesson Summary 440

3.4 The Correlation Coefficient 441

3.4.0 Lesson Overview 441
3.4.1 Comparing Scatter Plots with Linear and Nonlinear Trends 441
3.4.2 Recognizing Differences in Scatter Plots 442
3.4.3 Finding the Correlation Coefficient with Technology 446
3.4.4 Matching Correlation Coefficients 449
3.4.5 Using Correlation Coefficient to Determine Best Fit 458
3.4.6 Practice 459
3.4.7 Lesson Summary 463

3.5 Using the Correlation Coefficient 465

3.5.0 Lesson Overview 465
3.5.1 Using Bivariate Data in Context 465
3.5.2 Finding and Using Correlation Coefficient to Interpret the Strength of Linear Relationships 467
3.5.3 Using Correlation Coefficient to Describe Relationships between Two Variables 471
3.5.4 Interpreting the Correlation Coefficient 473
3.5.5 Practice 474
3.5.6 Lesson Summary 476

3.6 Causal Relationships 477

3.6.0 Lesson Overview 477
3.6.1 Describe the Relationship between Variables 477
3.6.2 Describing How Two Variables Are Related 477
3.6.3 Using the Term Causal Relationship 481
3.6.4 Determining the Type of Relationship 483
3.6.5 Practice 484
3.6.6 Lesson Summary 485

Project 3: Two-Variable Statistics 487

Project 3 Overview 487
Estimate Lengths 487
Analyze Data 487

Unit 3 Wrap Up 488

UNIT 4 FUNCTIONS

Inquiry Project: Defining Functions 489

Inquiry Project Overview 489
Simulating Relationships 489
What is a function? 490
Different Kinds of Functions 492

Unit 4 Overview and Readiness 495

Unit 4 Overview 495
Find Slope: Mini-Lesson Review 495
Describe Graphs: Mini-Lesson Review 499
Extend a Pattern: Mini-Lesson Review 502
Unit 4 Overview: Wrap Up 503

4.1 Describing and Graphing Situations 505

4.1.0 Lesson Overview 505
4.1.1 Contrasting Two Relationships by Reasoning 505
4.1.2 Reasoning Graphically about the Relationship between the Two Quantities 508
4.1.3 Examining Relations and Functions 510
4.1.4 Describing Functional Relationships 513
4.1.5 Modeling Relationships between Two Variables 515
4.1.6 Practice 516
4.1.7 Lesson Summary 518

4.2 Function Notation 519

4.2.0 Lesson Overview 519
4.2.1 Interpreting Graphs to Answer Questions 519
4.2.2 Interpreting Statements Written in Function Notation 521
4.2.3 Finding a Unique Output for Each Input 523
4.2.4 Understanding Function Notation Using Application Problems 526

4.2.5 Practice 527
4.2.6 Lesson Summary 529

4.3 Interpreting Using Function Notation 531

4.3.0 Lesson Overview 531
4.3.1 Comparing Function Values 531
4.3.2 Writing Statements from Function Notation 532
4.3.3 Comparing Statements in Function Notation 534
4.3.4 Using Function Notation with a Real-World Example 535
4.3.5 Practice 537
4.3.6 Lesson Summary 539

4.4 Using Function Notation to Describe Rules, Part 1 541

4.4.0 Lesson Overview 541
4.4.1 Using Function Notation in Tables of Values 541
4.4.2 Defining Functions by a Rule 542
4.4.3 Using Tables to Find Function Rules 544
4.4.4 Writing a Rule Using Function Notation 547
4.4.5 Practice 548
4.4.6 Lesson Summary 550

4.5 Using Function Notation to Describe Rules, Part 2 551

4.5.0 Lesson Overview 551
4.5.1 Finding Solutions of Equations 551
4.5.2 Using the Vertical Line Test 551
4.5.3 Comparing Functions 555
4.5.4 Using Function Notation and Graphing Technology 556
4.5.5 Using Graphical and Algebraic Approaches for Finding Unknowns of Linear Functions 558
4.5.6 Practice 559
4.5.7 Lesson Summary 561

4.6 Features of Graphs 563

4.6.0 Lesson Overview 563
4.6.1 Interpreting Statements in Function Notation 563
4.6.2 Analyzing Graphs of Functions 564
4.6.3 Connecting Graphical and Verbal Representations of a Function 566
4.6.4 Key Features of Linear Functions 569
4.6.5 Interpreting Graphs 572
4.6.6 Practice 573
4.6.7 Lesson Summary 576

4.7 Finding Slope 577

4.7.0 Lesson Overview 577
4.7.1 Evaluating Fractions 577
4.7.2 Finding Slope From Tables, Graphs, and Points 577
4.7.3 Writing Linear Equations 582
4.7.4 Writing Equations From Two Points 584

4.7.5 Practice 586
4.7.6 Lesson Summary 587

4.8 **Using Graphs to Find Average Rate of Change 589**

4.8.0 Lesson Overview 589
4.8.1 Comparing Changes in Output Based on Input 589
4.8.2 Average Rate of Change and Slope 589
4.8.3 Finding and Interpreting Average Rates of Change 593
4.8.4 Rate of Change of Linear Functions 595
4.8.5 Interpreting Average Rates of Change with Real-World Examples 599
4.8.6 Practice 600
4.8.7 Lesson Summary 603

4.9 **Interpreting and Creating Graphs 605**

4.9.0 Lesson Overview 605
4.9.1 Analyzing and Comparing the Properties of Graphs 605
4.9.2 Interpreting Graphs Without Units 607
4.9.3 Sketching Graphs of Functions 612
4.9.4 Representing Quantities in a Situation 614
4.9.5 Using Verbal Descriptions to Create Graphs of Functions 615
4.9.6 Practice 617
4.9.7 Lesson Summary 619

4.10 **Comparing Graphs 621**

4.10.0 Lesson Overview 621
4.10.1 Analyzing Graphs and Statements in Function Notation 621
4.10.2 Interpreting Graphs and Statements in Terms of a Situation 622
4.10.3 Comparing Functions Represented in Separate Graphs 624
4.10.4 Comparing Graphs and Statements that Represent Functions without a Context 628
4.10.5 Analyzing Functions Using Real-World Context 630
4.10.6 Practice 631
4.10.7 Lesson Summary 634

4.11 **Graphing a Function Using Transformations 635**

4.11.0 Lesson Overview 635
4.11.1 Graphing Linear Functions 635
4.11.2 Vertical Shifts 636
4.11.3 Horizontal Shifts 637
4.11.4 Vertical Stretches and Compressions 642
4.11.5 Horizontal Stretches and Compressions 645
4.11.6 Graphing Using Transformations 652
4.11.7 Practice 653
4.11.8 Lesson Summary 656

4.12 **Domain and Range, Part 1 657**

4.12.0 Lesson Overview 657
4.12.1 Determining Reasonable Inputs and Outputs 657
4.12.2 Domain: The Input of a Function 658

4.12.3 Range: The Output of a Function 660
4.12.4 Using Graphs to Determine the Domain of a Function 664
4.12.5 Using Functions to Answer Real-World Questions 664
4.12.6 Practice 665
4.12.7 Lesson Summary 667

4.13 Domain and Range, Part 2 669

4.13.0 Lesson Overview 669
4.13.1 Different Features of a Graph 669
4.13.2 Graphs of Functions 670
4.13.3 Finding Domain and Range Using a Graph 674
4.13.4 Real-World Domain and Range 677
4.13.5 Graph of a Function and the Domain and Range Connection 680
4.13.6 Practice 681
4.13.7 Lesson Summary 684

4.14 Sequences 685

4.14.0 Lesson Overview 685
4.14.1 Describe the Pattern 685
4.14.2 What Is a Sequence? 685
4.14.3 The Connection between Sequences and Terms 686
4.14.4 Rules for a Sequence 688
4.14.5 Practice 689
4.14.6 Lesson Summary 690

4.15 Introducing Geometric Sequences 691

4.15.0 Lesson Overview 691
4.15.1 What Are the Patterns? 691
4.15.2 Using Tables and Graphs to Represent Geometric Sequences 691
4.15.3 Complete the Sequence 693
4.15.4 Connecting the Common Ratio and the Term in a Sequence 695
4.15.5 Practice 696
4.15.6 Lesson Summary 698

4.16 Introducing Arithmetic Sequences 699

4.16.0 Lesson Overview 699
4.16.1 Interpreting Function Notation 699
4.16.2 What Is an Arithmetic Sequence? 699
4.16.3 A Sequence Is a Type of Function 701
4.16.4 Identify and Generate Sequences 702
4.16.5 Practice 703
4.16.6 Lesson Summary 705

4.17 Representing Sequences 707

4.17.0 Lesson Overview 707
4.17.1 Define a Sequence 707
4.17.2 What Is a Recursive Definition? 707
4.17.3 Ways to Represent a Sequence 711

4.17.4 Geometric Recursive Definitions 713
4.17.5 Practice 714
4.17.6 Lesson Summary 716

4.18 The n^{th} Term of a Sequence 717

4.18.0 Lesson Overview 717
4.18.1 Repeated Operations 717
4.18.2 Identifying Domain for a Function 717
4.18.3 Define an Arithmetic Sequence by the n^{th} Term 718
4.18.4 Define a Geometric Sequence by the nth Term 720
4.18.5 Interchanging between Formulas 722
4.18.6 Different Types of Equations 726
4.18.7 Practice 727
4.18.8 Lesson Summary 729

Project 4: Using Functions to Model Battery Power 731

Project 4 Overview 731
Estimate Time to Charge a Battery 731
Analyze Percent Charge Over Time 731
Write an Equation to Model Data 732

Unit 4 Wrap Up 734

UNIT 5 INTRODUCTION TO EXPONENTIAL FUNCTIONS

Unit 5 Overview and Readiness 735

Unit 5 Overview 735
Write Linear Functions from a Graph: Mini-Lesson Review 736
Interpret Linear Functions: Mini-Lesson Review 738
Evaluate Exponential Expressions: Mini-Lesson Review 739
Unit 5 Overview: Wrap Up 741

5.1 Properties of Exponents 743

5.1.0 Lesson Overview 743
5.1.1 Understanding Exponents 743
5.1.2 Using Product and Quotient Properties for Exponents 743
5.1.3 Using Zero Exponent Property and Negative Exponents 747
5.1.4 Using Power Properties for Exponents 749
5.1.5 Simplifying Exponents 753
5.1.6 Practice 755
5.1.7 Lesson Summary 757

5.2 Rational Exponents 759

5.2.0 Lesson Overview 759
5.2.1 Perfect Squares Revisited 759
5.2.2 The nth Root 759
5.2.3 Radicals and Rational Exponents 761
5.2.4 Properties of Exponents with Rationals 763

5.2.5 Practice 764
5.2.6 Lesson Summary 766

5.3 **Patterns of Growth 767**

5.3.0 Lesson Overview 767
5.3.1 Creating a Pictorial Representation of Exponential Growth 768
5.3.2 Exploring Different Growth Patterns 768
5.3.3 Compare and Contrast Two Patterns 771
5.3.4 Match Expressions and Tables with Situations 775
5.3.5 Describe Mathematically Changes in Growth 777
5.3.6 Practice 778
5.3.7 Lesson Summary 781

5.4 **Representing Exponential Growth 783**

5.4.0 Lesson Overview 783
5.4.1 Exponent Rules 783
5.4.2 Zero Exponent Rule 783
5.4.3 Exponential Change: The Growth Factor 785
5.4.4 Graphing Exponential Expressions 788
5.4.5 Interpret the Growth Factor of the Equation 792
5.4.6 Practice 793
5.4.7 Lesson Summary 796

5.5 **Representing Exponential Decay 797**

5.5.0 Lesson Overview 797
5.5.1 Identifying Linear and Exponential Quantities 797
5.5.2 Exponential Decay 797
5.5.3 Using Graphs to Represent Exponential Decay 799
5.5.4 Justifying Exponential Models 804
5.5.5 Writing Equations to Represent Exponential Decay 806
5.5.6 Practice 807
5.5.7 Lesson Summary 810

5.6 **Negative Exponents and Scientific Notation 811**

5.6.0 Lesson Overview 811
5.6.1 Checking Exponent Rules 811
5.6.2 Interpreting Negative Exponents in Exponential Growth 812
5.6.3 Interpreting Negative Exponents in Exponential Decay 814
5.6.4 Using Scientific Notation 818
5.6.5 Applying Scientific Notation 821
5.6.6 Interpreting Negative Exponents and Scientific Notation 824
5.6.7 Practice 825
5.6.8 Lesson Summary 827

5.7 **Analyzing Graphs 829**

5.7.0 Lesson Overview 829
5.7.1 Patterns in Real Numbers 829
5.7.2 Comparing Graphs 829

5.7.3 Describing Graphs 833
5.7.4 Quantities that Change Exponentially 837
5.7.5 Practice 839
5.7.6 Lesson Summary 842

5.8 **Exponential Situations as Functions 843**

5.8.0 Lesson Overview 843
5.8.1 Meaning of a Function Presented Graphically 843
5.8.2 Represent a Function as a Table of Values, Graph, and Equation 843
5.8.3 Using Function Language and Notation 845
5.8.4 Function Notation 847
5.8.5 Practice 848
5.8.6 Lesson Summary 850

5.9 **Interpreting Exponential Functions 851**

5.9.0 Lesson Overview 851
5.9.1 Evaluating Expressions for Different Values of x 851
5.9.2 Analyze Underlying Relationship in a Graph of a Function 851
5.9.3 Graph Equations to Solve Problems 853
5.9.4 Deciding on Graphing Window 855
5.9.5 Interpret Graphs to Find Approximate Values 856
5.9.6 Practice 858
5.9.7 Lesson Summary 859

5.10 **Looking at Rates of Change 861**

5.10.0 Lesson Overview 861
5.10.1 Calculate an Average Rate of Change from Two Points 861
5.10.2 Explore Average Rates of Change in an Exponential Growth Function 862
5.10.3 Explore Average Rates of Change in an Exponential Decay Context 865
5.10.4 Average Rate of Change 867
5.10.5 Practice 868
5.10.6 Lesson Summary 870

5.11 **Modeling Exponential Behavior 871**

5.11.0 Lesson Overview 871
5.11.1 Changing the Graphing Window 871
5.11.2 Choosing an Appropriate Model 872
5.11.3 Modeling with Exponential Functions 875
5.11.4 Examine Exponential Decay in Context 876
5.11.5 Using Relationship Models to Answer Questions 880
5.11.6 Practice 881
5.11.7 Lesson Summary 882

5.12 **Reasoning about Exponential Graphs, Part 1 883**

5.12.0 Lesson Overview 883
5.12.1 Modeling Exponential Decay 883
5.12.2 Equations and Their Graphs 883
5.12.3 Graphs Representing Exponential Decay 885

5.12.4 Possible Equation for a Function on a Graph 888
5.12.5 Practice 890
5.12.6 Lesson Summary 893

5.13 **Reasoning about Exponential Graphs, Part 2 895**

5.13.0 Lesson Overview 895
5.13.1 Comparing Functions 895
5.13.2 Creating Exponential Functions from Graphs 895
5.13.3 Comparing Exponential Functions 899
5.13.4 Interpreting Features of an Exponential Graph 901
5.13.5 Practice 902
5.13.6 Lesson Summary 904

5.14 **Which One Changes Faster? 905**

5.14.0 Lesson Overview 905
5.14.1 Distinguish Linear and Exponential Growth 905
5.14.2 Using Tables to Compare Linear and Exponential Growth Functions 906
5.14.3 Compare Linear Functions with Exponential Functions 909
5.14.4 Using Function Notation to Compare Linear and Exponential Functions 913
5.14.5 Practice 914
5.14.6 Lesson Summary 916

5.15 **Changes Over Equal Intervals 917**

5.15.0 Lesson Overview 917
5.15.1 Writing Equivalent Expressions 917
5.15.2 Linear Function Constant Rate of Change 917
5.15.3 Growth Rate of an Exponential Function 919
5.15.4 Comparing Linear and Exponential Growth 922
5.15.5 Practice 924
5.15.6 Lesson Summary 926

Project 5: Introduction to Exponential Functions 927

Project 5 Overview 927
Analyze Data 927
Write Equations to Model Populations 927
Open-ended Modeling with Data Investigation 928

Unit 5 Wrap Up 930

UNIT 6 WORKING WITH POLYNOMIALS

Unit 6 Overview and Readiness 931

Unit 6 Overview 931
Simplify Expressions Using Properties of Exponents: Mini-Lesson Review 931
Simplify Expressions Using the Distributive Property: Mini-Lesson Review 934
Find the Greatest Common Factor: Mini-Lesson Review 935
Unit 6 Overview: Wrap Up 937

Unit 6 Inquiry Project: Area Model Multiplication 939

Inquiry Project Overview 939
Discovering an Area Model 939
Understanding an Area Model 940
Using an Area Model 941
Applying an Area Model to Polynomials 942

6.1 Add and Subtract Polynomials 945

6.1.0 Lesson Overview 945
6.1.1 Understanding Polynomials 945
6.1.2 Adding and Subtracting Polynomials 946
6.1.3 Evaluating a Polynomial Function for a Given Value 949
6.1.4 Adding and Subtracting Polynomial Functions 951
6.1.5 Reviewing Polynomial Addition and Subtraction 954
6.1.6 Practice 955
6.1.7 Lesson Summary 956

6.2 Multiplying Polynomials 957

6.2.0 Lesson Overview 957
6.2.1 Multiplying Monomials 957
6.2.2 Multiplying Binomials 958
6.2.3 Multiplying a Polynomial by a Polynomial 963
6.2.4 Multiplying Special Products 965
6.2.5 Multiplying Polynomial Functions 968
6.2.6 Practice 969
6.2.7 Lesson Summary 971

6.3 Dividing Polynomials 973

6.3.0 Lesson Overview 973
6.3.1 Dividing by Monomials 973
6.3.2 Dividing Polynomials Using Long Division 974
6.3.3 Dividing Polynomials Using Synthetic Division 979
6.3.4 Dividing Polynomial Functions and the Remainder Theorem 982
6.3.5 Using the Factor Theorem 985
6.3.6 Practice 987
6.3.7 Lesson Summary 989

6.4 Greatest Common Factor and Factor by Grouping 991

6.4.0 Lesson Overview 991
6.4.1 Finding Factors 991
6.4.2 Finding the GCF of Two or More Expressions 991
6.4.3 Factoring the GCF from Polynomials 993
6.4.4 Factoring Polynomials by Grouping 996
6.4.5 Can You Factor the Polynomial? 998
6.4.6 Practice 999
6.4.7 Lesson Summary 1000

6.5 Factor Trinomials 1001

6.5.0 Lesson Overview 1001
6.5.1 Undoing FOIL 1001
6.5.2 Factoring Trinomials with Leading Coefficients of 1 1002
6.5.3 Factoring Trinomials Using Trial and Error 1007
6.5.4 Factoring Trinomials Using the "ac" Method and Substitution 1011
6.5.5 Choosing Your Own Method to Factor Trinomials 1014
6.5.6 Practice 1015
6.5.7 Lesson Summary 1017

6.6 Factor Special Products 1019

6.6.0 Lesson Overview 1019
6.6.1 Identifying Perfect Square Trinomials 1019
6.6.2 Factoring Perfect Square Trinomials 1019
6.6.3 Factoring the Difference of Squares 1024
6.6.4 Sorting Polynomials by Factoring Methods 1027
6.6.5 Practice 1029
6.6.6 Lesson Summary 1031

6.7 General Strategy for Factoring Polynomials 1033

6.7.0 Lesson Overview 1033
6.7.1 Factoring Strategies for Polynomials 1033
6.7.2 Reviewing General Strategies to Factor a Polynomial 1033
6.7.3 Implementing General Strategies for Factoring 1036
6.7.4 Is the Polynomial Completely Factored? 1039
6.7.5 Practice 1040
6.7.6 Lesson Summary 1042

Project 6: Polynomials and Rectangles 1043

Project 6 Overview 1043
Areas of Rectangles 1043
Dimensions and Areas of Rectangles 1044
Create Your Own Rectangles 1044

Unit 6 Wrap Up 1046

UNIT 7 INTRODUCTION TO QUADRATIC FUNCTIONS

Unit 7 Overview and Readiness 1047

Unit 7 Overview 1047
Multiply Binomials: Mini-Lesson Review 1047
Factor Trinomials: Mini-Lesson Review 1049
Compare Linear and Exponential Functions: Mini-Lesson Review 1051
Unit 7 Overview: Wrap Up 1055

7.1 Patterns of Change 1057

7.1.0 Lesson Overview 1057
7.1.1 Noticing New Patterns of Change 1058

7.1.2 The Relationship between Length and Area 1059
7.1.3 Plotting Measurements 1060
7.1.4 The Input and Output of a Function 1062
7.1.5 Practice 1063
7.1.6 Lesson Summary 1065

7.2 Introduction to Quadratic Relationships 1067

7.2.0 Lesson Overview 1067
7.2.1 Squares in a Geometric Pattern 1067
7.2.2 Constant and Exponential Change 1068
7.2.3 Quadratic Expressions 1071
7.2.4 Quadratic Relationships 1073
7.2.5 Practice 1075
7.2.6 Lesson Summary 1077

7.3 Determining if a Function is Quadratic 1079

7.3.0 Lesson Overview 1079
7.3.1 Quadratic Expressions and Area 1079
7.3.2 Writing Equations for Patterns with Squares 1080
7.3.3 Quadratic Sequences 1082
7.3.4 Is the Pattern a Quadratic Function? 1084
7.3.5 Practice 1085
7.3.6 Lesson Summary 1087

7.4 Comparing Quadratic and Exponential Functions 1089

7.4.0 Lesson Overview 1089
7.4.1 Comparing Expressions 1089
7.4.2 Exploring Linear, Exponential, and Quadratic Growth 1089
7.4.3 Comparing Exponential and Quadratic Functions 1092
7.4.4 Comparing Exponential and Quadratic Expressions 1094
7.4.5 Practice 1095
7.4.6 Lesson Summary 1097

7.5 Building Quadratic Functions to Describe Situations, Part 1 1099

7.5.0 Lesson Overview 1099
7.5.1 Numerical Patterns 1099
7.5.2 A Time-Distance Quadratic Model 1099
7.5.3 Distance as a Quadratic Function of Elapsed Time 1101
7.5.4 A Quadratic Expression that Represents the Relationship between Distance and Time 1103
7.5.5 Practice 1104
7.5.6 Lesson Summary 1106

7.6 Building Quadratic Functions to Describe Situations, Part 2 1107

7.6.0 Lesson Overview 1107
7.6.1 Using Linear Functions to Describe Constant Speed 1107
7.6.2 The Force of Gravity Change in Quadratic Functions 1108
7.6.3 Using Quadratic Functions to Describe Height 1109
7.6.4 Interpreting Graphs of Quadratic Functions 1111

7.6.5 Practice 1112
7.6.6 Lesson Summary 1114

7.7 **Domain, Range, Vertex, and Zeros of Quadratic Functions 1115**

7.7.0 Lesson Overview 1115
7.7.1 Comparing Graphs of Functions 1116
7.7.2 Modeling Real-World Data with Quadratic Functions 1116
7.7.3 The Domain, Vertex, and Zero of Quadratic Functions 1119
7.7.4 Relate the Domain of a Function to Its Graph 1123
7.7.5 Practice 1124
7.7.6 Lesson Summary 1127

7.8 **Equivalent Quadratic Expressions 1129**

7.8.0 Lesson Overview 1129
7.8.1 Area Diagrams 1129
7.8.2 Using the Distributive Property to Write Equivalent Expressions 1129
7.8.3 Using Diagrams to Find Equivalent Quadratic Expressions 1131
7.8.4 Writing Equivalent Expressions 1133
7.8.5 Practice 1135
7.8.6 Lesson Summary 1137

7.9 **Standard Form and Factored Form 1139**

7.9.0 Lesson Overview 1139
7.9.1 Solving Equations Using Opposite Operations 1139
7.9.2 Finding Products of Differences 1139
7.9.3 Standard and Factored Forms of Quadratic Expressions 1141
7.9.4 Expressions in Standard Form 1142
7.9.5 Practice 1143
7.9.6 Lesson Summary 1144

7.10 **Graphs of Functions in Standard and Factored Forms 1145**

7.10.0 Lesson Overview 1145
7.10.1 Graphs of Linear Equations 1145
7.10.2 Quadratic Forms and Their Graphs 1146
7.10.3 The *x*- and *y*-intercepts of Quadratic Expressions 1148
7.10.4 Using the Standard Form of an Equation to Find the *x*- and *y*-intercepts 1152
7.10.5 Practice 1153
7.10.6 Lesson Summary 1154

7.11 **Graphing from the Factored Form 1155**

7.11.0 Lesson Overview 1155
7.11.1 Finding Coordinates 1155
7.11.2 The Factored Form of a Quadratic Expression and the x-intercepts of Its Graph 1156
7.11.3 Sketching a Graph of a Quadratic Function Using at Least Three Identifiable Points 1157
7.11.4 Using the Vertex and Axis of Symmetry of Quadratics 1159
7.11.5 Graphs of Quadratic Functions 1161
7.11.6 Practice 1162
7.11.7 Lesson Summary 1164

7.12 Graphing the Standard Form, Part 1 1165

7.12.0 Lesson Overview 1165
7.12.1 Using Coefficients and Constant Terms to Identify Graphs of Linear Equations 1165
7.12.2 Transformations with Quadratic Functions 1166
7.12.3 Understanding the Behaviors of a Graph in Relation to Its Quadratic Expression 1170
7.12.4 Representations of Quadratic Functions 1177
7.12.5 Identifying Graphs of Quadratic Equations 1179
7.12.6 Practice 1180
7.12.7 Lesson Summary 1184

7.13 Graphing the Standard Form, Part 2 1185

7.13.0 Lesson Overview 1185
7.13.1 Equivalent Expressions 1185
7.13.2 The Linear Term in a Quadratic Expression 1186
7.13.3 Writing Equations that Represent a Graph 1188
7.13.4 Writing Quadratic Equations from Real Solutions 1190
7.13.5 Graphing Quadratic Equations 1191
7.13.6 Practice 1193
7.13.7 Lesson Summary 1197

7.14 Graphs That Represent Situations 1199

7.14.0 Lesson Overview 1199
7.14.1 Evaluating Quadratic Functions in the Real World 1199
7.14.2 Interpreting a Functional Relationship between Two Quantities 1200
7.14.3 Analyzing Functions Using Different Representations 1202
7.14.4 Using Quadratic Functions to Solve Problems 1203
7.14.5 Identifying Key Features of Graphed Functions 1204
7.14.6 Practice 1205
7.14.7 Lesson Summary 1207

7.15 Vertex Form 1209

7.15.0 Lesson Overview 1209
7.15.1 Analyzing Two Sets of Equations 1209
7.15.2 The Vertex Form 1209
7.15.3 The Coordinates of the Vertex 1212
7.15.4 The Graph of a Function in Vertex Form 1214
7.15.5 Practice 1215
7.15.6 Lesson Summary 1217

7.16 Graphing from the Vertex Form 1219

7.16.0 Lesson Overview 1219
7.16.1 Three Forms of Quadratic Expressions 1219
7.16.2 Graph Functions in Vertex Form 1219
7.16.3 Quadratic Equations and Graphs 1222
7.16.4 Sketching A Graph of an Equation 1226
7.16.5 Practice 1227
7.16.6 Lesson Summary 1229

7.17 Changing the Vertex 1231

7.17.0 Lesson Overview 1231
7.17.1 Quadratic Expressions in Different Forms 1231
7.17.2 Modify Expressions to Translate Graphs 1232
7.17.3 Changing the Parameters of a Quadratic Expression 1236
7.17.4 Transforming the Parent Function 1238
7.17.5 Equations of Graph Translations 1239
7.17.6 Practice 1240
7.17.7 Lesson Summary 1243

Project 7: Design a Fountain 1245

Project 7 Overview 1245
Introduction to Fountain Design 1245
Design a Fountain 1245

Unit 7 Wrap Up 1248

UNIT 8 QUADRATIC EQUATIONS

Unit 8 Overview and Readiness 1249

Unit 8 Overview 1249
Write Equivalent Expressions: Mini-Lesson Review 1249
Find Intercepts: Mini-Lesson Review 1250
Match Quadratic Graphs and Their Equations: Mini-Lesson Review 1252
Unit 8 Overview: Wrap Up 1254

8.1 Finding Unknown Inputs 1255

8.1.0 Lesson Overview 1255
8.1.1 Understanding a Situation with Quadratic Equations 1255
8.1.2 Modeling a Quadratic Problem 1256
8.1.3 Formulating a Quadratic Equation to Represent the Model 1257
8.1.4 Interpreting a Solution 1260
8.1.5 Practice 1261
8.1.6 Lesson Summary 1263

8.2 When and Why Do We Write Quadratic Equations? 1265

8.2.0 Lesson Overview 1265
8.2.1 Interpreting Equations in Context 1265
8.2.2 Exploring Non-Graphing Strategies to Solve Quadratic Equations 1266
8.2.3 Solving a Quadratic Equation Set Equal to Zero 1267
8.2.4 Solving a Real-World Problem Using Quadratic Equations 1269
8.2.5 Practice 1270
8.2.6 Lesson Summary 1272

8.3 Solving Quadratic Equations by Reasoning 1273

8.3.0 Lesson Overview 1273
8.3.1 Determining the Number of Solutions 1273
8.3.2 Recognizing Pairs of Solutions 1273

8.3.3 Solving More Complex Quadratic Equations 1276
8.3.4 Finding the Solutions 1278
8.3.5 Practice 1279
8.3.6 Lesson Summary 1281

8.4 Solving Quadratic Equations with the Zero Product Property 1283

8.4.0 Lesson Overview 1283
8.4.1 Introducing the Zero Product Property 1283
8.4.2 Solving Equations of Increasing Complexity Using Reasoning 1283
8.4.3 Applying the Zero Product Property to Solve a Real-World Projectile Problem 1285
8.4.4 Solving Equations Using the Zero Product Property and Reasoning 1287
8.4.5 Practice 1288
8.4.6 Lesson Summary 1290

8.5 How Many Solutions? 1291

8.5.0 Lesson Overview 1291
8.5.1 Reviewing Zero Product Property 1291
8.5.2 Interpreting Graphs to Solve Quadratic Equations 1291
8.5.3 Choosing an Effective Strategy to Solve Quadratic Equations 1295
8.5.4 Analyzing Errors When Solving Quadratic Equations 1297
8.5.5 Practice 1298
8.5.6 Lesson Summary 1300

8.6 Rewriting Quadratic Expressions in Factored Form, Part 1 1301

8.6.0 Lesson Overview 1301
8.6.1 Finding and Reasoning Unknown Factors 1301
8.6.2 Using Diagrams to Understand Equivalent Expressions 1302
8.6.3 Rewriting Quadratic Expressions in Standard Form 1306
8.6.4 Finding the Missing Numbers in Pairs of Equivalent Expressions 1308
8.6.5 Practice 1309
8.6.6 Lesson Summary 1311

8.7 Rewriting Quadratic Expressions in Factored Form, Part 2 1313

8.7.0 Lesson Overview 1313
8.7.1 Understanding Sums and Products of Integers 1313
8.7.2 Interpreting Negative Constant Terms When Factoring Quadratic Expressions 1313
8.7.3 Analyzing Factors of 100 and –100 1316
8.7.4 Finding the Missing Terms 1320
8.7.5 Practice 1321
8.7.6 Lesson Summary 1323

8.8 Rewriting Quadratic Expressions in Factored Form, Part 3 1325

8.8.0 Lesson Overview 1325
8.8.1 Evaluating Expressions Using Mental Math 1325
8.8.2 Recognizing the Expanded Product of the Difference of Two Squares 1325
8.8.3 Factoring Quadratic Equations without a Linear Term 1328
8.8.4 Determining if an Expression Can Be Rewritten in Factored Form 1330

8.8.5 Practice 1331
8.8.6 Lesson Summary 1333

8.9 Solving Quadratic Equations by Using Factored Form 1335

8.9.0 Lesson Overview 1335
8.9.1 Finding a Solution through Substitution 1335
8.9.2 Using Factored Form and the Zero Product Property to Solve Quadratic Equations 1335
8.9.3 Writing an Equation to Represent a Quadratic Function with Only One Solution 1337
8.9.4 Solving More Quadratic Equations 1339
8.9.5 Practice 1340
8.9.6 Lesson Summary 1342

8.10 Rewriting Quadratic Expressions in Factored Form, Part 4 1343

8.10.0 Lesson Overview 1343
8.10.1 Analyzing Various Quadratic Expressions 1343
8.10.2 Working with Quadratic Factored Form with Leading Coefficient Other Than One 1343
8.10.3 Using Technology to Find Rational Factors 1347
8.10.4 Finding the Factors of Quadratic Expressions in Standard Form 1349
8.10.5 Solving Quadratic Equations by Any Method 1351
8.10.6 Practice 1353
8.10.7 Lesson Summary 1355

8.11 Writing Quadratic Equations Given Real Solutions 1357

8.11.0 Lesson Overview 1357
8.11.1 Relating a Quadratic Function and Its Zeros 1357
8.11.2 Finding a Quadratic Function from Its Zeros 1357
8.11.3 Finding a Quadratic Function from Its Zeros and a Point 1360
8.11.4 Writing Quadratics When Given Any Format 1364
8.11.5 Practice 1365
8.11.6 Lesson Summary 1367

8.12 Using Technology to Find the Quadratic Regression 1369

8.12.0 Lesson Overview 1369
8.12.1 Graphing a Quadratic Data Set Using Technology 1369
8.12.2 Finding the Curve of Best Fit 1370
8.12.3 Making Predictions Using a Quadratic Model 1373
8.12.4 Finding the Missing Data in the Set 1375
8.12.5 Practice 1377
8.12.6 Lesson Summary 1380

Project 8: Modeling Rocket Flight 1381

Project 8 Overview 1381
Relate Zeros to the Factored Form of Quadratic Equations 1381
Convert between Factored Form and Standard Form 1382
Model a Rocket Flight Using a Quadratic Equation 1382

Unit 8 Wrap Up 1384

UNIT 9 MORE QUADRATIC EQUATIONS

Unit 9 Overview and Readiness 1385

Unit 9 Overview 1385
Simplifying Radicals: Mini-Lesson Review 1385
Factoring Perfect Square Trinomials: Mini-Lesson Review 1387
Transformations: Mini-Lesson Review 1389
Unit 9 Overview: Wrap Up 1390

9.1 What Are Perfect Squares? 1391

9.1.0 Lesson Overview 1391
9.1.1 Equations with Quadratic Expressions on Both Sides of the Equal Sign 1391
9.1.2 Recognizing Structure in Perfect-Square Expressions 1391
9.1.3 Solving Multi Step Quadratic Equations Using Perfect Squares 1393
9.1.4 Explain Why an Expression Is a Perfect Square 1395
9.1.5 Practice 1396
9.1.6 Lesson Summary 1398

9.2 Completing the Square, Part 1 1399

9.2.0 Lesson Overview 1399
9.2.1 Different Forms of Perfect Squares 1399
9.2.2 Standard and Factored Forms of Perfect Squares 1399
9.2.3 Completing the Square 1401
9.2.4 Making a Perfect Square 1404
9.2.5 Practice 1405
9.2.6 Lesson Summary 1408

9.3 Completing the Square, Part 2 1409

9.3.0 Lesson Overview 1409
9.3.1 Solving Equations with Fractions 1409
9.3.2 Using Completing the Square to Solve Equations 1409
9.3.3 Solve by Completing the Square 1412
9.3.4 Completing the Square with Fractions 1414
9.3.5 Practice 1415
9.3.6 Lesson Summary 1418

9.4 Completing the Square, Part 3 1419

9.4.0 Lesson Overview 1419
9.4.1 Perfect Squares with Coefficients Other than 1 1419
9.4.2 Rewrite Squared Expressions 1419
9.4.3 Standard Form and Squared Factors 1421
9.4.4 Three Different Methods for Solving an Equation 1423
9.4.5 Perfect Square and Squared Factors 1424
9.4.6 Practice 1425
9.4.7 Lesson Summary 1427

9.5 Quadratic Equations with Irrational Solutions 1429

9.5.0 Lesson Overview 1429
9.5.1 Roots of Squares 1429

9.5.2 Solutions Written as Square Roots 1429
9.5.3 Finding Irrational Solutions by Completing the Square 1431
9.5.4 Finding Exact Solutions 1433
9.5.5 Practice 1434
9.5.6 Lesson Summary 1438

9.6 The Quadratic Formula 1439

9.6.0 Lesson Overview 1439
9.6.1 Evaluating Expressions 1439
9.6.2 Choosing the Best Method to Solve an Equation 1439
9.6.3 The Quadratic Formula 1441
9.6.4 Solving Equations Using the Quadratic Formula 1443
9.6.5 Practice 1444
9.6.6 Lesson Summary 1446

9.7 Applying the Quadratic Formula 1447

9.7.0 Lesson Overview 1447
9.7.1 Evaluating Variable Expressions 1447
9.7.2 Common Calculation Errors When Using the Quadratic Formula 1447
9.7.3 Different Methods of Checking Solutions of Quadratic Equations 1450
9.7.4 Practice Spotting Calculation Errors 1452
9.7.5 Practice 1453
9.7.6 Lesson Summary 1455

9.8 Deriving the Quadratic Formula 1457

9.8.0 Lesson Overview 1457
9.8.1 Deriving the Quadratic Formula, Part 1 1458
9.8.2 Deriving the Quadratic Formula, Part 2 1460
9.8.3 Understanding That the Quadratic Formula Is the Combined Steps of Completing the Square 1462
9.8.4 Practice 1464
9.8.5 Lesson Summary 1466

9.9 Writing Quadratics in Different Forms 1467

9.9.0 Lesson Overview 1467
9.9.1 Finding the Vertex in Vertex Form 1467
9.9.2 Different Forms of Quadratics 1467
9.9.3 Vertex Form Given a Vertex and Point 1469
9.9.4 Rewriting Quadratics 1470
9.9.5 Practice 1472
9.9.6 Lesson Summary 1473

9.10 Rewriting Quadratic Expressions in Vertex Form 1475

9.10.0 Lesson Overview 1475
9.10.1 The Vertex and Intercepts of a Function 1475
9.10.2 Expanding from Factored Form to Standard Form 1476
9.10.3 Converting from Standard Form to Vertex Form 1477
9.10.4 Rewriting Expressions in Vertex Form 1478

9.10.5 Different Forms of Quadratic Expressions 1479
9.10.6 Vertex Form and Coordinates of the Vertex 1480
9.10.7 Practice 1481
9.10.8 Lesson Summary 1484

9.11 Using Quadratic Expressions in Vertex Form to Solve Problems 1485

9.11.0 Lesson Overview 1485
9.11.1 Maximum and Minimum Value of a Function 1485
9.11.2 Does the Vertex Represent the Minimum or Maximum Value? 1486
9.11.3 Comparing Maximums between Quadratics 1489
9.11.4 Maximum, Minimum, and Analyzing the Vertex 1491
9.11.5 Practice 1492
9.11.6 Lesson Summary 1494

Project 9: Using Quadratic Equations to Model Situations and Solve Problems 1495

Project 9 Overview 1495
Equations of Two Lines and a Curve 1495
Analyze a Dive 1495
A Linear Function and a Quadratic Function 1495
Profit from a River Cruise 1496

Unit 9 Wrap Up 1498

Index 1499

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OpenStax *Algebra 1* is a complete curriculum and an open education resource (OER) that is standards-aligned, easy to implement, supportive of diverse learners, and engaging for students. *Algebra 1* is supported by instructional resources, including implementation guidance, pacing suggestions, and ancillary engagement materials.

Senior Contributing Authors:

Lynn Marecek, Santa Ana College
MaryAnne Anthony-Smith, Formerly of Santa Ana College
Andrea Honeycutt Mathis, Northeast Mississippi Community College
Jay Abramson, Arizona State University
Sharon North, St. Louis Community College
Amy Baldwin, University of Central Arkansas
Alyssa Howell, OpenStax, Rice University

Errata

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STUDENTS START HERE

Quick Start

Welcome to your *Algebra 1* module! We're genuinely excited for you to join our community and embark on a transformative learning journey!

This user-friendly guide is your roadmap to navigating the features and resources of our unique *Algebra 1* content. Let's raise the bar on your algebra skills—get started with confidence today!

I. Where to Start

Algebra 1 has nine units, each of which begins with a special lesson called Overview and Readiness. These lessons provide a miniature review of the knowledge you need for the unit ahead. Completing these lessons will help you feel confident in your ability to handle what comes next.

Units 4 and 6 also have inquiry projects, meant to revisit previous knowledge while introducing you to upcoming topics. We recommend completing these alongside their respective Overview and Readiness lessons.

II. Unit Structure

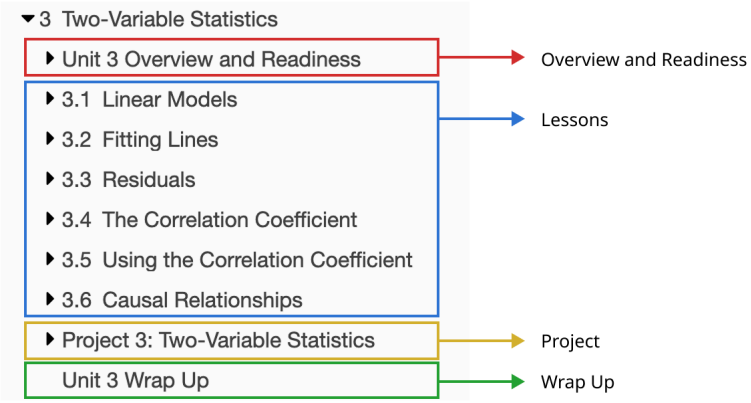


Image 1

After completing the **Overview and Readiness** lesson, you can progress through the regular **Lessons**.

Each Lesson has multiple pages to guide you through learning new skills and reinforcing old ones.

At the end of each unit, you will also find a **Project** and **Wrap Up** lesson covering the topics in the unit. Use these lessons to cement understanding of the skills you just learned.

III. Lesson Structure

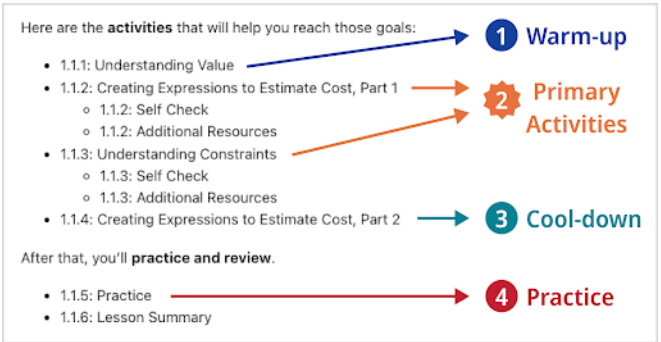


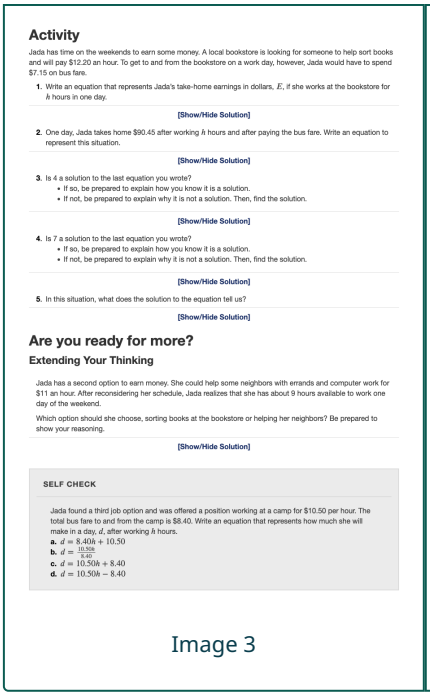
Image 2

Lesson Activities

The first page of a lesson is called “Lesson Overview.” Here you will find the different types of activities included in the lesson (Image 2).

- **Warm up:** This is the first activity in a lesson.
- **Primary activities:** These are unique and need extra focus. We'll take a close look at them in the next part of this guide.
- **Cool-down:** This is the last activity in a lesson.
- **Practice:** This is a practice page with problems on the topics you learned in the lesson.

IV. Primary Activities



Activity

Jada has time on the weekends to earn some money. A local bookstore is looking for someone to help sort books and will pay $12.20 an hour. To get to and from the bookstore on a work day, however, Jada would have to spend $7.15 on bus fare.

1. Write an equation that represents Jada's take-home earnings in dollars, E, if she works at the bookstore for h hours in one day.

[Show/Hide Solution]

2. One day, Jada takes home $90.45 after working h hours and after paying the bus fare. Write an equation to represent this situation.

[Show/Hide Solution]

3. Is 4 a solution to the last equation you wrote?
   - If so, be prepared to explain how you know it is a solution.
   - If not, be prepared to explain why it is not a solution. Then, find the solution.

[Show/Hide Solution]

4. Is 7 a solution to the last equation you wrote?
   - If so, be prepared to explain how you know it is a solution.
   - If not, be prepared to explain why it is not a solution. Then, find the solution.

[Show/Hide Solution]

5. In this situation, what does the solution to the equation tell us?

[Show/Hide Solution]

Are you ready for more?

Extending Your Thinking

Jada has a second option to earn money. She could help some neighbors with errands and computer work for $11 an hour. After reconsidering her schedule, Jada realizes that she has about 9 hours available to work one day of the weekend.

Which option should she choose, sorting books at the bookstore or helping her neighbors? Be prepared to show your reasoning.

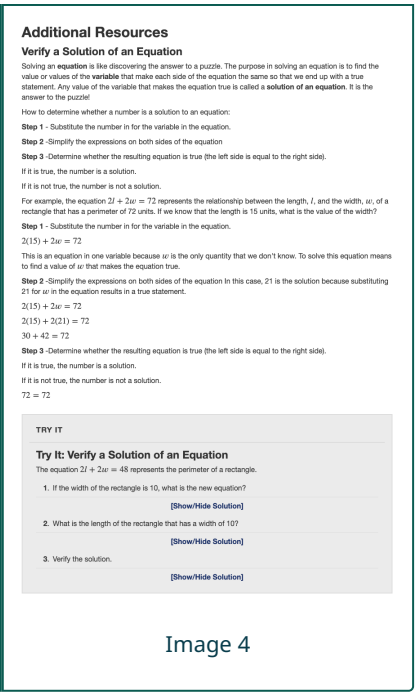
[Show/Hide Solution]

SELF CHECK

Jada found a third job option and was offered a position working at a camp for $10.50 per hour. The total bus fare to and from the camp is $8.40. Write an equation that represents how much she will make in a day, d, after working h hours.

a. $d = 8.40h + 10.50$
b. $d = \frac{10.50h}{8.40}$
c. $d = 10.50h + 8.40$
d. $d = 10.50h - 8.40$

Image 3



Additional Resources

Verify a Solution of an Equation

Solving an **equation** is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the **variable** that make each side of the equation the same so that we end up with a true statement. Any value of the variable that makes the equation true is called a **solution of an equation**. It is the answer to the puzzle!

How to determine whether a number is a solution to an equation:

Step 1 - Substitute the number in for the variable in the equation.

Step 2 -Simplify the expressions on both sides of the equation

Step 3 -Determine whether the resulting equation is true (the left side is equal to the right side).

If it is true, the number is a solution.

If it is not true, the number is not a solution.

For example, the equation $2l + 2w = 72$ represents the relationship between the length, l, and the width, w, of a rectangle that has a perimeter of 72 units. If we know that the length is 15 units, what is the value of the width?

Step 1 - Substitute the number in for the variable in the equation.

$2(15) + 2w = 72$

This is an equation in one variable because w is the only quantity that we don't know. To solve this equation means to find a value of w that makes the equation true.

Step 2 -Simplify the expressions on both sides of the equation In this case, 21 is the solution because substituting 21 for w in the equation results in a true statement.

$2(15) + 2w = 72$

$2(15) + 2(21) = 72$

$30 + 42 = 72$

Step 3 -Determine whether the resulting equation is true (the left side is equal to the right side).

If it is true, the number is a solution.

If it is not true, the number is not a solution.

$72 = 72$

TRY IT

Try It: Verify a Solution of an Equation

The equation $2l + 2w = 48$ represents the perimeter of a rectangle.

1. If the width of the rectangle is 10, what is the new equation?

[Show/Hide Solution]

2. What is the length of the rectangle that has a width of 10?

[Show/Hide Solution]

3. Verify the solution.

[Show/Hide Solution]

Image 4

Primary Activities

Here's an example of one of these unique types of activities. Each lesson has 2-3 of these.

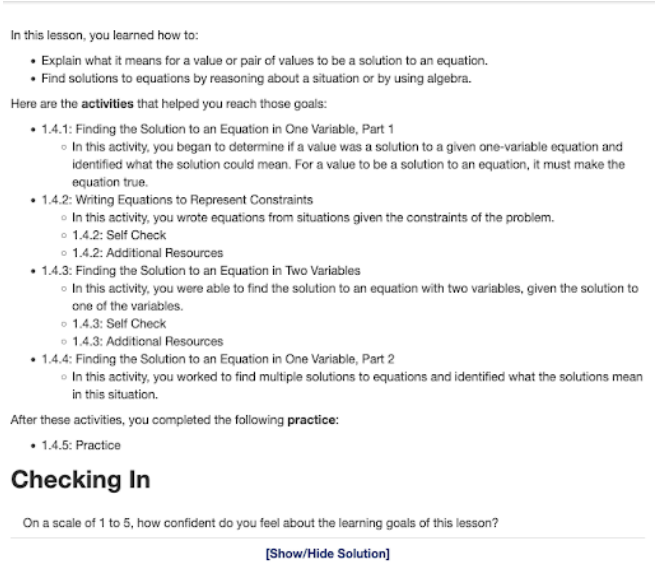
Some activities have a section called **Are you ready for more?** This section, meant to expand your thinking, is optional unless otherwise stated by your teacher (Image 3).

After you have finished the activity, you will do a multiple choice question called a **Self Check** (Image 3). If you feel confident in your skills, you may move on to the next lesson.

Additional Resources

If you want a refresher on the skills tested in the Self Check, read through the **Additional Resources** section, then complete the **Try It** section to test yourself on what you've learned. This section is optional unless otherwise stated by your teacher (Image 4).

V. End of Lesson



In this lesson, you learned how to:

- Explain what it means for a value or pair of values to be a solution to an equation.
- Find solutions to equations by reasoning about a situation or by using algebra.

Here are the **activities** that helped you reach those goals:

- 1.4.1: Finding the Solution to an Equation in One Variable, Part 1
  - In this activity, you began to determine if a value was a solution to a given one-variable equation and identified what the solution could mean. For a value to be a solution to an equation, it must make the equation true.
- 1.4.2: Writing Equations to Represent Constraints
  - In this activity, you wrote equations from situations given the constraints of the problem.
  - 1.4.2: Self Check
  - 1.4.2: Additional Resources
- 1.4.3: Finding the Solution to an Equation in Two Variables
  - In this activity, you were able to find the solution to an equation with two variables, given the solution to one of the variables.
  - 1.4.3: Self Check
  - 1.4.3: Additional Resources
- 1.4.4: Finding the Solution to an Equation in One Variable, Part 2
  - In this activity, you worked to find multiple solutions to equations and identified what the solutions mean in this situation.

After these activities, you completed the following **practice**:

- 1.4.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

[Show/Hide Solution]

Image 5

Lesson Summary

Here you will review the topics covered and the activities you completed (Image 5). You will also have an opportunity to check-in on your progress with the learning goals from the lesson.

After you've finished the Unit, you can use the **Wrap Up** lesson to check in on the important skills from each lesson. If there are any skills you want to brush up on, use the Lesson Summary page to find the most relevant activities to revisit.

Useful Resources

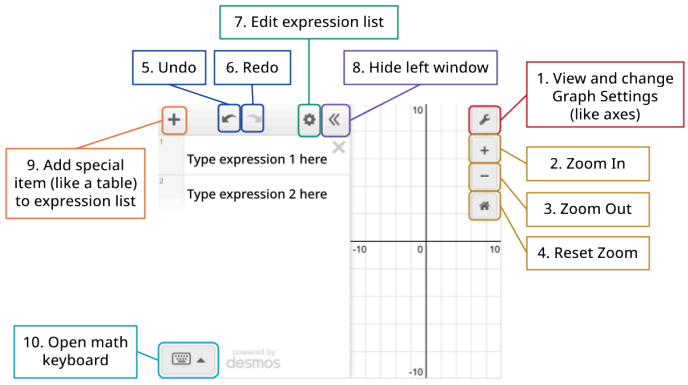
Want some study tips? Forgot the meaning of a word? Check out the sections **Students Start Here**, **Supporting All Learners**, and the **Index** for our curated resources.

Using Desmos

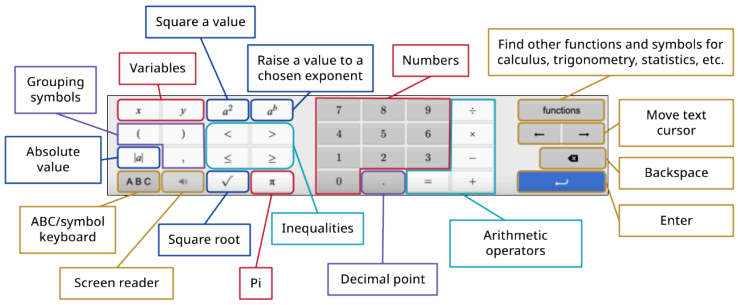
Throughout the *Algebra 1* curriculum, you will find Desmos activities and graphing calculators embedded in lessons. Desmos activities are interactive online problem sets located on the Desmos website. They can be assigned to your students and graded automatically if you have a Desmos account. Desmos graphing calculators are interactive online graphs embedded under graphing problems in the *Algebra 1* course. Note, the embedded calculator adheres to instructional requirements, not assessment requirements.

We have created a how-to guide with helpful screenshots and links to help you and your students use Desmos. If you need additional help, view the official Desmos guide (https://openstax.org/r/desmos-graphing).

I. Understanding Desmos Tools



1. **View and change graph settings**
   - Change limits of the x- and y-axis and the scale for each
   - Toggle display color and font size
   - Toggle between radians and degrees
   - Turn on Braille mode
2. **Zoom in:** focus on one part of the graph and see less of the rest of it
3. **Zoom out:** see less of one part of the graph and more of the rest of it
4. **Reset zoom:** return to default window where $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$.
5. **Undo:** undo your last action
6. **Redo:** redo your last action
7. **Edit expression list:**
   - Quickly copy or delete expressions in the expression list
   - Convert expressions to tables
   - Select the colored dot to the left of the expression to change the line's color, thickness, opacity, and line type
   - Quickly clear graph
8. **Hide left window:** collapse or expand the window containing the expression list
9. **Add special item to expression list:** add an
   - expression, an equation of a line;
   - note, a text description or note about the graph (a good scaffold to support differentiation);
   - table, a basic table where you can input x- and y-coordinates;
   - image, a resizable image from your computer
10. **Open math keyboard:** open a digital keypad that allows you to quickly add special math functions and symbols, like sin or ÷, to your expression. View our guide to the Desmos math keyboard below.
    - Tip: You can use your physical, non-Desmos keyboard to type math and interact with the graph. To learn more check out these guides Math Notation Guide by PurpleMath (https://openstax.org/r/PurpleMath) and Desmos Keyboard Shortcuts (https://help.desmos.com/hc/en-us/articles/4405966811021-Keyboard-Shortcuts)



II. Graphing Regressions Using the Desmos Graphing Calculator

You can use Desmos to graph a regression, regression line, regression curve, scatter plot, stat plot, and create a line of best fit. Check out our guide to using these features in Desmos, if you have additional questions, view the official Desmos regressions guide (https://openstax.org/r/desmos-regressions).

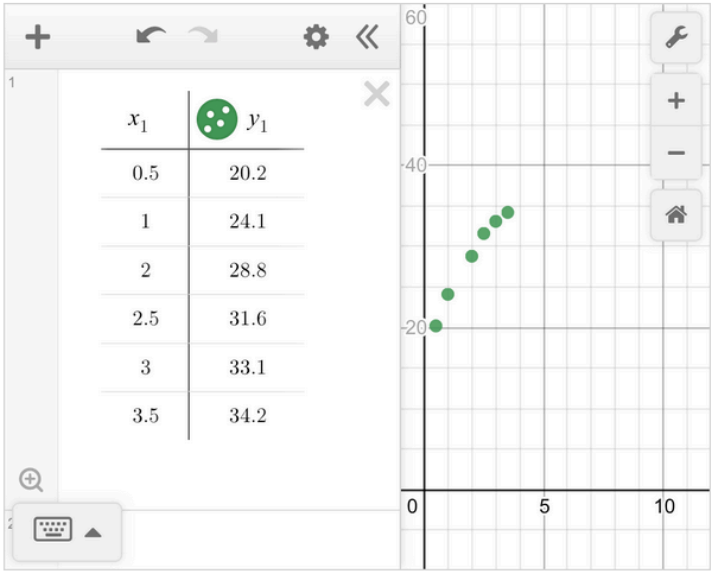


Image 1

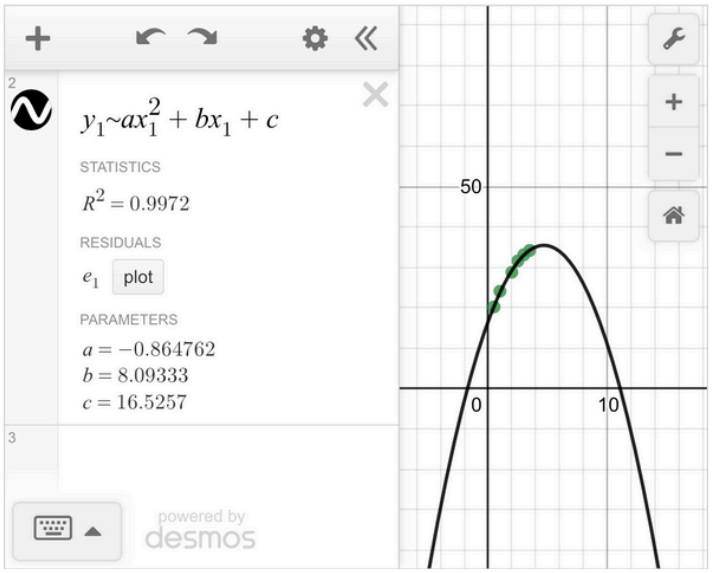


Image 2

Creating a Scatter Plot from a Table (Image 1)

1. On the Desmos graphing calculator, click the + button in the top left corner and add a table.
2. You can copy and paste an existing table in, or enter it manually. On the left side of the table, type the first x-coordinate from your reference table. On the right side, type the first y-coordinate. This will create a point on your graph.
3. Go to the next line of the table and repeat until all coordinates have been entered. Inputting all the coordinates will create a graph called a scatter plot, pictured above and left.
    - Tip: If you do not see your point graphed in the Desmos window, you may need to "Zoom" or change the graph settings for the window.

Creating a Line of Best Fit on a Scatter Plot (Image 2)

1. To create a line or curve that approximates the data trend, calculate a regression. Depending on the type of data (linear, exponential, or quadratic), the format of the regression equation will differ:
    - Linear: $y_1 \sim mx_1 + b$
    - Exponential: $y_1 \sim ax_1 b$
    - Quadratic: $y_1 \sim ax_1^2 + bx_1 + c$

2. Click the pointer below the table into entry box #2 to enter a new function. The cursor should start blinking. Enter the correct equation you found in step 4 using the following tips:
   - Notice that when you type the "1" after y or x, Desmos automatically places it as subscript.
   - Also notice that we are not using an equal sign. The symbol ~ means approximately and tells Desmos to approximate a line or curve of best fit. On your keyboard, you can type it with Shift+` (in the top left corner). If you are using the Desmos on-screen keyboard, it will be in the ABC/symbol keyboard.
   - To type an exponent, use the ^ symbol on your physical keyboard. You can type it using Shift+6. You can also use the Desmos keyboard button a^2 or a^b.
3. Once you have entered the regression equation, you can press Enter. The graph of the line or curve of best fit will appear. Pictured above and right is a quadratic curve of best fit. Desmos will also report correlation statistics such as r and R^2, as well as the a value, b value, and c value in the expression list below the quadratic equation you entered.
   - Tip: The window should be appropriately set to display the points in the table, but sometimes you have to adjust the window of the graph in order to properly view your data set. Click and drag the mouse to shift the viewing window. To zoom in or out, use the buttons in the top right or the scrolling wheel on the mouse. Or, edit the graph settings.
   - Tip: If you are having difficulty finding the correlation coefficient, r, type the command '$corr(x_1, y_1)$' on another expression line. It may also be helpful to check the accuracy of the data entered into the x_1 and y_1 columns of the table.

Additional Resources

You may find it helpful to keep the following resources on hand as you progress through the course:

- English-Spanish Glossary (https://openstax.org/r/english-spanish-glossary)
- Algebra 1 STAAR Reference Materials (https://openstax.org/r/STAAR-ref)
- Algebra 1 STAAR Graph Paper (https://openstax.org/r/STAAR-graph-paper)

UNIT 1 OVERVIEW AND READINESS

Unit 1 Overview

Why Should I Care?

Do you know what a budget is? Have you ever used one? Watch the video to learn a bit about how City Manager Arthur Noriega uses linear equations to manage the budget for the city of Miami.

Algebraic Thinking and Budgets (https://openstax.org/r/algebraic-budgets)

In this unit, you will learn about using variables, expressions, and equations to model relationships. You will learn that you can represent a constraint, or limit on what is possible in a situation, using expressions, equations, and inequalities. This will be helpful whether you want to know how soon you'll be able to save for a new bicycle, or if you can afford to go to a concert with your friends in two weeks.

You will also learn to transform one equation into an equivalent equation. Sometimes you want to write an equation in a different form so you can solve for a certain variable or so you can write the equation in a form that makes it easier to identify information about the relationship. You will learn how the form and parts of linear equations are related to the features of its graph. A graph can be helpful to model a relationship and to make sense of the constraints.

Building Character: Social Intelligence

As a city manager, Arthur Noriega must work with others to develop budgets. Similarly, in this course, you will need to work with your classmates to solve problems. Working well with others is a skill that will be useful to you in this course, as well as outside of the classroom. Most jobs require the ability to work well, or collaborate, with other people.

Throughout this unit, you will work on developing your **social intelligence**. Social intelligence is the ability to connect with other people. Having this ability allows you to interact in a more positive manner with others. Being empathetic, or sensitive to what others are feeling, shows that you care about others. In fact, empathetic people are less likely to experience anxiety, depression, and addictions later in life. They are also more likely to be hired, promoted, earn more money, and have happier marriages and better-adjusted children.

Think about your current sense of social intelligence. How many of the following statements are true for you?

- I have a lot of relationships that are mutually beneficial, enjoyable, and supportive.
- Most of the time, I can tell how other people feel and have a good idea about how to respond appropriately.
- My relationships make me feel good about myself.
- The people in my life help me be my best.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

Am I Ready to Learn This?

Getting Ready for Unit 1

To be ready for this unit, you need to be able to:

- Solve linear equations using a general strategy.
- Classify equations.
- Identify points on a graph.

Next, you will answer some questions that will help you see how ready you are for this unit. If you need help, some mini-lessons will help you brush up on anything you need to work on.

Solve Linear Equations: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #1: Solve Linear Equations

Solve: $2(m-6)+3=m-1$.

a. $m=-10$

b. $m=14$

c. $m = 8$
d. $m = 2$

Solving Linear Equations Using a General Strategy

A solution of an equation is a value of a variable that makes a true statement when substituted into the equation. For example, in $x + 2 = 3$, the value of the variable x is 1.

To find the solution to an equation in one variable, the goal is to isolate the variable on one side of the equation. For example,

$3x + 1 = 9 - x$

$+x + x$

$4x + 1 = 9$

$-1 - 1$

$4x = 8$

You can check the solution by substituting the value into the equation.

$3(2) + 1 = 9 - 2$

$6 + 1 = 7$

$7 = 7$

Equations may take several steps to solve, so it is helpful to have a clear and organized strategy. The following table shows a general strategy to solve any **linear equation** in one variable. You may not need to use every step.

General Strategy for Solving Linear Equations

Step 1

Simplify each side of the equation as much as possible.

Use the **Distributive Property** to remove any parentheses.

Combine like **terms** .

Step 2

Collect all the variable terms on one side of the equation.

Use the **Addition or Subtraction Property of Equality**.

Step 3

Collect all the constant terms on the other side of the equation.

Use the Addition or Subtraction Property of Equality.

Step 4

Make the **coefficient** of the variable term to equal to 1.

State the solution to the equation.

Step 5

Check the solution.

Substitute the solution into the original equation to make sure the result is a true statement.

TRY IT **Solving a Linear Equation Using a General Strategy**

Solve the following using a general strategy:

Solve $2(3x-8)=19-x$.

Check Your Understanding

Solve: $7(n-3)-8=-15$.

a. $n=-\frac{4}{7}$

b. $n=-2$

c. $n=2$

d. $n=-3\frac{5}{7}$

Videos: Solving Linear Equations

Khan Academy: Equations with Parentheses

Watch this video to see how to use the Distributive Property to solve an equation with parentheses.

A clever application of the distributive property to solve a multi-step equation | Khan Academy (https://www.youtube.com/embed/PL9UYj2awDc)

Khan Academy: Equations with Variables on Both Sides: Fractions

Watch this video to see how to solve an equation with fractional coefficients and variables on both sides.

Introduction to solving an equation with variables on both sides | Algebra I | Khan Academy (https://www.youtube.com/embed/f15zA0PhSek)

Classify Equations: Mini-Lesson Review

MINI LESSON QUESTION **Question #2: Classify Equations**

Which of the following correctly classifies the equation and the solution?

$6(2n-1)+3=2n-8+5(2n+1)$

a. contradiction; no solution

b. conditional equation; $1n=-\frac{1}{4}$

c. conditional equation; $n=0$

d. identity; all real numbers

Use Solutions to Classify Equations

An equation that is true for one or more values of the variable and false for all other values of the variable is a **conditional equation**.

Consider the equation $7x+8=-13$.

Subtract 8 to get the constants on one side.	$7x = -21$
Divide by 7 to make the coefficient of the variable 1.	$x = -3$

The solution is $x = -3$. This means the equation $7x + 8 = -13$ is true when we replace the variable, x, with the value -3, but it is not true when we replace x with any other value. Whether the equation $7x + 8 = -13$ is true depends on the value of the variable. The equation is a conditional equation.

An equation that is true for any value of the variable is called an **identity**. The solution of an identity is all real numbers.

Consider the equation $2y + 6 = 2(y + 3)$. Do you recognize that the left side and the right side are equivalent? Let's see what happens when we solve for y.

Distribute.	$2y + 6 = 2y + 6$
Subtract $2y$ to get the variables on one side.	$6 = 6$

But $6 = 6$ is true. This means the equation $2y + 6 = 2(y + 3)$ is true for any value of y. The equation is an identity, and we say the solution is all real numbers.

An equation that is false for all values of the variable is called a **contradiction**. A contradiction has no solution.

Consider the equation $5z = 5z - 1$.

Subtract $5z$ to get the variables on one side.	$0 \neq -1$

The table summarizes the types of equations and solutions.

Type of Equation	What happens when you solve it?	Solution
Conditional Equation	True for one or more values of the variables and false for all other values	One or more values
Identity	**True** for any value of the variable	All real numbers
Contradiction	**False** for all values of the variable	No solution

TRY IT **Use Solutions to Classify Equations**

Classify the following: $5m + 3(9 + 3m) = 2(7m - 11)$.

Check Your Understanding

Which of the following correctly classifies the equation and the solution?$4 + 9(3x - 7) = -42x - 13 + 23(3x - 2)$

a. identity; all real numbers

b. conditional equation; $x = 0$

c. conditional equation; $x = 2\frac{5}{27}$

d. contradiction; no solution

Video: Solving Linear Equations

Khan Academy: Number of Solutions to Linear Equations

Watch the video to see how a linear equation may have one solution, no solution, or infinite solutions.

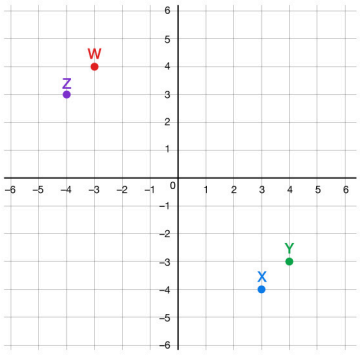
Number of solutions to linear equations | Linear equations | Algebra I | Khan Academy (https://www.youtube.com/embed/qsL_5Y8uWPU)

Find Coordinates: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #3: Find Coordinates

Which point has coordinates $(3, -4)$?



a. Point Z

b. Point Y

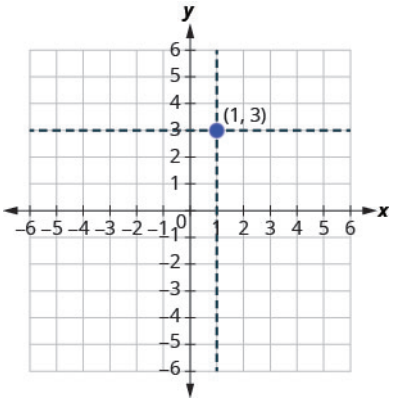
c. Point W

d. Point X

Identifying Coordinates of a Point on a Graph

In the rectangular coordinate system, every point is represented by an **ordered pair**. The first number in the ordered pair is the x**-coordinate** of the point, and the second number is the y**-coordinate** of the point. An ordered pair, (x, y), gives the coordinates of a point in a rectangular coordinate system. The first number is the x-coordinate. The second number is the y-coordinate. The phrase "ordered pair" means the order is important. What is the ordered pair of the point where the axes cross? At that point, both coordinates are zero, so its ordered pair is $(0, 0)$. The point $(0, 0)$ has a special name. It is called the origin.

We use the coordinates to locate a point on the xy-plane. Let's plot the point $(1, 3)$ as an example. First, locate 1 on the x-axis and lightly sketch a vertical line through $x = 1$. Then, locate 3 on the y-axis and sketch a horizontal line through $y = 3$. Now, find the point where these two lines meet—that is the point with coordinates $(1, 3)$.

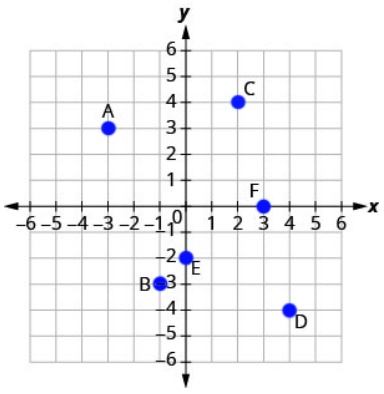


Notice that the vertical line through $x = 1$ and the horizontal line through $y = 3$ are not part of the graph. We just used them to help us locate the point $(1, 3)$.

In algebra, being able to identify the coordinates of a point shown on a graph is just as important as being able to plot points. To identify the x-coordinate of a point on a graph, read the number on the x-axis directly above or below the point. To identify the y-coordinate of a point, read the number on the y-axis directly to the left or right of the point. Remember, when you write the ordered pair, use the correct order, (x, y).

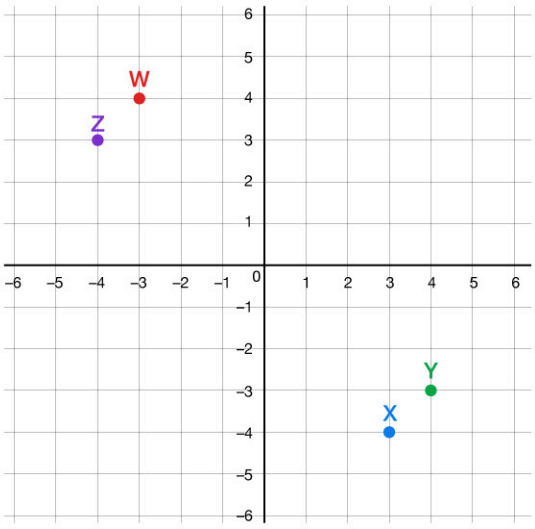
TRY IT **Identifying Coordinates of a Point on a Graph**

Name the ordered pair of each point shown in the rectangular coordinate system.



Check Your Understanding

Which point has coordinates $(-4, 3)$?



a. Point W

b. Point Z

c. Point X

d. Point Y

Video: Points on the Coordinate Plane

Khan Academy: Points on the Coordinate Plane

Watch this video to see how you can use a coordinate plane to plot points and to identify the coordinates of a point plotted on a coordinate plane.

Coordinate plane examples | Linear equations and functions | 8th grade | Khan Academy (https://www.youtube.com/embed/VhNkWdLGpmA)

Unit 1 Overview: Wrap Up

- Solve linear equations using a general strategy.
- Classify equations.
- Identify points on a graph.

If you are still struggling, be sure to reach out to your teacher for additional help. Nobody is good at problems like this on the first try. Keep at it, and it will get easier.

1.1 EXPLORING EXPRESSIONS AND EQUATIONS

1.1.0 Lesson Overview

In this lesson, you will experiment with expressions and equations to model a situation: planning a pizza party. For instance, what are the relevant quantities when planning for a pizza party? Are these quantities fixed or do they vary? How they might relate to one another? Are there any limitations or constraints on how much pizza should be ordered?

When you finish this lesson, you will be able to:

- Tell which quantities in a situation can vary and which ones cannot.
- Explain the meaning of the term "constraint."
- Use letters and numbers to write expressions representing the quantities in a situation.

Here are the **activities** that will help you reach those goals:

- 1.1.1: Understanding Value
- 1.1.2: Creating Expressions to Estimate Cost, Part 1
  - 1.1.2: Self Check
  - 1.1.2: Additional Resources
- 1.1.3: Understanding Constraints
  - 1.1.3: Self Check
  - 1.1.3: Additional Resources
- 1.1.4: Creating Expressions to Estimate Cost, Part 2

After that, you'll **practice and review**.

- 1.1.5: Practice
- 1.1.6: Lesson Summary

1.1.1 Understanding Value

Warm Up

Here are some letters and what they represent. All costs are in dollars.

- m represents the cost of a main dish.
- n represents the number of side dishes.
- s represents the cost of a side dish.
- t represents the total cost of a meal.

For questions 1 – 4, discuss what each **equation** means with a partner. Then, describe the situation in words.

1. $m = 7.50$

2. $m = s + 4.50$

3. $ns = 6$

4. $m + ns = t$

5. Write a new equation that could be true in this same situation.

6. Describe what your new equation means in words.

1.1.2 Creating Expressions to Estimate Cost, Part 1

Activity

Imagine your class is having a pizza party.

Work with your group to plan what to order and to estimate what the party would cost.



1. What is your favorite type of pizza to order?
2. How many toppings are on your favorite type of pizza?
3. What type of crust does your favorite pizza have?

For questions 4 – 9, use the following group instructions.

> Work with a group to research the cost of ordering pizza from a local restaurant. Then, determine what to order for the class pizza party and determine how much it would cost. Answer the following questions regarding your group's plan.

4. What is the cost estimate for your group's planned class party?
5. Write down one **expression** that show how your group's cost estimate was calculated.
6. In your expression, which quantities, if any, might change on the day of the party?
7. Rewrite your expression, replacing the quantities that might change with letters.
8. What do the letters represent?
9. How would you convince the class to go with your group's plan?

Are you ready for more?

Extending Your Thinking

A local pizzeria sells gourmet pizzas for the prices posted below. Remember the sizes are determined by the diameter of each pizza. Compare the cost per square inch of the sizes.

Pizza size	10" Small	12" Medium	14" Large	16" XLarge
Price	$12.99	$15.99	$19.99	$21.99

1. How did you determine the cost per square inch of each pizza?
2. Which size pizza is the better deal?
3. Which size pizza is the most expensive per square inch?
4. What is the difference in price per square inch between the most and least expensive sizes?

Self Check

A ninth-grade class is ordering pizza for lunch. There are n students in the class. Each slice of pizza costs $3 and the delivery fee is $8. Which expression represents the total cost if each student gets 2 slices of pizza?

a. $3n + 8$
b. $(2)3n + 8$
c. $(2)3n$
d. $3n$

Additional Resources

Translate an English Phrase to an Algebraic Expression

You can use the following operation symbols to translate English phrases into algebraic expressions.

Look closely at these phrases using the four operations:

The **SUM** of a and b $a + b$
The **DIFFERENCE** of a and b $a - b$
The **PRODUCT** of a and b $a \cdot b = ab = (a)(b)$
The **QUOTIENT** of a and b $\frac{a}{b} = a \div b$

CLASSROOM CONNECTION Log into student.desmos.com using the information provided by your teacher to complete the activity.

Identify other phrases that represent the same algebraic concepts by categorizing the following list according to the four operations.

b added to a	a divided by b	b subtracted from a
The product of a times b	a sets of b	The sum of a and b
a plus b	The ratio of a and b	a decreased by b
b less than a	a increased by b	b divided into a
The quotient of a and b	b more than a	The total of a and b
a times b	a minus b	The difference of a and b

Each phrase tells us to operate on two numbers. Look for the words "of" and "and" to find the numbers.

Later in this course, we'll apply our skills in algebra to solving applications. The first step will be to translate an English phrase to an algebraic expression.

EXAMPLE

The length of a rectangle is 14 less than the width. Let w represent the width of the rectangle. Write an expression for the length of the rectangle.

Step 1 - Write a phrase about the length of the rectangle.

14 less than the width

Step 2 - Substitute w for "the width."

14 less than w

Step 3 - Rewrite **less than** as **subtracted from**.

14 subtracted from w

Step 4 - Translate the phrase into algebra.

$w - 14$

TRY IT **Translate an English Phrase to an Algebraic Expression**

Translate the following into an algebraic expression:

June has dimes and quarters in her purse. The number of dimes is seven less than four times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

1.1.3 Understanding Constraints

Activity

A **constraint** is something that limits what is possible or reasonable in a situation.

For example, one constraint in a pizza party might be the number of slices of pizza each person could have, s. We can write $s < 4$ to say that each person gets fewer than 4 slices.

Look at the expressions you wrote when planning the pizza party earlier. Choose an expression that uses one or more letters.

1. What is the expression you are examining?
2. What does the first letter represent?
3. What values would be reasonable for this first letter? (For instance, could the value be greater than 50? Is it possible for the letter to be a non-whole number? A negative number?)
4. What does the second letter represent?
5. What values would be reasonable for this second letter? (For instance, could the value be greater than 50? Is it possible for the letter to be a non-whole number? A negative number?)
6. Write equations or inequalities that represent some constraints in your pizza party plan. If a quantity must be an exact value, use the = symbol. If it must be greater than or less than a certain value to be reasonable, use the $>$ or $<$ symbol.

Video: Writing an Inequality for the Constraint

Watch the following video to learn more about constraints.

Writing an Inequality for the Constraint (https://openstax.org/r/inequality-constraint)

Self Check

Marcus works s hours per week stocking shelves. He works m hours per week mowing lawns. Marcus works at least 8

hours per week stocking shelves. He works at most a total of 20 hours per week. Which of the following represent constraints for this situation?

a. $s + m \leq 20; s \leq 8$
b. $s + m \geq 20; s \leq 8$
c. $s + m \leq 20; s \geq 8$
d. $s + m \geq 20; s \geq 8$

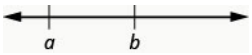
Additional Resources

Inequalities

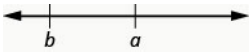
One way to represent a **constraint** is to use an inequality symbol. For example, you may decide that at most 3 toppings should be on each pizza. This could be represented by t < 3, if t represents the number of toppings.

On the number line, the numbers get larger as they go from left to right. The number line can be used to explain the symbols "<" and ">."

$a < b$ is read "a is less than b." a is to the left of b on the number line.



$a > b$ is read "a is greater than b." a is to the right of b on the number line.



The expressions $a < b$ or $a > b$ can be read from left to right or right to left, though in English we usually read from left to right.

- For example, $7 < 11$ is equivalent to $11 > 7$.
- For example, $17 > 4$ is equivalent to $4 < 17$.

Inequality symbols	Words
$a \neq b$	a is not equal to b
$a < b$	a is less than b
$a \leq b$	a is less than or equal to b
$a > b$	a is greater than b
$a \geq b$	a is greater than or equal to b

Let's look at some specific examples. Translate the following statements into English phrases.

1. $17 \leq 26$

2. $12 > 27 \div 3$

3. $y + 7 < 19$

TRY IT **Inequalities**

Translate the following statements into English phrases.

1. $14 \leq 27$

2. $12 > 4 \div 2$

3. $x - 7 < 1$

1.1.4 Creating Expressions to Estimate Cost, Part 2

Cool Down

At the end of the school year, all students in the ninth grade are invited to an ice cream party. Use the skills you learned from planning a pizza party to answer the following:

1. Identify and describe at least one letter that you will use to represent the cost of something you need to buy for the party.
2. Write an expression that could represent an estimated cost for the party using that letter. Note, you may use other letters, too, if needed.
3. Describe your expression in words.
4. Using the letter you identified in question 1, describe the values that would be reasonable for the quantity that the letter represents.

1.1.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 5, use the following scenario to identify the expression, equation, or inequality described.

> To support a local senior citizens center, a student club sent a flier home to the n students in the school. The flier said, "Please bring in money to support the senior citizens center. Paper money and coins accepted!" Their goal is to raise T dollars.

Choose the expression, equation, or inequality, from the list, that describes the given quantity.

1. The dollar amount the club would have if they reached half of their goal.

a. $T - 0.5n$
b. $0.5n$
c. $0.25n$
d. $0.5T$
e. $T + 50$

2. The dollar amount the club would have if every student at the school donated 50 cents to the cause.

a. $T - 0.5n$
b. $0.5n$
c. $0.25n$
d. $0.5T$
e. $T + 50$

3. The dollar amount the club could donate if they made $50 more than their goal.

a. $T - 0.5n$
b. $0.5n$
c. $0.25n$
d. $0.5T$
e. $T + 50$

4. The dollar amount the club would still need to raise to reach its goal after every student at the school donated 50 cents.

a. $T - 0.5n$
b. $0.5n$
c. $0.25n$
d. $0.5T$
e. $T + 50$

5. The dollar amount the club would have if half of the students at the school each gave 50 cents.

a. $T - 0.5n$
b. $0.5n$
c. $0.25n$
d. $0.5T$
e. $T + 50$

6. Each of the 10 students in the baking club made 2 chocolate cakes for a fundraiser. They all used the same recipe, using c cups of flour in total.

Write an expression that represents the amount of flour required for one cake.

a. $\frac{2c}{10}$
b. $c - 20$
c. $\frac{c}{20}$
d. $20c$

For questions 7 - 8, determine if each is an expression or an equation that represents the English phrase

7. $3(x - 7) = 27$

a. equation
b. expression

8. $5(4y - 2) - 7$

a. equation
b. expression

For questions 9 - 10, choose the algebraic expression that represents the English phrase

9. difference of $14x^2$ and 13

a. $\frac{14x^2}{13}$
b. $13 - 14x^2$
c. $14x^2 - 13$
d. $14x^2 + 13$

10. the quotient of $12x$ and 2

a. $\frac{2}{12x}$
b. $\frac{12x}{2}$
c. $2 - 12x$
d. $12x - 2$

1.1.6 Lesson Summary

In this lesson, you learned how to:

- Tell which quantities in a situation can vary and which ones cannot.
- Explain the meaning of the term “constraint.”
- Use letters and numbers to write expressions representing the quantities in a situation.

Here are the **activities** that helped you reach those goals:

- 1.1.1: Understanding Value
  - In this activity, you wrote equations to solve problems about a meal. The big thing you learned is that an equation is a statement that an expression has the same value as another expression. An equation can have letters, numbers, or a mix of letters and numbers.
- 1.1.2: Creating Expressions to Estimate Cost, Part 1
  - In this activity, you learned how expressions could represent the quantities in a situation like planning a pizza party. You also saw how expressions could change if quantities changed.
  - 1.1.2: Self Check
  - 1.1.2: Additional Resources
- 1.1.3: Understanding Constraints
  - In this activity, you continued working with the pizza party and learned that a constraint is something that limits what is possible or reasonable in a situation.
  - 1.1.3: Self Check
  - 1.1.3: Additional Resources
- 1.1.4: Creating Expressions to Estimate Cost, Part 2
  - In this activity, you wrote expressions and examined constraints for the cost of an ice cream party for the ninth-grade class.

After these activities, you completed the following **practice:**

- 1.1.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

1.2 WRITING EQUATIONS TO MODEL RELATIONSHIPS, PART 1

1.2.0 Lesson Overview

In this lesson, you will begin to learn how to write equations to model various situations. For instance, modeling scenarios with equations can help you figure out how much you need to pay for one pound of blueberries or how much tax you need to pay when buying a meal at a restaurant.

When you finish this lesson, you will be able to:

- Tell which quantities in a situation can vary and which ones cannot.
- Use letters and numbers to write equations representing the relationships in a situation.

Here are the **activities** that will help you reach those goals:

- 1.2.1: Finding the Percent of 200
- 1.2.2: Modeling with Equations to Find Edges in Platonic Solids
  - 1.2.2: Self Check
  - 1.2.2: Additional Resources
- 1.2.3: Writing Equations to Represent Relationships
  - 1.2.3: Self Check
  - 1.2.3: Additional Resources
- 1.2.4: Writing Equations to Represent Relationships with Percentages
  - 1.2.4: Self Check
  - 1.2.4: Additional Resources
- 1.2.5: Writing Equations to Represent T-Shirts

After that, you'll **practice and review**.

- 1.2.6: Practice
- 1.2.7: Lesson Summary

1.2.1 Finding the Percent of 200

Warm Up

You will be asked to evaluate the percentage of a number mentally. Sometimes, a visual is helpful to do this. If needed, use the GeoGebra activity to help you determine the requested values. Then, look for a pattern that can be used to describe how to find the percentage of any number.

Access multimedia content (http://openstax.org/books/algebra-1/pages/1-2-1-finding-the-percent-of-200)

1. What is 25% of 200?

2. What is 12% of 200?

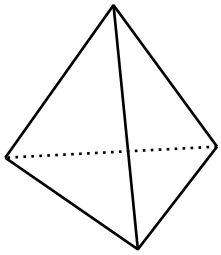
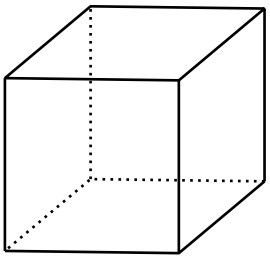
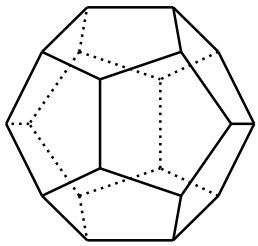
3. What is 8% of 200?

4. How would you determine p% of 200?

1.2.2 Modeling with Equations to Find Edges in Platonic Solids

Activity

These three figures are called Platonic solids.

Tetrahedron	Cube	Dodecahedron
		

	faces	vertices	edges
tetrahedron	4	4	6
cube			
dodecahedron	12	20	30

The table shows the number of faces, vertices, and edges for the tetrahedron and dodecahedron.

Determine the following values regarding the cube.

1. Number of faces.

2. Number of edges.

3. Number of vertices.

There are some interesting relationships between the number of faces (F), edges (E), and vertices (V) in all Platonic solids.

4. What do you notice about the relationships between the numbers of faces, edges, and vertices of the Platonic solids?

5. What do you wonder about the relationships between the numbers of faces, edges, and vertices of the Platonic solids?

6. There is a relationship that can be expressed with an equation using all three values: F, V, E. Write the equation that represents how the three values are related.

Are you ready for more?

Extending Your Thinking

There are two more Platonic solids: an octahedron, which has 8 faces that are all triangles, and an icosahedron, which has 20 faces that are all triangles.

1. How many edges would each of these solids have? (Keep in mind that each edge is used in two faces.)

a. The octahedron has how many edges?

b. The icosahedron has how many edges?

2. Use your discoveries from the activity to determine how many vertices each of these solids would have.

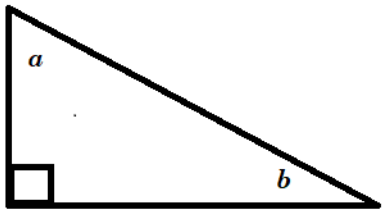
a. The octahedron has how many vertices?

b. The icosahedron has how many vertices?

3. For all 5 Platonic solids, determine how many faces meet at each vertex.

Self Check

The sum of the measures of the angles of a triangle is $180°$. The right triangle shown has angle measures a, b, and $90°$. Which equation could model this relationship?



a. $90ab = 180$
b. $a + b + 90 = 180$
c. $a + b - 90 = 180$
d. $ab + 90 = 180$

Additonal Resources

Writing Equations Using Symbols

How to write statements using algebraic symbols

Step 1 - Read the problem.

Step 2 - Identify the variables and known values. If needed, sketch a picture of the scenario.

Step 3 - Write a sentence using the relationship among the values.

Step 4 - Translate the sentence into an equation.

Remember that **variables** represent values in the problem that can change.

Let's examine two examples of how to write statements algebraically using symbols.

EXAMPLE 1

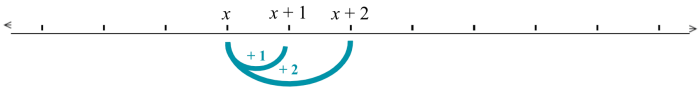
We want to express the following statement using symbolic language:

The sum of three consecutive integers is 372.

Step 1 - Read the problem.
The sum of three consecutive integers is 372.

Step 2 - Identify the variables and known values. If needed, sketch a picture of the scenario.

Let x = the first integer. There are two more numbers that follow x.
Each number is 1 more than the number before it: $x + 1$ and $x + 2$
The sum of all three numbers is 372.



Step 3 - Write a sentence using the relationship among the values.
The sum of x, $x + 1$, and $x + 2$ is 372.

Step 4 - Translate the sentence into an equation.
$x + (x + 1) + (x + 2) = 372$
$3x + 3 = 372$

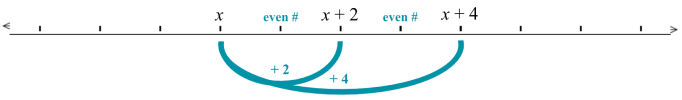
EXAMPLE 2

We want to express the following statement using symbolic language:
The sum of three consecutive odd integers is 93.

Step 1 - Read the problem.
The sum of three consecutive odd integers is 93.

Step 2 - Identify the variables and known values. If needed, sketch a picture of the scenario.

Let x = the first integer. There are two more ODD numbers that follow x.
Each number is 2 more than the number before it: $x + 2$ and $x + 4$
The sum of all three numbers is 93.



Step 3 - Write a sentence using the relationship among the values.
The sum of x, $x + 2$, and $x + 4$ is 93.

Step 4 - Translate the sentence into an equation.
$x + (x + 2) + (x + 4) = 93$
$3x + 6 = 93$

TRY IT **Writing Equations Using Symbols**

Now it's your turn. Write the statement below algebraically using symbols.

The sum of three consecutive integers is 1,623.

1.2.3 Writing Equations to Represent Relationships

Activity

Write an **equation** to represent each situation.

1. Blueberries are \$4.99 a pound. Diego buys b pounds of blueberries and pays \$14.95.
2. Blueberries are \$4.99 a pound. Jada buys p pounds of blueberries and pays c dollars.
3. Blueberries are d dollars a pound. Lin buys q pounds of blueberries and pays t dollars.
4. Noah earned n dollars over the summer. Mai earned \$275, which is \$45 more than Noah did.
5. Noah earned v dollars over the summer. Mai earned m dollars, which is 45 dollars more than Noah did.
6. Noah earned w dollars over the summer. Mai earned x dollars, which is y dollars more than Noah did.

7. How are the equations you wrote for the blueberry purchases like the equations you wrote for Mai's and Noah's summer earnings? How are they different?

Self Check

Max and Jules are renting a car for their 3-day trip. The rental company charges $50 per day and $0.40 per mile driven. Which is a possible equation for their cost of the rental car?

a. $C = 50(3) + 0.40m$
b. $C = 50(3) + 40(3) + 0.40m$
c. $C = 50m + 40(3)$
d. $C = 50(3) + 40m$

Additional Resources

Writing an Equation to Represent a Real-World Problems

Here are the steps to writing an equation to represent a real-world scenario:

Step 1 - Read the problem.

Step 2 - Identify the variables and known values. If needed, sketch a picture of the scenario.

Step 3 - Write a sentence using the relationship among the values.

Step 4 - Translate the sentence into an equation.

EXAMPLE

Write the following statement algebraically using symbols: Peanuts cost $3.59 per pound. Max buys p pounds of peanuts and pays $10.77.

Step 1 - Read the problem.

Peanuts cost $3.59 per pound. Max buys p pounds of peanuts and pays $10.77.

Step 2 - Identify the variables and known values. If needed, sketch a picture of the scenario.

$3.59 = cost per pound of peanuts

$10.77 = total cost of peanuts bought

p = number of pounds of peanuts bought

Step 3 - Write a sentence using the relationship among the values.

The cost per pound times the number of pounds bought equals the total cost of peanuts

Step 4 - Translate the sentence into an equation.

$3.59(p) = 10.77$

TRY IT **Writing an Equation to Represent a Real-World Problem**

Translate the following scenario into an equation:

A married couple together earns $110,000 a year. The wife earns $16,000 less than twice what her husband earns. What does the husband earn?

1. After reading the problem, what are some of the known variables identified in the description?
2. Identify a variable and describe what it represents for this scenario.
3. What sentence can be used to describe the relationship among the values?
4. What equation can be used to represent the scenario?

Video: Writing an Equation to Represent a Real-World Problem

Watch the following video to learn more about how to write an equation to represent a real-world problem.

Writing an Equation to Represent a Real-World Problem (https://openstax.org/r/real-world-equation)

1.2.4 Writing Equations to Represent Relationships with Percentages

Activity

The tax on the sale of a car in Michigan is 6%. At a dealership in Ann Arbor, a car purchase also involves $120 in miscellaneous charges.

There are several quantities in this situation: the original car price, sales tax, miscellaneous charges, and total price.

For questions 1 – 4, write an equation to describe the relationship between all the quantities when given specific information.

1. The original car price is $9,500.
2. The original car price is $14,699.
3. The total price is $22,480.
4. The original price is p.
5. How would each equation you wrote change if the tax on car sales is r% and the miscellaneous charges are m dollars?

Self Check

Sweet's Candy Store charges $5.00 to fill a bag of candy. There is also a sales tax of 7% per purchase. Mrs. Smith buys 4 bags of candy for her children for Christmas. What is an equation that can represent the cost of the candy that Mrs. Smith buys?

a. $C = 0.07 + 4(5)$
b. $C = 4(5)$
c. $C = 0.07(4(5))$
d. $C = 0.07(4(5)) + 4(5)$

Additional Resources

Writing an Equation to Represent a Real-World Problem with Percentages

The steps for writing an equation to represent a real-world scenario with percentages are the same as before. Just be sure to change the given percent to a decimal before you use it in the equation.

Writing an Equation

Step 1 - Read the problem.

Step 2 - Identify the variables and known values. If needed, sketch a picture of the scenario.

Step 3 - Write a sentence using the relationship among the values.

Step 4 - Translate the sentence into an equation.

For example, what is an equation that can represent this scenario?

The label on Audrey's yogurt said that one serving provided 12 grams of protein, which is 24% of the recommended daily amount. What is the total recommended daily amount of protein?

Here's how to use the steps:

Step 1 - Read the problem.

The label on Audrey's yogurt said that one serving provided 12 grams of protein, which is 24% of the recommended daily amount. What is the total recommended daily amount of protein?

Step 2 - Identify the variables and known values. If needed, sketch a picture of the scenario.

1 serving = 12 grams of protein
1 serving is 24% of recommended daily amount
Let a = total amount of protein (in grams).

Step 3 - Write a sentence using the relationship among the values.

12 grams is 24% of the total amount

Step 4 - Translate the sentence into an equation.

$12 = 0.24a$

TRY IT **Writing an Equation to Represent a Real-World Problem with Percentages**

Analyze the situation and then write an equation that can represent the scenario.

One serving of wheat square cereal has 7 grams of fiber, which is 28% of the recommended daily amount. What is the total recommended daily amount of fiber?

1. After reading the problem, what are some of the known variables identified in the description?
2. Identify a variable and describe what it represents for this scenario.
3. What sentence can be used to describe the relationship among the values?
4. What equation can be used to represent the scenario?

1.2.5 Writing Equations to Represent T-Shirts

Cool Down

A school choir needs to make T-shirts for its 75 members and has set aside some money in their budget to pay for them. The members of the choir decided to order from a printing company that charges $3 per shirt, plus a $50 fee for each color to be printed on the shirts.

1. Write an equation that represents the relationship between the number of T-shirts ordered, the number of colors on the shirts, and the total cost of the order. If you use a **variable**, specify what it represents.
2. In this situation, which quantities do you think can vary? Which might be fixed?

1.2.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Translate to an equation: Which equation means nine more than x is equal to 52?

a. $\frac{52}{x} = 9$
b. $9 - x = 52$
c. $9x = 52$
d. $x + 9 = 52$

2. Translate to an equation: Which equation means ten less than m is −14?

a. $-14 - m = 10$
b. $m - 10 = -14$
c. $m - 10 = 14$
d. $10 - m = -14$

3. Translate to an equation: Which equation means the sum of y and −30 is 40?

a. $y + (-30) = 40$
b. $y + 40 = -30$
c. $y = 40 + (-30)$
d. $-30y = 40$

4. Translate to an equation: For a family birthday dinner, Celeste bought a turkey that weighed 5 pounds less than the one she bought for Thanksgiving. The birthday turkey weighed 16 pounds. Which equation represents how much the Thanksgiving turkey weighed?

a. $-x + 5 = 16$
b. $x + 5 = 16$
c. $x - 5 = 16$
d. $5 - x = 16$

5. Translate to an equation: Which equation represents 45% of 120?

a. $x = 45 + 120$
b. $x = 0.45(120)$
c. $x = 4.5(120)$
d. $x = 45(120)$

6. 250% of 65 can also be written as:

a. $x = 250(65)$
b. $x = 25(65)$
c. $x = 0.25(65)$
d. $x = 2.5(65)$

7. Translate to an equation: Geneva treated her parents to dinner at their favorite restaurant. The bill was $74.25. Geneva wants to leave 16% of the total bill as a tip. How much should the tip be?

a. $t = 16 + (74.25)$
b. $t = 1.6(74.25)$
c. $t = 0.16(74.25)$
d. $t = 16(74.25)$

8. Large cheese pizzas cost $5 each, and large one-topping pizzas cost $6 each.
Choose the equation that represents the total cost, T, of c large cheese pizzas and d large one-topping pizzas.

a. $T = 6c - 5d$
b. $T = 5c - 6d$

c. $T = 5c + 6d$
d. $T = 6c + 5d$

9. Jada plans to serve milk and healthy cookies for a book club meeting. She is preparing 12 ounces of milk and 4 cookies per person. Including herself, there are 15 people in the club. A package of cookies contains 24 cookies and costs $4.50. A 1-gallon jug of milk contains 128 ounces and costs $3. Let n represent the number of people in the club, m represent the ounces of milk, c represent the number of cookies, and b represent Jada's budget in dollars.

Select **three** of the equations that could represent the quantities and constraints in this situation.

a. $m = 12(15)$
b. $3m + 4.5c = b$
c. $4n = c$
d. $4(4.50) = c$
e. $b = 2(3) + 3(4.50)$

1.2.7 Lesson Summary

In this lesson, you learned how to:

- Tell which quantities in a situation can vary and which ones cannot.
- Use letters and numbers to write equations representing the relationships in a situation.

Here are the **activities** that helped you reach those goals:

- 1.2.1: Finding the Percent of 200
  - In this activity, you found different percentages of 200 by changing the percent to a decimal and writing an expression.
- 1.2.2: Modeling with Equations to Find Edges in Platonic Solids
  - In this activity, you modeled the relationship between the edges, vertices, and faces of Platonic solids with an equation.
  - 1.2.2: Self Check
  - 1.2.2: Additional Resources
- 1.2.3: Writing Equations to Represent Relationships
  - In this activity, you worked to write more equations that modeled different situations.
  - 1.2.3: Self Check
  - 1.2.3: Additional Resources
- 1.2.4: Writing Equations to Represent Relationships with Percentages
  - In this activity, you continued to write equations to model given situations but incorporated percentages.
  - 1.2.4: Self Check
  - 1.2.4: Additional Resources
- 1.2.5: Writing Equations to Represent T-Shirts
  - In this activity, you wrote an equation about the cost of T-shirts for a choir group. You also identified which quantities were fixed in the situation and which quantities could change.

After these activities, you completed the following **practice**:

- 1.2.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

1.3 WRITING EQUATIONS TO MODEL RELATIONSHIPS, PART 2

1.3.0 Lesson Overview

In this lesson, you will continue to develop your ability to identify, describe, and model relationships with mathematics.

When you finish this lesson, you will be able to:

- Use words and equations to describe the patterns in a table of values or in a set of calculations.
- Use representations, like diagrams and tables, to help make sense of a described situation and write equations for it.

Here are the **activities** that will help you reach those goals:

- 1.3.1: Finding a Relationship between x and y
- 1.3.2: Describing Relationships Using Words and Equations
  - 1.3.2: Self Check
  - 1.3.2: Additional Resources
- 1.3.3: Identifying and Representing Relationships
  - 1.3.3: Self Check
  - 1.3.3: Additional Resources
- 1.3.4: Writing an Equation to Represent a Relationship

After that, you'll **practice and review**.

- 1.3.5: Practice
- 1.3.6: Lesson Summary

1.3.1 Finding a Relationship between x and y

Warm Up

Here is a table of values. The two quantities, x and y, are related.

x	y
1	0
3	8
5	24
7	48

What are some strategies you could use to find a relationship between x and y? Brainstorm as many ways as possible.

1.3.2 Describing Relationships Using Words and Equations

Activity

Create a math story that describes how the two quantities in each table are related.

1. **Table A**

Number of Laps, x	0	1	2.5	6	9
Meters Run, y	0	400	1,000	2,400	3,600

2. **Table B**

Meters From Home, x	0	75	128	319	396
Meters From School, y	400	325	272	81	4

3. **Table C**

Electricity Bills in Dollars, x	85	124	309	816
Total Expenses in Dollars, y	485	524	709	1,216

4. **Table D**

Monthly Salary in Dollars, x	872	998	1,015	2,110
Amount Deposited in Dollars, y	472	598	615	1,710

For questions 5 – 8, match each table from 1 - 4 to an equation that represents the relationship.

5. Equation 1: $400 + x = y$

6. Equation 2: $x - 400 = y$

7. Equation 3: $x + y = 400$

8. Equation 4: $400 \cdot x = y$

Are you ready for more?

Extending Your Thinking

Express every number between 1 and 20 at least one way using exactly four 4s and any operations. For example, 1 could be written as $\frac{4}{4} + 4 - 4$.

Video: Describing Relationships

Watch the following video to learn more about describing relationships using words and equations, specifically looking at the base and height of different rectangles.

Describing Relationships (https://openstax.org/r/describing-relationships)

Self Check

Miles from Home, x	0	5	10
Miles from School, y	30	25	20

Using the table above, what is an equation that shows the relationship between miles from home and miles from school?

a. $y = 30x$
b. $y = 30 - x$
c. $y = 20 + x$
d. $y = 30 + x$

Additional Resources

Modeling Linear Equations

Given a real-world problem, model a linear equation to fit it using these steps:

Step 1 - Read the problem.

Step 2 - Identify the variables and known values. If needed, sketch a picture of the scenario.

Step 3 - Write a sentence or create a table using the relationship among the values.

Step 4 - Translate the sentence or table into an equation

Let's look at an example. Using the steps above, we can model a linear equation to solve a real-world application.

EXAMPLE

Cell phone Company A charges a monthly service fee of $34 plus $0.05/minute talk time.

Step 1 - Read the problem.
Cell phone Company A charges a monthly service fee of $34 plus $0.05/minute talk time.

Step 2 - Identify the variables and known values. If needed, sketch a picture of the scenario.
t: total cost (in dollars)
m: number of minutes

Step 3 - Write a sentence or create a table using the relationship among the values.

Minutes Talked in One Month	0	1	2	3	100
Total for One Month (Dollars)	34	34.05	34.10	34.15	39

Company A: The total cost, t, equals the fee, 34, plus 0.05 per minute, $0.05m$.

Step 4 - Translate the sentence or table into an equation.
$t = 34 + 0.05m$

TRY IT **Modeling Linear Equations**

Cell phone Company B charges a monthly service fee of $40 plus $0.04/min talk-time. Write an equation to describe the relationship between the number of minutes and the total cost.

1.3.3 Identifying and Representing Relationships

Activity

1. The table represents the relationship between the base length and the height of some parallelograms. Both measurements are in inches.

1	48
2	24
3	16
4	12
6	8

What is the relationship between the base length and the height of these parallelograms?

2. Visitors to a carnival are invited to guess the number of beans in a jar. The person who guesses the correct number wins $300. If multiple people guess correctly, the prize will be divided evenly among them.
What is the relationship between the number of people who guess correctly and the amount of money each person will receive?

3. A $\frac{1}{2}$-gallon jug of milk can fill 8 cups, while 32 fluid ounces of milk can fill 4 cups.
What is the relationship between the number of gallons and ounces? If you get stuck, try creating a table.

Building Character: Social Intelligence



Social intelligence is the ability to connect with other people. Think about your current sense of social intelligence. Are the following statements true for you??

- I have a lot of relationships that are mutually beneficial, enjoyable, and supportive.
- Most of the time, I can tell how other people feel and have a good idea about how to respond appropriately.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

Self Check

Number of guests, g	1	2	3	12
Number of cupcakes per guest, n	24	12	8	2

What is one way to write the relationship between the number of guests and the number of cupcakes per guest from the table above?

a. $n = 24 - g$
b. $n = \frac{g}{24}$
c. $n = \frac{24}{g}$
d. $n = 24g$

Additional Resources

Finding Relationships between Quantities

There are times when the relationship between quantities may not be obvious. In some cases, the relationship between quantities might take a bit of work to figure out, by doing calculations several times or by looking for a pattern.

Here are two examples.

EXAMPLE 1

A plane departed from New Orleans and is heading to San Diego. The table shows its distance from New Orleans, x, and its distance from San Diego, y, at some points along the way.

miles from New Orleans	miles from San Diego
100	1,500
300	1,300
500	1,100
	1,020
900	700
1,450	
x	y

1. What is the relationship between the two distances?
2. Do you see any patterns in how each quantity is changing?
3. What is the value of x when y = 1020?
4. What is the value of y when x = 1450?

EXAMPLE 2

A company decides to donate $50,000 to charity. It will select up to 20 charitable organizations, as nominated by its employees. Each selected organization will receive an equal donation amount.

What is the relationship between the number of students, s, and the dollar amount each student will receive, d? To begin, let's examine some specific values to help uncover the pattern.

5. If 5 organizations are selected, how much will each charity receive?
6. If 10 organizations are selected, how much will each charity receive?
7. If 20 organizations are selected, how much will each charity receive?

Do you notice a pattern here? 10,000 is $\frac{50{,}000}{5}$, 5,000 is $\frac{50{,}000}{10}$, and 2,500 is $\frac{50{,}000}{20}$.

We can generalize that the amount each organization receives is 50,000 divided by the number of selected organizations, or $d = \frac{50{,}000}{n}$.

TRY IT **Finding Relationships between Quantities**

A local business is going to hand out $20,000 in scholarships to students at local high schools.

What is the relationship between the number of students, s, and the dollar amount each student will receive, d?

1. If 2 students are selected, what is the amount of the scholarship they will receive?
2. If there are 5 students selected, what is the amount of the scholarship they will receive?
3. If there are 10 students selected, what is the amount of the scholarship they will receive?
4. If there are 20 students selected, what is the amount of the scholarship they will receive?
5. What equation can be used to model the relationship between the number of students, s, receiving scholarships and the dollar amount, d, they receive?

1.3.4 Writing an Equation to Represent a Relationship

Cool Down

Clare volunteers at a local library during the summer. Her work includes putting labels on 750 books.

1. How many minutes will she need to finish labeling all books if she takes no breaks and labels 10 books a minute?
2. How many minutes will she need to finish labeling all books if she takes no breaks and labels 15 books a minute?
3. Suppose Clare labels the books at a constant speed of s books per minute. Write an equation that represents the relationship between her labeling speed and the number of minutes it would take her to finish labeling.

1.3.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 2, use the following scenario to identify the equation described.

Two different telephone carriers offer the following plans that a person is considering. Company A has a monthly fee of $20 and charges $0.05/minute for calls. Company B has a monthly fee of $5 and charges $0.10/minute for calls.

1. Find the model for the entire cost, c, when Company A's plan is used for m minutes.

a. $c = 5 + 0.05m$
b. $c = 20 + 0.10m$
c. $c = 5 + 0.10m$
d. $c = 20 + 0.05m$

2. Find the model for the entire cost, c, when Company B's plan is used for m minutes.

a. $c = 5 + 0.05m$
b. $c = 20 + 0.10m$
c. $c = 5 + 0.10m$
d. $c = 20 + 0.05m$

3. If a carpenter sawed a 10-ft board into two sections and one section was n ft long, how long would the other section, S, be in terms of n?

a. $S = \frac{10}{2}n$
b. $S = 10 - 2n$
c. $S = 10 - n$
d. $S = \frac{10}{n}$

Use the table below for problems 4 and 5:

Tyler needs to complete this table for his consumer science class. He knows that 1 tablespoon contains 3 teaspoons and that 1 cup contains 16 tablespoons.

Number of Teaspoons	Number of Tablespoons	Number of Cups
a. ____	b. ____	2
36	12	c. ____
d. ____	48	3

4. For each of the missing spaces in the table above, a-d, select the missing value that helps complete the table.

a. _____

a. 144
b. 96
c. 64
d. 32
e. $\frac{3}{4}$

b. _____

a. 144
b. 96

c. 64
d. 32
e. $\frac{3}{4}$

c. _____

a. 144
b. 96
c. 64
d. 32
e. $\frac{3}{4}$

d. _____

a. 144
b. 96
c. 64
d. 32
e. $\frac{3}{4}$

5. Choose the equation that represents the number of teaspoons, t, contained in a cup, C.

a. $c = \frac{t}{3}$
b. $c = t - 48$
c. $c = \frac{t}{48}$
d. $c = 48t$

6. The volume of dry goods, like apples or peaches, can be measured using bushels, pecks, and quarts. A bushel contains 4 pecks, and a peck contains 8 quarts.
What is the relationship between the number of bushels, b, and the number of quarts, q?

a. $b = 32q$
b. $b = \frac{q}{64}$
c. $b = \frac{q}{12}$
d. $b = \frac{q}{32}$

1.3.6 Lesson Summary

In this lesson, you learned how to:

- Use words and equations to describe the patterns in a table of values or in a set of calculations.
- Use representations, like diagrams and tables, to help make sense of a described situation and write equations for it.

Here are the **activities** that helped you reach those goals:

- 1.3.1: Finding a Relationship between x and y
  - In this activity, you brainstormed ways to find the relationship between variables in a table.
- 1.3.2: Describing Relationships Using Words and Equations
  - In this activity, you matched tables with the equations that could model the relationships in each table.
  - 1.3.2: Self Check
  - 1.3.2: Additional Resources
- 1.3.3: Identifying and Representing Relationships
  - In this activity, you created tables from a situation and then wrote an equation to model the relationship.
  - 1.3.3: Self Check
  - 1.3.3: Additional Resources
- 1.3.4: Writing an Equation to Represent a Relationship
  - In this activity, you wrote an equation to model a relationship in a situation and also began to solve an equation given one of the quantities in the equation.

After these activities, you completed the following **practice** :

- 1.3.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

1.4 EQUATIONS AND THEIR SOLUTIONS

1.4.0 Lesson Overview

In middle school, you learned that a solution to an equation is a value or values that make the equation true. In this lesson, you will revisit what you learned about solutions to equations in one variable and two variables. You will also continue to practice modeling relationships with equations and to make sense of equations and their solutions in context.

When you finish this lesson, you will be able to:

- Explain what it means for a value or pair of values to be a solution to an equation.
- Find solutions to equations by reasoning about a situation or by using algebra.

Here are the **activities** that will help you reach those goals:

- 1.4.1: Finding the Solution to an Equation in One Variable, Part 1
- 1.4.2: Writing Equations to Represent Constraints
  - 1.4.2: Self Check
  - 1.4.2: Additional Resources
- 1.4.3 Finding the Solution to an Equation in Two Variables
  - 1.4.3: Self Check
  - 1.4.3: Additional Resources
- 1.4.4: Finding the Solution to an Equation in One Variable, Part 2

After that, you'll **practice and review**.

- 1.4.5: Practice
- 1.4.6: Lesson Summary

1.4.1 Finding the Solution to an Equation in One Variable, Part 1

Warm Up

A granola bite contains 27 calories. Most of the calories come from c grams of carbohydrates. The rest come from other ingredients. One gram of carbohydrate contains 4 calories.

The equation $4c + 5 = 27$ represents the relationship between these quantities.

1. What could the 5 represent in this situation?

2. Priya said that neither 8 nor 3 could be the solution to the equation. Is she correct?

3. Marcus said the solution is 5.5. Is he correct? Explain.

1.4.2 Writing Equations to Represent Constraints

Activity

Jada has time on the weekends to earn some money. A local bookstore is looking for someone to help sort books and will pay $12.20 an hour. To get to and from the bookstore on a work day, however, Jada would have to spend $7.15 on bus fare.

1. Write an equation that represents Jada's take-home earnings in dollars, E, if she works at the bookstore for h hours in one day.

2. One day, Jada takes home $90.45 after working h hours and after paying the bus fare. Write an equation to represent this situation.

3. Is 4 a solution to the last equation you wrote?
   - If so, be prepared to explain how you know it is a solution.
   - If not, be prepared to explain why it is not a solution. Then, find the solution.
4. Is 7 a solution to the last equation you wrote?
   - If so, be prepared to explain how you know it is a solution.
   - If not, be prepared to explain why it is not a solution. Then, find the solution.
5. In this situation, what does the solution to the equation tell us?

Are you ready for more?

Extending Your Thinking

Jada has a second option to earn money. She could help some neighbors with errands and computer work for $11 an hour. After reconsidering her schedule, Jada realizes that she has about 9 hours available to work one day of the weekend.

Which option should she choose, sorting books at the bookstore or helping her neighbors? Be prepared to show your reasoning.

Self Check

Jada found a third job option and was offered a position working at a camp for $10.50 per hour. The total bus fare to and from the camp is $8.40. Write an equation that represents how much she will make in a day, d, after working h hours.

a. $d = 8.40h + 10.50$
b. $d = \frac{10.50h}{8.40}$
c. $d = 10.50h + 8.40$
d. $d = 10.50h - 8.40$

Additional Resources

Verify a Solution of an Equation

Solving an **equation** is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the **variable** that make each side of the equation the same so that we end up with a true statement. Any value of the variable that makes the equation true is called a **solution of an equation**. It is the answer to the puzzle!

How to determine whether a number is a solution to an equation:

Step 1 - Substitute the number in for the variable in the equation.

Step 2 - Simplify the expressions on both sides of the equation

Step 3 - Determine whether the resulting equation is true (the left side is equal to the right side).

- If it is true, the number is a solution.
- If it is not true, the number is not a solution.

For example, the equation $2l + 2w = 72$ represents the relationship between the length, l, and the width, w, of a rectangle that has a perimeter of 72 units. If we know that the length is 15 units, what is the value of the width?

Step 1 - Substitute the number in for the variable in the equation.

$2(15) + 2w = 72$

This is an equation in one variable because w is the only quantity that we don't know. To solve this equation means to find a value of w that makes the equation true.

Step 2 - Simplify the expressions on both sides of the equation In this case, 21 is the solution because substituting 21 for

w in the equation results in a true statement.

$2(15) + 2w = 72$

$2(15) + 2(21) = 72$

$30 + 42 = 72$

Step 3 - Determine whether the resulting equation is true (the left side is equal to the right side).

- If it is true, the number is a solution.
- If it is not true, the number is not a solution.

$72 = 72$

TRY IT **Verify a Solution of an Equation**

The equation $2l + 2w = 48$ represents the perimeter of a rectangle.

1. If the width of the rectangle is 10, what is the new equation?
2. What is the length of the rectangle that has a width of 10?
3. Verify the solution.

1.4.3 Finding the Solution to an Equation in Two Variables

Activity

Use the following scenario for questions 1 – 3:

One gram of protein contains 4 calories. One gram of fat contains 9 calories. A snack has 60 calories from p grams of protein and f grams of fat.

1. Determine if 5 grams of protein and 2 grams of fat could be the number of grams of protein and fat in the snack. Explain your reasoning.
2. Determine if 10.5 grams of protein and 2 grams of fat could be the number of grams of protein and fat in the snack. Explain your reasoning.
3. Determine if 8 grams of protein and 4 grams of fat could be the number of grams of protein and fat in the snack. Explain your reasoning.
4. If there are 6 grams of fat in the snack, how many grams of protein are there? Be prepared to show your reasoning.
5. In this situation, what does a solution to the equation $4p + 9f = 60$ tell us? Give an example of a solution.

Video: Working Through the Equation

Watch the following video to learn more about how to determine a solution to this particular equation: $4p + 9f = 60$.

Working Through the Equation (https://openstax.org/r/equation-workthrough)

Self Check

Which of the following is a solution (x, y) to $2x + 3y = 7$?

a. $(-2, 4)$
b. $(4, -1)$
c. $(1, 2)$
d. $(2, 1)$

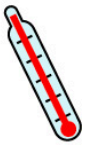
Additional Resources

Solutions to Equations in Two Variables

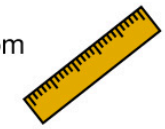
An equation that contains two unknown quantities or two quantities that vary is called an equation in two variables.

Equations in two variables:

$F = \frac{9}{5}C + 32$ Converts temperature from degrees Fahrenheit (F) to degrees Celsius (C)



$i = 2.54c$ Converts a measurement from centimeters (c) to inches (i)



A solution to such an equation is a pair of numbers that makes the equation true.

EXAMPLE

Suppose Tyler spends $45 on T-shirts and socks. A T-shirt costs $10, and a pair of socks costs $2.50. If t represents the number of T-shirts and p represents the number of pairs of socks that Tyler buys, we can represent this situation with the equation:

$10t + 2.50p = 45$

This is an equation in two variables. More than one pair of values for t and p make the equation true.

Which pair of values makes the equation $10t + 2.50p = 45$ true?

1. $t = 3$ and $p = 6$
2. $t = 4$ and $p = 2$
3. $t = 1$ and $p = 10$

In this situation, one **constraint** is that the combined cost of shirts and socks must equal $45.

Solutions to the equations are pairs of t and p values that satisfy this constraint, such as in questions 1 – 2.

Combinations such as $t = 1$ and $p = 10$, as in question 3, are not solutions because they don't meet the constraint. When these pairs of values are substituted into the equation, they result in statements that are false.

TRY IT **Solutions to Equations in Two Variables**

Is $a = 3$ and $b = 5$ a solution to $6a - 3b = 3$?

1.4.4 Finding the Solution to an Equation in One Variable, Part 2

Cool Down

An empty shipping box weighs 250 grams. The box is then filled with T-shirts. Each T-shirt weighs 132.5 grams.

The equation $W = 250 + 132.5T$ represents the relationship between the quantities in this situation, where W is the weight, in grams, of the filled box and T is the number of shirts in the box.

1. Name two possible solutions to the equation $W = 250 + 132.5T$.
2. What do the solutions mean in this situation?

3. Consider the equation $2{,}900 = 250 + 132.50T$. In this situation, what does the solution to this equation tell us?

1.4.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Is $y = \frac{4}{3}$ a solution of $9y + 2 = 6y + 3$?

a. Yes
b. No

2. Is $y = \frac{7}{5}$ a solution of $5y + 3 = 10y - 4$?

a. Yes
b. No

3. Is $y = \frac{5}{3}$ a solution of $6y + 10 = 12y$?

a. Yes
b. No

4. Is $u = -\frac{1}{2}$ a solution of $8u - 1 = 6u$?

a. Yes
b. No

5. Tickets to the museum cost \$4.50 for children and \$9.00 for adults. The equation $63 = 4.5c + 9a$ represents the total cost for c children's tickets and a adult tickets. How many children's tickets were sold if 4 adult tickets were sold?

For questions 6 and 7, use the following scenario: T-shirts are on sale for \$12.50. Isla has a coupon for \$10.00 off.

6. Choose the equation that represents the total cost, c, if she buys n T-shirts.

a. $c = 12.5(10n)$
b. $c = 12.5(n - 10)$
c. $c = 10n - 12.5$
d. $c = 12.5n - 10$

7. How many T-shirts can she buy if she spends all of her \$40?

8. An artist is selling children's crafts. Necklaces cost \$2.25 each, and bracelets cost \$1.50 per each. Select **three** combinations of necklaces and bracelets that the artist could sell for exactly \$12.00

a. 5 necklaces and 1 bracelet
b. 2 necklaces and 5 bracelets
c. 3 necklaces and 3 bracelet
d. 4 necklaces and 2 bracelets
e. 3 necklaces and 5 bracelets
f. 6 necklaces and no bracelets
g. No necklaces and 8 bracelets

9. Volunteer drivers are needed to bring 80 students to the championship baseball game. Drivers either have cars, which can seat 4 students, or vans, which can seat 6 students. The equation $4c + 6v = 80$ describes the relationship between the number of cars, c, and number of vans, v, that can transport exactly 80 students. Select **four** statements that are true about the situation.

a. If 12 cars go, then 2 vans are needed.
b. $c = 14$ and $v = 4$ and are a pair of solutions to the equation.
c. If 6 cars go and 11 vans go, there will be extra space.
d. 10 cars and 8 vans isn't enough to transport all the students.
e. If 20 cars go, no vans are needed.
f. 8 vans and 8 cars are numbers that meet the constraints in this situation.

Use the following scenario for questions 11 and 12:

The drama club is printing T-shirts for its members. The printing company charges a certain amount for each T-shirt plus a setup fee of $40. There are 21 students in the drama club.

10. If there are 21 students in the club and the T-shirt order costs a total of $187, how much does each T-shirt cost?

11. The equation $201.50 = f + 6.50(21)$ represents the cost of printing the T-shirts at a second printing company. Find the solution to the equation.

1.4.6 Lesson Summary

In this lesson, you learned how to:

- Explain what it means for a value or pair of values to be a solution to an equation.
- Find solutions to equations by reasoning about a situation or by using algebra.

Here are the **activities** that helped you reach those goals:

- 1.4.1: Finding the Solution to an Equation in One Variable, Part 1
  - In this activity, you began to determine if a value was a solution to a given one-variable equation and identified what the solution could mean. For a value to be a solution to an equation, it must make the equation true.
- 1.4.2: Writing Equations to Represent Constraints
  - In this activity, you wrote equations from situations given the constraints of the problem.
  - 1.4.2: Self Check
  - 1.4.2: Additional Resources
- 1.4.3: Finding the Solution to an Equation in Two Variables
  - In this activity, you were able to find the solution to an equation with two variables, given the solution to one of the variables.
  - 1.4.3: Self Check
  - 1.4.3: Additional Resources
- 1.4.4: Finding the Solution to an Equation in One Variable, Part 2
  - In this activity, you worked to find multiple solutions to equations and identified what the solutions mean in this situation.

After these activities, you completed the following **practice**:

- 1.4.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

1.5 EQUATIONS AND THEIR GRAPHS

1.5.0 Lesson Overview

In this lesson, you will analyze points on and off a graph and interpret them in context. In explaining correspondences between equations, verbal descriptions, and graphs, you will hone your skill at making sense of problems.

When you finish this lesson, you will be able to:

- Use graphing technology to graph linear equations and identify solutions to the equations.
- Explain how the coordinates of the points on the graph of a linear equation are related to the equation.
- Explain the meaning of points on a graph in terms of the situation it represents when given the graph of a linear equation.

Here are the **activities** that will help you reach those goals:

- 1.5.1: Analyzing Graphs of Linear Equations
- 1.5.2: Graphing Linear Functions in Two Variables
  - 1.5.2: Self Check
  - 1.5.2: Additional Resources
- 1.5.3: Examining an Equation in Two Variables and Its Graph, Part 1
  - 1.5.3: Self Check
  - 1.5.3: Additional Resources
- 1.5.4: Writing, Graphing, and Solving a Linear Equation
  - 1.5.4: Self Check
  - 1.5.4: Additional Resources
- 1.5.5: Examining an Equation in Two Variables and Its Graph, Part 2

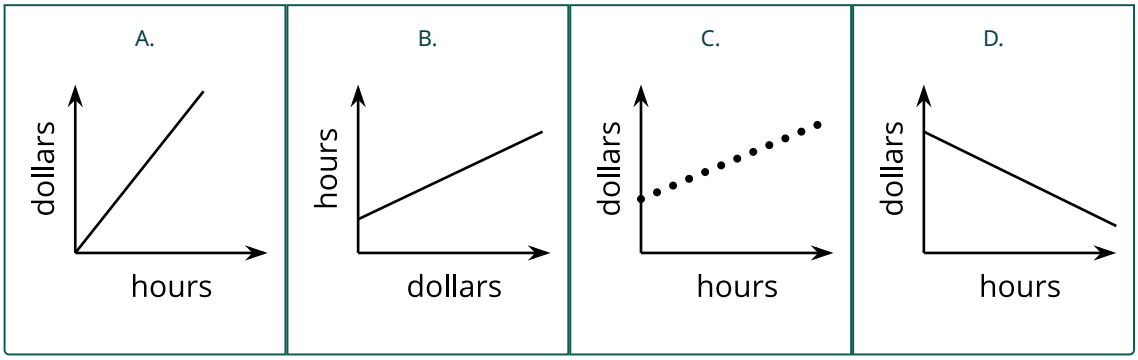
After that, you'll **practice and review**.

- 1.5.6: Practice
- 1.5.7: Lesson Summary

1.5.1 Analyzing Graphs of Linear Equations

Warm Up

Which one doesn't belong?



1.5.2 Graphing Linear Functions in Two Variables

Activity

For questions 1 – 7, use the equation $3x + 4y = 12$

1. Find the value of y when $x = 0$.

2. Find the value of x when $y = 0$.

3. Plot both of those points on a coordinate grid and use a ruler to draw a line.

4. What is the coordinate of the x-intercept?

5. What is the coordinate of the y-intercept?

6. What are two other names for the x-intercept?

7. What is the slope of the line?

Self Check

Graph the equation $5x - 2y = 10$. Which of the following describes the intercepts and slope of the graph?

a. x-intercept: $x = -5$; y-intercept: $y = 2$; Slope: $\frac{5}{2}$
b. x-intercept: $x = 2$; y-intercept: $y = -5$; Slope: $\frac{5}{2}$
c. x-intercept: $x = -2$; y-intercept: $y = 5$; Slope: $-\frac{5}{2}$
d. x-intercept: $x = 2$; y-intercept: $y = -5$; Slope: $\frac{5}{2}$

Additional Resources

Graphing a Line and Describing Characteristics

There are many ways to graph an equation of a line. One way is to use the x-intercepts and y-intercepts as two points on the line.

EXAMPLE

Graph $3x - 6y = 18$ using the intercepts, then identify key characteristics.

Step 1 - Find the x-intercept.
To find the x-intercept, let $y = 0$.

$$\begin{aligned} 3x - 6y &= 18 \\ 3x - 6(0) &= 18 \\ 3x &= 18 \\ x &= 6 \end{aligned}$$

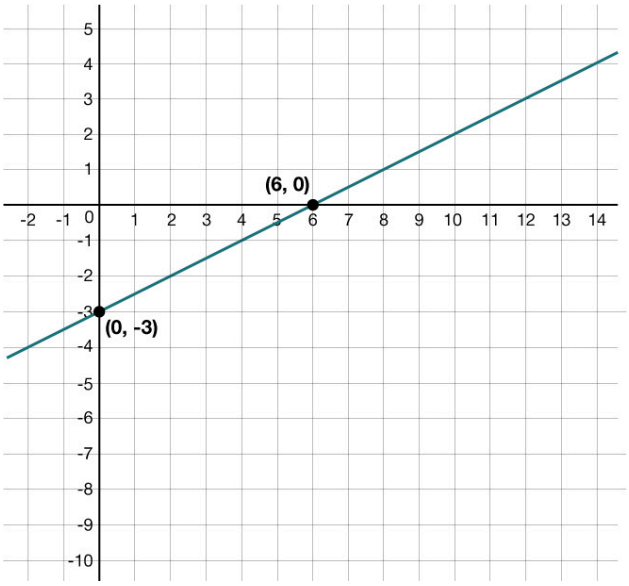
$(6, 0)$ is the x-intercept
The x-intercept is also called a solution or zero.

Step 2 -Find the y-intercept.
To find the y-intercept, let $x = 0$.

$$\begin{aligned} 3x - 6y &= 18 \\ 3(0) - 6y &= 18 \\ -6y &= 18 \\ y &= -3 \end{aligned}$$

$(0, -3)$ is the y-intercept

Step 3 -Graph the line by connecting the y-intercepts.



Step 4 - Find the slope of the line.

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{3}{6} = \frac{1}{2}$$

TRY IT **Graphing a Line and Describing Characteristics**

For questions 1 – 2, use the equation $7x - 3y = 21$.

1. Graph the equation using the intercepts.

2. What is the slope of the line?

1.5.3 Examining an Equation in Two Variables and Its Graph, Part 1

Activity

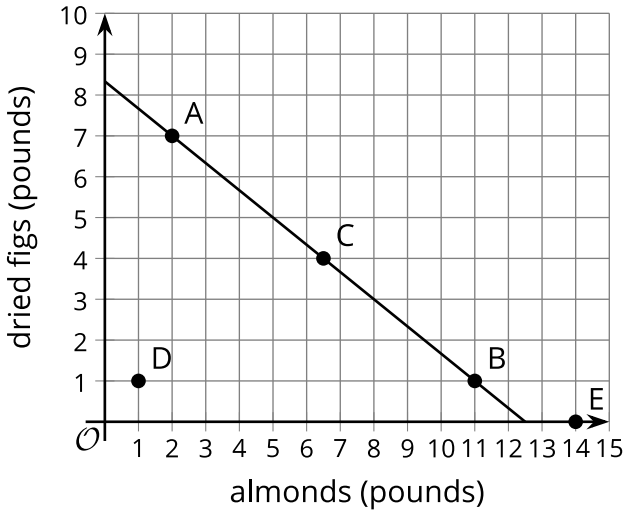
To get snacks for a class trip, Clare went to the "bulk" section of the grocery store, where she could buy any quantity of a product and the prices are usually good.

Clare purchased some salted almonds at $6 a pound and some dried figs at $9 per pound. She spent $75 before tax.



1. If she bought 2 pounds of almonds, how many pounds of figs did she buy?
2. If she bought 1 pound of figs, how many pounds of almonds did she buy?
3. Write an **equation** that describes the relationship between pounds of figs and pounds of almonds that Clare bought and the dollar amount that she paid. Be sure to specify what the **variables** represent.

For questions 4 and 5, use the graph that represents the quantities in the situation.



4. Choose any point on the line, state its coordinates, and explain what it tells us.
5. Choose any point that is not on the line, state its coordinates, and explain what it tells us.

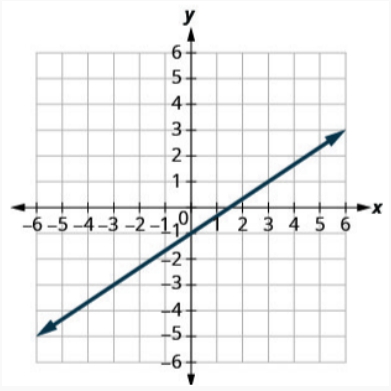
Video: Using a Graph to Solve an Equation

Watch the following video to further explore the above scenario.

Using a Graph to Solve an Equation (https://openstax.org/r/graph-equation-2)

Self Check

Which point is a solution to the line graphed below?



a. $(-3, 3)$
b. $(1, 3)$
c. $(0, 1)$
d. $(3, 1)$

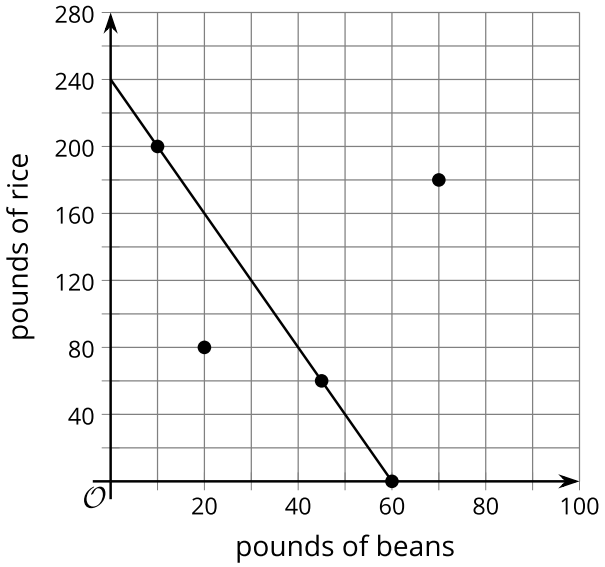
Additional Resources

Determining the Meaning of Solutions of a Graphed Line

Suppose we are buying beans and rice to feed a large gathering of people, and we plan to spend \$120 on the two ingredients. Beans cost \$2 a pound, and rice costs \$0.50 a pound.

If x represents pounds of beans and y pounds of rice, the equation $2x + 0.50y = 120$ can represent the constraints in this situation.

The graph of $2x + 0.50y = 120$ shows a straight line.



Each point on the line is a pair of x- and y-values that make the equation true and is thus a solution. It is also a pair of values that satisfy the constraints in the situation.

- The point $(10, 200)$ is on the line. If we buy 10 pounds of beans and 200 pounds of rice, the cost will be $2(10) + 0.50(200)$, which equals 120.
- The points $(60, 0)$ and $(45, 60)$ are also on the line. If we buy only beans—60 pounds of them—and no rice, we will spend \$120. If we buy 45 pounds of beans and 60 pounds of rice, we will also spend \$120.

What about points that are *not* on the line? They are not solutions because they don't satisfy the constraints, but they still have meaning in the situation.

- The point $(20, 80)$ is not on the line. Buying 20 pounds of beans and 80 pounds of rice costs $2(20) + 0.50(80)$ or 80, which does not equal 120. This combination costs less than what we intend to spend.
- The point $(70, 180)$ means that we buy 70 pounds of beans and 180 pounds of rice. It will cost $2(70) + 0.50(180)$ or 230, which is over our budget of 120.

Let's look at an example.

EXAMPLE

1. Looking at the graph about purchasing beans and rice, is the point (30, 120) a solution?

Solution

Yes, because $2(3) + 0.50(12) = 60 + 60 = 120$.

2. What does the point (30, 120) mean in this situation?

Solution

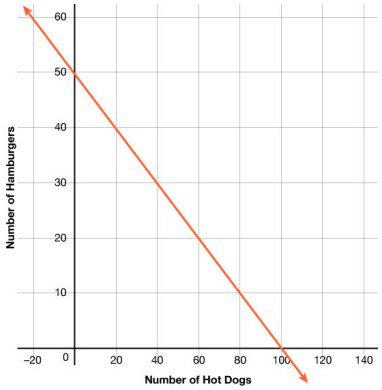
Compare your answer:

The point means that 30 pounds of beans and 120 pounds of rice were purchased. Since the point is on the line, the total spent was \$120.

TRY IT **Determining the Meaning of Solutions of a Graphed Line**

At a high school baseball game, the concession stand sold hot dogs and hamburgers. Hot dogs were $1.50, and hamburgers cost $3. The goal is to make $150 for the game.

What is the meaning of the point $(40, 30)$ on the graph?



1.5.4 Writing, Graphing, and Solving a Linear Equation

Activity

Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

Use the following scenario for questions 1 – 5:
A student has a savings account with $475 in it. She deposits $125 of her paycheck into the account every week. Her goal is to save $7000 for college.

1. How much will be in the account after 3 weeks?
2. How many weeks will it take until she has $1350?
3. Write an equation that represents the relationship between the dollar amount in her account and the number of weeks of saving.

Access multimedia content (http://openstax.org/books/algebra-1/pages/1-5-4-writing-graphing-and-solving-a-linear-equation)

4. Graph your equation using graphing technology. Mark the points on the graph that represent the amount after 3 weeks and the week she has $1350. Write down the coordinates. Use the graphing tool or technology outside the course. Graph the equation that represents this scenario using the Desmos tool above.
5. What is the x-intercept?
6. What other information does the x-intercept identify in the function?
7. What is the the y-intercept?
8. What is the slope?

Are you ready for more?

Extending Your Thinking

Use the following information to answer questions 1 – 5. A 450-gallon tank full of water is draining at a rate of 20 gallons per minute.

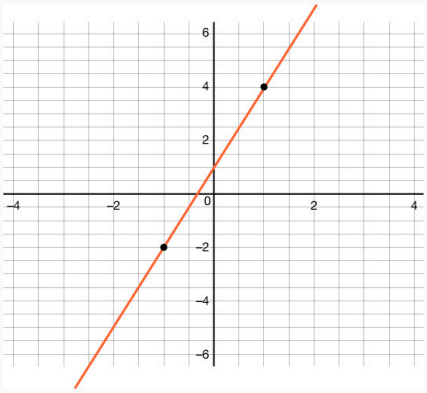
1. Write an equation that represents the relationship between the gallons of water in the tank and hours the tank has been draining.
2. Write an equation that represents the relationship between the gallons of water in the tank and seconds the tank has been draining.

Access multimedia content (http://openstax.org/books/algebra-1/pages/1-5-4-writing-graphing-and-solving-a-linear-equation)

3. Graph each of your new equations. Use the graphing tool or technology outside the course. Graph the equation that represents this scenario using the Desmos tool above.
4. In what way are all of the graphs the same? In what way are they all different?
5. How would these graphs change if we used quarts of water instead of gallons? What would stay the same?

Self Check

Which of the following points is a solution to the graph below?



a. $(-2,-6)$
b. $(7,2)$
c. $(0,3)$
d. $(2,7)$

Additional Resources

Writing Equations Using Graphs in Situations

An equation that contains two unknown quantities or two quantities that vary is called an equation in two variables. A solution to such an equation is a pair of numbers that makes the equation true.

Suppose Tyler spends $40 on T-shirts and socks. A T-shirt costs $10 and a pair of socks costs $2.50. If t represents the number of T-shirts and p represents the number of pairs of socks that Tyler buys, what is an equation that represents the equation?

EXAMPLE 1

Step 1 - Create a two-variable equation.
The cost is $10 per t-shirt $(10t)$ plus $2.50 per pair of socks $(2.50p)$ which equals $40.
$10t + 2.50p = 40$

Now, we have to graph the equation. We will let $t = x$ and $p = y$.

Step 2 - Find the x-intercept.
To find the x-intercept, let $p = 0$.

$10t + 2.50p = 40$
$10t + 2.50(0) = 40$
$10t + 0 = 40$
$t = 4$

$(4, 0)$ is the x-intercept.
The x-intercept is also called a solution or zero.

In this scenario, 4 represents the number of T-shirts Tyler can buy if he doesn't purchase any socks with $40.

Step 3 - Find the y-intercept.
To find the y-intercept, let $t = 0$.

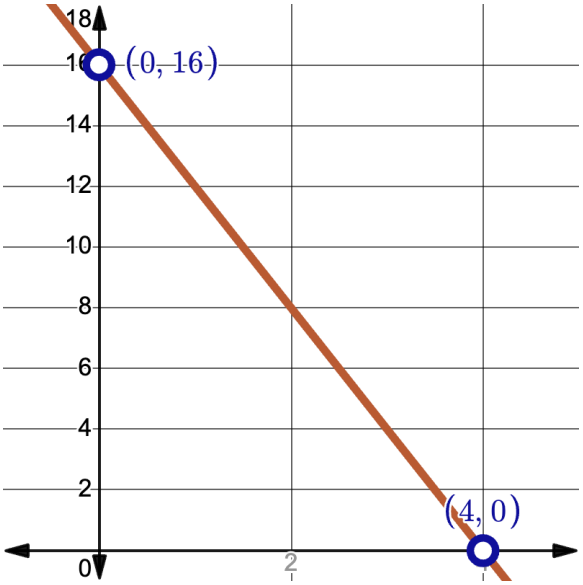
$10t + 2.50p = 40$
$10(0) + 2.50p = 40$
$(0) + 2.50p = 40$
$2.50p = 16$

$(0, 16)$ is the y-intercept.

In this scenario, 16 represents the number of socks Tyler can buy if he doesn't purchase any T-shirts with $40.

Step 4 - Graph the line by connecting the intercepts.

Let's look at the graph of this equation:



Let's reflect about the graph and what it means.

EXAMPLE 2

What is the slope of the graph?

Solution

$m = -4$

EXAMPLE 3

What does the point $(4, 6)$ mean on this graph?

Solution

Compare your answer:

If Tyler bought 4 T-shirts and 6 pairs of socks, it would cost more than $40.

TRY IT **Writing Equations Using Graphs in Situations**

Use Desmos or a graphing calculator to create a graph for $40x + 20y = 180$.

Access multimedia content (http://openstax.org/books/algebra-1/pages/1-5-4-writing-graphing-and-solving-a-linear-equation)

1. If x represents the number of pairs of shoes and y represents the number of pairs of jeans, what is one combination that is a solution?
2. What does the combination you identified mean on the graph?

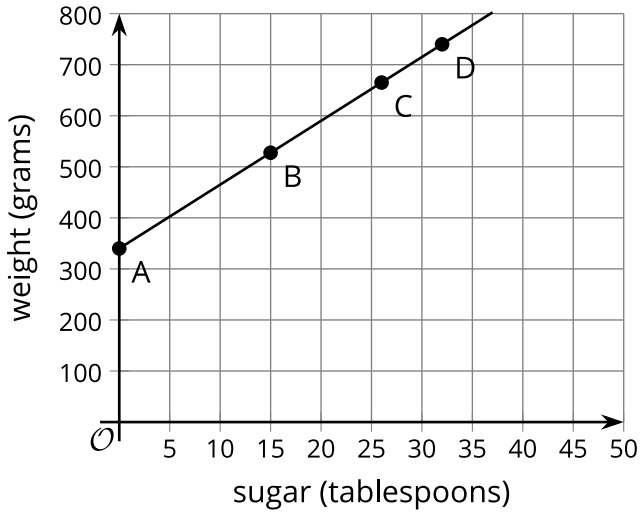
1.5.5 Examining an Equation in Two Variables and Its Graph, Part 2

Cool Down

A ceramic sugar bowl weighs 340 grams when empty. It is then filled with sugar. One tablespoon of sugar weighs 12.5 grams.

1. Write an equation to represent the relationship between the total weight of the bowl in grams, W, and the tablespoons of sugar, T.
2. When the sugar bowl is full, it weighs 740 grams. How many tablespoons of sugar can the bowl hold? Be prepared to show your reasoning.

The graph represents the relationship between the number of tablespoons of sugar in the bowl and the total weight of the bowl.

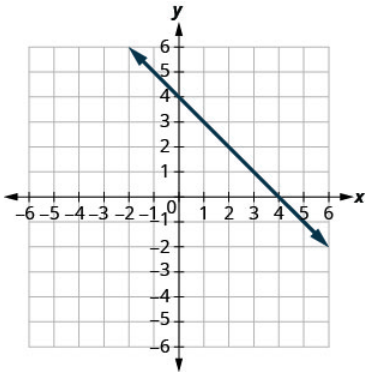


3. Which point on the graph could represent your answer to the previous question?
4. About how many tablespoons of sugar are in the bowl when the total weight is 600 grams?

1.5.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

Use the graph for questions 1 - 2.



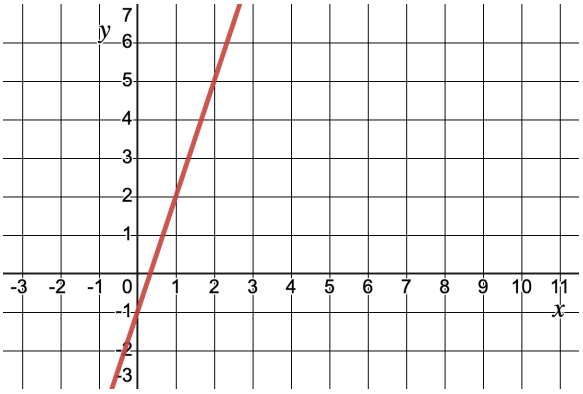
1. Which of the following ordered pairs are on the line? Select **three** answers.

a. $(0, 4)$
b. $(2, 2)$
c. $(-2, 6)$
d. $(-1, 3)$
e. $(-4, 0)$

2. Which of the following ordered pairs are solutions to the equation? Select **three** solutions.

a. $(0, 4)$
b. $(2, 2)$
c. $(-2, 6)$
d. $(-1, 3)$
e. $(-4, 0)$

For questions 3 and 4, use the graph of $y = 3x - 1$ as shown.



3. Which of the following ordered pairs are on the line? Select **three** solutions.

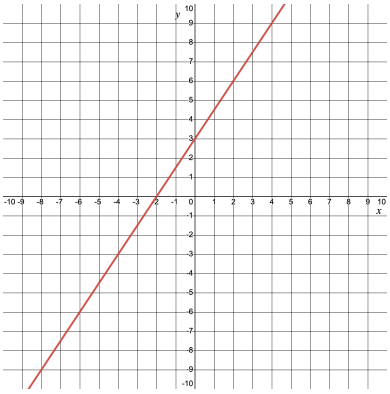
a. $(0, -1)$
b. $(1, 2)$
c. $(2, 5)$
d. $(1, 1)$
e. $(0, 1)$

4. Which of the following ordered pairs are solutions to the equation? Select **three** solutions.

a. $(0, -1)$
b. $(1, 2)$

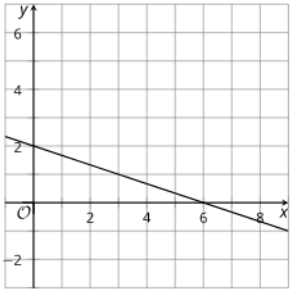
c. $(2, 5)$
d. $(1, 1)$
e. $(0, 1)$

5. Select three points that are on the graph of the equation $4y - 6x = 12$.



a. $(-4, -3)$
b. $(-1, 1.5)$
c. $(0, 3)$
d. $(0, -2)$
e. $(6, 4)$

6. Here is a graph of the equation $x + 3y = 6$.



Select three coordinate pairs that represent a solution to the equation.

a. $(0, 2)$
b. $(6, 0)$
c. $(2, 6)$
d. $(3, 1)$
e. $(8, -1)$
f. $(-1, 2)$

1.5.7 Lesson Summary

In this lesson, you learned how to:

- Use graphing technology to graph linear equations and identify solutions to the equations.
- Explain how the coordinates of the points on the graph of a linear equation are related to the equation.
- Explain the meaning of points on a graph in terms of the situation it represents when given the graph of a linear equation.

Here are the **activities** that helped you reach those goals:

- 1.5.1: Analyzing Graphs of Linear Equations
  - In this activity, you compared the characteristics of similar, yet different, graphs. In graphs, slopes, intercepts, axis labels, and where points are located can all create different meanings.
- 1.5.2: Graphing Linear Functions in Two Variables
  - In this activity, you reviewed how to graph points on a coordinate plane and then used your understanding to find and graph the x- and y-intercepts of an equation written in standard form. With these intercepts, you calculated the slope of the line.
  - 1.5.2: Self Check
  - 1.5.2: Additional Resources
- 1.5.3: Examining an Equation in Two Variables and Its Graph, Part 1
  - In this activity, you determined if a point was on the graph of a linear equation and what the meaning of that point was in a given situation
  - 1.5.3: Self Check
  - 1.5.3: Additional Resources
- 1.5.4: Writing, Graphing, and Solving a Linear Equation
  - In this activity, you wrote equations, graphed the equations, and used the graph to help solve the equations and find the meaning of solutions.
  - 1.5.4: Self Check
  - 1.5.4: Additional Resources
- 1.5.5: Examining an Equation in Two Variables and Its Graph, Part 2
  - In this activity, you wrote equations to express relationships. You also looked at graphs and determined the meaning of points on the graph to answer questions.

After these activities, you completed the following **practice**:

- 1.5.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

1.6 EQUIVALENT EQUATIONS

1.6.0 Lesson Overview

In this lesson, you will learn that equivalent equations are equations with identical solutions. Specifically, you will explore how applying properties of real numbers can create equivalent equations. Additionally, you will use context to interpret the solution to equivalent equations.

When you finish this lesson, you will be able to:

- Tell whether two expressions are equivalent and explain why or why not.
- Identify the moves that can be made to transform an equation into an equivalent one.
- Explain what it means for two equations to be equivalent and how equivalent equations can be used to describe the same situation in different ways.

Here are the **activities** that will help you reach those goals:

- 1.6.1: Exploring Equivalent Expressions
- 1.6.2: Expressing Relationships as Equations
  - 1.6.2: Self Check
  - 1.6.2: Additional Resources
- 1.6.3: Exploring Related Equations
  - 1.6.3: Self Check
  - 1.6.3: Additional Resources
- 1.6.4: Recognizing Related Equations

After that, you'll **practice and review**.

- 1.6.5: Practice
- 1.6.6: Lesson Summary

1.6.1 Exploring Equivalent Expressions

Warm Up

Your teacher will assign you one of these expressions:

$$\frac{n^2-9}{2(4-3)} \quad \text{or} \quad (n+3)\cdot\frac{n-3}{8-3\cdot 2}$$

For questions 1 – 4, evaluate your expression when n is the given value.

1. $n = 5$

2. $n = 7$

3. $n = 13$

4. $n = -1$

5. What did you discover when you compared your answers? Explain your rationale.

1.6.2 Expressing Relationships as Equations

Activity

In questions 1 – 3, write as many equations as possible that could represent the relationship between the ages of the two children in each family described. Be prepared to explain what each part of your equation represents.

1. In Family A, the youngest child is 7 years younger than the oldest, who is 18.

2. In Family B, the middle child is 5 years older than the youngest child.

3. Tyler thinks that the relationship between the ages of the children in Family B can be described with $2m - 2y = 10$, where m is the age of the middle child and y is the age of the youngest. Describe how Tyler came up with this equation.

4. **Select three** equations that are equivalent to $3a + 6 = 15$.
   a. $a + 3 = 12$
   b. $3a = 9$
   c. $\frac{1}{3}a = 1$
   d. $a + 2 = 5$

5. Explain your reasoning for the equivalent equations in number 4.

Are you ready for more?

Extending Your Thinking

Here is a puzzle:

- $m + m = N$
- $N + N = p$
- $m + p = Q$
- $p + Q = ?$

Write two expressions that are equivalent to $p + Q$.

Self Check

Which equation is equivalent to $3x - 6 = 18$?
a. $3x + 12 = 0$
b. $x - 2 = 18$
c. $3x = 12$
d. $3x = 24$

Additional Resources

Finding Equivalent Equations

Equivalent equations are equations that have the same solutions. They are often found by using inverse operations or by multiplying each term in the equation by the same value.

EXAMPLE

Write three equivalent equations to $4x - 2 = 8$ using one of three strategies.

- Using an inverse operation
- Using division
- Using multiplication

Using an inverse operation:
An equivalent equation can be determined by adding 2 to both sides.

$4x - 2 = 8$
$4x - 2 + 2 = 8 + 2$
$4x = 10$
$(4x - 2 = 8)$ and $(4x = 10)$ are equivalent.

Using division:
An equivalent equation can be determined by dividing by 2 (or multiplying by $\frac{1}{2}$).

$4x - 2 = 8$
$(\frac{1}{2})(4x - 2) = (\frac{1}{2})(8)$
$2x - 1 = 4$

$(4x - 2 = 8)$ and $(2x - 1 = 4)$ are equivalent.

Using multiplication:
An equivalent equation can be determined by multiplying the same number to both sides.

What is the equivalent equation if you multiply each term in the equation $4x - 2 = 8$ by 3?

TRY IT **Finding Equivalent Equations**

1. Determine the equivalent equation if you subtract 5 from each side of the equation, $3x + 5 = 15$.
2. Explain how $(3x = 10)$ is related to $(x = \frac{10}{3})$.
3. Multiply every term by the same number to determine an equivalent equation to $3x + 5 = 15$.
4. Is $(3x + 5 = 15)$ equivalent to $(6x + 15 = 60)$?
5. Explain your reasoning for your answer to number 4 and whether the equations $(3x + 5 = 15)$ and $(6x + 15 = 60)$ are equivalent.

1.6.3 Exploring Related Equations

Activity

With a partner, discuss the following situations and equations. For each, consider:

- What does the solution mean in the context of the situation?
- Are the given values solutions?

If operations are applied correctly, the solution to an equation is also the solution to all equations equivalent to it. If operations are incorrectly applied, the solution of each equation is different.

For questions 1 and 2, use the following situation and equation:

Noah is buying a pair of jeans and using a coupon for 10% off. The total price is $56.70, which includes $2.70 in sales tax. Noah's purchase can be modeled by the equation:

$x - 0.1x + 2.70 = 56.70$

1. What does the solution to the equation mean in this situation?
2. How can you verify that 70 is not a solution but 60 is the solution?
3. What was done to Noah's equation to make this one?
4. What is the interpretation of this new equation?
5. Is the solution the same?
6. What was done to Noah's equation to make this one?
7. What is the interpretation of this new equation?
8. Is the solution the same?
9. What was done to Noah's equation to make this one?
10. What is the interpretation of this new equation?
11. Is the solution the same?

For questions 12 – 14, use the equation $x - 0.1x = 56.70$.

12. What was done to Noah's equation to make this one?
13. What is the interpretation of this new equation?
14. Is the solution the same?

For questions 15 – 17, use the equation $x - 0.1x = 59.40$

15. What was done to Noah's equation to make this one?

16. What is the interpretation of this new equation?

17. Is the solution the same?

For questions 18–20, use the equation $2(x - 0.1x + 2.70) = 56.70$.

18. What was done to Noah's equation to make this one?

19. What is the interpretation of this new equation?

20. Is the solution the same?

21. Based on your work above, which of the six equations are equivalent to the original equation, $x - 0.1x + 2.70 = 56.70$? Select the **three** equations that are equivalent to the original.

a. $100x - 10x + 270 = 5670$
b. $x - 0.1x = 54$
c. $0.9x + 2.70 = 56.70$
d. $x - 0.1x = 56.70$
e. $x - 0.1x = 59.40$
f. $2(x - 0.1x + 2.70) = 56.70$

Video: Looking at Equivalent Equations

Watch the following video to learn more about why these are equivalent equations.

Looking at Equivalent Equations (https://openstax.org/r/equivalent-equations)

Self Check

Which of the following is a correct next step to create an equation equivalent to $x - 0.2x - 4.2 = 13.4$?

a. $2(x - 0.2x) = 17.6$
b. $x - 0.2x = 9.2$
c. $10(x - 0.2x - 4.2) = 13.4$
d. $0.8x - 4.2 = 13.4$

Additional Resources

Properties of Equality

When you add, subtract, multiply, or divide the same quantity from both sides of an equation, you still have equality.

Subtraction Property of Equality For any real numbers a, b, and c, if $a = b$, then $a - c = b - c$.	Addition Property of Equality For any real numbers a, b, and c, if $a = b$, then $a + c = b + c$.
Division Property of Equality For any real numbers a, b, and c, and $c \neq 0$, if $a = b$, then $\frac{a}{c} = \frac{b}{c}$.	Multiplication Property of Equality For any real numbers a, b, and c, if $a = b$, then $ac = bc$.

For equations to be equivalent, inverse operations are used and must be applied to both sides of an equation so it remains balanced.

Inverse operations are operations that "undo" other operations.

EXAMPLE 1

Solve $3x = 15$.

Since 3 is being multiplied by x, to solve for x, divide both sides by 3.

$\frac{3x}{3} = \frac{15}{3}$

$x = 5$

EXAMPLE 2

Solve $\frac{x}{2} + 4 = 12$.

Step 1 - Subtract 4 from both sides.
$\frac{x}{2} + 4 - 4 = 12 - 4$

Step 2 - Simplify.
$\frac{x}{2} = 8$

Step 3 - Multiply both sides by 2.
$2 \times \frac{x}{2} = 8 \times 2$

Step 4 - Simplify.
$x = 16$

TRY IT **Properties of Equality**

For questions 1 – 2, use the scenario:

Denae bought 6 pounds of grapes for $10.74.

1. Write an equation for the situation.
2. Solve the equation.

1.6.4 Recognizing Related Equations

Cool Down

A cardboard box, which weighs 0.6 pound when empty, is filled with 15 bags of beans and a 4-pound bag of rice. The total weight of the box and the contents inside it is 25.6 pounds. One way to represent this situation is with the equation $0.6 + 15b + 4 = 25.6$.

1. In this situation, what does the solution to the equation represent?
2. Identify the **three** equations that are equivalent to 0.6+15b+4=25.6.
   a. $15b + 4 = 25.6$
   b. $15b + 4 = 25$
   c. $3(0.6 + 15b + 4) = 76.8$
   d. $15b = 25.6$
   e. $15b = 21$

1.6.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. **Select three** equations that have the same solution as the equation $3x - 12 = 24$.

   a. $15x - 60 = 120$
   b. $3x = 12$
   c. $3x = 36$
   d. $x - 4 = 8$
   e. $12x - 12 = 24$

2. **Select three** equations that have the same solution as $2x - 5 = 15$.

   a. $2x = 10$
   b. $2x = 20$
   c. $2(x - 5) = 15$
   d. $2x - 20 = 0$
   e. $4x - 10 = 30$
   f. $15 = 5 - 2x$

3. **Select four** of the equations that are equivalent to: $x + 0.4x - 4.2 = 12.8$.

   a. $1.4x - 4.2 = 12.8$
   b. $1.4x = 8.6$
   c. $1.4x = 17$
   d. $14x - 42 = 128$
   e. $10x + 4x - 42 = 12.8$
   f. $14x = 170$

4. **Select two** of the following that are equivalent to $3x - 6 = 12$.

   a. $3x = 6$
   b. $3(x - 2) = 12$
   c. $x - 2 = 4$
   d. $x - 2 = 12$
   e. $x - 6 = 4$

5. Identify which step the mistake is made in finding an equivalent equation to $2(x - 4) = 16$.

   Step 1: $2x - 4 = 16$
   Step 2: $2x = 20$
   Step 3: $x = 10$

   a. Step 1
   b. Step 2
   c. Step 3

6. Identify which step the mistake is made in finding an equivalent equation to $2x + 3x - 4 = 20$.

   Step 1: $5x - 4 = 20$
   Step 2: $5x = 16$
   Step 3: $x = 16/5$

   a. Step 1
   b. Step 2
   c. Step 3

7. Identify which step the mistake is made in finding an equivalent equation to $7x - 4x - 5 = 16$.

   Step 1: $3x - 5 = 16$
   Step 2: $3x = 21$

Step 3: $x = 62$

a. Step 1
b. Step 2
c. Step 3

8. Fill in the blank to complete the equivalent equation to $3(x - 5) = 30$.

$3x -$ ___ $= 30$

9. Fill in the blank to complete the equivalent equation to $\frac{x}{2} - 5 = 11$.

$x -$ _____ $= 22$

10. Fill in the blank to complete the equivalent equation to $3(x - 9) = 24$.

$x -$ ___ $= 8$

1.6.6 Lesson Summary

In this lesson, you learned how to:

- Tell whether two expressions are equivalent and explain why or why not.
- Identify the moves that can be made to transform an equation into an equivalent one.
- Explain what it means for two equations to be equivalent, and how equivalent equations can be used to describe the same situation in different ways.

Here are the **activities** that helped you reach those goals:

- 1.6.1 Exploring Equivalent Expressions
  - In this activity, you substituted values into equivalent expressions to show they are equivalent.
- 1.6.2 Expressing Relationships as Equations
  - In this activity, you wrote equivalent equations given different relationships.
  - 1.6.2 Self Check
  - 1.6.2 Additional Resources
- 1.6.3 Exploring Related Equations
  - In this activity, you identified changes that were made to create equivalent equations then determined if other equations were also equivalent after certain moves were made to the equation.
  - 1.6.3 Self Check
  - 1.6.3 Additional Resources
- 1.6.4 Recognizing Related Equations
  - In this activity, you determined which equations in a given group were all equivalent to each other.

After these activities, you completed the following **practice**:

- 1.6.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

1.7 EXPLAINING STEPS FOR REWRITING EQUATIONS

1.7.0 Lesson Overview

In this lesson, you will further develop the idea of equivalent equations by applying the concept of acceptable moves. Additionally, you will encounter equations that have no solutions with moves that appear to be acceptable.

When you finish this lesson, you will be able to:

- Explain why some algebraic moves create equivalent equations but some do not.
- Describe how equivalent equations are related to the steps of solving equations.
- Explain what it means for an equation to have no solutions and recognize such an equation.

Here are the **activities** that will help you reach those goals:

- 1.7.1: Determining if Zero Is a Solution
- 1.7.2: Explaining Acceptable Moves Used to Solve an Equation
  - 1.7.2: Self Check
  - 1.7.2: Additional Resources
- 1.7.3: Understanding Equations with No Solution or Infinitely Many
  - 1.7.3: Self Check
  - 1.7.3: Additional Resources
- 1.7.4: Solving Equations by Dividing Values

After that, you'll **practice and review**.

- 1.7.5: Practice
- 1.7.6: Lesson Summary

1.7.1 Determining if Zero Is a Solution

Warm Up

1. Is 0 a **solution of an equation** $4(x+2)=10$? Explain.
2. Is 0 a solution for the equation $12-8x=3(x+4)$? Explain.
3. Is 0 a solution for the equation $5x=\frac{1}{2}x$? Explain.
4. Is 0 a solution for the equation $\frac{6}{x}+1=8$? Explain.

1.7.2 Explaining Acceptable Moves to Solve an Equation

Activity

Here are some pairs of equations. While one partner listens, the other partner should:

- Choose a pair of equations from column A. Explain why, if x is a number that makes the first equation true, then it also makes the second equation true.
- Choose a pair of equations from column B. Explain why the second equation is no longer true for a value of x that makes the first equation true.

Then, switch roles until you run out of time or you run out of pairs of equations.

Some principles you might use to complete this activity include the following:

- Distributive Property
- Commutative Property
- Addition Property of Equality
- Additive Inverse
- Subtraction Property of Equality
- Multiplication Property of Equality

- Division Property of Equality

	A	B
1.	$16 = 4(9 - x)$ $16 = 36 - 4x$	$9x = 5x + 4$ $14x = 4$
2.	$5x = 24 + 2x$ $3x = 24$	$\frac{1}{2}x - 8 = 9$ $x - 8 = 18$
3.	$-3(2x + 9) = 12$ $2x + 9 = -4$	$6x - 6 = 3x$ $x - 1 = 3x$
4.	$5x = 3 - x$ $5x = -x + 3$	$-11(x - 2) = 8$ $x - 2 = 8 + 11$
5.	$18 = 3x - 6 + x$ $18 = 4x - 6$	$4 - 5x = 24$ $5x = 20$

After you have discussed these equations with your partner, answer questions 1 – 2.

1. Which pair of equations did you select from Column A? Explain your reasoning.

2. Which pair of equations did you select from Column B? Explain your reasoning.

Self Check

Select all of the following equations that are equivalent to $3(x - 2) = 18$.

a. $3x = 12$
b. $3x - 6 = 18$
c. $3x - 2 = 18$
d. $x - 18 = 6$

Additional Resources

Solving Equations and Creating Equivalent Equations

One way to create equivalent equations is to solve the equation. In each step of solving, an equivalent equation is created.

Here is a general strategy for solving linear equations:

Step 1 - Simplify each side of the equation as much as possible.

Use the Distributive Property to remove any parentheses. Combine like terms.

Step 2 - Collect all the variable terms on one side of the equation.

Use the Addition or Subtraction Property of Equality.

Step 3 - Collect all the constant terms on the other side of the equation.

Use the Addition or Subtraction Property of Equality.

Step 4 - Make the coefficient of the variable term equal to 1.

Use the Multiplication or Division Property of Equality.

State the solution to the equation.

Step 5 - Check the solution.

Substitute the solution into the original equation to make sure the result is a true statement.

EXAMPLE 1

Here is an example of solving equations. Notice that each step is an acceptable move that creates an equivalent equation.

Step 1 - Simplify each side of the equation as much as possible.

Use the Distributive Property to remove any parentheses.

$$\begin{aligned} -6(x+3) &= 24 \\ -6x-18 &= 24 \end{aligned}$$

Step 2 - Collect all the variable terms on one side of the equation.

Nothing to do. All the variables of x are on the left side.

Step 3 - Collect all the constant terms on the other side of the equation.

Use the Addition or Subtraction Property of Equality.

$$\begin{aligned} -6x-18+18 &= 24+18 \\ -6x &= 42 \end{aligned}$$

Step 4 - Make the coefficient of the variable term equal to 1.

Use the Multiplication or Division Property of Equality.

$$\begin{aligned} \frac{-6x}{-6} &= \frac{42}{-6} \\ x &= -7 \end{aligned}$$

Step 5 - Check the solution.

Substitute the solution into the original equation to make sure the result is a true statement.

$$\begin{aligned} -6(x+3) &= 24 \\ -6(-7+3) &= 24 \\ -6(-4) &= 24 \\ 24 &= 24\checkmark \end{aligned}$$

EXAMPLE 2

Find the solution for $3(4x-1)-2=8x+3$.

Step 1 - Simplify each side of the equation as much as possible.

$$\begin{aligned} 3(4x-1)-2 &= 8x+3 \\ 12x-3-2 &= 8x+3 \\ 12x-5 &= 8x+3 \end{aligned}$$

Step 2 - Collect all the variable terms on one side of the equation.

$$\begin{aligned} 12x-5-8x &= 8x+3-8x \\ 4x-5 &= 3 \end{aligned}$$

Step 3 - Collect all the constant terms on the other side of the equation.

$$\begin{aligned} 4x-5+5 &= 3+5 \\ 4x &= 8 \end{aligned}$$

Step 4 - Make the coefficient of the variable term equal to 1.

$$\begin{aligned} \frac{4x}{4} &= \frac{8}{4} \\ x &= 2 \end{aligned}$$

Step 5 - Check the solution.

$$\begin{aligned} 3(4x-1)-2 &= 8x+3? \\ 3(4(2)-1)-2 &= 8(2)+3? \\ 3(8-1)-2 &= 16+3? \\ 3(7)-2 &= 19? \\ 21-2 &= 19? \\ 19 &= 19\checkmark \end{aligned}$$

EXAMPLE 3

Solve $12x+3(x+7)=10x-24$.

Step 1 - Simplify each side of the equation as much as possible.

$$\begin{aligned} 12x+3(x+7) &= 10x-24 \\ 12x+3x+21 &= 10x-24 \\ 15x+21 &= 10x-24 \end{aligned}$$

Step 2 - Collect all the variable terms on one side of the equation.

$$\begin{aligned} 15x+21-10x &= 10x-24-10x \\ 5x+21 &= -24 \end{aligned}$$

Step 3 - Collect all the constant terms on the other side of the equation.

$$\begin{aligned} 5x+21-21 &= -24-21 \\ 5x &= -45 \end{aligned}$$

Step 4 - Make the coefficient of the variable term equal to 1.

$$\begin{aligned} \frac{5x}{5} &= -\frac{45}{5} \\ x &= -9 \end{aligned}$$

Step 5 - Check the solution.

$$\begin{aligned} 12x+3(x+7) &= 10x-24? \\ 12(-9)+3(-9+7) &= 10(-9)-24? \\ -108+3(-2) &= -90-24? \\ -108+-6 &= -86? \\ -114 &= -114\checkmark \end{aligned}$$

Video: Acceptable Moves

Watch the following video to learn more about acceptable moves.

Simplifying Equations with Inverse Operations (https://openstax.org/r/simplify-equations)

TRY IT **Solving Equations and Creating Equivalent Equations**

1. Write two equivalent equations to $2(x+4)=12$.
2. Solve $2(x+4)=12$.
3. Solve $-12+8(x-5)=-4+3(5x-2)$

1.7.3 Understanding Equations with No Solution or Infinitely Many

Activity

Noah is having trouble solving two equations. In each case, he took steps that he thought were acceptable but ended up with statements that are clearly not true.

Analyze Noah's work on each equation and the moves he made. Were they acceptable moves? Why do you think he

ended up with a false equation?

Discuss your observations with your group and be prepared to share your conclusions. If you get stuck, consider solving each equation.

EXAMPLE 1

Step 1 - Original equation.

$x + 6 = 4x + 1 - 3x$

Step 2 - Apply the Commutative Property.

$x + 6 = 4x - 3x + 1$

Step 3 - Combine like terms.

$x + 6 = x + 1$

Step 4 - Subtract x from each side.

$6 = 1$

EXAMPLE 2

Step 1 - Original equation.

$2(5 + x) - 1 = 3x + 9$

Step 2 - Apply the Distributive Property.

$10 + 2x - 1 = 3x + 9$

Step 3 - Subtract 10 from each side.

$2x - 1 = 3x - 1$

Step 4 - Add 1 to each side.

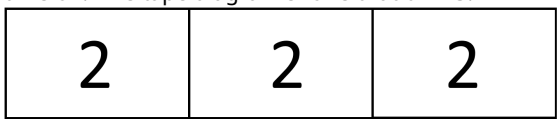
$2x = 3x$

1. Why did Noah have no solution for either equation?
2. Why did Noah arrive at no solution for the equation solved in Example 2?

Are you ready for more?

Extending Your Thinking

1. We cannot divide the number 100 by zero because dividing by zero is undefined. Instead, try dividing 100 by 10, then 1, then 0.1, then 0.01. What do you notice happens as you divide by smaller numbers?
2. Now try dividing the number -100 by 10, by 1, by 0.1, 0.01. What is the same and what is different when you compare to question 1?
3. In middle school, you used tape diagrams to represent division. This tape diagram shows that 6÷2=3.



   Draw a tape diagram that shows why $6 \div \frac{1}{2} = 12$.

4. Try to draw a tape diagram that represents 6÷0. Explain why this is so difficult.

Self Check

Which of the following equations has a solution?

a. $10x + 10 = 7x + 3x$

b. $x + 3 = x - 2$
c. $2x = 2x + 1$
d. $3x = x + 18$

Additional Resources

Equations Without Solutions

What happens when we solve $5z = 5z - 1$?

Step 1 - Subtract $5z$ to get the z terms together on one side.

$5z - 5z = 5z - 5z - 1$

Step 2 - Simplify and the z terms are gone!.

$0 = 1$

Since 0 does not equal -1, this equation led to a false statement. There are no solutions to this equation.

EXAMPLE

Solve $5m + 3(9 + 3m) = 2(7m - 11)$.

Step 1 - Simplify each side of the equation as much as possible. Use the Distributive Property to remove any parentheses. Combine like terms.

$5m + 27 + 9m = 14m - 22$

$14m + 27 = 14m - 22$

Step 2 - Collect all the variable terms on one side of the equation.

Use the Addition or Subtraction Property of Equality.

$14m - 14m + 27 = 14m - 14m - 22$

$27 = -22$

Step 3 - Check the solution.

The equation $27 = -22$ is NOT a true statement. The equation 5m+3(9+3m)=2(7m−11) has no solutions.

Infinitely Many Solutions

What if an equation results in a true statement without variables?

The equations $-4 = -4$, $5 = 5$, and $0 = 0$ are all true statements because one side is equal to the other. This means that any value of the variable in the original equation would be a solution. When this happens in the final step of an equation, there are infinitely many solutions.

EXAMPLE

Solve $2y + 6 = 2(y + 3)$

Step 1 - Simplify each side of the equation as much as possible. Use the Distributive Property to remove any parentheses. Combine like terms.

$2y + 6 = 2(y + 3)$

Step 2 - Collect all the variable terms on one side of the equation. Use the Addition or Subtraction Property of Equality.

$2y - 2y + 6 = 2y - 2y + 6$

$6 = 6$

$2y + 6 = 2y + 6$

Step 3 - Check the solution. We say the solution to the equation is all of the real numbers.An equation that is true for any value of the variable is called an **identity**.

The solution of an identity is all real numbers.

TRY IT **Equations with No Solutions and Infinitely Many Solutions**

1. Solve $10x - 4 = 2(5x - 2)$and determine the number of solutions.
2. Solve $3x - 4 = 3(x - 2)$ and determine the number of solutions.

1.7.4 Solving Equations by Dividing Values

Cool Down

1. The equation $4(x - 2) = 100$ is a true equation for a particular value of x. Explain why $2(x - 2) = 50$ is also true for the same value of x.

For questions 2 – 4, use the following situation:

To solve the equation $7.5 = 2.5d$, Lin divides each side by $2.5d$, and Elena subtracts $2.5d$ from each side.

2. Will Lin's move lead to a solution? Explain your reasoning.
3. Will Elena's move lead to a solution? Explain your reasoning.
4. What is the solution to the equation?

1.7.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 3, choose an equivalent equation.

1. Which equation is equivalent to $3x + 6 = 4x + 7$?

a. $3x = 4x + 7$
b. $3x - 1 = 4x$
c. $9x = 4x + 7$
d. $3x + 18 = 4x + 7$
e. $7x = 1$

2. Which equation is equivalent to $3(x + 6) = 4x + 7$?

a. $3x = 4x + 7$
b. $3x - 1 = 4x$
c. $9x = 4x + 7$
d. $3x + 18 = 4x + 7$
e. $7x = 1$

3. Which equation is equivalent to $4x + 3x = 7 - 6$?

a. $3x = 4x + 7$
b. $3x - 1 = 4x$
c. $9x = 4x + 7$
d. $3x + 18 = 4x + 7$
e. $7x = 1$

4. **Select three** equations that have 0 as a solution.

a. $2x + 10 = 4x + 10$
b. $4x = 9x$
c. $2x - 3 + 4x = 4 - 7$
d. $5x + 7 = 0$
e. $3(x - 2) = 9$

5. Which equation is equivalent to $6(x - 3) = 24$?

a. $6x = 27$
b. $6x = 21$
c. $x - 3 = 4$
d. $6x - 3 = 24$

6. Which of the following sets of equations are equivalent to each other?

a. $6x - 4 = 11$ and $6x = 7$
b. $5x - 2 = 7x + 4$ and $12x - 2 = 4$
c. $7(x + 2) = 21$ and $7x + 2 = 21$
d. $10(x + 9) = 50$ and $2(x + 9) = 10$

7. Using the distributive property, choose which equation would be an equivalent equation to $3(x - 2) = 7$.

a. $3x = 9$
b. $x - 2 = 4$
c. $3x - 6 = 7$
d. $3x - 2 = 7$

1.7.6 Lesson Summary

In this lesson, you learned how to:

- Explain why some algebraic moves create equivalent equations but some do not.
- Illustrate that dividing by a variable is not used in solving equations because it can lead to equations that have fewer solutions than the original equation.
- Demonstrate that equations that are not true for any value of the variable(s) do not have solutions.

Here are the **activities** that helped you reach those goals:

- 1.7.1: Determining if Zero Is a Solution
  - In this activity, you substituted 0 into equations to determine for which equations 0 was a solution.
- 1.7.2: Explaining Acceptable Moves Used to Solve an Equation
  - In this activity, you learned how acceptable moves applied to an equation can create an equivalent equation. Through these discussions you had the opportunity to construct logical arguments.
  - 1.7.2: Self Check
  - 1.7.2: Additional Resources
- 1.7.3: Understanding Equations with No Solution or Infinitely Many
  - In this activity, you explored why equations may not have a solution and if the moves made to solve the equations were acceptable.
  - 1.7.3: Self Check
  - 1.7.3: Additional Resources
- 1.7.4: Solving Equations by Dividing Values
  - In this activity, you used division to find equivalent equations and also learned that you can not divide by 0.

After these activities, you completed the following **practice**:

- 1.7.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

1.8 CHOOSING THE CORRECT VARIABLE TO SOLVE FOR, PART 1

1.8.0 Lesson Overview

In this lesson, you will manipulate equations and solve for a variable. You will also begin to determine which forms of equations are more useful than others.

When you finish this lesson, you will be able to:

- Solve for a particular variable, given an equation.
- Determine the more useful form of an equation.
- Explain the phrase "to solve for a variable."

Here are the **activities** that will help you reach those goals:

- 1.8.1: Expressing Relationships Between Two Quantities
- 1.8.2: Identifying the Most Useful Form of an Equation
  - 1.8.2: Self Check
  - 1.8.2: Additional Resources
- 1.8.3: Writing Equations and Isolating Variables
  - 1.8.3: Self Check
  - 1.8.3: Additional Resources
- 1.8.4: Writing Equations Given Dimensions of Shapes

After that, you'll **practice and review.**

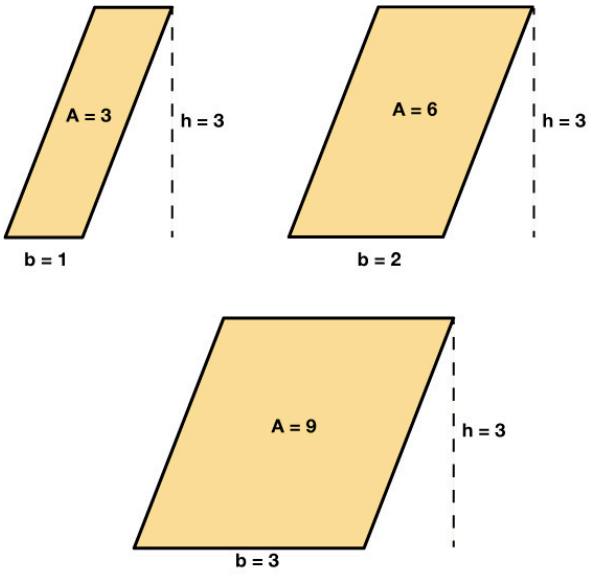
- 1.8.5: Practice
- 1.8.6 Lesson Summary

1.8.1 Expressing Relationships between Two Quantities

Warm Up

In the image below, all the parallelograms have the same height. Base length is measured in inches, and area is measured in square inches.

A table can be used to show the relationship between the base length, b, and the area, A, of the parallelograms.



b (inches)	A (square inches)
1	3
2	6
3	9
4.5	a. ____
$\frac{11}{2}$	b. ____
c. ____	36
d. ____	46.5

In the following activity, you will complete the table by entering the missing information.

CLASSROOM CONNECTION Log into student.desmos.com using the information provided by your teacher to complete the activity.

What is your answer for a?

What is your answer for b?

What is your answer for c?

What is your answer for d?

1.8.2 Identifying the Most Useful Form of an Equation

Activity

For questions 1 – 4, use the situation below:

> After a parade, a group of volunteers is helping to pick up the trash along a 2-mile stretch of a road. The group decides to divide the length of the road so that each volunteer is responsible for cleaning up equal-length sections.
>
> Find the length of a road section for each volunteer if there are the following numbers of volunteers. Be prepared to show your reasoning.



1. 8 volunteers
2. 10 volunteers
3. 25 volunteers
4. 36 volunteers
5. Write an equation that would make it easy to find l, the length of a road section in miles for each volunteer, if there are n volunteers.

For questions 6 – 9, find the number of volunteers in the group if each volunteer cleans up a section of the following lengths. Be prepared to show your reasoning.

6. 0.4 mile
7. $\frac{2}{7}$ mile
8. 0.125 mile
9. $\frac{6}{45}$ mile
10. Write an equation that would make it easy to find the number of volunteers, n, if each volunteer cleans up a section that is l miles.

Are you ready for more?

Extending Your Thinking

Let's think about the graph of the equation $y = \frac{2}{x}$.

1. Create a list (x, y) pairs that will help you graph the equation. Make sure to include some negative numbers for x and some numbers that are not integers.

Access multimedia content (http://openstax.org/books/algebra-1/pages/1-8-2-identifying-the-most-useful-form-of-an-equation)

2. Use the graphing tool or technology outside the course. Graph the equation that represents this scenario using the Desmos tool above.
3. What do you think the graph looks like when x is between 0 and $\frac{1}{2}$? Continue to use the graphing tool or technology outside the course to graph the function $y = \frac{2}{x}$ with your points. Try some values of x to test your idea. (Students were provided access to Desmos.)
4. What is the largest value y can ever be? $y = \frac{2}{x}$ with your points. Try some values of x to test your idea. In Desmos, zoom in on your graph.

Self Check

Use formula for the area of a triangle, $A = \frac{1}{2}bh$, where b is the base of the triangle and h is the height of the triangle. Which equation would best be used to find the base of the triangle?

a. $\frac{1}{2}b = \frac{A}{h}$
b. $2A = bh$
c. $h = \frac{2A}{b}$
d. $b = \frac{2A}{h}$

Additional Resources

Solve a Formula for a Specific Variable

We have all probably worked with some geometric formulas in our study of mathematics. Formulas are used in many fields, so it is important to recognize formulas and be able to manipulate them easily.

It is often helpful to solve a formula for a specific variable. If you need to put a formula in a spreadsheet, it is not unusual to have to solve it for a specific variable first. We isolate that variable on one side of the equal sign with a coefficient of 1, and all other variables and constants are on the other side of the equal sign.

Geometric formulas often need to be solved for another variable, too. The formula $V = \frac{1}{3}\pi r^2 h$ is used to find the volume of a right circular cone when given the radius of the base and height. In the next example, we will solve this formula for the height.

EXAMPLE 1

Solve $V = \frac{1}{3}\pi r^2 h$ for h.

Step 1 - Remove the fraction on the right.

Multiply both sides by 3.

$$3 \cdot V = 3 \cdot \frac{1}{3}\pi r^2 h$$

Step 2 - Simplify.

$$3V = \pi r^2 h$$

Step 3 - Divide both sides by πr^2.

$$\frac{3V}{\pi r^2} = h$$

We could now use this formula to find the height of a right circular cone when we know the volume and the radius of the base, by using the formula $h = \frac{3V}{\pi r^2}$

In the sciences, we often need to change temperature from Fahrenheit to Celsius or vice versa. If you travel in a foreign country, you may want to change the Celsius temperature to the more familiar Fahrenheit temperature.

EXAMPLE 2

Solve

$C = \frac{5}{9}(F - 32)$ for F.

Step 1 - Remove the fraction on the right.

Multiply both sides by $\frac{9}{5}$.

$$\frac{9}{5}C = \frac{9}{5} \cdot \frac{5}{9}(F - 32)$$

Step 2 - Simplify.

$$\frac{9}{5}C = (F - 32)$$

Step 3 - Add 32 to both sides.

$$\frac{9}{5}C + 32 = F$$

EXAMPLE 3

Solve the formula $8x + 7y = 15$ for y.

Step 1 - Subtract $8x$ from both sides to isolate the term with y.

$$8x - 8x + 7y = 15 - 8x$$

Step 2 - Simplify.

$$7y = 15 - 8x$$

Step 3 - Divide both sides by 7 to make the coefficient of y equal to 1.

$$\frac{7y}{7} = \frac{15 - 8x}{7}$$

Step 4 - Simplify.

EXAMPLE 4

$\frac{2d}{t} = (v_0 + v)$

Now, try to solve $d = \frac{1}{2}(v_0 + v)t$ for v_0.

Answer:

Step 1 - Remove the fraction on the right.

Multiply by 2.

$2 \times d = 2 \times \frac{1}{2}(v_0 + v)t$

Step 2 - Simplify.

$2d = (v_0 + v)t$

Step 3 - Divide both sides by t.

$\frac{2d}{t} = \frac{(v_0+v)t}{t}$

Step 4 - Simplify.

$\frac{2d}{t} = (v_0 + v)$

Step 5 - Subtract v from both sides to solve for v_0.

$\frac{2d}{t} - v = v_0 + v - v$

$\frac{2d}{t} - v = v_0$

TRY IT **Solve a Formula for a Specific Variable**

Solve $8x + 7y = 15$ for x.

1.8.3 Writing Equations and Isolating Variables

Activity

Use the following scenario for questions 1 – 7:

Tank A initially contained 124 liters of water. It is then filled with more water, at a constant rate of 9 liters per minute.

1. How many liters of water are in Tank A after 4 minutes?
2. How many liters of water are in Tank A after 80 seconds?
3. How many liters of water are in Tank A after 4 minutes?
4. How many minutes, m, have passed when Tank A contains 151 liters of water?
5. How many minutes, m, have passed when Tank A contains 191.5 liters of water?
6. How many minutes, m, have passed when Tank A contains 270.25 liters of water?
7. How many minutes, m, have passed when Tank A contains p liters of water?

For questions 8 – 14, use the following scenario:

Tank B, which initially contained 80 liters of water, is being drained at a rate of 2.5 liters per minute.

8. How many liters of water are in Tank B after 30 seconds?
9. How many liters of water are in Tank B after 7 minutes?
10. How many liters of water are in Tank B after t minutes?
11. For how many minutes, t, has the water been draining when Tank B contains 75 liters of water?
12. For how many minutes, t, has the water been draining when Tank B contains 32.5 liters of water?
13. For how many minutes, t, has the water been draining when Tank B contains 18 liters of water?
14. For how many minutes, t, has the water been draining when Tank B contains v liters of water?

Video: Isolating the Variable

Watch the following video to further explore the above situation.

Isolating the Variable (https://openstax.org/r/isolate-variable)

Self Check

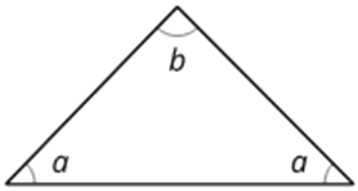
A pool is filled with 15,000 gallons of water. It loses 200 gallons per day. How many days, d, have passed when the pool has g gallons of water?

a. $\frac{g}{200d}$
b. $g - 200d$
c. $15,000 - 200d$
d. $\frac{15,000-g}{200}$

Additional Resources

Choosing Which Equation to Use

A relationship between quantities can be described in more than one way. Some ways are more helpful than others, depending on what we want to find out. Let's look at the angles of an isosceles triangle, for example.



The two angles near the horizontal side have equal measurement in degrees, a.

The sum of angles in a triangle is $180°$, so the relationship between the angles can be expressed as:

$a + a + b = 180$

EXAMPLE 1

Suppose we want to find a when b is $20°$.

$a + a + b = 180$

$2a + 20 = 180$

$2a = 180 - 20$

$2a = 160$

$a = 80$

EXAMPLE 2

What is the value of b if a is $72.5°$?

$a + a + b = 180$

$72.5 + 72.5 + b = 80$

$145 + b = 180$

$b = 35$

Notice that when b is given, we did the same calculation repeatedly to find a: We substituted b into the first equation, subtracted b from 180, and then divided the result by 2.

Instead of taking these steps over and over whenever we know b and want to find a, we can rearrange the equation to isolate a:

$a + a + b = 180$

$2a + b = 180$

$2a = 180 - b$

$a = \frac{180-b}{2}$

$a = 90 - \frac{1}{2}b$

This equation is equivalent to the first one. To find a, we can now simply substitute any value of b into this equation and evaluate the expression on the right side.

Likewise, we can write an equivalent equation to make it easier to find b when we know a:

$a + a + b = 180$

$2a + b = 180$

$b = 180 - 2a$

Rearranging an equation to isolate one variable is called solving for a variable. In this example, we have solved for a and

for b. All three equations are equivalent. Depending on what information we have and what we are interested in, we can choose a particular equation to use.

Rewrite the equation $a + a + b = 180$ as an equivalent equation that could be used to easily find b.

TRY IT **Choosing Which Equation to Use**

Jake and Bryce are saving their allowance. Jake saved for one week, and Bryce saved for 2 weeks. Together they saved $15. Write an equation to find Bryce's allowance.

1.8.4 Writing Equations Given Dimensions of Shapes

Cool Down

The perimeter of a rectangle is 48 centimeters. The relationship between the length, the width, and the perimeter of the rectangle can be described with the equation: $2 \cdot \text{length} + 2 \cdot \text{width} = 48$.

Find the length, in centimeters, if the width is:

1. 10 centimeters

2. 3.6 centimeters

3. w centimeters

1.8.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 4 use the problem below:

> Priya is buying raisins and almonds to make trail mix. Almonds cost $5.20 per pound and raisins cost $2.75 per pound. Priya spent $11.70 buying almonds and raisins. The relationship between pounds of almonds, pounds of raisins, and the total cost is represented by the equation $5.2a + 2.75r = 11.7$.

How many pounds of raisins did Priya buy if she bought the following amount of almonds?

1. 2 pounds of almonds

a. 0.473
b. 0.946
c. 2.75
d. 1.192

2. 1.06 pounds of almonds

a. 2.811
b. 5.5
c. 1.689
d. 2.25

3. 0.64 pounds of almonds

a. 8.371
b. 3.044
c. 1.911
d. 9.94

4. a pounds of almonds

a. $r = \frac{11.7-5.2a}{2.75}$
b. $r = \frac{2.75}{11.7-5.2a}$
c. $r = \frac{11.7-2.75a}{5.2}$
d. $r = \frac{11.7+5.2a}{275}$

For questions 5 – 6 use the scenario below:

> A car traveled 180 miles at a constant rate.

5. For each of the missing spaces in the table, a-d, select the missing value that completes the table to show the rate at which the car was traveling if it completed the same distance in each number of hours.

Travel Time (Hours)	Rate of Travel (Miles Per Hour)
5	a. ____
4.5	b. ____
3	c. ____
2.25	d. ____

a._____

a. 60
b. 36
c. 80
d. 40

b. _____
a. 60
b. 36
c. 80
d. 40

c. _____
a. 60
b. 36
c. 80
d. 40

d. _____
a. 60
b. 36
c. 80
d. 40

6. Choose the equation that would make it easy to find the rate at which the car was traveling in miles per hour r, if it traveled for t hours.

a. $r - 180 - t$
b. $r = 180t$
c. $r = \frac{180}{t}$
d. $r = \frac{t}{180}$

7. Bananas cost $0.50 each, and apples cost $1.00 each.
Select all three combinations of bananas and apples that Elena could buy for exactly $3.50.

a. 2 bananas and 2 apples
b. 3 bananas and 2 apples
c. 1 banana and 2 apples
d. 1 banana and 3 apples
e. 5 bananas and 2 apples
f. 5 bananas and 1 apple

8. Choose the best answer to solve the formula $A = P + Prt$ for t.

a. $t = \frac{r(A-P)}{P}$
b. $t = \frac{A+P}{Pr}$
c. $t = \frac{A-P}{Pr}$
d. $t = \frac{P-A}{Pr}$

1.8.6 Lesson Summary

In this lesson, you learned how to:

- Solve for a particular variable, given an equation.
- Determine the more useful form of an equation.
- Explain the phrase “to solve for a variable.”

Here are the **activities** that helped you reach those goals:

- 1.8.1: Expressing Relationships Between Two Quantities
  - In this activity, you completed a table by using the relationship between the values already in the table.
- 1.8.2: Identifying the Most Useful Form of an Equation
  - In this activity, you worked to recognize the most useful form of an equation. You also wrote an equation in the most useful form that would be needed to find specific information.
  - 1.8.2: Self Check
  - 1.8.2: Additional Resources
- 1.8.3: Writing Equations and Isolating Variables
  - In this activity, you used equations to solve for specific variables that helped answer questions about given situations.
  - 1.8.3: Self Check
  - 1.8.3: Additional Resources
- 1.8.4: Writing Equations Given Dimensions of Shapes
  - In this activity, you found the length of a rectangle given the perimeter as well as the width as both a constant and a variable.

After these activities, you completed the following **practice**:

- 1.8.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

1.9 CHOOSING THE CORRECT VARIABLE TO SOLVE FOR, PART 2

1.9.0 Lesson Overview

In this lesson, you will practice solving for a specific variable to obtain an expression defined in terms of the other variables. Through the lesson's activities you will see that this process allows you to solve problems more efficiently.

When you finish this lesson, you will be able to:

- Solve an equation for one of the variables.

Here are the **activities** that will help you reach those goals:

- 1.9.1: Rearranging an Equation in Three Variables
- 1.9.2: Writing an Equation to Represent a Constraint
  - 1.9.2: Self Check
  - 1.9.2: Additional Resources
- 1.9.3: Writing and Rearranging Equations in Two Variables
  - 1.9.3: Self Check
  - 1.9.3: Additional Resources
- 1.9.4: Solving for a Specified Variable

After that, you'll **practice and review**.

- 1.9.5: Practice
- 1.9.6: Lesson Summary

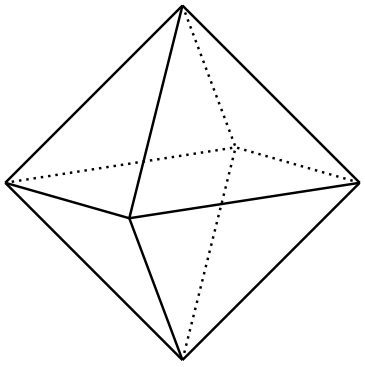
1.9.1 Rearranging an Equation in Three Variables

Warm Up

In an earlier lesson, you saw the equation $V + F - 2 = E$, which relates the number of vertices, faces, and edges in a Platonic solid.

In questions 1 – 2, write an equation that makes it easier to find the number of vertices in each of the Platonic solids described:

1. An octahedron (shown here), which has 8 faces



2. An icosahedron, which has 30 edges

3. A Buckminsterfullerene (also called a “Buckyball”) is a polyhedron with 60 vertices. It is not a Platonic solid, but the numbers of faces, edges, and vertices are related the same way as those in a Platonic solid.
   Write an equation that makes it easier to find the number of faces a Buckyball has if we know how many edges it has.

Why Should I Care?



Alex's brother can use algebra to budget his paycheck from his job at a pizza place. He uses some money for food, some for going out with his friends, and some, he saves for college.

If he uses variables to represent the different ways he uses his money, he can put those variables into an equation to make sure he has enough money to last until his next paycheck.

1.9.2 Writing an Equation to Represent a Constraint

Activity

An automobile manufacturer is preparing a shipment of cars and trucks on a cargo ship that can carry 21,600 tons.

The cars weigh 3.6 tons each, and the trucks weigh 7.5 tons each.



1. Write an equation that represents the weight constraint of a shipment. Let c be the number of cars and t be the number of trucks.

For questions 2 – 5, use the following situation:

For one shipment, trucks are loaded first and cars are loaded afterward. (Even though trucks are bulkier than cars, a shipment can consist of all trucks as long as it is within the weight limit.)

2. Find the number of cars that can be shipped if the cargo already has 480 trucks.
3. Find the number of cars that can be shipped if the cargo already has 1500 trucks.
4. Find the number of cars that can be shipped if the cargo already has 2736 trucks.
5. Find the number of cars that can be shipped if the cargo already has t trucks

Use the following information for questions 6 – 8. For a different shipment, cars are loaded first, and then trucks are loaded afterward.

6. Write an equation you could enter into a calculator or a spreadsheet tool to find the number of trucks that can be shipped if the number of cars is known.
7. Use your equation and a calculator or a computer to find the number of trucks that can be shipped if the cargo already has 1000 cars.
8. Use your equation and a calculator or a computer to find the number of trucks that can be shipped if the cargo already has 4250 cars.

Are you ready for more?

Extending Your Thinking

For questions 1 and 2, use the following situation:

For yet another shipment, the manufacturer is also shipping motorcycles, which weigh 0.3 ton each.

1. Write an equation that you could enter into a calculator or a spreadsheet tool to find the number of motorcycles that can be shipped, m, if the number of cars and trucks are known.
2. Use your equation to find the number of motorcycles that can be shipped if the cargo already contains 1,200 trucks and 3,000 cars.

Self Check

The total revenue a high school makes from a play is T dollars. The revenue is made from selling tickets at $3.25 per

child ticket, c, and $7.50 for each adult ticket, a. Which equation is best to find the number of child tickets sold?

a. $c = \frac{T}{3.25}$
b. $a = \frac{T-3.25c}{7.50}$
c. $T = 3.25c + 7.50a$
d. $C = \frac{T-7.50a}{3.25}$

Additional Resources

Write Constraint Resources

Review the following information from a previous lesson.

Writing Equations Using Symbols

When writing statements using algebraic symbols, the following steps may be helpful.

Step 1 - Read the problem.

Step 2 - Identify the variables and known values. If needed, sketch a picture of the scenario.

Step 3 - Write a sentence using the relationship among the values.

Step 4 - Translate the sentence into an equation.

EXAMPLE

You are buying chicken and steak for a barbeque. Chicken costs $1.29/lb and steak costs $3.49/lb.

1. What variables might you need to use in this scenario?
2. What sentence might you use to describe the total cost of the barbeque?
3. Write an equation to show the total cost, T, of the meat.
4. You have $100 to spend on the barbeque. Use this information and rewrite the equation $T = 1.29c + 3.49s$ to find the number of pounds of chicken, c, that you can buy.
5. You have $100 to spend on the barbeque. Use this information and rewrite the equation $T = 1.29c + 3.49s$ to find the number of pounds of steak, s, that you can buy.

TRY IT **Write Constraint Equations**

On a canoeing trip, you walk 4 mph then canoe at 7 mph and cover 30 miles. Write an equation to find how many miles you travel while canoeing.

1.9.3 Writing and Rearranging Equations in Two Variables

Activity

The Department of Streets of a city has a budget of $1,962,800 for resurfacing roads and hiring additional workers this year.

The cost of resurfacing a mile of 2-lane road is estimated at $84,000. The average starting salary of a worker in the department is $36,000 a year.



1. Write an equation that represents the relationship between the miles of 2-lane roads the department could resurface, m, and the number of new workers it could hire, p, if it spends the entire budget.
2. Use the equation you wrote in the first question to solve for p.
3. Explain what the solution to the equation in question 2 represents in this situation.
4. Use the equation you wrote in the first question to solve for m.
5. Explain what the solution to the equation in question 4 represents in this situation.
6. The city is planning to hire 6 new workers and use its entire budget. Which equation should be used to find how many miles of 2-lane roads it could resurface?
   a. $m = \frac{1{,}962{,}800 - 36{,}000p}{84{,}000}$
   b. $84{,}000m + 36{,}000p = 1{,}962{,}800$
   c. $p = \frac{1{,}962{,}800 - 84{,}000m}{36{,}000}$
7. Explain your reasoning.
8. To the nearest hundredth, how many miles of 2-lane roads could the city resurface if it hires 6 new workers?

Video: Rewriting Equations for a Specific Variable

Watch the following video for more help on how to write an equation for a specific variable and why it is helpful to do so.

1.9.3 (https://openstax.org/r/writing-constraints-inequalities)

Self Check

A farmer makes money selling pumpkins and apples in the fall. His revenue, M, is made from selling pumpkins, p, for $4.00 and apples, a, for $0.75. Which equation is solved to help him find out how many pumpkins he sold?

a. $p = \frac{m}{4.00}$
b. $a = \frac{M - 4.00p}{1.75}$
c. $M = 4.00p + 0.75a$
d. $p = \frac{M - 0.75a}{4.00}$

Additional Resources

Solve a Formula for a Specific Variable

Solving for a variable in a formula is actually finding equivalent equations.

To solve a formula for a specific variable means to isolate that variable on one side of the equals sign with a coefficient of 1. All other variables and constants are on the other side of the equals sign. To see how to solve a formula for a specific

variable, we will start with the distance, rate and time formula.

EXAMPLE 1

Solve the formula $d = rt$ for t when $d = 520$ and $r = 65$.

Step 1 - Write the formula.

$d = rt$

Step 2 - Substitute.

$520 = 65t$

Step 3 - Divide, to isolate t.

$\frac{520}{65} = \frac{65t}{65}$

Step 4 - Simplify.

$t = 8$

EXAMPLE 2

Solve the formula $d = rt$ for t.

Step 1 - Write the formula.

$d = rt$

Step 2 - Substitute.

(step not needed)

Step 3 - Divide, to isolate t.

$\frac{d}{r} = \frac{rt}{r}$

Step 4 - Simplify.

$\frac{d}{r} = t$

EXAMPLE 3

Solve the formula $A = \frac{1}{2}bh$ for h, when $A = 90$ and $b = 15$.

Step 1 - Write the formula.

$A = \frac{1}{2}bh$

Step 2 - Substitute.

$90 = \frac{1}{2}(15)h$

Step 3: Multiply to clear the fraction.

$2 \times 90 = 2 \times \frac{1}{2}(15)h$

Step 4 - Simplify.

$180 = 15h$

Step 5 - Solve for h.

$12 = h$

EXAMPLE 4

Solve the formula $A = \frac{1}{2}bh$ for h.

Step 1 - Write the formula.

$A = \frac{1}{2}bh$

Step 2 - Substitute.

(step not needed)

Step 3 - Multiply to clear the fraction.

$2 \times A = 2 \times \frac{1}{2}bh$

Step 4 - Simplify.

$2A = bh$

Step 5 - Solve for h.

$\frac{2A}{b} = h$

EXAMPLE 5

Solve the formula $3x + 2y = 18$ for y when $x = 4$.

Step 1 - Write the formula.

$3x + 2y = 18$

Step 2 - Substitute.

$3(4) + 2y = 18$

Step 3 - Simplify.

$12 + 2y = 18$

Step 4 - Isolate the y-term using subtraction.

$12 - 12 + 2y = 18 - 12$

Step 5 - Simplify.

$2y = 6$

Step 6 - Divide.

$\frac{2y}{2} = \frac{6}{2}$

Step 7 - Simplify.

$y = 3$

EXAMPLE 6

Solve the formula $3x + 2y = 18$ for y.

Step 1 - Write the formula.

$3x + 2y = 18$

Step 2 - Substitute.

(step not needed)

Step 3 - Simplify.

(step not needed)

Step 4 - Isolate the y-term using subtraction.

$3x - 3x + 2y = 18 - 3x$

Step 5 - Simplify.

$2y = 18 - 3x$

Step 6 - Divide.

$\frac{2y}{2} = \frac{18-3x}{2}$

Step 7 - Simplify.

$y = -\frac{3}{2}x + 9$

TRY IT **Solve a Formula for a Specific Variable**

Solve $2x - 4y = 20$ for x.

1.9.4 Solving for a Specified Variable

Cool Down

A school is holding a carnival and hopes to raise $500. Child tickets cost $3, and adult tickets cost $5. If the school sells x child tickets and y adult tickets, then the equation $3x + 5y = 500$ expresses the fact that the school raised exactly $500 from ticket sales.

1. Solve the equation for x.
2. Explain when it might be helpful to rewrite the equation this way.

1.9.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For numbers 1 – 9, use the following scenario:
A car has a 16-gallon fuel tank. When driven on a highway, it has a gas mileage of 30 miles per gallon. The gas mileage (also called "fuel efficiency") tells us the number of miles the car can travel for a particular amount of fuel (one gallon of gasoline, in this case). After filling the gas tank, the driver got on a highway and drove for a while.

1. How many miles has the car traveled if it has 15 gallons left in the tank?
2. How many miles has the car traveled if it has 10 gallons left in the tank?
3. How many miles has the car traveled if it has 2.5 gallons left in the tank?
4. Choose the equation that represents the relationship between the distance the car has traveled in miles, d, and the amount of gas left in the tank in gallons, x.

   a. $x = \frac{30}{d-480}$
   b. $d = 30(16 - x)$
   c. $x = \frac{d-480}{-30}$
   d. $d = 16(30 - x)$

5. How many gallons are left in the tank when the car has traveled 90 miles?
6. How many gallons are left in the tank when the car has traveled 246 miles?
7. Choose an equation that makes it easier to find the amount of gas left in the tank, x, if we know the car has traveled d miles.

   a. $x = \frac{30}{d-480}$
   b. $d = 30(16 - x)$
   c. $x = \frac{d-480}{-30}$
   d. $d = 16(30 - x)$

8. Noah was helping to collect the entry fees at his school's football game. Student entry costs $2.75 each and adult entry costs $5.25 each. By the end of the game, Diego had collected $281.25.

Select two equations that could represent the relationship between the number of students, s, the number of adults, a, and the dollar amount collected at the game.

   a. $281.25 - 5.25a = 2.75s$
   b. $a = 53.57 - \frac{2.75}{5.25}s$
   c. $x - 2 = 4$
   d. $x - 2 = 12$
   e. $x - 6 = 4$

1.9.6 Lesson Summary

In this lesson, you learned how to:

- Solve an equation for one of the variables.

Here are the **activities** that helped you reach those goals:

- 1.9.1: Rearranging an Equation in Three Variables
  - In this activity, you rearranged the equation, $V + F - 2 + E$, to find the number of faces, vertices, or edges of different platonic solids.
- 1.9.2: Writing an Equation to Represent a Constraint
  - In this activity, you wrote an equation to represent the constraints of shipments of cars and trucks. You then rearranged the equation to find the number of cars and number of trucks that could be in the shipment.
  - 1.9.2: Self Check
  - 1.9.2: Additional Resources
- 1.9.3: Writing and Rearranging Equations in Two Variables
  - In this activity, you wrote an equation about the budget of a city for road resurfacing and worker salaries. You rearranged the equations to help solve for the number of workers that could be hired and the number of miles of roads that could be resurfaced. Solving for different variables in an equation can help you answer different questions.
  - 1.9.3: Self Check
  - 1.9.3: Additional Resources
- 1.9.4: Solving for a Specified Variable
  - In this activity, you worked to solve an equation for a specific variable. Solving for the specific variable would help you find the number of child or adult tickets sold at the carnival.

After these activities, you completed the following **practice**:

- 1.9.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

1.10 CONNECTING EQUATIONS TO GRAPHS, PART 1

1.10.0 Lesson Overview

In this lesson, you will learn how the parameters and variables relate to features of the graphs of two-variable linear equations. In addition, you will think about how different forms of two-variable equations affect the information we could gain about the relationships between the quantities and about the graphs.

When you finish this lesson, you will be able to:

- Describe the connections between an equation of the form $ax + by = c$, the features of its graph, and the rate of change in the situation.
- Graph linear equations of the form $ax + by = c$.
- Demonstrate that rewriting the equation for a line in different forms can make it easier to find certain kinds of information about the relationship and about the graph.

Here are the **activities** that will help you reach those goals

- 1.10.1: Interpreting Equations in Context
- 1.10.2: Graphing Equations in Context
  - 1.10.2: Self Check
  - 1.10.2: Additional Resources
- 1.10.3: Writing and Graphing Equations in Standard Form
  - 1.10.3: Self Check
  - 1.10.3: Additional Resources
- 1.10.4: Matching Equations and Graphs

After that, you'll **practice and review**.

- 1.10.5: Practice
- 1.10.6: Lesson Summary

1.10.1 Interpreting Equations in Context

Warm Up



Jada has $20 to spend on games and rides at a carnival.

- Games cost $1 each.

- Rides are $2 each.

In the following activity, you will complete a series of questions regarding this scenario.

CLASSROOM CONNECTION Log into student.desmos.com using the information provided by your teacher to complete the activity.

1. Which equation represents the relationship between the number of games, x, and the number of rides, y, Jada could do if she spends all $20?

a. $x + y = 20$
b. $2x + y = 20$
c. $x + 2y = 20$

2. Explain what each of the other two equations could mean in this situation.

1.10.2 Graphing Equations in Context

Activity



For questions 1 – 7, use the equation $x + y = 20$ which represents the relationship between the number of games, x, the number of rides, y, and the dollar amount a student is spending on games and rides at an amusement park.

1. What's the number of rides the student could get on if they don't play any games?

2. What would the coordinate be on the graph to represent the answer to question 1?

3. What's the number of games the student could play if they don't get on any rides?

4. What would the coordinate be on the graph to represent the answer to question 3?

Self Check

Which of the following equations correctly solved $2x + 4y = 8$ for y?

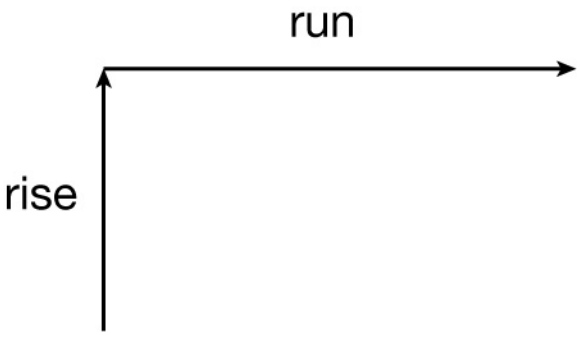
a. $y = \frac{-1}{2}x + 8$
b. $y = \frac{1}{2}x + 2$
c. $y = -2x + 4$
d. $y = \frac{-1}{2}x + 2$

Additional Resources

Find the Slope of a Line

When you graph linear equations, you may notice that some lines tilt up as they go from left to right and some lines tilt down. Some lines are very steep and some lines are flatter.

In mathematics, the measure of the steepness of a line is called the of slope the line. The concept of slope has many applications in the real world. In construction the pitch of a roof, the slant of the plumbing pipes, and the steepness of the stairs are all applications of slope. As you ski or jog down a hill, you definitely experience a slope. We can assign a numerical value to the slope of a line by finding the ratio of the rise and run. The rise is the amount the vertical distance changes while the run measures the horizontal change, as shown in this illustration. Slope is a rate of change.



Slope of a Line

The slope of a line is $m = \frac{rise}{run}$. The rise measures the vertical change and the run measures the horizontal change. To find the slope of a line, we locate two points on the line whose coordinates are integers. Then we sketch a right triangle where the two points are vertices and one side is horizontal and one side is vertical. To find the slope of the line, we measure the vertical distance is called the *rise* and the horizontal distance is called the *run*.

How To Find the Slope from a Graph

Find the slope of a line from its graph using $m = \frac{rise}{run}$

Step 1 - Locate two points on the line whose coordinates are integers.

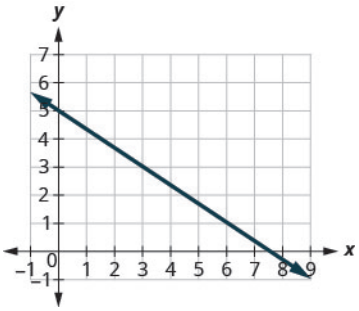
Step 2 - Starting with one point, sketch a right triangle, going from the first point to the second point.

Step 3 - Count the rise and the run on the legs of the triangle.

Step 4 - Take the ratio of rise to run to find the slope: $m = \frac{rise}{run}$.

EXAMPLE 1

Find the slope of the line shown.

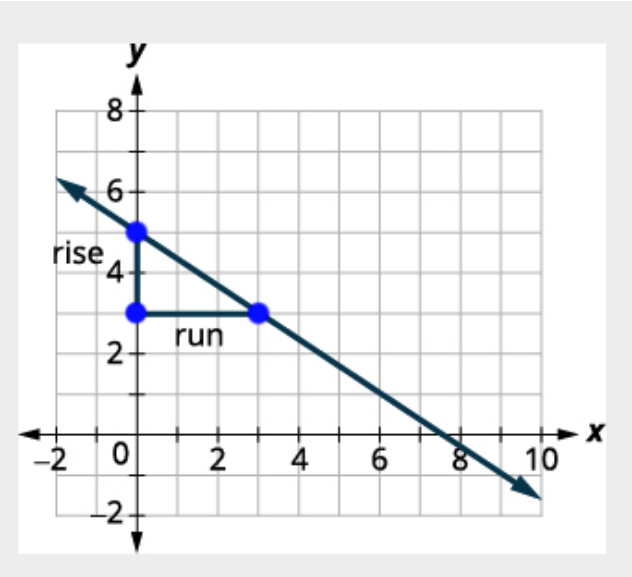


Solution

Step 1 - Locate two points on the graph whose coordinates are integers.

$(0, 5)$ and $(3, 3)$

Step 2 - Starting at $(0, 5)$, sketch a right triangle to $(3, 3)$ as shown in this graph.



Step 3 - Count the rise(how many spaces up or down)— since it goes down, it is negative.

The rise is -2.

Step 4 - Count the run (how many spaces to the right or left).

The run is 3.

Step 5 - Use the slope formula.

$m = \frac{rise}{run}$

Step 6 - Substitute the values of the rise and run.

$m = \frac{-2}{3}$

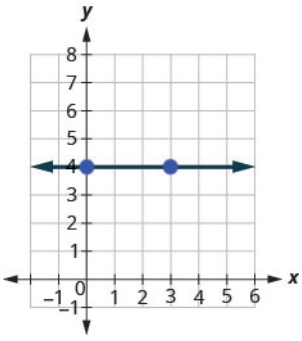
Step 7 - Simplify.

$m = -\frac{2}{3}$

So y decreases by 2 units as x increases by 3 units.

Slopes of Horizontal and Vertical Lines

How do we find the slope of horizontal and vertical lines? To find the slope of the horizontal line, $y = 4$, we could graph the line, find two points on it, and count the rise and the run. Let's see what happens when we do this, as shown in the graph below.



Step 1 - What is the rise?

The rise is 0.

Step 2 - What is the run?

The run is 3.

Step 3 - What is the slope?

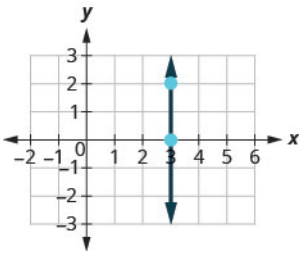
$m = \frac{rise}{run}$

$m = \frac{0}{3}$

$m = 0$

So The slope of the horizontal line $y = 4$ is 0.

Let's also consider a vertical line, the line $x = 3$, as shown in the graph below.



Step 1 - What is the rise?

The rise is 2.

Step 2 - What is the run?

The run is 0.

Step 3 - What is the slope?

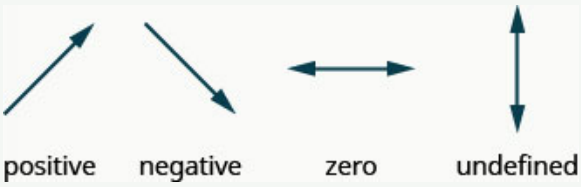
$m = \frac{rise}{run}$

$m = \frac{2}{0}$

The slope is undefined since division by zero is undefined. So we say that the slope of the vertical line is undefined.

- All horizontal lines have slope 0. When the y–coordinates are the same, the rise is 0.
- The slope of any vertical line is undefined. When the x–coordinates of a line are all the same, the run is 0.

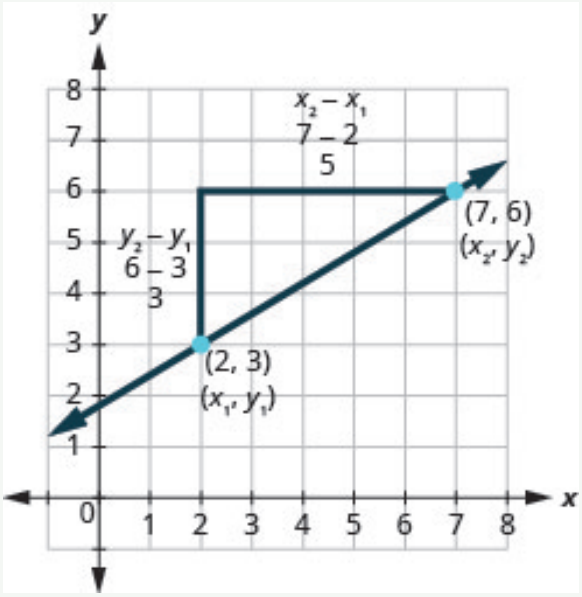
Quick Guide to the Slopes of Lines



Sometimes we'll need to find the slope of a line between two points when we don't have a graph to count out the rise and the run. We could plot the points on grid paper, then count out the rise and the run, but as we'll see, there is a way to find the slope without graphing. Before we get to it, we need to introduce some algebraic notation.

We have seen that an ordered pair (x, y) gives the coordinates of a point. But when we work with slopes, we use two points. How can the same symbol (x, y) be used to represent two different points? Mathematicians use subscripts to distinguish the points.

We will use (x_1, y_1) to identify the first point and (x_2, y_2) to identify the second point. Let's see how the rise and run relate to the coordinates of the two points by taking another look at the slope of the line between the points $(2, 3)$, and $(7, 6)$, as shown in this graph.



Step 1 - Since we have two points, we will use subscript notation.

$\begin{pmatrix} x_1, y_1 \\ 2,3 \end{pmatrix} \begin{pmatrix} x_2, y_2 \\ 7,6 \end{pmatrix}$

Step 2 - What is the slope?

The rise is 3.

The run is 5.

Step 3 - What is the slope?

$m = \frac{rise}{run}$

On the graph, we counted the rise of 3 and the run of 5. $m = \frac{3}{5}$

Notice that the rise of 3 can be found by subtracting the y–coordinates, 6 and 3, and the run of 5 can be found by subtracting the x–coordinates, 7 and 2.

Step 4 - We rewrite the rise and run by putting in the coordinates.

$m = \frac{6-3}{7-2}$

But 6 is y_2, the y–coordinate of the second point and 3 is y_1, the y–coordinate of the first point. So we can rewrite the slope using subscript notation.

$m = \frac{y_2 - y_1}{7-2}$

Also, 7 is x_2, the x–coordinate of the second point and 2 is x_1, the x–coordinate of the first point. So again, we rewrite the slope using subscript notation

$m = \frac{y_2 - y_1}{x_2 - x_1}$

We've shown that $m = \frac{y_2 - y_1}{x_2 - x_1}$ is really another version of $m = \frac{rise}{run}$. We can use this formula to find the slope of a line when we have two points on the line.

How to Find the Slope from Two Points

The slope of the line between two points (x_1, y_1) and (x_2, y_2) is: $m = \frac{y_2 - y_1}{x_2 - x_1}$

EXAMPLE 2

Step 1 - We'll call (-2, -3) point #1 and (-7, 4) point #2.

$$\begin{pmatrix} x_1, y_1 \\ -2, -3 \end{pmatrix} \begin{pmatrix} x_2, y_2 \\ -7, 4 \end{pmatrix}$$

Step 2 - Use the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Step 3 - Substitute the values.

y of the second point minus y of the first point.

x of the second point minus x of the first point

$$m = \frac{4 - (-3)}{-7 - (-2)}$$

Step 4 - Simplify.

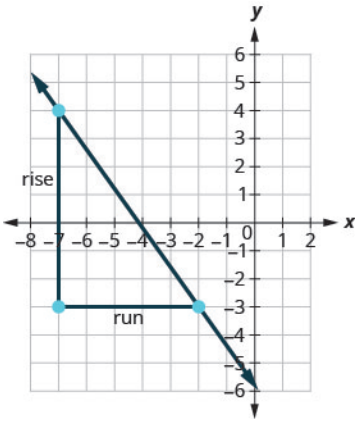
$$m = \frac{4+3}{-7+2}$$

$$m = \frac{7}{-5}$$

$$m = -\frac{7}{5}$$

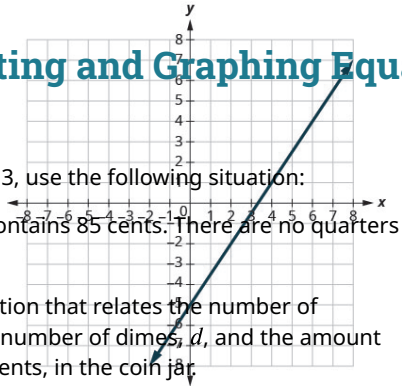
Step 5 - Let's verify this slope on the graph shown.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



TRY IT **Find the Slope of a Line**

Find the slope of the line graph below.



1.10.3 Writing and Graphing Equations in Standard Form

Activity

For questions 1 – 3, use the following situation:

Andre's coin jar contains 85 cents. There are no quarters or pennies in the jar, so the jar has all nickels, all dimes, or some of each.

1. Write an equation that relates the number of nickels, n, the number of dimes, d, and the amount of money, in cents, in the coin jar.
2. Graph your equation on the coordinate plane. Be sure to label the axes. Use the graphing tool or technology outside the course. Graph the equation that represents this scenario using the Desmos tool below.

Access multimedia content (http://openstax.org/books/algebra-1/pages/1-10-3-writing-and-graphing-equations-in-standard-form)

3. How many nickels are in the jar if there are no dimes?
4. How many dimes are in the jar if there are no nickels?

Are you ready for more?

Extending Your Thinking

What are all the different ways the coin jar could have 85 cents if it could also contain quarters?

Self Check

A coin jar contains 75 cents and only contains quarters and nickels. Write an equation that relates the number of quarters, q, the number of nickels, n, and the amount of money, in cents, in the coin jar.

a. $0.05n - 0.25q = 0.75$
b. $25q - 5n = 75$
c. $25q + 5n = 0.75$
d. $25q + 5n = 75$

Additional Resources

Standard Form of Linear Equation

A linear equation is in **standard form** when it is written $Ax + By = C$.

Most people prefer to have A, B, and C be integers and $A \geq 0$ when writing a linear equation in standard form, although it is not strictly necessary.

- Linear equations have infinitely many solutions.
- For every number that is substituted for x there is a corresponding y value.
- This pair of values is a solution to the linear equation and is represented by the ordered pair (x, y).
- When we substitute these values of x and y into the equation, the result is a true statement, because the value on the left side is equal to the value on the right side.

Solution of a Linear Equation in Two Variables

An ordered pair (x, y) is a solution of the linear equation $Ax + By = C$, if the equation is a true statement when the x– and y–values of the ordered pair are substituted into the equation.

Linear equations have infinitely many solutions. We can plot these solutions in the rectangular coordinate system. The points will line up perfectly in a straight line. We connect the points with a straight line to get the graph of the equation. We put arrows on the ends of each side of the line to indicate that the line continues in both directions.

A graph is a visual representation of all the solutions of the equation. It is an example of the saying, "A picture is worth a thousand words." The line shows you *all* the solutions to that equation. Every point on the line is a solution of the

equation. And, every solution of this equation is on this line. This line is called the graph of the equation. Points *not* on the line are not solutions!

When a linear equation in standard form is solved for y, it is then easy to pick out the slope of the line and where the graph of the line crosses the y–axis.

EXAMPLE

Solve $9x - 3y = 15$ for y

Step 1 - Move the x term to isolate the y term.

Subtract $9x$ from both sides.

$9x - 9x - 3y = 15 - 9x$

Step 2 - Simplify.

$-3y = 15 - 9x$

Step 3 - Divide by the coefficient of y.

Divide both sides by -3

$\frac{-3y}{-3} = \frac{15-9x}{-3}$

Step 4 - Simplify.

$y = 3x - 5$

In the new equation, $y = 3x - 5$, the slope is 3 and the graph of the line would cross the y–axis at -5.

TRY IT **Understanding Standard Form**

Solve $6x + 2y = 14$ for y.

1.10.4 Matching Equations and Graphs

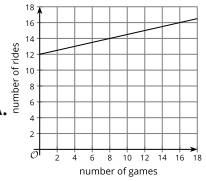
Cool Down

Kiran is spending $12 on games and rides at another carnival, where a game costs $0.25 and a ride costs $1.

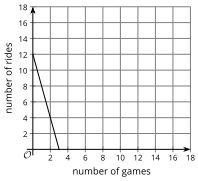
1. Write an equation to represent the relationship between the dollar amount Kiran is spending and the number of games, x, and the number of rides, y, he could do.

2. Which graph represents the relationship between the quantities in this situation?

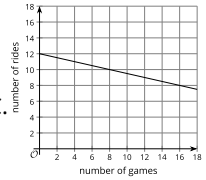
A.



B.



C.



3. Explain how you determined your answer to the previous question.

1.10.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 - 4, use the problem and graph below.

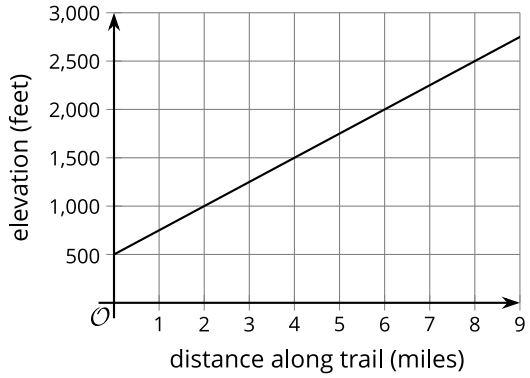
Andre bought a new bag of cat food. The next day, he opened it to feed his cat. The graph shows how many ounces were left in the bag on the days after it was bought.



1. How many ounces of food were in the bag 12 days after Andre bought it?
2. How many days did it take for the bag to contain 16 ounces of food?
3. How much did the bag weigh before it was opened?
4. Approximately many days did it take for the bag to be empty?

For 5 - 7, use the problem and graph below.

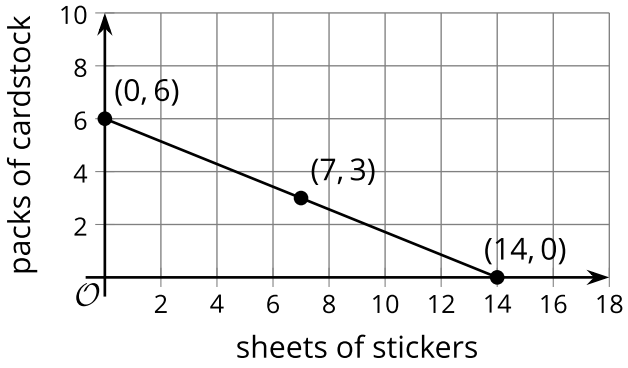
A group of hikers is progressing steadily along an uphill trail. The graph shows their elevation (or height above sea level), in feet, at each distance from the start of the trail, in miles.



5. What is the slope of the graph?
6. What does the slope tell us about this situation?
   a. Only the elevation (feet).
   b. Only the distance along the trail (miles).
   c. The change in miles as the elevation increases.
   d. The change in elevation as the hikers hike more miles.
7. What is the equation that represents the relationship between the hikers' distance from the start of the trail, x, and their elevation, y?
8. A kindergarten teacher bought \$21 worth of stickers and cardstock for his class. The stickers cost \$1.50 a sheet and the cardstock cost \$3.50 per pack. The equation $1.5s + 3.5c = 21$ represents the relationship between sheets

of stickers, s, packs of cardstocks, c, and the dollar amount a kindergarten teacher spent on these supplies.

What does the vertical intercept, (0,6), mean in this situation?



a. 14 sheets of stickers and 6 packs of cardstock were bought.
b. 7 sheets of stickers and 3 packs of cardstock were bought.
c. 6 sheets of stickers and 0 packs of cardstock were bought.
d. 0 sheets of stickers and 6 packs of cardstock were bought.

1.10.6 Lesson Summary

In this lesson, you learned how to:

- Describe the connections between an equation of the form $ax + by = c$, the features of its graph, and the rate of change in the situation.
- Graph a linear equation of the form $ax + by = c$.
- Demonstrate that rewriting the equation for a line in different forms can make it easier to find certain kinds of information about the relationship and about the graph.

Here are the **activities** that helped you reach those goals:

- 1.10.1 Interpreting Equations in Context
  - In this activity, you explained how equations matched given situations.
- 1.10.2 Graphing Equations in Context
  - In this activity, you used a graph of an equation to look at possible solutions to the situation.
  - 1.10.2 Self Check
  - 1.10.2 Additional Resources
- 1.10.3 Writing and Graphing Equations in Standard Form
  - In this activity, you explored writing and graphing equations in standard form. You then worked on solving the equations for y and recognized where the slope and y-intercept are located in the new equations.
  - 1.10.3 Self Check
  - 1.10.3 Additional Resources
- 1.10.4 Matching Equations and Graphs
  - In this activity, you wrote an equation to represent the number of rides and games that Kiran could do and matched the graph to this equation given the amount he could spend.

After these activities, you completed the following **practice**:

- 1.10.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

1.11 CONNECTING EQUATIONS TO GRAPHS, PART 2

1.11.0 Lesson Overview

In this lesson, you will continue to practice relating the structure of equations to the situation and corresponding graphs. You will use your understanding of constraints, equations, and points on a graph to explain whether a graph represents an equation and a situation. Along the way, you will practice reasoning quantitatively and abstractly and constructing logical arguments.

When you finish this lesson, you will be able to:

- Find the slope and vertical intercept of a line with equation $ax + by = c$.
- Take an equation of the form $ax + by = c$ and rearrange it into the equivalent form $y = mx + b$.
- Use a variety of strategies to find the slope and vertical intercept of the graph of a linear equation given in different forms.

Here are the **activities** that will help you reach those goals:

- 1.11.1: Applying the Distributive Property to Rewrite Expressions
- 1.11.2: Relating Two-Variable Equations, Their Graphs, and Situations
  - 1.11.2: Self Check
  - 1.11.2: Additional Resources
- 1.11.3:Reasoning Symbolically and Abstractly about Linear Equations
  - 1.11.3: Self Check
  - 1.11.3: Additional Resources
- 1.11.4: Identifying the Features of a Graph

After that, you'll **practice and review**.

- 1.11.5: Practice
- 1.11.6: Lesson Summary

1.11.1 Applying the Distributive Property to Rewrite Expressions

Warm Up

Rewrite each quotient as a sum or a difference.

1. $\frac{4x - 10}{2}$

2. $\frac{1 - 50x}{-2}$

3. $\frac{5(x + 10)}{25}$

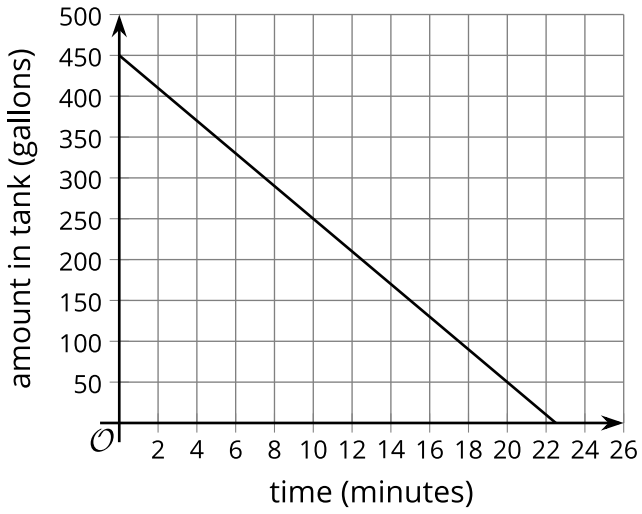
4. $\frac{-\frac{1}{5}x + 5}{2}$

1.11.2 Relating Two-Variable Equations, Their Graphs, and Situations

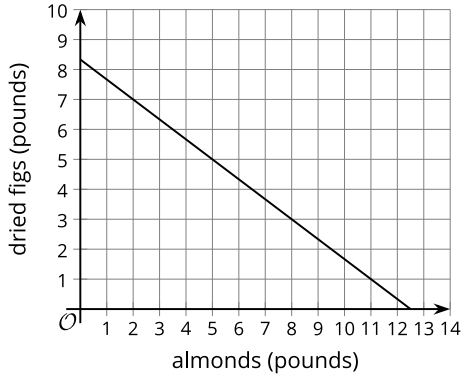
Activity

For this activity, you will use two graphs that represent situations you saw in other activities.

For questions 1 – 3, use the graph below: The graph represents $a = 450 - 20t$, which describes the relationship between gallons of water in a tank and time in minutes.

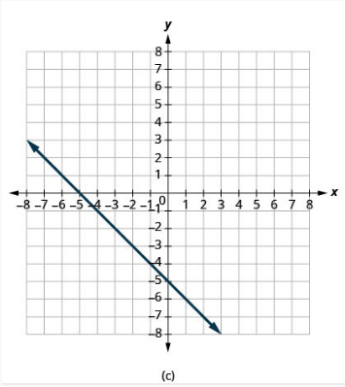


1. Where on the graph can we see the 450?
2. Where can we see the −20?
3. What do the numbers 450 and −20 mean in the situation that is graphed?
4. The graph represents $6x + 9y = 75$. It describes the relationship between pounds of almonds and figs and the dollar amount Clare spent on them. Suppose a classmate says, "I am not sure the graph represents $6x + 9y = 75$ because I don't see the 6, 9, or 75 on the graph." How would you show your classmate that the graph indeed represents this equation?



Self Check

For the line graphed below, what is the y-intercepts?



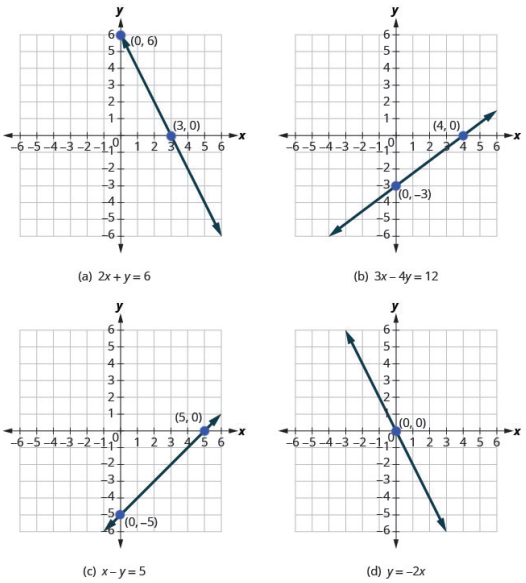
a. $(5, 0)$
b. $(0, 5)$
c. $(-5, 0)$
d. $(0, -5)$

Additional Resources

Finding Intercepts from Graphs and Equations

The points where a line crosses the x-axis and the y-axis are called the intercepts of the line.

Let's look at the graphs of the lines.



First, notice where each of these lines crosses the x-axis.

Now, let's look at the points where these lines cross the y-axis.

Figure	The Line Crosses the x -axis at:	Ordered Pair for this Point	The Line Crosses the y -axis at:	Ordered Pair for this Point
Figure (a)	3	$(3, 0)$	6	$(0, 6)$
Figure (b)	4	$(4, 0)$	−3	$(0, -3)$
Figure (c)	5	$(5, 0)$	−5	$(0, 5)$

Figure	The Line Crosses the x -axis at:	Ordered Pair for this Point	The Line Crosses the y -axis at:	Ordered Pair for this Point
Figure (d)	0	$(0, 0)$	0	$(0, 0)$
General Figure	a	$(a, 0)$	b	$(0,b)$

X-INTERCEPT and Y-INTERCEPT of a Line

The x -intercept is the point $(a, 0)$ where the line crosses the x-axis.

The y-intercept is the point $(0, b)$ where the line crosses the y-axis.

- The x-intercept occurs when y is zero.
- The y-intercept occurs when x is zero.

x	y
a	0
0	b

Find the x- and y-intercepts from the Equation of a Line

Use the equation of the line. To find:

- the x-intercept of the line, let $y = 0$ and solve for x.
- the y-intercept of the line, let $x = 0$ and solve for y.

EXAMPLE

Find the intercepts of $2x + y = 8$

Solution

To find the x-intercept:

Step 1 - Let $y = 0$.

$2x + 0 = 8$

Step 2 - Solve for x.

$x = 4$

Step 3 - Write the intercept as a point.

$(4, 0)$

To find the y-intercepts:

Step 1 - Let $x = 0$.

$2(0) + y = 8$

Step 2 - Solve for y.

$y = 8$

Step 3 - Write the intercept as a point.

$(0, 8)$

TRY IT **Finding Intercepts from Graphs and Equations**

Find the x- and y-intercept of $x + 4y = 8$.

1.11.3 Reasoning Symbolically and Abstractly about Linear Equations

Activity

For questions 1 – 5, select the slope and intercept that corresponds to each equation:

1. $-4x + 3y = 3$
   a. $m = 3$, y-int $= (0, 1)$
   b. $m = \frac{4}{3}$, y-int $= (0, 1)$
   c. $m = \frac{4}{3}$, y-int $= (0, -2)$
   d. $m = -4$, y-int $= (0, 8)$
   e. $m = 3$, y-int $= (0, -2)$

2. $12x - 4y = 8$
   a. $m = 3$, y-int $= (0, 1)$
   b. $m = \frac{4}{3}$, y-int $= (0, 1)$
   c. $m = \frac{4}{3}$, y-int $= (0, -2)$
   d. $m = -4$, y-int $= (0, 8)$
   e. $m = 3$, y-int $= (0, -2)$

3. $8x + 2y = 16$
   a. $m = 3$, y-int $= (0, 1)$
   b. $m = \frac{4}{3}$, y-int $= (0, 1)$
   c. $m = \frac{4}{3}$, y-int $= (0, -2)$
   d. $m = -4$, y-int $= (0, 8)$
   e. $m = 3$, y-int $= (0, -2)$

4. $-x + \frac{1}{3}y = \frac{1}{3}$
   a. $m = 3$, y-int $= (0, 1)$
   b. $m = \frac{4}{3}$, y-int $= (0, 1)$
   c. $m = \frac{4}{3}$, y-int $= (0, -2)$
   d. $m = -4$, y-int $= (0, 8)$
   e. $m = 3$, y-int $= (0, -2)$

5. $-4x + 3y = -6$
   a. $m = 3$, y-int $= (0, 1)$
   b. $m = \frac{4}{3}$, y-int $= (0, 1)$
   c. $m = \frac{4}{3}$, y-int $= (0, -2)$
   d. $m = -4$, y-int $= (0, 8)$
   e. $m = 3$, y-int $= (0, -2)$

Using Standard Form and Intercepts to Graph

Standard form, $Ax + By = C$, is helpful for finding the x-intercepts and y-intercepts of an equation.

For questions 6 – 11, use the equation $6x - 4y = 24$.

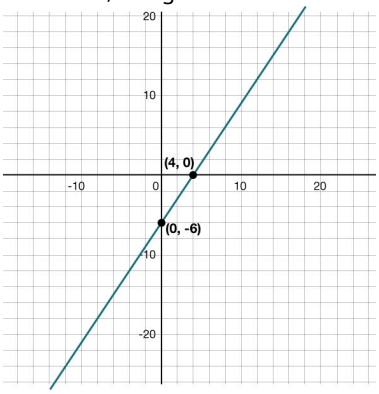
6. What is the value of y at the x-intercept?

7. What is the x-intercept of the equation?

8. What is the value of x at the y-intercept?

9. What is the y-intercept of the equation?

10. Use the graphing tool or technology outside the course. Plot the two intercepts and the equation of the line, using the Desmos tool below.



Access multimedia content (http://openstax.org/books/algebra-1/pages/1-11-3-reasoning-symbolically-and-abstractly-about-linear-equations)

11. Using the graph from question 10, what is the slope of the line in simplest form?

Are you ready for more?

Extending Your Thinking

For questions 1-2, use the equations below:

These are the equations from the activity. Each equation is in the form $Ax + By = C$.
$4x - 3y = -3$
$12x - 4y = 8$
$8x + 2y = 16$
$3x - y = -1$
$4x - 3y = 6$

1. Observe the graph of the equation $-4x + 3y = 3$ where it is graphed, on the same coordinate plane, with another line passing through $(0, 0)$ and (A, B). In this example $A = -4$ and $B = 3$, so the other line would pass through $(0, 0)$ and $(-4, 3)$. What do you notice about the lines?
Access multimedia content (http://openstax.org/books/algebra-1/pages/1-11-3-reasoning-symbolically-and-abstractly-about-linear-equations)

2. Use the graphing tool or technology outside the course. Graph the equation that represents this scenario using the Desmos tool above. Then make another observation about the lines.

Self Check

What is the slope of the line $4x + 3y = 12$?

a. $\frac{4}{3}$
b. $-\frac{3}{4}$
c. $\frac{3}{4}$
d. $-\frac{4}{3}$

Additional Resources

Video: Using Standard Form

Watch the following video to learn more about using standard form to find slope and y-intercepts.

Using Standard Form (https://openstax.org/r/standard-form)

TRY IT **Using Standard Form**

1. Find the slope for $5x - y = 15$.
2. Find the y-intercept for $5x - y = 15$.
3. Explain your reasoning.

1.11.4 Identifying the Features of a Graph

Cool Down

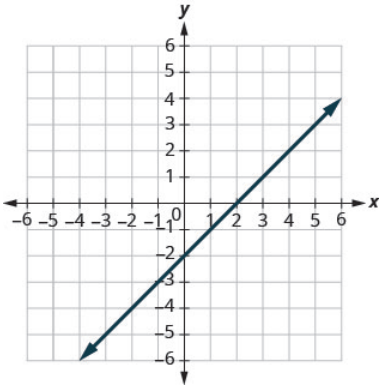
For questions 1 – 3, consider the equation $1.5x + 4.5y = 18$.

1. If we graph the equation, what is the slope of the graph?
2. Where does the graph intersect the y-axis?
3. Where does it intersect the x-axis?

1.11.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For numbers 1 – 2, use the graph below:



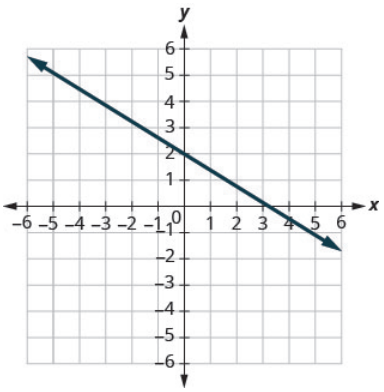
1. What are the coordinates of the x-intercept of the graph?

a. $(-2, 0)$
b. $(0, -2)$
c. $(2, 0)$
d. $(0, 2)$

2. What are the coordinates of the y-intercept of the graph?

a. $(-2, 0)$
b. $(0, -2)$
c. $(2, 0)$
d. $(0, 2)$

For numbers 3 – 4, use the graph below:



3. What are the coordinates of the x-intercept of the graph?

a. $(3, 0)$
b. $(0, 3)$
c. $(2, 0)$
d. $(0, 2)$

4. What are the coordinates of the y-intercept of the graph?

a. $(-2, 0)$
b. $(0, -2)$
c. $(2, 0)$
d. $(0, 2)$

5. What are the coordinates of the x-intercept of the equation $3x + y = 12$?

a. $(0, 4)$
b. $(4, 0)$
c. $(12, 0)$
d. $(0, 12)$

6. What are the coordinates of the y-intercept of the equation $3x + y = 12$?

a. $(0, 4)$
b. $(4, 0)$
c. $(12, 0)$
d. $(0, 12)$

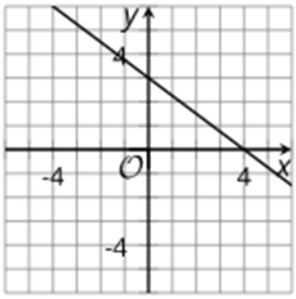
7. What is the slope of the line $3x - 2y = 6$?

8. What is the y-intercept of $y = 6x + 2$?

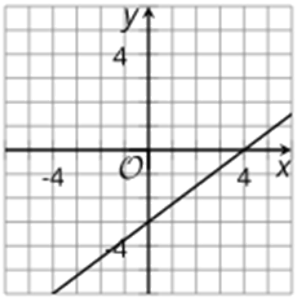
9. What is the y-intercept of $10x + 5y = 30$?

10. Which graph represents the equation $12 = 3x + 4y$? Explain how you know.

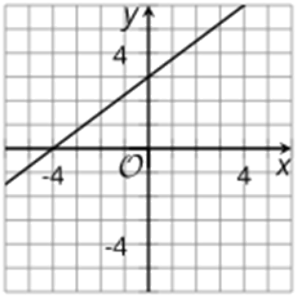
a.



b.



c.



1.11.6 Lesson Summary

In this lesson, you learned how to:

- Find the slope and vertical intercept of a line with equation $ax + by = c$.
- Take an equation of the form $ax + by = c$ and rearrange it into the equivalent form $y = mx + b$.
- Use a variety of strategies to find the slope and vertical intercept of the graph of a linear equation given in different forms.

Here are the **activities** that helped you reach those goals:

- 1.11.1: Applying the Distributive Property to Rewrite Expressions
  - In this activity, you rewrote expressions and discussed your strategies, including the distributive property.
- 1.11.2: Relating Two-Variable Equations, Their Graphs, and Situations
  - In this activity, you looked at two graphs from previous lessons and which numbers were slopes and intercepts. You also interpreted the meaning of those numbers in the situations graphed.
  - 1.11.2: Self Check
  - 1.11.2: Additional Resources
- 1.11.3: Reasoning Symbolically and Abstractly about Linear Equations
  - In this activity, you matched equations in standard form to their slopes and y-intercepts.
  - 1.11.3: Self Check
  - 1.11.3: Additional Resources
- 1.11.4: Identifying the Features of a Graph
  - In this activity, you used an equation to identify features of the graph of the equation including the slope and where the line would cross the x-axis and y-axis.

After these activities, you completed the following **practice**:

1.11.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

1.12 WRITING THE EQUATION OF A LINE

1.12.0 Lesson Overview

In this lesson, you will continue to examine and create linear models with linear equations relating two variables. You will also look at several ways to write the equation of a line based on the information given.When you finish this lesson, you will be able to:

- Write an equation of the line given the slope and y-intercept.
- Write an equation of the line given the slope and a point.
- Write an equation of the line given two points.

Here are the **activities** that will help you reach those goals:

- 1.12.1: Using the Slope Formula
- 1.12.2: Writing an Equation Given the Slope and y-Intercept
  - 1.12.2: Self Check
  - 1.12.2: Additional Resources
- 1.12.3: Writing an Equation Given the Slope and a Point
  - 1.12.3: Self Check
  - 1.12.3: Additional Resources
- 1.12.4: Writing Equations in Different Forms
  - 1.12.4: Self Check
  - 1.12.4: Additional Resources
- 1.12.5: Writing an Equation Given Two Points
  - 1.12.5: Self Check
  - 1.12.5: Additional Resources
- 1.12.6: Choosing a Method to Write an Equation

After that, you'll **practice and review**.

- 1.12.7: Practice
- 1.12.8: Lesson Summary

1.12.1 Using the Slope Formula

Warm Up

Use the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$, to find the slope of the line through the given points.

1. $(1, -2)$ and $(3, 4)$

2. $(-2, -3)$ and $(-7, 4)$

1.12.2 Writing an Equation Given the Slope and y-intercept

Activity

Recall that the slope-intercept form of the equation of a line is $y = mx + b$, where m is the slope and b is the y-intercept. The y-intercept is the point where the graph crosses the y-axis. For y-intercept b, the coordinates of the point are $(0, b)$.

In the following activity, you will complete a series of questions regarding this information.

CLASSROOM CONNECTION Log into student.desmos.com using the information provided by your teacher to complete the activity.

For questions 1 – 3, determine the slope, m, and the y-intercept, b.

1. $y = 2x + 5$

2. $y = -3x + 2$

3. $y = -\frac{1}{5}x - 1$

For questions 4 – 7, match the correct slope and y-intercept to the given equations.

a. slope: $-\frac{1}{2}$, y-intercept: -3
b. slope: 2, y-intercept: -3
c. slope: -2, y-intercept: 3
d. slope: 3, y-intercept: -2

4. What is the correct slope and y-intercept of $y = -2x + 3$?

5. What is the correct slope and y-intercept of $y = 3x - 2$?

6. What is the correct slope and y-intercept of $y = 2x - 3$?

7. What is the correct slope and y-intercept of $y = -\frac{1}{2}x - 3$?

8. Write the equation of a line, in slope-intercept form, that has a slope of -9 and a y-intercept at $(0, -4)$.

9. Write the equation of a line, in slope-intercept form, that has a slope of $\frac{2}{5}$ and a y-intercept at $(0, -4)$.

Self Check

Write the equation of a line in slope-intercept form with a slope of $-\frac{3}{4}$ and a y-intercept of -2.

a. $y = \frac{3}{4}x - 2$
b. $3x + 4y = -8$
c. $y = -\frac{3}{4}x + 2$
d. $y = -\frac{3}{4}x - 2$

Additional Resources

Write Equations Given a Slope and y-Intercept

We can determine the slope and intercept of a line if the equation is written in slope-intercept form, $y = mx + b$. Now we will do the reverse—we will start with the slope and y-intercept and use them to find the equation of the line.

EXAMPLE 1

Find the equation of a line with slope -1 and y-intercept $(0, -3)$.

Solution

Step 1 - Name the slope.

$m = -1$

Step 2 - Name the y-intercept.

$b = -3$

Step 3 - Substitute the values into $y = mx + b$.

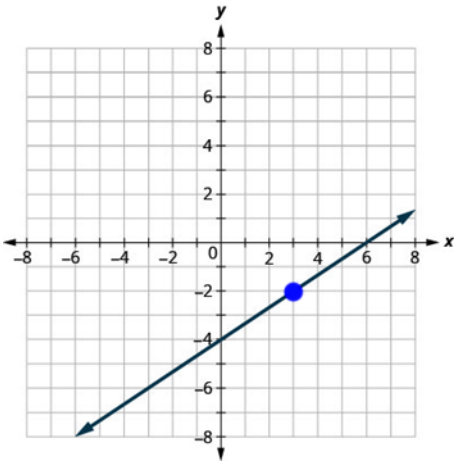
$y = -1x + -3$

$y = -x - 3$

The same information can be picked up from a graph of a line.

EXAMPLE 2

Find the equation of the line shown in the graph.



Solution

Step 1 - Find the slope by counting the rise and run.

$m = \frac{rise}{run}$

From $(0, -4)$ to $(3, -2)$, the slope is $m = \frac{2}{3}$.

Step 2 - Find the y-intercept.

y-intercept $(0, -4)$

$b = -4$

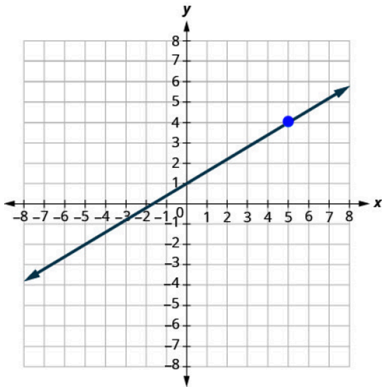
Step 3 - Substitute the values into $y = mx + b$.

$y = mx + b$

$y = \frac{2}{3}x - 4$

TRY IT **Write Equations Given a Slope and y-Intercept**

Write the equation of the line graphed below in slope-intercept form:



1.12.3 Writing an Equation Given the Slope and a Point

Activity

What if the point you are given isn't a y-intercept? Point-slope is a great form to use when you are given the slope and any point.

First, work to derive the point-slope formula by completing the following steps:

Start with the slope formula as $m = \frac{y-y_1}{x-x_1}$.

Step 1 - Multiply both sides of the equation by $x - x_1$.

Step 2 - Simplify.

Step 3 - Rewrite the equation with the y terms on the left.

With your partner, you will find the equation of the line in point-slope form that goes through the point $(3, 4)$ and has a slope of 2 by answering the following questions.

1. Use the point and slope to graph the line on a coordinate plane.
2. Write the point-slope form.
3. Name the point (x_1, y_1) and the slope, m.
4. Substitute the point and slope into the point-slope form.
5. Now, with your partner, each solve for y to write the equation in slope-intercept form. Compare your answers for any possible errors. Come to an agreement on your equation. You may also choose to graph the line on a coordinate plane using a graphing calculator or Desmos with your final equation to check your answer.

What is the simplified equation in slope-intercept form $(y = mx + b)$?

6. Convert slope-intercept form to standard form.

Building Character: Social Intelligence



Being empathetic, or sensitive to what others are feeling, shows that you care about others. In fact, empathetic people are less likely to experience anxiety, depression, and addictions later in life. They are also more likely to be hired, promoted, earn more money, and have happier marriages and better-adjusted children.

Think about your current sense of social intelligence. Are the following statements true for you??

- My relationships make me feel good about myself.
- The people in my life help me be my best.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

Self Check

Which equation represents the slope-intercept form of the equation of a line with slope $m = -\frac{3}{4}$ that contains the point $(4, -7)$?

a. $y = -\frac{3}{4}x + 10$
b. $y = -\frac{3}{4}x - 11$
c. $y = -\frac{3}{4}x - \frac{5}{4}$
d. $y = -\frac{3}{4}x - 4$

Additional Resources

Write the Equation of a Line Given the Slope and a Point

Writing an equation using the slope-intercept form works well when you are given the slope and y-intercept. If you are given the slope and another point that is not the y-intercept, we need another form of the equation of a line.

The point-slope form of an equation of a line can be derived from the formula for slope. Suppose you have a line that has slope m and that contains some specific point (x_1, y_1) as point 1 and some other point, (x, y), as point 2. You can write the slope of this line and then change it to a different form.

The point-slope form of an equation of a line with slope m and containing the point (x_1, y_1) is: $y - y_1 = m(x - x_1)$. You can use this form to find the equation of a line when you are given the slope and a point.

EXAMPLE 1

Write the equation of a line with slope $m = -\frac{1}{3}$ that contains the point $(6, -4)$. Write the equation in slope-intercept form and standard form.

Step 1 - Identify the slope.

The slope is given.

$m = \frac{1}{3}$

Step 2 - Identify the point.

The point is given.

(x_1, y_1)

$(6, -4)$

Step 3 - Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$. Simplify.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-4) &= \tfrac{1}{3}(x - 6) \\ y + 4 &= \tfrac{1}{3}x + 2 \end{aligned}$$

Step 4 - Write the equation in slope-intercept form.

$y = \frac{1}{3}x - 2$

Step 5 - Convert slope-intercept form to standard form.

$y = \frac{1}{3}x - 2$

$3y = 3\frac{1}{3}x - (3)2$

$3y = x - 6$

$-x + 3y = -6$

$(-1)(-x + 3y = -6)$

$x - 3y = 6$

To find an equation of a line given the slope and a point, follow these steps:

Step 1 - Identify the slope.

Step 2 - Identify the point.

Step 3 - Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.

Step 4 - Write the equation in slope-intercept form.

Step 5 - Convert slope-intercept form to standard form.

EXAMPLE 2

Find the equation of a horizontal line that contains the point $(-2, -6)$. Write the equation in slope-intercept form.

Step 1 - Identify the slope. Every horizontal line has slope 0.

$m = 0$

Step 2 - Identify the point.

(x_1, y_1)

$(-2, -6)$

Step 3 - Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$. Simplify.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-6) &= 0(x - (-2)) \\ y + 6 &= 0 \end{aligned}$$

Step 4 - Write the equation in slope-intercept form.

$y = -6$

$y = 0x - 6$

Step 5 - Convert slope-intercept form to standard form.

$y = 0x - 6$

$0x + y = -6$

TRY IT **Try It: Writing the Equation of a Line Given the Slope and Point**

Write the equation of a line with slope $m = -\frac{2}{5}$ that contains the point $(10, -5)$. Write the equation in slope-intercept form.

1.12.4 Writing Equations in Different Forms

Activity

Three Different Forms of Linear Equations

Work with a partner to answer questions 1 – 3.

1. What is the slope-intercept form of a linear equation?
2. What is the point-slope form of a linear equation?
3. What is the standard form of a linear equation?

Using the Different Forms to Write Equivalent Equations

Linear equations can be written in the different forms to create linear equations. Use the linear equation forms from 1 - 3 to help you complete the rest of the activity. Remember to show all of your work on your paper and to use inverse operations to move terms to different sides of the equation.

For questions 4 – 5, use the equation $y - 4 = 3(x + 1)$.

4. Write the equation in slope-intercept form.
5. Write the equation in standard form.
6. Write $4x - 6y = 12$ in slope-intercept form.

Self Check

Which of the following is equivalent to $y = 4x + 5$ in standard form?

$4x - y = -5$
$-4x + y = 5$
$x = 4y + 5$
$y - 4x = 5$

Additional Resources

Writing Equivalent Linear Forms

Write an equation of a line in all three forms that has a slope of 2 and a y-intercept of 7.

Slope-Intercept Form

- Slope-intercept form is $y = mx + b$ where m is the slope and b is the y-intercept.

Substitute $m = 2$ and $b = 7$ to have the equation $y = 2x + 7$.

Standard Form

Write the same equation in standard form.

- Standard form is $Ax + By = C$.
- Remember that A can not be negative and there should not be any fractions or decimals.

For this form, rearrange slope-intercept form to lead with the x term:

Step 1 - Bring the x and y terms on the left side.

$y - 2x = 2x - 2x + 7$

Step 2 - Simplify.

$-2x + y = 7$

Step 3 - Divide all terms by -1 so that A is not negative.

$\frac{-2x}{-1} + \frac{y}{-1} = 7 - 1$

Step 4 - Simplify.

$2x - y = -7$

Point-Slope Form

- Point-Slope form is $y - y_1 = m(x - x_1)$.

- Substitute the slope in for m and the point in for (x_1, y_1).

$m = 2, x_1 = 0, y_1 = 7$

$y - 7 = 2(x - 0)$

Notice this can easily be written in slope-intercept form by solving for y:

$$\begin{aligned} y - 7 &= 2(x - 0) \\ y - 7 &= 2x - 0 \\ y - 7 + 7 &= 2x - 0 + 7 \\ y &= 2x + 7 \end{aligned}$$

Standard Form to Slope-Intercept Form

Standard form can also be rearranged into slope-intercept form.

Write $3x - 5y = 14$ in slope-intercept form.

Work to solve for y.

Step 1 - Bring the x term to the right side with the constant.

$3x - 3x - 5y = 14 - 3x$

Step 2 - Simplify.

$-5y = 14 - 3x$

Step 3 - Divide by the coefficient of y.

$\frac{-5y}{-5} = \frac{14}{5} - \frac{3}{5}x$

Step 4 - Simplify and write in the correct order.

$y = -\frac{3}{5}x + \frac{14}{5}$

TRY IT **Writing Equivalent Linear Forms**

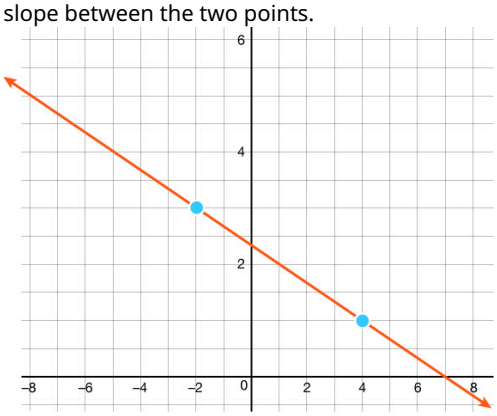
Write the equation $3x + 8y = 24$ in slope-intercept form.

1.12.5 Writing an Equation Given Two Points

Activity

Answer the following questions.

1. Use the graph below to pick two points and find the slope between the two points.



2. Use your two points in the slope formula to confirm the slope is correct.

3. Using the slope and a point, write your equation in point-slope form.
4. Now write your equation in slope-intercept form.
5. Convert slope-intercept form to standard form.
6. Look at your equation and the original graph. What do you notice? How can you use a graph to check your equation?

Self Check

Which equation represents the slope-intercept form of the equation of a line containing the points $(-2, -4)$ and $(1, -3)$?

a. $y = 3x - 6$
b. $y = 7x + 10$
c. $y = \frac{1}{3}x - \frac{7}{3}$
d. $y = \frac{1}{3}x - \frac{10}{3}$

Additional Resources

Find an Equation of the Line Given Two Points

So far, you have two options for finding an equation of a line: slope-intercept or point-slope. When you start with two points, it makes more sense to use the point-slope form.

But then you need the slope. You can find the slope with just two points and then use it and one of the given points to find the equation.

EXAMPLE

Write the equation of a line that contains the points $(-3, -1)$ and $(2, -2)$. Write the equation in slope-intercept form and standard form.

Step 1 - Find the slope using the given points.

Find the slope of the line through $(-3, -1)$ and $(2, -2)$.

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$m = \frac{-2-(-1)}{2-(-3)}$

$m = \frac{-1}{5}$

$m = -\frac{1}{5}$

Step 2 - Choose one point.

Choose either point.

(x_1, y_1)

$(2, -2)$

Step 3 - Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.

Simplify

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-2) &= -\tfrac{1}{5}(x - 2) \\ y + 2 &= -\tfrac{1}{5}x + \tfrac{2}{5} \end{aligned}$$

Step 4 - Write the equation in slope-intercept form.

$y = -\frac{1}{5}x - \frac{8}{5}$

Step 5 - Convert slope-intercept form to standard form.

$y = -\frac{1}{5}x - \frac{8}{5}$

$\frac{1}{5}x + y = -\frac{8}{5}$

$5(\frac{1}{5}x + y) = 5(-\frac{8}{5})$

$(1x + 5y) = -8$

Use this table for easy reference to find an equation of a line given two points.

Step 1 - Find the slope using the given points.

$m = \frac{y_2 - y_1}{x_2 - x_1}$

Step 2 - Choose one point.

Step 3 - Substitute the values into the point-slope form: $y - y_1 = m(x - x_1)$.

Step 4 - Write the equation in slope-intercept form.

Step 5 - Convert slope-intercept form to standard form.

To find an equation of a line given the slope and a point, follow these steps:

Step 1 - Identify the slope.

Step 2 - Identify the point.

Step 3 - Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.

Step 4 - Write the equation in slope-intercept form.

Step 5 - Convert slope-intercept form to standard form.

TRY IT **Writing the Equation of a Line Given Two Points**

Write the equation of a line that contains the points $(-3, 5)$ and $(-3, 4)$.

1.12.6 Choosing a Method to Write an Equation

Cool Down

You have seen that you can use either the slope-intercept form or the point-slope form to find an equation of a line. Which form you use will depend on the information you are given.

The table can be used as a reference.

If given:	Use:	Form:
Slope and y-intercept	slope-intercept	$y = mx + b$
Slope and a point	point-slope	$y - y_1 = m(x - x_1)$
Two points	point-slope	$y - y_1 = m(x - x_1)$

1. Each partner should find a slip labeled “Points.” Place the remaining equation slips in a pile face down to be used in the second step.
   Find the slope of the line containing the points on your slip.
   Take turns sharing with your partner the slope of your line and the method used to find the slope.

2. Partners take turns choosing slips from the pile to see if the equation on the slip represents the line formed by the points on their Points slip.
   - One partner chooses an Equation slip from the pile. Decide if the equation represents a line through the points on your Points slip. If it does, explain to your partner how you know. Keep the slip. If it does not represent a line through your points, give the Equation slip to your partner.
   - Your partner should verify the equation represents a line through the points on your partner’s Points slip. Discuss and reach an agreement. If the equation does not represent a line through either partner’s points, set it aside.
   - Take turns choosing Equation slips until they are all gone. (There are 6 Equation slips.) Each partner should end with a total of 2 Equation slips that match. There will be 2 Equations slips that do not match either set of points.
   - Which equations represent a line through your points?

3. Is identifying the slope enough to match your set of points to an equation?

1.12.7 Practice

Complete the following questions to practice the skills you have learned in this lesson.

In 1 – 2, find the equation of a line with given slope and y-intercept. Write the equation in slope-intercept form.

1. slope 3 and y-intercept $(0, 5)$

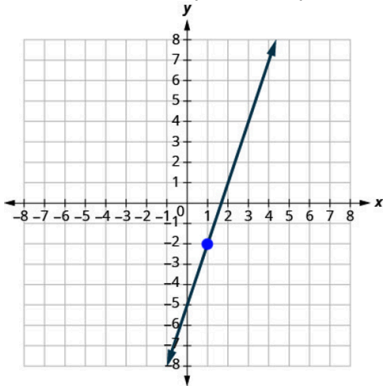
a. $y = 0x + 3$
b. $y = 3x + 5$
c. $y = 3x + 0$
d. $y = 5x + 3$

2. slope -3 and y-intercept $(0, -1)$

a. $y = 0x - 3$
b. $y = -x + 0$
c. $y = -3x - 1$
d. $y = -x - 3$

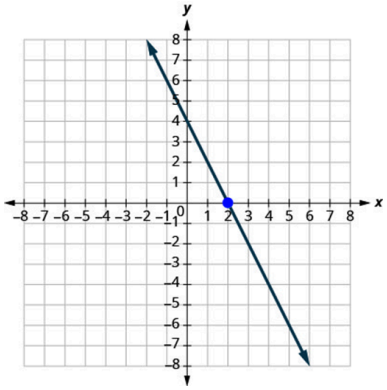
In 3 – 4, Choose the slope-intercept form of the equation of the line, shown in each graph.

3.



a. $y = 3x - 5$
b. $y = -5x + 3$
c. $y = 5x - 3$
d. $y = -3x + 5$

4.



a. $y = \frac{1}{2}x + 2$
b. $y = \frac{1}{2}x - 4$
c. $y = 2x + 4$
d. $y = -2x + 4$

5. Choose the equation of a line with the given slope and containing the given point that best represents the line in slope-intercept form.

$m = -7$, point $(-1, -3)$

a. $y = -7x - 10$
b. $y = -3x + 10$
c. $y = -7x - 3$
d. $y = -7x - 1$

6. Choose the equation of a line containing the given points that best represents the line in slope-intercept form.

$(0, 4)$ and $(2, -3)$

a. $y = -\frac{7}{2}x + 4$
b. $y = -\frac{2}{7}x + 4$
c. $y = \frac{7}{2}x + 4$
d. $y = -\frac{7}{2}x - 3$

7. Choose the equation of a line containing the given points that best represents the line in slope-intercept form.

$(3, -4)$ and $(5, -4)$

a. $x = -2$
b. $x = 2$
c. $y = 0$
d. $y = -4$

8. What is $y - 5 = 3(x - 2)$ written in standard form?

a. $y = 3x - 11$
b. $3x - y = 1$
c. $-3x + y = -1$
d. $y = 3x - 1$

9. Write the equation of a line that contains the points $(-3, -1)$ and $(2, -2)$. Write the equation in point-slope form.

a. $y - 4 = 3(x - 2)$
b. $y + 2 = \frac{1}{3}(x - 4)$
c. $y - 4 = -3(x + 2)$
d. $y - 2 = 3(x - 4)$

10. Write the equation of a line that contains the point $(2, -3)$ with a slope of 8. Write the equation in standard form.

a. $19x + y = 8$
b. $8x - y = 19$
c. $8x + y = 19$
d. $8x - y = -19$

11. Write the equation of a line that contains the points $(3, 1)$ and $(-1, 5)$. Write the equation in standard form.

a. $x + y = -4$
b. $-x + y = 4$
c. $x + y = 4$
d. $x - y = 4$

1.12.8 Lesson Summary

In this lesson, you learned how to:

- Write an equation of the line given the slope and y-intercept.
- Write an equation of the line given the slope and a point.
- Write an equation of the line given two points.

Here are the **activities** that helped you reach those goals:

- 1.12.1: Using the Slope Formula
  - In this activity, you reviewed using the slope formula to find the slope between two points.
- 1.12.2: Writing an Equation Given the Slope and y-Intercept
  - In this activity, you used the given slope and y-intercept to write the equation of a line in slope-intercept form. You also worked to find the slope and y-intercept from a graph to then write the equation of the line in slope-intercept form.
  - 1.12.2: Self Check
  - 1.12.2: Additional Resources
- 1.12.3: Writing an Equation Given the Slope and a Point
  - In this activity, you used point-slope form to write the equation of a line given a slope and a point. You then wrote the equation in slope-intercept form.
  - 1.12.3: Self Check
  - 1.12.3: Additional Resources
- 1.12.4: Writing Equations in Different Forms
  - In this activity, you reviewed three possible forms of a linear equation, slope-intercept form, point-slope form, and standard form, and how to convert between the different forms. You then wrote the equation for a line in all three different formats when given a variety of information.
  - 1.12.4: Self Check
  - 1.12.4: Additional Resources
- 1.12.5: Writing an Equation Given Two Points
  - In this activity, you found the slope between two points and confirmed it with the slope formula. You then used one point and the slope to write the equation of a line in point-slope form. From here, you wrote the equation in slope-intercept form and used the graph to check your equation.
  - 1.12.5: Self Check
  - 1.12.5: Additional Resources
- 1.12.6: Choosing a Method to Write an Equation
  - In this activity, you discussed when to use each method of writing an equation. With a partner, you worked to find slopes and equations of lines using different methods.

After these activities, you completed the following **practice**:

1.12.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

1.13 LINES FROM TABLES AND GRAPHS

1.13.0 Lesson Overview

In this lesson, you will not only write equations given a point and slope or two points but also from verbal descriptions and from tables.

When you finish this lesson, you will be able to:

- Write equations from a point and slope.
- Write equations from two points.
- Write equations from a verbal description.
- Write equations from a table.

Here are the **activities** that will help you reach those goals:

- 1.13.1: Writing Linear Equations
- 1.13.2: Creating Tables from Verbal Descriptions
  - 1.13.2: Self Check
  - 1.13.2: Additional Resources
- 1.13.3: Matching Tables, Equations, and Graphs
  - 1.13.3: Self Check
  - 1.13.3: Additional Resources
- 1.13.4: Creating a Table from an Equation

After that, you'll **practice and review**.

- 1.13.5: Practice
- 1.13.6: Lesson Summary

1.13.1 Writing Linear Equations

Warm Up

For questions 1 – 3, write the equation of:

1. A line with a slope of 5 and a y-intercept of $\frac{1}{2}$.

2. A line with a slope of 2 that goes through $(-1, 3)$.

3. A line that passes through $(2, 4)$ and $(1, -2)$.

1.13.2 Creating Tables from Verbal Descriptions

Activity

Creating a Table of Values

For each situation in numbers 1 – 3, create a table of values with at least five entries to represent the situation.

1. Julio works at a clothing store and makes $14 per hour.

2. Jake had cavities at his last dentist visit. The dentist told him to brush his teeth for two minutes each time he brushes.

3. For a fundraiser, Celia ran 5 miles on Monday (day 0) and then 3 miles every day after that.

Using a Table of Values to Write Equations

Tables of values can be used to write equations. The tables from **Creating a Table of Values** all represent linear equations.

For questions 4 – 8, revisit the table from question 1 and write an equation to represent the table of values. Notice that

columns for the x and y values are added.

x	0	1	2	3	4	5
y	0	14	28	42	56	70

4. What is the starting value (when $x = 0$, what does y equal)?

5. How would you describe this value on the graph?

6. What is the rate of change, or the change between the y-values over the change between the x-values? Worked Solution 1.13.2 (https://www.youtube.com/embed/nZiRu4PpXDo)

7. How would you describe this value on the graph?

8. Using the information in questions 4 – 7, what would the equation be for the values represented by the table? Use slope-intercept form.

9. Write an equation from the table of values created in question 2 of **Creating a Table of Values**.
Hint: Work backward to find the value where $x = 0$ using the pattern set in the table of values.

x	0	1	2	3	4	5
y	?	2	4	6	8	10

10. Write an equation from the table of values created in question 3 of **Creating a Table of Values**

x	0	1	2	3	4	5
y	?	2	4	6	8	10

Are you ready for more?

Extending Your Thinking

1. Kylie pays $20 to rent a scooter plus $0.40 per mile she drives the scooter. Create a table of values to represent Kylie's situation starting with $x = 0$.

2. Use the table to determine how much she has to pay after 5 miles.

Video: Writing an Equation from a Table of Values

Watch the following video to learn more about writing equations from a table of values.

Worked Solution 1.13.2 (https://www.youtube.com/embed/nZiRu4PpXDo)

Create Your Own Situation

Think of your own situation that could be modeled by a linear equation. Create a table of values that represents that situation. Write your problem down then switch with a partner. Make a table of values for each other's problems. Then, check your work together.

Next, plot the points of each table of values in a coordinate plane. Answer the questions after discussing with your partner:

1. Describe the scenario created by your partner.
2. Use Desmos to graph the table of values to represent your partner's scenario.
   Use the graphing tool or technology outside the course. Graph the equation that represents this scenario using the Desmos tool below.
   Access multimedia content (http://openstax.org/books/algebra-1/pages/1-13-2-creating-tables-from-verbal-descriptions)
3. Does your partner's data model a linear relationship or not? Describe how you determined your answer.

Self Check

Which of the following equations could be represented by this table of values?

x	0	1	2	3
y	20	23	26	29

a. $y = 20x + 3$
b. $y = 20x$
c. $y = \frac{1}{3}x + 20$
d. $y = 3x + 20$

Additional Resources

Writing Linear Equations from Tables

When writing linear equations from tables, it is important to identify the slope and the y-intercepts from the table.

EXAMPLE

Given the table, write an equation in slope-intercept form.

x	0	1	2	3	4
y	12	16	20	24	28

Step 1 - Identify the y-intercept.

The y-intercept is the starting value, when $x = 0$.

Looking at the table, when $x = 0$, the y-value is 12, so this is the y-intercept.

Step 2 - Identify the slope.

Slope is the change in y over the change in x.

The y-values change by 4 each time. The x-values change by 1 each time.

The slope is $\frac{4}{1}$ or 4.

Step 3 - Write the equation in slope-intercept form.

Slope intercept form is $y = mx + b$, where m is the slope and b is the y-intercept.

Substitute the values, $y = 4x + 12$.

TRY IT **Writing Linear Equations from Tables**

Given the table, write an equation in slope-intercept form.

x	0	1	2	3
y	6	11	16	21

1.13.3 Matching Tables, Equations, and Graphs

Activity

Take turns with your partner to match a graph with each set of matching cards (https://openstax.org/r/matching-cards). Eventually all the cards will be sorted into groups of four cards (an equation, situation, table, and graph).

For each match that you find, explain to your partner how you know it's a match. Ask your partner if they agree with your thinking.

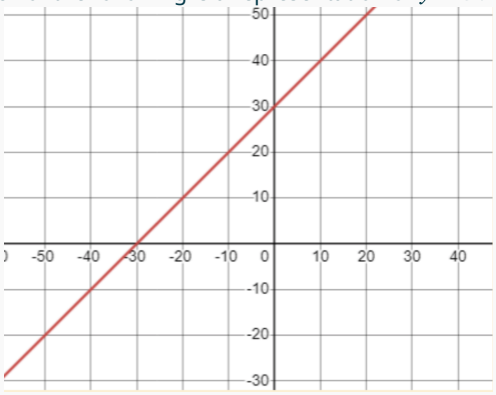
For each match that your partner finds, listen carefully to their explanation. If you disagree, discuss your thinking and work to reach an agreement.

1. What did you and your partner agree are the cards that match the equation on card 1?
2. What did you and your partner agree are the cards that match the equation on card 2?
3. What did you and your partner agree are the cards that match the equation on card 3?
4. What did you and your partner agree are the cards that match the equation on card 4?
5. What did you and your partner agree are the cards that match the equation on card 5?
6. What did you and your partner agree are the cards that match the equation on card 6?
7. What did you and your partner agree are the cards that match the equation on card 7?
8. What did you and your partner agree are the cards that match the equation on card 8?

Self Check

Which of the following is a representation of $y = 30x$?

a.



b.

x	0	1	2	3
y	30	35	40	45

c. Eric was paid $30 to cut grass in July plus $5 each time he came to cut the grass.

d. Josie paid $30 per pair of jeans.

Additional Resources

Linear Equations and Their Representations

Linear equations can be written from situations, tables of values, and graphs.

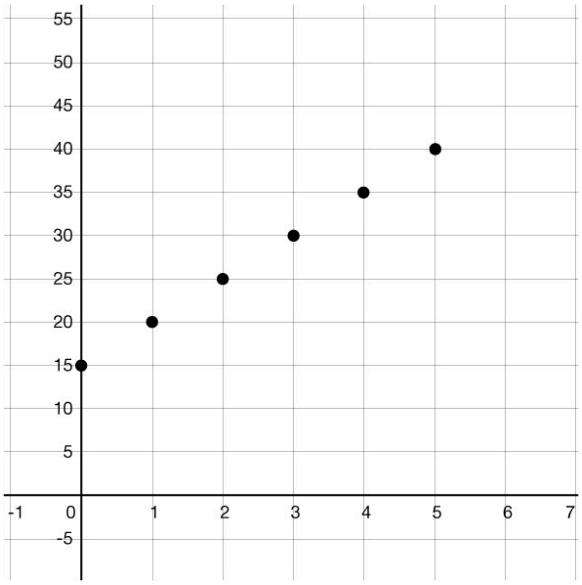
Here is a linear situation:

Sadie pays $15 a month for ballet classes plus $5 each lesson she takes.

First, make a table of values to represent how much Sadie pays for the first 5 lessons.

Number of Lessons	0	1	2	3	4	5
Total Cost	15	20	25	30	35	40

Next, graph all of the points:



Looking at the slope and y-intercepts of the table and graph, write an equation.

Step 1 - Identify the y-intercept.

When $x = 0$, $y = 15$ so the y-intercept is 15.

Step 2 - Identify the slope.

Slope is the change in y over the change in x. For this table, the change in y is 43 and the change in x is 1.

The slope is 3.

Step 3 - Write the equation in slope-intercept form.

In the slope-intercept form, $y = mx + b$, m is the slope and b is the y-intercept.

The equation is $y = 5x + 15$.

TRY IT **Linear Equations and Their Representations**

Write an equation from the table:

x	0	1	2	3	4
y	7	11	15	19	23

1.13.4 Creating a Table from an Equation

Cool Down

Using the equation $m = \frac{x-4}{6}$, complete the table of values.

x	0	1	2	3	4
m					

1.13.5 Practice

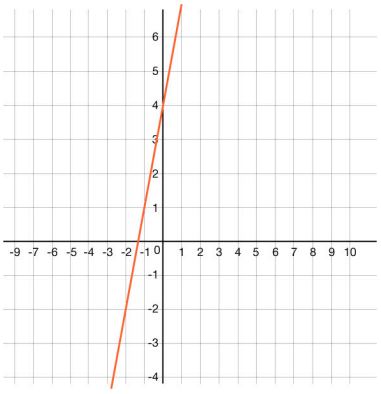
Complete the following questions to practice the skills you have learned in this lesson.

1. What is the slope of the line that goes through the points in the given table of values?

x	1	2	3	4
y	6	11	16	21

2. In the equation, $y = 4x + 2$, what is the value of the y-intercept?

3. Identify the slope in the given graph.



4. Which of the following is an equation for the table of values?

x	0	1	2	3
y	2	7	12	17

a. $y = 5x - 2$
b. $y = 7x - 2$
c. $y = 5x + 2$
d. $y = 2x + 5$

5. For the equation $y = -2x + 3$, what is the value when $x = 2$?

6. Jonah started the month with $40 and earned $15 each week walking dogs for neighbors. How much does he have after w weeks?

a. $y = 40(15w)$
b. $y = 40w + 15$
c. $y = 40w$
d. $y = 15w + 40$

7. Write the slope of the given table.

x	0	2	4
y	5	11	17

8. For the equation $g = \frac{w+3}{2}$, what is the value of g when $w = 7$?

9. Identify the y-intercept of the given table.

x	1	2	3
y	3	5	7

10. Identify the slope of the line between $(3, 6)$ and $(-2, 7)$.

1.13.6 Lesson Summary

In this lesson, you learned how to:

- Write equations from a point and slope.
- Write equations from two points.
- Write equations from a verbal description.
- Write equations from a table.

Here are the **activities** that helped you reach those goals:

- 1.13.1: Writing Linear Equations
  - In this activity, you wrote linear equations in slope-intercept form given a point and slope or two points.
- 1.13.2: Creating Tables from Verbal Descriptions
  - In this activity, you learned how to create tables of values from verbal situations then how to write a linear equation from the table.
  - 1.13.2: Self Check
  - 1.13.2: Additional Resources
- 1.13.3: Matching Tables, Equations, and Graphs
  - In this activity, you continued working with linear equations and their representations by matching verbal descriptions, graphs, tables of values, and their equations.
  - 1.13.3: Self Check
  - 1.13.3: Additional Resources
- 1.13.4: Creating a Table from an Equation
  - In this activity, you worked backward and created a table of values from a given equation.

After these activities, you completed the following **practice**:

- 1.13.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

1.14 WRITING EQUATIONS OF PARALLEL AND PERPENDICULAR LINES

1.14.0 Lesson Overview

In this lesson, you will continue to examine methods of writing an equation of a line. In recent lessons, you learned that the method you use to write the equation of a line is determined by the information given. You will continue to use those methods along with the relationship between slopes of lines.

When you finish this lesson, you will be able to:

- Write an equation of a line parallel to a given line.
- Write an equation of a line perpendicular to a given line.
- Write an equation of a line parallel or perpendicular to an axis.
- Write an equation of a line in slope-intercept form given two points.

Here are the **activities** that will help you reach those goals:

- 1.14.1: Using Point-Slope Form to Write the Equation of a Line
- 1.14.2: Writing Equations for Point-Slope and Slope-Intercept
  - 1.14.2: Self Check
  - 1.14.2: Additional Resources
- 1.14.3: Writing an Equation of a Line Parallel to a Given Line
  - 1.14.3: Self Check
  - 1.14.3: Additional Resources
- 1.14.4: Writing an Equation of a Line Perpendicular to a Given Line
  - 1.14.4: Self Check
  - 1.14.4: Additional Resources
- 1.14.5: Writing an Equation of a Line Parallel or Perpendicular to an Axis
  - 1.14.5: Self Check
  - 1.14.5: Additional Resources
- 1.14.6: Drawing Parallel and Perpendicular Lines

After that, you'll **practice and review**.

- 1.14.7: Practice
- 1.14.8: Lesson Summary

1.14.1 Using Point-Slope Form to Write the Equation of a Line

Warm Up

Your teacher will assign you one pair of the points shown.

- Points for Line A: $(-1, 5)$ and $(1, 1)$
- Points for Line B: $(-1, -1)$ and $(1, -5)$

Use your assigned points to find the following:

1. Find the slope of the line.
2. Write the equation of the line in slope-intercept form.
3. Compare your equation to your partner's equation. What were some things that were the same? What were some things that were different

1.14.2 Writing Equations for Point-Slope and Slope-Intercept

Activity

Writing Equations Given Two Points

For questions 1 – 3, use the point $(3, 4)$ and $(-2, 5)$.

1. Find the slope of the line that goes through the two points.
2. Find the equation in point slope-form of the line that goes through the two points.
3. Write the equation from question 2 into slope-intercept form.
(Hint: Solve for y.)

Writing Equations in Slope-Intercept Form Given a Point and Slope

For questions 4 – 7 use the point $(3, -2)$ and the slope -4.

4. Substitute the slope into slope-intercept form.
5. Substitute the given point in for (x, y) in the equation.
6. Solve for b in slope-intercept form.
7. Write the equation in slope-intercept form.

With a partner, work to go directly from two points to an equation in slope-intercept form, without using point-slope form.

For questions 8 – 12, use the two points $(5, 2)$ and $(3, -1)$

8. Find the slope between the two points.
9. Substitute the slope into slope-intercept form.
10. Substitute the one point in for (x, y) in the equation.
11. Solve for b in slope-intercept form.
12. Write the equation in slope-intercept form.

Self Check

Which of the following is the equation of a line that has a slope of 3 and goes through the point $(1, -2)$?

$y = 1x - 2$
$y = 3x - 2$
$y = 3x + 1$
$y = 3x - 5$

Additional Resources

Find an Equation Given Two Points

Method 1: Using Point-Slope Form

Find an equation of a line that contains the points $(5, 4)$ and $(3, 6)$. Write the equation in slope-intercept form.

Step 1 - Find the slope between the two points.

$m = \frac{6-4}{3-5} = \frac{2}{-2} = -1$

Step 2 - Choose a point to substitute into point-slope form.

$y - y_1 = m(x - x_1)$

$y - 4 = -1(x - 5)$ OR $y - 6 = -1(x - 3)$

Step 3 - Write the equation in slope-intercept form.

$$\begin{aligned} y-4 &= -1(x-5) \\ y-4 &= -1x+5 \\ y &= -x+9 \end{aligned}$$

OR

$$\begin{aligned} y-6 &= -1(x-3) \\ y-6 &= -1x+3 \\ y &= -x+9 \end{aligned}$$

Method 2: Writing an Equation in Slope-Intercept Form by finding b.

Find the equation of the line containing the points $(3, 1)$ and $(5, 6)$ in slope-intercept form.

Step 1 - Find the slope between the two points.

$m = \frac{6-1}{5-3} = \frac{5}{2}$

Step 2 - Choose a point to substitute into slope-intercept form then solve for b.

$y = mx + b$

$$\begin{aligned} y &= \tfrac{5}{2}x + b \\ 1 &= \tfrac{5}{2}(3) + b \\ \tfrac{2}{2} &= \tfrac{15}{2} + b \\ b &= \tfrac{13}{2} \end{aligned}$$

OR

$$\begin{aligned} y &= \tfrac{5}{2}x + b \\ 6 &= \tfrac{5}{2}(5) + b \\ \tfrac{12}{2} &= \tfrac{25}{2} + b \\ b &= \tfrac{13}{2} \end{aligned}$$

Step 3 - Write the equation in slope-intercept form.

$y = \frac{5}{2}x + \frac{13}{2}$

TRY IT **Find an Equation Given Two Points**

Find the equation of a line that consists of the points $(1, 4)$ and $(6, 2)$ in slope-intercept form.

1.14.3 Writing an Equation of a Line Parallel to a Given Line

Activity

1. Find the equation of the line using point-slope form: $m = 2$, $(3, 1)$
2. Rewrite your equation from Question 1 in slope-intercept form.
3. Find the equation of the line using point-slope form: $m = 2$, $(-1, 4)$
4. Rewrite your equation from Question 3 in slope-intercept form.
5. Find the equation of the line using point-slope form: $m = -4$, $(2, 1)$
6. Rewrite your equation from Question 5 in slope-intercept form.
7. Find the equation of the line using point-slope form: $m = -4$, $(3, -2)$
8. Rewrite your equation from Question 7 in slope-intercept form.

Use the graphing tool or technology outside the course.

Access multimedia content (http://openstax.org/books/algebra-1/pages/1-14-3-writing-an-equation-of-a-line-parallel-to-

a-given-line)

9. Graph your equations from questions 2 and 4 on the same graph using the Desmos tool.

Use the graphing tool or technology outside the course.

Access multimedia content (http://openstax.org/books/algebra-1/pages/1-14-3-writing-an-equation-of-a-line-parallel-to-a-given-line)

10. Graph your equations from questions 6 and 8 on the same graph using the Desmos tool.

11. What is the same about the equations given in questions 2 and 4?

12. What is the same about the equations given in questions 6 and 8?

Self Check

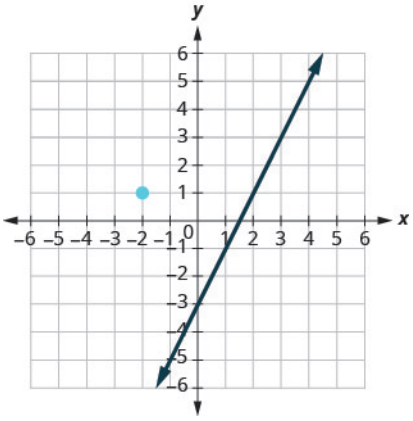
Which equation represents an equation of a line parallel to the line $y = \frac{1}{2}x - 3$ that contains the point $(6, 4)$?

a. $y = \frac{1}{2}x - 2$
b. $y = \frac{1}{2}x - 7$
c. $y = \frac{1}{2}x + 4$
d. $y = \frac{1}{2}x + 1$

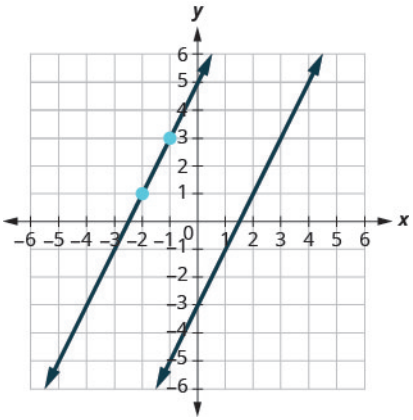
Additional Resources

Find the Line Parallel to a Given Line

In this section, you will use what you know about writing the equation of a line given the slope and a point to write the equation of parallel lines. The graph shows the equation of the line $y = 2x - 3$. The point $P\,(-2, 1)$ is also plotted. You can use the fact that parallel lines have the same slope to find a line that is parallel to line m and goes through point $P\,(-2, 1)$.



From the equation, you know the slope of the line is 2. The second line will pass through $(-2, 1)$ and have slope 2. To graph the line, start at $(-2, 1)$ and count out the rise and run. The slope is $\frac{rise}{run} = \frac{2}{1}$. Count out the rise, 2, and run, 1, and plot the point. You can graph the line as shown. Line m is parallel to line l and goes through point $P\,(-2, 1)$.



To find the equation of a line parallel to a line through a given point algebraically, you can use what you know about finding the equation of a line given the slope and a point.

EXAMPLE

Write an equation of a line parallel to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope-intercept form.

Step 1 - Find the slope of the given line.

The line is in slope-intercept form $y = 2x - 3$

$m = 2$

Step 2 - Find the slope of the parallel line

Parallel lines have the same slope

$m_1 = 2$

Step 3 - Identify the point

The given point is $(-2, 1)$

$(x_1, y_1) = (-2, 1)$

Step 4 - Substitute values into the point-slope form $y - y_1 = m(x - x_1)$

Simplify

$$\begin{aligned} y - y_1 &= (x - x_1) \\ y - 1 &= (x - (-2)) \\ y - 1 &= (x + 2) \\ y - 1 &= x + 4 \end{aligned}$$

Step 5 - Write the equation in slope-intercept form

$y = 2x + 5$

Look at the graph with the parallel lines shown previously. Does this equation make sense? What is the y-intercept of the line? What is the slope?

You can use this table as a reference for writing an equation of a line parallel to a given line.

Write the equation in slope-intercept form: $y = mx + b$.

Step 1 - Find the slope of the given line.

Step 2 - Find the slope of the parallel line.

Step 3 - Identify the point.

Step 4 - Substitute the values into the point-slope form: $y - y_1 = m(x - x_1)$.

Step 5 - Simplify.

Step 6 - Write the equation in slope-intercept form: $y = mx + b$.

TRY IT **Find the Line Parallel to a Given Line**

Write an equation of a line parallel to the line $y = 3x + 1$ that contains the point $(4, 2)$. Write the equation in slope-intercept form.

1.14.4 Writing an Equation of a Line Perpendicular to a Given Line

Activity

1. Find the equation of the line using point-slope form: $m = 2$, $(3, 1)$
2. Rewrite your equation from Question 1 in slope-intercept form.
3. Find the equation of the line using point-slope form: $m = -\frac{1}{2}$, $(-1, 4)$
4. Rewrite your equation from Question 3 in slope-intercept form.
5. Find the equation of the line using point-slope form: $m = -4$, $(2, 1)$
6. Rewrite your equation from Question 5 in slope-intercept form.
7. Find the equation of the line using point-slope form: $m = \frac{1}{4}$, $(3, 2)$
8. Rewrite your equation from Question 7 in slope-intercept form.

Use the graphing tool or technology outside the course.

Access multimedia content (http://openstax.org/books/algebra-1/pages/1-14-4-writing-an-equation-of-a-line-perpendicular-to-a-given-line)

9. Graph your equations from questions 2 and 4 on the same graph using the Desmos tool.

Use the graphing tool or technology outside the course.

Access multimedia content (http://openstax.org/books/algebra-1/pages/1-14-4-writing-an-equation-of-a-line-perpendicular-to-a-given-line)

10. Graph your equations from questions 6 and 8 using the Desmos tool.
11. What is the slope of the line perpendicular to a line with a slope of $\frac{1}{4}$?
12. What is the slope of the line perpendicular to a line with a slope of -8?
13. What is the slope of the line perpendicular to a line with a slope of $-\frac{6}{11}$?

Self Check

Which equation represents an equation of a line perpendicular to the line $y = \frac{1}{2}x - 3$ that contains the point $(6, 4)$?

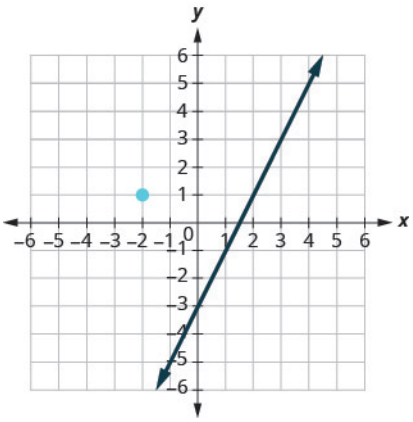
a. $y = -2x - 2$
b. $y = 2x - 8$
c. $y = -2x + 14$
d. $y = -2x + 16$

Additional Resources

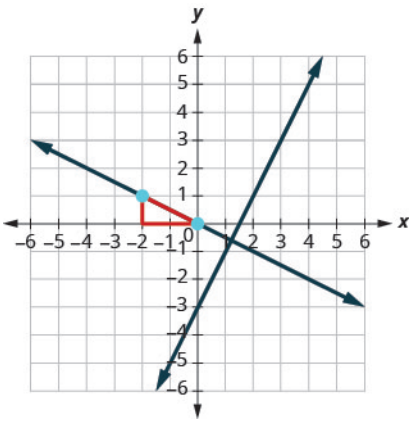
Finding a Perpendicular Line Given an Equation and Point

In this section, you will use what you know about writing the equation of a line given the slope and a point to write the

equation of perpendicular lines. The graph shows the equation of the line $y = 2x - 3$. The point $P(-2, 1)$ is also plotted. You can use the fact that perpendicular lines have slopes that are opposite reciprocals to find a line that is perpendicular to line l and goes through point $P(-2, 1)$.



From the equation, you know the slope of the line is 2. The second line will pass through $P(-2, 1)$ and have slope $-\frac{1}{2}$. To graph the line, start at $(-2, 1)$ and count out the rise and run. The slope is $\frac{rise}{run} = \frac{-1}{2}$. Count out the rise, -1, and run, 2, and plot the point. You can graph the line as shown. Line n is perpendicular to line l and goes through point $(-2, 1)$.



To find the equation of a line perpendicular to a line through a given point algebraically, you can use what you know about finding the equation of a line given the slope and a point.

EXAMPLE

Write an equation of a line perpendicular to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope-intercept form.

Step 1 - Find the slope of the given line.

The line is in slope-intercept form $y = 2x - 3$

$m = 2$

Step 2 - Find the slope of the perpendicular line perpendicular lines have the inverse slope

$m_1 = -\frac{1}{2}$

Step 3 - Identify the point

The given point is $(-2, 1)$

$(x_1, y_1) = (-2, 1)$

Step 4 - Substitute values into the point-slope form $y - y_1 = m(x - x_1)$

Simplify

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= -\tfrac{1}{2}(x - (-2)) \\ y - 1 &= -\tfrac{1}{2}(x + 2) \\ y - 1 &= -\tfrac{1}{2}x - 1 \end{aligned}$$

Step 5 - Write the equation in slope-intercept form

$y = -\frac{1}{2}x$

Use this table for reference to write an equation of a line perpendicular to a given line:

Step 1 - Find the slope of the given line.

Step 2 - Find the slope of the perpendicular line.

Step 3 - Identify the point.

Step 4 - Substitute the values into the point-slope form:

$y - y_1 = m(x - x_1)$.

Step 5 - Simplify.

Step 6 - Write the equation in slope-intercept form ($y = mx + b$).

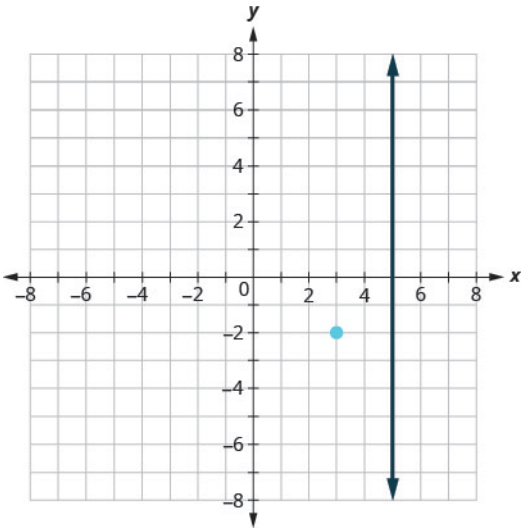
TRY IT **Finding a Perpendicular Line Given an Equation and Point**

Find an equation of a line perpendicular to the line $y = 3x + 1$ that contains the point $(4, 2)$. Write the equation in slope-intercept form.

1.14.5 Writing an Equation of a Line Parallel or Perpendicular to an Axis

Activity

The line $x = 5$ and the point $(3, -2)$ are shown on the graph.



Work with a partner to complete the tasks:

1. Identify the slope of the original line, l.

Use the graphing tool or technology outside the course.

Access multimedia content (http://openstax.org/books/algebra-1/pages/1-14-5-writing-an-equation-of-a-line-parallel-or-perpendicular-to-an-axis)

2. Graph line m through the point $(3, -2)$ and parallel to line l using the Desmos tool.

Then,

3. Identify the slope of the new line, m.

4. Write the equation of the new line, m.

Use the graphing tool or technology outside the course.

Access multimedia content (http://openstax.org/books/algebra-1/pages/1-14-5-writing-an-equation-of-a-line-parallel-or-perpendicular-to-an-axis)

5. Graph line n through the point $(3, -2)$ and perpendicular to line l using the Desmos tool.

Then,

6. Identify the slope of the new line, n.

7. Write the equation of the new line, n.

The original line l and the line m that you created are both parallel to the y-axis and perpendicular to the x-axis. It follows, then, that line n is parallel to the x-axis and perpendicular to the y-axis.

8. Write the equation of a line that is parallel to the y-axis and passing through $(-2, -5)$.

9. Write the equation of a line that is perpendicular to the y-axis and passing through $(1, 5)$.

10. Write the equation of a line parallel to the x-axis and passing through $(3, 4)$.

11. How does the line described as "parallel to the x-axis and passing through $(3, 4)$" relate to the y-axis? Will it always, sometimes, or never be perpendicular to it?

Self Check

Which is the equation that is perpendicular to the line $x = 1$ that contains the point $(-5, 2)$.

a. $y = -2$
b. $y = 1$
c. $x = -5$
d. $y = 2$

Additional Resources

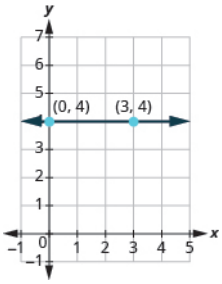
Find Equations of Lines Perpendicular to an Axis

Lines that are horizontal or vertical are special for two reasons. First, their equations have just one variable.

- Horizontal line $y = b$; because all the y-coordinates are the same and the slope equals zero.
- Vertical line $x = a$; because all the x-coordinates are the same and the slope is undefined.

Second, they have a special relationship with the x- and y-axes because they are either parallel or perpendicular to them.

Examine the line that passes through $(0, 4)$ and $(3, 4)$.



What is the rise? The rise is 0.

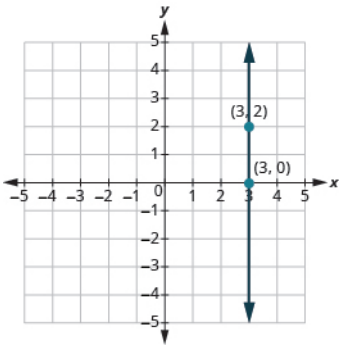
What is the run? The run is 3.

What is the slope? $m = \frac{\text{rise}}{\text{run}} = \frac{0}{3} = 0$

So, the horizontal line passing through $(0, 4)$ and $(3, 4)$ has a slope of 0. For all of the points on this line, the y-values will always equal 4. So, the equation is $y = 4$.

Also notice that it is parallel to the x-axis and perpendicular to the y-axis.

Now consider the vertical line passing through $(3, 2)$ and $(3, 0)$.



What is the rise? The rise is 2.

What is the run? The run is 0.

What is the slope? $m = \frac{\text{rise}}{\text{run}} = \frac{2}{0}$ (undefined)

So, the vertical line passing through $(3, 2)$ and $(3, 0)$ has an undefined slope. And, for all of the points on this line, the x-values will always equal 3. So, the equation is $x = 3$.

Also notice that it is parallel to the y-axis and perpendicular to the x-axis.

EXAMPLE 1

Find the equation of the line parallel to the x-axis and passing through $(8, 1)$.

Step 1 - Identify the point.

$(8, 1)$

Step 2 - Identify the slope of the line.

Since the line is parallel to the x-axis, the line is horizontal. This means the slope of the line is 0.

$m = 0$

Step 3 - Substitute the values into an equation.

From here, you may know the equation, but if you don't, then we can use the point-slope equation.

$y - y_1 = m(x - x_1)$

$y - 1 = 0(x - 8)$

Step 4 - Simplify.

$y - 1 = 0$

$y = 1$

Thus, the equation of the line parallel to the x-axis and passing through $(8, 1)$ is $y = 1$.

EXAMPLE 2

Find the equation of the line perpendicular to the x-axis and passing through $(8, 1)$.

Step 1 - Identify the point.

$(8, 1)$

Step 2 - Identify the slope of the line.

Since the line is perpendicular to the x-axis, the line is vertical. This means the slope of the line is undefined!

$m = \text{undefined}$

Step 3 - Substitute the values into an equation.

Since the slope is undefined, we cannot use the point-slope equation. But, we know that if $(8, 1)$ is on the line, then another point vertically above this point is $(8, 2)$. Another point vertically below this point is $(8, 0)$. On a vertical line, the x-values do not change.

So, $x = x_1$

$x = 8$

EXAMPLE 3

Find an equation of a line perpendicular to $x = 5$ that contains the point $(3, -2)$. Write the equation in slope-intercept form.

Again, since we know one point, the point-slope option seems more promising than the slope-intercept option. We need the slope to use this form, and we know the new line will be perpendicular to $x = 5$. This line is vertical, so its perpendicular will be horizontal. This tells us the $m_\perp = 0$.

Step 1 - Identify the point.

$(3, -2)$

Step 2 - Identify the slope of the perpendicular line.

$m_\perp = 0$

Step 3 - Substitute the values into an equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-2) &= 0(x - 3) \end{aligned}$$

Step 4 - Simplify

$$\begin{aligned} y + 2 &= 0 \\ y &= -2 \end{aligned}$$

EXAMPLE 4

Find an equation of a line that is perpendicular to $y = 4$ that contains the point $(2, 3)$. Write the equation in slope-intercept form.

The line $y = 4$ is a horizontal line. Any line perpendicular to it must be vertical, in the form $x = a$. Since the perpendicular line is vertical and passes through $(2, 3)$, every point on it has an x-coordinate of 2. The equation of the perpendicular line is $x = 2$.

TRY IT **Find Equations of Lines Perpendicular to an Axis**

Find an equation of a line that is perpendicular to $y = -3$ that contains the point $(-3, 5)$.

1.14.6 Drawing Parallel and Perpendicular Lines

Cool Down

On the same coordinate grid:

- Partner A draws a vertical line, and Partner B draws a horizontal line.
- Each partner labels their line with its equation in slope-intercept form.

Pass your paper to your partner. Check your partner's graph and equation.

1. Explain if your partner graphed the equation correctly and labeled the line with the correct equation.

For question 2, complete the following tasks:

- Draw a point on your partner's paper that is not on the line.
- Label the point P.
- Draw a line parallel to the original line through point P.

2. What is the equation of the line that passes through point P and is parallel to your partner's original line?

3. Draw a line perpendicular to the original line that goes through point P. What is the equation of the line that passes through point P and is perpendicular to your partner's original line?

1.14.7 Practice

Complete the following questions to practice the skills you have learned in this lesson.

In questions 1 – 5, choose an equation of a line parallel to the given line and contains the given point. The choices are in slope-intercept form.

1. Which line is parallel to $y = 4x + 2$ and contains the point $(1, 2)$?

a. $y = -\frac{1}{4}x + 2$
b. $y = -\frac{1}{4}x - 2$
c. $y = 4x - 2$
d. $y = 4x + 2$

2. Which line is parallel to $y = -3x - 1$ and contains the point $(2, -3)$?

a. $y = \frac{1}{3}x - 3$
b. $y = \frac{1}{3}x + 3$
c. $y = -3x - 3$
d. $y = -3x + 3$

3. Which line is parallel to $2x - y = 6$ and contains the point $(3, 0)$?

a. $y = 2x - 6$
b. $y = -2x$
c. $y = -2x + 6$
d. $y = 2x$

4. Which line is parallel to $x = -4$ and contains the point $(-3, -5)$?

a. $y = -4$
b. $x = -5$
c. $y = -5$
d. $x = -3$

5. Which line is parallel to $y = 5$ and contains the point $(2, -2)$?

a. $y = 2$
b. $y = -2$
c. $x = 5$
d. $x = 2$

In questions 6 – 10, find an equation of a line that is perpendicular to the given line and contains the given point. The choices are in slope-intercept form.

6. Which line is perpendicular to $y = -2x + 3$ and contains the point $(2, 2)$?

a. $y = \frac{1}{2}x + 1$
b. $y = \frac{1}{2}x + 3$
c. $y = -\frac{1}{2}x + 3$
d. $y = -2x + 6$

7. Which line is perpendicular to $y = -x + 5$ and contains the point$(3, 3)$?

a. $y = -x + 3$
b. $y = x + 5$
c. $y = x$
d. $y = -x + 6$

8. Which line is perpendicular to $y = \frac{3}{4}x - 2$ and contains the point $(-3, 4)$?

a. $y = \frac{3}{4}x + \frac{17}{4}$
b. $y = \frac{3}{4}x + 4$
c. $y = -\frac{4}{3}x + 4$

d. $y = -\frac{4}{3}x$

9. Which line is perpendicular to $x = -5$ and contains the point $(1, -2)$?

a. $x = -2$
b. $y = -2$
c. $x = 1$
d. $y = -5$

10. Which line is perpendicular to $y - 3 = 0$ and contains the point $(-2, -4)$?

a. $x = -2$
b. $x = 3$
c. $y = -4$
d. $y = 33$

1.14.8 Lesson Summary

In this lesson, you learned how to:

- Write an equation of a line parallel to a given line.
- Write an equation of a line perpendicular to a given line.
- Write an equation of a line parallel or perpendicular to an axis.
- Write an equation of a line in slope-intercept form given two points.

Here are the **activities** that helped you reach those goals:

- 1.14.1: Using Point-Slope Form to Write the Equation of a Line
  - In this activity, you used the slope formula to find the slope of a line. Using the slope and a point, you wrote equations in point-slope form and slope-intercept form.
- 1.14.2: Writing Equations for Point-Slope and Slope-Intercept
  - In this activity, you wrote an equation of a line in slope-intercept form given two points.
  - 1.14.2: Self Check
  - 1.14.2: Additional Resources
- 1.14.3: Writing an Equation of a Line Parallel to a Given Line
  - In this activity, you explored parallel lines that have the same slope. You also wrote the equation of lines parallel to given lines that went through a specific point.
  - 1.14.3: Self Check
  - 1.14.3: Additional Resources
- 1.14.4: Writing an Equation of a Line Perpendicular to a Given Line
  - In this activity, you explored perpendicular lines that have slopes that are opposite reciprocals of each other. You wrote the equations of lines that were perpendicular to a given line and went through a specific point.
  - 1.14.4: Self Check
  - 1.14.4: Additional Resources
- 1.14.5: Writing an Equation of a Line Parallel or Perpendicular to an Axis
  - In this activity, you wrote equations of lines parallel or perpendicular to an axis through a given point, given a line in the form $x = a$ or $y = b$.
  - 1.14.5: Self Check
  - 1.14.5: Additional Resources
- 1.14.6: Drawing Parallel and Perpendicular Lines
  - In this activity, you worked with a partner to draw parallel and perpendicular lines on a coordinate plane.

After these activities, you completed the following **practice**:

- 1.14.7: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

1.15 DIRECT VARIATION

1.15.0 Lesson Overview

In this lesson, you will review proportions and proportional relationships. Understanding proportions will help you learn how to recognize direct variation situations. You will then set up direct variation equations and work to solve problems. Direct variation is seen in everyday life in a variety of proportional relationships. Examples include the amount a paycheck increases with more hours of work; the distance a car travels compared to its rate of speed; and the cost of an electric bill based on the amount of electricity used.

When you finish this lesson, you will be able to:

- Set up and solve direct variation problems.

Here are the **activities** that will help you reach those goals:

- 1.15.1: Solving Proportions
- 1.15.2: Modeling Equations Using Direct Variation
  - 1.15.2: Self Check
  - 1.15.2: Additional Resources
- 1.15.3: Using Direct Variation to Solve Application Problems, Part 1
  - 1.15.3: Self Check
  - 1.15.3: Additional Resources
- 1.15.4: Using Direct Variation to Solve Application Problems, Part 2

After that, you'll **practice and review**.

- 1.15.5: Practice
- 1.15.6: Lesson Summary

1.15.1 Solving Proportions

Warm Up

Solve each of the following proportions:

1. $\frac{x}{56} = \frac{7}{8}$

2. $\frac{56}{72} = \frac{y}{9}$

3. $\frac{98}{154} = \frac{-7}{p}$

1.15.2 Modeling Equations Using Direct Variation

Activity

In your group, work through your assigned problem. Then be ready to share.

Group 1. When Raoul runs on the treadmill at the gym, the number of calories, c, he burns varies directly with the number of minutes, m, he uses the treadmill. He burned 315 calories when he used the treadmill for 18 minutes. Write the equation that relates c and m.

Group 2. The number of calories, c, burned varies directly with the amount of time, t, spent exercising. Arnold burned 312 calories in 65 minutes exercising. Write the equation that relates c and t.

Group 3. The distance a moving body travels, d, varies directly with time, t, it moves. A train travels 100 miles in 2 hours. Write the equation that relates d and t.

What is Direct Variation?

All of these are examples of direct variation.

Two variables vary directly if one is the product of a constant and the other.

For any two variables x and y, y varies directly with x if

$y = kx$, where $k \neq 0$.

The constant k is called the constant of variation.

Look back at the problem you solved. Use the direct variation equation to solve it again using these steps:

Step 1 - Write the equation for direct variation.

Step 2 - Substitute the given values for the variables.

Step 3 - Solve for the constant of variation.

Step 4 - Write the equation that relates x and y using the constant of variation.

Let's look at example 3 again and solve it with the direct variation equation.

The distance a moving body travels, d, varies directly with time, t, it moves. A train travels 100 miles in 2 hours. Write the equation that relates d and t.

Step 1 - Write the equation for direct variation.

$y = kx$

Step 2 - Substitute the given values for the variables.

$100 = k(2)$

Step 3 - Solve for the constant of variation.

$k = 50$

Step 4 - Write the equation that relates x and y using the constant of variation.

$y = 50x$

Self Check

The price, P, that Eric pays for gas varies directly with the number of gallons, g, he buys. It costs him \$65 to buy 25 gallons of gas. Which equation expresses the price of any amount of gas, P, in terms of the number of gallons, g?

a. $P = \frac{5}{13}g$
b. $P = \frac{65}{g}$
c. $P = 25g$
d. $P = \frac{13}{5}g$

Additional Resources

Modeling Situations Using Direct Variation

When two quantities are related by a proportion, we say they are proportional to each other. Another way to express this relation is to talk about the variation of the two quantities. We will discuss direct variation here.

Lindsay gets paid \$15 per hour at her job. If we let s be her salary, in dollars per hour, and h be the number of hours she has worked, we could model this situation with the equation

$s = 15h$

Lindsay's salary is the product of a constant, 15, and the number of hours she works. We say that Lindsay's salary varies

directly with the number of hours she works. Two variables vary directly if one is the product of a constant and the other.

Direct Variation

For any two variables x and y, y varies directly with x if

$y = kx$, where $k \neq 0$

The constant k is called the constant of variation.

In applications using direct variation, generally we will know values of one pair of the variables and will be asked to find the equation that relates x and y.

Here's how to model an equation using direct variation for application problems.

Step 1 - Write the formula for direct variation.

Step 2 - Substitute the given values for the variables.

Step 3 - Solve for the constant of variation.

Step 4 - Write the equation that relates x and y using the constant of variation.

Let's look at an application problem:

The cost of a pizza, y, varies directly with its diameter, x, and an 8-inch pizza costs $12. What is the equation that relates x and y?

Step 1 - Write the formula for direct variation.

$y = kx$

Step 2 - Substitute the given values for the variables.

$12 = k(8)$

Step 3 - Solve for the constant of variation.

$k = \frac{12}{8} = \frac{3}{2}$

Step 4 - Write the equation that relates x and y using the constant of variation.

$y = \frac{3}{2}x$

TRY IT **Modeling Situations Using Direct Variation**

The distance a moving body travels, d, varies directly with the amount of time, t, it moves. A train travels 60 miles in 2 hours. What is the equation that relates d and t?

1.15.3 Using Direct Variation to Solve Application Problems, Part 1

Activity

Return to your original group and work through this next part of your assigned problem. Then be ready to share.

When Raoul runs on the treadmill at the gym, the number of calories, c, he burns varies directly with the number of minutes, m, he uses the treadmill. He burned 315 calories when he used the treadmill for 18 minutes.

How many calories would he burn if he ran on the treadmill for 25 minutes?

The number of calories, c, burned varies directly with the amount of time, t, spent exercising. Arnold burned 312 calories in 65 minutes exercising.

How many calories would he burn if he exercises for 90 minutes?

The distance a moving body travels, d, varies directly with time, t, it moves. A train travels 100 miles in 2 hours.

How many miles would it travel in 5 hours?

Are you ready for more?

Extending Your Thinking

For example 1, how could you find out how many minutes Raoul would need to use the treadmill to burn a certain number of calories?

Video: Using Direct Variation to Solve Application Problems

Watch the following video to learn more about how to use direct variation to solve an application problem.

Using Direct Variation to Solve Application Problems (https://openstax.org/r/graph-equation)

Self Check

Sophia is making parts for a model ship. The number of parts, p, she can make in a day varies directly with the number of hours, h, she works. She can make 28 parts in 4 hours. How many parts would Sophia make in 9 hours?

a. 252
b. 79
c. 16
d. 63

Additional Resources

Solve Application Problems Using Direct Variation

In applications using direct variation, generally we will know values of one pair of the variables and will be asked to find the equation that relates x and y. Then we can use that equation to find values of y for other values of x.

Here's how to solve applications of direct variation problems.

Step 1 - Write the equation for direct variation. Identify the variables.

Step 2 - Substitute the given values for the variables.

Step 3 - Solve for the constant of variation.

Step 4 - Write the equation that relates x and y using the constant of variation.

Step 5 - Substitute the value for the variable.

Step 6 - Solve.

Let's look at an application problem:

If the cost of a pizza varies directly with its diameter, and an 8-inch pizza costs $12, how much would a 6-inch pizza cost?

Step 1 - Write the equation for direct variation. Identify the variables.

$y = kx$

y: cost of pizza, x: diameter

Step 2 - Substitute the given values for the variables.

$12 = k(8)$

Step 3 - Solve for the constant of variation.

$k = \frac{3}{2}$

Step 4 - Write the equation that relates x and y using the constant of variation.

$y = \frac{3}{2}x$

Step 5 - Substitute the value for the variable.

$y = \frac{3}{2}(6)$

Step 6 - Solve.

$y = 9$

What does this mean in the context of the problem?

A pizza that has a diameter of 6 inches costs $9.

TRY IT **Try It: Solve Application Problems Using Direct Variation**

The number of buckets of water, b, filled by a spring varies directly with the number of minutes, m, passed. The spring fills 3 buckets of water in 14 minutes.

How many buckets would be filled in 42 minutes?

1.15.4 Using Direct Variation to Solve Application Problems, Part 2

Cool Down

Jesse's car gets 30 miles per gallon of gas.

1. Write an equation using direct variation for distance, d, and number of gallons of gas, g for this problem.
2. If Jesse travels 150 miles he uses 5 gallons of gas. Find the constant of variation.
3. Now, write the equation if Jesse travels d miles with a car that uses 30 miles per gallon of gas.
4. If Las Vegas is 285 miles away, how many gallons of gas are needed to get there?

Why Should I Care?



Alex's mother can use algebra to budget her paycheck from her job as an accountant. She uses some money for food, rent, medicine, and things like water and electricity.

If she uses variables to represent the different ways that she uses her money, she can put those variables into an equation to make sure she has enough money to pay for all those things.

1.15.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. If y varies directly as x and $y = 14$, when $x = 3$, find the equation that relates x and y.

2. The price, P, that Eric pays for gas varies directly with the number of gallons, g, he buys. It costs him $50 to buy 20 gallons of gas. Choose the equation that relates P and g.

a. $P = \frac{g}{2.5}$
b. $G = \frac{2.5}{p}$
c. $P = \frac{2.5}{g}$
d. $P = 2.5g$

3. Joseph is traveling on a road trip. The distance, d, he travels before stopping for lunch varies directly with the speed, v, he travels. He can travel 120 miles at a speed of 60 mph. How many miles would he travel before stopping for lunch at a rate of 65 mph?

4. The mass of a liquid, M, varies directly with its volume, V. A liquid with mass 16 kilograms has a volume of 2 liters. What is the volume, in liters, of this liquid if its mass is 128 kilograms?

5. The length, L, that a spring stretches varies directly with a weight, W, placed at the end of the spring. When Sarah placed a 10-pound watermelon on a hanging scale, the spring stretched 5 inches. What weight of watermelon, in pounds, would stretch the spring 6 inches?

6. The maximum load a beam, M, will support varies directly with the square of the diagonal of the beam's cross-section, d^2. A beam with diagonal 6 inch will support a maximum load of 108 pounds. What load, in pounds, will a beam with a 10-inch diagonal support?

7. The number of calories, c, burned varies directly with the amount of time, t, spent exercising. Arnold burned 312 calories in 65 minutes exercising. How many calories would he burn if he exercises for 90 minutes?

1.15.6 Lesson Summary

In this lesson, you learned how to:

- Set up and solve direct variation problems.

Here are the **activities** that helped you reach those goals:

- 1.15.1: Solving Proportions
  - In this activity, you recalled how to solve proportions for variables.
- 1.15.2: Modeling Equations Using Direct Variation
  - In this activity, you found the constant of variation given a situation and wrote a direct variation equation.
  - 1.15.2: Self Check
  - 1.15.2: Additional Resources
- 1.15.3: Using Direct Variation to Solve Application Problems, Part 1
  - In this activity, you continued to find the constant of variation and write direct variation equations. In addition, you solved the equations for a given constraint in application problems.
  - 1.15.3: Self Check
  - 1.15.3: Additional Resources
- 1.15.4: Using Direct Variation to Solve Application Problems, Part 2
  - In this activity, you looked at a direct variation situation involving the miles a car can travel given the gallons of gas and distance.

After these activities, you completed the following **practice**:

- 1.15.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

PROJECT 1: SLOPES AND INTERCEPTS

Project 1 Overview

In this culminating project, you will integrate ideas from the unit. In the first activity, you will analyze points on a graph to identify your meaning in context. In the second activity, you will identify slopes and intercepts from graphs and match equations to graphs. Finally, you will create your own scenarios to match a linear equation.

Project Activities

1. Notice and Wonder: Sugar and Flour
2. Matching Matching

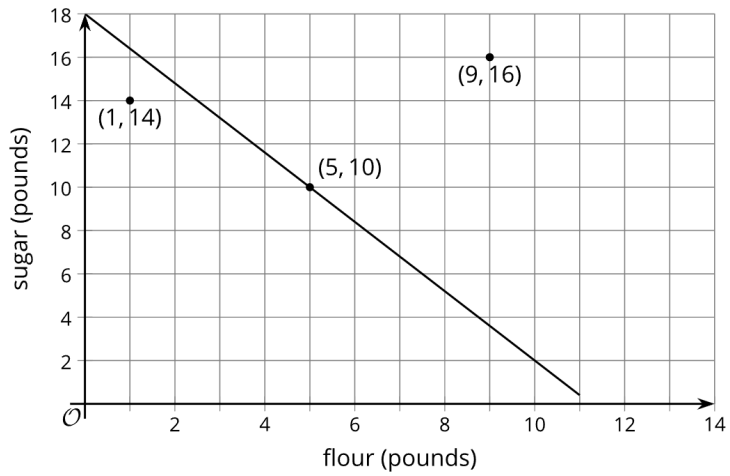
Learning Targets

When you finish this project, you will have demonstrated how to:

- Identify slopes and intercepts from a graph.
- Identify an equation that matches a graph.
- Create a real-world scenario from a graph.

Notice and Wonder Graph Features

Examine the following graph. Take a minute to think of at least one thing you notice and one thing you wonder.



Matching Graph and Equations

Round 1: Key Features

Deal out the graph cards, face up. Take turns with your partner picking a graph. Then, answer questions 1 – 2.

1. For each graph you find, tell your partner the slope, x-intercept, and y-intercept.
2. For each graph your partner picks, see if you agree with their thinking. If you agree, write their answers down on the graph for them. If you disagree, discuss your thinking and work to reach an agreement.

Round 2: Equation Matches

Deal out the equation cards face up, leaving the graph cards face up as well. Take turns with your partner to match each graph with an equation. Then, answer questions 3 – 4.

3. For each match that you find, explain to your partner how you know it's a match.
4. For each match that your partner finds, listen carefully to their explanation. If you disagree, discuss your thinking and work to reach an agreement.

Round 3: Scenarios

Write a real-world problem that could be modeled by one of the equations or graphs.
5. Choose a point that is on the graph or satisfies the equation and explain to your partner what it means in the context of your problem.
6. Explain to your partner what the slope and intercepts represent in the context of your problem.

Unit 1 Wrap Up

Throughout this unit, you learned that Arthur Noriega, Miami City Manager, has to consider constraints when planning a budget for various departments. He has to examine the relationships among quantities. He can use expressions, equations, and inequalities to represent the constraints for each department — such as the number of workers needed, the salaries of the employees, or the cost of supplies — to make sure he stays in budget. He can use graphs of equations to make sense of constraints and identify values that satisfy them.

Similarly, in this unit, you learned how to use expressions, equations, and inequalities to represent constraints on quantities. You also used graphs of equations to make sense of constraints and identify values to solve them.

You also learned to recognize moves that transform one equation into an equivalent equation. You used these moves to manipulate and solve equations. You also used these moves to write equations in point-slope form, standard form, and point-slope form.

Finally, you learned how the form and parts of a linear equation in two variables are related to the features of its graph. You identified lines that are parallel and perpendicular using the relationship between slopes.



Access our Unit 1 Student Self-Assessment (https://openstax.org/r/unit-1-self-assesment) to reflect on the skills you learned during this unit. Congratulations on completing Unit 1!

Building Character: Social Intelligence

Throughout this unit, you learned that **social intelligence** is important for developing relationships that are mutually beneficial and supportive. You learned that you can encourage social intelligence in others by waiting your turn before speaking, acknowledging other's points of view, encouraging teamwork and loyalty over hierarchy and competition, and by creating opportunities for everyone to feel equal.

UNIT 2 OVERVIEW AND READINESS

Unit 2 Overview

Why should I care?

Do you ever set goals for yourself? Perhaps you have a goal to make your school's basketball team, or you have a goal to improve your GPA next semester.

Just as you might set goals for yourself, businesses set goals for themselves. These goals often relate to producing a quality product, staying on budget, and making a profit for the company.

Watch the video to discover how entrepreneur Nafy Flatley has worked to achieve her goal of bringing the food of Senegal to the world, and how she might use linear inequalities to create a healthy and affordable product.

Linear Inequalities in Business (https://openstax.org/r/business-inequalities)

In this unit, you will learn about solving systems of linear equations, linear inequalities, and systems of linear inequalities. Understanding how to solve these types of equations, inequalities, and systems can help you in your daily life. For example, solving systems of linear equations can help you with deciding which of two phone plans with different flat fees and different costs per text charges is the better deal based on your usage.

Building Character: Growth Mindset

As an entrepreneur, Nafy Flatley must embrace challenges and try new strategies as she works to make her business a success. Similarly, in this course, you may be faced with challenges as you work through new skills and concepts. Being able to learn from mistakes and struggles is an important skill to have both in and out of the classroom.

Throughout this unit, you will work on developing a growth mindset. Having a growth mindset means believing you can improve your abilities. Compared to a fixed mindset, a growth mindset encourages you to embrace challenges, sustain effort, and try new strategies. Of course, no one embodies only a growth or fixed mindset; we are all a mixture of the two, and we can learn to recognize what triggers a fixed versus growth mindset. The brain is like a muscle—something you can strengthen.

Think about your current sense of having a growth mindset. How many of the following statements are true for you?

- After I make a mistake, I always look for ways to learn from it.
- I love challenges because they make me smarter.
- I truly believe that people can change.
- I can always change how intelligent I am.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

Am I ready to learn this?

Getting Ready for Unit 2

To be ready for this unit, you need to be able to:

- Determine if an ordered pair is a solution to a linear equation.
- Work with slope-intercept form of a linear equation.
- Find the x- and y-intercepts of a line when given its equation.

Next, you will answer some questions that will help you see how ready you are for this unit. If you need help, some mini-lessons will help you brush up on anything you need to work on.

Determine the Ordered Pair: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #1: Determine the Ordered Pair

Which ordered pair is a solution for $y = \frac{2}{3}x - 4$?

a. $(-2, -3)$

b. $(0, 6)$

c. $(-3, -2)$
d. $(6, 0)$

Evaluating Ordered Pairs as Solutions to a Linear Equation

An ordered pair (x, y) is a solution of the linear equation $Ax + By = C$, if the equation is a true statement when the x- and y-values of the ordered pair are substituted into the equation.

Linear equations have infinitely many solutions. We can plot these solutions in the rectangular coordinate system. The points will line up perfectly in a straight line. We connect the points with a straight line to get the graph of the equation. We put arrows on the ends of each side of the line to indicate that the line continues in both directions.

A graph is a visual representation of all the solutions of the equation. It is an example of the saying, "A picture is worth a thousand words." The line shows you all the solutions to that equation. Every point on the line is a solution of the equation. And, every solution of this equation is on this line. This line is called the graph of the equation. Points not on the line are not solutions!

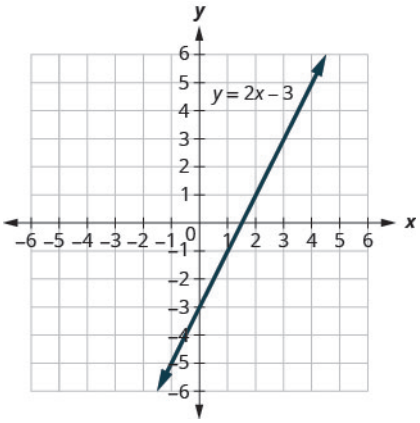
The graph of a linear equation $Ax + By = C$ is a straight line.

- Every point on the line is a solution of the equation.
- Every solution of this equation is a point on this line.

Let's examine the equation $2x - y = 3$.

To analyze the equation, we may want to graph the line. If we change the equation to slope-intercept form, the equation is $y = 2x - 3$.

The graph of $y = 2x - 3$ is shown.



For each ordered pair, decide:

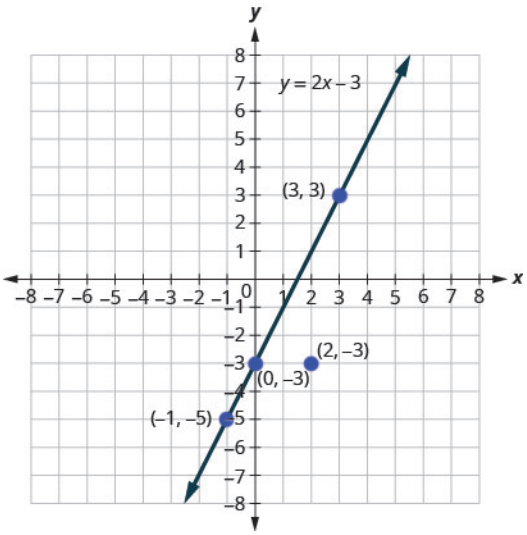
- Is the ordered pair a solution to the equation?
- Is the point on the line?

1. (0, −3)
2. (3, 3)
3. (2, −3)
4. (−1, −5)

To answer the first question, substitute the $x-$ and $y-$ values into the equation to check if the ordered pair is a solution to the equation.

A: $(0, -3)$	B: $(3, 3)$	C: $(2, -3)$	D: $(-1, -5)$
$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$
$-3 \stackrel{?}{=} 2(0) - 3$	$3 \stackrel{?}{=} 2(3) - 3$	$-3 \stackrel{?}{=} 2(2) - 3$	$-5 \stackrel{?}{=} 2(-1) - 3$
$-3 = -3$ ✓	$3 = 3$ ✓	$-3 \neq 1$	$-5 = -5$ ✓
$(0, -3)$ is a solution.	$(3, 3)$ is a solution.	$(2, -3)$ is not a solution.	$(-1, -5)$ is a solution.

To answer the second question, examine the graph:



The points $(0, 3)$, $(3, -3)$, and $(-1, -5)$ are on the line $y = 2x - 3$, and the point $(2, -3)$ is not on the line.

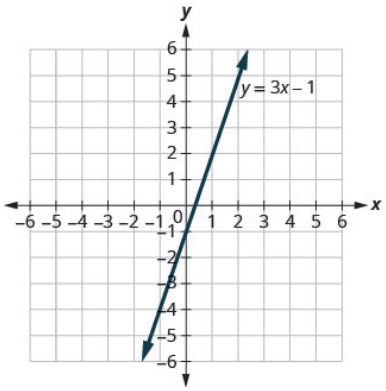
The points that are solutions to $y = 2x - 3$ are on the line, but the point that is not a solution is not on the line.

TRY IT **Evaluating Ordered Pairs as Solutions to a Linear Equation**

Use the graph of $y = 3x - 1$. For each ordered pair, decide:

Is the ordered pair a solution to the equation?
Is the point on the line?

1. (0, −1)
2. (2, 5)



Check Your Understanding

Which ordered pair is a solution for $y = -2x + 5$?

a. $(1, 2)$

b. $(-2, 1)$

c. $(2, 1)$

d. $(-1, 3)$

Video: Ordered Pairs and Solutions to 2-Variable Equations

Khan Academy: Solutions to 2-Variable Equations

Watch this video to learn about ordered pairs and how to test solutions to 2-variable equations.

Checking ordered pair solutions to equations example 1 | Algebra I | Khan Academy (https://www.youtube.com/embed/qk69pR91R00)

Find the Slope and y-intercept: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #2: Find the Slope and y-intercept

What is the slope and y-intercepts of the line $3x - y = 12$?

a. $m = \frac{3}{12}$; y-intercept $= (3, -12)$

b. $m = \frac{12}{3}$; y-intercept $= (3, -1)$

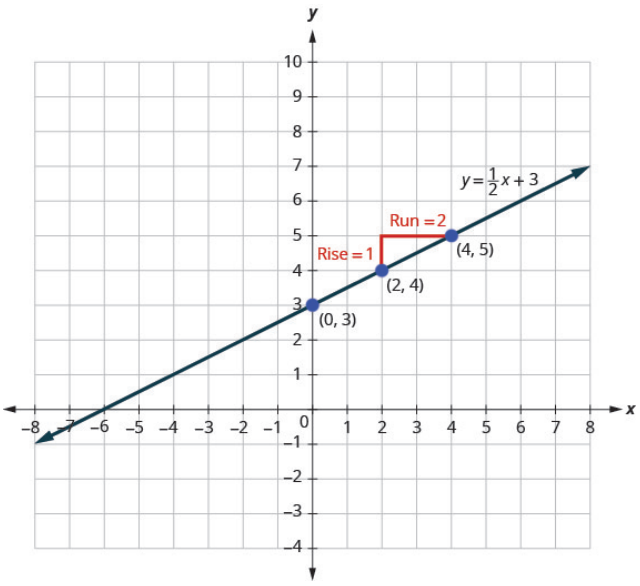
c. $m = \frac{1}{3}$; y-intercept $= (-12, 0)$

d. $m = \frac{3}{1}$; y-intercept $= (0, -12)$

Graph a Line Using Its Slope and y-intercepts

We have graphed linear equations by plotting points, using intercepts, recognizing horizontal and vertical lines, and using one point and the slope of the line. Once we see how an equation in slope–intercept form and its graph are related, we'll have one more method we can use to graph lines.

Let's look at the graph of the equation $y = \frac{1}{2}x + 3$ and find its slope and y-intercept.



The red lines in the graph show us the rise is 1 and the run is 2. Substituting into the slope formula:

$m = \frac{rise}{run}$

$m = \frac{1}{2}$

The y-intercept is 3.

Look at the equation of this line.

$y = \frac{1}{2}x + 3$

Look at the slope and y-intercepts.

slope $m = \frac{1}{2}$ and y-intercept $(0, 3)$.

Slope m equals 1 divided by 2 and y-intercept (0, 3). The 1 divided by 2 is emphasized in red. The 3 is emphasized in blue.

When a linear equation is solved for y, the coefficient of the x term is the slope and the constant term is the y-coordinate of the y-intercept. We say that the equation $y = \frac{1}{2}x + 3$ is in slope–intercept form. Sometimes the slope–intercept form is called the "y-form."

$m = \frac{1}{2}$; y-intercept is $(0, 3)$

$y = \frac{1}{2}x + 3$

$y = mx + b$

The slope–intercept form of an equation of a line with slope m and y-intercept, $(0,b)$ is $y = mx + b$.

Let's practice finding the values of the slope and y-intercept from the equation of a line.

EXAMPLE

Identify the slope and y-intercept of the line from the equation:

$y = -\frac{4}{7}x - 2$

Write the slope-intercept form of the equation of the line.	$y = mx + b$
Write the equation of the line.	$y = -\frac{4}{7}x - 2$
Identify the slope.	$m = -\frac{4}{7}$
Identify the y -intercept.	y-intercept is (0, −2)

TRY IT **Identifying the Slope and y-intercepts of a Line**

Identify the slope and y-intercept of the line from the equation:

$y = -\frac{1}{3}x + 3$

Check Your Understanding

What is the slope and y-intercepts of the line $6x + 2y = 18$?

a. $m = -3$; y-intercepts = $(9, 0)$

b. $m = 6$; y-intercepts = $(0, 18)$

c. $m = -6$; y-intercepts = $(0, 18)$

d. $m = -3$; y-intercepts = $(0, 9)$

Find the x- and y-intercepts: Mini-Lesson Review

MINI LESSON QUESTION **Question #3: Find x- and y-intercepts**

Select the x- and y- intercepts for the line $2x - 3y = 12$.

a. x-intercept $(0, 6)$; y-intercept $(0, 6)$

b. x-intercept $(-4, 0)$; y-intercept $(-4, 0)$

c. x-intercept $(0, -4)$; y-intercept $(6, 0)$

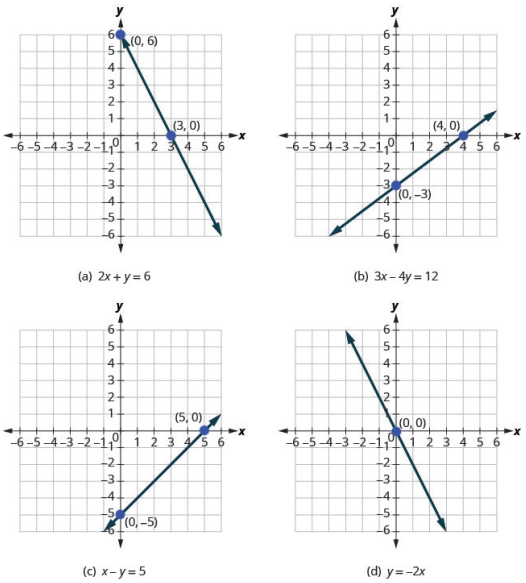
d. x-intercept $(6, 0)$; y-intercept $(0, -4)$

Finding Intercepts of a Line

Every linear equation can be represented by a unique line that shows all the solutions of the equation. We have seen that when graphing a line by plotting points, you can use any three solutions to graph. This means that two people graphing the line might use different sets of three points.

At first glance, their two lines might not appear to be the same, since they would have different points labeled. But if all the work was done correctly, the lines should be exactly the same. One way to recognize that they are indeed the same line is to look at where the line crosses the x-axis and the y-axis. These points are called the intercepts of a line.

Let's look at the graphs of the lines.



First, notice where each of these lines crosses the *x*-axis.

Now, let's look at the points where these lines cross the *y*-axis.

Figure	The line crosses the x -axis at:	Ordered pair for this point	The line crosses the y -axis at:	Ordered pair for this point
Figure (a)	3	(3,0)	6	(0,6)
Figure (b)	4	(4,0)	-3	(0,-3)
Figure (c)	5	(5,0)	-5	(0,-5)
Figure (d)	0	(0,0)	0	(0,0)
General figure	*a*	(*a*,0)	*b*	(0,*b*)

For each line, the *y*-coordinate of the point where the line crosses the x-axis is zero. The point where the line crosses the x-axis has the form (*a*, 0) and is called the x-intercepts of the line. The x-intercepts occurs when *y* is zero.

In each line, the *x*-coordinate of the point where the line crosses the y-axis is zero. The point where the line crosses the y-axis has the form $(0, b)$ and is called the y-intercepts of the line. The y-intercepts occurs when *x* is zero.

Recognizing that the x-intercepts occurs when *y* is zero and that the y-intercepts occurs when *x* is zero, gives us a method to find the intercepts of a line from its equation. To find the x-intercepts, let $y = 0$ and solve for *x*. To find the y-intercepts, let $x = 0$ and solve for *y*.

We can use this method to find the intercepts of the equation $2x + y = 8$

To find the x-intercept, let y=0.

Step 1 - Let $y = 0$. $2x + 0 = 8$

Step 2 - Simplify. $2x = 8$

Step 3 - Simplify. $x = 4$

Step 4 - The x-intercept is: $(4, 0)$

To find the y-intercepts, let $x = 0$.

Step 1 - Let $y = 0$. $2(0) + y = 8$

Step 2 - Simplify. $0 + y = 8$

Step 3 - Simplify. $y = 8$ The x-intercept is: $(0, 8)$

The intercepts are the points (4,0) and (0,8).

TRY IT **Finding Intercepts of a Line**

Find the intercepts of the equation $3x + y = 12$.

Check Your Understanding

Select the *x*- and *y*- intercepts for the line $-5y = 10x - 20$.

a. x-intercepts $(0, 4)$; y-intercepts $(2, 0)$

b. x-intercepts $(2, 0)$; y-intercepts $(0, 4)$

c. x-intercepts $(0, 2)$; y-intercepts $(4, 0)$

d. x-intercepts $(4, 0)$; y-intercepts $(0, 2)$

Video: Finding Intercepts

Khan Academy: Finding Intercepts

Watch this video to see how to find the *x*- and *y*-intercepts of an equation.

Finding intercepts from an equation | Algebra I | Khan Academy (https://www.youtube.com/embed/xGmef7lFc5w)

Unit 2 Overview: Wrap Up

How did you do? Take a moment to reflect on how many answers you got right and wrong on the previous pages. Did you complete the mini-lesson reviews? If not, completing these resources can help you get a better understanding of these concepts:

- Determine if an ordered pair is a solution to a linear equation.
- Use the slope-intercept form of a linear equation to find its slope and *y*-intercepts.
- Find the *x*- and *y*-intercepts of a line when given its equation.

If you are still struggling, be sure to reach out to your teacher for additional help. Nobody is good at problems like this the first try. Keep at it and it will get easier.

2.1 WRITING AND GRAPHING SYSTEMS OF EQUATIONS

2.1.0 Lesson Overview

In this lesson, you will write a system of equations to represent the quantities and constraints in each of several situations, find a solution that meets multiple constraints by graphing, and then interpret the solution in context.

When you finish this lesson, you will be able to:

- Explain "the solution to a system of linear equations" and how the solution is represented graphically.
- Refer to two equations as a system of equations.
- Use tables and graphs to solve systems of equations.

Here are the **activities** that will help you reach those goals:

- 2.1.1: Understanding Constraints and Values
- 2.1.2: Writing and Graphing Equations
  - 2.1.2: Self Check
  - 2.1.2: Additional Resources
- 2.1.3: Meeting Constraints
  - 2.1.3: Self Check
  - 2.1.3: Additional Resources
- 2.1.4: Exploring a System of Equations

After that, you'll **practice and review**.

- 2.1.5: Practice
- 2.1.6: Lesson Summary

2.1.1 Understanding Constraints and Values

Warm Up

Diego bought some raisins and walnuts to make trail mix.

Raisins cost $4 a pound, and walnuts cost $8 a pound. Diego spent $15 on both ingredients.



Decide if each pair of values could be a combination of raisins and walnuts that Diego bought.

1. 4 pounds of raisins and 2 pounds of walnuts

2. 1 pound of raisins and 1.5 pounds of walnuts

3. 2.25 pounds of raisins and 0.75 pound of walnuts

4. 3.5 pounds of raisins and 1 pound of walnuts

2.1.2 Writing and Graphing Equations

Activity

Use the situation you saw earlier to answer questions 1 – 3.

Diego bought some raisins and walnuts to make trail mix. Raisins cost $4 a pound and walnuts cost $8 a pound. Diego spent $15 on both ingredients..

1. Write an equation to represent this constraint. Let x be the pounds of raisins and y be the pounds of walnuts.

2. Use the graphing tool or technology outside the course. Graph your equation from question 2 that represents this scenario using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-1-2-writing-and-graphing-equations)

3. Complete the table with the amount of one ingredient Diego could have bought given the other. Be prepared to show your reasoning.

Raisins (lbs)	Walnuts (lbs)
0	
0.25	
	1.375
	1.25
1.75	
3	

Use the following new piece of information to answer questions 4 – 6.

Diego bought a total of 2 pounds of raisins and walnuts combined.

4. Write an equation to represent this constraint. Let x be the pounds of raisins and y be the pounds of walnuts.

5. Use the graphing tool or technology outside the course. Graph your equation from question 5 that represents this scenario using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-1-2-writing-and-graphing-equations)

6. On your own paper, complete the table with the amount of one ingredient Diego could have bought given the other. Be prepared to show your reasoning.

Raisins (lbs)	Walnuts (lbs)
0	
0.25	
	1.375
	1.25
1.75	
3	

Use the following information to answer questions 7 – 9.

Diego spent $15 and bought exactly 2 pounds of raisins and walnuts.

7. How many pounds of raisins did he buy?

8. How many pounds of walnuts did he buy?

9. Explain or show how you know how many pounds he purchased of each type.

Self Check

A school group bought 12 tickets to visit a history museum.

- Adult tickets are $5 each.
- Student tickets are $8 each.

The school paid a total of $90. How many adult tickets, x, and student tickets, y, did the school buy?

a. 11 student tickets and 1 adult ticket
b. 10 student tickets and 2 adult tickets
c. 9 student tickets and 3 adult tickets
d. 5 student tickets and 10 adult tickets

Additional Resources

Solve Systems of Linear Equations

In Unit 1, you learned how to solve linear equations with one variable. Now we will work with two or more linear equations grouped together, which is known as a system of linear equations.

An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

A linear equation in two variables, such as $2x + y = 7$, has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation, and every solution to the equation is a point on the line.

To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into

each equation. If the ordered pair makes both equations true, it is a solution to the system.

To solve a system of two linear equations, we want to find the values of the variables that are solutions to both equations. In other words, we are looking for the ordered pairs (x, y) that make both equations true. These are called the solutions to a system of equations.

Determining if Ordered Pairs Solve a System

Step 1 - Substitute the values of both variables into the first equation and simplify. Verify if the equation is true.

Step 2 - Substitute the values of both variables into the second equation and simplify. Verify if the equation is true.

Step 3 - If the ordered pair makes both equations true, then it is a solution to the system.

EXAMPLE 1

Determine whether the ordered pair $(-2, -1)$ is a solution to the system $\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$.

Here is how to do this:

$$\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$$

We substitute $x = -2$ and $y = -1$ into both equations.

$$\begin{array}{ll} x - y = -1 & 2x - y = -5 \\ -2 - (-1) \stackrel{?}{=} -1 & 2(-2) - (-1) \stackrel{?}{=} -5 \\ -1 = -1 \checkmark & -3 \neq -5 \end{array}$$

$(-2, -1)$ does not make both equations true.

$(-2, -1)$ is not a solution.

EXAMPLE 2

Determine whether the ordered pair $(-4, -3)$ is a solution to the system $\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$.

Here is how to do this:

We substitute $x = -4$ and $y = -3$ into both equations.

$$\begin{array}{ll} x - y = -1 & 2x - y = -5 \\ -4 - (-3) \stackrel{?}{=} -1 & 2(-4) - (-3) \stackrel{?}{=} -5 \\ -1 = -1 \checkmark & -5 = -5 \checkmark \end{array}$$

$(-4, -3)$ is true for both equations. It is a solution.

TRY IT **Solve Systems of Linear Equations**

Determine whether the ordered pair is a solution to the system

$3x + y = 0$

$x + 2y = -5$

1. Is $(1, -3)$ a solution to the system of equations?
2. Explain how you know if it is or is not a solution of the system.

3. Is $(0, 0)$ a solution to the system of equations?

4. Explain how you know if it is or is not a solution of the system.

Solve a System of Linear Equations by Graphing

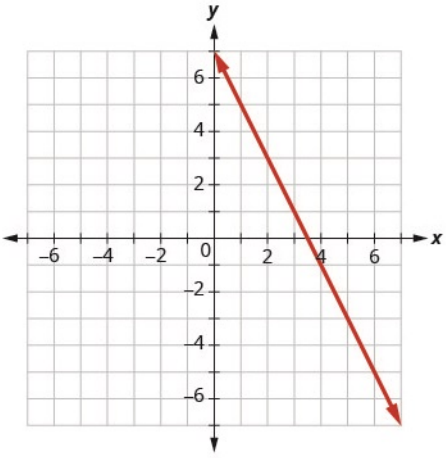
The graph of a linear equation is a line. Each point on the line is a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And by finding what the lines have in common, we'll find the solution to the system.

EXAMPLE

Solve the system by graphing $\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$

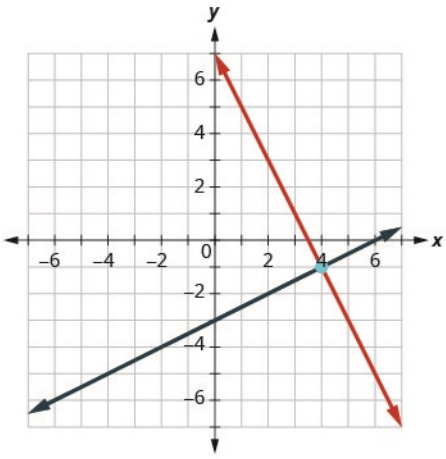
Step 1 - Graph the first equation.

To graph the first line, write the equation in slope-intercept form. $2x + y = 7$ $y = -2x + 7$ So, the slope is -2 and the y-intercept is 7.



Step 2 - Graph the second equation on the same rectangular coordinate system.

This line may be easier to graph using the intercepts. To find the x-intercept, substitute $y = 0$ into the equation. $x - 2y = 6$ $x - 2(0) = 6$ $x = 6$ To find the y-intercept, substitute $x = 0$ into the equation. $x - 2y = 6$ $0 - 2y = 6$ $-2y = 6$ $y = -3$ So, the intercepts are $(6, 0)$ and $(0, -3)$.



Step 3 - Determine whether the lines intersect, are parallel, or are the same line.

Look at the graph of the lines. In this case, the lines intersect.

Step 4- Estimate the solution to the system.

- If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system.
- If the lines are parallel, the system has no solution.
- If the lines are the same, the system has an infinite number of solutions.

The lines intersect at $(4, -1)$.

Step 5 - Check the solution in both equations.

$2x + y = 7$

$2(4) + (-1) \stackrel{?}{=} 7$

$8 - 1 \stackrel{?}{=} 7$

$7 = 7$✔

$x - 2y = 6$

$4 - 2(-1) \stackrel{?}{=} 6$

$6 = 6$✔

The steps to use to solve a system of linear equations by graphing are shown here.

Using a Graph to Solve a System

Step 1 - Graph the first equation.

Step 2 - Graph the second equation on the same rectangular coordinate system.

Step 3 - Determine whether the lines intersect.

Step 4 - Estimate the solution to the system.

If the lines intersect, identify the point of intersection. This is the solution to the system.

Step 5 - Check the solution in both equations.

TRY IT **Solve a System of Linear Equations by Graphing**

Solve the system by graphing $\begin{cases} x - 3y = -3 \\ x + y = 5 \end{cases}$

2.1.3 Meeting Constraints

Activity

Use the following information to answer questions 1 – 3.

A dining hall had a total of 25 tables: some long rectangular tables and some round ones. Long tables can seat 8 people. Round tables can seat 6 people. On a busy evening, all 190 seats at the tables are occupied.

1. How many long tables x are there?

2. How many round tables, y, are there?

3. Explain or show how you know how many different size tables there are.

Use the following information to answer questions 4 – 6.

A family bought a total of 16 adult and child tickets to a magic show. Adult tickets are $10.50 each, and child tickets are $7.50 each. The family paid a total of $141.

4. How many adult tickets, a did they buy?

5. How many child tickets, c did they buy?

6. Explain or show how you know the prices of the tickets.

Use the following information to answer questions 7 - 9

At a poster shop, Han paid $16.80 for 2 large posters and 3 small posters of his favorite band. Kiran paid $14.15 for 1 large poster and 4 small posters of his favorite TV shows. Posters of the same size have the same price.

7. Find the price of a large poster, l.

8. Find the price of a small poster, s.

9. Explain or show how you know the prices of the posters.

Are you ready for more?

Extending Your Thinking

1. Make up an equation for x whose two lines that intersect at $(4, 1)$.

2. Make up an equation for y whose two lines that intersect at $(4, 1)$.

3. Make up equations for three lines whose intersection points form a triangle with vertices at $(-4, 0)$,$(2, 9)$, and $(6, 5)$.

Video: Meeting Constraints

Watch the following video to learn more about constraints.

Meeting Constraints (https://openstax.org/r/meeting-constraints)

Self Check

An automotive store is shipping car parts to a mechanic.

- Each shock weighs 7 pounds.
- Each brake pad weighs 4 pounds.

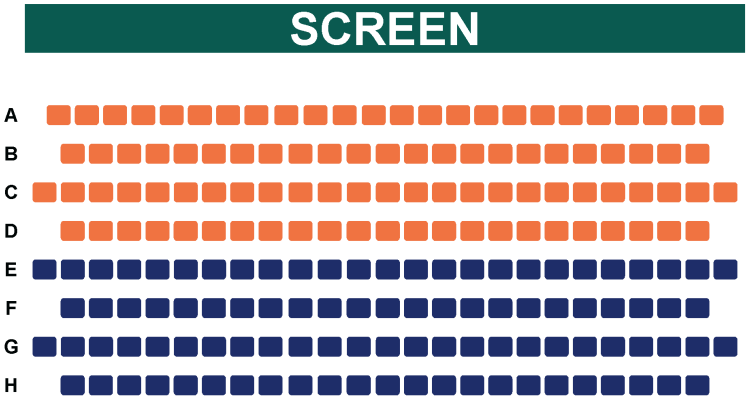
The store shipped 20 parts weighing a total of 104 pounds. How many shocks, x, and brake pads, y, did the mechanic buy?

Additional Resources

Meeting Constraints in a Real-World Situation

Let's look at a situation similar to the activity and discuss how to find values that meet the constraints given.

THEATER SEATS



A movie theater sells two types of tickets for a movie showing. Stadium recliner seats cost $18 each, and regular seats cost $15 each. The theater has a total of 40 seats. When every seat is sold, the theater earns $648 in ticket sales. How many stadium recliner seats, x, and how many regular seats, y, are there?

To solve this problem, we first need to recognize that there are two constraints in this situation. The constraints are the total number of seats in the theater and the total money earned from sales. Let's write an equation to represent each constraint.

Constraint #1: the total number of seats

There are x stadium recliner seats and y regular seats representing a total of 40 seats.

This is represented by the equation $x + y = 40$.

Constraint #2: the total sales from the seats

Stadium recliner seats cost $18 and regular seats cost $15, for a total of $648.

This is represented by the equation $18x + 15y = 648$.

There are many ways to solve these equations: graphing and algebraic solving, guessing and checking, or using tables.

TRY IT

Meeting Constraints in a Real-World Situation

Let's try creating a table. We know the total number of seats must equal 40. So we can try different values of seat types to determine if they match the total sales constraint, $648.

If the total sales value is too low, try more stadium recliner seats, since they are more expensive.

Number of Stadium Recliner Seats	Number of Regular Seats	Total Seats	Total Sales ($)
		40	
		40	
		40	
		40	

2.1.4 Exploring a System of Equations

Cool Down

At a fabric store, fabrics are sold by the yard. A dressmaker spent $36.35 on 4.25 yards of silk and cotton fabrics for a

dress. Silk is $16.90 per yard, and cotton is $4 per yard.

Here is a system of equations that represents the constraints in the situation.

$$\begin{cases} x + y = 4.25 \\ 16.90x + 4y = 36.35 \end{cases}$$

1. What does the solution to the system represent?

2. Find the solution to the system of equations.

3. Explain or show how you know the solution solves the system.

2.1.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

In questions 1 – 4, determine if the following points are solutions to the given system of equations.

1. Determine if the point is a solution to the system of equations: $\begin{cases} 2x - 6y = 0 \\ 3x - 4y = 5 \end{cases}$

a. $(1, \frac{3}{4})$
b. $(\frac{8}{7}, \frac{6}{7})$
c. $(-3, 4)$
d. $(3, 1)$

2. Determine if the point is a solution to the system of equations: $\begin{cases} x - 6y = 0 \\ 3x - 4y = 5 \end{cases}$

a. $(1, \frac{3}{4})$
b. $(\frac{8}{7}, \frac{6}{7})$
c. $(-3, 4)$
d. $(3, 1)$

3. Determine if the point is a solution to the system of equations: $\begin{cases} x + y = 2 \\ y = \frac{3}{4}x \end{cases}$

a. $(1, \frac{3}{4})$
b. $(\frac{8}{7}, \frac{6}{7})$
c. $(-3, 4)$
d. $(3, 1)$

4. Determine if the point is a solution to the system of equations: $\begin{cases} 4x + 4y = 7 \\ y = \frac{3}{4}x \end{cases}$

a. $(1, \frac{3}{4})$
b. $(\frac{8}{7}, \frac{6}{7})$
c. $(-3, 4)$
d. $(3, 1)$

5. Solve the system of equations by graphing: $\begin{cases} 3x + y = -3 \\ 2x + 3y = 5 \end{cases}$

a. $(4, -1)$
b. $(-3, 6)$
c. $(0, -3)$
d. $(-2, 3)$

Access multimedia content (http://openstax.org/books/algebra-1/pages/2-1-5-practice)

6. Solve the system of equations by graphing: $\begin{cases} y = x + 2 \\ y = -2x + 2 \end{cases}$

a. $(-2, 0)$
b. $(0, 2)$

c. $(-1, 4)$
d. $(2, 0)$

Access multimedia content (http://openstax.org/books/algebra-1/pages/2-1-5-practice)

7. Solve the system of equations by graphing: $\begin{cases} y = \frac{3}{2}x + 1 \\ y = -\frac{1}{2}x + 5 \end{cases}$

a. $(2, 4)$
b. $(0, 5)$
c. $(0, 1)$
d. $(4, 3)$

Access multimedia content (http://openstax.org/books/algebra-1/pages/2-1-5-practice)

8. Solve the system of equations by graphing: $\begin{cases} x + y = -4 \\ -x + 2y = -2 \end{cases}$

a. $(-2, -2)$
b. $(4, 1)$
c. $(-1, 0)$
d. $(2, 0)$

Access multimedia content (http://openstax.org/books/algebra-1/pages/2-1-5-practice)

9. Solve the system of equations by graphing: $\begin{cases} -2x + 3y = 3 \\ x + 3y = 12 \end{cases}$

a. $(0, 1)$
b. $(3, 3)$
c. $(4, 0)$
d. $(6, 2)$

Access multimedia content (http://openstax.org/books/algebra-1/pages/2-1-5-practice)

10. Solve the system of equations by graphing: $\begin{cases} 2x - 4y = 8 \\ x + 3y = 9 \end{cases}$

a. $(8, 2)$
b. $(-2, 0)$
c. $(4, 0)$
d. $(6, 1)$

Access multimedia content (http://openstax.org/books/algebra-1/pages/2-1-5-practice)

11. A bakery charges $2 per croissant and $4 per muffin. One day, they charge $42 in total and sell 15 baked goods. How many croissants and muffins did they sell?

a. 8 croissants and 7 muffins
b. 10 croissants and 5 muffins
c. 9 croissants and 6 muffins
d. 5 croissants and 10 muffins

12. A pet store sells mice for $7 each and gerbils for $9 each. They sell 17 animals for a total of $137. How many mice and gerbils did they sell?

a. 6 mice and 11 gerbils
b. 7 mice and 10 gerbils
c. 8 mice and 9 gerbils
d. 15 mice and 2 gerbils

2.1.6 Lesson Summary

In this lesson, you learned how to:

- Explain “the solution to a system of linear equations” and how the solution is represented graphically.
- Refer to two equations as a system of equations.
- Use tables and graphs to solve systems of equations.

Here are the **activities** that helped you reach those goals:

- 2.1.1: Understanding Constraints and Values
  - In this activity, you analyzed different combinations of trail mix to determine if they met a certain cost constraint. You learned that writing an equation is one way to check if these values meet the constraint.
- 2.1.2: Writing and Graphing Equations
  - In this activity, you wrote equations to represent two different constraints covering the cost and amount of trail mix. These equations represent a system of linear equations. Then you graphed the system of equations and used the graph to complete a table. These are important methods when solving a system of equations.
  - 2.1.2: Self Check
  - 2.1.2: Additional Resources
- 2.1.3: Meeting Constraints
  - In this activity, you analyzed a situation and found a pair of values that met two different constraints in that situation. You were then able to explain your reasoning in how you solved the systems of equations.
  - 2.1.3: Self Check
  - 2.1.3: Additional Resources
- 2.1.4: Exploring a System of Equations
  - In this activity, you solved a system of equations using your preferred method of choice. Systems of equations can be solved by graphing, algebraically, guessing and checking, or using tables.

After these activities, you completed the following **practice**:

- 2.1.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

2.2 WRITING SYSTEMS OF EQUATIONS

2.2.0 Lesson Overview

In this lesson, you will write a system of equations from a table of values, graphs, and situations. You will also be able to graph systems by hand and use graphs of systems of equations to estimate real world solutions.

When you finish this lesson, you will be able to:

- Write a system of equations from tables of values and graphs.
- Graph systems of equations by hand.
- Estimate real-world solutions by graphing systems of equations.

Here are the **activities** that will help you reach those goals:

- 2.2.1: Interpreting Graphs of Systems of Equations
- 2.2.2: Writing Systems of Equations from Tables
  - 2.2.2: Self Check
  - 2.2.2: Additional Resources
- 2.2.3: Graphs of Systems of Equations
  - 2.2.3: Self Check
  - 2.2.3: Additional Resources
- 2.2.4: Systems of Equations in the Real-World

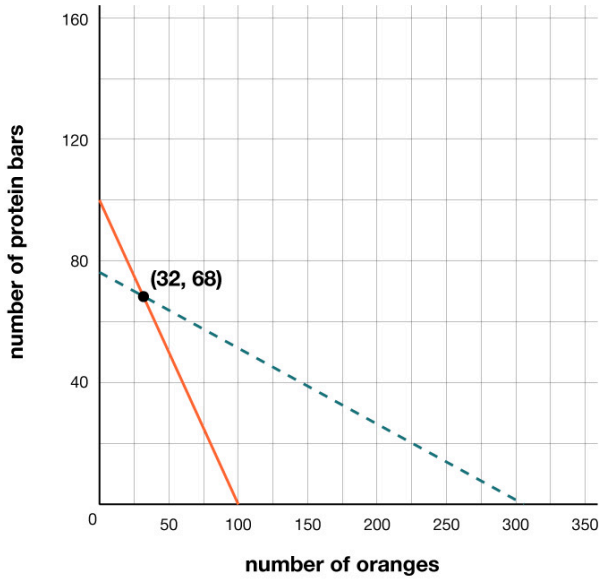
After that, you'll **practice and review**.

- 2.2.5: Practice
- 2.2.6: Lesson Summary

2.2.1 Interpreting Graphs of Systems of Equations

Warm Up

For questions 1 – 6, use the given graph and scenario:



A club is selling snacks at a track meet. Oranges cost $1 each and protein bars cost $4 each. They sell a total of 100 items, and collect $304.

1. What do you notice?

2. What do you wonder?

3. What is the approximate y–intercept of the green dashed line?
4. What is the approximate slope of the green dashed line?
5. What is the approximate y-intercept of the orange solid line?
6. What is the approximate slope of the orange solid line?

2.2.2 Writing Systems of Equations from Tables

Activity

This summer, the Farmer's Almanac is calling for very high temperatures in central Texas. Mr. and Mrs. Flores were talking about ways to keep their four children cool during the summer and also ways that the family could have fun together. They decided that this summer they would purchase a family membership to the community pool.

Mrs. Flores called the community pool to find out different membership plans for a family of six. The receptionist at the community center told her that there are two options. Option A is to pay $100 for the summer and $20 per visit. The second option, Option B, is to pay $250 for the summer and $5 per visit.

At dinner, the Flores family talked about how many times a week they think they will visit the pool and how many times total in the summer they will visit.

How can the Flores family decide which option to choose? Which option do you think they will pick and why?

1. Make a table of values for Option A for 0–10 visits.
Option A:

0	1	2	3	4	5	6	7	8	9	10

2. Make a table of values for Option B for 0–10 visits.
Option B:

0	1	2	3	4	5	6	7	8	9	10

3. For Option A, identify the y-intercept (where $x = 0$).
4. For Option A, identify the slope between any two points.
5. For Option A, write an equation of the line in slope-intercept form.
6. For Option B, identify the y-intercept (where $x = 0$).
7. For Option B, identify the slope between any two points.
8. For Option B, write an equation of the line in slope-intercept form.
9. Write the system of equations.
10. Use the graphing tool or technology outside the course. Graph your system of equations from question 9 that represents this scenario using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-2-2-writing-systems-of-equations-from-tables)
11. What is the point of intersection of the two lines?
12. Talk with your partner. What does this intersection mean in the context of the problem?
13. Write a paragraph to describe when Option A is better and when Option B is better. Share your paragraph with your partner. Read each other's paragraphs and then discuss if you agree or disagree.

Self Check

A streaming service offers two options for subscription plans. Option A charges $10.99 a month plus $2 per movie.

Option B charges \$5.99 a month plus \$3 per movie. The table below describes this. Which of the following systems of equations represents the two options for one month?

x (Number of movies	0	1	2	3
Option A	10.99	12.99	14.99	16.99
Option B	5.99	8.99	11.99	14.99

a. $y = 2x + 10.99$
$y = 3x + 5.99$

b. $y = 3x + 10.99$
$y = 2x + 5.99$

c. $y = 2x - 10.99$
$y = 3x - 5.99$

d. $y = 10.99x + 2$
$y = 5.99x + 3$

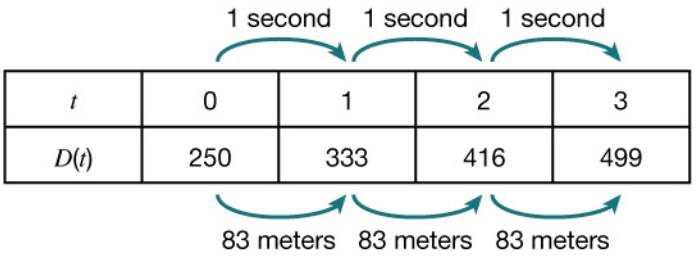
Additional Resources

Representing a Linear Function in Tabular Form

A system of equations is two or more equations. Often, each equation in a system of equations represents a situation. Like other equations, the equations that make up the system can be represented by a table. When writing any equation from a table into slope-intercept form, two things are needed: a slope and point, often the y-intercept.

Example

Suppose a maglev train travels a long distance, and maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. Here is a table that shows this constant speed.



Step 1 - Find the y-intercept.

Notice that when $t = 0$, $D(t) = 250$. This is the y-intercept.

Step 2 - Find the slope.

To find the slope, find the change of y over the change of x: $m = \frac{83}{1} = 83$.

Step 3 - Write the equation in slope-intercept form.

$y = mx + b$, where m is the slope and b is the y-intercept.

$y = 83x + 250$

TRY IT **Representing a Linear Function in Tabular Form**

A new plant food was introduced to a young tree to test its effect on the height of the tree. The table shows the height of the tree, in feet, x months since the measurements began. Write a linear function, $H(x)$, where x is the number of months since the start of the experiment.

x	0	2	4	8	12
$H(x)$	12.5	13.5	14.5	16.5	18.5

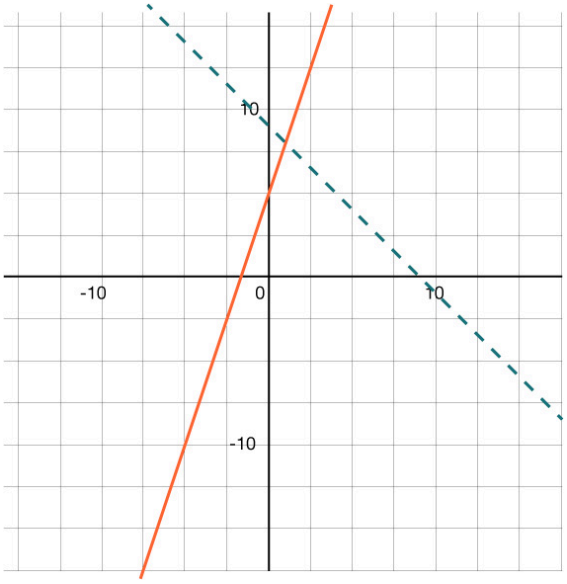
2.2.3 Graphs of Systems of Equations

Activity

Now that you have written equations from tables, it is time to write systems of equations from graphs.

Part 1: Writing a System of Equations from a Given Graph

For questions 1 – 8, use the given graph:



1. For the green dashed line what is the y-value of the y-intercept?
2. For the green dashed line what is the slope?
3. What is the equation of the green dashed line in slope-intercept form?
4. For the orange solid line what is the y-value of the y-intercept?
5. For the orange solid line what is the slope?
6. What is the equation of the orange solid line in slope-intercept form?
7. Write the two equations as a system of linear equations.
8. What is the solution of the system of equations?

Are you ready for more?

Extending Your Thinking

Graph the following system of equations on a coordinate plane:

$y = 2x + 5$

$4x - 2y = 10$

Find the solution.

(Hint: For the second equation, solve for y first.)

Video: Graphing a System of Equations

Watch the following video to learn more about writing systems of equations from graphs.

Worked Solution 2.2.3 (https://www.youtube.com/embed/RQHhdqkJEo0)

Part 2: Use a Graph to Solve a Given System

Now that you wrote a system of equations from a given graph, how can you work backward and graph a given system of equations by hand? Discuss this with a partner.

For questions 1 – 2, use this system of equations:

$y = 4x - 4$

$y = -2x + 8$

1. On the same coordinate plane, graph $y = 4x - 4$ and $y = -2x + 8$ by hand.

2. What is the solution to the system of equations?

For questions 3 and 4, consider the system of equations:

$y = -\frac{3}{4}x + 5$

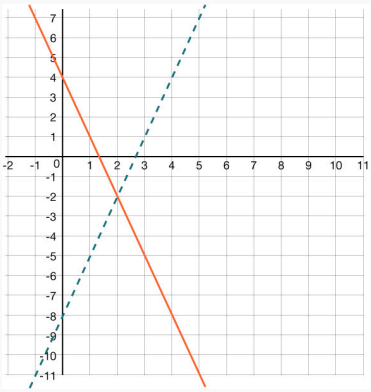
$y = \frac{1}{2}x - 5$

3. On the same coordinate plane, graph the system of equations by hand.

4. What is the solution to the system of equations?

Self Check

Which of the following systems of equations is graphed:



a. $y = 3x + 4$

$y = 3x + 8$

b. $y = 4x - 4$

$y = 8x + 3$

c. $y = 3x + 4$

$y = -3x + 8$

d. $y = -3x + 4$

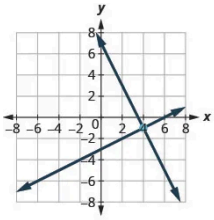
$y = 3x + 8$

Additional Resources

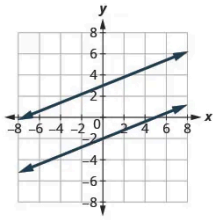
Solving Systems by Graphing

The graph of a linear equation is a line. Each point on the line is a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what the lines have in common, we'll find the solution to the system.

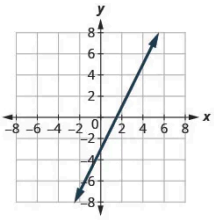
Most linear equations in one variable have one solution, but we saw that some equations, called contradictions, have no solutions and for other equations, called identities, all numbers are solutions.



The lines intersect.
Intersecting lines have one point in common.
There is one solution to this system.



The lines are parallel.
Parallel lines have no points in common.
There is no solution to this system.



Both equations give the same line.
Because we have just one line, there are infinitely many solutions.

Part 1:Writing a System of Equations from a Given Graph

How to write a system of linear equations from graphs.

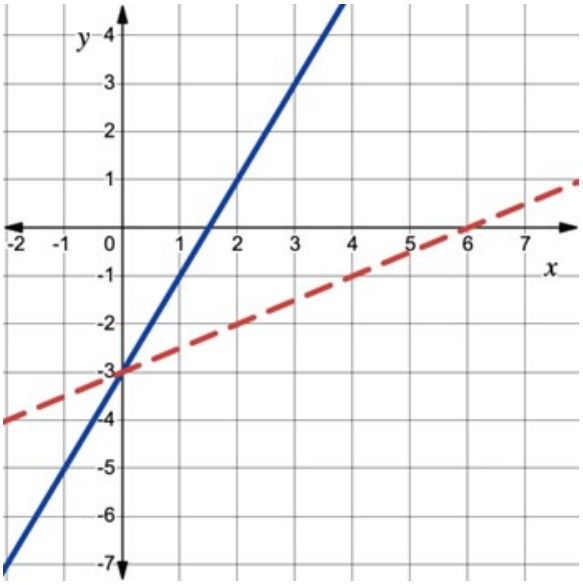
Step 1 - Find the slope and y-intercept of the first line.

Step 2 - Find the slope and y-intercept of the second line.

Step 3 - Write the two equations for a system of equations.

EXAMPLE 1

Write a system of equations from the graph.



Step 1 - Find the slope and y-intercept of the red dotted line.

$m = \frac{(y_1 - y_2)}{(x_1 - x_2)} = \frac{(0-(-3))}{(6-0)} = \frac{3}{6} = \frac{1}{2}$

y-intercept = $(0, -3)$

Step 2 - Find the slope and y-intercept of the blue solid line.

$m = \frac{(y_1 - y_2)}{(x_1 - x_2)} = \frac{(3-(-3))}{(3-0)} = \frac{6}{3} = \frac{2}{1}$

y-intercept = $(0, -3)$

Step 3 - Write the two equations for a system of equations.

$y = mx + b$

$y = \frac{1}{2}x - 3$

$y = 2x - 3$

Part 2: Use a Graph to Solve a Given System

How to solve a system of linear equations by graphing.

Step 1 - Find the slope and y-intercept of the first equation.

Step 2 - Find the slope and y-intercept of the second equation.

Step 3 - Graph the two equations on the same rectangular coordinate system.

Step 4 - Estimate the solution to the system.

If the lines intersect, estimate the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system.

- If the lines are parallel, the system has no solution.
- If the lines are the same, the system has an infinite number of solutions.

EXAMPLE 2

Solve the system by graphing:

$y = 2x + 1$

$y = 4x - 1$

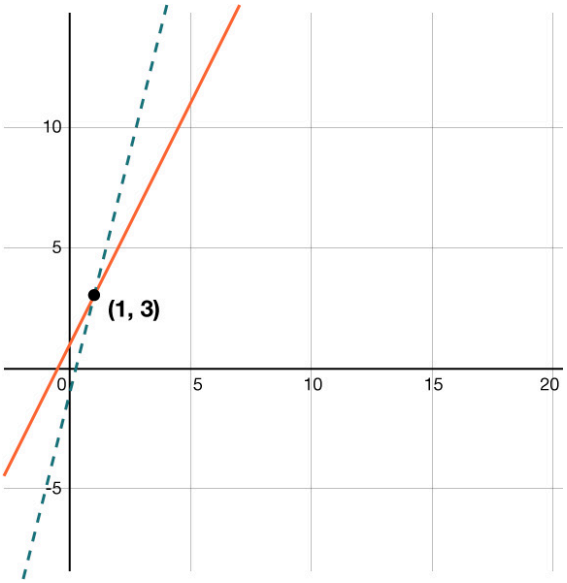
Step 1 - Find the slope and /(y/)-intercept of the first equation.

In $y = 2x + 1$, $m = 2$, $b = 1$

Step 2 - Find the slope and y-intercept of the second equation.

In $y = 4x - 1$, $m = 4$, $b = -1$

Step 3 - Graph the two equations on the same rectangular coordinate system.



Step 4 - Estimate the solution to the system.

The point where the lines intersect is:

$(1, 3)$

Check to make sure the estimated point, $(1, 3)$ solves the system by substituting the values for x and y into both equations.

Check the first equation: $y = 2x + 1$

$x = 1$

$y = 3$

$y = 2x + 1$

$(3) = 2(1) + 1$

$3 = 2 + 1$

$3 = 3$

So, $(1, 3)$ solves $y = 2x + 1$

Check the second equation: $y = 4x - 1$

$y = 4x - 1$

$(3) = 4(1) - 1$

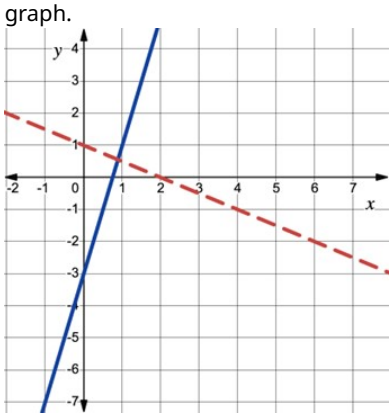
$3 = 4 - 1$

$3 = 3$

So, $(1, 3)$ solves $y = 4x - 1$

TRY IT **Solving Systems by Graphing**

1. Write a system of equations from the graph.



2. Solve the system by graphing:
$y = 3x + 3$
$y = -x + 7$

2.2.4 Systems of Equations in the Real-World

Cool Down

Translate to a system of equations and then solve:

When Jenna spent 10 minutes on the elliptical trainer and then lifted weights for 20 minutes, her fitness app said she burned 278 calories. When she spent 20 minutes on the elliptical trainer and 30 minutes lifting weights she burned 473 calories. How many calories does she burn for each minute on the elliptical trainer? How many calories does she burn for each minute of lifting weights?

Let $x =$ calories burned per minute on the elliptical and $y =$ calories burned per minute lifting weights.

1. Write a system of equations.

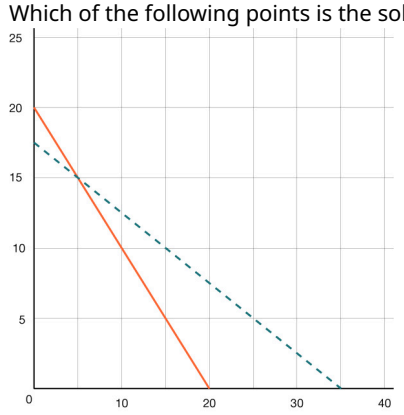
2. Use the graphing tool or technology outside the course. Graph the equation that represents this scenario using the Desmos tool below.
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3. Use the graph to estimate the solution.

2.2.5 Practice

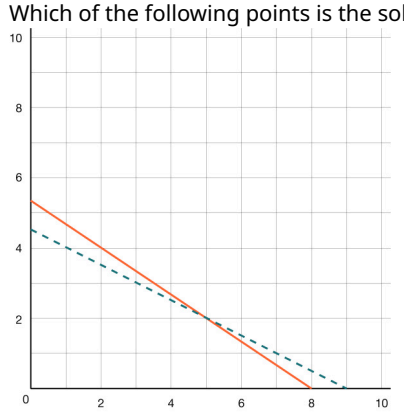
Complete the following questions to practice the skills you have learned in this lesson.

1. Which of the following points is the solution to the system of equations that is graphed?



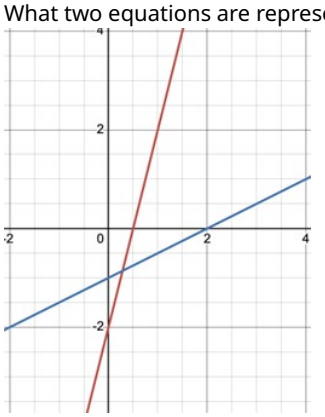
a. $(5, 15)$
b. $45, 0)$
c. $(0, 20)$
d. $(20, 0)$

2. Which of the following points is the solution to the system of equations that is graphed?



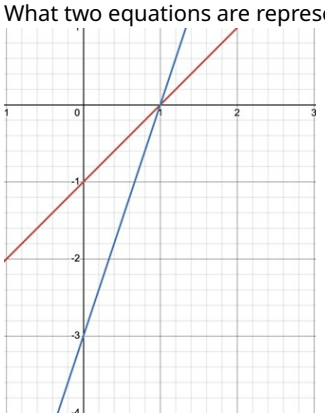
a. $(5.2, 0)$
b. $(8, 0)$
c. $(2, 5)$
d. $(5, 2)$

3. What two equations are represented by the graph? Select **two**.



a. $y = \frac{1}{2}x - 1$
b. $y = 4x - 2$
c. $y = \frac{1}{4}x + 2$
d. $y = \frac{-1}{2}x + 1$

4. What two equations are represented by the graph? Select **two**.



a. $y = x - 1$
b. $y = x + 1$
c. $y = \frac{1}{3}x + 2$
d. $y = 3x - 2$

For questions 5 – 8, use the following scenario and matching table.

A streaming service has two different membership plans. Option A charges $10 per month plus $5 for each movie. Option B charges $20 a month then $2 a movie.

Number of movies in a month, x	0	1	2	3	4
Option A	10	15	20	25	30
Option B	20	22	24	26	28

5. What is the y-intercept for Option A and Option B?

a. Option A: 2 Option B: 5
b. Option A: 5 Option B: 2
c. Option A: 20 Option B: 10
d. Option A: 10 Option B: 20

6. What is the slope of Option A and Option B?

a. Option A: $m = 5$ Option B: $m = 2$
b. Option A: $m = \frac{1}{2}$ Option B: $m = \frac{1}{5}$
c. Option A: $m = 2$ Option B: $m = 5$
d. Option A: $\frac{52}{x} = 9$ Option B: $m = \frac{1}{2}$

7. What is the equation of Option A?

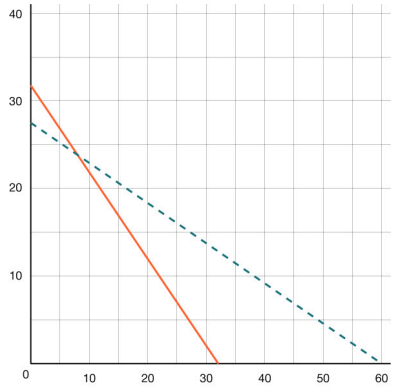
a. $y = 10x - 5$
b. $y = 5x + 10$
c. $y = 5x - 10$
d. $y = 10x + 5$

8. What is the equation of Option B?

a. $y = 2x + 10$
b. $y = 5x + 10$
c. $y = 2x + 20$
d. $y = 5x + 20$

For questions 9 - 12, use the following situation and graph.

A family buys a total of 32 tickets at a carnival. Ride tickets cost $1.50 each and food tickets cost $3.25 each. The family pays a total of $90 for the tickets. The situation is graphed below.



The x-axis represents the number of ride tickets and the y-axis represents the number of food tickets.

9. Which line depicts the equation that represents the statement "A family buys a total of 32 tickets at a carnival"?

a. The blue, dashed line
b. The orange, solid line

10. The remaining line represents the statements "Ride tickets cost $1.50 each and food tickets cost $3.25 each. The family pays a total of $90 for the tickets." Which equation corresponds to these statements and the remaining line?

a. $(1.50 + 3.25)y = 90x$
b. $(1.50 + 3.25)x = 90y$
c. $3.25x + 1.50y = 90$
d. $1.50x + 3.25y = 90$

11. What is the approximate number of ride tickets the family purchased?

a. 24
b. 8
c. 60
d. 32

12. What is the approximate number of food tickets the family purchased?

a. 24
b. 8
c. 60
d. 32

2.2.6 Lesson Summary

In this lesson, you learned how to:

- Write a system of equations from tables of values and graphs.
- Graph systems of equations by hand.
- Estimate real-world solutions by graphing systems of equations.

Here are the **activities** that helped you reach those goals:

- 2.2.1: Interpreting Graphs of Systems of Equations
  - In this activity, you looked at a graph of a system of equations and thought about what you noticed and wondered. You then estimated the y-intercept and slope of each equation that was graphed.
- 2.2.2: Writing Systems of Equations from Tables
  - In this activity, you created a table of values for a given situation, then wrote a system of equations from two tables of values.
  - 2.2.2: Self Check
  - 2.2.2: Additional Resources
- 2.2.3: Graphs of Systems of Equation
  - In this activity, you wrote a system of equations from a graph, then worked backward to graph a system of equations by hand given the equations.
  - 2.2.3: Self Check
  - 2.2.3: Additional Resources
- 2.2.4: Systems of Equations in the Real-World
  - In this activity, you wrote a system of equations for a given situation, then graphed the system and estimated the solution from the graph.

After these activities, you completed the following **practice**:

- 2.2.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

2.3 SOLVING SYSTEMS BY SUBSTITUTION

2.3.0 Lesson Overview

In this lesson, you will activate and build on what you have learned about solving by **substitution**, or replacing a variable with a number or with an expression. Various substitutions can be done to solve the same system. You will also begin to build an awareness of the kinds of systems that are conducive to being solved by substitution.

When you finish this lesson, you will be able to:

- Solve systems of equations by substituting a variable or an expression.
- Know more than one way to perform a substitution and decide which way or what to substitute based on how the given equations are written.

Here are the **activities** that will help you reach those goals:

- 2.3.1: Finding Connections between Graphs and Equations
- 2.3.2: Checking Solutions in Systems
  - 2.3.2: Self Check
  - 2.3.2: Additional Resources
- 2.3.3: Solving Equations Using Substitution
  - 2.3.3: Self Check
  - 2.3.3: Additional Resources
- 2.3.4: Solving More Equations

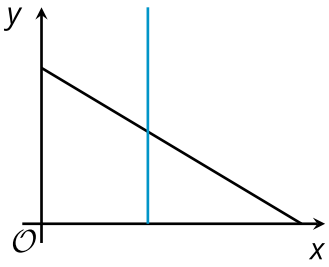
After that, you'll **practice and review**.

- 2.3.5: Practice
- 2.3.6: Lesson Summary

2.3.1 Finding Connections between Graphs and Equations

Warm Up

Use the following information to answer questions 1 – 8. Here are graphs of two equations in a system.



Determine if each of these systems could be represented by the graphs. Be prepared to explain how you know.

1. $\begin{cases} x + 2y = 8 \\ x = -5 \end{cases}$

2. How do you know if the system of equations in question 1 could be represented by the graph?

3. $\begin{cases} y = -7x + 13 \\ y = -1 \end{cases}$

4. How do you know if the system of equations in question 3 could be represented by the graph?

5. $\begin{cases} 3x = 8 \\ 3x + y = 15 \end{cases}$

6. How do you know if the system of equations in question 5 could be represented by the graph?

7. $\begin{cases} y = 2x - 7 \\ 4 + y = 12 \end{cases}$

8. ow do you know if the system of equations in question 7 could be represented by the graph?

2.3.2 Checking Solutions in Systems

Activity

Here are the four systems of equations you saw in the previous exercise. Solve each system. Then, check your solutions by substituting them into the original equations to see if the equations are true.

Use this system of equations to answer questions 1 – 3.

$$\begin{cases} x + 2y = 8 \\ x = -5 \end{cases}$$

1. What is the x value?

2. What is the y value?

3. Explain your solution.

Use this system of equations to answer questions 4 – 6.

$$\begin{cases} y = -7x + 13 \\ y = -1 \end{cases}$$

4. What is the x value?

5. What is the y value?

6. Explain your solution.

Use this system of equations to answer questions 7 – 9.

$$\begin{cases} 3x = 8 \\ 3x + y = 15 \end{cases}$$

7. What is the x value?

8. What is the y value?

9. Explain your solution.

Use this system of equations to answer questions 10 – 12.

$$\begin{cases} y = 2x - 7 \\ 4 + y = 12 \end{cases}$$

10. What is the x value?

11. What is the y value?

12. Explain your solution.

Video: Checking Solutions in Systems

Watch the following video to learn more about systems.

Checking Solutions in Systems (https://openstax.org/r/system-solutions)

Self Check

Solve the following system of equations. Check to see if your solution makes each equation true.

$$\begin{cases} 3x + y = 6 \\ x + 2y = 7 \end{cases}$$

a. $x = 1, y = 3$
b. $x = 3, y = -3$
c. $x = -3, y = 5$
d. $x = 3, y = 1$

Additional Resources

Solving Systems of Linear Equations

Let's solve this system of linear equations using the substitution method.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

We will first solve one of the equations for either x or y. We can choose either equation and solve for either variable—but we'll try to make a choice that will keep the work easy.

Then we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it makes both equations true.

How to Solve a System of Equations by Substitution

Step 1 - Solve one of the equations for either variable.

Step 2 - Substitute the expression from Step 1 into the other equation.

Step 3 - Solve the resulting equation.

Step 4 - Substitute the solution from Step 3 into one of the original equations to find the other variable.

Step 5 - Write the solution as an ordered pair.

Step 6 - Check that the ordered pair is a solution to both original equations.

Solve the system by substitution: $\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$

Solution

Step 1 - Solve one of the equations for either variable.

We'll solve the first equation for y.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

$$\begin{aligned} 2x + y &= 7 \\ y &= 7 - 2x \end{aligned}$$

Step 2 - Substitute the expression from Step 1 into the other equation.

$$\begin{aligned} x - 2y &= 6 \\ x - 2(7 - 2x) &= 6 \end{aligned}$$

Step 3 - Solve the resulting equation.

Now we have an equation with just 1 variable. We know how to solve this!

$$\begin{aligned} x - 2(7 - 2x) &= 6 \\ x - 14 + 4x &= 6 \\ 5x &= 20 \\ x &= 4 \end{aligned}$$

Step 4 - Substitute the solution from Step 3 into one of the original equations to find the other variable.

We'll us the first equation and replace x with 4.

$$\begin{aligned} 2x + y &= 7 \\ 2(4) + y &= 7 \\ 8 + y &= 7 \\ y &= -1 \end{aligned}$$

Step 5 - Write the solution as an ordered pair.

The ordered pair is (x, y).

$(4, -1)$

Step 6 - Check that the ordered pair is a solution to both original equations.

Substitute $x = 4$, $y = -1$ into both equations and make sure they are both true.

$$\begin{array}{ll} rvr2x + y = 7 & x - 2y = 6 \\ 2(4) + (-1) \stackrel{?}{=} 7 & 4 - 2(-1) \stackrel{?}{=} 6 \\ 7 = 7\checkmark & 6 = 6\checkmark \end{array}$$

Both equations are true.

$(4, -1)$ is the solution to the system.

TRY IT **Solving Systems of Linear Equations**

Solve the system by substitution, then choose the correct ordered pair. $\begin{cases} -2x + y = -11 \\ x + 3y = 9 \end{cases}$

a. $(9, -11)$

b. $(1, 6)$

c. $(6, 1)$

d. $(-11, 9)$

2.3.3 Solving Equations Using Substitution

Activity

Solve each system without graphing.

Use this system of equations to answer questions 1 – 3.

$$\begin{cases} x + 2y = 8 \\ x = -5 \end{cases}$$

Here are the four systems of equations you saw in the previous exercise. Solve each system. Then, check your solutions by substituting them into the original equations to see if the equations are true.

1. What is the x value?

2. What is the y value?

3. Explain your solution.

Use this system of equations to answer questions 4 – 6.

$$\begin{cases} 2m - 2p = -6 \\ p = 2m + 10 \end{cases}$$

4. What is the m value?

5. What is the p value?

6. Explain your solution.

Use this system of equations to answer questions 7 – 9.

$$\begin{cases} 2d = 8f \\ 18 - 4f = 2d \end{cases}$$

7. What is the d value?

8. What is the f value?

9. Explain your solution.

Use this system of equations to answer questions 10 - 12.

$$\begin{cases} w + \frac{1}{7}z = 4 \\ z = 3w - 2 \end{cases}$$

10. What is the w value?

11. What is the z value?

12. Explain your solution.

Are you ready for more?

Extending Your Thinking

Solve this system with four equations.

Use this system of equations to answer questions 1 – 4.

$$\begin{cases} 3x + 2y - z + 5w = 20 \\ y = 2z - 3w \\ z = w + 1 \\ 2w = 8 \end{cases}$$

1. What is the x value?

2. What is the y value?

3. What is the z value?

4. What is the w value?

Self Check

Solve the systems of equations by substitution.

$$\begin{cases} 2x + y = -4 \\ 3x - 2y = -6 \end{cases}$$

a. $x = -6$, $y = 6$

b. $x = -4$, $y = 0$

c. $x = 0$, $y = 3$

d. $x = -2$, $y = 0$

Additional Resources

Reinforcing the Substitution Method

Solve the system by substitution:

$$\begin{cases} 4x + 2y = 4 \\ 6x + -y = 8 \end{cases}$$

We need to solve one equation for one variable. We will solve the first equation for y.

Step 1 - Solve the first equation for y.

$$\begin{aligned} 4x + 2y &= 4 \\ 2y &= -4x + 4 \\ y &= -2x + 2 \end{aligned}$$

Step 2 - Substitute $-2x + 2$ for y in the second equation.

$$\begin{aligned} 6x - y &= 8 \\ 6x - (-2x + 2) &= 8 \end{aligned}$$

Step 3 - Solve the equation for x.

$$\begin{aligned} 6x-(-2x+2) &= 8 \\ 6x+2x-2 &= 8 \\ 8x-2 &= 8 \\ 8x &= 10 \\ x &= \tfrac{10}{8} \\ x &= \tfrac{5}{4} \end{aligned}$$

Step 4 - Substitute $x = \frac{5}{4}$ into $4x + 2y = 4$ to find y.

$$\begin{aligned} 4x+2y &= 4 \\ 4(\tfrac{5}{4})+2y &= 4 \\ 5+2y &= 4 \\ 2y &= -1 \\ y &= \tfrac{-1}{2} \end{aligned}$$

Step 5 - Check the ordered pair. $(\frac{5}{4}, \frac{-1}{2})$ in both equations.

$$\begin{aligned} 4x+2y=4 \qquad & 6x-y=8 \\ & 6(\tfrac{5}{4})-(-\tfrac{1}{2}) \stackrel{?}{=} 8 \\ & \tfrac{30}{4}-(-\tfrac{1}{2}) \stackrel{?}{=} 8 \\ 4(-\tfrac{1}{2})+2(\tfrac{5}{4}) \stackrel{?}{=} 4 \qquad & \tfrac{15}{2}-(-\tfrac{1}{2}) \stackrel{?}{=} 8 \\ 5-1 \stackrel{?}{=} 4 \qquad & \tfrac{15}{2}+\tfrac{1}{2} \stackrel{?}{=} 8 \\ 4=4\checkmark \qquad & 8=8\checkmark \end{aligned}$$

TRY IT **Reinforcing the Substitution Method**

1. Solve the system by substitution:

$$\begin{cases} x - 4y = -4 \\ -3x + 4y = 0 \end{cases}$$

2.3.4 Solving More Equations

Cool Down

Solve this system of equations without graphing. Be prepared to show your reasoning. Use this system of equations to answer questions 1 – 3.

$$\begin{cases} 5x + y = 7 \\ y = -20x + 2 \end{cases}$$

1. What is the x value in the solution?

2. What is the y value in the solution?

3. Explain how you determined your solution.

Building Character: Growth Mindset



Having a growth mindset means believing you can improve your abilities. Compared to a fixed mindset, a growth mindset encourages you to embrace challenges, sustain effort, and try new strategies. Think about your current sense of having a growth mindset. Are the following statements true for you??

- After I make a mistake, I always look for ways to learn from it.
- I love challenges because they make me smarter.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

2.3.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Lin is solving this system of equations:

$$\begin{cases} 6x - 5y = 34 \\ 3x + 2y = 8 \end{cases}$$

She starts by rearranging the second equation to isolate the y variable: $y = 4 - 1.5x$. She then substitutes the expression $4 - 1.5x$ for y in the first equation, as shown:

$6x - 5(4 - 1.5x) = 34$
$6x - 20 - 7.5x = 34$

$-1.5x = 54$
$x = -36$

$y = 4 - 1.5x$

$y = 4 - 1.5 \cdot (-36)$

$y = 58$

Does Lin's solution of $(-36, 58)$ makes both equations in the system true?

a. No; Her solution does not make both equations true.
b. yes; Her solution makes both equations true.

2. Which of the following is true of Lin's solution?

a. Lin's solution of $(-36, 58)$ is correct.
b. Lin made an addition error. The correct answer is $x = 0$ and $y = 4$, or $(0, 4)$.
c. Lin made a multiplication error. The correct answer is $x = 9$ and $y = 4$, or $(9, 4)$.
d. Lin made a multiplication error. The correct answer is $x = 4$ and $y = -2$, or $(4, -2)$.

In questions 3 –9, solve the systems of equations by substitution.

3. $\begin{cases} 2x + y = -2 \\ 3x - y = 7 \end{cases}$

a. $(2, 6)$
b. $(1, -4)$
c. $(0, -2)$
d. $(-1, 0)$

4. $\begin{cases} x - 3y = -9 \\ 2x + 5y = 4 \end{cases}$

a. $(-3, 2)$
b. $(0, 3)$
c. $(2, 0)$
d. $(-6, 1)$

5. $\begin{cases} -2x + 2y = 6 \\ 2y = -6x + 14 \end{cases}$

a. $(-3, 0)$
b. $(2, 1)$
c. $(0, 7)$
d. $(1, 4)$

6. $\begin{cases} 3x + 4y = 1 \\ y = -\frac{2}{5}x + 2 \end{cases}$

a. $(-5, 4)$
b. $(-1, 1)$
c. $(0, 2)$
d. $(5, 0)$

7. $\begin{cases} 3x + y = 10 \\ x - 2y = -20 \end{cases}$

a. $(4, 12)$
b. $(-4, 8)$
c. $(0, 10)$
d. $(2, 2)$

8. $\begin{cases} y = x - 6 \\ y = -\frac{3}{2}x + 4 \end{cases}$

a. $(0, 4)$
b. $(4, -2)$
c. $(2, 1)$
d. $(6, 0)$

9. $\begin{cases} 2x - 16y = 8 \\ -x - 8y = -4 \end{cases}$

a. $(-4, -1)$
b. $(-4, 1)$
c. $(4, 0)$
d. $(12, 1)$

10. $\begin{cases} y = -\frac{2}{3}x + 6 \\ 2x - y = 2 \end{cases}$

a. $(6, 2)$
b. $(3, 4)$
c. $(-2, 0)$
d. $(0, 6)$

11. Tyler and Han are trying to solve this system by substitution:

$\begin{cases} x + 3y = -5 \\ 9x + 3y = 3 \end{cases}$

- Tyler's first step is to isolate x in the first equation to get $x = -5 - 3y$.
- Han's first step is to isolate 3y in the first equation to get $3y = -5 - x$.

Solve using both methods to show that both first steps can be used to solve the system and will yield the same solution.

What is the solution to the system?

a. $(-5, 0)$
b. $(4, -3)$
c. $(0, 1)$
d. $(1, -2)$

2.3.6 Lesson Summary

In this lesson, you learned how to:

- Solve systems of equations by substituting a variable or an expression.
- Identify more than one way to perform a substitution and decide how to substitute based on how the given equations are written.

Here are the **activities** that helped you reach those goals:

- 2.3.1: Finding Connections between Graphs and Equations
  - In this activity, you analyzed the relationship between systems of equations and their graphs. You determined whether a given graph could represent a specific system of equations using your knowledge of horizontal and vertical lines and the slopes and intercepts of those lines.
- 2.3.2: Checking Solutions in Systems
  - In this activity, you solved different systems of equations using any method available and then checked your solution by substituting the values found back into the original equations to verify their accuracy.
  - 2.3.2: Self Check
  - 2.3.2: Additional Resources
- 2.3.3: Solving Equations Using Substitution
  - In this activity, you used the substitution method to solve systems of equations. By isolating a variable in one of the equations, you found an equivalent expression that was substituted into the second equation. Using this method, the solution to one variable was found. This value was then substituted back into the first equation to find the value of the remaining unknown variable
  - 2.3.3: Self Check
  - 2.3.3: Additional Resources
- 2.3.4: Solving More Equations
  - In this activity, you continued to solve systems of equations using substitution, but you went a step further by showing your reasoning for each step in the process.

After these activities, you completed the following **practice**.

- 2.3.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

2.4 SOLVING SYSTEMS BY ELIMINATION, PART 1

2.4.0 Lesson Overview

In this lesson, you will add equations in standard form to eliminate one of the variables. This will allow you to solve for the remaining variable. You can then use that value to determine the value of the other variable. Ultimately, you will be able to see the relationship between the graph and solution of the system of equations.

When you finish this lesson, you will be able to:

- Solve systems of equations by adding or subtracting them to eliminate a variable.
- Know that adding or subtracting equations in a system creates a new equation, where one of the solutions to this equation is the solution to the system.

Here are the **activities** that will help you reach those goals:

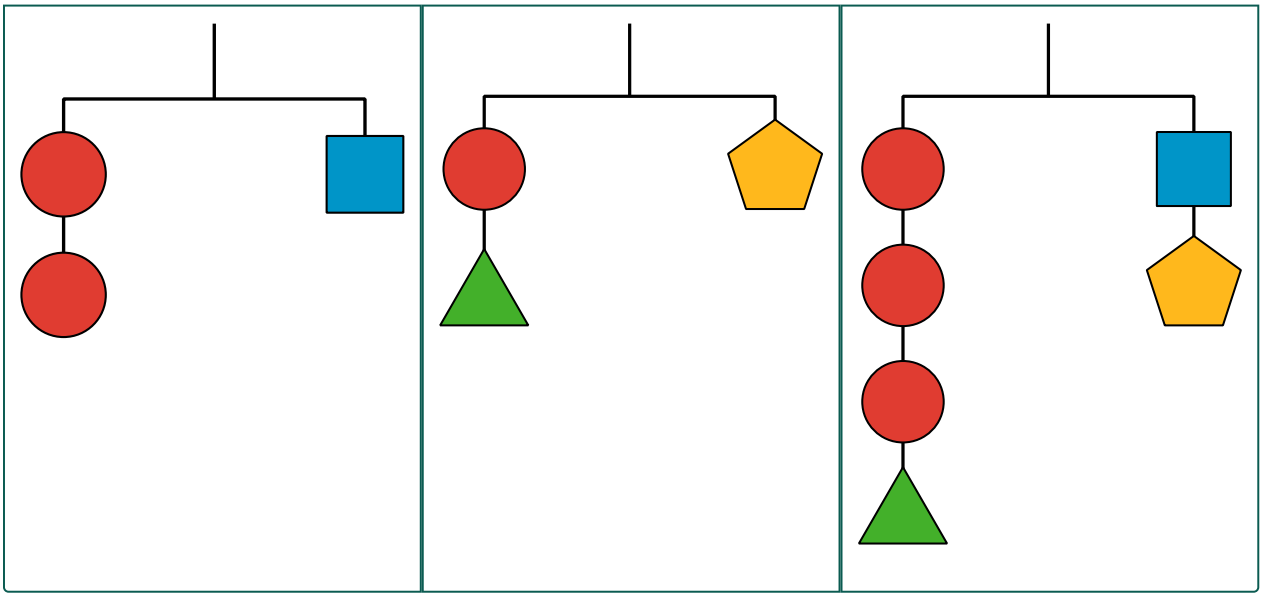
- 2.4.1: Combining Two True Equations
- 2.4.2: Adding Equations
  - 2.4.2: Self Check
  - 2.4.2: Additional Resources
- 2.4.3: Adding and Subtracting Systems of Equations and Using Their Graphs
  - 2.4.3: Self Check
  - 2.4.3: Additional Resources
- 2.4.4: Determining the Best Method for Solving a System of Equations

After that, you'll **practice and review**.

- 2.4.5: Practice
- 2.4.6: Lesson Summary

2.4.1 Combining Two True Equations

Warm Up



1. What do you notice?

2. What do you wonder?

2.4.2 Adding Equations

Activity

Make sense of Diego's work and discuss with a partner, then answer questions 1 – 3.

$$\begin{cases} 2x - 3y = -4 \\ 2x + 3y = 8 \end{cases}$$

$4x = 4$
$x = 1$
$2(1) + 3y = 8$
$3y = 6$
$y = 2$

1. What did Diego do to solve the system?

2. Is the pair of x- and y-values that Diego found actually a solution to the system?

3. How do you know?

For questions 4 – 7 determine if Diego's method will work for solving the system. Be prepared to show your reasoning. If Diego's method does not work, use a different method to solve and get the answers.

$$\begin{cases} 2x + y = 4 \\ x - y = 11 \end{cases}$$

4. Will Diego's method work for solving the system?

5. What is the x value?

6. What is the y value?

7. Explain your solutions.

For questions 8 – 11 determine if Diego's method will work for solving the system. Be prepared to show your reasoning. If Diego's method does not work, use a different method to solve and get the answers.

$$\begin{cases} 8x + 11y = 37 \\ 8x + y = 7 \end{cases}$$

8. Will Diego's method work for solving the system?

9. What is the x value?

10. What is the y value?

11. Explain your solutions.

Self Check

Which of the following equations can be found by adding the equations in the system of equations shown?

$$\begin{cases} 3x + 5y = -9 \\ -3x - 7y = -21 \end{cases}$$

a. $-2y = -30$
b. $-6x = -30$
c. $6x + 12y = 30$
d. $-2x = -30$

Additional Resources

The Elimination method of Solving Systems of Equations

The **Elimination** Method is based on the **Addition Property of Equality**. The Addition Property of Equality says that when you add the same quantity to both sides of an equation, you still have equality. We will extend the Addition Property of Equality to say that when you add equal quantities to both sides of an equation, the results are equal.

For any expressions a, b, c, and d:

$$\begin{aligned} \text{if} \quad & a = b \\ \text{and} \quad & c = d \\ \text{then} \quad & a + c = b + d. \end{aligned}$$

To solve a **system of equations** by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites so that we can add the equations together and eliminate that variable

Notice how that works when we add these two equations together:

$$\begin{array}{l} \left\{ \begin{array}{l} 3x + y = 5 \\ 2x - y = 0 \end{array} \right. \\ \hline 5x = 5 \end{array}$$

The y's add to zero, and we have one equation with one variable.

Let's try another one:

$$\left\{ \begin{array}{rcl} 4p - 2s & = & -26 \\ -5p + 2s & = & 29 \end{array} \right.$$

We can add the equations, so s will be eliminated when we add these two equations.

$$\begin{array}{l} \left\{ \begin{array}{rcl} 4p - 2s & = & -26 \\ -5p + 2s & = & 29 \end{array} \right. \\ \hline \begin{array}{rcl} -p & = & 3 \\ p & = & -3 \end{array} \end{array}$$

Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations.

EXAMPLE

Solve the system by elimination:

$$\left\{ \begin{array}{l} 4x + 8y = 12 \\ 4x - 5y = 38 \end{array} \right.$$

Step 1 - Write both equations in standard form. If any coefficients are fractions, clear them. $4x + 8y = 12$ $4x - 5y = 38$

Step 2 - Check to see if the coefficients of one variable are opposites or equivalent. The coefficients on the x-variables are equivalent. We can subtract the equations.

Step 3 - Subtract the equations to eliminate one variable.

$$\begin{array}{l} \begin{array}{rcl} 4x + 8y & = & 12 \\ 4x - 5y & = & 38 \end{array} \\ \hline 13y = -26 \end{array}$$

Step 4 - Solve for the remaining variable.

$$\begin{array}{rcl} 13y & = & -26 \\ y & = & -2 \end{array}$$

Step 5 - Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.

$$\begin{array}{rcl} 4x + 8y & = & 12 \\ 4x + 8(-2) & = & 12 \\ 4x - 16 & = & 12 \\ 4x & = & 28 \\ x & = & 7 \end{array}$$

Step 6 - Write the solution as an ordered pair. $(7, -2)$

Step 7 - Check that the ordered pair is a solution to both original equations.

$4x + 8y = 12$ $\quad$ $4x - 5y = 38$

$4(7) + 8(-2) = 12$ $\quad$ $4(7) - 5(-2) = 38$

$28 - 16 = 12$ $\quad$ $28 + 10 = 38$

$12 = 12$✔ $\quad$ $38 = 38$✔

TRY IT **The Elimination Method of Solving Systems of Equations**

Solve the system by elimination:

$$\begin{cases} 3x + y = 5 \\ -3x - 7y = 1 \end{cases}$$

2.4.3 Adding and Subtracting Systems of Equations and Using Their Graphs

Activity

Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

Below are three systems of equations you saw earlier.

Use System A to answer questions 1 – 6.

System A

$$\begin{cases} 4x + 3y = 10 \\ -4x + 5y = 6 \end{cases}$$

1. Use the graphing tool or technology outside the course. Graph the equations for System A that represent this scenario using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-4-3-adding-and-subtracting-systems-of-equations-and-using-their-graphs)

2. Identify the x coordinate of the solution.

3. Identify the y coordinate of the solution.

4. Find the sum or difference of the two original equations that would enable the system to be solved.

5. Use the graphing tool or technology outside the course. Graph your third equation and the System A equations using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-4-3-adding-and-subtracting-systems-of-equations-and-using-their-graphs)

6. Make an observation about the graph.

Use System B to answer questions 7 – 12.

System B

$$\begin{cases} 2x + y = 4 \\ x - y = 11 \end{cases}$$

7. Use the graphing tool or technology outside the course. Graph the equations for System B that represent this scenario using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-4-3-adding-and-subtracting-systems-of-equations-and-using-their-graphs)

8. Identify the x coordinate of the solution.

9. Identify the y coordinate of the solution.

10. Find the sum or difference of the two original equations that would enable the system to be solved.

11. Use the graphing tool or technology outside the course. Graph your third equation and the System B equations using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-4-3-adding-and-subtracting-systems-of-equations-and-using-their-graphs)

12. Make an observation about the graph.

Use System C to answer questions 13–18.

System C

$$\begin{cases} 8x + 11y = 37 \\ 8x + y = 7 \end{cases}$$

13. Use the graphing tool or technology outside the course. Graph the equations for System C that represent this scenario using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-4-3-adding-and-subtracting-systems-of-equations-and-using-their-graphs)

14. Identify the x coordinate of the solution.

15. Identify the y coordinate of the solution.

16. Find the sum or difference of the two original equations that would enable the system to be solved.

17. Use the graphing tool or technology outside the course. Graph your third equation and the System C equations using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-4-3-adding-and-subtracting-systems-of-equations-and-using-their-graphs)

18. Make an observation about the graph.

Are you ready for more?

Extending Your Thinking

Mai wonders what would happen if we multiply equations. That is, we multiply the expressions on the left side of the two equations and set them equal to the expressions on the right side of the two equations.

Use System B to answer questions 1 – 4.

System B

$$\begin{cases} 2x + y = 4 \\ x - y = 11 \end{cases}$$

1. Write out an equation that you would get if you multiply the two equations in this manner.
2. Does your original solution still work in this new equation?
3. Use the graphing tool or technology outside the course. Graph your third equation and the System B equations using the Desmos tool below. Access multimedia content (http://openstax.org/books/algebra-1/pages/2-4-3-adding-and-subtracting-systems-of-equations-and-using-their-graphs)
4. Explain if the approach of graphing the new equation is or is not particularly helpful.

Self Check

Solve the system of equations using elimination. Check the solution by graphing.

$$\begin{cases} 2x - 4y = -26 \\ -3x + 4y = 29 \end{cases}$$

a. $(-13, 0)$
b. $(0, 6.25)$
c. $(-3, 5)$
d. $(-8, 2.5)$

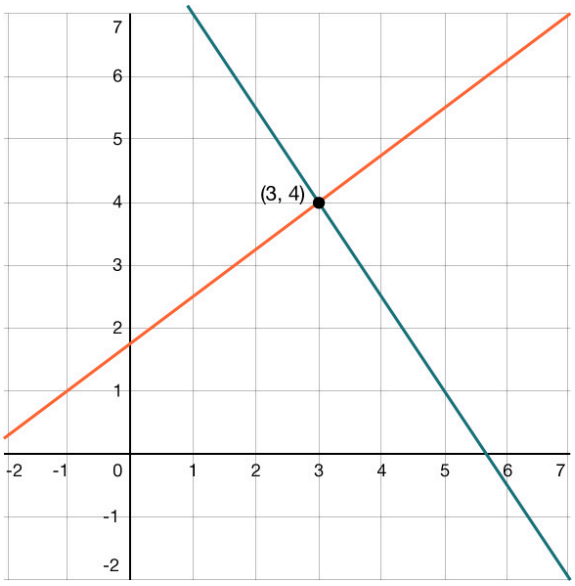
Additional Resources

Adding Systems of Equations and Using Their Graphs

Look at the system of equations shown here:

$$\begin{cases} -3x + 4y = 7 \\ 3x + 2y = 17 \end{cases}$$

Let's graph each equation on the coordinate plane. Identify the point of intersection.

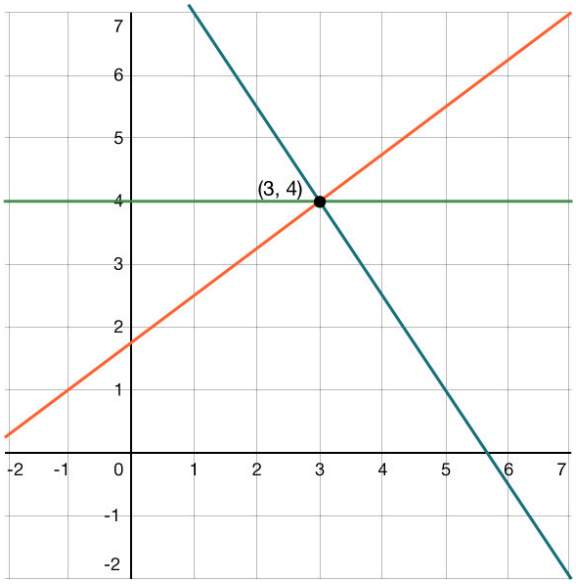


The lines intersect at $(3, 4)$.

Add the two equations together. Simplify the resulting equation.

$$\begin{array}{r} -3x+4y=7 \\ 3x+2y=17 \\ \hline 6y=24 \\ y=4 \end{array}$$

Graph the resulting equation on the same coordinate plane.



All three lines intersect at $(3, 4)$.

You verified that by adding the two equations in the system, you created a third equation that is also true.

The solution to the system is $x = 3$, $y = 4$.

TRY IT **Adding Systems of Equations and Using Their Graphs**

Add the equations to find the resulting equation. Then graph all three equations to find the ordered pair where they intersect.

Use this system of equations to answer questions 1 – 4.

$$\begin{cases} -2x-3y=7 \\ 2x+y=3 \end{cases}$$

1. Add the equations to find the resulting equation.
2. Use the graphing tool or technology outside the course. Graph your third equation and the system of equations for this scenario using the Desmos tool below. Access multimedia content (http://openstax.org/books/algebra-1/pages/2-4-3-adding-and-subtracting-systems-of-equations-and-using-their-graphs)
3. Identify the x coordinate of the solution.
4. Identify the y coordinate of the solution.

2.4.4 Determining the Best Method for Solving a System of Equations

Cool Down

Graphing technology should not be used in this cool down.

Use this linear equation to answer questions 1 – 2.

$$\begin{cases} 2x + \frac{1}{2}y = 7 \\ 6x - \frac{1}{2}y = 5 \end{cases}$$

1. Which would be a more helpful method for solving the system?
2. Explain or show how you know.

Solve the system without graphing to answer questions 3 – 5.

3. Identify the x coordinate of the solution.
4. Identify the y coordinate of the solution.
5. Explain or show your reasoning.

Video: Determining the Best Method for Solving a System of Equations

Watch the following video to learn more about finding the best method to solve a system of equations.

Determining the Best Method for Solving a System of Equations (https://openstax.org/r/system-solving-method)

2.4.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 8, solve the systems of equations using the elimination method.

1. $\begin{cases} 5x + 2y = 2 \\ -6x - 2y = 0 \end{cases}$

a. $(-2, 6)$
b. $(2, -4)$
c. $(0, 1)$
d. $(-3, 9)$

2. $\begin{cases} 2x - 5y = 7 \\ -15x + 5y = -85 \end{cases}$

a. $(-4, -3)$
b. $(6, 1)$
c. $(5, -2)$
d. $(7, 4)$

3. $\begin{cases} 6x - 10y = -18 \\ 25x + 10y = 80 \end{cases}$

a. $(4, -2)$
b. $(0, 8)$
c. $(7, 6)$
d. $(2, 3)$

4. $\begin{cases} -6x - 16y = 6 \\ 6x + 15y = -9 \end{cases}$

a. $(7, -2)$
b. $(1, -1)$
c. $(-9, 3)$
d. $(-14, 5)$

5. $\begin{cases} -15x - 40y = -335 \\ 15x + 9y = 180 \end{cases}$

a. $(6, 10)$
b. $(17, 2)$
c. $(9, 5)$
d. $(12, 0)$

6. $\begin{cases} x - 3y = -9 \\ x + \frac{5}{2}y = 2 \end{cases}$

a. $(-9, 0)$
b. $(2, 0)$
c. $(0, 3)$
d. $(-3, 2)$

7. $\begin{cases} x + \frac{1}{3}y = 3 \\ 2x + \frac{1}{3}y = 5 \end{cases}$

a. $(2, 3)$
b. $(0, 15)$
c. $(0, 9)$
d. $(3, 0)$

8. $\begin{cases} 3x - 8y = -35 \\ -5x + 8y = 45 \end{cases}$

a. $(-1, 4)$
b. $(-5, 2.5)$
c. $(7, 7)$
d. $(-9, 0)$

For questions 9 – 10, add the equations in the following systems of equations. What is the resulting new equation?

9. $\begin{cases} 4x - 6y = 14 \\ -4x + y = -19 \end{cases}$

10. $\begin{cases} 3x - 5y = 18 \\ 7x + 5y = -8 \end{cases}$

2.4.6 Lesson Summary

In this lesson, you learned how to:

- Solve systems of equations by adding or subtracting them to eliminate a variable.
- Demonstrate that adding or subtracting equations in a system creates a new equation, where one of the solutions to this equation is the solution to the system.

Here are the **activities** that helped you reach those goals:

- 2.4.1: Combining Two True Equations
  - In this activity, you analyzed a model representing the process of combining two different equations. You learned that when two equations that are true are combined, the resulting equation is also true.
- 2.4.2: Adding Equations
  - In this activity, you learned how to add two equations together into a third equation. By using elimination, you created a third equation containing only one variable. This third equation can be solved to find the value of one of the variables. Using this value, you can find the value of the remaining unknown variable. You used solutions to the first two equations to test that they were still true for the third equation.
  - 2.4.2: Self Check
  - 2.4.2: Additional Resourcess
- 2.4.3: Adding and Subtracting Systems of Equations and Using Their Graphs
  - In this activity, you added two equations to form a third equation. You then graphed the original system of equations alongside the new third equation. By determining that all three equations intersect at the same point, you verified that the solution to the first two equations is the same as the solution to the third equation.
  - 2.4.3: Self Check
  - 2.4.3: Additional Resourcess
- 2.4.4: Determining the Best Method for Solving a System of Equations
  - In this activity, you analyzed systems of equations and evaluated whether adding or subtracting them would allow you to solve the system. By using coefficients of variables with similar absolute values, the equations can be added or subtracted to eliminate that variable in the third equation.

After these activities, you completed the following **practice**:

- 2.4.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

2.5 SOLVING SYSTEMS BY ELIMINATION, PART 2

2.5.0 Lesson Overview

In this lesson, you will continue to develop your understanding of solving systems by elimination. You will be given a system that represents the quantities and constraints in a situation and learn to interpret, in context, the solutions to the individual equations and to the system. Ultimately, the context will be used to make sense of the sum of the two equations and why it shares a solution with the equations in the given system.

When you finish this lesson, you will be able to:

- Explain why adding or subtracting two equations that share a solution results in a new equation that also shares the same solution.

Here are the **activities** that will help you reach those goals:

- 2.5.1: Writing and Evaluating New Equations
- 2.5.2: Adding Two Equations in a System
  - 2.5.2: Self Check
  - 2.5.2: Additional Resources
- 2.5.3: Solving Systems of Linear Equations in Two Variables
  - 2.5.3: Self Check
  - 2.5.3: Additional Resources
- 2.5.4: Solving Systems of Equations Using Real-World Examples

After that, you'll **practice and review**.

- 2.5.5: Practice
- 2.5.6: Lesson Summary

2.5.1 Writing and Evaluating New Equations

Activity

Look at the following equation: $50 + 1 = 51$.

Perform each of the following operations and answer these questions: What does each resulting equation look like? Is it still a true equation?

1. Add 12 to each side of the equation.

2. Add $10 + 2$ to the left side of the equation and 12 to the right side.

3. Add the equation $4 + 3 = 7$ to the equation $50 + 1 = 51$.

4. Write a new equation that, when added to $50 + 1 = 51$, gives a sum that is also a true equation.

5. Write a new equation that, when added to $50 + 1 = 51$, gives a sum that is a false equation.

2.5.2 Adding Two Equations in a System

Activity

A teacher purchased 20 calculators and 10 measuring tapes for her class and paid $495. Later, she realized that she didn't order enough supplies. She placed another order of 8 of the same calculator and 1 of the same measuring tape and paid $178.50.



This system represents the constraints in this situation. Discuss questions 1 – 4 with a partner, then answer the questions.

$$\begin{cases} 20c + 10m = 495 \\ 8c + m = 178.50 \end{cases}$$

1. In this situation, what do the solutions to the first equation mean?
2. What do the solutions to the second equation mean?
3. For each equation, how many possible solutions are there? Explain how you know.
4. In this situation, what does the solution to the system mean?
5. Find the value of c.
6. Find the value of m.
7. Explain your reasoning for the solution.

Use the following information to answer questions 8 – 11:

To be reimbursed for the cost of the supplies, the teacher recorded, "Items purchased: 28 calculators and 11 measuring tapes. Amount: $673.50."

8. Write an equation to represent the relationship between the numbers of calculators and measuring tapes, the prices of those supplies, and the total amount spent.
9. How is this equation related to the first two equations?
10. In this situation, what do the solutions of this equation mean?
11. How many possible solutions does this equation have? How many solutions make sense in this situation? Be prepared to show your reasoning.

Self Check

At a toy store, a family bought 3 toy cars, c, and 2 teddy bears, b, for

$$\begin{cases} 3c + 2b = 46 \\ 7c + 2b = 70 \end{cases}$$

Which statement must be true?

a. $10c + 4b = 116$ is the same line as one of the equations in the system.
b. The solution of the system is not a solution of $10c + 4b = 116$.
c. The solution of the system is a solution of $10c + 4b = 116$.
d. $10c + 4b = 116$ represents the solutions of the system.

Additional Resources

Adding Two Equations in a Real-World Situation

Let's look at the situation described and write a system of equations.

Paulina went to the pool 8 times and the skating rink 4 times this month. She spent a total of $128 on admission to both places. Last month, she went to the pool 3 times and the skating rink 4 times. Last month she spent $83 on admission to both places.

Let p represent the cost of the pool and s represent the cost of the skating rink.

This month	Last month
$8p + 4s = 128$	$3p + 4s = 83$

So, this system represents the constraints in this situation:

$$\begin{cases} 8p + 4s = 128 \\ 3p + 4s = 83 \end{cases}$$

Over both months, she went to the pool 11 times and the skating rink 8 times. She spent a total of $128 + $83, or $211, over both months.

The equation that represents this total is $11p + 8s = 211$. Notice this is the equivalent of adding the like terms of the two equations in the system of equations.

Let's solve the original system and determine if the solution also applies to this new equation.

Using substitution, we find that the solution is $(9, 14)$, or the pool costs $9 and the skating rink costs $14.

Substitute these values into the new equation:

$$\begin{aligned} 11p + 8s &= 211 \\ 11(9) + 8(14) &= 211 \\ 99 + 112 &= 211 \\ 211 &= 211 \checkmark \end{aligned}$$

Adding the equations together produces a new equation that has the same solution.

TRY IT **Adding Two Equations in a Real-World Situation**

Alicia is a chef for a restaurant. This week, she ordered 32 steaks and 45 chickens. The total cost this week was $840.

Last week, she ordered 32 steaks and 24 chickens. The total cost was $672.

Let s represent the cost of a steak and c represent the cost of a chicken.

Use this information to answer questions 1 – 2.

1. Write a system of equations to represent the constraints given.
2. Write a new equation that represents the total cost for both weeks.

2.5.3 Solving Systems of Linear Equations in Two Variables

Activity

Solve each system of equations without graphing, show your reasoning, and check your solutions.

System A

$$\begin{cases} 2x + 3y = 7 \\ -2x + 4y = 14 \end{cases}$$

Use System A to answer questions 1 – 3.

1. What is the value of x?
2. What is the value of y?
3. Show your reasoning and check the solution.

System B

$$\begin{cases} 2x + 3y = 7 \\ 3x - 3y = 3 \end{cases}$$

Use System B to answer questions 4 – 6.

4. What is the value of x?
5. What is the value of y?
6. Show your reasoning and check the solution.

System C

$$\begin{cases} 2x + 3y = 5 \\ 2x + 4y = 9 \end{cases}$$

Use System C to answer questions 7 – 9.

7. What is the value of x?
8. What is the value of y?
9. Show your reasoning and check the solution.

System D

$$\begin{cases} 6x + 9y = 48 \\ 6x - 5y = 20 \end{cases}$$

Use System D to answer questions 10 – 12.

10. What is the value of x?
11. What is the value of y?
12. Show your reasoning and check the solution.

Are you ready for more?

Extending Your Thinking

This system has three equations:

$$\begin{cases} 3x + 2y - z = 7 \\ -3x + 5y + 2z = -14 \\ 3x + y - z = 10 \end{cases}$$

1. Add the first two equations to get a new equation.
2. Add the second two equations to get a new equation.
3. What is the value of y?
4. What is the value of z?

Solve the original system of equations, then answer questions 5 – 7.

5. What is the value of x?

6. What is the value of y?

7. What is the value of z?

Self Check

A mechanic's garage purchased 8 brake pads and 5 headlight sets for inventory and paid $1,460. Later, the mechanic purchased another order of 6 of the same brake pads and 5 of the same headlight sets and paid $1,220. How much does each part cost?

a. brake pads: $80; headlight sets: $148
b. brake pads: $80; headlight sets: $140
c. brake pads: $100; headlight sets: $100
d. brake pads: $120; headlight sets: $100

Additonal Resources

Solving a System of Equations by Elimination

Let's reinforce the steps to solve a system of equations using elimination.

Step 1 - Write both equations in standard form. If any coefficients are fractions, clear them.

Step 2 - Check to see if the coefficients of one variable are opposites or equivalent.

Step 3 - Add or subtract the equations to eliminate one variable.

Step 4 - Solve for the remaining variable.

Step 5 - Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.

Step 6 - Write the solution as an ordered pair.

Step 7 - Check that the ordered pair is a solution to both original equations.

EXAMPLE

Let's look at an example. Solve the following system of equations using elimination:

$$\begin{cases} 3x-3y=-24 \\ -3x+5y=30 \end{cases}$$

Step 1 - Write both equations in standard form. If any coefficients are fractions, clear them. $\begin{cases} 3x-3y=-24 \\ -3x+5y=30 \end{cases}$

Step 2 - Check to see if the coefficients of one variable are opposites or equivalent. The coefficients on the x-variables are opposites. We can add the equations.

Step 3 - Add the equations to eliminate one variable.

$$\begin{array}{rcl} 3x-3y & = & -24 \\ -3x+5y & = & 30 \\ 2y & = & 6 \end{array}$$

Step 4 - Solve for the remaining variable.

$$\begin{array}{rcl} 2y & = & 6 \\ y & = & 3 \end{array}$$

Step 5 - Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
\begin{array}{rcl}3x-3y&=&-24\\3x-3(3)&=&-24\\3x-9&=&-24\\3x&=&-15\\x&=&-5\end{array}

Step 6 - Write the solution as an ordered pair. $(-5, 3)$

Step 7 - Check that the ordered pair is a solution to both original equations.

$$\begin{aligned} 3x-3y &= -24 \\ 3(-5)-3(3) &= -24 \\ -15-9 &= -24 \\ -24 &= -24\checkmark \end{aligned}$$

$$\begin{aligned} -3x+5y &= 30 \\ -3(-5)+5(3) &= 30 \\ 15+15 &= 30 \\ 30 &= 30\checkmark \end{aligned}$$

TRY IT **Solving a System of Equations by Elimination**

Solve the following system of equations using elimination:

$$\begin{cases} 4x-6y=8 \\ x+6y=17 \end{cases}$$

2.5.4 Solving Systems of Equations Using Real-World Examples

Cool Down

On a family outing, Marlon bought 5 cups of hot cocoa and 4 pretzels for $18.40. Some of his family members would like a second serving, so he went back to the same food stand and bought another 2 cups of hot cocoa and 4 pretzels for $11.20.

Here is a system of equations that represent the quantities and constraints in this situation.

$$\begin{cases} 5c+4p=18.40 \\ 2c+4p=11.20 \end{cases}$$

1. What does the solution to the system, (c, p), represent in this situation?
2. If we add the second equation to the first equation, we have a new equation: $7c+8p=29.60$. Explain why the same (c, p) pair that is a solution to the two original equations is also a solution to this new equation.
3. Does the equation $7c+8p=29.60$ help us solve the original system? If you think so, explain how it helps. If you don't think so, explain why not and what would help us solve the system.

Video: Solving Systems of Equations Using Real-World Examples

Watch the following video to learn more about when addition is useful for solving systems of equations.

Determining the Best Method for Solving a System of Equations (https://openstax.org/r/system-solving-method)

2.5.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 4, add the equations in the system to find a new equation.

1. $\begin{cases} 6x - 5y = -1 \\ -6x - 3y = -39 \end{cases}$

a. $-8x = -40$
b. $-8y = -40$
c. $-2y = -40$
d. $2y = -38$

2. $\begin{cases} 5x - 3y = -1 \\ -6x + 3y = -6 \end{cases}$

a. $-11x = -7$ or $11x = 7$
b. $-6x = -7$ or $6x = 7$
c. $-x = -7$ or $x = 7$
d. $x = -7$ or $-x = 7$

3. $\begin{cases} 4x - 3y = 3 \\ -4x - 10y = 62 \end{cases}$

a. $-13x = 65$
b. $-8y = 65$
c. $-8x = 65$
d. $-13y = 65$

4. $\begin{cases} 6x + 16y = -6 \\ -6x - 10y = 6 \end{cases}$

a. $6y = 12$ or $y = 6$
b. $12x = 0$ or $x = 0$
c. $6x = 0$ or $x = 0$
d. $6y = 0$ or $y = 0$

For questions 5 – 10, solve each system of equations.

5. $\begin{cases} 6x + 27y = -12 \\ -6x - 26y = 14 \end{cases}$

a. $(-20, 4)$
b. $(7, -1)$
c. $(-11, 2)$
d. $(2.5, -1)$

6. $\begin{cases} 3x + 4y = 8 \\ -3x + 6y = 42 \end{cases}$

a. $(0, 7)$
b. $(-4, 5)$
c. $(-14, 0)$
d. $(-10, 2)$

7. $\begin{cases} 5x - 9y = 68 \\ -5x + 6y = -47 \end{cases}$

a. $(1, -7)$

b. $(19, 3)$
c. $(-13, 3)$
d. $(7, -2)$

8. $\begin{cases} -3x + 5y = -57 \\ 8x - 5y = 77 \end{cases}$

a. $(-1, -17)$
b. $(14, -3)$
c. $(4, -9)$
d. $(19, 0)$

9. $\begin{cases} 4x - 2y = -28 \\ -7x + 2y = 46 \end{cases}$

a. $(-6, 2)$
b. $(-8, -5)$
c. $(4, 6)$
d. $(0, 14)$

10. $\begin{cases} -2x + 3y = 23 \\ 2x - 6y = -56 \end{cases}$

a. $(-8, 13)$
b. $(8, 12)$
c. $(2, 9)$
d. $(5, 11)$

2.5.6 Lesson Summary

In this lesson, you learned how to:

- Explain why adding or subtracting two equations that share a solution results in a new equation that also shares the same solution.

Here are the **activities** that helped you reach those goals:

- 2.5.1: Writing and Evaluating New Equations
  - In this activity, you added equivalent values to both sides of an equation to evaluate if the equation was still true. By determining that the equation was still true, you discovered that adding equivalent values to each side of an equation maintains its truth.
- 2.5.2: Adding Two Equations in a System
  - In this activity, you wrote equations to represent constraints in a real-world situation. You then determined that adding these equations together results in a third equation with the same solution.
  - 2.5.2: Self Check
  - 2.5.2: Additional Resources
- 2.5.3: Solving Systems of Linear Equations in Two Variables
  - In this activity, you continued solving systems of equations in two variables by using an algebraic method of your own choice. You learned that some systems of equations are more suited to be solved by substitution and others by elimination.
  - 2.5.3: Self Check
  - 2.5.3: Additional Resources
- 2.5.4: Solving Systems of Equations Using Real-World Examples
  - In this activity, you analyzed a system of equations derived from a real-world situation. You reinforced that to solve a system using elimination by adding, the coefficients of one of the variables must be the same value and have opposite signs. If this is not the case, then no variable is eliminated when the resulting equation is created.

After these activities, you completed the following **practice**:

- 2.5.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

2.6 SOLVING SYSTEMS BY ELIMINATION, PART 3

2.6.0 Lesson Overview

In this lesson, you will explore the idea that you can multiply one or both equations in the system by a factor to make it possible to eliminate a variable. In addition, you will learn whenever you multiply equations in a system by a factor, add or subtract the equations, or otherwise manipulate the equations, you are essentially creating an equivalent system that will help you get closer to finding the solution of the original system.

When you finish this lesson, you will be able to:

- Explain how multiplying equations by a factor can help solve systems of linear equations.

Here are the **activities** that will help you reach those goals:

- 2.6.1: Multiplying Equations by a Number
- 2.6.2: Writing a New System to Solve a Given System
  - 2.6.2: Self Check
  - 2.6.2: Additional Resources
- 2.6.3: Finding Solutions to Unordered Sets of Equivalent Systems
  - 2.6.3: Self Check
  - 2.6.3: Additional Resources
- 2.6.4: Building Equivalent Systems

After that, you'll **practice and review.**

- 2.6.5: Practice
- 2.6.6: Lesson Summary

2.6.1 Multiplying Equations by a Number

Warm Up

Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

Consider two equations in a system:

$$\begin{cases} 4x + y = 1 & \text{Equation A} \\ x + 2y = 9 & \text{Equation B} \end{cases}$$

1. Use the graphing tool or technology outside the course. Graph the system of equations using the Desmos tool below.

Access multimedia content (http://openstax.org/books/algebra-1/pages/2-6-1-multiplying-equations-by-a-number)

2. What is the x value?

3. What is the y value?

For questions 4 – 6, use the following equation:

Equation A: $4x + y = 1$

4. Multiply the equation by 2, show the resulting equivalent equation, and label it A1.

5. Multiply the equation by -5, show the resulting equivalent equation, and label it A2.

6. Multiply the equation by $\frac{1}{2}$, show the resulting equivalent equation, and label it A3.

For questions 7 – 9, use the following equations:

The new equations A1, A2, and A3 and Equation B: $x + 2y = 9$

7. Use the graphing tool or technology outside the course. Graph the system of equations A1 and B using the Desmos tool below.

8. Use the graphing tool or technology outside the course. Graph the system of equations A2 and B using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-6-1-multiplying-equations-by-a-number)

Access multimedia content (http://openstax.org/books/algebra-1/pages/2-6-1-multiplying-equations-by-a-number)

9. Use the graphing tool or technology outside the course. Graph the system of equations A3 and B using the Desmos tool below.

Access multimedia content (http://openstax.org/books/algebra-1/pages/2-6-1-multiplying-equations-by-a-number)

10. Make a couple of observations about the graphs of Equations A1, A2, and A3.

2.6.2 Writing a New System to Solve a Given System

Activity

Here is a system you solved by graphing earlier.

$$\begin{cases} 4x + y = 1 & \text{Equation A} \\ x + 2y = 9 & \text{Equation B} \end{cases}$$

To start solving the system, Elena wrote:

$$\begin{aligned} 4x + y &= 1 \\ 4x + 8y &= 36 \end{aligned}$$

And then she wrote:

$$\begin{aligned} 4x + y &= 1 \\ -(4x + 8y &= 36) \\ \hline -7y &= -35 \end{aligned}$$

1. What was Elena's first move?

2. What was Elena's second move?

3. What might be possible reasons for those moves?

4. Complete the solving process algebraically. Show that the solution is indeed $x = -1$, $y = 5$.

Video: Writing a New System to Solve a Given System

Watch the following video to learn more about writing a new system of equations to solve a situation of constraints.

Writing a New System to Solve a Given System (https://openstax.org/r/new-system)

Self Check

Solve the system of equations using elimination.

$$\begin{cases} 6x - 5y = -34 \\ 2x + 6y = 4 \end{cases}$$

a. (2, 0)
b. (-4, 2)

c. (-10, 4)
d. (5, -1)

Additional Resources

Solving a System of Equations by Elimination Using Multiplication

Let's look at an example where we need to multiply both equations by constants in order to make the coefficients of one variable opposites.

Solve the system by elimination: $\begin{cases} 4x-3y=9 \\ 7x+2y=-6 \end{cases}$

Solution

In this example, we cannot multiply just one equation by any constant to get opposite coefficients. So we will strategically multiply both equations by different constants to get the opposites.

Step 1 - Write both equations in standard form. If any coefficients are fractions, clear them. Both equations are in standard form.

$4x-3y=9$

$7x+2y=-6$

Step 2 - Check to see if the coefficients of one variable are opposites or equivalent. To get opposite coefficients of y, we will need to multiply the first equation by 2 and the second equation by 3. Multiply the first equation by 2 and the second equation by 3.

$2(4x-3y)=2(9)$

$3(7x+2y)=3(-6)$

Step 3 - Simplify

$8x-6y=18$

$21x+6y=-18$

Step 4 - Add the equations to eliminate one variable.

$$\begin{array}{r} 8x-6y=18 \\ \underline{21x+6y=-18} \\ 29x=0 \end{array}$$

Step 5 - Solve for the remaining variable.

$x=0$

Step 6 - Substitute the solution into one of the original equations. Then solve for the other variable.

$$\begin{aligned} 7x+2y &= -6 \\ 7(0)+2y &= -6 \\ 2y &= -6 \\ y &= -3 \end{aligned}$$

Step 7 - Check that the ordered pair is a solution to both original equations.

$$\begin{array}{ll} 4x-3y=9 & 7x+2y=-6 \\ 4(0)-3(-3)\stackrel{?}{=}9 & 7(0)+2(-3)\stackrel{?}{=}-6 \\ 9=9\checkmark & -6=-6\checkmark \end{array}$$

The solution is $(0,-3)$.

TRY IT **Solving a System of Equations by Elimination Using Multiplication**

Solve the following system of equations using elimination:

$$\begin{cases} 7x + 8y = 4 \\ 3x - 5y = 27 \end{cases}$$

2.6.3 Finding Solutions to Unordered Sets of Equivalent Systems

Activity

Your teacher will give you some slips of paper (https://openstax.org/r/slips-paper) with systems of equations written on them. Each system represents a step in solving this system:

$$\begin{cases} \frac{4}{5}x + 6y = 15 \\ -x + 18y = 11 \end{cases}$$

Arrange the slips in the order that would lead to a solution. Be prepared to:

1. Describe what move takes one system to the next system.
2. Explain why the systems are equivalent between the original system and step 1.
3. Explain why the systems are equivalent between the system in step 1 to the system in step 2.
4. Explain why the systems are equivalent between the system in step 2 to the system in step 3.
5. Explain why the systems are equivalent between the system in step 3 to the system in step 4.
6. Explain why the systems are equivalent between the system in step 4 to the system in step 5.
7. Explain why the systems are equivalent between the system in step 5 to the system in step 6.
8. Explain why the systems are equivalent between the system in step 6 to the system in step 7.

Are you ready for more?

Extending Your Thinking

This system of equations has the solution $(5, -2)$: $\begin{cases} Ax - By = 24 \\ Bx + Ay = 31 \end{cases}$

1. Find the missing value of A.
2. Find the missing value of B.

Self Check

Solve the system of equations using elimination.

$$\begin{cases} 7x - 2y = 3 \\ 4x + 5y = 3.25 \end{cases}$$

a. $\left(\frac{1}{2}, \frac{1}{4}\right)$
b. $(0, -1.5)$
c. $(1, 2)$
d. $(7, -5)$

Additional Resources

Solving a System of Equations by Elimination Involving Fractions

You just ordered the steps for solving a system of equations using elimination. Try using these steps when the system has equations containing fractions.

When the system of equations contains fractions, we will first clear the fractions by multiplying each equation by the least common denominator (LCD) of all the fractions in the equation.

Solve the system by elimination: $\begin{cases} x+\frac{1}{2}y=6 \\ \frac{3}{2}x+\frac{2}{3}y=\frac{17}{2} \end{cases}$

Step 1 - Check to see if each equation is in standard form $Ax+By=C$. Both equations are in standard form.

$x+\frac{1}{2}y=6$

$\frac{3}{2}x+\frac{2}{3}y=\frac{17}{2}$

Step 2 - To clear the fractions, multiply each equation by its LCD (Least Common Denominator).

$2(x+\frac{1}{2}y=6)$

$6(\frac{3}{2}x+\frac{2}{3}y=\frac{17}{2})$

Step 3 - Simplify the equations.

$2x+y=12$

$9x+4y=51$

Step 4 - Eliminate one of the variables and simplify the equation.

$-4(2x+y=12)=-8x-4y=-48$

Step 5 - Add the two equations to eliminate y.

$$\begin{array}{r} -8x-4y=-48 \\ 9x+4y=51 \\ \hline x=3 \end{array}$$

Step 6 - Solve for y by substituting x back into one of the equations.

$$\begin{aligned} 2x+y&=12 \\ 2(3)+y&=12 \\ 6+y&=12 \\ y&=6 \end{aligned}$$

Step 7 - Write the solution as an ordered pair.

The ordered pair is $(3,6)$.

Step 8 - Check that the ordered pair is a solution to **both** original equations.

$$\begin{aligned} x+\tfrac{1}{2}y&\stackrel{?}{=}6 & \tfrac{3}{2}x+\tfrac{2}{3}y&\stackrel{?}{=}\tfrac{17}{2} \\ 3+\tfrac{1}{2}(6)&\stackrel{?}{=}6 & \tfrac{9}{2}+4&\stackrel{?}{=}\tfrac{17}{2} \\ 3+3&\stackrel{?}{=}6 & \tfrac{9}{2}+\tfrac{8}{2}&\stackrel{?}{=}\tfrac{17}{2} \\ 6&=6\checkmark & \tfrac{17}{2}&=\tfrac{17}{2}\checkmark \end{aligned}$$

The solution is $(3,6)$.

TRY IT **Solving a System of Equations by Elimination Involving Fractions**

Solve the following system of equations.

$$\begin{cases} \frac{1}{3}x-\frac{1}{2}y=1 \\ \frac{3}{4}x-y=\frac{5}{2} \end{cases}$$

2.6.4 Building Equivalent Systems

Cool Down

Here is a system of equations:

$$\begin{cases} 12a + 5b = -15 \\ 8a + b = 11 \end{cases}$$

To solve this system, Diego wrote these equivalent systems for his first two steps.

Step 1

$$\begin{cases} 12a + 5b = -15 \\ -40a + -5b = -55 \end{cases}$$

Step 2

$$\begin{cases} 12a + 5b = -15 \\ -28a = -70 \end{cases}$$

1. Describe the move that Diego made between the original system and step 1.

2. How do you know the system of equations in step 1 is equivalent to the original system?

3. Describe the move that Diego made between step 1 and step 2.

4. How do you know the system of equations in step 2 is equivalent to the ones in step 1 (and to the original system)

5. Write another set of equivalent systems (different from Diego's first two steps) that will allow one variable to be eliminated and enable you to solve the original system. Be prepared to describe the moves you make to create each new system and to explain why each one has the same solution as the original system.

6. Use your equivalent systems to solve the original system. Then, check your solution by substituting the pair of values into the original system.

2.6.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

Solve each system of equations to practice the skills you have learned in this lesson.

1. $\begin{cases} -2x + 5y = -42 \\ 7x + 2y = 30 \end{cases}$

a. $(6, -6)$
b. $(1, -8)$
c. $(11, -4)$
d. $(0, 15)$

2. $\begin{cases} 6x - 5y = -34 \\ 2x + 6y = 4 \end{cases}$

a. $(-9, -4)$
b. $(-4, 2)$
c. $(1, 8)$
d. $(2, 0)$

3. $\begin{cases} 5x - y = -2.6 \\ -4x - 6y = 1.4 \end{cases}$

a. $(-0.5, 0.1)$
b. $(-3, -12)$
c. $(0.5, 0.1)$
d. $(7, -4.9)$

4. $\begin{cases} 7x - 2y = 3 \\ 4x + 5y = 3.25 \end{cases}$

a. $(3, 9)$
b. $(-1, -5)$
c. $(0.5, 0.25)$
d. $(1, 2)$

5. $\begin{cases} 5x + 9y = 16 \\ x + 2y = 4 \end{cases}$

a. $(8, -2)$
b. $(-8, 6)$
c. $(3.2, 0)$
d. $(-4, 4)$

6. $\begin{cases} \frac{3}{2}x - \frac{1}{4}y = 5 \\ \frac{1}{3}x + \frac{1}{6}y = 2 \end{cases}$

a. $(-4, -5)$
b. $(-\frac{1}{5}, \frac{2}{3})$
c. $(1, -5)$
d. $(4, 4)$

7. $\begin{cases} \frac{1}{3}x + \frac{1}{9}y = 2 \\ -\frac{1}{2}x + \frac{4}{5}y = -3 \end{cases}$

a. $(-2, -5)$
b. $(8, -5)$
c. $(7, -3)$
d. $(6, 0)$

8. $\begin{cases} 6x - 8y = \frac{16}{5} \\ 3x + 2y = \frac{14}{5} \end{cases}$

a. $(2, 1.1)$
b. $(0.8, 0.2)$
c. $(0, -0.4)$
d. $(4, -4.6)$

9. $\begin{cases} 5x - 2y = 2 \\ 7x - 4y = 2.5 \end{cases}$

a. $(0.5, 0.25)$
b. $(0, -1)$
c. $(2, 4)$
d. $(2.4, 5)$

10. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = \frac{1}{3} \\ \frac{1}{3}x + \frac{1}{4}y = -\frac{1}{8} \end{cases}$

a. $(12, -17)$
b. $(6, -8)$
c. $(0, 1)$
d. $(9, -12.5)$

2.6.6 Lesson Summary

In this lesson, you learned how to:

- Explain how multiplying equations by a factor can help us solve systems of linear equations.

Here are the **activities** that helped you reach those goals:

- 2.6.1: Multiplying Equations by a Number
  - In this activity, you use technology to graph a system of equations. Then you created new equations by multiplying any of the original equations by the same number. After graphing these new equations, you learned that multiplying an entire equation by a number does not change its solution.
- 2.6.2: Writing a New System to Solve a Given System
  - In this activity, you reviewed the process of multiplying one equation in a system of equations by a single number to form a new system of equations in order to solve by elimination. By solving the new system of equations, you showed that the solution to the system of equations was not changed from the original system of equations.
  - 2.6.2: Self Check
  - 2.6.2: Additional Resources
- 2.6.3: Finding Solutions to Unordered Sets of Equivalent Systems
  - In this activity, you used your knowledge of solving systems of equations to correctly order the steps used when solving a system by elimination when multiplication is needed in both original equations.
  - 2.6.3: Self Check
  - 2.6.3: Additional Resources
- 2.6.4: Building Equivalent Systems
  - In this activity, you showed that there are multiple ways to solve a system of equations using elimination. Since any one of the original equations can be multiplied by any number without changing the validity of the equation, there are many ways to cancel out a variable when combining the equations in the system.

After these activities, you completed the following **practice**:

- 2.6.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

2.7 SYSTEMS OF LINEAR EQUATIONS AND THEIR SOLUTIONS

2.7.0 Lesson Overview

In this lesson, you will revisit the idea that not all systems of linear equations have a single solution. Next, you will investigate different ways to determine the number of solutions to a system of linear equations.

When you finish this lesson, you will be able to:

- Determine how many solutions a system of equations could have.

Here are the **activities** that will help you reach those goals:

- 2.7.1: Systems of Linear Equations with Infinitely Many Solutions
- 2.7.2: Systems of Linear Equations with No Solution
  - 2.7.2: Self Check
  - 2.7.2: Additional Resources
- 2.7.3: Sorting Systems of Equations Based on Number of Solutions
  - 2.7.3: Self Check
  - 2.7.3: Additional Resources
- 2.7.4: Writing Consistent and Inconsistent Systems of Equations

After that, you'll **practice and review**.

- 2.7.5: Practice
- 2.7.6: Lesson Summary

2.7.1 Systems of Linear Equations with Infinitely Many Solutions

Warm Up

Andre is trying to solve this system of equations:

$$\begin{cases} x + y = 3 \\ 4x = 12 - 4y \end{cases}$$

Looking at the first equation, he thought, "The solution to the system is a pair of numbers that add up to 3. I wonder which two numbers they are."

1. Choose any two numbers that add up to 3. Let the first one be the x-value and the second one be the y-value.
2. The pair of values you chose is a solution to the first equation. Check if it is also a solution to the second equation. Then, pause for a brief discussion with your group.
3. How many solutions does the system have? Use what you know about equations and solving systems to show that you are right.

2.7.2 Systems of Linear Equations with No Solution

Activity

A recreation center is offering special prices on its pool passes and gym memberships for the summer. On the first day of the offering, a family paid $96 for 4 pool passes and 2 gym memberships. Later that day, an individual bought a pool pass for herself, a pool pass for a friend, and 1 gym membership. She paid $72.

1. Write a system of equations that represents the relationships between pool passes, gym memberships, and the costs. Be sure to state what each variable represents.
2. Find the price of a pool pass and the price of a gym membership by solving the system algebraically. Be prepared to show your reasoning.
3. Use the graphing tool or technology outside the course. Graph the equations in the system using the Desmos tool below.
   Access multimedia content (http://openstax.org/books/algebra-1/pages/2-7-2-systems-of-linear-equations-with-no-solution)
4. Then make 1 – 2 observations about your graphs.
5. How do your observations about the graph of this system relate to the fact that the system was determined to not have a solution?

Video: Analyzing Systems of Linear Equations with No Solution

Watch the following video to learn more about systems that have no solution.

Analyzing Systems of Linear Equations with No Solution (https://openstax.org/r/no-solution-systems)

Self Check

Solve the system of equations.

$$\begin{cases} -0.1x - 0.2y = 0.6 \\ -5x - 10y = 1 \end{cases}$$

Additional Resources

Solving an Inconsistent System of Equations

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y-intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12 = 0$.

Solve the following system of equations.

$x = 9 - 2y$

$x + 2y = 13$

Solution

We can approach this problem in two ways. Because one equation is already solved for x, the most obvious step is to use substitution.

$$\begin{aligned} x + 2y &= 13 \\ (9 - 2y) + 2y &= 13 \\ 9 + 0y &= 13 \\ 9 &= 13 \end{aligned}$$

Clearly, this statement is a contradiction because $9 \neq 13$. Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows:

$x = 9 - 2y$

$2y = -x + 9x$

$y = -\frac{1}{2}x + \frac{9}{2}$

We then convert the second equation expressed to slope-intercept form.

$$\begin{aligned} x+2y &= 13 \\ 2y &= -x+13 \\ y &= -\tfrac{1}{2}x+\tfrac{13}{2} \end{aligned}$$

Comparing the equations, we see that they have the same slope but different y-intercepts. Therefore, the lines are parallel and do not intersect.

$y=-\frac{1}{2}x+\frac{9}{2}$

$y=-\frac{1}{2}x+\frac{13}{2}$

TRY IT **Solving an Inconsistent System of Equations**

Solve the following system of equations. If there is no solution, explain how you know.

$9x+3y=-24$

$6x+2y=14$

2.7.3 Sorting Systems of Equations Based on Number of Solutions

Activity

Your teacher will give you a set of cards (https://openstax.org/r/set-cards). Each card contains a system of equations.

Sort the systems into three groups based on the number of solutions each system has. Be prepared to explain how you know where each system belongs.

Are you ready for more?

Extending Your Thinking

1. In the cards, for each system with no solution, change a single constant term so that there are infinitely many solutions to the system.
2. For each system with infinitely many solutions, change a single constant term so that there are no solutions to the system.
3. Explain why in these situations it is impossible to change a single constant term so that there is exactly one solution to the system.

Self Check

Which of the following is true of the following system of equations?

$$\begin{cases} 7x+6y=2 \\ -28x-24y=-8 \end{cases}$$

a. If you write both equations in slope-intercept form, they have the same slope.
b. equations in slope-intercept form, they have different y-intercepts.
c. The system of equations has exactly one solution.
d. The system of equations has no solutions.

Additional Resources

Classifying and Rewriting Systems of Equations

Find a solution to the system of equations using the addition method.

$x+3y=2$

$3x+9y=6$

Solution

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating x. If we multiply both sides of the first equation by -3, then we will be able to eliminate the x-variable.

Now add the equations:

$$\begin{aligned} x+3y &= 2 \\ (-3)(x+3y) &= (-3)(2) \\ -3x-9y &= -6 \end{aligned}$$

$$\begin{aligned} -3x-9y &= -6 \\ +3x+9y &= 6 \\ 0 &= 0 \end{aligned}$$

We can see that there will be an infinite number of situations that satisfy both equations.

TRY IT **Classifying and Rewriting Systems of Equations**

Change a single constant term in the system from above so that there are no solutions.

$$\begin{aligned} x+3y &= 2 \\ 3x+9y &= 6 \end{aligned}$$

Show that you were successful in changing the system to have no solution.

2.7.4 Writing Consistent and Inconsistent Systems of Equations

Cool Down

Here is an equation: $5x-2y=10$.

Create a second equation that would make a system of equations with:

1. One solution
2. No solutions
3. Infinitely many solutions

2.7.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 8, solve each system of equations to determine if there is one solution, no solution, or infinitely many solutions.

1. $\begin{cases} 5x - 3y = 7 \\ x - 0.6y = 1.4 \end{cases}$

a. infinitely many solutions
b. no solution
c. one solution

2. $\begin{cases} -x + 2y = -1 \\ 5x - 10y = 6 \end{cases}$

a. infinitely many solutions
b. no solution
c. one solution

3. $\begin{cases} 7x + 6y = 2 \\ -28x - 24y = -8 \end{cases}$

a. infinitely many solutions
b. no solution
c. one solution

4. $\begin{cases} -0.2x + 0.4y = 0.6 \\ x - 2y = -3 \end{cases}$

a. infinitely many solutions
b. no solution
c. one solution

5. $\begin{cases} -0.1x + 0.2y = 0.6 \\ 5x - 10y = 1 \end{cases}$

a. infinitely many solutions
b. no solution
c. one solution

6. $\begin{cases} 3x + 6y = 11 \\ 2x + 4y = 9 \end{cases}$

a. infinitely many solutions
b. no solution
c. one solution

7. $\begin{cases} -x + 2y = 4 \\ 2x - 4y = 1 \end{cases}$

a. infinitely many solutions
b. no solution
c. one solution

8. $\begin{cases} 3x - 2y = 5 \\ -9x + 6y = -15 \end{cases}$

a. infinitely many solutions
b. no solution
c. one solution

9. Which system of equations has infinitely many solutions?

a. $\begin{cases} -4x + 3y = -12 \\ 8x + 6y = -24 \end{cases}$

b. $\begin{cases} -4x + 3y = 12 \\ 8x - 6y = 24 \end{cases}$

c. $\begin{cases} 4x + 3y = 12 \\ 8x + 6y = 12 \end{cases}$

d. $\begin{cases} 4x + 3y = 12 \\ 8x + 6y = -24 \end{cases}$

10. Which system of equations has no solution?

a. $\begin{cases} 9x + 21y = 270 \\ 3x + 7y = 90 \end{cases}$

b. $\begin{cases} 3x - 7y = 15 \\ 3x + 7y = 40 \end{cases}$

c. $\begin{cases} 3x + 7y = 15 \\ 6x + 14y = 30 \end{cases}$

d. $\begin{cases} 3x + 7y = 15 \\ 3x + 7y = 40 \end{cases}$

2.7.6 Lesson Summary

In this lesson, you learned how to:

- Determine how many solutions a system of equations could have.

Here are the **activities** that helped you reach those goals:

- 2.7.1: Systems of Linear Equations with Infinitely Many Solutions
  - In this activity, you analyzed a system of equations that has infinitely many solutions. When solving by graphing, the graphs of the equations are the same line. When solving by substitution or elimination, you end up with an equation, such as $4x = 4x$, that is always true no matter what value is used.
- 2.7.2: Systems of Linear Equations with No Solution
  - In this activity, you analyzed a system of equations that has no solution. This means there are no values that make both equations in the system true. When solving by graphing, the graphs of the equations are parallel lines that never intersect. When solving by substitution or elimination, you end up with an equation, such as $0 = 24$, that is never true.
  - 2.7.2: Self Check
  - 2.7.2: Additional Resources
- 2.7.3: Sorting Systems of Equations Based on Number of Solutions
  - In this activity, you solved many different systems of equations in order to sort which systems had one solution, infinitely many solutions, or no solution.
  - 2.7.3: Self Check
  - 2.7.3: Additional Resources
- 2.7.4: Writing Consistent and Inconsistent Systems of Equations
  - In this activity, you wrote new systems of equations to have one solution, infinitely many solutions, or no solution. By writing new systems of equations from each category, you demonstrated an awareness of the characteristics of each type of system.

After these activities, you completed the following **practice**:

- 2.7.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

2.8 REPRESENTING SITUATIONS WITH INEQUALITIES

2.8.0 Lesson Overview

Prior to this point, you have worked primarily with equations in one and two variables. In this lesson, you will shift your attention to inequalities. You will interpret, write, and find solutions to inequalities in one variable.

When you finish this lesson, you will be able to:

- Write inequalities that represent the constraints in a situation.

Here are the **activities** that will help you reach those goals:

- 2.8.1: Understanding Inequality Symbols
- 2.8.2: Identifying Constraints in Inequalities
  - 2.8.2: Self Check
  - 2.8.2: Additional Resources
- 2.8.3: Writing Inequalities to Represent Constraints
  - 2.8.3: Self Check
  - 2.8.3: Additional Resources

After that, you'll **practice and review**.

- 2.8.5: Practice
- 2.8.6: Lesson Summary

2.8.1 Understanding Inequality Symbols

Warm Up

For questions 1 – 3, match each inequality to the meaning of a symbol within it.

1. $h > 50$

a. less than or equal to

b. greater than

c. greater than or equal to

2. $h \leq 20$

a. less than or equal to

b. greater than

c. greater than or equal to

3. $30 \geq h$

a. less than or equal to

b. greater than

c. greater than or equal to

4. Is 25 a solution to any of the inequalities? Which one(s)?

5. Is 40 a solution to any of the inequalities? Which one(s)?

6. Is 30 a solution to any of the inequalities? Which one(s)?

2.8.2 Identifying Constraints in Inequalities

Activity

Seniors in a high school student council are trying to develop a budget for the Senior Dance. The revenue from ticket sales needs to cover the cost of the meals and entertainment and also make a profit of at least $200 to be contributed to the school.

Here is some information they have gathered:

- Last year, 120 people attended. It was a success and is expected to be even bigger this year. Up to 200 people will attend.

- There needs to be at least 1 chaperone for every 20 students.
- The ticket price cannot exceed $20 per person.

Here are some inequalities the seniors wrote about the situation. Each letter stands for one quantity in the situation. Determine what is meant by each letter in 1 – 4.

1. $t \leq 20$

2. $120 \leq p \leq 200$

3. $pt - m \geq 200$

4. $c \geq \frac{p}{20}$

Are you ready for more?

Extending Your Thinking

Kiran says we should add the constraint $t \geq 0$.

1. What is the reasoning behind this constraint?

2. What other "natural constraint" like this should be added?

Self Check

A zookeeper feeds the tigers a meal of rabbits and chickens. The average rabbit weighs 4 pounds, and the average chicken weighs 5 pounds. Each tiger must eat at least 15 pounds of meat per day.The zookeeper writes the inequality $4x + 5y \geq 15$ to represent a tiger's daily diet. What is meant by the letter y in the inequality above?

Additional Resources

Translate to an Inequality

To translate English sentences into inequalities, we need to recognize the phrases that indicate the inequality. Some words are easy, like "more than" and "less than." But others are not as obvious. The table shows some common phrases that indicate inequalities.

$>$	$\geq$	$<$	$\leq$
is greater than is more than is larger than exceeds	is greater than or equal to is at least is no less than is the minimum	is less than is smaller than has fewer than is lower than	is less than or equal to is at most is no more than is the maximum

EXAMPLE

Translate each phrase to an inequality.

Twenty-seven less than $\underbrace{x \text{ is at least}}$ 48 $x - 17 \geq 48$

"is at least" means $\geq$

5 more than $\underbrace{z \text{ is lower than}}$ 52 $z + 5 < 52$

"is at least" means $\geq$

TRY IT **Translate to an Inequality**

Translate the following sentences into inequalities.

1. Nineteen less than p is no less than 47.

2. Four more than a is at most 15.

2.8.3 Writing Inequalities to Represent Constraints

Activity

An elevator car in a skyscraper can hold at most 15 people. For safety reasons, each car can carry a maximum of 1,500 kg. On average, an adult weighs 70 kg and a child weighs 35 kg. Assume that each person carries 4 kg of gear with them.

1. Write as many equations and inequalities as you can think of to represent the constraints in this situation. Be sure to specify the meaning of any letters that you use. (Avoid using the letters z, m, or g.)
2. After you have finished the following tasks, explain the adjustments you made to the equations and inequalities so that they are communicated more clearly.
   Trade your work with a partner and read each other's equations and inequalities.
   - Explain to your partner what you think their statements mean, and listen to their explanation of yours.
   - Make adjustments to your equations and inequalities so that they are communicated more clearly.
3. Rewrite your equations and inequalities so that they would work for a different building where:
   - an elevator car can hold at most z people
   - each car can carry a maximum of m kilograms
   - each person carries d kg of gear

Video: Writing Inequalities to Represent Constraints Solution

Watch the following video to further explain the solution to the activity you just completed.

Writing Inequalities to Represent Constraints Solution (https://openstax.org/r/constraints-inequalities)

Self Check

Serena is running a juice shop. Oranges cost $2 per pound. She can spend at most $64 per day on oranges. If p represents the number of pounds of oranges Serena can buy per day, which of the following inequalities is true?

$2p \geq 64$
$2p \leq 64$
$2p > 64$
$2p < 64$

Additional Resources

Applications with Linear Inequalities

Many real-life situations require us to solve inequalities. The method we will use to solve applications with linear inequalities is very much like the one we used when we solved applications with equations.

We will read the problem and make sure all the words are understood. Next, we will identify what we are looking for and assign a variable to represent it. We will restate the problem in one sentence to make it easy to translate into an inequality.

EXAMPLE

Dawn won a mini-grant of $4,000 to buy tablet computers for her classroom. The tablets she would like to buy cost $254.12 each, including tax and delivery. Write an inequality to represent the maximum number of tablets Dawn can buy.

Solution

Step 1 - Read the problem.

Step 2 - Identify what you are looking for. the maximum number of tablets Dawn can buy

Step 3 - Name what you are looking for.

Choose a variable to represent that quantity. Let $n =$ the number of tablets.

Step 4 - Translate. Write a sentence that gives the information to find it.

$254.12 times the number of tablets is no more than $4,000. Translate into an inequality. $254.12n \leq 4000$

TRY IT **Applications with Linear Inequalities**

1. Angie has $20 to spend on juice boxes for her son's preschool picnic. Each pack of juice boxes costs $2.63. Write an inequality to represent the maximum number of packs, p, she can buy.
2. Daniel wants to surprise his girlfriend with a birthday party at her favorite restaurant. It will cost $42.75 per person for dinner, including tip and tax. His budget for the party is $500. Write an inequality to represent the maximum number of people, n, Daniel can have at the party.

2.8.4 Writing More Inequalities

Cool Down

Han has a budget of $25 to buy grapes. Write inequalities to represent the number of pounds of grapes that Han could buy in each situation:

1. Grapes cost $1.99 per pound.
2. Grapes cost $2.49 per pound.
3. Grapes cost $\$c$ per pound.

2.8.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 2, use the following scenario:

> A runner is training for a marathon. She runs more than 4 days each week. Each day, she runs between 12 and 15 miles. The marathon is 3 months away at the most.

1. What does the inequality $d > 4$ represent?

a. the number of days the runner runs each week
b. the number of months away the marathon is
c. the number of miles the runner runs each day
d. the number of miles in a marathon

2. What does the inequality $12 \leq m \leq 15$ represent?

a. the number of days the runner runs each week
b. the number of months away the marathon is
c. the number of miles the runner runs each day
d. the number of miles in a marathon

For questions 3 – 8, write an inequality to represent the constraint given.

3. There are less than 4 patients, p, in the waiting room.

a. $p \geq 4$
b. $p \leq 4$
c. $p > 4$
d. $p < 4$

4. There are no more than 6 cats, c, in the yard.

a. $c \geq 6$
b. $c < 6$
c. $c \leq 6$
d. $c > 6$

5. There are no less than 7 markers, m, in the drawer.

a. $m \geq 7$
b. $m < 7$
c. $m \leq 7$
d. $m > 7$

6. The basket has between 12 and 19 eggs, e.

a. $12 \leq e \leq 19$
b. $e \leq 19$
c. $12 \geq e$ and $e \geq 19$
d. $12 > e$ and $e > 19$

7. The total sales of tickets to a show is more than \$68. An adult ticket, a, costs \$8 and a children's ticket, c, is \$3.

a. $8a + 3c > 68$
b. $8a + 3c < 68$
c. $8a + 3c \geq 68$
d. $8a + 3c \leq 68$

8. There needs to be at least 1 pound of clay, c, for every 4 students, s.

a. $c \leq \frac{s}{4}$

b. $c < \frac{s}{4}$
c. $c > \frac{s}{4}$
d. $c \geq \frac{s}{4}$

For questions 9 – 11, use the following passage:

A doctor sees between 7 and 12 patients each day. On Mondays and Tuesdays, the appointment times are 15 minutes. On Wednesdays and Thursdays, they are 30 minutes long. On Fridays, they are one hour long. The doctor works for no more than 8 hours a day. Here are some inequalities that represent this situation.

$0.25 \leq y \leq 1 \qquad 7 \leq x \leq 12 \qquad xy \leq 8$

9. What does the variable x represent?

a. the number of patients the doctor sees in a week
b. the total time the doctor spends working with patients each day
c. the part of an hour that each appointment lasts
d. the number of patients the doctor sees each day

10. What does the variable y represent?

a. the number of patients the doctor sees in a week
b. the total time the doctor spends working with patients each day
c. the part of an hour that each appointment lasts
d. the number of patients the doctor sees each day

11. What does the expression xy in the last inequality mean in this situation?

a. the number of patients the doctor sees in a week
b. the total time the doctor spends working with patients each day
c. the part of an hour that each appointment lasts
d. the number of patients the doctor sees each day

For questions 12 – 14, use the following scenario:

Arnav wants to build a dog house. He makes a list of the materials needed:

- At least 60 square feet of plywood for the surfaces
- At least 36 feet of wood planks for the frame of the dog house
- Between 1 and 2 quarts of paint

Arnav's budget is $65. Plywood costs $0.70 per square foot, planks of wood cost $0.10 per foot, and paint costs $8 per quart.

12. Which inequality represents the square feet of plywood needed?

a. $1 \leq p \leq 2$
b. $p \geq 36$
c. $p \geq 60$

13. Which inequality represents the number of wood planks needed?

a. $w \geq 36$
b. $1 \leq w \leq 2$
c. $w \geq 60$

14. Which inequality represents the number of quarts of paint needed?

a. $q \geq 36$
b. $1 \leq q \leq 2$
c. $q \geq 60$

2.8.6 Lesson Summary

In this lesson, you learned how to:

- Write inequalities that represent the constraints in a situation.

Here are the **activities** that helped you reach those goals:

- 2.8.1: Understanding Inequality Symbols
  - In this activity, you recalled the meaning of inequality symbols (<, >, ≤, and ≥) and the meaning of "solutions to an inequality." You were reminded that an inequality in one variable can have a range of values that make the statement true. You also explored the value that is at the boundary of an inequality and consider whether it is or isn't a solution to an inequality.
- 2.8.2: Identifying Constraints in Inequalities
  - In this activity, you were prompted to interpret several inequalities that represent the constraints in a situation. To explain what the letters in the inequalities mean in the given context, you could not simply match the numbers in the verbal descriptions and those in the inequalities. You worked to attend carefully to the symbols and any operations, and reason both quantitatively and abstractly.
  - 2.8.2: Self Check
  - 2.8.2: Additional Resources
- 2.8.3: Writing Inequalities to Represent Constraints
  - Previously, you interpreted given inequalities and made sense of them in terms of a situation. In this activity, you wrote inequalities to represent the constraints in a situation. You identified key quantities and relationships, and thought about ways to represent them.
  - 2.8.3: Self Check
  - 2.8.3: Additional Resources
- 2.8.4: Writing More Inequalities
- In this activity, you wrote inequalities to represent the number of pounds of grapes that a person could buy in a given situation.

After these activities, you completed the following **practice**:

- 2.8.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

2.9 SOLUTIONS TO INEQUALITIES

2.9.0 Lesson Overview

In this lesson, you will investigate different ways to find the solution set to an inequality—by reasoning about the quantities and relationships in context; by guessing some values, substituting them into the inequality, and checking them to see if you make the inequality true; and by first solving a related equation in one variable.

When you finish this lesson, you will be able to:

- Graph the solution to an inequality in one variable.
- Solve one-variable inequalities.
- Interpret the solutions to one-variable inequalities in terms of the situation.

Here are the **activities** that will help you reach those goals:

- 2.9.1: Writing Solutions to Inequalities
- 2.9.2: Graphing Inequalities
  - 2.9.2: Self Check
  - 2.9.2: Additional Resources
- 2.9.3: Understanding the Meaning of an Inequality
  - 2.9.3: Self Check
  - 2.9.3: Additional Resources
- 2.9.4: Making Sense of Inequalities and Their Solutions
- 2.9.5: Comparing Equality and Inequality
  - 2.9.5: Self Check
  - 2.9.5: Additional Resources
- 2.9.6 Using Technology to Visualize Inequality Solutions
- 2.9.7 Identifying Inequality Solutions on a Graph

After that, you'll **practice and review**.

- 2.9.8: Practice
- 2.9.9: Summary

2.9.1 Writing Solutions to Inequalities

Warm Up

1. Write some solutions to the inequality $y \leq 9.2$. Be prepared to explain what makes a value a solution to this inequality.
2. Write one solution to the inequality $7(3 - x) > 14$. Be prepared to explain your reasoning.

2.9.2 Graphing Inequalities

Activity

Work with a partner to match each of the following inequalities to its correct number line.

1. 

a. x<2
b. x≥2
c. x>2
d. x≤2

2. 

a. x<2
b. x≥2
c. x>2
d. x≤2

3. 

a. x<2
b. x≥2
c. x>2
d. x≤2

4. 

a. x<2
b. x≥2
c. x>2
d. x≤2

Why Should I Care?



Nafy Flatley has worked to achieve her goal of bringing the food of Senegal to the world through her snack mix company, TERANGA.

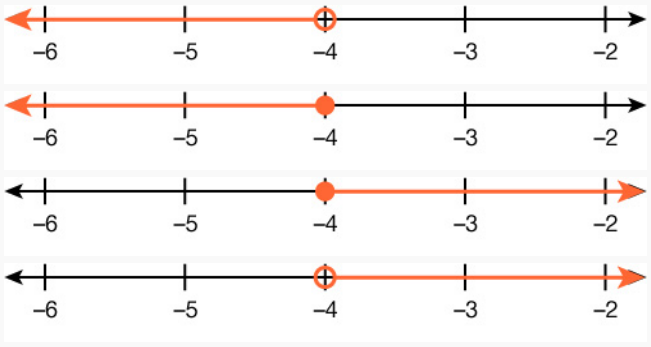
Coming up with snack mixes is not as simple as just combining ingredients. They have a lot of things to consider: What ingredients will they use? How much does each ingredient cost? What are the nutrititional values of each ingredient? How much does each ingredient weigh?

If they can translate each question in a linear inequality, then they can use math to find mixes that meet their goals.

Self Check

Which number line is represented by the inequality $x \geq -4$?

a.
b.
c.
d.



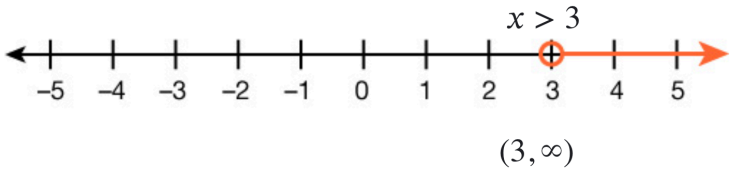
Additional Resources

Graphing Linear Inequalities

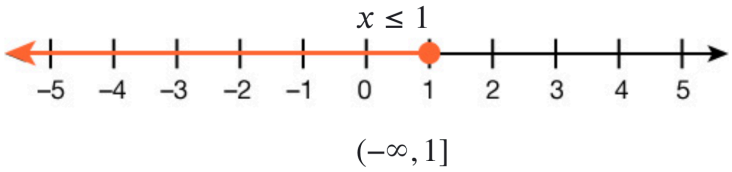
What number would make the inequality $x > 3$ true? Are you thinking, "x could be 4"? That's correct, but x could be 6, too, or 37, or even 3.001. Any number greater than 3 is a solution to the inequality $x > 3$. We show all the solutions to the inequality $x > 3$ on the number line by shading in all the numbers to the right of 3, to show that all numbers greater than 3 are solutions. Because the number 3 itself is not a solution, we put an open circle at 3.

We can also represent inequalities using interval notation. There is no upper end to the solution to this inequality. In interval notation, we express $x > 3$ as $(3, \infty)$. The symbol ∞ is read as "infinity." It is not an actual number.

The figure below shows both the number line and the interval notation.

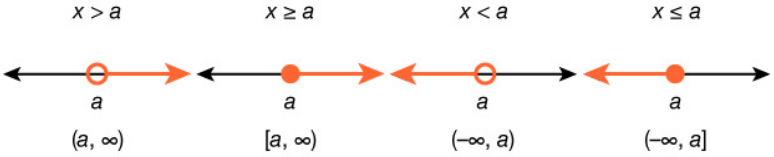


The inequality $x \leq 1$ means all numbers less than or equal to 1. Here we need to show that 1 is a solution, too. We do that by putting a closed circle at $x = 1$. We then shade in all the numbers to the left of 1, to show that all numbers less than 1 are solutions. See the figure below.



There is no lower end to those numbers. We write $x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as "negative infinity." The figure below shows both the number line and interval notation.

Inequalities, Number Lines, and Interval Notation



TRY IT **Graphing Linear Inequalities**

For questions 1 – 2 use the following inequality:

$$x > -3$$

1. Write the inequality on a number line.
2. Write the inequality in interval notation.

For questions 3 and 4 use the following inequality:

$$x \leq 5$$

3. Write the inequality in interval notation.
4. Write the inequality on a number line.

2.9.3 Understanding the Meaning of an Inequality

Activity

For questions 1 – 3, use the information about the orchards to answer each question.

A teacher is choosing between two options for a class field trip to an orchard. At each orchard, the same price applies to both chaperones and students.

- At Orchard A, admission costs $9 per person, and 3 chaperones are required.
- At Orchard B, the cost is $10 per person, but only 1 chaperone is required.



1. Which orchard would be cheaper to visit if the class has 8 students?
2. Which orchard would be cheaper to visit if the class has 12 students?
3. Which orchard would be cheaper to visit if the class has 30 students?

Use the following information to answer questions 4 – 7.

To help her compare the cost of her two options, the teacher first writes the equation $9(n+3) = 10(n+1)$, and then she writes the inequality $9(n+3) < 10(n+1)$.

4. What does n represent in each statement?
5. In this situation, what does the equation $9(n+3) = 10(n+1)$ mean?
6. What does the solution to the inequality $9(n+3) < 10(n+1)$ tell us?
7. The teacher needs a visual aid to show the school budgeting committee. Graph the solution set to the inequality on the number line. Be prepared to show or explain your reasoning.

Video: Understanding the Meaning of an Inequality

Watch the following video to learn more about the meaning of an inequality.

Understanding the Meaning of an Inequality (https://openstax.org/r/inequality-meaning)

Self Check

A water park charges $17 per entry plus a $40 fee for raft rentals. An amusement park charges a $21 entry fee.

For the number of entries, e, which inequality represents the total cost of the water park being less than the total cost of the amusement park?

a. $17e + 40 > 21e$
b. $17e < 21e + 40$
c. $17e + 40 < 21e$
d. $17e - 40 < 21e$

Additional Resources

Translating Situations into Inequalities

Many real-life situations require us to use inequalities. Translating the details of these situations into inequalities is the first step in understanding them. Let's look at how we can translate these situations into mathematical sentences.

EXAMPLE 1

Imani won a mini-grant of $4,000 to buy tablet computers for her classroom. The tablets she would like to buy cost $254.12 each, including tax and delivery. She can only spend up to the amount of the mini-grant.

Choose a variable to represent the quantity of tablet computers:

Let n = the number of tablets.

Translate: Write a sentence that gives the information provided.

$254.12 times the number of tablets is no more than $4,000.

Translate into an inequality:

$254.12n \leq 4000$

Imani's situation is represented by the inequality $254.12n \leq 4000$.

EXAMPLE 2

Taleisha's phone plan costs her $28.80 a month plus $0.20 per text message. Her monthly bill can be no more than $50.

Choose a variable to represent the quantity of text messages:

Let t = the number of text messages.

Translate: Write a sentence that gives the information provided.

$28.80 plus $0.20 times the number of text messages is less than or equal to $50.

Translate into an inequality:

$28.80 + 0.20t \leq 50$

Taleisha's situation is represented by the inequality $28.80 + 0.20t \leq 50$.

TRY IT **Translating Situations into Inequalities**

Read each problem below and write an inequality to represent the situation.

1. Angie has at most $20 to spend on juice boxes for her son's preschool picnic. Each pack of juice boxes costs $2.63. She will also buy one bag of chips for $3.99.
2. Jose wants to surprise his girlfriend with a birthday party at her favorite restaurant. It will cost $42.75 per person for dinner, including tip and tax. He must also pay $5 for parking. His budget for the party is less than $500.

2.9.4 Making Sense of Inequalities and Their Solutions

Activity

Use this information to answer questions 1 – 2.

To help pay for his tuition, a college student plans to work in the evenings and on weekends. He has been offered two part-time jobs: working in the guest-services department at a hotel and waiting tables at a popular restaurant.

- The job at the hotel pays $18 an hour and offers $33 in transportation allowance per month.
- The job at the restaurant pays $7.50 an hour plus tips. The entire wait staff typically collects about $50 in tips each hour. Tips are divided equally among the 4 wait staff members who share a shift.

The equation $7.50h + \frac{50}{4}h = 18h + 33$ represents a possible constraint about a situation for 1 month.

1. Solve the equation and check your solution.
2. Add a scale on the number line so that the point marked with a circle represents the solution to the equation you found in question 1.

3. Which job pays better if he works fewer hours than the solution you found in question 1?
4. Explain how you know.
5. Which job pays better if he works more hours than the solution you found in question 1?
6. Explain how you know.
7. Draw a number line, including a scale, so that the point marked with a circle represents the solution to the inequality: $7.50h + \frac{50}{4}h < 18h + 33$.

8. Draw a number line, including a scale, so that the point marked with a circle represents the solution to the inequality: $7.50h + \frac{50}{4}h > 18h + 33$.

9. Explain how you know.

2.9.5 Comparing Equality and Inequality

Activity

Use this equation for questions 1 – 2:

$$-\frac{4(x+3)}{5} = 4x - 12$$

1. Solve this equation. What is the value of x?
2. Check your solution.

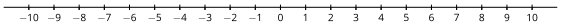
Use this inequality for questions 3 and 6:

$$-\frac{4(x+3)}{5} \leq 4x - 12$$

3. Choose a couple of values less than 2 for x. Are they solutions to the inequality?
4. Choose a couple of values greater than 2 for x. Are they solutions to the inequality?

5. Choose 2 for x. Is it a solution to the inequality?

6. Graph the solution to the inequality on the number line.



Are you ready for more?

Extending Your Thinking

Here is a different type of inequality: $x^2 \leq 4$.

1. Which of the following values solve the inequality: 1, 3 or -3?

2. Describe all solutions to this inequality. (If you like, you can graph the solutions on a number line.)

3. Describe all solutions to the inequality $x^2 \geq 9$. Test several numbers to make sure your answer is correct.

Self Check

Which number line represents the solution to the inequality?

$\frac{1}{2}(-2z + 2) \geq 12$

a.



b.



c.



d.



Additional Resources

Comparing Equality and Inequality

We can use related equations to help find solutions to inequalities.

EXAMPLE 1

Here's an inequality with one variable on one side of the equation.

$2x - 3 < 5$

We can solve its related equation.

$$\begin{aligned} 2x - 3 &= 5 \\ 2x &= 8 \\ x &= 4 \end{aligned}$$

Now, let's verify which solutions verify the original inequality.

Step 1 - Verify if the solution to the related equation satisfies the original inequality.

Does $x = 4$ satisfy the inequality?

$2(4) - 3 \overset{?}{<} 5$

$5 \overset{?}{<} 5$

No, 4 is not a solution since it the $5 \not< 5$.

Step 2 - Verify if a value greater than the solution to the related equation satisfies the original inequality.

Let's try a number bigger than 4. For example, try the value of $x = 8$.

$2(8) - 3 \overset{?}{<} 5$

$13 \overset{?}{<} 5$

So, 8 does not satisfy the inequality since $13 \not< 5$. That means x is not a value greater than 4.

Step 3 - Verify if a value less than the solution to the related equation satisfies the original inequality.

Let's try a number smaller than 4, such as $x = 0$.

$2(0) - 3 \overset{?}{<} 5$

$-3 \overset{?}{<} 5$

Thus, 0 satisfies the inequality since $-3 < 5$. Since 0 is a number less than 4, this means the solution set is $x < 4$.

EXAMPLE 2

Let's solve an inequality with the same variable on both sides of the equation.

$3y - 7 \geq 8 - 2y$

We can solve its related equation.

$$\begin{aligned} 3y - 7 &= 8 - 2y \\ 5y &= 15 \\ y &= 3 \end{aligned}$$

Step 1 - Verify if the solution to the related equation satisfies the original inequality.

Does y = 3 satisfy the inequality?

$3(3) - 7 \overset{?}{\geq} 8 - 2(3)$

$2 \overset{?}{\geq} 2$

Yes, $y = 3$ satisfies the inequality because 2 is greater than or equal to 2.

Step 2 - Verify if a value greater than the solution to the related equation satisfies the original inequality.

Let's try a number greater than 3 such as $y = 5$.

$3(5) - 78 - 2(5)$

$1 \overset{?}{\geq} -2$

So, 5 satisfies the inequality since the value of 1 is greater than or equal to the value of -2.

Step 3 - Verify if a value less than the solution to the related equation satisfies the original inequality.

Let's try a number less than $y = 3$ such as 0.

$3(0) - 7 \overset{?}{\geq} 8 - 2(0)$

$-7 \overset{?}{\geq} 8$

So, $y = 0$ does not satisfy the inequality. This means that values less than $y = 3$ are not part of the solution set.

We verified that $y = 3$ and values greater than 3 satisfy the inequality $(y > 3)$, so the solution set is $y \geq 3$.

EXAMPLE 3

Look at this more complex inequality:

$-\frac{3x+7}{4} < -3x + 5$

We can solve its related equation $-\frac{3x+7}{4} = -3x + 5$.

After solving, we know that $x = 3$.

Does 3 satisfy the inequality?

$-\frac{3x+7}{4} < -3x + 5$

$-\frac{3(3)+7}{4} \stackrel{?}{<} -3(3) + 5$

$-4 \not< -4$

So, 3 is not a solution.

Let's try a number bigger than 3, such as 5.

$-\frac{3x+7}{4} < -3x + 5$

$-\frac{3(5)+7}{4} \stackrel{?}{<} -3(5) + 5$

$-\frac{11}{2} \not< -10$

So, 5 is not a solution

Let's try a number smaller than 3, such as 0.

$-\frac{3x+7}{4} < -3x + 5$

$-\frac{3(0)+7}{4} \stackrel{?}{<} -3(0) + 5$

$-\frac{7}{4} < 5$

So, 0 is a solution.

This means that the solution set is $x < 3$.

TRY IT **Comparing Equality and Inequality**

Use the related equation to find the solution set to the inequality.

$$\frac{5x-3}{2} < 3x - 4$$

1. What is the related equation?
2. What is the solution to the related equation?
3. What values did you test on the inequality?
4. What is the solution set to the inequality?

2.9.6 Using Technology to Visualize Inequality Solutions

Activity

Consider the inequality $-\frac{1}{2}x + 6 < 4x - 3$. Let's look at another way to find its solutions.

1. Use the graphing tool or technology outside the course. Graph $y = -\frac{1}{2}x + 6$and $y = 4x - 3$ on the same coordinate plane using the Desmos tool below.
   Access multimedia content (http://openstax.org/books/algebra-1/pages/2-9-6-using-technology-to-visualize-inequality-solutions)

Use your graphs to answer the following questions:

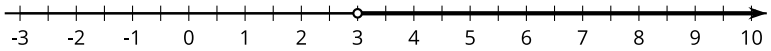
2. Find the values of $-\frac{1}{2}x + 6$ and $4x - 3$ when x is 1.
3. What value of x makes $-\frac{1}{2}x + 6$ and $4x - 3$ equal?
4. For what values of x is $-\frac{1}{2}x + 6$ less than $4x - 3$?
5. For what values of x is $-\frac{1}{2}x + 6$ greater than $4x - 3$?
6. What is the solution to the inequality $-\frac{1}{2}x + 6 < 4x - 3$? Be prepared to explain how you know.

2.9.7 Identifying Inequality Solutions on a Graph

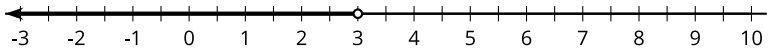
Cool Down

Which graph correctly shows the solution to the inequality $\frac{7x - 3}{9} \geq 8 - 2x$? Show or explain your reasoning.

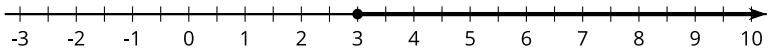
A



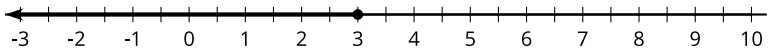
B



C



D



2.9.8 Practice

Complete the following questions to practice the skills you have learned in this lesson.

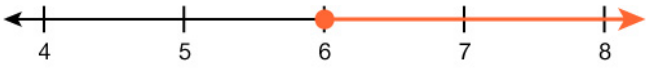
1. Here is an inequality: $\frac{7x+6}{2} \leq 3x + 2$

   Select **two** of the values that are a solution to the inequality.

   a. $x = -3$
   b. $x = -2$
   c. $x = -1$
   d. $x = 0$
   e. $x = 1$
   f. $x = 2$
   g. $x = 3$

2. Which number line is represented by the inequality $x \leq 6$?

   a.
   

   b.
   

   c.
   

   d.
   

For questions 3 – 4, use the following information:

Scooter rental company A charges $9 per hour plus a $11 initiation fee. Scooter rental company B charges $15 per hour.

3. A customer writes the following inequality to represent the situation: $9h + 11 > 15h$.
   Which of the following is true about the given inequality?

   a. h represents the number of scooters rented.
   b. Company A costs the same as Company B.
   c. Company A costs less than Company B.
   d. Company A costs more than Company B.

4. For which number of hours does Company B cost less?

   a. 4 hours
   b. 3 hours
   c. 2 hours
   d. 1 hour

For questions 5 – 7, use the following information:

Cycle Rentals offers 20 membership points for each day that a customer rents a bike. Bikes Unlimited offers 5 membership points for each day that a customer rents a bike plus 60 points for joining the membership club.

5. The equation $20x = 5x + 60$ represents a possible constraint about the situation. Solve for x. x = ___.

6. Which shop offers more membership points if you rent a bike for longer than the constraint you found in the previous question?

   a. Cycle Rentals
   b. Bikes Unlimited

7. Which number line represents the solution to the inequality, $20x > 5x + 60$, representing the situation when you earn more membership points at Cycle Rentals?

a.



b.



c.

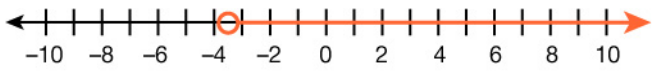


d.

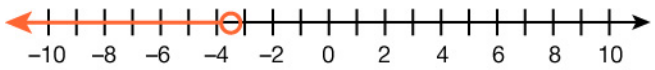


8. Which number line represents the solution to $5 + 8x < 3(2x + 4)$?

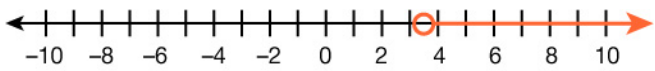
a.



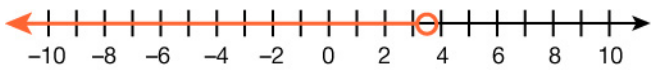
b.



c.



d.



9. Find the solution set to this inequality by solving the related equation: $2x - 3 > \frac{2x-5}{2}$.

a. $x \geq \frac{1}{2}$
b. $x \leq \frac{1}{2}$
c. $x > \frac{1}{2}$
d. $x < \frac{1}{2}$

10. Here is an inequality: $\frac{-10+x}{4} + 5 \geq \frac{7x-5}{3}$. What value of x will produce equality (or make the two sides equal)?

x = ____

2.9.9 Lesson Summary

In this lesson, you learned how to:

- Graph the solution to an inequality in one variable.
- Solve one-variable inequalities.
- Interpret the solutions to one-variable inequalities in terms of the situation.

Here are the **activities** that helped you reach those goals:

- 2.9.1: Writing Solutions to Inequalities
  - In this activity, you activated what you know about the solutions to an inequality and ways to find the solutions. For the first time, you referred to the solutions as the solution set of the inequality.
- 2.9.2: Graphing Inequalities
  - In this activity, you learned how to graph an inequality on a number line. You learned how to represent an inequality using set notation. You know that an open circle is used to represent $<$ and $>$, while a closed circle represents $\leq$ and $\geq$.
  - 2.9.2: Self Check
  - 2.9.2: Additional Resources
- 2.9.3: Understanding the Meaning of an Inequality
  - In this activity, you interpreted an inequality and its solution set in terms of a situation. Adding context can help you intuit why the solutions to an inequality are often represented by a ray on the number line. The activity also prompted you to think about the solutions to an inequality in terms of a related equation.
  - 2.9.3: Self Check
  - 2.9.3: Additional Resources
- 2.9.4: Making Sense of Inequalities and Their Solutions
  - In this activity, you received another opportunity to make sense of inequalities and their solutions in context. The context helped reinforce why it makes sense for the solutions to an inequality to be represented by a set of points on one side of a particular value on the number line.
- 2.9.5: Comparing Equality and Inequality
- ◦ In this activity, you further developed the idea that we can solve an inequality by first solving a related equation. Previously, you reasoned about the solutions to an inequality in context. Here, you transitioned to solving an inequality without a context.
  - 2.9.5: Self Check
  - 2.9.3: Additional Resources
- 2.9.6: Using Technology to Visualize Inequality Solutions
  - In this activity, you visualized an inequality in one variable in another way: by graphing the expressions on each side and comparing the values of the expressions at different values of the variable. This allows you to see the values at which the inequality is satisfied.
- 2.9.7: Identifying Inequality Solutions on a Graph
  - In this activity, you identified a graph that correctly showed the solution to an inequality.

After these activities, you completed the following **practice**:

- 2.9.8: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

2.10 WRITING AND SOLVING INEQUALITIES IN ONE VARIABLE

2.10.0 Lesson Overview

In this lesson, you will write and solve inequalities to answer questions about a situation. You will also practice finding the solution set to an inequality by reasoning about its composition and parts.

When you finish this lesson, you will be able to:

- Analyze the structure of an inequality in one variable to help determine if the solution is greater or less than the solution to the related equation.
- Write and solve inequalities to answer questions about a situation.

Here are the **activities** that will help you reach those goals:

- 2.10.1: Writing an Inequality to Represent a Constraint
- 2.10.2: Using Inequalities to Solve a Problem
  - 2.10.2: Self Check
  - 2.10.2: Additional Resources
- 2.10.3: Different Ways of Solving an Inequality
  - 2.10.3: Self Check
  - 2.10.3: Additional Resources
- 2.10.4: Matching Inequalities and Solutions
- 2.10.5: Solving Inequalities
  - 2.10.5: Self Check
  - 2.10.5: Additional Resources
- 2.10.6: Writing Inequalities to Represent a Situation

After that, you'll **practice and review**.

- 2.10.7: Practice
- 2.10.8: Lesson Summary

2.10.1 Writing an Inequality to Represent a Constraint

Warm Up

Kiran is getting dinner for his drama club on the evening of their final rehearsal. He plans to buy some prepared dishes from a supermarket.

- The budget for the dinner is $60.
- The prepared dishes are sold by the pound, at $5.29 per pound.
- He also plans to buy two large bottles of sparkling water at $2.49 each.

1. Represent the constraints in the situation mathematically. If you use variables, specify what each one means.

2. How many pounds of prepared dishes can Kiran buy? Be prepared to show your reasoning.

2.10.2 Using Inequalities to Solve a Problem

Activity

Use the following information to answer questions 1 – 2.

> Han is about to mow some lawns in his neighborhood. His lawn mower has a 5-gallon fuel tank, but Han is not sure how much gasoline is in the tank.
> He knows, however, that the lawn mower uses 0.4 gallon of gasoline per hour of mowing.



1. What are all the possible values for x, the number of hours Han can mow without refilling the lawn mower?
2. Write one or more inequalities to represent your response. Be prepared to show your reasoning.

Video: Using Inequalities to Solve a Problem

Watch the following video to learn more about how to solve a problem using inequalities.

Using Inequalities to Solve a Problem (https://openstax.org/r/inequalities-problem)

Self Check

Samantha is feeding the horses at her barn. She has 6 buckets of oats. If each horse eats $\frac{3}{4}$ bucket of oats, how many horses can she feed at most?

a. $x \leq 10$, Samantha can feed a maximum of 10 horses.
b. $x \leq 8$, Samantha can feed a maximum of 8 horses.
c. $x \leq 4$, Samantha can feed a maximum of 4 horses.
d. $x \leq 6$, Samantha can feed a maximum of 6 horses.

Additional Resources

Using Inequalities to Solve a Problem

Let's look at an example where we might be able to solve a problem using inequalities.

A school is repainting its classroom walls. The school has at most 30 gallons of paint. The painter needs $\frac{5}{2}$ gallons of paint for each classroom. How many classrooms can the school paint at most?

We know that the answer is greater than or equal to 0 classrooms, so $x \geq 0$.

We also know that each classroom requires $\frac{5}{2}$ or 2.5 gallons of paint.

Let's take a guess at how many classrooms and check if we are close to the solution. How many gallons of paint would we need to paint 10 classrooms?

$10 \times 2.5 = 25$, and since $25 < 30$, it could be possible for the school to paint 10 classrooms.

We can try 11 classrooms.

$11 \times 2.5 = 27.5$, and since $27.5 < 30$, it could be possible for the school to paint 11 classrooms.

We can try 12 classrooms.

$12 \times 2.5 = 30$, and since $30 = 30$, the most classrooms the school could paint is 12 classrooms.

TRY IT **Using Inequalities to Solve a Problem**

Use the following information for questions 1 – 2.

While working in a car factory, a worker can use a drill for as long as the batteries last. The worker has at most 9 fully charged batteries. Each car the worker helps assemble uses $\frac{3}{4}$ of a full battery.

1. What are all of the possible values for c, the number of cars the work can help assemble? Write one or more inequalities to represent your response.
2. Explain the meaning of the inequalities.

2.10.3 Different Ways of Solving an Inequality

Activity

Use the following scenario to answer questions 1 – 2.

Andre and Priya used different strategies to solve the following inequality, but they reached the same solution.
$2(2x + 1.5) \leq 18 - x$
Make sense of each strategy until you can explain what each student has done.

Andre	Priya
$2(2x + 1.5) = 18 - x$ $4x + 3 = 18 - x$ $4x - 15 = -x$ $-15 = -5x$ $3 = x$ Testing to see if $x = 4$ is a solution: $2(2 \cdot 4 + 1.5) < 18 - 4$ $2(9.5) < 14$ $19 < 14$ The inequality is false, so 4 is not a solution. If a number greater than 3 is not a solution, the solution must be less than 3, or $x < 3$.	$2(2x + 1.5) = 18 - x$ $4x + 3 = 18 - x$ $5x + 3 = 18$ $5x = 15$ $x = 3$ In $4x + 3 = 18 - x$, there is $4x$ on the left and $-x$ on the right. If x is a negative number, $4x + 3$ could be positive or negative, but $18 - x$ will always be positive. For $4x + 3 < 18 - x$ to be true, x must include negative numbers or x must be less than 3.

1. Summarize the strategy Andre used to solve the inequality.
2. Summarize the strategy Priya used to solve the inequality.
3. Examine four inequalities:
   Inequality A: $\frac{1}{5}p > -10$
   Inequality B: $4(x + 7) \leq 4(2x + 8)$
   Inequality C: $-9n < 36$
   Inequality D: $\frac{c}{3} < -2(c - 7)$
   Work with a partner to decide on at least two inequalities to solve. Solve one inequality using Andre's strategy (by testing values on either side of the given solution), while your partner uses Priya's strategy (by reasoning about the parts of the inequality). Switch strategies for the other inequality.

Are you ready for more?

Extending Your Thinking

1. Using positive integers between 1 and 9 and each positive integer at most once, fill in values to get two constraints so that $x = 7$ is the only integer that will satisfy both constraints at the same time.
$\square x + \square < \square x + \square$

2. Using positive integers between 1 and 9 and each positive integer at most once, fill in values to get two constraints so that $x = 7$ is the only integer that will satisfy both constraints at the same time.
$\square x + \square > \square x + \square$

Self Check

Solve the following inequality by solving the related equation and reasoning what the answer may be given the situation.

$\frac{1}{3}p < 4$

a. $p < \frac{4}{3}$; The related equation is $p = \frac{4}{3}$. Since the solution set must contain negative numbers, p must be less than $\frac{4}{3}$.
b. $p < \frac{3}{4}$; The related equation is $p = frac34$. Since the solution set must contain negative numbers, p must be less than $\frac{3}{4}$.
c. $p < 12$; The solution to the related equation is $p = 12$. Since $\frac{1}{3}p$ must be less than 4, the solution cannot include large positive numbers, so numbers larger than 12 cannot be included in the solution.
d. $p > 12$; The solution to the related equation is $p = 12$. For $\frac{1}{3}p$ to be less than 4, the solution must contain large positive numbers.

Additional Resources

Reasoning about Solution Sets to Linear Inequalities

You can derive a lot of information about the solution set to an inequality by studying the inequality itself.

Let's look at the inequality shown.

$12x > -3(x - 1)$

The left side of the inequality contains $12x$. The right side of the inequality contains $-3x$.

We can think to ourselves: "What values of x would make $12x$ greater than $-3x$?"

If x is a large positive number, then $12x$ will still be positive. If x is a large positive number, then $-3x$ will be a negative number. This makes the inequality true, so the solution set must include large positive numbers.

Let's solve the related equation.

$$\begin{aligned} 12x &> -3(x-1) \\ 12x &= -3(x-1) \\ 12x &= -3x+3 \\ 15x &= 3 \\ x &= \frac{3}{15} \end{aligned}$$

Since we know that the solution set must include large positive numbers, then $x > \frac{3}{15}$ must be the solution to the inequality.

TRY IT **Reasoning about Solution Sets to Linear Inequalities**

For questions 1 – 2, solve the related equation for each inequality. Use what you know about the inequality to determine the solution set.

1. $-5y \geq 30$

2. $4p < -4$

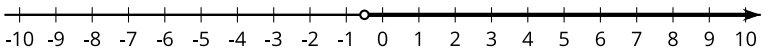
2.10.4 Matching Inequalities and Solutions

Activity

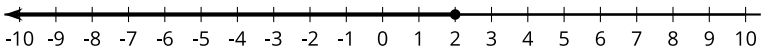
Match each inequality to a graph that represents its solutions. Be prepared to show your reasoning.

- $6x \leq 3x$
- $\frac{1}{4}x > -\frac{1}{2}$
- $5x + 4 \geq 7x$
- $8x - 2 < -4(x - 1)$
- $\frac{4x-1}{3} > -1$
- $\frac{12}{5} - \frac{x}{5} \leq x$

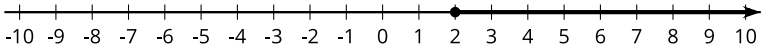
Graph A



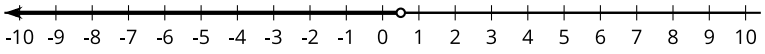
Graph B



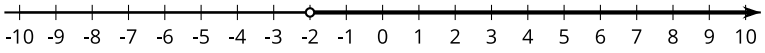
Graph C



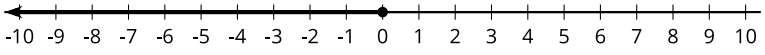
Graph D



Graph E



Graph F



Use this inequality for questions 1 – 2:

$6x \leq 3x$

1. Match the graph to the equation.

2. Explain or show your reasoning.

Use this inequality for questions 3 and 4:

$\frac{1}{4}x > -\frac{1}{2}$

3. Match the graph to the equation.

4. Explain or show your reasoning.

Use this inequality for questions 5 – 6:

$5x + 4 \geq 7x$

5. Match the graph to the equation.

6. Explain or show your reasoning.

Use this inequality for questions 7 and 8:

$8x - 2 < -4(x - 1)$

7. Match the graph to the equation.

8. Explain or show your reasoning.

Use this inequality for questions 9 – 10:

$\frac{4x-1}{3} > -1$

9. Match the graph to the equation.

10. Explain or show your reasoning.

Use this inequality for questions 11 and 12:

$\frac{12}{5} - \frac{x}{5} \leq x$

11. Match the graph to the equation.

12. Explain or show your reasoning.

2.10.5 Solving Inequalities

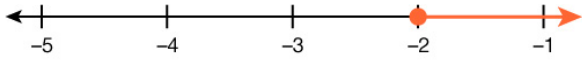
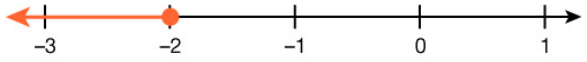
Activity

In this activity, we're going to solve inequalities. Work with a partner to solve each inequality given. Then match it with the number line representing its solution.

As you complete the activity, discuss with your partner how you know each number line is the solution to its inequality.

When you think you've found the correct number line for an inequality, substitute a test point that is on the highlighted part of the number line into the inequality. Is it true?

Try a test point that is not highlighted. Is the inequality false?

1. $x - 4 \leq -6$	Graph A 
2. $x + 6 \geq 8$	Graph B 
3. $4x \leq 8$	Graph C 
4. $-\frac{1}{2}x \leq 1$	Graph D 

Use the table to choose the matching graph to the inequality.

1. $x - 4 \leq -6$

2. $x + 6 \geq 8$

3. $4x \leq 8$

4. $-\frac{1}{2}x \leq 1$

For each of the inequalities above, match it to the property you used to solve.

5. $x - 4 \leq -6$	A. Multiplication Property of Inequality
6. $x + 6 \geq 8$	B. Addition Property of Inequality

7. $4x \leq 8$	C. Division Property of Inequality
8. $-\frac{1}{2}x \leq 1$	D. Subtraction Property of Inequality

5. $x - 4 \leq -6$

6. $x + 6 \geq 8$

7. $4x \leq 8$
Select **two** properties that could be used to solve the inequality

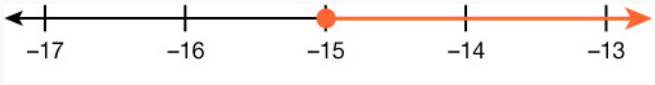
8. $-\frac{1}{2}x \leq 1$
Select **two** properties that could be used to solve the inequality

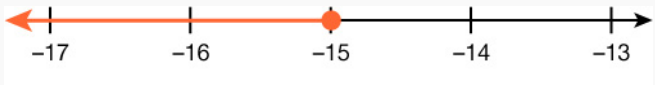
9. Is there anything unique about the method used to solve $-\frac{1}{2}x \leq 1$? Explain.

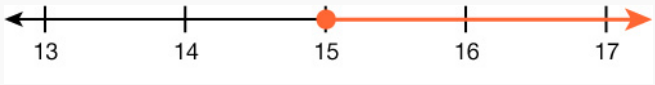
Self Check

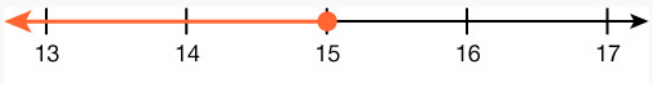
Solve the inequality. Which number line represents the solution?

$-\frac{1}{3}m \leq 5$

a.


b.


c.


d.


Additional Resources

Solving Linear Inequalities

A linear inequality is much like a linear equation—but the equal sign is replaced with an inequality sign. So, when we solve linear equations, we are able to use the properties of equality to add, subtract, multiply, or divide both sides and still keep the equality. Similar properties hold true for inequalities.

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality. For example:

$$\begin{array}{ll} -4 < 2 & -4 < 2 \\ -4 - 5 < 2 - 5 & -4 + 7 < 2 + 7 \\ -9 < -3 \text{ True} & 3 < 9 \text{ True} \end{array}$$

Notice that the inequality sign stayed the same.

This leads us to the Addition and Subtraction Properties of Inequality.

Addition and Subtraction Properties of Inequality

For any numbers a, b, and c, if $a < b$, then

$a + c < b + c$

$a - c < b - c$

For any numbers a, b, and c, if $a > b$, then

$a + c > b + c$

$a - c > b - c$

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality.

What happens to an inequality when we divide or multiply both sides by a constant?

Let's first multiply and divide both sides by a positive number.

$10 < 15$ $\quad$ $10 < 15$

$10(5) < 15(5)$ $\quad$ $\frac{10}{5} < \frac{15}{5}$

$50 < 75$ True $\quad$ $2 < 3$ True

The inequality signs stayed the same.

Does the inequality stay the same when we divide or multiply by a negative number?

$10 < 15$ $\quad$ $10 < 15$

$10(-5) \;?\; 15(-5)$ $\quad$ $\frac{10}{-5} \;?\; \frac{15}{-5}$

$-50 \;?\; -75$ $\quad$ $-2 \;?\; -3$

$-50 > -75$ $\quad$ $-2 > -3$

Notice that when we filled in the inequality signs, the inequality signs reversed their direction.

When we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we divide or multiply an inequality by a negative number, the inequality sign reverses.

This gives us the Multiplication and Division Properties of Inequality.

Multiplication and Division Properties of Inequality

For any numbers a, b, and c,

multiply or divide by a positive

if $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$

if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$

Multiply or divide by a negative

if $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$

if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$

When we divide or multiply an inequality by a:

- positive number, the inequality stays the same.
- negative number, the inequality reverses.

Sometimes when solving an inequality, as in the next example, the variable ends up on the right. We can rewrite the inequality in reverse to get the variable to the left.

$x > a$ has the same meaning as $a < x$.

Think about it as "If Xander is taller than Andy, then Andy is shorter than Xander."

In Examples 1 and 2: solve the inequality, graph the solution on the number line, and write the solution in interval notation.

EXAMPLE 1

$x - \frac{3}{8} \leq \frac{3}{4}$

Solution

Step 1 - Add $\frac{3}{8}$ to both sides of the inequality.

$x - \frac{3}{8} + \frac{3}{8} \leq \frac{3}{4} + \frac{3}{8}$

Step 2 - Simplify.

$x \leq \frac{9}{8}$

Step 3 - Graph the solution on the number line.



Step 4 - Write the solution in interval notation.

$(-\infty, \frac{9}{8}]$

EXAMPLE 2

$-15 < \frac{3}{5}z$

Solution

Step 1 - Multiply both sides of the inequality by $\frac{5}{3}$. Since $\frac{5}{3}$ is positive, the inequality stays the same.

$(\frac{5}{3})(-15) < (\frac{5}{3})(\frac{3}{5}z)$

Step 2 - Simplify.

$-25 < z$

Step 3 - Rewrite with the variable on the left.

$z > -25$

Step 4 - Graph the solution on the number line.



Step 5 - Write the solution in interval notation.

$(-25, \infty)$

TRY IT **Solving Linear Inequalities**

Solve the following linear inequalities:

1. $9y < 54$
2. $-13m \geq 65$

2.10.6 Writing Inequalities to Represent a Situation

Cool Down

Use the following scenerio for questions 1 – 2.

> Lin's job pays $8.25 an hour plus $10 of transportation allowance each week. She has to work at least 5 hours a week to keep the job, and she can earn up to $175 per week, including $10 to cover transportation per week.

1. Represent this situation mathematically. If you use variables, specify what each one means.
2. How many hours per week can Lin work? Be prepared to show your reasoning.

2.10.7 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Xander is loading a pallet with boxes that each weighs 45 pounds. The pallet can safely support no more than 900 pounds. How many boxes can he safely load onto the pallet?
   ___ boxes or less

2. The elevator in Yehire's apartment building has a sign that says the maximum weight is 2100 pounds. If the average weight of one person is 150 pounds, how many people can safely ride the elevator?
   a maximum of ___ people

3. Jaden is looking at apartments with three of his friends. They want the monthly rent to be no more than $2360. If the roommates split the rent evenly among the four of them, what is the maximum rent each will pay?
   $___

4. Melody got a $20 gift card for the coffee shop. Her favorite iced drink costs $3.79. What is the maximum number of drinks she can buy with the gift card?
   ___ drinks

5. Teegan likes to play golf. He has budgeted $60 next month for the driving range. It costs him $10.55 for a bucket of balls each time he goes. What is the maximum number of times he can go to the driving range next month?
   ___ times

6. Ryan charges his neighbors $17.50 to wash their car. How many cars must he wash next summer if his goal is to earn at least $1500?
   ___ cars

7. Alejandro's water bill is $24.80 per month plus $2.20 per ccf (hundred cubic feet) of water. What is the maximum number of ccf Alejandro can use if he wants his bill to be no more than $60?
   ___ ccf

8. Look at the inequality $5m + 6 < 14 - 2m$. Which of the following is NOT true?

   a. If m is a negative number, then $14 - 2m$ will always be negative.
   b. If m is a positive number, then $5m + 6$ will always be positive.
   c. If m is a negative number, then $14 - 2m$ will always be positive.
   d. If m is a negative number, then $5m + 6$ could be a positive or negative number.

9. Solve the inequality: $x + 8 > -12$

   a. $x > 4$
   b. $x > -4$
   c. $x > 20$
   d. $x > -20$

10. Solve the inequality: $x - 15 < -3$

    a. $x < -18$
    b. $x < 12$
    c. $x < -12$
    d. $x < 18$

11. Solve the inequality: $x + 19 \leq 23$

    a. $x \leq 4$
    b. $x \leq 42$
    c. $x \leq -4$
    d. $x \geq 4$

12. Solve the inequality: $7x < -56$

a. $x < -49$
b. $x < -63$
c. $x < -8$
d. $x > -8$

13. Solve the inequality: $\frac{x}{2} \leq \frac{5}{2}$

a. $x \geq 5$
b. $x \leq 2$
c. $x \leq 5$
d. $x \leq \frac{5}{2}$

14. Solve the inequality: $-3x > 21$

a. $x < -7$
b. $x > -7$
c. $x < -24$
d. $x < 18$

2.10.8 Lesson Summary

In this lesson, you learned how to:

- Analyze the structure of an inequality in one variable to help determine if the solution is greater or less than the solution to the related equation.
- Write and solve inequalities to answer questions about a situation.

Here are the **activities** that helped you reach those goals:

- 2.10.1: Writing an Inequality to Represent a Constraint
  - In this activity, you practiced writing an inequality to represent a constraint, reasoning about its solutions, and interpreting the solutions.
- 2.10.2: Using Inequalities to Solve a Problem
  - In this activity, you used inequalities to solve a problem about a situation whose constraints might be unfamiliar and in which multiple quantities are unknown.
  - 2.10.2: Self Check
  - 2.10.2: Additional Resources
- 2.10.3: Different Ways of Solving an Inequality
  - In this activity, you explored some ways to decide whether the solution set to an inequality is greater than or less than a particular boundary value (identified by solving a related equation).
  - 2.10.3: Self Check
  - 2.10.3: Additional Resources
- 2.10.4: Matching Inequalities and Solutions
  - In this activity, you received additional practice in reasoning about the solutions to inequalities without any context.
- 2.10.5: Solving Inequalities
  - In this activity, you used the Addition and Subtraction Properties of Inequality to solve different inequalities. You also used the Multiplication and Division Properties of Inequality to solve inequalities. You learned that when using the Multiplication and Division Properties of Inequality, you must reverse the symbol when you multiply or divide by a negative number.
  - 2.10.5: Self Check
  - 2.10.5: Additional Resources
- 2.10.6: Writing Inequalities to Represent a Situation
  - In this activity, you wrote inequalities to represent a situation.

After these activities, you completed the following **practice**:

- 2.10.7: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

2.11 GRAPHING LINEAR INEQUALITIES IN TWO VARIABLES

2.11.0 Lesson Overview

In this lesson, you will write, solve, and graph linear inequalities in two variables.

When you finish this lesson, you will be able to:

- Determine which side of the line the solutions to the inequality will fall, given a two-variable inequality and the graph of the related equation.
- Describe the graph that represents the solutions to a linear inequality in two variables.

Here are the **activities** that will help you reach those goals:

- 2.11.1: Using Computation and Reasoning
- 2.11.2: Finding Solutions to Inequalities on the Coordinate Plane
  - 2.11.2: Self Check
  - 2.11.2: Additional Resources
- 2.11.3: Sketching Solutions to Inequalities
  - 2.11.3: Self Check
  - 2.11.3: Additional Resources
- 2.11.4: Shading Boundary Line Regions

After that, you'll **practice and review**.

- 2.11.5: Practice
- 2.11.6: Lesson Summary

2.11.1 Using Computation and Reasoning

Warm Up

Here is an expression:

$2x + 3y$

Decide if the values in each ordered pair, (x, y), make the value of the expression less than, greater than, or equal to 12.

1. $(0, 5)$

2. $(6, 0)$

3. $(-1, -1)$

4. $(-5, 10)$

2.11.2 Finding Solutions to Inequalities on the Coordinate Plane

Activity

Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

Here are four inequalities. Study each inequality assigned to your group and work together to:

- Use the graphing tool or technology outside the course to graph each inequality. (The Desmos tool is provided).
- Find some coordinate pairs that represent solutions to the inequality.
- Find some coordinate pairs that do not represent solutions.
- Plot both sets of points. Either use two different colors or two different symbols like X and O.
- Plot enough points until you start to see the region that contains solutions and the region that contains non-

solutions. Look for a pattern describing the region where solutions are plotted.

1. $x \geq y$
   Access multimedia content (http://openstax.org/books/algebra-1/pages/2-11-2-finding-solutions-to-inequalities-on-the-coordinate-plane)
2. $-2y \geq -4$
   Access multimedia content (http://openstax.org/books/algebra-1/pages/2-11-2-finding-solutions-to-inequalities-on-the-coordinate-plane)
3. $3x < 0$
   Access multimedia content (http://openstax.org/books/algebra-1/pages/2-11-2-finding-solutions-to-inequalities-on-the-coordinate-plane)
4. $x + y > 10$
   Access multimedia content (http://openstax.org/books/algebra-1/pages/2-11-2-finding-solutions-to-inequalities-on-the-coordinate-plane)

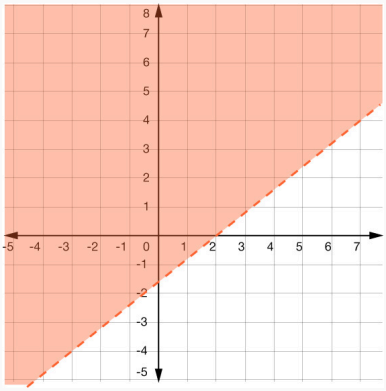
Video: Finding Solutions to Inequalities on the Coordinate Plane

Watch the following video to learn more about solutions to inequalities on the coordinate plane.

Finding Solutions to Inequalities on the Coordinate Plane (https://openstax.org/r/inequalities-coordinate-plane)

Self Check

Which ordered pair is a solution to the inequality graphed here?



$y > \frac{4}{5}x - \frac{8}{5}$

a. (2, -2)
b. (-4, 0)
c. (4, 0)
d. (0, -4)

Additional Resources

Verify Solutions to an Inequality in Two Variables

Previously we learned to solve inequalities with only one variable. We will now learn about inequalities containing two variables. In particular we will look at linear inequalities in two variables which are very similar to linear equations in two variables.

Linear inequalities in two variables have many applications. If you ran a business, for example, you would want your revenue to be greater than your costs—so that your business made a profit.

Linear Inequality

A **linear inequality** is an inequality that can be written in one of the following forms:

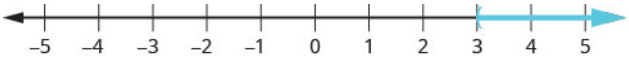
$Ax + By > C$

$Ax + By \geq C$

$Ax + By < C$

$$Ax + By \leq C$$

Recall that an inequality with one variable had many solutions. For example, the solution to the inequality $x > 3$ is any number greater than 3. We showed this on the number line by shading in the number line to the right of 3, and putting an open parenthesis at 3. See the figure below.



Similarly, linear inequalities in two variables have many solutions. Any ordered pair(x, y) that makes an inequality true when we substitute in the values is a solution to a linear inequality.

Solution to a Linear Inequality

An ordered pair(x, y) is a solution to a linear inequality if the inequality is true when we substitute the values of x and y.

EXAMPLE

Determine whether each ordered pair is a solution to the inequality $y > x + 4$:

1. $(0, 0)$
2. $(1, 6)$
3. $(2, 6)$
4. $(-5, -15)$
5. $(-8, 12)$

Solution

1. No; Substitute 0 for x and 0 for y.

$y > x + 4$

$0 > 0 + 4$

$0 \not> 4$

2. Yes; Substitute 1 for x and 6 for y.

$y > x + 4$

$6 > 1 + 4$

$6 > 5$

3. No; Substitute 2 for x and 6 for y.

$y > x + 4$

$6 > 2 + 4$

$6 \not> 6$

4. No; Substitute -5 for x and -15 for y.

$y > x + 4$

$-15 > -5 + 4$

$-15 \not> 6 - 1$

5. Yes; Substitute -8 for x and 12 for y.

$y > x + 4$

$12 > -8 + 4$

$12 > -4$

TRY IT **Verify Solutions to an Inequality in Two Variables**

Determine whether each ordered pair is a solution to the inequality $y > x - 3$.

1. $(0, 0)$

2. $(4, 9)$

3. $(-2, 1)$

4. $(-5, -3)$

5. $(5, 1)$

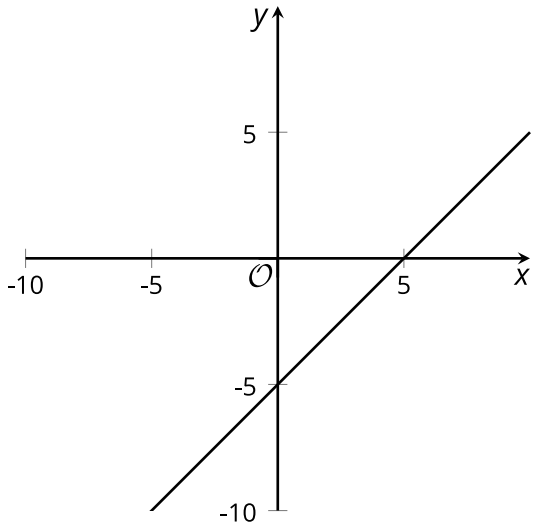
2.11.3 Sketching Solutions to Inequalities

Activity

Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

Here is a graph that represents solutions to the equation $x - y = 5$.

Using this graph as a guide, sketch a quick graph that represents the solutions to each of the inequalities in questions 1 – 4. Use the graphing tool or technology outside the course. Graph the equation that represents each scenario using the Desmos tool provided.



1. $x - y < 5$
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-11-3-sketching-solutions-to-inequalities)

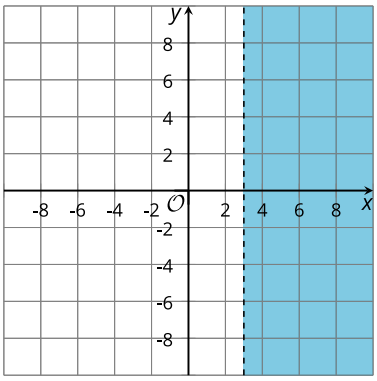
2. $x - y \leq 5$
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-11-3-sketching-solutions-to-inequalities)

3. $x - y > 5$
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-11-3-sketching-solutions-to-inequalities)

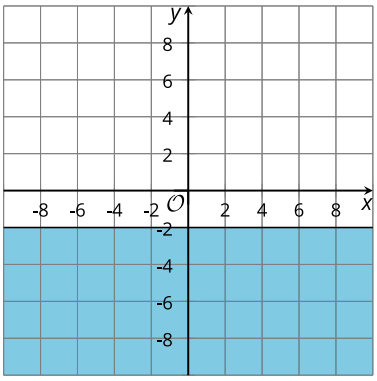
4. $x - y \geq 5$
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-11-3-sketching-solutions-to-inequalities)

For each graph in questions 5 – 8, write an inequality whose solutions are represented by the shaded part of the graph.

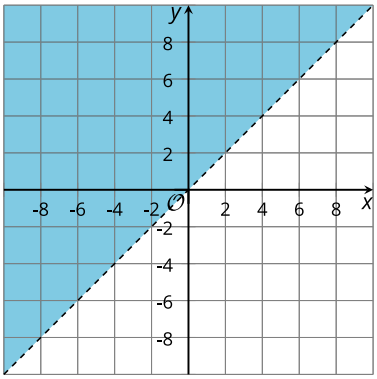
5.



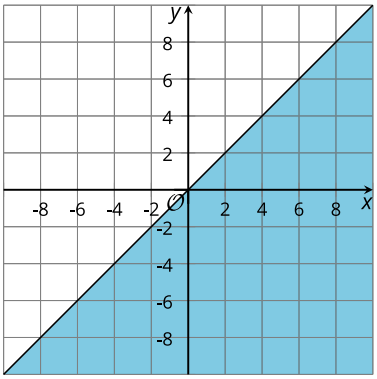
6.



7.

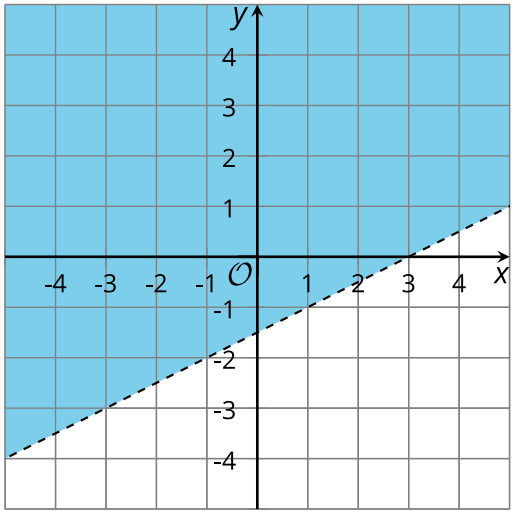


8.



Are you ready for more?

Extending Your Thinking



This is a graph of the inequality $x - 2y < 3$. The points $(7, 3)$ and $(7, 5)$ are both in the solution region for this inequality. Use this information to answer questions 1 – 5.

1. Compute $x - 2y$ for both of these points.

2. Which point comes closest to satisfying the equation $x - 2y = 3$? That is, for which (x, y) pair is $x - 2y$ closest to 3?

3. The points $(3, 2)$ and $(5, 2)$ are also in the solution region. Which of these points comes closest to satisfying the equation $x - 2y = 3$?
4. Find a point in the solution region that comes even closer to satisfying the equation $x - 2y = 3$. What is the value of $x - 2y$?
5. For the points $(5, 2)$ and $(7, 3)$, $x - 2y = 1$. Find another point in the solution region for which $x - 2y = 1$.

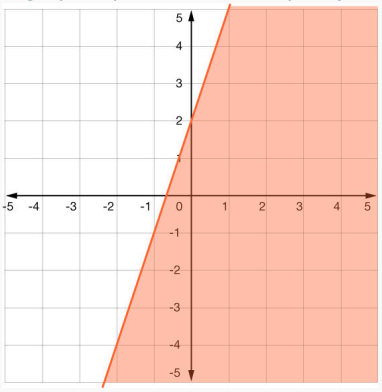
Use the point $(5, 3)$ to answer question 6.

6. Find the value of $x - 2y$.
7. Find two other points that give the same answer.

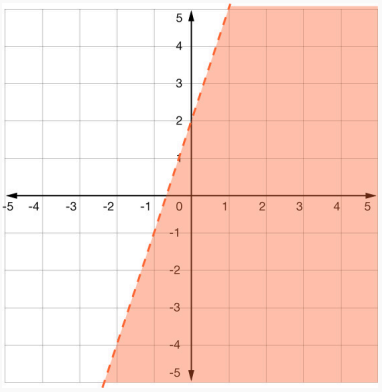
Self Check

Which graph represents the inequality $y \leq 3x + 2$?

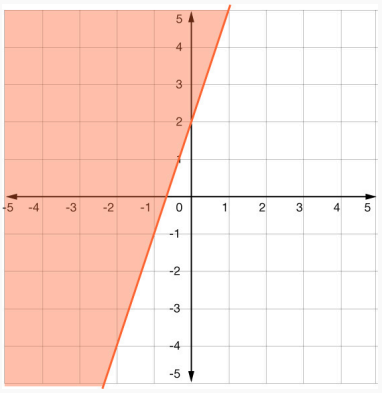
a.



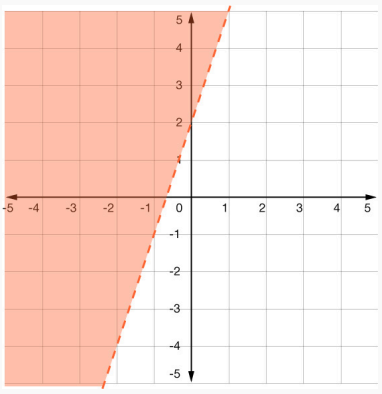
b.



c.



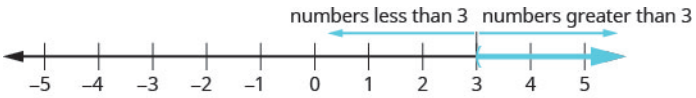
d.



Additional Resources

Recognize the Relation Between the Solutions of an Inequality and its Graph

Now, we will look at how the solutions of an inequality relate to its graph. Let's think about the number line shown previously again. The point $x = 3$ separated that number line into two parts. On one side of 3 are all the numbers less than 3. On the other side of 3 all the numbers are greater than 3. See the figure below.



The solution of $x > 3$ is the shaded part of the number line to the right of $x = 3$

Similarly, the line $y = x + 4$ separates the plane into two regions. On one side of the line are points with $y < x + 4$. On the other side of the line are the points with $y > x + 4$. We call the line $y = x + 4$ a **boundary line.**

Boundary Line

The line with equation $Ax + By = C$ is the boundary line that separates the region where $Ax + By > C$ from the region where $Ax + By < C$.

For an inequality in one variable, the endpoint is shown with a parenthesis or a bracket depending on whether or not a is included in the solution:

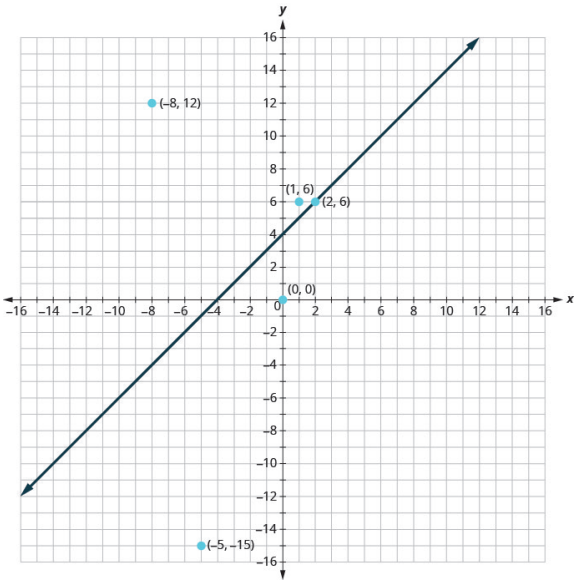


Similarly, for an inequality in two variables, the boundary line is shown with a solid or dashed line to show whether or not it the line is included in the solution.

$Ax + By < C$	$Ax + By \leq C$
$Ax + By > C$	$Ax + By \geq C$
Boundary line is $Ax + By = C$	Boundary line is $Ax + By = C$
Boundary line is not included in solution.	Boundary line is included in solution.
Boundary line is dashed.	**Boundary line is solid.**

Now, let's take a look at what we found in the example above. We'll start by graphing the line $y = x + 4$, and then we'll

plot the five points we tested, as shown in the graph. See the figure below.



In the previous example, we found that some of the points were solutions to the inequality $y > x + 4$ and some were not.

Which of the points we plotted are solutions to the inequality $y > x + 4$?

The points $(1, 6)$ and $(-8, 12)$ are solutions to the inequality $y > x + 4$. Notice that they are both on the same side of the boundary line $y > x + 4$.

The two points $(0, 0)$ and $(-5, -15)$ are on the other side of the boundary line $y = x + 4$, and they are not solutions to the inequality $y > x + 4$. For those two points. $y < x + 4$.

What about the point $(2, 6)$? The point $(2, 6)$ is a solution to the boundary line equation $y = x + 4$, because $6 = 2 + 4$. However, $(2, 6)$ is not a solution to the inequality, because the boundary line is not included in the solution to the inequality $y > x + 4$.

Let's take another point above the boundary line and test whether or not it is a solution to the inequality $y > x + 4$. The point $(0, 10)$ clearly looks to above the boundary line, doesn't it? Is it a solution to the inequality?

$y > x + 4$

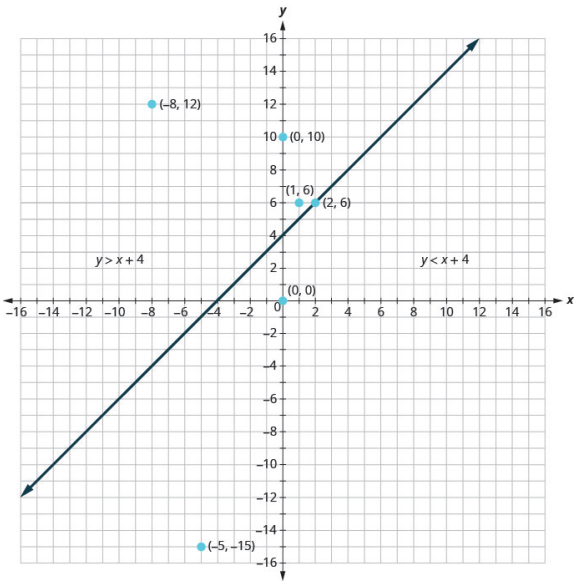
$10 > 0 + 4$

$10 > 4$

So, $(0, 10)$ is a solution to $y > x + 4$.

Any point you choose above the boundary line is a solution to the inequality $y > x + 4$. All points above the boundary line are solutions.

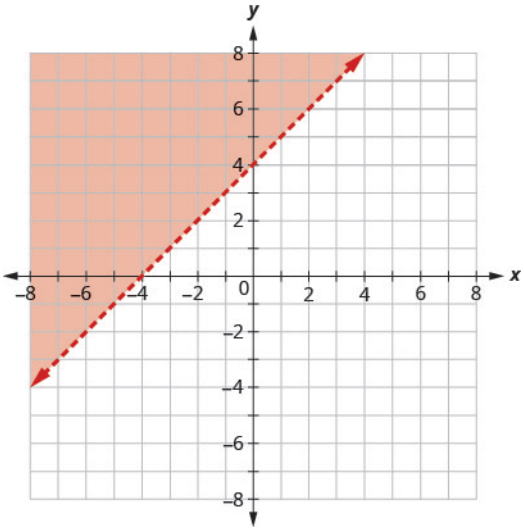
Similarly, all points below the boundary line, the side with $(0, 0)$ and $(-5, -15)$, are not solutions to $y > x + 4$, as shown in the figure below.



The graph of the inequality $y > x + 4$ is shown in below.

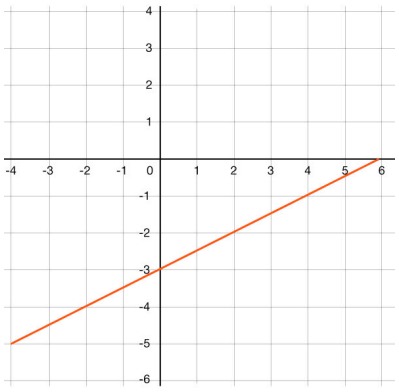
The line $y = x + 4$ divides the plane into two regions. The shaded side shows the solutions to the inequality $y > x + 4$.

The points on the boundary line, those where $y = x + 4$, are not solutions to the inequality $y > x + 4$, so the line itself is not part of the solution. We show that by making the line dashed, not solid.



TRY IT **Recognize the Relation Between the Solutions of an Inequality and its Graph**

The graph of $y = \frac{1}{2}x - 3$ is shown here.



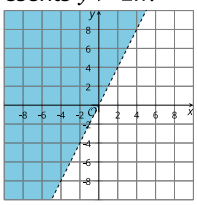
Use the graphing tool or technology outside the course. Graph the inequality that represents $y < \frac{1}{2}x - 3$. using the Desmos tool below.

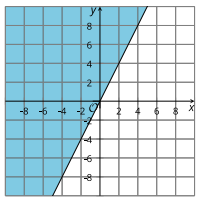
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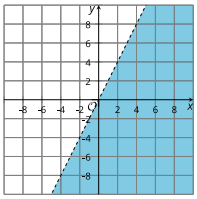
2.11.4 Shading Boundary Line Regions

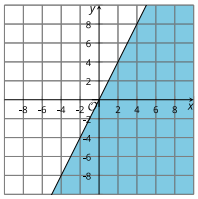
Cool Down

1. The line in each shaded graph represents $y = 2x$. Which graph including the shaded region represents $y > 2x$?

A. 

B. 

C. 

D. 

2. Explain your reasons for choosing that graph.

2.11.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 3, determine whether each ordered pair is a solution to the inequality $y > x - 1$.

1. $(0, 1)$

a. Yes
b. No

2. $(-4, -1)$

a. Yes
b. No

3. $(4, 2)$

a. Yes
b. No

For questions 4 - 6, determine whether each ordered pair is a solution to the inequality $y > x - 3$.

4. $(0, 0)$

a. Yes
b. No

5. $(2, 1)$

a. Yes
b. No

6. $(-1, -5)$

a. Yes
b. No

For questions 7 and 8, determine whether each ordered pair is a solution to the inequality $2x + 3y > 2$.

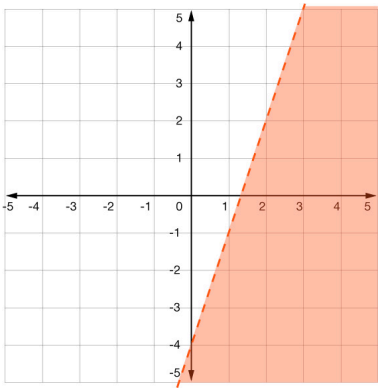
7. $(1, 1)$

a. Yes
b. No

8. $(4, -3)$

a. Yes
b. No

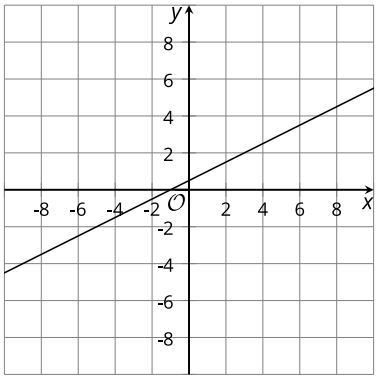
9. Which inequality represents the graph?



a. $y \geq 3x - 4$
b. $y > 3x - 4$

c. $y < 3x - 4$
d. $y \leq 3x - 4$

For questions 10 and 11, use this graph of the equation $2y - x = 1$ as a guide to answer the questions.



10. Is the point $(0, \frac{1}{2})$ a solution to the equation?

a. Yes
b. No

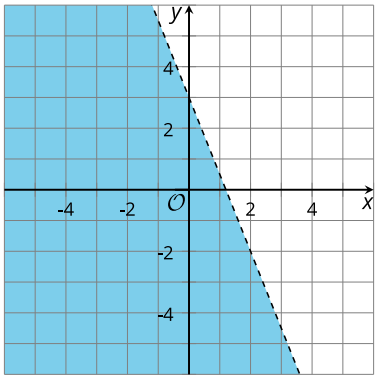
11. Is the point $(0, 2)$ a solution to the inequality $2y - x > 1$?

a. Yes
b. No

12. Consider the linear equation $2y - 3x = 5$. The pair $(-1, 1)$ is a solution to the equation. Which of the following is also a solution to the equation?

a. $(-3, -1)$
b. $(5, 11)$
c. $(3, 7)$
d. $(1, 5)$

13. The boundary line on the graph represents the equation $5x + 2y = 6$. Write an inequality that is represented by the graph.



a. $5x + 2y < 6$
b. $5x + 2y < 2$
c. $-5x - 2y < 6$
d. $5x + 2y < -6$

2.11.6 Lesson Summary

In this lesson, you learned how to:

- Determine which side of the line the solutions to the inequality will fall, given a two-variable inequality and the graph of the related equation.
- Describe the graph that represents the solutions to a linear inequality in two variables.

Here are the **activities** that helped you reach those goals:

- 2.11.1: Using Computation and Reasoning
  - In this activity, you were familiarized with the computation and reasoning that you will need later to determine the solution region of a linear inequality in two variables.
- 2.11.2: Finding Solutions to Inequalities on the Coordinate Plane
  - In this activity, you built on your understanding that a linear inequality in one variable has many solutions, which are represented by all the points on one side of a number on a number line. You also recalled that a linear equation in two variables has many solutions, which are represented by all the points on the graph of the equation.
  - 2.11.2: Self Check
  - 2.11.2: Additional Resources
- 2.11.3: Sketching Solutions to Inequalities
  - In this activity, you looked at the regions that represent solutions and non-solutions to inequalities. You recognized that the boundary between the two regions is the graph of an equation that is related to the inequality.
  - 2.11.3: Self Check
  - 2.11.3: Additional Resources
- 2.11.4: Shading Boundary Line Regions
  - In this activity, you considered how to make decisions about which side of the boundary line to shade.

After these activities, you completed the following **practice**:

- 2.11.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

2.12 USING LINEAR INEQUALITIES AS CONSTRAINTS

2.12.0 Lesson Overview

In this lesson, you will deepen your understanding of the solutions to linear inequalities by studying them in context. You will write inequalities in two variables to represent constraints, and interpret the points on a boundary line and on either side of it in terms of the situation.

When you finish this lesson, you will be able to:

- Interpret points in the coordinate plane and decide if they are solutions to the inequality, given a two-variable inequality that represents a situation.
- Find the solutions to a two-variable inequality by using the graph of a related two-variable equation.
- Write inequalities to describe the constraints in a situation.

Here are the **activities** that will help you reach those goals:

- 2.12.1: Sketching a Graph to Represent an Equation
- 2.12.2: Write an Inequality to Represent a Constraint
  - 2.12.2: Self Check
  - 2.12.2: Additional Resources
- 2.12.3: Graphing Solutions and Interpreting Points
  - 2.12.3: Self Check
  - 2.12.3: Additional Resources
- 2.12.4: Reasoning Points Along Boundary Line
- 2.12.5: Recognizing Points for an Inequality

After that, you'll **practice and review.**

- 2.12.6: Practice
- 2.12.7: Summary

2.12.1 Sketching a Graph to Represent an Equation

Warm Up

For questions 1 – 3 use the following information:

> A homeowner is making plans to landscape her yard. She plans to hire professionals to install grass sod in some parts of the yard and flower beds in other parts.
> Grass sod installation costs $2 per square foot and flower bed installation costs $12 per square foot. Her budget for the project is $3,000.



1. Write an equation that represents the square feet of grass sod, x, and the square feet of flower beds, y, that she could afford if she used her entire budget.
2. Use the graphing tool or technology outside the course. Graph your equation from question 1 that represents this scenario using the Desmos tool below.

Access multimedia content (http://openstax.org/books/algebra-1/pages/2-12-1-sketching-a-graph-to-represent-an-equation)

3. Explain your reasoning.

2.12.2 Writing an Inequality to Represent a Constraint

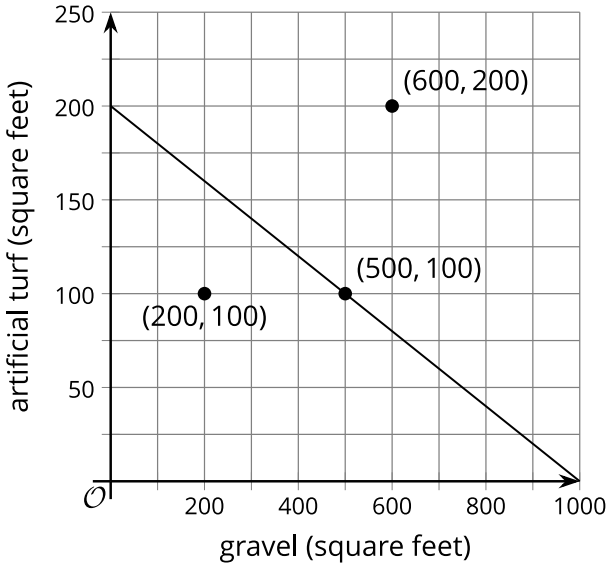
Activity

For questions 1 – 8, use the following scenario:

The homeowner is worried about the work needed to maintain a grass lawn and flower beds, so she is now looking at some low-maintenance materials.
She is considering a combination of the materials shown to cover the yard. Her budget is still $3,000.

- artificial turf: $15 per square foot
- gravel: $3 per square foot

Here is a graph representing some constraints in this situation.



1. The graph shows a line going through $(500, 100)$. In this situation, what does the point $(500, 100)$ mean?
2. Write an equation that the line represents.
3. What do the solutions to the equation mean?
4. The point $(600, 200)$ is located to the right and above the line. Does that combination of turf and gravel meet the homeowner's constraints? Be prepared to show your reasoning.
5. Choose another point in the same region (to the right and above the line). Check if the combination meets the homeowner's constraints.
6. The point $(200, 100)$ is located to the left and below the line. Does that combination of turf and gravel meet the homeowner's constraints? Be prepared to show your reasoning.
7. Choose another point in the same region (to the left and below the line). Check if the combination meets the homeowner's constraints.
8. Write an inequality that represents the constraints in this situation. Explain what the solutions mean and show the solution region on the graph.

Video: Writing an Inequality to Represent a Constraint

Watch the following video to learn more about representing constraints using inequalities.

Write an Inequality to Represent a Constraint (https://openstax.org/r/write-inequality)

Self Check

A builder is using a mix of different tiles to remodel a kitchen. The marble tiles cost, x, $15 per square and the ceramic tiles, y, cost $7 per square. The total amount allowed for tiles is at most $900. Which inequality represents this situation?

a. $22(x + y) \leq 900$
b. $15x - 7y < 900$
c. $15x + 7y < 900$
d. $15x + 7y \leq 900$

Additional Resources

Writing Inequalities to Represent Constraints

EXAMPLE

Elena needs to earn at least $450 a week during her summer break to pay for college. She works two jobs. One as a swimming instructor that pays $9 an hour and the other as an intern in a genetics lab for $22.50 per hour. How many hours does Elena need to work at each job to earn at least $450 per week?

Let x be the number of hours she works teaching swimming and let y be the number of hours she works as an intern.

Write an inequality that would model this situation.

She earns $9 per hour at the job as a swimming instructor and $22.50 in the genetics lab. At each job the number of hours multiplied by the hourly wage will give the amount earned at that job.

Amount earned as a swimming instructor	plus	the amount earned in the genetics lab	is at least $450
9x	+	22.5y	≥ 450

TRY IT **Writing Inequalities to Represent Constraints**

A baker sells pies and cakes at the bakery. Each pie sells for $14 and each cake sells for $23. Between all the sales, the baker wants to earn at least $525. How many pies and cakes does the baker need to sell to meet their goal?

Let p be the number of pies sold and let c be the number of cakes sold. Write an inequality that would model this situation.

2.12.3 Graphing Solutions and Interpreting Points

Activity

For questions 1 – 7, use the following scenario:

> A vendor at the Saturday Market makes $9 profit on each necklace she sells and $5 profit on each bracelet. Find a combination of necklaces and bracelets that she could sell to make the profit listed in each question.

1. Exactly $100 profit.

2. More than $100 profit.

3. Write an equation whose solution is the combination of necklaces (n) and bracelets (b) she could sell and make exactly $100 profit.

4. Write an inequality whose solutions are the combinations of necklaces (n) and bracelets (b) she could sell and make more than $100 profit.

5. Use the graphing tool or technology outside the course.Graph the linear inequality $y > \frac{5}{2}x - 4$ using the Desmos tool below.
[Access multimedia content (http://openstax.org/books/algebra-1/pages/2-12-3-graphing-solutions-and-interpreting-points)]
6. Is (3,18.6) a solution to the inequality?
7. Explain your reasoning.

Are you ready for more?

Extending Your Thinking

1. Write an inequality using two variables x and y where the solution would be represented by shading the entire coordinate plane.
2. Write an inequality using two variables x and y where the solution would be represented by not shading any of the coordinate plane.

Self Check

An artist is designing a sculpture to be displayed hanging in the lobby of a museum. She is using x wooden rods and y metal pipes.

- Each wooden rod, x, weighs 4 pounds.
- Each metal pipe, y, weighs 7 pounds.
- The total weight of these materials cannot be more than 510 pounds.

Which of these ordered pairs represents an amount of materials that meets the artist's constraints?

a. $(70, 20)$
b. $(50, 60)$
c. $(20, 70)$
d. $(60, 50)$

Additional Resources

Graph Linear Inequalities in Two Variables

Now that we know what the graph of a linear inequality looks like and how it relates to a boundary equation we can use this knowledge to graph a given linear inequality.

EXAMPLE

How to Graph a Linear Equation in Two Variables

Graph the linear inequality $y \geq \frac{3}{4}x - 2$.

Step 1 - Identify and graph the boundary line.

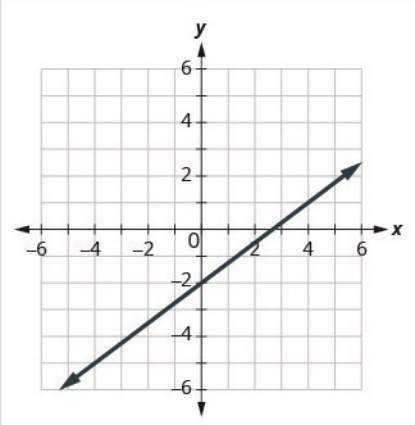
If the inequality is $\leq$ or $\geq$ boundary line is solid.

If the inequality is < or >, the boundary line is dashed.

Replace the inequality sign with an equal sign to find the boundary line

$y = \frac{3}{4}x - 2.$

The inequality sign is $\geq$, so we draw a solid line.



Step 2 - Test a point that is not on the boundary line. Is it a solution of the inequality?

We'll test $(0, 0)$.

Is it a solution of the inequality?

At $(0, 0)$, is $y \geq \frac{3}{4}x - 2$.?

$0 \stackrel{?}{\geq} \frac{3}{4}(0) - 2$

$0 \geq -2$

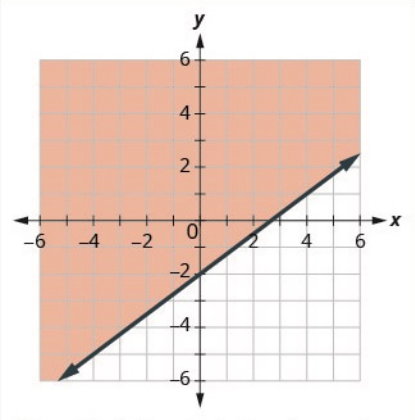
So, $(90, 0)$ is a solution.

Step 3 - Shade in one side of the boundary line.

If the test point is a solution, shade in the side that includes the point.

If the test point is not a solution, shade in the opposite side.

The test point $(0, 0)$ is a solution to $y \geq \frac{3}{4}x - 2$. So we shade in that side.



All points in the shaded region and on the boundary line represent the solutions to $y \geq \frac{3}{4}x - 2$.

TRY IT **Graph Linear Inequalities in Two Variables**

1. Graph the linear inequality $y > \frac{5}{2}x - 4$. Use the graphing tool or technology outside the course. Graph the inequality using the Desmos tool below.

Access multimedia content (http://openstax.org/books/algebra-1/pages/2-12-3-graphing-solutions-and-interpreting-points)

2.12.4 Recognizing Points Along Boundary Line

Activity

For questions 1 – 6, use the following scenario:

A popular band is trying to raise at least $20,000 for charity by holding multiple concerts at a park. It plans to sell tickets at $25 each. For each 2-hour concert, the band would need to pay the park $1,250 in fees for security, cleaning, and traffic services.

The band needs to find the combinations of number of tickets sold, t, and number of concerts held, c, that would allow it to reach its fundraising goal.

1. Write an inequality to represent the constraints in this situation.

2. Use the graphing tool or technology outside the course. Graph your inequality from question 1 that represents this scenario using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-12-4-recognizing-points-along-boundary-line)

3. Name two possible combinations of number of tickets sold and number of concerts held that would allow the band to meet its goal.

For questions 4 – 6, decide which combination of tickets and concerts would mean more money for charity:

4. 1,300 tickets and 10 concerts, or 1,300 tickets and 5 concerts?

5. 1,600 tickets and 16 concerts, or 1,200 tickets and 9 concerts?

6. 2,000 tickets and 4 concerts, or 2,500 tickets and 10 concerts?

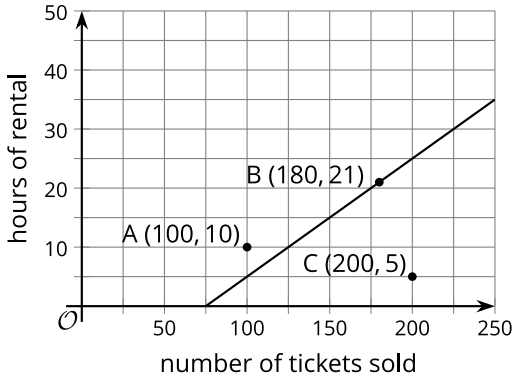
2.12.5 Recognizing Points for an Inequality

Cool Down

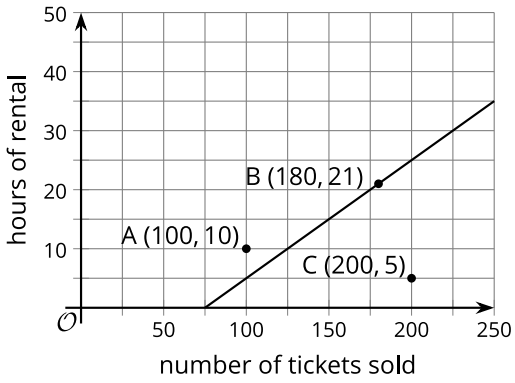
To raise money for after-school programs at an elementary school, a group of parents is holding a weekend of games in a community center. They charge $8 per person for entry into the event. The group would like to earn at least $600, after paying for the cost of renting the space, which is $40 an hour.

1. If x represents the number of entry tickets sold and y the hours of space rental, which inequality represents the constraints in the situation?
a. $8x - 40y < 600$
b. $8x - 40y \leq 600$
c. $8x - 40y > 600$
d. $8x - 40y \geq 600$

2. The line is the graph of $8x - 40y = 600$. Select all points whose (x, y) values represent the group reaching its fundraising goal. Be prepared to show your reasoning.



3. Complete the graph so that it represents solutions to an inequality that represents this situation. (Be clear about whether you want to use a solid or dashed line.) Use the graphing tool or technology outside the course. Graph the equation that represents this scenario using the Desmos tool below.



Access multimedia content (http://openstax.org/books/algebra-1/pages/2-12-5-recognizing-points-for-an-inequality)

Building Character: Growth Mindset



Compared to a fixed mindset, a growth mindset encourages you to embrace challenges, sustain effort, and try new strategies. Of course, no one embodies only a growth or fixed mindset; we are all a mixture of the two, and we can learn to recognize what triggers a fixed versus growth mindset. The brain is like a muscle—something you can strengthen.

Think about your current sense of having a growth mindset. Are the following statements are true for you?

- I truly believe that people can change.
- I can always change how intelligent I am.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

2.12.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 4, use the following scenario:

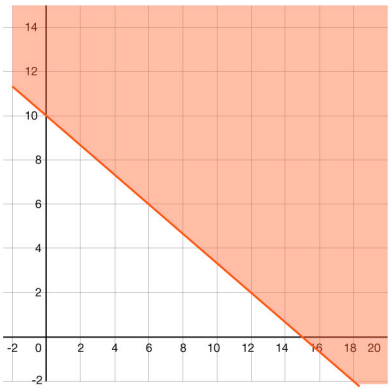
> Harrison works two part time jobs. One at a gas station that pays $11 an hour and the other is IT troubleshooting for $16.50 an hour. Between the two jobs, Harrison wants to earn at least $330 a week. How many hours does Harrison need to work at each job to earn at least $330?

1. Let x be the number of hours he works at the gas station and let y be the number of hours he works troubleshooting. Which inequality correctly models this situation?

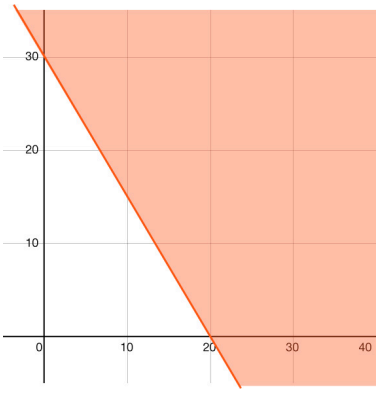
a. $16.5y + 11x > 330$
b. $11x + 16.5y \leq 330$
c. $16.5x + 11y \geq 330$
d. $11x + 16.5y \geq 330$

2. Which graph correctly represents the inequality from the previous question?

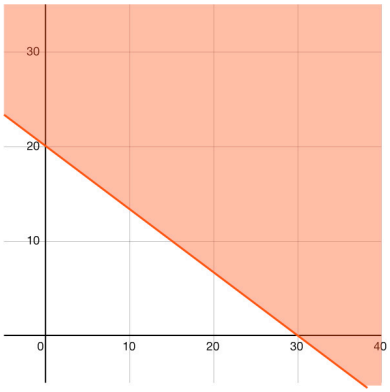
a.

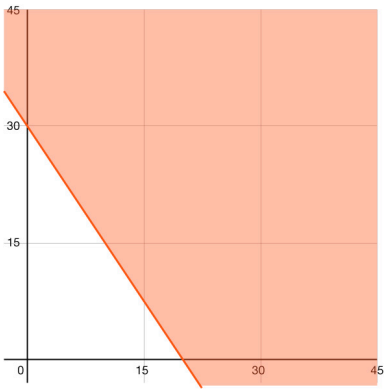


b.



c.



d. 

3. Which of the following ordered pairs is a solution to the inequality?

a. $(3, 17)$
b. $(15, 9)$
c. $(25, 2)$
d. $(10, 20)$

4. Which of the following is true about the solution you found in the previous question?

a. The solution represents a work schedule that earns him more than $330 per week.
b. The x-coordinate represents how many hours he works at the IT job.
c. The y-coordinate represents how many hours Harrison will work at the gas station.

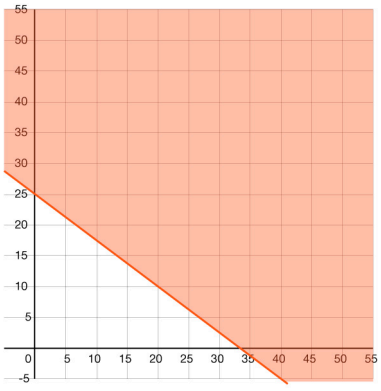
For questions 5 – 8, use the following passage:

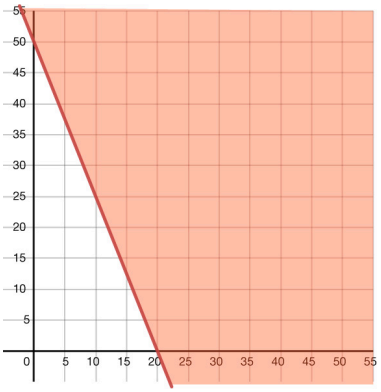
> The doctor tells Maya she needs to exercise enough to burn 500 calories each day. She likes to run which burns 15 calories per minute or bike which burns 10 calories per minute.

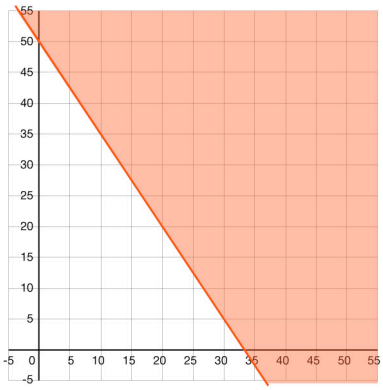
5. If x is the number of minutes that Maya runs and y is the number of minutes she bikes, find the inequality that models the situation.

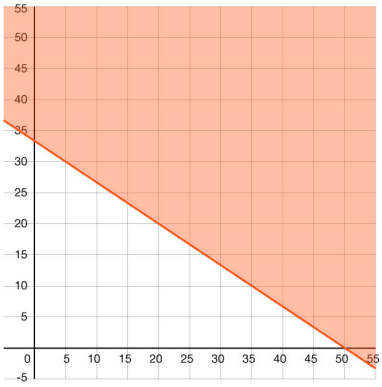
a. $15x + 10y \leq 500$
b. $15x + 10y > 500$
c. $15x + 10y \geq 500$
d. $15x + 10y < 500$

6. Which graph correctly represents the inequality from the previous question?

a. 

b. 

c. 

d. 

7. Which of the following ordered pairs is a solution to the inequality?

a. $(25, 10)$
b. $(20, 15)$
c. $(15, 30)$
d. $(5, 40)$

8. Which of the following is true about the solution you found in the previous question?

a. The solution represents a workout which allows Maya to burn at least 500 calories.
b. The y-coordinate represents the number of minutes that Maya runs.
c. The x-coordinate represents the number of minutes that Maya bikes.

For questions 9 – 12, use the following scenario:

Armando's workouts consist of kickboxing and swimming. While kickboxing, he burns 10 calories per minute and he burns 7 calories a minute while swimming. He wants to burn at least 600 calories each day.

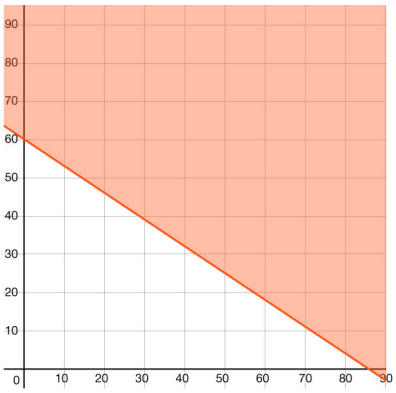
9. If x is the number of minutes that Armando will kickbox and y is the number of minutes he will swim, find the inequality that will help Armando create a workout for today.

a. $10x + 7y \geq 600$

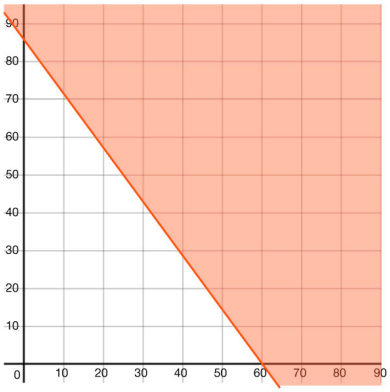
b. $10x + 7y > 600$
c. $7x + 10y < 600$
d. $7x + 10y \geq 600$

10. Which graph correctly represents the inequality from the previous question?

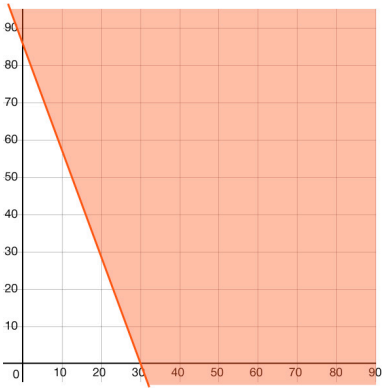
a.



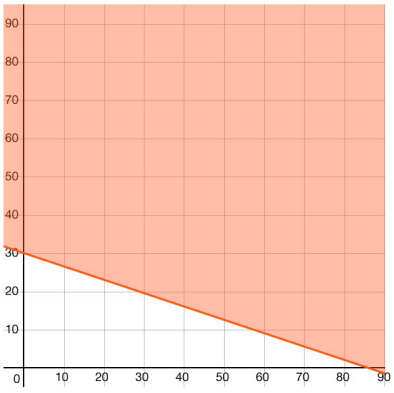
b.



c.



d.



11. Which of the following ordered pairs is a solution to the inequality?

a. $(40, 20)$
b. $(30, 40)$
c. $(10, 70)$
d. $(20, 70)$

12. Which of the following is true about the solution you found in the previous question?

a. The solution represents a workout which allows Armando to burn at least 500 calories.
b. The y-coordinate represents the number of minutes that Armando swims.
c. The x-coordinate represents the number of minutes that Armando swims.

2.12.7 Lesson Summary

In this lesson, you learned how to:

- Interpret points in the coordinate plane and decide if they are solutions to the inequality, given a two-variable inequality that represents a situation.
- Find the solutions to a two-variable inequality by using the graph of a related two-variable equation.
- Write inequalities to describe the constraints in a situation.

Here are the **activities** that helped you reach those goals:

- 2.12.1: Sketching a Graph to Represent an Equation
  - In this activity, you reviewed writing and graphing an equation that represents a situation. This prepared you to write linear inequalities that represent constraints in situations and graphing the solution regions.
- 2.12.2: Writing an Inequality to Represent a Constraint
  - In this activity, you wrote an inequality to represent a constraint in a situation. You graphed a related equation, interpreted the coordinate pairs of points on the graph and on either side of the graph, and tested the pairs of values to see if they make the inequality true. You then used these observations to determine the solution region to the inequality.
  - 2.12.2: Self Check
  - 2.12.2: Additional Resources
- 2.12.3: Graphing Solutions and Interpreting Points
  - In this activity, you continued to work with inequalities in two variables in context. You wrote an inequality that represents the constraints in a situation, graphed its solutions, and interpreted points in the solution region
  - 2.12.3: Self Check
  - 2.12.3: Additional Resources
- 2.12.4: Recognizing Points Along Boundary Line
  - In this activity, you had an additional opportunity for writing a linear inequality in two variables to represent constraints and for graphing and interpreting the solutions.
- 2.12.5: Recognizing Points for an Inequality
  - In this activity, you had more opportunities to understand the mathematical ideas and receive guidance for what to look for to identify when an inequality represents the constraints in a situation.

After these activities, you completed the following **practice:**

- 2.12.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

2.13 SOLVING PROBLEMS WITH INEQUALITIES IN TWO VARIABLES

2.13.0 Lesson Overview

In this lesson, you will learn to use graphing technology to find the solution set of a linear inequality in two variables. You will also write linear inequalities to represent the constraints in situations and then use the representations (including the graphs of the solutions) to answer questions about the situations.

When you finish this lesson, you will be able to:

- Use graphing technology to find the solution to a two-variable inequality.
- Connect the different representations and interpret them in terms of the situation, when given inequalities, graphs, and descriptions that represent the constraints in a situation.

Here are the **activities** that will help you reach those goals:

- 2.13.1: Graphing Inequalities with Technology
- 2.13.2: Solving Problems with Inequalities in Two Variables
  - 2.13.2: Self Check
  - 2.13.2: Additional Resources
- 2.13.3: Matching Representations of Inequalities
  - 2.13.3: Self Check
  - 2.13.3: Additional Resources
- 2.13.4: Solving Inequality Problems Using Tables
  - 2.13.4: Self Check
  - 2.13.4: Additional Resources
- 2.13.5: Graphing Solutions to an Inequality

After that, you'll **practice and review.**

- 2.13.6: Practice
- 2.13.7: Summary

2.13.1 Graphing Inequalities with Technology

Warm Up



Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

Graph each inequality. Adjust the graphing window as needed to show meaningful information.

1. Use the graphing tool or technology outside the course. Graph the inequality $y > x$ using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-13-1-graphing-inequalities-with-technology)

2. Use the graphing tool or technology outside the course. Graph the inequality $y \geq x$ using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-13-1-graphing-inequalities-with-technology)

3. Use the graphing tool or technology outside the course. Graph the inequality $y < -8$ using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-13-1-graphing-inequalities-with-technology)

4. Use the graphing tool or technology outside the course. Graph the inequality $-x + 8 \leq y$ using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-13-1-graphing-inequalities-with-technology)

5. Use the graphing tool or technology outside the course. Graph the inequality $y < 10x - 200$ using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-13-1-graphing-inequalities-with-technology)

6. Use the graphing tool or technology outside the course. Graph the inequality $2x + 3y > 60$ using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-13-1-graphing-inequalities-with-technology)

2.13.2 Solving Problems with Inequalities in Two Variables

Activity

Here are three types of situations. Answer the questions about each situation.

Bank Accounts

For questions 1 – 4, use the following scenario:

> A customer opens a checking account and a savings account at a bank. A maximum of $600 will be deposited into the two accounts, some in the checking account and some in the savings account. (They might not deposit all of it and keep some of the money as cash.)

1. Write an inequality to describe the constraints. Specify what each variable represents.

2. Rewrite your inequality using slope-intercept form. Then, use the graphing tool or technology outside the course. Graph the inequality that represents this scenario using the Desmos tool below.
Hint: To restrict the x and y values to the constraints of the problem, enter $\{0 \leq x \leq 600\}\{0 \leq y \leq 600\}$ after your equation.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-13-2-solving-problems-with-inequalities-in-two-variables)

3. Name one solution to the inequality and explain what it represents in that situation.

4. If the customer deposits $200 in their checking account, what can you say about the amount they deposit in their savings account?

For questions 5 – 8, use the following scenario:

> A customer opens a checking account and a savings account at a bank. The bank requires a minimum balance of $50 in the savings account. It does not matter how much money is kept in the checking account.

5. Write an inequality to describe the constraints. Specify what each variable represents

6. Rewrite your inequality using slope-intercept form. Then, use the graphing tool or technology outside the course. Graph the inequality that represents this scenario using the Desmos tool below.
Hint: To restrict the x values to the constraints of the problem (the checking account should never have less than $0), enter $\{x \geq 0\}$ after your equation.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-13-2-solving-problems-with-inequalities-in-two-variables)

7. Name one solution to the inequality and explain what it represents in that situation.
8. If the customer deposits no money in the checking account but is able to maintain both accounts without penalty, what can you say about the amount deposited in the savings account?

Concert Tickets

For questions 1 – 4, use the following scenario:

> Two kinds of tickets to an outdoor concert were sold: lawn tickets and seat tickets. Fewer than 400 tickets in total were sold.

1. Write an inequality to describe the constraints. Specify what each variable represents.
2. Rewrite your inequality using slope-intercept form. Then, use the graphing tool or technology outside the course. Graph the inequality that represents this scenario using the Desmos tool below.
Hint: To restrict the x and y values to the constraints of the problem, enter $\{0 \leq x\}\{0 \leq y\}$ after your equation (because you cannot have a negative number of tickets).
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-13-2-solving-problems-with-inequalities-in-two-variables)
3. Name one solution to the inequality and explain what it represents in that situation.
4. If you know that exactly 100 lawn tickets were sold, what can you say about the number of seat tickets?

For questions 5 – 8, use the following scenario:

> Lawn tickets cost $30 each and seat tickets cost $50 each. The organizers made at least $14,000 from ticket sales.

5. Write an inequality to describe the constraints. Specify what each variable represents.
6. Rewrite your inequality using slope-intercept form. Then, use the graphing tool or technology outside the course. Graph the inequality that represents this scenario using the Desmos tool below.
Hint: To restrict the x and y values to the constraints of the problem, enter $\{0 \geq x\}\{0 \geq y\}$ after your equation (because you cannot have a negative number of tickets).
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-13-2-solving-problems-with-inequalities-in-two-variables)
7. Name one solution to the inequality and explain what it represents in that situation.
8. If you know that exactly 200 seat tickets were sold, what can you say about the number of lawn tickets?

Advertising Packages

For questions 1 – 4, use the following scenario:

> An advertising agency offers two packages for small businesses who need advertising services. A basic package includes only design services. A premium package includes design and promotion. The agency's goal is to sell at least 60 packages in total.

1. Write an inequality to describe the constraints. Specify what each variable represents.
2. Rewrite your inequality using slope-intercept form. Then, use the graphing tool or technology outside the course. Graph the inequality that represents this scenario using the Desmos tool below.
   Hint: To restrict the x and y values to the constraints of the problem, enter $\{x \geq 0\}\{y \geq 0\}$ after your equation (because the business will not have a negative number of packages).
   Access multimedia content (http://openstax.org/books/algebra-1/pages/2-13-2-solving-problems-with-inequalities-in-two-variables)
3. Name one solution to the inequality and explain what it represents in that situation.
4. If the agency sells exactly 45 basic packages, what can you say about the number of premium packages it needs to sell to meet its goal?

For questions 5 – 8, use the following scenario:

The basic advertising package has a value of $1000 and the premium package has a value of $2500. The goal of the agency is to sell more than $60,000 worth of small-business advertising packages.

5. Write an inequality to describe the constraints. Specify what each variable represents.
6. Rewrite your inequality using slope-intercept form. Then, use the graphing tool or technology outside the course. Graph the inequality that represents this scenario using the Desmos tool below.
   Hint: To restrict the x and y values to the constraints of the problem, enter $\{x \geq 0\}\{y \geq 0\}$ after your equation.
   Access multimedia content (http://openstax.org/books/algebra-1/pages/2-13-2-solving-problems-with-inequalities-in-two-variables)
7. Name one solution to the inequality and explain what it represents in that situation.
8. If you know that exactly 10 premium packages were sold, what can you say about the number of basic packages the agency needs to sell to meet its goal?

Are you ready for more?

Extending Your Thinking

This activity will require a partner and either graph paper or Desmos.

1. Without letting your partner see it, write an equation of a line so that both the x-intercepts and the y-intercepts are each between -3 and 3. Graph your equation.
2. Still without letting your partner see it, rewrite your equation as an inequality. That is, write an inequality for which your equation is the related equation. In other words, your line should be the boundary between the solution region and non-solution region. Shade the solutions on your graph.
3. Take turns stating coordinates of points. Your partner will tell you whether your guess is a solution to their inequality. After each partner has stated a point, each may guess what the other's inequality is. If neither guesses correctly, play continues. Use another coordinate system to keep track of your guesses.
   What was your process for determining your partner's inequality?

Video: Solving Problems with Inequalities in Two Variables

Watch the following video to learn more about solving inequalities in two variables.

Solving Problems with Inequalities in Two Variables (https://openstax.org/r/inequalities-two-variables)

Self Check

Use graphing technology to solve the following problem.

Jacob charges $20 to mow a lawn in his neighborhood and $100 to paint a room.

Which combination of jobs earns Jacob more than $1,000?

a. Jacob mows 5 lawns and paints 9 rooms.
b. Jacob mows 35 lawns and paints 1 room.
c. Jacob mows 8 lawns and paints 10 rooms.
d. Jacob mows 10 lawns and paints 8 rooms.

Additional Resources

Graphing Linear Inequalities in Two Variables to Solve Applications

Many fields use linear inequalities to model a problem. While our examples may be about simple situations, they give us an opportunity to build our skills and to get a feel for how they might be used.

For questions 1 – 3, use the following scenario to write, graph, and find solutions to an inequality:

EXAMPLE

Hilaria works two part-time jobs in order to earn enough money to meet her obligations of at least $240 a week. Her job in food service pays $10 an hour and her tutoring job on campus pays $15 an hour. How many hours does Hilaria need to work at each job to earn at least $240?

1. Let x be the number of hours she works at the job in food service and let y be the number of hours she works tutoring. Write an inequality that would model this situation.

Solution

We let x be the number of hours she works at the job in food service and let y be the number of hours she works tutoring. She earns $10 per hour at the job in food service and $15 an hour tutoring. At each job, the number of hours multiplied by the hourly wage will gives the amount earned at that job.

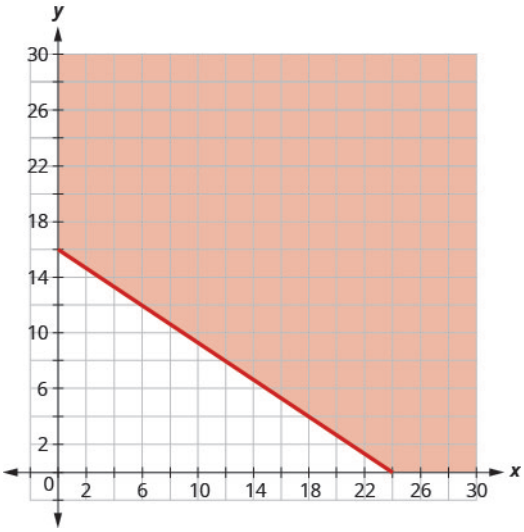
Amount earned at the food service job plus the amount earned tutoring is at least $240

$$10x + 15y \geq 240$$

2. Graph the inequality.

To graph the inequity, we put it in slope-intercept form.

$$10x + 15y \geq 240$$
$$15y \geq -10x + 240$$
$$y \geq -\frac{2}{3}x + 16$$



3. Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Hilaria.

From the graph, we see that the ordered pairs $(15, 10)$, $(0, 16)$, $(24, 0)$ represent three of infinitely many solutions. Check the values in the inequality.

(15, 10)	(0, 16)	(24, 0)
$10x + 15y \geq 240$	$10x + 15y \geq 240$	$10x + 15y \geq 240$
$10(15) + 15(10) \stackrel{?}{\geq} 240$	$10(0) + 15(16) \stackrel{?}{\geq} 240$	$10(24) + 15(0) \stackrel{?}{\geq} 240$
$300 \geq 240$ True	$240 \geq 240$ True	$240 \geq 240$ True

For Hilaria, it means that to earn at least $240, she can work 15 hours tutoring and 10 hours at her fast-food job, earn all her money tutoring for 16 hours, or earn all her money while working 24 hours at the job in food service.

TRY IT **Graphing Linear Inequalities in Two Variables to Solve Applications**

For questions 1 – 3, use the following scenario to write, graph, and find solutions to an inequality:

> Elijah works two part-time jobs. One is at a grocery store that pays $10 an hour and the other is babysitting for $13 per hour. Between the two jobs, Elijah wants to earn at least $260 a week. How many hours does Elijah need to work at each job to earn at least $260?

1. Let x be the number of hours he works at the grocery store and let y be the number of hours he works babysitting. Write an inequality that would model this situation.
2. Graph the inequality.
   Access multimedia content (http://openstax.org/books/algebra-1/pages/2-13-2-solving-problems-with-inequalities-in-two-variables)
3. Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Elijah.

2.13.3 Matching Representations of Inequalities

Activity

Your teacher will give you a set of cards (https://openstax.org/r/set-cards-2-13-3). Take turns with your partner to match a group of 4 cards that contain: a situation, an inequality that represents it, a graph that represents the solution region, and a solution written as a coordinate pair.

For each match that you find, explain to your partner how you know it's a match.

For each match that your partner finds, listen carefully to their explanation. If you disagree, discuss your thinking and work to reach an agreement.

Record your matches.

Group 1

- situation: perimeter of a rectangle
- inequality:
- a solution:
- sketch of graph:

Group 2

- situation: jar of coins
- inequality:
- a solution:
- sketch of graph:

Group 3

- situation: honey and jam
- inequality:
- a solution:
- sketch of graph:

Group 4

- situation: a school trip
- inequality:
- a solution:
- sketch of graph:

Self Check

Veronica works two part time jobs in order to earn enough money to meet her obligations of at least $280 a week. Her job at the day spa pays $10 an hour and her administrative assistant job on campus pays $17.50 an hour.

Which of the following is a solution that allows Veronica to earn enough money?

a. 5 hours at the day spa and 15 hours as an administrative assistant
b. 15 hours at the day spa and 5 hours as an administrative assistant
c. 10 hours at the day spa and 10 hours as an administrative assistant
d. 5 hours at the day spa and 10 hours as an administrative assistant

Additional Resources

Connecting Representations of Inequalities

In order to connect the constraints of a situation with an inequality, its graph, and possible solutions, we need a mastery of each representation.

EXAMPLE

Christy sells her photographs at a booth at a street fair. Each small photo she displays costs her $4 and each large photo costs her $10. She doesn't want to spend more than $200 on photos to display.

Let x be the number of small photos and y be the number of large photos that Christy displays.

We know that each small photo costs $4 to display and each large photo costs $10. We also know that she doesn't want

to spend more than $200.

The inequality that represents this situation is $4x + 10y \leq 200$.

To be able to connect this inequality with possible solutions, translating it into a graph is an important step.

TRY IT **Connecting Representations of Inequalities**

1. Graph the linear inequality: $4x + 10y \leq 200$.
2. Is it possible for Christy to display 15 small paintings and 15 large paintings and maintain her budget of $200?

2.13.4 Solving Inequality Problems Using Tables

Activity



For questions 1 – 5, use the following scenario to graph an inequality from a table:

Andre is making trail mix with dried fruit and mixed nuts. Dried fruit costs $3 per pound and mixed nuts cost $6 per pound. He wants to spend no more than $54 on the trail mix. The inequality $y \leq -2x + 18$ represents the number of pounds of dried fruit, y, he can buy for x pounds of nuts.

1. The related equation $y = -2x + 18$ represents the boundary line of the inequality. Complete the table of values to find points on the boundary line of the inequality.

x	y
0	
2	
4	
6	

2. Use the graphing tool or technology outside the course. Graph the data from your table of values that represents this scenario using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-13-4-solving-inequality-problems-using-tables)

3. Is the boundary line for the inequality $y \leq -2x + 18$ a solid line or a dashed line? Explain how you know.

4. Is the point $(0, 0)$ a solution to the inequality?

5. Use the graph from number 2. Enter the inequality $y \leq -2x + 18$ to represent the data. Notice the shading for the solution region of this inequality.

In problems 6–10, use the data from the table to graph the inequality and write an inequality with two variables.

x	y
-2	-5
0	-6
2	1
3	5

6. Use the graphing tool or technology outside the course. Graph the data from your table of values that represents this scenario using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-13-4-solving-inequality-problems-using-tables)

7. Now, in the Desmos window graph $y = x$.

8. How does the line need to move to capture all of the points?

9. Using desmos, adjust the slope and/or the y-intercept to find an inequality that could be a possibility.

10. Use the graph from number 7. Shade the correct side of the boundary line to represent the inequality $y > 3x - 2$.

Self Check

Which table of values can be used to graph the boundary line of the inequality $y > -3x + 1$?

x	y
-2	7
0	1
1	-5
3	-11

x	y
-2	7
0	1
1	-2
3	-8

x	y
-2	-5
0	1
1	4
3	10

x	y
-2	8
0	2
1	-1
3	-8

Additional Resources

Connecting Representations of Inequalities

Tables can be used to help you to write and solve inequalities in two variables.

Part 1: Writing Inequalities in Two Variables Given A Set of Data in a Table

Write a linear inequality in two variables based on the given data:

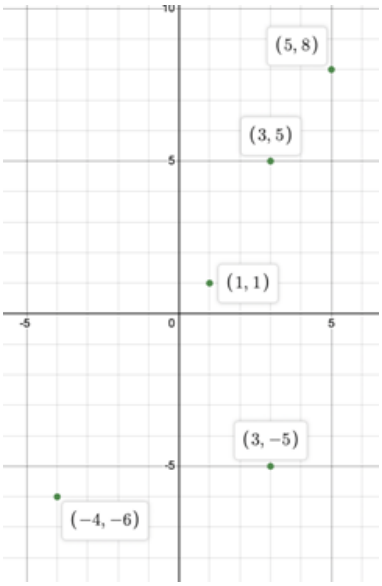
x	y
-4	-3
-2	-2
0	-1
2	0

Remember that for an inequality the points could be on the line, above the the line, or below the line. The type of inequality will differ based on these conditions.

On the line	$\leq$ or $\geq$
Above the line	$>$
Below the line	$<$

If you are given a set of possible points like the table above, there could be multiple lines that the data points could represent. Let's figure out a few possibilities for this given set.

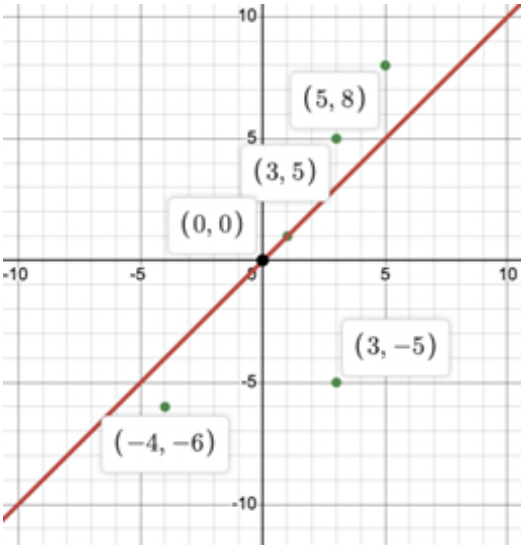
Step 1 - Graph the points on a coordinate plane.



Step 2 - Determine possible lines.

Remember that the dots can be on the line, but they all have to be either above or below to create an inequality. If the line has points on both sides, then the line isn't a possibility.

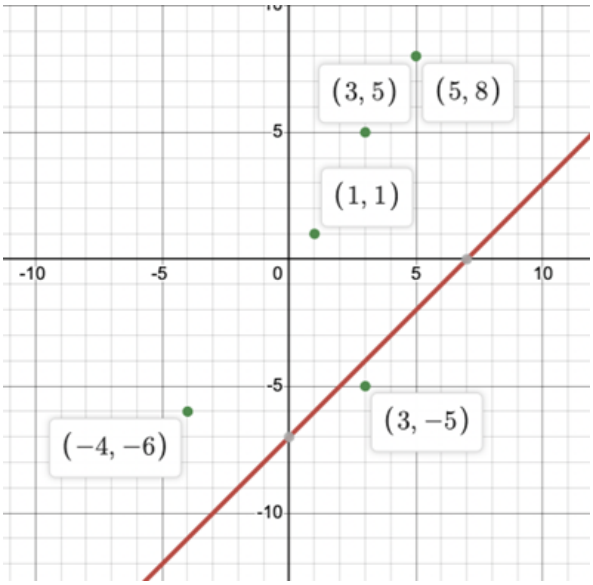
Let's try a few possibilities. We can start with the equation $y = x$ to see if the points are above the line or below the line.



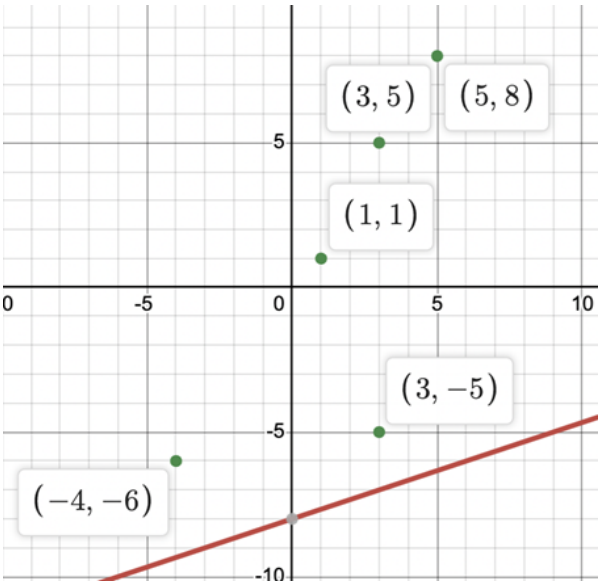
The equation $y = x$ cannot be a possibility because there are points above and below the line.

Step 3 - Adjust the slope or y-intercept

We can change the slope and the y-intercept to see how we can shift the line to be able to capture all the points. Let's move the line to below the $(3, -5)$ and $(-4, -6)$. We can leave the slope as 1 and change the y-intercept to -7.



There is still one point below the line. We could decrease the slope of the line to $m = 13$.



Now all of the points are above the line. This means that the value of y is always greater than the x value.

So, one of the possible solutions could be $y > 13x - 7$.

Part 2: Using Tables to Solve Inequalities in Two Variables

Graph the linear inequality $y < \frac{1}{2}x - 1$ using a table.

Step 1 - Create a table of values using the related equation $y = \frac{1}{2}x - 1$ to identify points on the boundary line of the inequality.

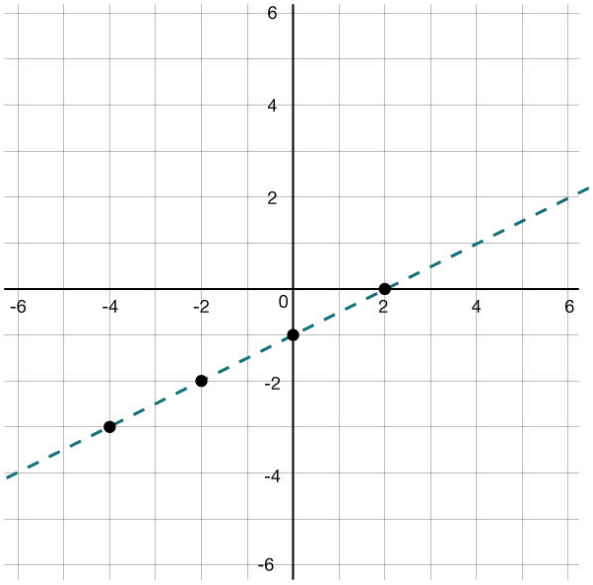
x	y
−4	−3
−2	−2
0	-1
2	0

Step 2 - Plot the points and sketch the boundary line.

If the inequality is $\leq$ or $\geq$, the boundary line is solid.

If the inequality is $<$ or $>$, the boundary line is dashed.

The inequality sign is $<$, so we draw a dashed line.



Step 3 - Test a point that is not on the boundary line. Is it a solution of the inequality?

Test $(0, 0)$ as it is not on the boundary line of the inequality.

$y \overset{?}{<} \frac{1}{2}x - 1$

$(0) < \frac{1}{2}(0) - 1$

$0 < -1$ is False.

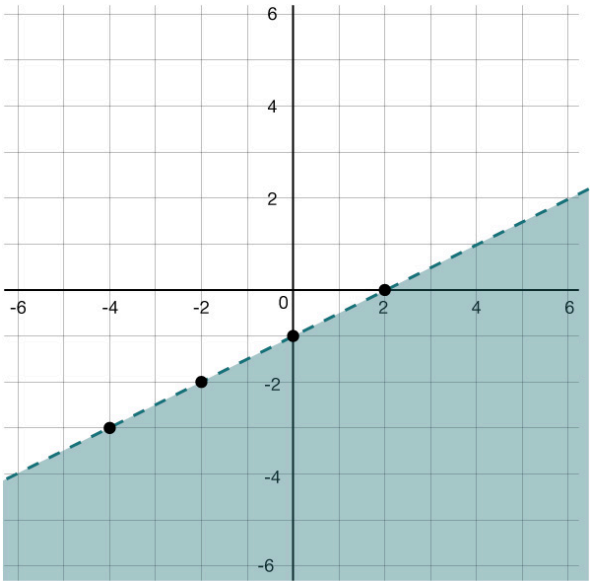
So, $(0, 0)$ is not a solution.

Step 4 - Shade in one side of the boundary line.

If the test point is a solution, shade in the side that includes the point.

If the test point is not a solution, shade in the opposite side.

The test point $(0, 0)$ is not a solution, so shade in the side of the boundary line that does not include $(0, 0)$.



All points in the shaded region and not on the boundary line represent the solutions to $y < \frac{1}{2}x - 1$.

TRY IT **Graph Linear Inequalities in Two Variables**

1. Is the inequality $y < 3x + 6$ a solution for the table of values? Explain your reasoning.

x	y
-4	-3
-2	-2
0	-1
2	0

2. Use a table to graph the inequality $y = 2x + 3$.

2.13.5 Graphing Solutions to an Inequality

Cool Down

For questions 1 – 4, use the following scenario to write, graph, and find solutions to inequalities.

A band is playing at an auditorium with floor seats and balcony seats. The band wants to sell the floor tickets for $15 each and balcony tickets for $12 each. They want to make at least $3,000 in ticket sales.

1. How much money will they collect for selling x floor tickets?
2. How much money will they collect for selling y balcony tickets?
3. Write an inequality whose solutions are the number of floor and balcony tickets sold if they make at least $3,000 in ticket sales.
4. Use the graphing tool or technology outside the course. Graph the inequality that represents this scenario using the Desmos tool below. Access multimedia content (http://openstax.org/books/algebra-1/pages/2-13-5-graphing-solutions-to-an-inequality)

5. Write 3 linear inequalities in two variables given this table of values:

x	y
4	4
1	5
-2	3
0	0

2.13.6 Practice

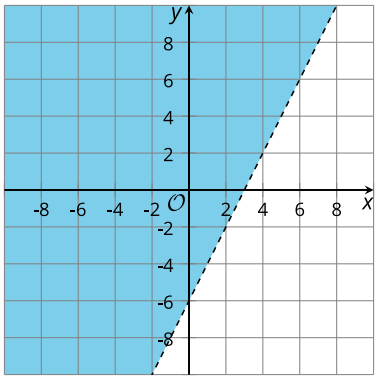
Complete the following questions to practice the skills you have learned in this lesson.

This year, students in the 9th grade are collecting dimes and quarters for a school fundraiser. They are trying to collect more money than the students who were in the 9th grade last year. The students in 9th grade last year collected $143.88.

1. Using d to represent the number of dimes collected and q to represent the number of quarters, which statement best represents this situation?

a. $0.25q + 0.1d > 143.88$
b. $0.25d + 0.1q > 143.88$
c. $0.25q + 0.1d < 143.88$
d. $0.25d + 0.1q \geq 143.88$

2. Which inequality is represented by the graph?



a. $4x + 2y < 12$
b. $4x + 2y > 12$
c. $4x - 2y < 12$
d. $4x - 2y > 12$

For questions 3 – 4, use the scenario below to answer the questions:

A farmer is creating a budget for planting soybeans and wheat. Planting soybeans costs $200 per acre and planting wheat costs $500 per acre. He wants to spend no more than $100,000 planting soybeans and wheat.

3. Let s represent the number of acres of soybeans and w represent the number of acres of wheat. Write an inequality to describe the situation.

a. $200s + 500w > 100,000$
b. $200s + 500w \leq 100,000$
c. $500s + 200w \leq 100,000$
d. $500s + 200w < 100,000$

4. Name one solution to the inequality and explain what it represents in that situation.

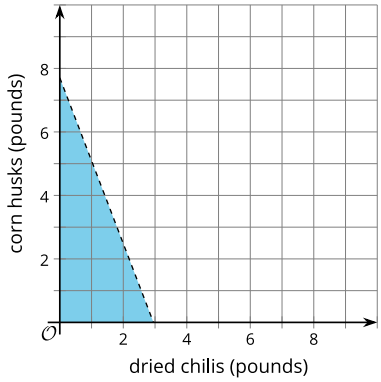
a. $(300, 90)$
b. $(170, 140)$
c. $(250, 100)$
d. $(200, 130)$

For questions 5 – 7, use the passage below to answer the questions:

Amira is ordering dried chili peppers and corn husks for her cooking class. Chili peppers cost $16.95 per pound and corn husks cost $6.49 per pound.

Amira spends less than $50 on d pounds of dried chili peppers and h pounds of corn husks.

Here is a graph that represents this situation.



5. Write an inequality that represents this situation.

a. $16.95d + 6.49h > 50$
b. $16.95d + 6.49h < 50$
c. $16.95d - 6.49h < 50$
d. $16.95d + 6.49h \geq 50$

6. Can Amira purchase 2 pounds of dried chili peppers and 4 pounds of corn husks and spend less than $50? Be prepared to show your reasoning.

a. Yes
b. No

7. Can Amira purchase 1.5 pounds of dried chili peppers and 3 pounds of corn husks and spend less than $50? Be prepared to show your reasoning.

a. Yes
b. No

For questions 8 – 10, use the scenario below to answer the questions:

A beach store rents surfboards for $15 per hour and beach umbrellas for $10 per hour. The store wants to make at least $200 per day on renting beach equipment. Let s be the number of hours that a surfboard is rented and u be the number of hours an umbrella is rented.

8. Which inequality represents this situation?

a. $15s + 10u < 200$
b. $15s + 10u \geq 200$
c. $15s + 10u \leq 200$
d. $15s - 10u \geq 200$

9. Can the beach store rent surfboards for 7 total hours and rent umbrellas for 9 total hours and reach their goal of making $200? Be prepared to show your reasoning.

a. Yes
b. No

10. Can the beach store rent surfboards for 10 total hours and rent umbrellas for 4 total hours and reach their goal of making $200? Be prepared to show your reasoning.

a. Yes
b. No

11. Mariah has data in a table. Plot the data. Which inequality in two variables represents these ordered pairs as solutions?

x	y
−4	7
−2	3
0	-1
2	-5

a. $y < -2x - 4$
b. $y \leq -2x - 4$
c. $y > -2x - 4$

2.13.7 Lesson Summary

In this lesson, you learned how to:

- Use graphing technology to find the solution to a two-variable inequality.
- Connect the different representations and interpret them in terms of the situation, when given inequalities, graphs, and descriptions that represent the constraints in a situation.
- Solve inequality problems using tables.

Here are the **activities** that helped you reach those goals:

- 2.13.1: Graphing Inequalities with Technology
  - In this activity, you used graphing technology to graph simple linear inequalities in two variables. You practiced adjusting the graphing window until the solution regions become visible and give useful information.
- 2.13.2: Solving Problems with Inequalities in Two Variables
  - In this activity, you integrated several ideas and skills from the past few lessons. You wrote inequalities in two variables to represent constraints in situations, used technology to graph the solutions, interpreted points in the solution regions, and used the inequalities and the graphs to answer contextual questions.
  - 2.13.2: Self Check
  - 2.13.2: Additional Resources
- 2.13.3: Matching Representations of Inequalities
  - In this activity, you practiced interpreting inequalities in context and reasoning about your solutions graphically and numerically. This sorting and matching task gave you opportunities to analyze representations, statements, and structures closely and to make connections.
  - 2.13.3: Self Check
  - 2.13.3: Additional Resources
- 2.13.4: Solving Inequality Problems Using Tables
  - 2.13.4 Self Check
  - 2.13.4 Additional Resources
- 2.13.5: Solving Inequalities with Tables
  - 2.13.5 Self Check
  - 2.13.5 Additional Resources
- 2.13.5: Graphing Solutions to an Inequality
  - In this activity, you used technology to graph the solutions to your inequality and determined which side of the boundary line to shade.

After these activities, you completed the following **practice:**

- 2.13.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

2.14 SOLUTIONS TO SYSTEMS OF LINEAR INEQUALITIES IN TWO VARIABLES

2.14.0 Lesson Overview

In this lesson, you will find the solutions to systems of linear inequalities in two variables.

When you finish this lesson, you will be able to:

- Write a system of inequalities to describe a situation, find the solution by graphing, and interpret points in the solution.
- Explain what is meant by "the solutions to a system of inequalities" and describe the graphs that represent the solutions.
- Find values that satisfy each constraint individually, and values that satisfy both constraints at once, when given descriptions and graphs that represent two different constraints.

Here are the **activities** that will help you reach those goals:

- 2.14.1: Modeling Equations to Solve a Riddle
- 2.14.2: Finding a Pair of Values that Satisfies Multiple Inequalities
  - 2.14.2: Self Check
  - 2.14.2: Additional Resources
- 2.14.3: Writing Systems of Inequalities that Represent Situations
  - 2.14.3: Self Check
  - 2.14.3: Additional Resources
- 2.14.4: Graphing Solutions of Systems of Inequalities
- 2.14.5: Writing Inequalities for Situations and Representing on a Graph

After that, you'll **practice and review**.

- 2.14.6: Practice
- 2.14.7: Lesson Summary

2.14.1 Modeling Equations to Solve a Riddle

Warm Up

For questions 1 – 4, use the following scenario to use equations to solve a riddle:

> Here is a riddle: "I am thinking of two numbers that add up to 5.678. The difference between them is 9.876. What are the two numbers?"

1. Name any pair of numbers whose sum is 5.678.

2. Name any pair of numbers whose difference is 9.876.

3. The riddle can be represented with two equations. Write the equations.

4. Solve the riddle. Be prepared to show your reasoning.

2.14.2 Finding a Pair of Values that Satisfies Multiple Inequalities

Activity

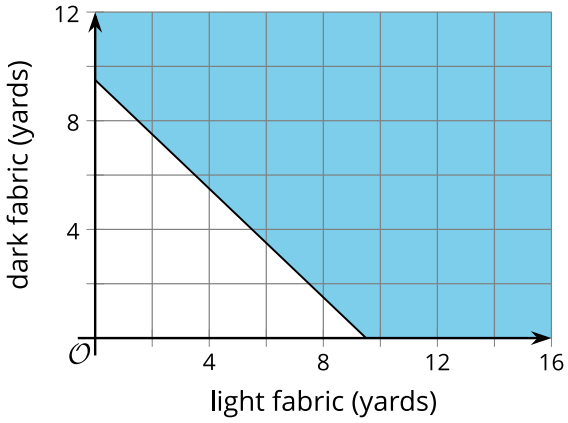
To make a quilt, a quilter is buying fabric in two colors, light and dark. He needs at least 9.5 yards of fabric in total.

The light color costs $9 a yard. The dark color costs $13 a yard. The quilter can spend up to $110 on fabric.

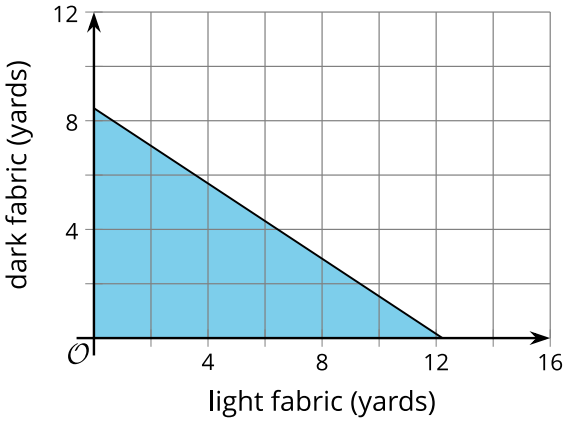


Here are two graphs that represent the two constraints.

Length



Cost



1. Write an inequality to represent the length constraint. Let x represent the yards of light fabric and y represent the yards of dark fabric.

2. Select **three** pairs that satisfy the length constraint.
   - $(5, 5)$
   - $(2.5, 4.5)$
   - $(7.5, 3.5)$
   - $(12, 10)$

3. Write an inequality to represent the cost constraint.

4. Select **three** the pairs that satisfy the cost constraint.
   - $(4, 5)$
   - $(10, 1)$
   - $(8, 3)$
   - $(1, 1)$

5. Explain why $(2, 2)$ satisfies the cost constraint but not the length constraint.

6. Find at least one pair of numbers that satisfies both constraints. Be prepared to explain how you know.

7. What does the pair of numbers represent in this situation?

8. Write the two inequalities together as a **system of inequalities**.

Self Check

A singer sold two types of tickets to a performance, seated tickets and standing-room tickets.

- Seated tickets cost $25.
- Standing-room tickets cost $15.
- More than 60 tickets were sold.
- More than $1,200 was made from selling tickets.

Which of the following ticket combinations is possible given the situation described here?

a. 20 seated tickets and 50 standing-room tickets
b. 10 seated tickets and 60 standing-room tickets
c. 40 seated tickets and 19 standing-room tickets
d. 14 seated tickets and 55 standing-room tickets

Additional Resources

Determine Whether an Ordered Pair Is a Solution of a System of Linear Inequalities

The definition of a **system of linear inequalities** is very similar to the definition of a **system of linear equations**.

System of Linear Inequalities

Two or more linear inequalities grouped together form a system of linear inequalities.

A system of linear inequalities looks like a system of linear equations, but it has inequalities instead of equations. A system of two linear inequalities is shown here.

$$\begin{cases} x+4y \geq 10 \\ 3x-2y < 12 \end{cases}$$

To solve a system of linear inequalities, we will find values of the variables that are solutions to both inequalities. We solve the system by using the graphs of each inequality and show the solution as a graph. We will find the region on the plane that contains all ordered pairs (x, y) that make both inequalities true.

Solutions of A System of Linear Inequalities

Solutions of a system of linear inequalities are the values of the variables that make all the inequalities true.

The **solution of a system of linear inequalities** is shown as a shaded region in the (x, y) coordinate system that includes all the points whose ordered pairs make the inequalities true.

To determine if an ordered pair is a solution to a system of two inequalities, we substitute the values of the variables into each inequality. If the ordered pair makes both inequalities true, it is a solution to the system.

EXAMPLE

Determine whether the ordered pair is a solution to this system:

$$\begin{cases} x+4y \geq 10 \\ 3x-2y < 12 \end{cases}$$

a.) $(-2, 4)$ b.) $(3, 1)$

a.) Is the ordered pair $(-2, 4)$ a solution?

We substitute $x = -2$ and $y = 4$ into both inequalities.

$$x + 4y \geq 10 \qquad 3x - 2y < 12$$

$$-2 + 4(4) \stackrel{?}{\geq} 10 \qquad 3(-2) - 2(4) \stackrel{?}{<} 12$$

$$14 \geq 10 \text{ true} \qquad -14 < 12 \text{ true}$$

The ordered pair $(-2, 4)$ made both inequalities true. Therefore $(-2, 4)$ is a solution to this system.

b.) Is the ordered pair $(3, 1)$ a solution?

We substitute $x = 3$ and $y = 1$ into both inequalities.

$$\begin{array}{ll} x + 4y \geq 10 & 3x - 2y < 12 \\ 3 + 4(1) \stackrel{?}{\geq} 10 & 3(3) - 2(1) \stackrel{?}{<} 12 \\ 7 \geq 10 \text{ false} & 7 < 12 \text{ true} \end{array}$$

The ordered pair $(3, 1)$ made one inequality true, but it made the other one false. Therefore, $(3, 1)$ is not a solution to this system.

TRY IT **Determine Whether an Ordered Pair is a Solution of a System of Linear Inequalities**

Determine whether the ordered pair is a solution to the system:

$$\begin{cases} y > 4x - 2 \\ 4x - y < 20 \end{cases}$$

a. $(-2, 1)$

b. $(4, -1)$

2.14.3 Writing Systems of Inequalities that Represent Situations

Activity

Here are some situations you have seen before. Your teacher will give you one situation. Answer the questions for one situation. You will then meet with others in a group. You each will explain the situation and your answers.

Bank Accounts

- A customer opens a checking account and a savings account at a bank. They will deposit a maximum of $600 will be deposited into the two accounts, some in the checking account and some in the savings account. (They might not deposit all of it and keep some of the money as cash.)
- The bank requires a minimum balance of $50 in the savings account. It does not matter how much money is kept in the checking account.

Concert Tickets

- Two kinds of tickets to an outdoor concert were sold: lawn tickets and seat tickets. Fewer than 400 tickets in total were sold.
- Lawn tickets cost $30 each, and seat tickets cost $50 each. The organizers want to make at least $14,000 from ticket sales.

Advertising Packages

- An advertising agency offers two packages for small businesses that need advertising services. A basic package includes only design services. A premium package includes design and promotion. The agency's goal is to sell at least 60 packages in total.
- The basic advertising package has a value of $1,000, and the premium package has a value of $2,500. The goal of the agency is to sell more than $60,000 worth of small-business advertising packages.

Based on the discussions you had with your group members, answer the following questions about each situation.

1. Write a system of inequalities to represent the constraints. Specify what each variable represents for the Bank Accounts situation.
2. Write a system of inequalities to represent the constraints. Specify what each variable represents for the Concert Tickets situation.
3. Write a system of inequalities to represent the constraints. Specify what each variable represents for the Advertising Packages situation.

For questions 4 – 6, use technology or paper to graph the systems of inequalities you identified in the previous questions. Be sure to alter the scale along each axis to fit the data in the scenario.

4. Use the graphing tool or technology outside the course. Graph the system of inequalities that represents the Bank Accounts scenario using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-14-3-writing-systems-of-inequalities-that-represent-situations)

5. Use the graphing tool or technology outside the course. Graph the system of inequalities that represents the Concert Tickets scenario using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-14-3-writing-systems-of-inequalities-that-represent-situations)

6. Use the graphing tool or technology outside the course. Graph the system of inequalities that represents the Advertising Packages scenario using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-14-3-writing-systems-of-inequalities-that-represent-situations)

For questions 7 – 9, review the graphs you created for each system of inequalities in the questions in the previous questions. Use these graphs to help find the solutions to each scenario.

7. Identify a solution to the system that represents the Bank Accounts situation. Explain what the numbers mean in the situation.

8. Identify a solution to the system that represents the Concert Tickets situation. Explain what the numbers mean in the situation.

9. Identify a solution to the system that represents the Advertising Packages situation. Explain what the numbers mean in the situation.

Video: Concert Tickets

Watch the following video to learn more about writing systems of inequalities to represent situations.

Concert Tickets (https://openstax.org/r/concert-tickets)

Self Check

A music store is shipping guitars and snare drums.

- Each guitar weighs 6 pounds.
- Each snare drum weighs 14 pounds.
- The maximum weight of guitars and snare drums cannot be more than 120 pounds.
- The store needs to ship more than 10 total instruments.

Which of the following systems of inequalities describes the situation if x is the number of guitars and y is the number of snare drums?

a. $6x + 14y \leq 120$
$x + y > 10$
b. $14x + 6y \geq 120$
$x + y < 10$
c. $14x + 6y \leq 120$
$x + y > 10$
d. $6x + 14y \geq 120$
$x + y > 10$

Additional Resources

Write a System of Inequalities for a Situation

For questions 1 – 4, use the following scenario to write, graph, and find solutions to a system of inequalities.

> Christy sells her photographs at a booth at a street fair. At the start of the day, she wants to have at least 25 photos to display at her booth. Each small photo she displays costs her $4, and each large photo costs her $10.

She doesn't want to spend more than $200 on photos to display.

1. Write a system of inequalities to model this situation.

Step 1 - Identify the Variables.

Let $x =$ the number of small photos.

Let $y =$ the number of large photos.

Step 2 - To find the system of equations, translate the information. She wants to have at least 25 photos. The number of small plus the number of large should be at least 25.

$$x + y \geq 25$$

$4 for each small and $10 for each large must be no more than $200.

$$4x + 10y \leq 200$$

To find the system of equations, translate the information.

She wants to have at least 25 photos.

The number of small plus the number of large should be at least 25.

$$x + y \geq 25$$

$4 for each small and $10 for each large must be no more than $200.

$$4x + 10y \leq 200$$

Step 3 - Identify constraints based on the context.

The number of small photos must be greater than or equal to 0.

$$x \geq 0$$

The number of large photos must be greater than or equal to 0.

$$y \geq 0$$

Step 4 - Write the system of inequalities.

We have our system of equations.

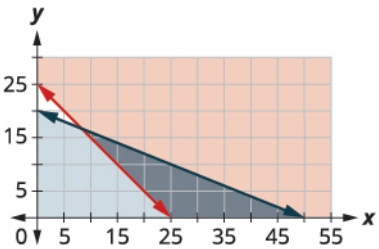
$$\begin{cases} x + y \geq 25 \\ 4x + 10y \leq 200 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

2. Graph the system

Since $x \geq 0$ and $y \geq 0$ (both are greater than or equal to zero), all solutions will be in the first quadrant. As a result, our graph shows only quadrant one.

To graph $x + y \geq 25$, graph $x + y = 25$ as a solid line. Choose $(0, 0)$ as a test point. Since it does not make the inequality true, shade (red) the side that does not include the point $(0, 0)$.

To graph $4x + 10y \leq 200$, graph $4x + 10y = 200$ as a solid line. Choose $(0, 0)$ as a test point. Since it does make the inequality true, shade (blue) the side that includes the point $(0, 0)$.



3. Could she display 10 small and 20 large photos?

The solution of the system is the region of the graph that is shaded the darkest. The boundary line sections that border the darkly shaded section are included in the solution, as are the points on the x-axis from $(25, 0)$ to $(50, 0)$.

To determine if 10 small and 20 large photos would work, we look at the graph to see if the point $(10, 20)$ is in the solution region. We could also test the point to see if it is a solution of both equations.

It is not, so Christy would not display 10 small and 20 large photos.

4. Could she display 20 large and 10 small photos?

To determine if 20 small and 10 large photos would work, we look at the graph to see if the point $(20, 10)$ is in the solution region. We could also test the point to see if it is a solution of both equations.

It is, so Christy could choose to display 20 small and 10 large photos.

Notice that we could also test the possible solutions by substituting the values into each inequality.

TRY IT **Write a System of Inequalities for a Situation**

A trailer can carry a maximum weight of 160 pounds and a maximum volume of 15 cubic feet. A microwave oven, m, weighs 30 pounds and has 2 cubic feet of volume, while a printer, p, weighs 20 pounds and has 3 cubic feet of volume.

Write a system of inequalities to model this situation.

2.14.4 Graphing Solutions of Systems of Inequalities

Activity

Members of a high school math club have a scavenger hunt. Three items are hidden in the park, which is a rectangle that measures 50 meters by 20 meters.

- The clues are written as systems of inequalities. One system has no solutions.
- The locations of the items can be narrowed down by solving the systems. A coordinate plane can be used to describe the solutions.

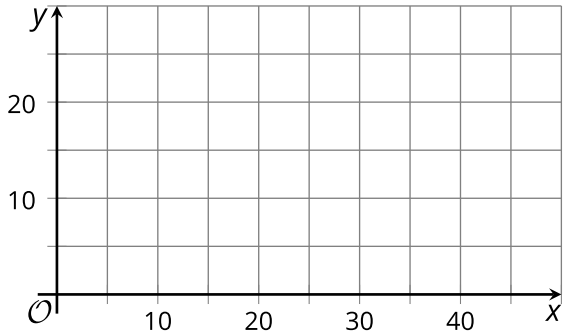
Can you find the hidden items? Sketch a graph to show where each item could be hidden.

You will be provided with a sheet of graph paper to help solve this scavenger hunt without the use of technology. For each clue, graph each inequality separately on the same coordinate grid.

Clue 1:

$y > 14$

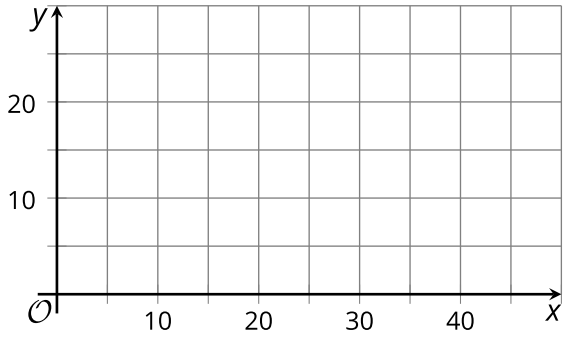
$x < 10$



Clue 2:

$x + y < 20$

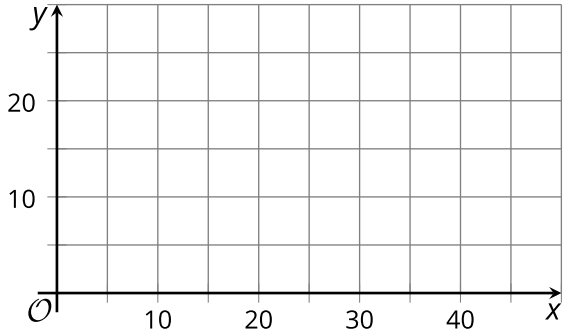
$x > 6$



Clue 3:

$y < -2x + 20$

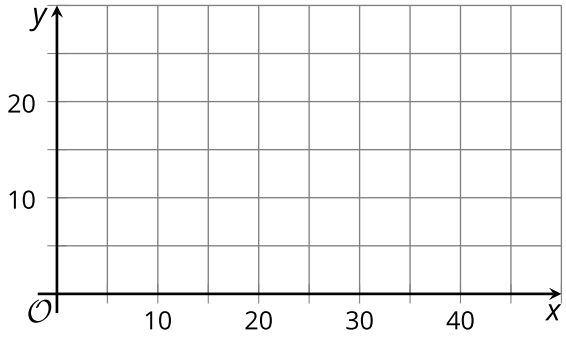
$y < -2x + 10$



Clue 4:

$y \geq x + 10$

$x > y$



Reflect: When you graphed each pair of inequalities, you created a system of inequalities. Where did the possible solutions lie on the graph?

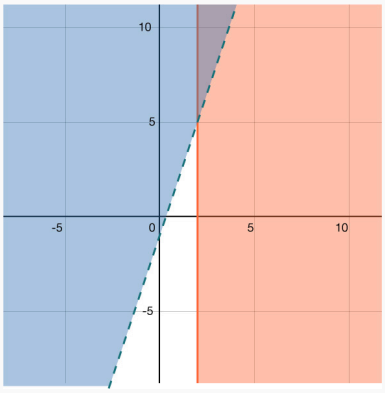
Are you ready for more?

Extending Your Thinking

1. Two non-negative numbers x and y satisfy $x + y \leq 1$. Find a second inequality, also using x and y values greater than or equal to zero, to make a system of inequalities with exactly one solution.
2. Two non-negative numbers x and y satisfy $x + y \leq 1$. Find as many ways to identify the answers to this question as you can.

Self Check

Which two inequalities are graphed in the system of inequalities below?



a. $x > 2$
$y \geq 3x - 1$

b. $x \geq 2$
$y < 3x - 1$

c. $x \geq 2$
$y > 3x - 1$

d. $x \leq 2$
$y > 3x - 1$

Additional Resources

Solve a System of Linear Inequalities by Graphing

The solution to a single linear inequality is the region on one side of the boundary line that contains all the points that make the inequality true. The solution to a system of two linear inequalities is a region that contains the solutions to both inequalities. To find this region, we will graph each inequality separately and then locate the region where they are both true. The solution is always shown as a graph.

EXAMPLE

Solve the system by graphing:

$$\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$$

Answer:

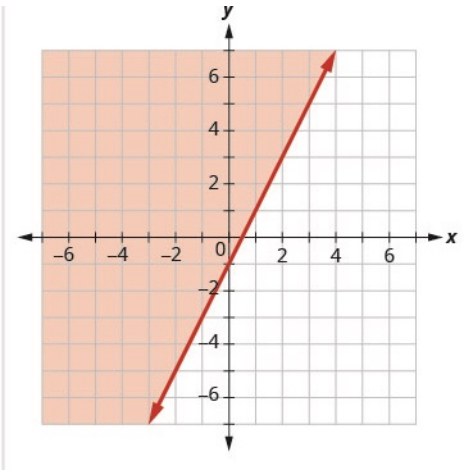
Step 1 - Graph the first inequality.

- Graph the boundary line.
- Shade in the side of the boundary line where the inequality is true.

We will graph $y \geq 2x - 1$

We graph the line $y = 2x - 1$. It is a solid line because the inequality sign is $\geq$.

We choose $(0, 0)$ as a test point. It is a solution to $y \geq 2x - 1$, so we shade in above the boundary line.



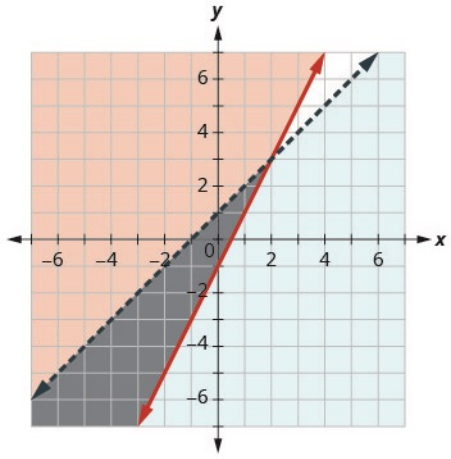
Step 2 - On the same grid, graph the second inequality.

- Graph the boundary line.
- Shade the side of that boundary line where the inequality is true.

We will graph $y < x + 1$ on the same grid.

We graph the line $y = x + 1$. It is a dotted line because the inequality sign is $<$.

Again, we use $(0, 0)$ as a test point. It is a solution so we shade in that side of the line $y = x + 1$.



Step 3 - The solution is the region where the shading overlaps.

The point where the boundary lines intersect is not a solution because it is not a solution to $y < x + 1$.

The solution is all points in the area shaded twice, which appears as the darkest shaded region.

Step 4 - Check by choosing a test point.

We'll use $(-1, -1)$ as a test point.

Is $(-1, 1)$ a solution to $y \geq 2x - 1$?

$-1 \stackrel{?}{\geq} 2(-1) - 1$

$-1 \geq -3$

✓

Is $(-1, -1)$ a solution to $y < x + 1$?

$-1 \stackrel{?}{\geq} -1 + 1$

$-1 < 0$

✓

The region containing $(-1, -1)$ is the solution to this system.

TRY IT **Solve a System of Linear Inequalities by Graphing**

1. Solve the system by graphing

$$\begin{cases} y < 3x + 2 \\ y > -x - 1 \end{cases}$$

2.14.5 Writing Inequalities for Situations and Representing on a Graph

Cool Down

For questions 1–3, use the following scenario to write inequalities and to identify solutions.

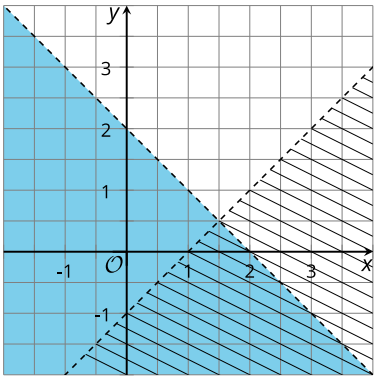
Here is another riddle:

- The sum of two numbers is less than 2.
- If we subtract the second number from the first, the difference is greater than 1.

What are the two numbers?

1. The riddle can be represented by a system of inequalities. Write an inequality for each statement.

2. These graphs represent the inequalities in the system.



3. Name a possible solution to the riddle. Explain or show how you know.

2.14.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

In questions 1 - 5, determine whether each ordered pair is a solution to the system.

1. $\begin{cases} 3x + y > 5 \\ 2x - y \leq 10 \end{cases}$

a. (3, -3)

a. Yes
b. No

b. $(7, 1)$

a. Yes
b. No

2. $\begin{cases} 4x - y < 10 \\ -2x + 2y > -8 \end{cases}$

a. (5, -2)

a. Yes
b. No

b. $(-1, 3)$

a. Yes
b. No

3. $\begin{cases} y > \frac{2}{3}x - 5 \\ x + \frac{1}{2}y \leq 4 \end{cases}$

a. (6, −4)

a. Yes
b. No

b. $(3, 0)$

a. Yes
b. No

4. $\begin{cases} y < \frac{3}{2}x + 3 \\ \frac{3}{4}x - 2y < 5 \end{cases}$

a. (−4, −1)

a. Yes
b. No

b. $(8, 3)$

a. Yes
b. No

5. $\begin{cases} 6x - 5y < 20 \\ -2x + 7y > -8 \end{cases}$

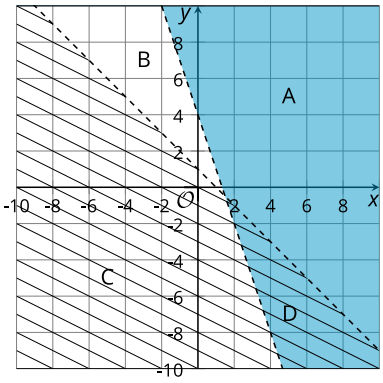
a. (1, −3)

a. Yes
b. No

b. $(-4, 4)$

a. Yes
b. No

6. Two inequalities are graphed on the same coordinate plane. Which region represents the solution to the system of the two inequalities?



a. Region A
b. Region B
c. Region C
d. Region D

2.14.7 Lesson Summary

In this lesson, you learned how to:

- Write a system of inequalities to describe a situation, find the solution by graphing, and interpret points in the solution.
- Explain what is meant by "the solutions to a system of inequalities" and describe the graphs that represent the solutions.
- Find values that satisfy each constraint individually, and values that satisfy both constraints at once, when given descriptions and graphs that represent two different constraints.

Here are the **activities** that helped you reach those goals:

- 2.14.1: Modeling Equations to Solve a Riddle
  - In this activity, you reviewed systems of equations and their solutions. You recalled that a solution to a linear equation in two variables is any pair of numbers that makes the equation true, and that a solution to a system of two equations in two variables is a pair of numbers that makes both equations true.
- 2.14.2: Finding a Pair of Values that Satisfy Multiple Inequalities
  - In this activity, you encountered a situation in which two constraints can be expressed with inequalities represented on two separate graphs, which made it challenging to find pairs of values that meet both constraints simultaneously. This served to motivate a desire to represent both constraints on the same graph.
  - 2.14.2: Self Check
  - 2.14.2: Additional Resources
- 2.14.3: Writing Systems of Inequalities that Represent Situations
  - In this activity, you wrote systems of inequalities that represent situations and found the solutions by graphing. Your focus was on thinking about each pair of constraints as a system, finding the solution region of each inequality, and identifying a point in the region where the graphs overlap as a solution to the system.
  - 2.14.3: Self Check
  - 2.14.3: Additional Resources
- 2.14.4: Graphing Solutions of Systems of Inequalities
  - In this activity, you reviewed the idea that the solutions to a system of inequalities can be effectively represented by a region on the graphs of the inequalities in the system. You began to graph systems of inequalities.
  - 2.14.4: Self Check
  - 2.14.4: Additional Resources
- 2.14.5: Writing Inequalities for Situations and Representing on a Graph
  - In this activity, you wrote an inequality for given statements and identified graphs that represented each inequality.

After these activities, you completed the following **practice**:

- 2.14.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

2.15 SOLVING PROBLEMS WITH SYSTEMS OF LINEAR INEQUALITIES IN TWO VARIABLES

2.15.0 Lesson Overview

In this lesson, you will take a closer look at whether points on the boundary lines of the system's solution region are included in the solutions. Analyzing graphs and communicating observations about them will require your attention to precision. You will also apply your insights to solve more challenging contextual problems.

When you finish this lesson, you will be able to:

- Explain how to tell if a point on the boundary of the graph of the solutions to a system of inequalities is a solution.

Here are the **activities** that will help you reach those goals:

- 2.15.1: Analyzing Graphs that Represent Linear Equations and Inequalities
- 2.15.2: Determining if Points on Boundary Lines Are Solutions to a System
  - 2.15.2: Self Check
  - 2.15.2: Additional Resources
- 2.15.3: Solving Problems to Satisfy Multiple Constraints Simultaneously
  - 2.15.3: Self Check
  - 2.15.3: Additional Resources
- 2.15.4: Identifying Graph Regions Included in a System's Solution

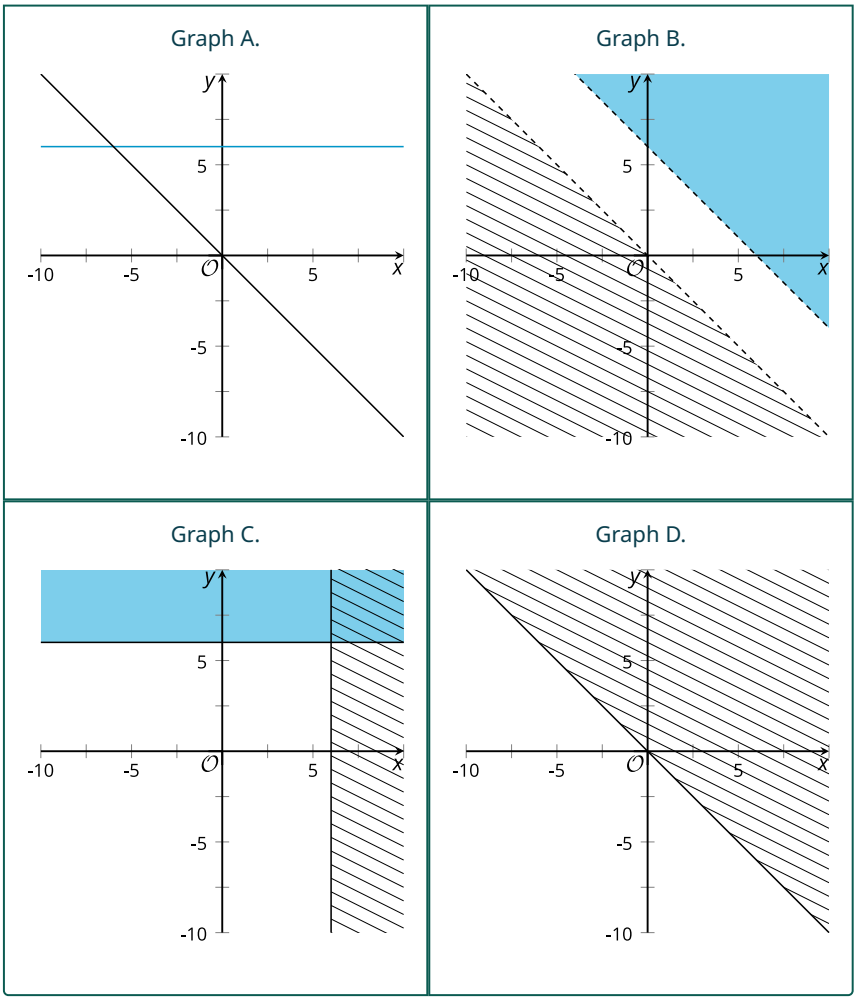
After that, you'll **practice and review**.

- 2.15.5: Practice
- 2.15.6: Lesson Summary

2.15.1 Analyzing Graphs that Represent Linear Equations and Inequalities

Warm Up

Compare each of the following graphs, then list a reason for each graph why it would not belong with the other three.

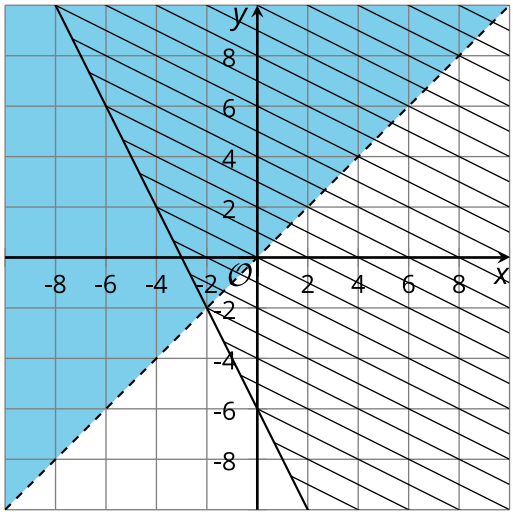


2.15.2 Determining if Points on Boundary Lines Are Solutions to a System

Activity

Here are the graphs of the inequalities in this system:

$$\begin{cases} x < y \\ y \geq -2x - 6 \end{cases}$$



Decide whether each point is a solution to the system. Be prepared to explain how you know.

1. $(3, -5)$

2. $(0, 5)$

3. $(-6, 6)$

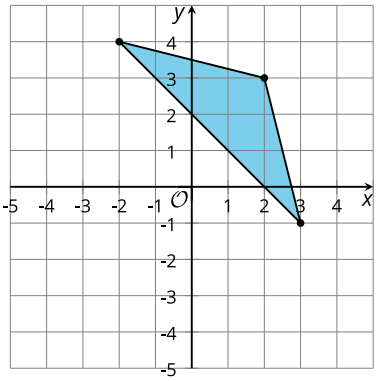
4. $(3, 3)$

5. $(-2, -2)$

Are you ready for more?

Extending Your Thinking

Find a system of inequalities with this triangle as its set of solutions:

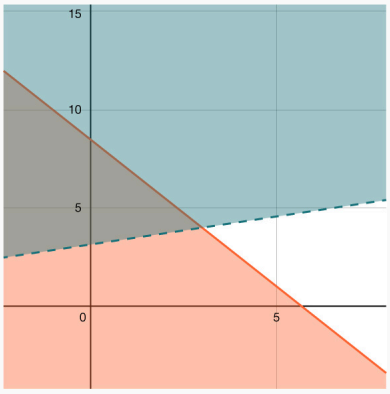


Video: Determining if Points on Boundary Lines Are Solutions to a System

Solving Systems of Inequalities (https://openstax.org/r/systems-inequalities)

Self Check

Here is the graph of this system of inequalities. Which of the following ordered pairs falls within the solution set?



$6x + 4y \leq 34$

$-2x + 7y > 22$

$(10, 6)$

Additional Resources

Determine Solutions of Systems of Inequalities on Boundary Lines

A solution to a system of inequalities only exists on a boundary line if the line is solid, not dashed.

EXAMPLE

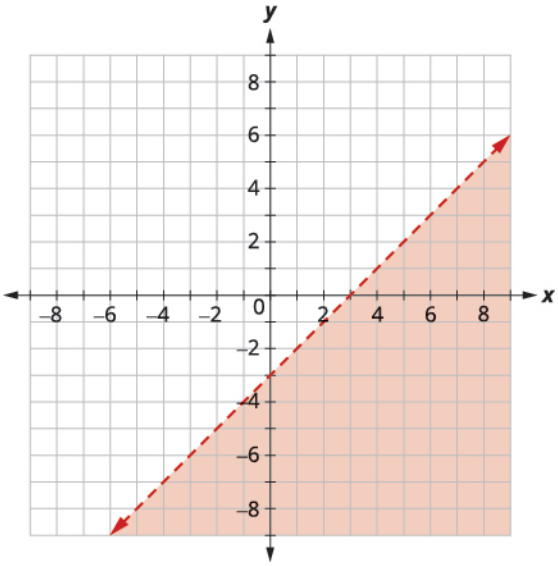
Solve the system by graphing:

$$\begin{cases} x - y > 3 \\ y < -\frac{1}{5}x + 4. \end{cases}$$

Solution

$$\begin{cases} x - y > 3 \\ y < -\frac{1}{5}x + 4 \end{cases}$$

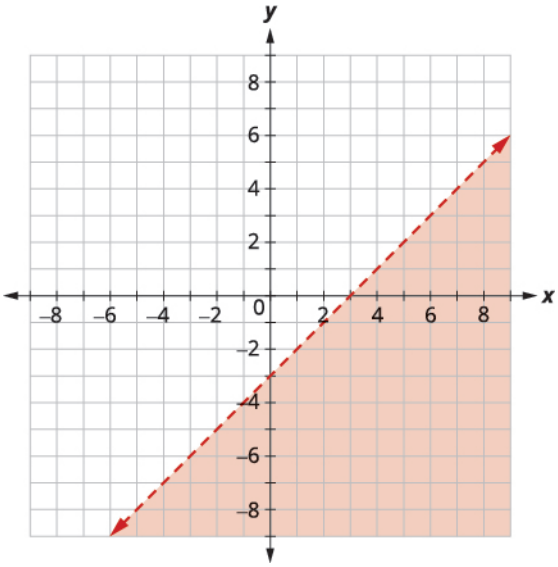
Graph $x - y > 3$, by graphing $x - y = 3$ and testing a point. The intercepts are $x = 3$ and $y = -3$, and the boundary line will be dashed. Test $(0, 0)$, which makes the inequality false, so shade (red) the side that does not contain $(0, 0)$.

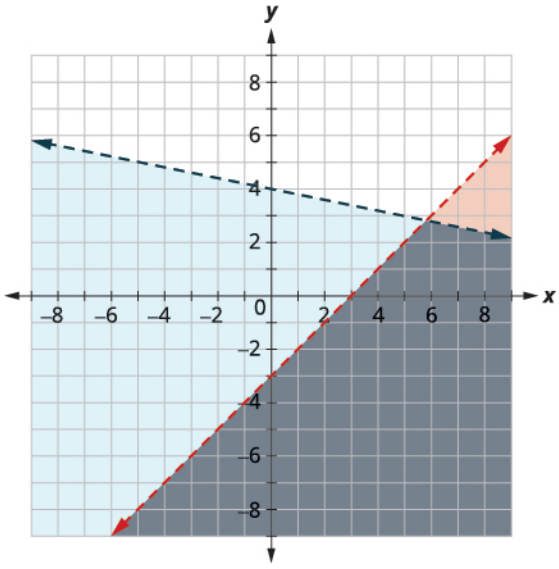


Graph $y < -\frac{1}{5}x + 4$ by graphing $y = -\frac{1}{5}x + 4$ using the slope $m = -15$ and y-intercept $b = 4$. The boundary line will be dashed

Test $(0, 0)$, which makes the inequality true, so shade (blue) the side that contains $(0, 0)$.

Choose a test point in the solution and verify that it is a solution to both inequalities.





The point of intersection of the two lines is not included because both boundary lines were dashed. The solution is the area shaded twice—which appears as the darkest shaded region.

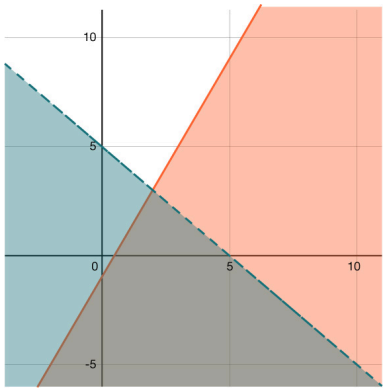
Notice that the point $(0, -3)$ is NOT a solution because it falls on the red dashed line. The point $(4, -2)$ IS a solution because it is included in the solutions to both inequalities.

TRY IT **Determine Solutions of Systems of Inequalities on Boundary Lines**

Which of the following are solutions of the graph below?

- $(2, 1)$
- $(2, 3)$
- $(5, 0)$
- $(3, -1)$
- $(-1, 6)$

- $(1, 6)$



2.15.3 Solving Problems to Satisfy Multiple Constraints Simultaneously

Activity

Your teacher will give you either a problem card or a data card. Do not show or read your card to your partner.

If your teacher gives you the data card:

1. Silently read the information on your card.
2. Ask your partner, "What specific information do you need?" and wait for your partner to ask for information. Only give information that is on your card. (Do not figure out anything for your partner!)
3. Before telling your partner the information, ask, "Why do you need to know (that piece of information)?"
4. Read the problem card and solve the problem independently.
5. Share the data card and discuss your reasoning.

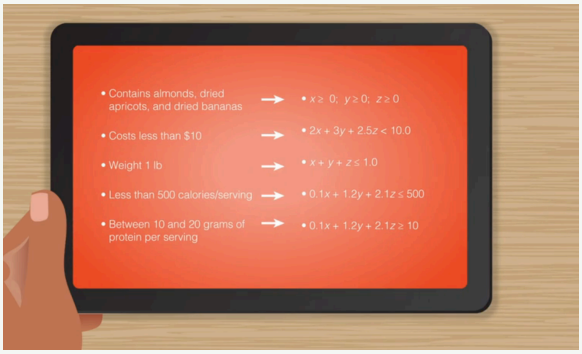
If your teacher gives you the problem card:

1. Silently read your card and think about what information you need to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain to your partner how you are using the information to solve the problem.
4. When you have enough information, share the problem card with your partner, and solve the problem independently.
5. Read the data card and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

1. Use the graphing tool or technology outside the course to help you determine the correct solution. Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-15-3-solving-problems-to-satisfy-multiple-constraints-simultaneously)

Why Should I Care?



Coming up with snack mixes is not as simple as just combining ingredients for the company TERANGA. They must meet a series of requirements to make sure the snack mix is affordable and healthy.

In this image, you can see the requirements of each snack mix translated into linear inequalities. TERANGA's team can use the linear inequalities to know exactly what combination of fruits and nuts to put into a mix that will not only be tasty, but will also be nutritious and affordable. Plus, the mix will make the company a nice profit.

Self Check

Which of the following ordered pairs falls within the solution set to the system of inequalities?

$-5x + 2y > -5$

$4x + 3y < 27$

a. $(1, 0)$
b. $(2, 5)$
c. $(0, 9)$
d. $(3, 5)$

Additional Resources

Solve Applications of Systems of Inequalities

When we use variables other than x and y to define an unknown quantity, we must change the names of the axes of the graph as well.

EXAMPLE

Use the following information to answer questions 1 – 4.

> Omar needs to eat at least 800 calories before going to his team practice. All he wants is hamburgers, h, and cookies, c, and he doesn't want to spend more than $5. At the hamburger restaurant near his college, each hamburger has 240 calories and costs $1.40. Each cookie has 160 calories and costs $0.50.

1. Write a system of inequalities to model this situation.

Solution

Let h equal the number of hamburgers.

Let c equal the number of cookies.

To find the system of equations, translate the information.

> The calories from hamburgers at 240 calories each, plus the calories from cookies at 160 calories each must be 800

or more. $240h + 160c \geq 800$

The amount spent on hamburgers at \$1.40 each, plus the amount spent on cookies at \$0.50 each must be no more than \$5.00. $1.40h + 0.50c \leq 5$

The number of hamburgers must be greater than or equal to 0. $h \geq 0$

The number of cookies must be greater than or equal to 0. $c \geq 0$

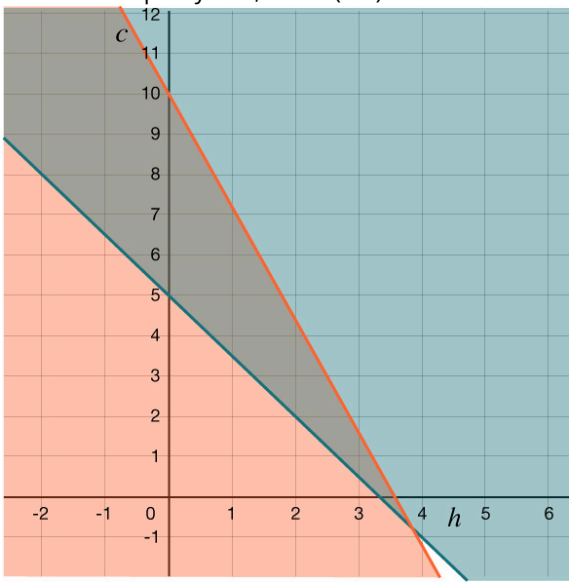
We have our system of equations. $\begin{cases} 240h + 160c \geq 800 \\ 1.40h + 0.50c \leq 5 \\ h \geq 0 \\ c \geq 0 \end{cases}$

2. Graph the system.

Solution

Since $h \geq 0$ and $c \geq 0$ all solutions will be in the first quadrant. As a result, our graph shows only quadrant one.

To graph $240h + 160c \geq 80$, graph $240h + 160c = 800$ as a solid line. Choose $(0, 0)$ as a test point. Since it does not make the inequality true, shade (red) the side that does not include the point $(0, 0)$.



Graph $1.40h + 0.50c \leq 5$. The boundary line is $1.40h + 0.50c = 5$. We test $(0, 0)$ and it makes the inequality true. We shade the side of the line that includes $(0, 0)$.
The solution of the system is the region of the graph that is shaded the darkest. The boundary line sections that border the darkly shaded section are included in the solution as are the points on the x-axis from $(5, 0)$ to $(10, 0)$.

3. Could he eat 3 hamburgers and 1 cookie?

Solution

To determine if 3 hamburgers and 1 cookie would meet Omar's criteria, we see if the point $(3, 2)$ is in the solution region. It is, so Omar might choose to eat 3 hamburgers and 1 cookie.

4. Could he eat 2 hamburgers and 4 cookies?

Solution

To determine if 2 hamburgers and 4 cookies would meet Omar's criteria, we see if the point $(2, 4)$ is in the solution region. It is, so Omar might choose to eat 2 hamburgers and 4 cookies.

We could also test the possible solutions by substituting the values into each inequality.

TRY IT **Solve Applications of Systems of Inequalities**

Use the following information for questions 1 – 4.

Philip's doctor tells him he should add at least 1,000 more calories per day to his usual diet. Philip wants to buy protein bars, p, that cost \$1.80 each and have 140 calories and juice, j, that costs \$1.25 per bottle and has 125 calories. He doesn't want to spend more than \$12.

1. Write a system of inequalities that models this situation.

2. Use the graphing tool or technology outside the course. Graph the system of inequalities from question 1 using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-15-3-solving-problems-to-satisfy-multiple-constraints-simultaneously)

3. Can he buy 3 protein bars and 5 bottles of juice?

4. Can he buy 5 protein bars and 3 bottles of juice?

2.15.4 Identifying Graph Regions Included in a System's Solution

Cool Down

Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

Use this information to answer questions 1 – 7.

A factory produces "widgets" and "zurls." The combined number of widgets and zurls made each day cannot be more than 12. The maximum number of widgets the factory can produce in a day is 4.
Let x be the number of widgets and y be the number of zurls.

1. Select **two** inequalities that represent this situation.
a. $x \geq 4$
b. $x + y \leq 12$
c. $x > 4$
d. $x + y \geq 12$
e. $x \leq 4$

2. Use the graphing tool or technology outside the course. Graph the equations $x = 4$ and $x + y = 12$ using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-15-4-identifying-graph-regions-included-in-a-systems-solution)

3. Use the graphing tool or technology outside the course. Graph the system of inequalities that shows all the allowable numbers of widgets and zurls that the factory can produce in one day using the Desmos tool below.
Access multimedia content (http://openstax.org/books/algebra-1/pages/2-15-4-identifying-graph-regions-included-in-a-systems-solution)

In questions 4 – 7 answer the question:

Does the ordered pair represent an allowable combination of widgets and zurls produced in one day?

4. (4, 5)

5. (11, 1)

6. (4, 12)

7. (3, 9)

2.15.5 Practice

For questions 1 – 4, use this scenario: Jada has p pennies and n nickels that add up to more than 40 cents. She has fewer than 20 coins altogether.

1. Which two inequalities could make up a system of inequalities that represents how many pennies and nickels that Jada could have? Select **two** inequalities.

a. $p + n < 20$
b. $p + n > 20$
c. $0.01p + 0.05n < 0.40$
d. $0.01p + 0.05n > 0.40$

For questions 2 – 4:

Determine if it is possible that Jada has each of the following combinations of coins. If so, explain or show how you know. If not, state which constraint—the amount of money or the number of coins—it does not meet.

2. 15 pennies and 5 nickels

a. Yes
b. No

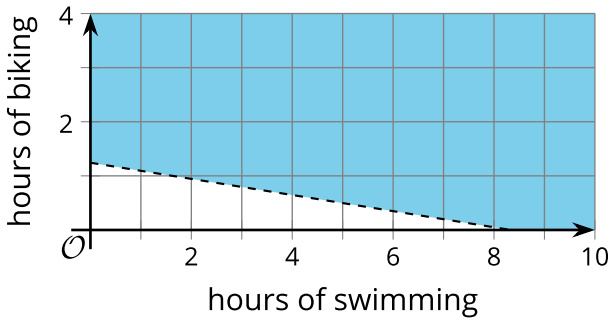
3. 16 pennies and 2 nickels

a. Yes
b. No

4. 10 pennies and 8 nickels

a. Yes
b. No

5. A triathlon athlete swims at an average rate of 2.4 miles per hour, and bikes at an average rate of 16.1 miles per hour. At the end of one training session, she has swam and biked more than 20 miles in total.
The inequality $2.4s + 16.1b > 20$ and this graph represent the relationship between the hours of swimming, s, the hours of biking, b, and the total distance the athlete could have traveled in miles.



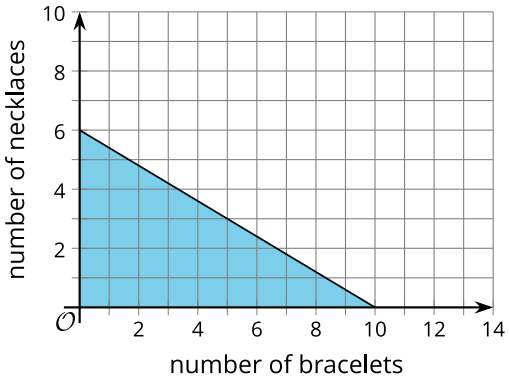
Mai said, "I'm not sure the graph is right. For example, the point $(10, 3)$ is in the shaded region, but it's not realistic for an athlete to swim for 10 hours and bike for 3 hours in a training session! I think triathlon athletes generally train for no more than 2 hours a day."

Which inequality represents Mai's last statement?

a. $s + b = 2$
b. $s + b > 2$
c. $s + b < 2$
d. $s + b \leq 2$

For questions 6 and 7, use the following situation and graph:

Elena is considering buying bracelets and necklaces as gifts for her friends. Bracelets, b, cost \$3, and necklaces, n, cost \$5. She can spend no more than \$30 on the gifts. Elena needs at least 7 gift items.
This graph represents the inequality $3b + 5n \leq 30$, which describes the cost constraint in this situation.



6. Let b represent the number of bracelets and n the number of necklaces.
Which inequality represents the number of gift items that Elena needs?

a. $b + n = 7$
b. $b + n < 7$
c. $b + n \geq 7$
d. $b + n > 7$

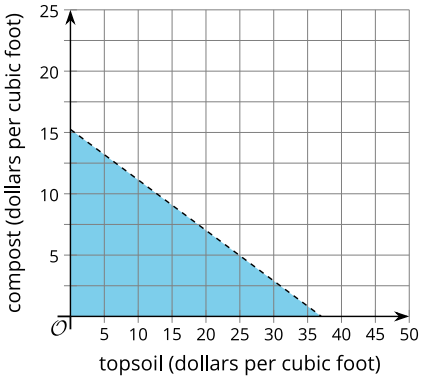
7. Which of the following could be solutions to the system of inequalities? Select **two** answers.

a. $(8, 5)$
b. $(1, 5)$
c. $(7, 1)$
d. $(6, 2)$

8. A gardener is buying some topsoil and compost to fill his garden. His budget is $70. Topsoil costs $1.89 per cubic foot, and compost costs $4.59 per cubic foot.
Select **three** statements or representations that correctly describe the gardener's constraints in this situation. Let t represent the cubic feet of topsoil and c the cubic feet of compost.

a: The combination of 7.5 cubic feet of topsoil and 12 cubic feet of compost is within the gardener's budget.

b: If the line on the graph represents the equation $1.89t + 4.59c = 70$, this graph represents the solutions to the gardener's budget constraint.

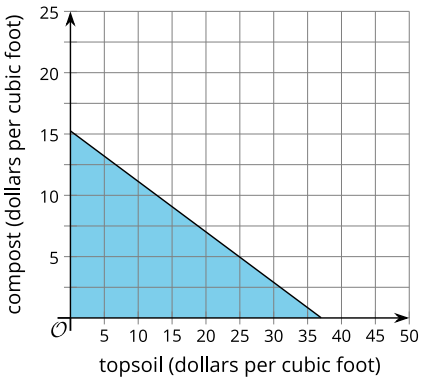


c: $1.89t + 4.59c \geq 70$

d: The combination of 5 cubic feet of topsoil and 20 cubic feet of compost is within the gardener's budget.

e: $1.89t + 4.59c \leq 70$

f: If the line on the graph represents the equation $1.89t + 4.59c = 70$, this graph represents the solutions to the gardener's budget constraint.


compost (dollars per cubic foot)
25
20
15
10
5
O
5 10 15 20 25 30 35 40 45 50
topsoil (dollars per cubic foot)

2.15.6 Lesson Summary

In this lesson, you learned how to:

- Explain how to tell if a point on the boundary of the graph of the solutions to a system of inequalities is a solution.

Here are the **activities** that helped you reach those goals:

- 2.15.1: Analyzing Graphs that Represent Linear Equations and Inequalities
  - In this activity, you carefully analyzed and compared graphs that represent linear equations and inequalities. Making comparisons prompted you to think about the solutions to the equations, inequalities, or systems that were represented. It also gave you an opportunity to use language precisely.
- 2.15.2: Determining if Points on Boundary Lines Are Solutions to a System
  - In this activity, you took a closer look at whether points that are on the boundary lines are solutions to the system.
  - 2.15.2: Self Check
  - 2.15.2: Additional Resources
- 2.15.3: Solving Problems to Satisfy Multiple Constraints Simultaneously
  - In this activity, you had an opportunity to use your understanding of systems of linear inequalities to solve problems that involve satisfying multiple constraints simultaneously.
  - 2.15.3: Self Check
  - 2.15.3: Additional Resources
- 2.15.4: Identifying Graph Regions Included in a System's Solution
  - In this activity, you determined all the inequalities that represent a given situation and shaded regions and adjusted line types to represent solutions.

After these activities, you completed the following **practice**:

- 2.15.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

PROJECT 2: MODELING WITH SYSTEMS OF INEQUALITIES IN TWO VARIABLES

Project 2 Overview

In this culminating project, you will integrate the ideas from the unit and engage in multiple aspects of mathematical modeling.

In the first activity, you interpret and analyze given models that represent the constraints and conditions in a situation. In the second activity, you create your own models after specifying quantities of interest, identifying relevant information, and setting the constraints.

Project Activities

1. Determining a Solution to Inequalities
2. Interpret a Set of Mathematical Models
3. Perform Mathematical Modeling

Learning Targets

When you finish this project, you will have demonstrated how to:

- Interpret inequalities and graphs in a mathematical model.
- Choose variables, specify the constraints, and write inequalities to create a mathematical model.

Determine a Solution to Inequalities

Is the ordered pair $(5.43, 0)$ a solution to all, some, or none of these inequalities? Be prepared to explain your reasoning.

- $x > 0$
- $y > 0$
- $x \geq 0$
- $y \geq 0$

Interpret a Set of Mathematical Models

Step 1 - Examine the following nutrition information for some trail mix ingredients.

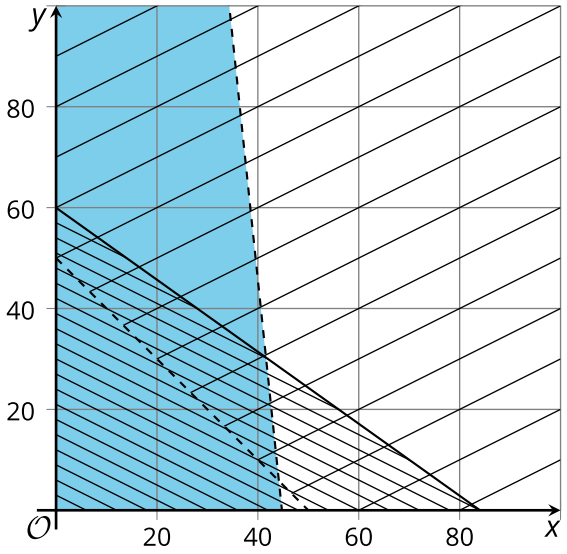
	Calories per gram (kcal)	Protein per gram (g)	Sugar per gram (g)	Fat per Gram (g)	Fiber per Gram (g)
Peanuts	5.36	0.21	0.04	0.46	0.07
Almonds	5.71	0.18	0.21	0.46	0.07
Raisins	3.00	0.03	0.60	0.00	0.05
Chocolate pieces	4.76	0.05	0.67	0.19	0.02
Shredded Coconut	6.67	0.07	0.07	0.67	0.13
Sunflower Seeds	5.50	0.20	0.03	0.47	0.10

	Calories per gram (kcal)	Protein per gram (g)	Sugar per gram (g)	Fat per Gram (g)	Fiber per Gram (g)
Dried Cherries	3.25	0.03	0.68	0.00	0.03
Walnuts	6.43	0.14	0.04	0.61	0.07

Step 2 - Tyler and Jada each designed their own custom trail mix using two of these ingredients. Examine the following inequalities they wrote and graphs they created to represent their constraints.

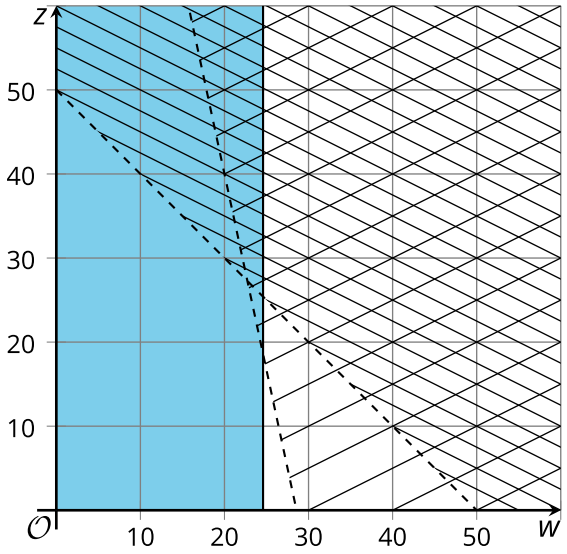
Tyler

- $x + y > 50$
- $4.76x + 6.67y \leq 400$
- $0.67x + 0.07y < 30$
- $x > 0$
- $y > 0$



Jada

- $w + z > 50$
- $0.14w + 0.03z > 4$
- $0.61w + 0z \leq 15$
- $w > 0$
- $z > 0$



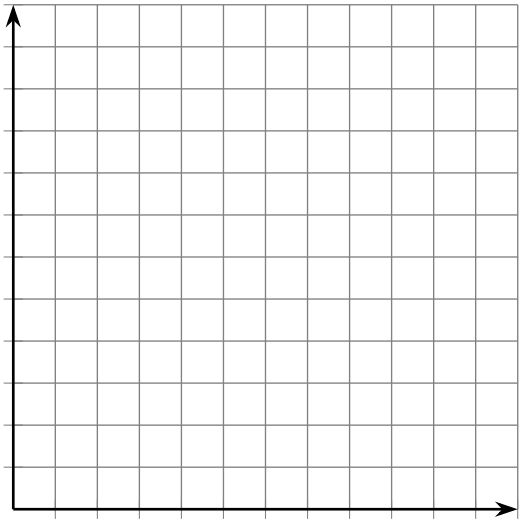
Step 3 - Use the inequalities and graphs to answer questions 1 – 5 about each student's trail mix. Be prepared to explain your reasoning.

1. Which two ingredients did they choose?
2. What do their variables represent?
3. What does each constraint mean?
4. Which graph represents which constraint?
5. Name one possible combination of ingredients for their trail mix.

Perform Mathematical Modeling

It's time to design your own trail mix!

1. Choose two ingredients that you like to eat. (You can choose from the ingredients in the previous activity, or you can look up nutrition information for other ingredients.)
2. Think about the constraints for your trail mix. What do you want to be true about its calories, protein, sugar, fat, or fiber?
3. Write inequalities to represent your constraints. Then, graph the inequalities.



4. Is it possible to make a trail mix that meets all your constraints using your ingredients? If not, make changes to your constraints or your ingredients and record them here.
5. Write a possible combination of ingredients for your trail mix.

Pause here so your teacher can review your work and give further instructions for displaying your work.

Unit 2 Wrap Up

Throughout this unit, you learned that Nafy, owner of Teranga's snack mixes, can use constraints to create recipes for snack mixes that are nutritious and will make a profit for the company. She can use a system of equations to examine the company's profits. Constraints can be used to represent the upper and lower limits of weight, cost, and nutritional values of the snack mixes. Inequalities can be used to represent those constraints. Inequalities in two variables can be used to find combinations of two ingredients that meet constraints. Systems of inequalities can be used to find combinations of ingredients that create snacks that meet the company's goals of nutrition and cost.

Similarly, in this unit, you learned how to write and solve systems of linear equations. You learned that the solution to a system of equations is a pair of coordinates that satisfies both equations. You used graphing and algebraic methods to find solutions.

You also learned to use constraints to represent upper and lower limits in real-world situations. You used inequalities in one variable to represent these constraints. You learned that the solution set to one-variable inequalities is a range of values that can be graphed on a number line.

Next, you wrote inequalities in two variables to represent constraints. You learned that the solution set is combinations of values that make the inequality true. The graph of the solution set is a region bounded by a line that may or not be part of the solution.

Finally, you learned that two linear inequalities that represent the constraints in the same situation form a system of inequalities. You learned that the solutions to the system include all pairs of numbers that satisfy both constraints simultaneously. You represented the solution set of the system graphically by identifying pairs of points in the region where the graphs of the individual inequalities overlap.



Access our Unit 2 Student Self-Assessment (https://openstax.org/r/unit-2-self-assesment) to reflect on the skills you learned during this unit. Congratulations on completing Unit 2!

Building Character: Growth Mindset

In this unit, you learned that a **growth mindset** helps you focus on developing your abilities rather than proving how smart you are. The brain is like a muscle that you can strengthen. Improvement comes from challenging yourself and exercising your brain. When you make a mistake, use that as an opportunity to learn the error in your thinking. You may look for a similar example or find someone who can help you.

UNIT 3 OVERVIEW AND READINESS

Unit 3 Overview

Why Should I Care?

Do you ever think about where your food comes from, or what it takes to get those vegetables you're eating from the ground to your plate? Watch the video to learn a bit about the organization Farms To Grow and how farmers use statistics to make sound decisions for their farms.

Statistics in Farming (https://openstax.org/r/farming-statistics)

In this unit, you will learn about how to graph two-dimensional data and create models of the data to make predictions. You'll also use residuals to judge how good a model is and use correlation to describe how variables are related to each other.

Building Character: Purpose

Through their organization, Farms to Grow, Gordon Reed and Dr. Gail Myers work to promote the sustainability and history of Black farmers. They have a strong sense of purpose to make a positive contribution to the world.

Throughout this unit, you will work on developing your **purpose**. Having purpose is showing a commitment to making a meaningful contribution to the world. Purpose drives you to make positive contributions in line with your own interests and strengths. And this pursuit gives your life direction and forward momentum; it motivates and guides your short-term goals and daily activities. With a strong sense of purpose, you flourish: you enjoy a more meaningful life, are healthier and more resilient to setbacks, live more energetically, and feel good about what you've accomplished.

Think about your current sense of purpose. How many of the following statements are true for you?

- I look for ways to have positive effects on others' lives.
- I often reflect on my life goals and the kind of person I want to be.
- When I plan out my day, I consider how my activities connect to what I want to accomplish with my life.
- I often think about what I can offer the world, taking into account what the world needs as well as my personal strengths and interests.
- I often think about what matters most to me and why it matters.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

Am I ready to learn this?

Getting Ready for Unit 3

To be ready for this unit, you need to be able to:

- Distinguish linear patterns from nonlinear patterns based on scatter plots.
- Identify positive and negative trends in a scatter plot.
- Interpret slopes and y-intercepts of graphs of linear equations.

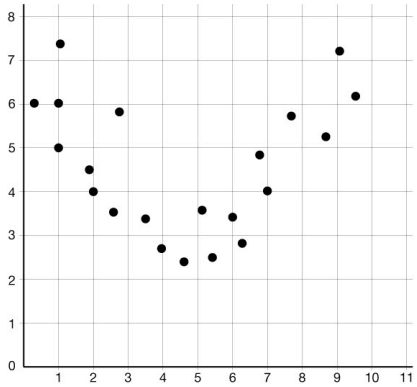
Next, you will answer some questions that will help you see how ready you are for this unit. If you need help, some mini-lessons will help you brush up on anything you need to work on.

Distinguish Patterns: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #1: Distinguish Patterns

Which statement best describes the scatter plot?

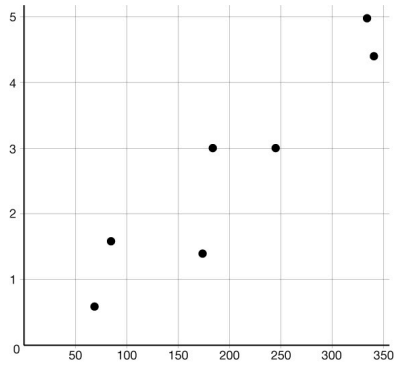
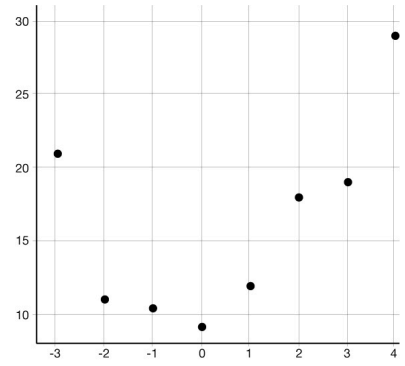
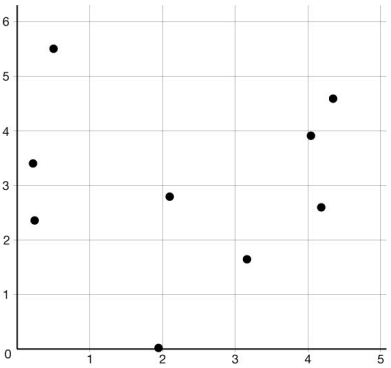


a. There is an increasing linear pattern.
b. There is a decreasing linear pattern.
c. There is no pattern.
d. There is a nonlinear pattern.

Distinguish Linear Patterns from Nonlinear Patterns

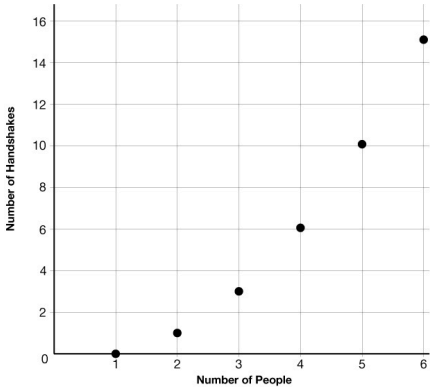
A scatter plot is a graph of numerical data with two variables. If two variables vary together in a predictable way, we say there is a statistical relationship between the two variables.

- When the points of the scatter plot seem to fall along a line, the scatter plot is linear.
- Nonlinear patterns seem to follow a curve.
- If the scatter plot shows no pattern, it means there is no relationship between the variables.

		
There is a relationship.	There is a relationship.	There is no relationship.
It appears to be an increasing linear pattern.	It appears to be nonlinear.	

TRY IT **Distinguish Linear Patterns from Nonlinear Patterns**

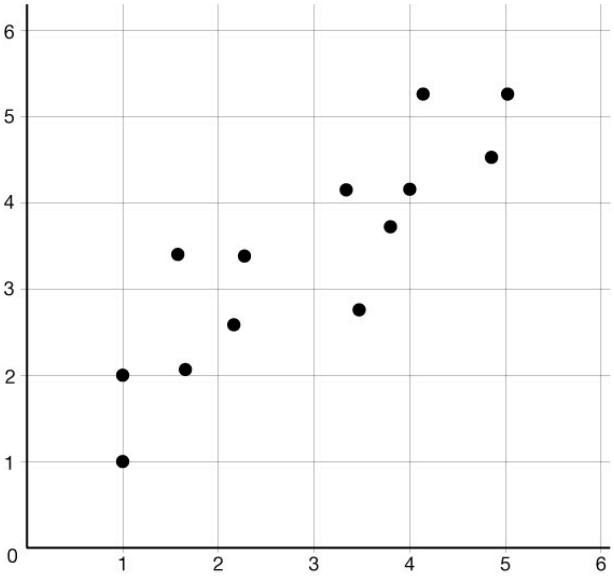
At a dinner party, every person shakes hands with every other person present. The scatter plot shows the number of people, x, and the number of handshakes, y.



Is the relationship linear or nonlinear? Be prepared to show your reasoning.

Check Your Understanding

Which statement best describes the scatter plot?



a. There is an increasing linear pattern.

b. There is no pattern.

c. There is a decreasing linear pattern.

d. There is a nonlinear pattern.

Video: Distinguishing Linear Patterns

Scatter Plots Example

Watch the video to see scatter plots that show the relationship between study time, shoe size, and test score.

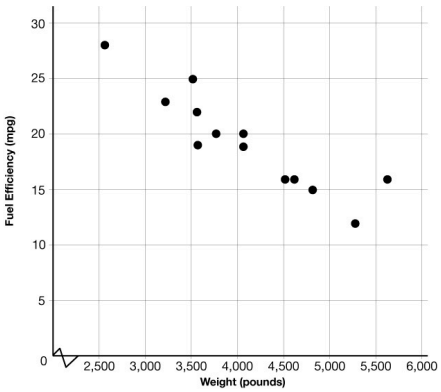
Studying, shoe size, and test scores scatter plots | Probability and Statistics | Khan Academy (https://www.youtube.com/embed/Jpbm5YgciqI)

Identify Positive and Negative Trends: Mini-Lesson Review

MINI LESSON QUESTION

Question #2: Identify Positive and Negative Trends

The scatter plot shows data on weight in pounds (x) and fuel efficiency in miles per gallon (y) for 13 cars.

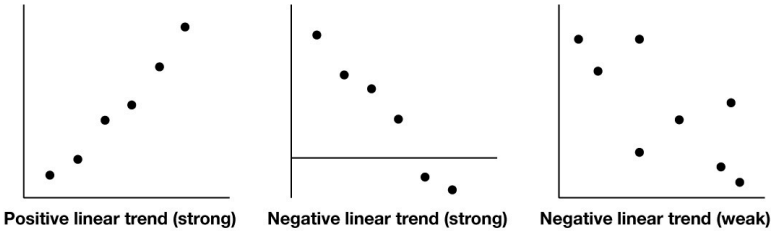


Which statement best describes the data?

a. There appears to be a positive relationship between fuel efficiency and weight. The cars with greater weight tend to have greater efficiency.
b. There appears to be a positive relationship between fuel efficiency and weight. The cars with greater weight tend to have lesser efficiency.
c. There appears to be a negative relationship between fuel efficiency and weight. The cars with greater weight tend to have greater efficiency.
d. There appears to be a negative relationship between fuel efficiency and weight. The cars with greater weight tend to have lesser efficiency.

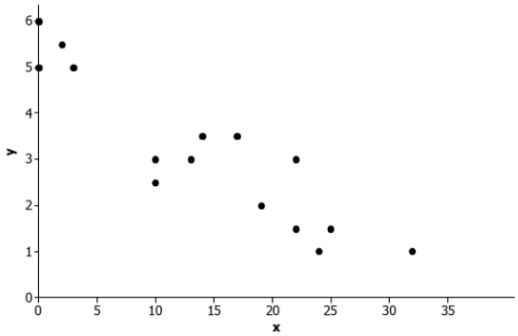
Identify Positive and Negative Trends

Scatter plots may have a strong linear trend or a weak linear trend. A strong positive trend means as one variable increases, the other variable increases, A strong negative trend means as one variable increases, the other variable decreases. A weak linear pattern means the points are more scattered.



Positive linear trend (strong) Negative linear trend (strong) Negative linear trend (weak)

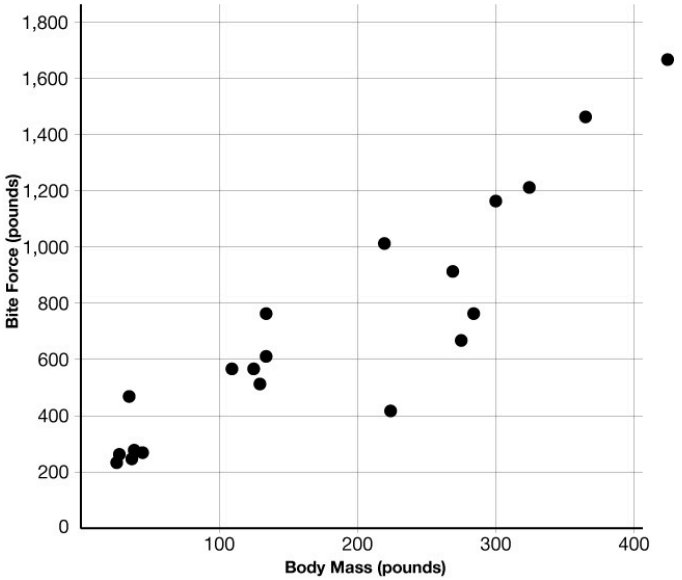
TRY IT **Identify Positive and Negative Trends**



Does the scatter plot show a positive or negative trend? Is the relationship weak or strong?

Check Your Understanding

The scatter plot shows the relationship between crocodiles body mass in pounds, x, and bite force in pounds, y. Which statement best describes the relationship between body mass and bite force?



a. There appears to be a positive relationship between bite force and body mass. The greater the body mass, the greater the bite force tends to be.

b. There appears to be a positive relationship between bite force and body mass. The greater the body mass, the lighter the bite force tends to be.

c. There appears to be a negative relationship between bite force and body mass. The greater the body mass, the greater the bite force tends to be.

d. There appears to be a negative relationship between bite force and body mass. The greater the body mass, the lighter the bite force tends to be.

Video: Trends in a Scatter Plot

Khan Academy: More on Scatter Plots

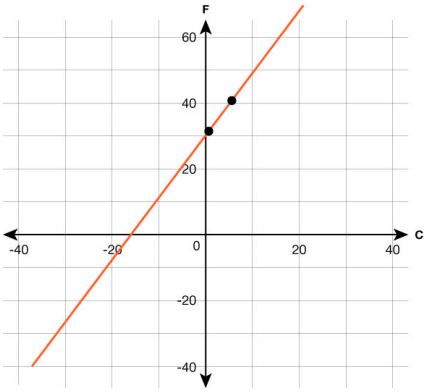
The following video shows how to identify positive and negative trends in a scatter plot.

People smoking less over time scatter plot | Regression | Probability and Statistics | Khan Academy (https://www.youtube.com/embed/F5uiFXECnVY)

Interpret Slope and y-intercept: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #3: Interpret Slope and y-Intercept



The figure shows the graph of the equation used to convert temperature in Celsius to temperature in Fahrenheit: $F = \frac{9}{5}C + 32$.

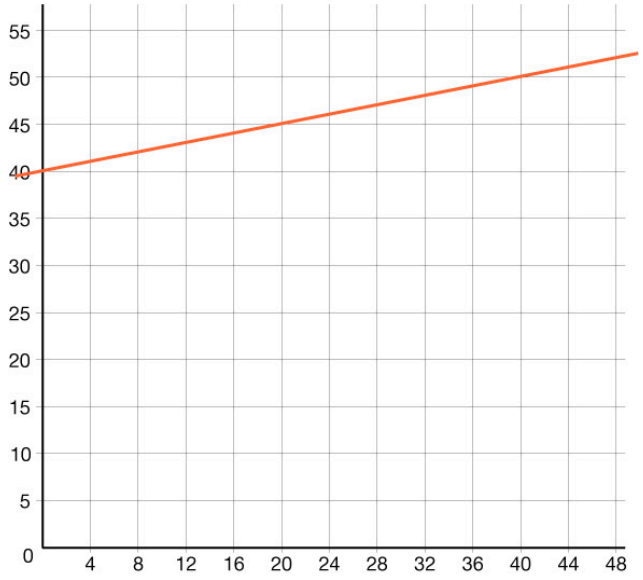
$F = \frac{9}{5}C + 32$ is used to convert temperatures, C, on the Celsius scale to temperatures, F, on the Fahrenheit scale. Identify and interpret the slope and F-intercept.

a. The slope is $\frac{9}{5}$, which means that the temperature Fahrenheit (F) increases 9 degrees when the temperature Celsius (C) increases 5 degrees. The F-intercept is -18, which means that when the temperature is 0° on the Celsius scale, it is 18° lower on the Fahrenheit scale.
b. The slope is 32, which means that for every 32 degrees Fahrenheit, it is 1 degree Celsius. The F-intercept is $\frac{9}{5}$, which means that when the temperature is 0° on the Celsius scale, it is $\frac{5}{9}°$ on the Fahrenheit scale.
c. The slope is $\frac{9}{5}$, which means that the temperature Celsius (C) increases 9 degrees when the temperature Fahrenheit (F) increases 5 degrees. The F -intercept is 32, which means that when the temperature is 0° on the Celsius scale, it is 32° on the Fahrenheit scale.
d. The slope is $\frac{9}{5}$, which means that the temperature Fahrenheit (F) increases 9 degrees when the temperature Celsius (C) increases 5 degrees. The F-intercept is 32, which means that when the temperature is 0° on the Celsius scale, it is 32° on the Fahrenheit scale.

Interpret Slope and y-Intercept

Many real-world applications are modeled by linear equations. Often, when a linear equation model uses real-world data, different letters are used for the variables, instead of using only x and y. The variable names remind us of what quantities are being measured.

The equation $T = \frac{1}{4}n + 40$ is used to estimate the temperature in degrees Fahrenheit, T, based on the number of cricket chirps, n, in one minute. The graph shows the relationship.



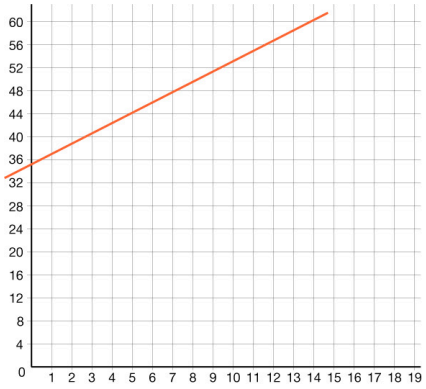
What do the slope and T-intercept represent?

Answer: The slope is $\frac{1}{4}$ and represents that for every 4 chirps per minute, the temperature increases by 1 degree. The T-intercept means when there are zero chirps, the temperature is 40 degrees.

TRY IT

Try It: Interpret Slope and y-intercepts

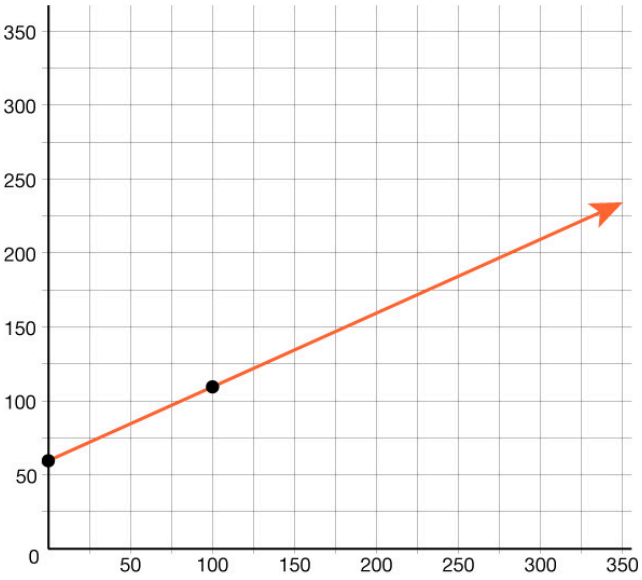
Loreen has a calligraphy business. The equation $C = 1.8n + 35$ models the relation between her weekly cost, C, in dollars and the number of wedding invitations, n, that she writes. The graph shows the relationship.



What do the slope and C-intercept represent?

Check Your Understanding

Sam drives a delivery van. The equation $C = 0.5m + 60$ models the relation between his weekly cost, C, in dollars and the number of miles, m, that he drives. The graph shows the relationship.



Find and interpret the meaning of the slope and C-intercept.

a. The slope, 0.5, means that the weekly cost, C, increases by \$0.50 when the number of miles driven, m, increases by 1. The C-intercept means that when the number of miles driven is 0, the weekly cost is \$60.

b. The slope, 0.5, means the weekly cost, C, increases \$0.50 a week. The C-intercept is 60, which means that every mile costs \$60.

c. The slope, 0.5, means that for every half mile, m, the cost, C, increases by 1. The C-intercept means that when the number of miles driven is 0, the weekly cost is \$60.

d. The slope, 0.5, means that the weekly cost, C, decreases by \$0.50 when the number of miles driven, m, increases by 1. The C-intercept means that when the number of miles driven is 60, the weekly cost is \$0.

Videos: Interpreting Slope and y-Intercept of Linear Equations

Khan Academy: Slope, x-Intercept, and y-Intercept Meaning in Context

Watch the following video to see how you can interpret slope, x-intercept, and y-intercept in context.

Slope, x-intercept, y-intercept meaning in context | Algebra I | Khan Academy (https://www.youtube.com/embed/XQlaZuh7E8w)

Khan Academy: Slope and Intercept Meaning in Context

Watch this video to see how you can interpret slope and intercept meaning in context.

Slope and intercept in tables (https://www.youtube.com/embed/TYbN--4KHIk)

Unit 3 Overview: Wrap Up

- Distinguish linear patterns from nonlinear patterns based on scatter plots.
- Identify positive and negative trends in a scatter plot.
- Interpret slopes and y-intercepts of graphs of linear equations.

If you are struggling, be sure to reach out to your teacher for additional help. Nobody is good at problems like this the

first try. Keep at it and it will get easier.

3.1 LINEAR MODELS

3.1.0 Lesson Overview

In this lesson, you will explore how a linear model is used to describe the relationship between two numerical variables and how to use a line of best fit to make predictions.

When you finish this lesson, you will be able to:

- Describe the rate of change and y-intercept for a linear model in everyday language.
- Draw a linear model that fits the data well and use the linear model to estimate values.

Here are the **activities** that will help you reach those goals:

- 3.1.1: Exploring Scatter Plots
- 3.1.2: Creating a Scatter Plot Using Data
  - 3.1.2: Self Check
  - 3.1.2: Additional Resources
- 3.1.3: Interpreting the Slope and Vertical Intercept of a Linear Model
  - 3.1.3: Self Check
  - 3.1.3: Additional Resources
- 3.1.4: Interpreting the Slope and Vertical Intercept of a Scatter Plot
  - 3.1.4: Self Check
  - 3.1.4: Additional Resources
- 3.1.5: Using an Equation for a Fit Line

After that, you'll **practice and review**.

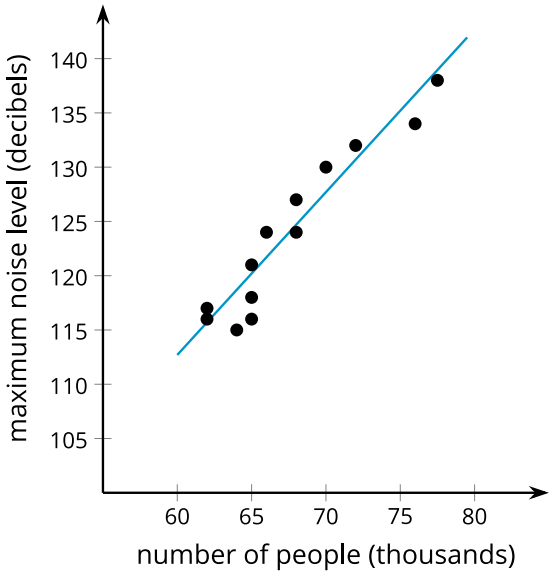
- 3.1.6: Practice
- 3.1.7: Summary

3.1.1 Exploring Scatter Plots

Warm Up

What do you notice? What do you wonder?

$y = 1.5x + 22.7$



3.1.2 Creating a Scatter Plot Using Data

Activity

Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

Watch the following video to help you answer the questions below.

Weighing Oranges in a Box (https://openstax.org/r/weigh-oranges)

1. To complete the table, record the weight for the number of oranges in the box from the video.

number of oranges	weight in kilograms
3	
4	
5	
6	
7	
8	
9	
10	

2. Create a scatter plot of the data. Access multimedia content (http://openstax.org/books/algebra-1/pages/3-1-2-creating-a-scatter-plot-using-data)

3. Approximate the equation of the line that best fits the data. Use the same graphing tool you used in the question above. Approximate the equation of the line that best fits the data.

4. Estimate a value for the slope of the line that you drew. What does the value of the slope represent?

5. Estimate the weight of a box containing 11 oranges. Will this estimate be close to the actual value? Be prepared to show your reasoning.

6. Estimate the weight of a box containing 50 oranges. Will this estimate be close to the actual value? Be prepared to show your reasoning.

When you use a linear **model** to estimate a value in the data given in a scatter plot, that is called **interpolation**. When you use a linear model to estimate the value outside the data given in the scatter plot, that is called **extrapolation**.

7. Estimate the coordinates for the vertical intercept of the line you drew. What might the y-coordinate for this point represent?

8. Which point(s) are best fit by your linear model? How did you decide?

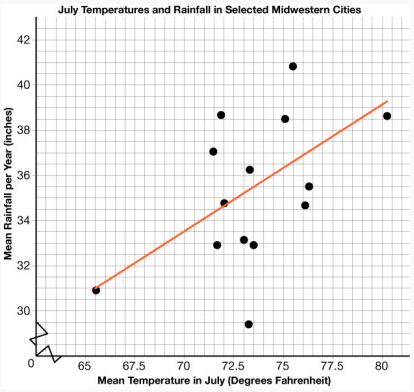
9. Which point(s) fit the least well by your linear model? How did you decide?

Compare your line with that of a partner. Discuss why you chose to put your lines where you did. How are your answers

different? Which of you has a line that is the better fit? Why?

Self Check

The plot below is a scatter plot of mean temperature in July and mean inches of rain per year for a sample of midwestern cities. A line is drawn to fit the data.



Use the line provided to predict the mean number of inches of rain per year for a city that has a mean temperature of 70°F in July.

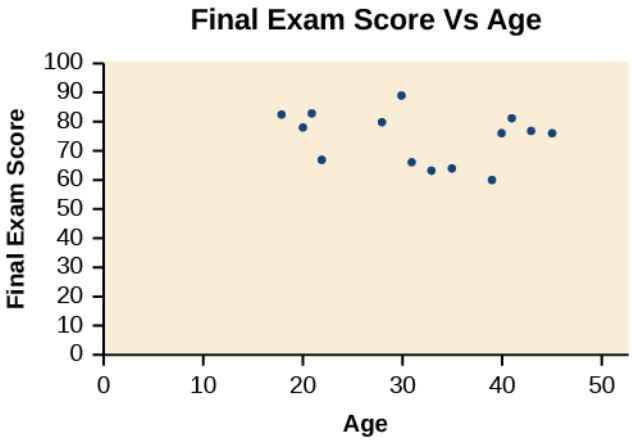
a. 31.5 inches per year
b. 34.5 inches per year
c. 33.5 inches per year
d. 32.5 inches per year

Additional Resources

Drawing and Interpreting Scatter Plots

A scatter plot is a graph of plotted points that may show a relationship between two sets of data. If the relationship is from a linear model or from a model that is nearly linear, the professor can draw conclusions using his knowledge of linear functions.

Here is an example of a scatter plot.



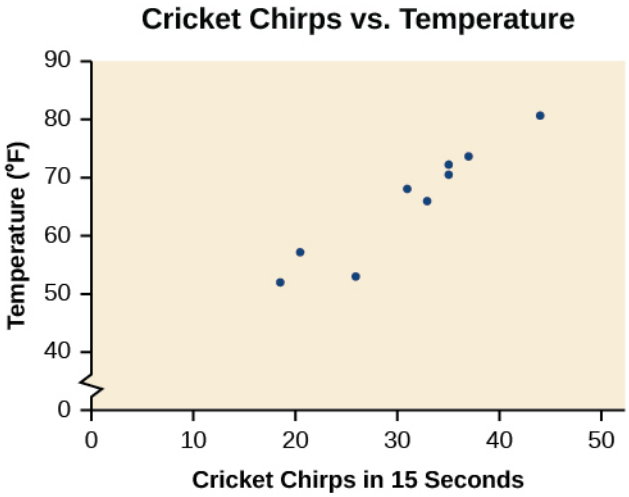
Notice this scatter plot does not indicate a linear relationship. The points do not appear to follow a trend. In other words, there does not appear to be a relationship between the age of the student and the score on the final exam.

EXAMPLE: USING A SCATTER PLOT TO INVESTIGATE CRICKET CHIRPS

The table shows the number of cricket chirps in 15 seconds for several different air temperatures in degrees Fahrenheit.

Chirps	44	35	20.4	33	31	35	18.5	37	26
Temperature	80.5	70.5	57	66	68	72	52	73.5	53

Plotting these data suggests that there may be a trend. We can see from the trend in the data that as the number of chirps increases every 15 seconds, the temperature increases. The trend appears to be roughly linear, though certainly not perfectly so.



Once we recognize a need for a linear function to model that data, the natural follow-up question is: "What is that linear function?" One way to approximate our linear function is to sketch the line that seems to best fit the data. Then we can extend the line until we can verify the y-intercept. We can approximate the slope of the line by extending it until we can estimate the $\frac{rise}{run}$.

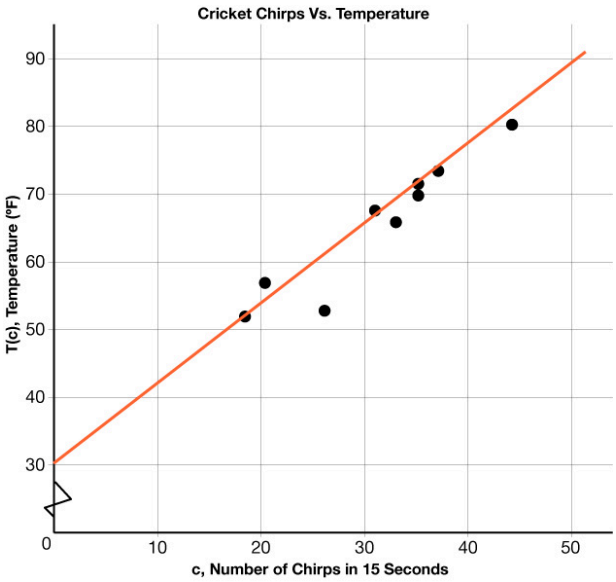
Continuing the Example: Finding a Line of Best Fit

Find a linear function that fits the data above by "eyeballing" a line that seems to fit.

On a graph, we could try sketching a line. Using the starting and ending points of our hand-drawn line, points $(0, 30)$ and $(50, 90)$, this graph has a slope of $m = \frac{60}{50} = 1.2$ and a y-intercept at 30.

The slope means that for every increase of 1 chirp per 15 seconds, the temperature increases 1.2 degrees.

The y-intercept stands for the temperature needed for crickets to start chirping.



Different methods of making predictions are used to analyze data. The method of interpolation involves predicting a value inside the domain and/or range of the data. The method of extrapolation involves predicting a value outside the domain and/or range of the data.

Model breakdown occurs at the point when the model no longer applies.

Continuing the Example: Understanding Interpolation and Extrapolation

Use the cricket data from the table to answer the following questions:

1. Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction and discuss whether it is reasonable.
2. Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction and discuss whether it is reasonable.

Solution

1. The number of chirps in the data provided varied from 18.5 to 44. A prediction at 30 chirps per 15 seconds is inside the domain of our data, so it would be interpolation. Using our model:

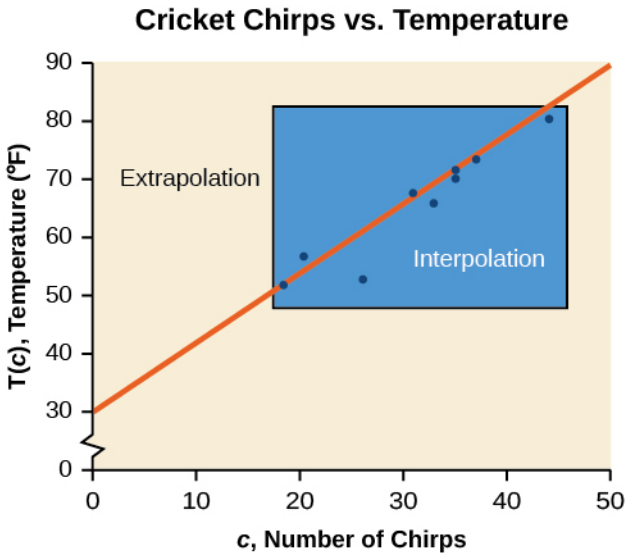
$$\begin{aligned} T(30) &= 30 + 1.2(30) \\ &= 66\ degrees \end{aligned}$$

Based on the data we have, this value seems reasonable.

2. The temperature values varied from 52 to 80.5. Predicting the number of chirps at 40 degrees is extrapolation because 40 is outside the range of our data. Using our model:

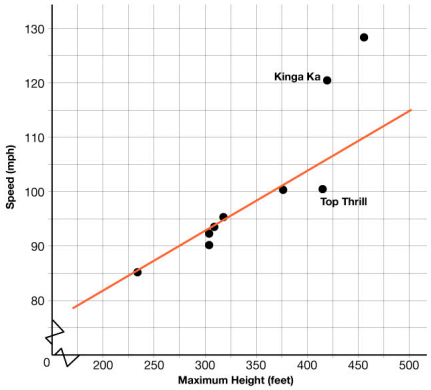
$$\begin{aligned} 40 &= 30 + 1.2c \\ 10 &= 1.2c \\ c &\approx 8.33 \end{aligned}$$

We can compare the regions of interpolation and extrapolation using the following figure.



TRY IT **Drawing and Interpreting Scatter Plots**

The scatter plot below shows the height and speed of some of the world's fastest roller coasters. A linear model was drawn to fit the data.

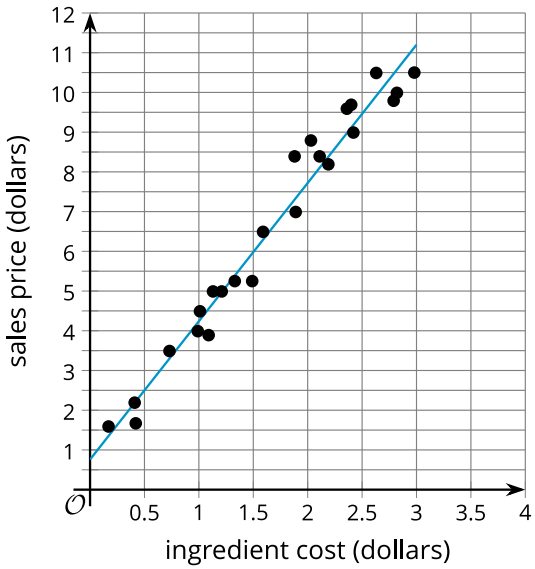


What would you expect the speed to be for a roller coaster that is 350 feet tall?

3.1.3 Interpreting the Slope and Vertical Intercept of a Linear Model

Activity

The scatter plot shows the sale price of several food items, y, and the cost of the ingredients used to produce those items, x, as well as a line that models the data. The line is also represented by the equation $y = 3.48x + 0.76$.



1. What is the predicted sale price of an item that has ingredients that cost $1.50? Be prepared to show your reasoning.
2. What is the predicted ingredient cost of an item that has a sale price of $7? Be prepared to show your reasoning.
3. What is the slope of the linear model?
4. What does the slope value mean in this situation?
5. What is the y-intercept of the linear model?
6. What does the value of the y-intercept mean in this situation?
7. Does the value of the y-intercept make sense in this situation?

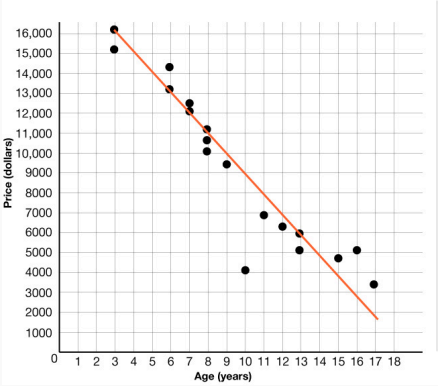
Video: Using Linear Models

Watch the following video to learn more about how to use linear models.

Linear Models (https://openstax.org/r/linear-models)

Self Check

The scatter plot below shows the age (in years) and the price (in dollars) of used compact cars advertised in the local newspaper.



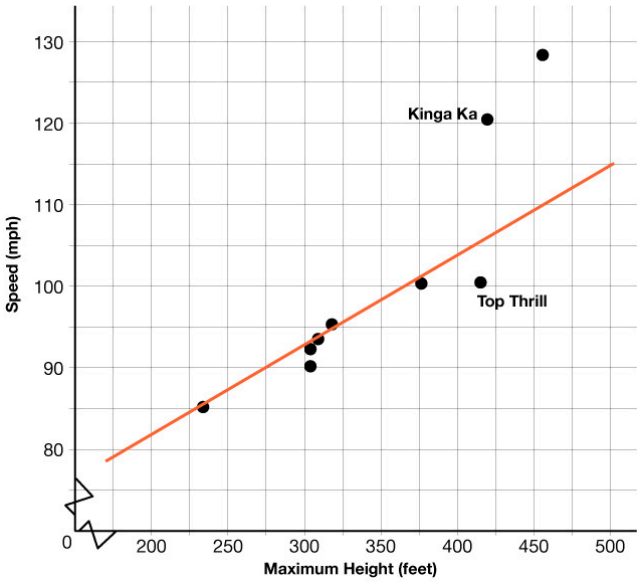
Nora drew a line to fit the data. What is the slope of her line, and what does it mean in the problem?

a. $-\frac{1}{1000}$, for every year older the car is, it decreases $\frac{1}{1000}$ of its value the year before.
b. 1000, for every year older the car is, its price increases 1000 dollars.
c. 1000, for every year older the car is, its price decreases 1000 dollars.
d. -1000, for every year older the car is, its price decreases 1000 dollars.

Additional Resources

Interpreting a Linear Model

The scatter plot below shows the height and speed of some of the world's fastest roller coasters. A line has been drawn to show a good fit for the data.



The equation of the line drawn is $S = 0.11h + 60$.

What does this slope mean in the problem?

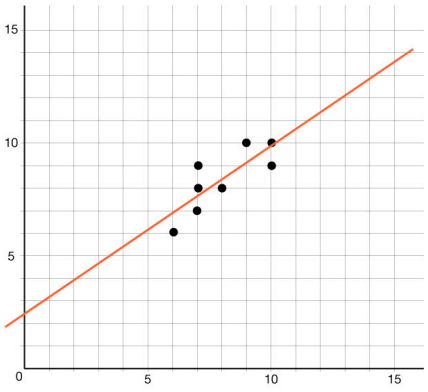
For every foot taller the roller coaster is, the speed increases 0.11 miles per hour.

TRY IT **Interpreting a Linear Model**

Eight students were asked to estimate their score on a 10-point quiz. Their estimated and actual scores are given in the table below. Plot the points. Then sketch a line that fits the data.

Predicted	Actual
6	6
7	7
7	8
8	8
7	9
9	10
10	10
10	9

Here are the points from the table plotted on a scatter plot, with the x-axis being the predicted scores and the y-axis being the actual scores.



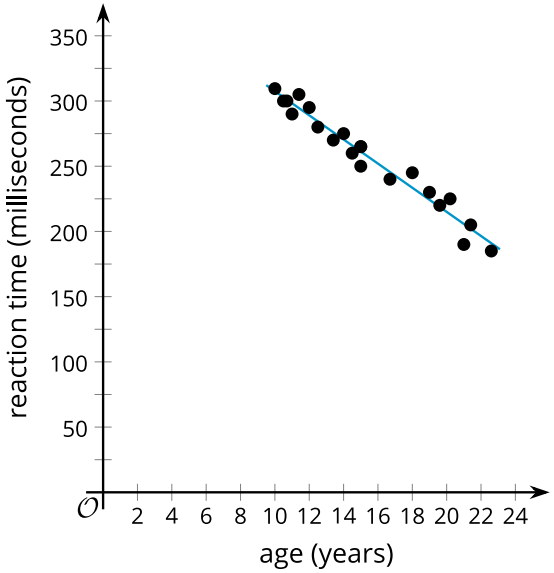
1. Tell what you would expect someone's grade to actually be if they predicted their grade to be a 5 out of 10.
2. What does the slope mean in this problem?

3.1.4 Interpreting the Slope and Vertical Intercept of a Scatter Plot

Activity

Here are several scatter plots.

Use scatter plot A to answer questions 1 – 2.



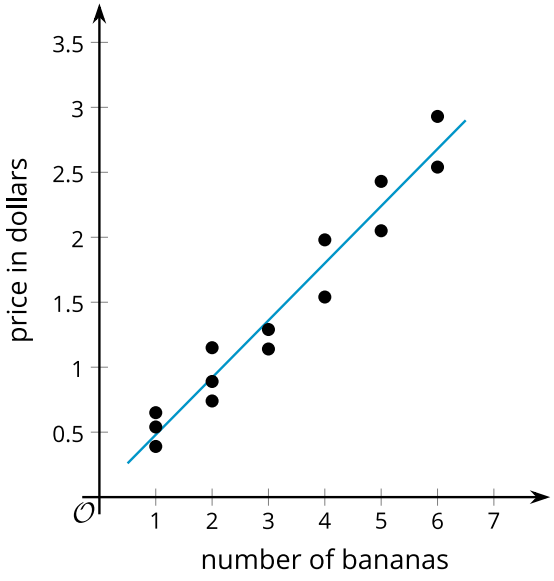
Scatter plot A: $y = -9.25x + 400$

1. Using the horizontal axis for x and the vertical axis for y, interpret the slope of the linear model in the situation shown.

2. If the linear relationship continues to hold for the situation, interpret the y-intercept of the linear model.

Use scatter plot B to answer questions 3 and 4.

Scatter plot B: $y = 0.44x + 0.04$

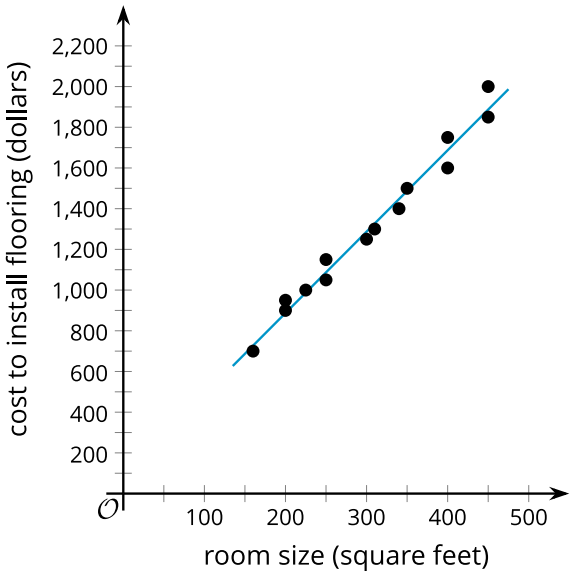


3. Using the horizontal axis for x and the vertical axis for y, interpret the slope of the linear model in the situation shown.

4. If the linear relationship continues to hold for the situation, interpret the y-intercept of the linear model.

Use scatter plot C to answer questions 5 – 6.

Scatter plot C: $y = 4x + 87$

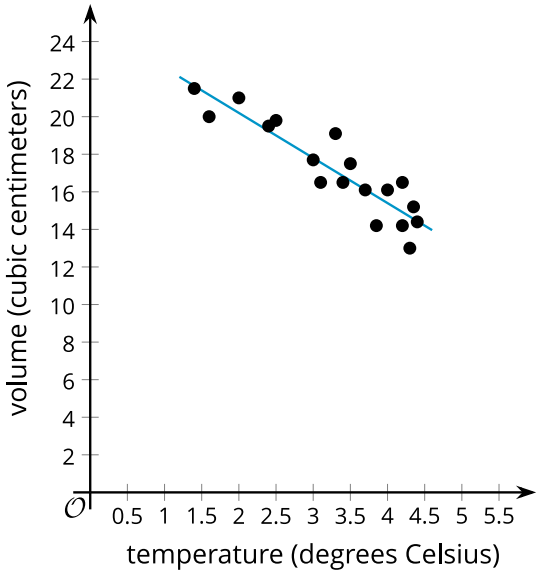


5. Using the horizontal axis for x and the vertical axis for y, interpret the slope of the linear model in the situation shown.

6. If the linear relationship continues to hold for the situation, interpret the y-intercept of the linear model.

Use scatter plot D to answer questions 7 and 8.

Scatter plot D: $y = -2.4x + 25.0$



7. Using the horizontal axis for x and the vertical axis for y, interpret the slope of the linear model in the situation shown.

8. If the linear relationship continues to hold for the situation, interpret the y-intercept of the linear model.

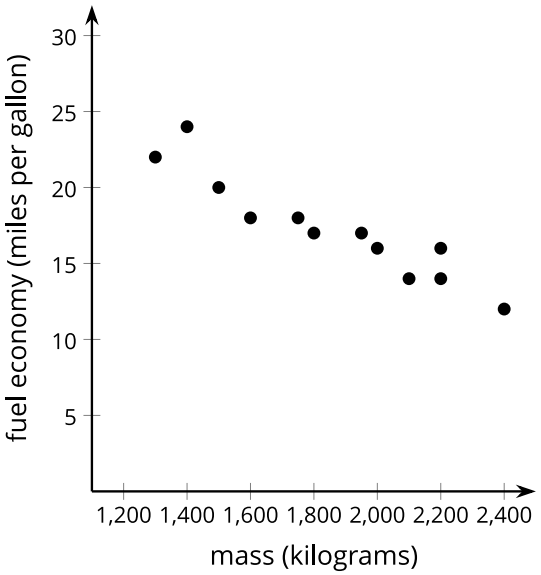
Are you ready for more?

Extending Your Thinking

Clare, Diego, and Elena collect data on the mass and fuel economy of cars at different dealerships.

Clare finds the line of best fit for data she collected for 12 used cars at a used car dealership. The line of best fit is $y = \frac{-9}{1{,}000}x + 34.3$ where x is the car's mass, in kilograms, and y is the fuel economy, in miles per gallon.

Diego made a scatter plot for the data he collected for 10 new cars at a different dealership.



Elena made a table for data she collected on 11 hybrid cars at another dealership.

mass (kilograms)	fuel economy (miles per gallon)
1,100	38
1,200	39
1,250	35
1,300	36
1,400	31
1,600	27
1,650	28

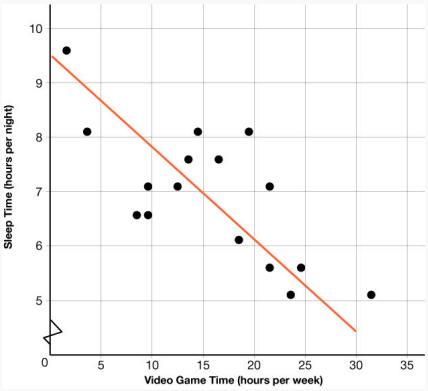
1. Interpret the slope and y-intercept of Clare's line of best fit in this situation.

2. Diego looks at the data for new cars and used cars. He claims that the fuel economy of new cars decreases as the mass increases. He also claims that the fuel economy of used cars increases as the mass increases. Do you agree with Diego's claims? Be prepared to show your reasoning.

3. Elena looks at the data for hybrid cars and correctly claims that the fuel economy decreases as the mass increases. How could Elena compare the decrease in fuel economy as mass increases for hybrid cars to the decrease in fuel economy as mass increases for new cars? Be prepared to show your reasoning.

Self Check

The scatter plot below shows the results of a survey of eighth-grade students who were asked to report the number of hours per week they spend playing video games and the typical number of hours they sleep each night.



Which of the following conclusions can you make looking at the data?

a. There is not a trend between time spent on video games and sleep time.
b. As video game time increases, sleep time decreases.
c. As video game time decreases, sleep time decreases.
d. As video game time increases, sleep time increases.

Additional Resources

Using Linear Models

The Monopoly board game is popular in many countries. The scatter plot below shows the distance from "Go" to a property (in number of spaces moving from "Go" in a clockwise direction) and the price of the properties on the Monopoly board. The equation of the line is $P = 8x + 40$, where P represents the price (in Monopoly dollars) and x represents the distance (in number of spaces).



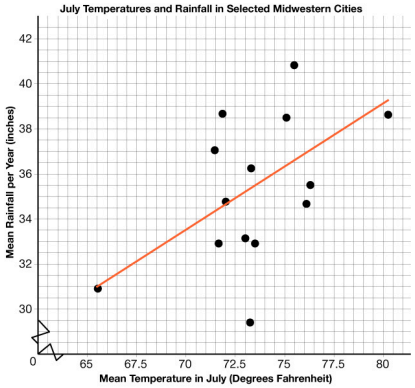
The model shows that as the distance from "Go" increases, the price of the property increases.

The slope of the equation of the linear model is 8. This means that for every space from "Go," the price of the property increases $8.

Notice this gives an estimate. The most expensive property is 39 spaces from "Go" and costs $400. The model would give the price to be $y = 8(39) + 40$ or $352.

The model gives a difference of $48 between the estimated price using the model and the actual price.

TRY IT **Using Linear Models**



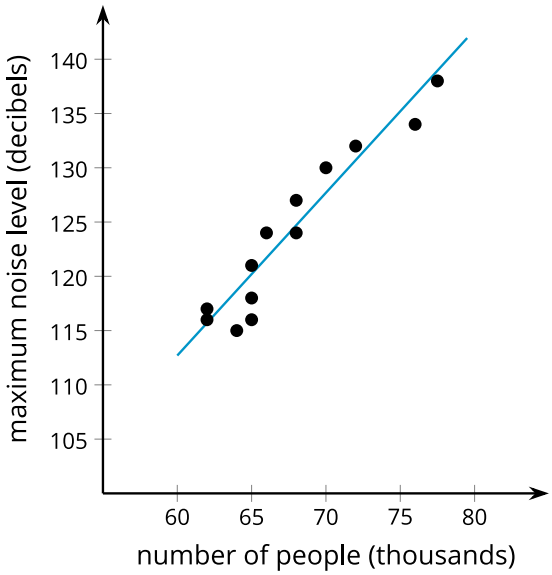
Looking at the linear model above, what claim can be made?

3.1.5 Using an Equation for a Fit Line

Cool Down

The scatter plot shows the maximum noise level when different numbers of people are in a stadium. The linear model is given by the equation $y = 1.5x + 22.7$, where y represents maximum noise level and x represents the number of people, in thousands, in the stadium.

$y = 1.5x + 22.7$



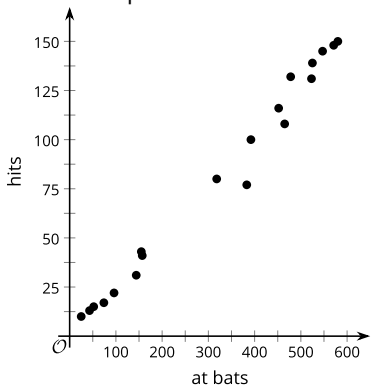
1. The slope of the linear model is 1.5. What does this mean in terms of the maximum noise level and the number of people?
2. A sports announcer states that there are 65,000 fans in the stadium. Estimate the maximum noise level. Be prepared to show your reasoning.

3. Is your estimate for the previous question reasonable?
4. What is the y-intercept of the linear model given? Be prepared to show your reasoning.
5. What does the value of the y-intercept mean in the context of the problem?
6. Is this value for the y-intercept reasonable?

3.1.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. The scatter plot shows the number of times a player came to bat and the number of hits they had.

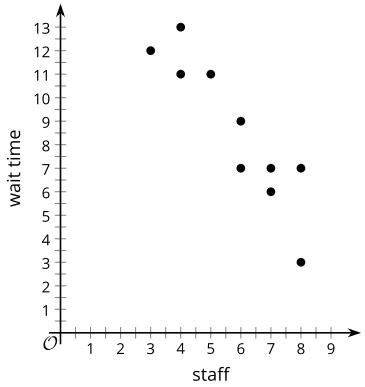


The scatter plot includes a point at $(318, 80)$. Describe the meaning of this point in this situation.

a. For every 80 at bats, there are 318 hits.
b. For every 318 at bats, there are 80 hits.

For questions 2–5, use the following to answer each question.

The scatter plot shows the number of minutes people had to wait for service at a restaurant and the number of waitstaff available at the time.



A line that models the data is given by the equation $y = -1.62x + 18$, where y represents the wait time, and x represents the number of waitstaff available.

2. The slope of the line is -1.62. What does this mean in this situation?

a. When there are 18 staff people, there is no wait time
b. When there are 0 staff people, the wait time is 18 minutes
c. For every wait time increase of 1 minute the staff people increase by 1.62
d. For every increase of 1 staff person, the wait time decreases 1.62 minutes

3. The slope of the line is -1.62. Is it realistic to conclude, for this slope, as x increases, y will decrease?

a. Yes
b. No

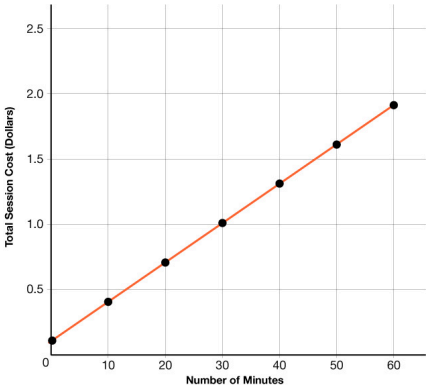
4. The y-intercept is $(0, 18)$. What does this mean in this situation?

a. When there are 18 staff people, there is no wait time
b. When there are 0 staff people, the wait time is 18 minutes
c. For every wait time increase of 1 minute the staff people increase by 1.62
d. For every increase of 1 staff person, the wait time decreases 1.62 minutes

5. The y-intercept is $(0, 18)$. Is this a realistic y-intercept point if y represents the wait time and x represents the number of staff available?

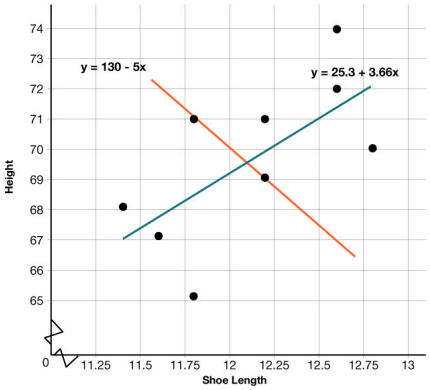
a. Yes
b. No

6. The company's pricing plan states that the usage rate is constant for any number of minutes connected to the Internet. The number of minutes is related to the cost in dollars and has a linear relationship. The equation of the line is $y = 0.03x + 0.10$. What does the slope represent in this situation?



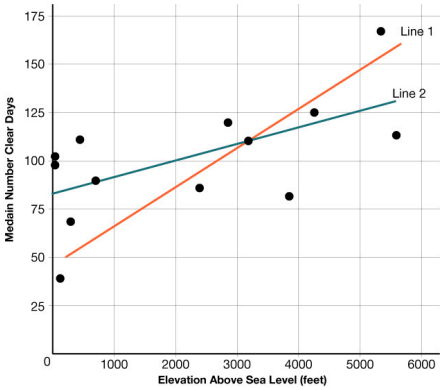
a. For every 3 minutes of internet usage, the price increases 1 cent.
b. For every minute of internet usage, the price increases 3 cents.

7. Below is a scatter plot of the data with two linear models, $y = 130 - 5x$ and $y = 25.3 + 3.66x$. Which of these two models does a better job of describing how shoe length x and height y (in inches) are related?



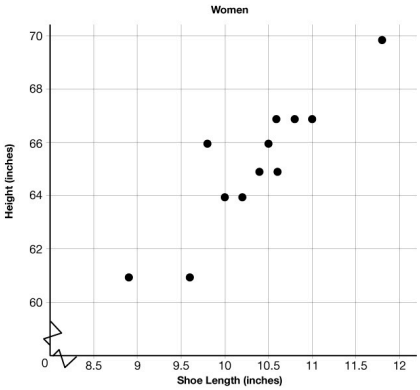
a. $y = 130 - 5x$
b. $y = 25.3 + 3.66x$

8. The scatter plot below displays the elevation and median number of clear days per year of 14 U.S. cities. Two lines are shown on the scatter plot. Which line fits the data better?



a. Line 1
b. Line 2

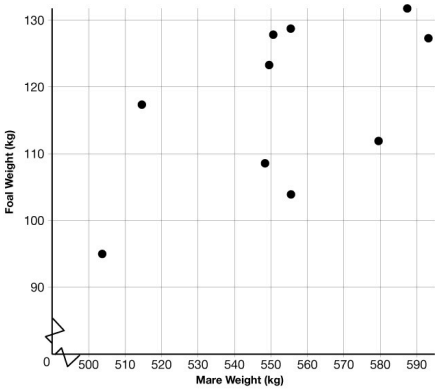
9. According to the scatter plot below, what claim can be made about women's shoe length and height?



a. As a woman's shoe length increases, their height also tends to increase.
b. As a woman's shoe length increases, their height decreases.

For questions 10 – 12, use the scatter plot below.

This scatter plot shows foal birth weight and mare's weight.



10. What does the slope mean of a linear model on this scatter plot?

a. the increases in mare weight in kg for each kg increase in foal weight
b. the increases in foal weight in kg for each kg increase in mare weight

11. Will the y-intercept of a linear model make sense in this problem?

a. Yes
b. No

12. What does the point $(548, 108)$ mean in this problem?

a. A mare weighing 108 kg has a foal weighing 548 g.
b. A mare weighing 548 kg has a foal weighing 108 kg.

3.1.7 Lesson Summary

In this lesson, you learned how to:

- Describe the rate of change and y-intercept for a linear model in everyday language.
- Draw a linear model that fits the data well and use the linear model to estimate values.

Here are the **activities** that helped you reach those goals:

- 3.1.1: Exploring Scatter Plots
  - In this activity, you recalled information about scatter plots, which was used when you expanded your understanding in a later activity. You noticed and wondered many things about the images, the relationship between the number of people and the maximum noise level, interpreting the line of best fit, and general information about scatter plots.
- 3.1.2: Creating a Scatter Plot Using Data
  - In this activity, you created a scatter plot from data given in context to interpret the slope and vertical intercept of the linear model and to use the linear model to make predictions.
  - 3.1.2: Self Check
  - 3.1.2: Additional Resources
- 3.1.3: Interpreting the Slope and Vertical Intercept of a Linear Model
  - In this activity, you interpreted in context the slope and vertical intercept of a linear model given a scatter plot and the equation for a linear model that fit the data well. The linear model was also used to interpolate and extrapolate information about the data in context.
  - 3.1.3: Self Check
  - 3.1.3: Additional Resources
- 3.1.4: Interpreting the Slope and Vertical Intercept of a Scatter Plot
  - In this activity, you interpreted the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. You were given scatter plots for different pairs of variables and the equation of a line of best fit. Then you used the line of best fit and its equation to describe the meaning of the vertical intercept and slope.
  - 3.1.4: Self Check
  - 3.1.4: Additional Resources
- 3.1.5: Using an Equation for a Fit Line
  - In this activity, you considered that we have to look at axes and demonstrate the usefulness of using an equation for a fit line, rather than sketching on a graph, to determine if the slope and y-intercept make sense for the linear model in the context of the problem.

After these activities, you completed the following **practice**:

- 3.1.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

3.2 FITTING LINES

3.2.0 Lesson Overview

In this lesson, you will assess the fit of various lines to data, use technology to find the line of best fit, and interpret the slope and vertical intercept of the linear model.

When you finish this lesson, you will be able to:

- Write linear models by hand from tables and graphs.
- Describe the rate of change and y-intercept for a linear model in everyday language.
- Use technology to find the line of best fit.

Here are the **activities** that will help you reach those goals:

- 3.2.1: Selecting the Best Line
- 3.2.2: Determining the Best Line That Fits the Data
  - 3.2.2: Self Check
  - 3.2.2: Additional Resources
- 3.2.3: Writing Linear Models without Technology
  - 3.2.3: Self Check
  - 3.2.3: Additional Resources
- 3.2.4: Assessing the Fit of a Linear Model
  - 3.2.4: Self Check
  - 3.2.4: Additional Resources
- 3.2.5: Working with Fitted Lines

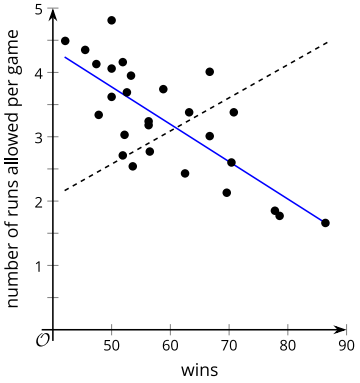
After that, you'll **practice and review**.

- 3.2.6: Practice
- 3.2.7: Lesson Summary

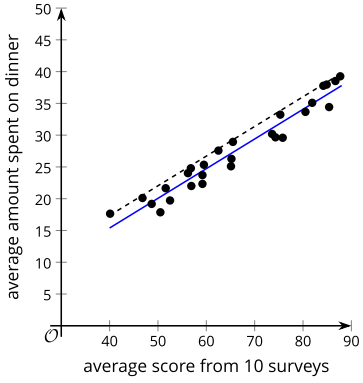
3.2.1 Selecting the Best Line

Warm Up

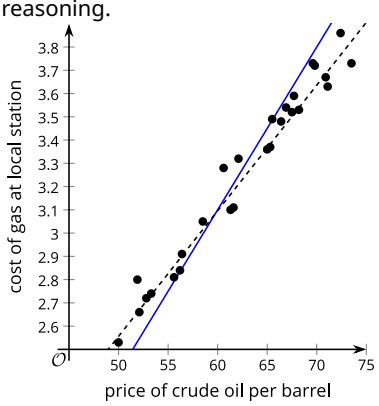
1. Which of the lines is the better fit for the data in each scatter plot? Be prepared to show your reasoning.



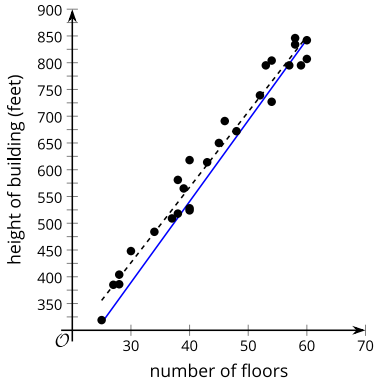
2. Which of the lines is the better fit for the data in each scatter plot? Be prepared to show your reasoning.



3. Which of the lines is the better fit for the data in each scatter plot? Be prepared to show your reasoning.



4. Which of the lines is the better fit for the data in each scatter plot? Be prepared to show your reasoning.



3.2.2 Determining the Best Line That Fits the Data

Activity

Your teacher will give you a set of cards (https://openstax.org/r/scatter-plots-3-2-2) that show scatter plots.

Arrange all the cards in three different ways. Ensure that you and your partner agree on each arrangement before moving on to the next one. Sort all the cards in order from:

- best to worst for representing with a linear model
- least to greatest slope of a linear model that fits the data well
- least to greatest vertical intercept of a linear model that fits the data well

Know that the **line of best fit** is the line that provides the best fit to the data. It should follow the trend of the data and go through the middle of the data with a similar number of points on each side of the line.

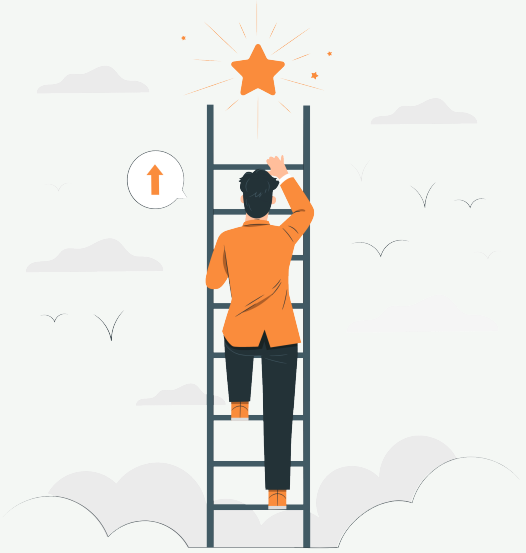
1. What is the order of the cards that displays the best to worst representations of a linear model?

2. What is the order of the cards that displays the least to greatest slope representations of a linear model that fits the data well?

3. What is the order of the cards that displays the least to greatest vertical intercept of a linear model that fits the data well?

4. For each card, write a sentence that describes how y changes as x increases and whether the linear model is a good fit for the data or not.

Building Character: Purpose



Having purpose is showing a commitment to making a meaningful contribution to the world. Purpose drives you to make positive contributions in line with your own interests and strengths.

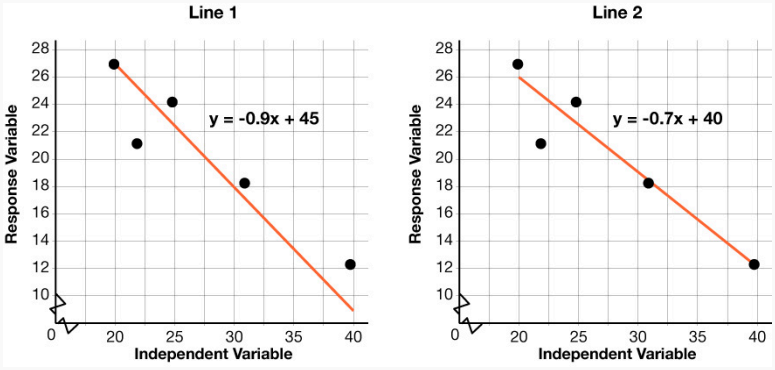
Think about your current sense of purpose. Are the following statements true for you??

- I often reflect on my life goals and the kind of person I want to be.
- I often think about what matters most to me and why it matters.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

Self Check

Tell which line is the best fit for the data and give an explanation why it is the best fit.



a. Line 1 is the best fit because it has a steeper slope.
b. Line 2 is the best fit because it goes through the middle of the data with about the same number of data points above and below the line.
c. Line 2 is the best fit because it touches the last point on the right of the scatter plot.
d. Line 1 is the best fit because it touches the first point on the left of the scatter plot.

Additional Resources

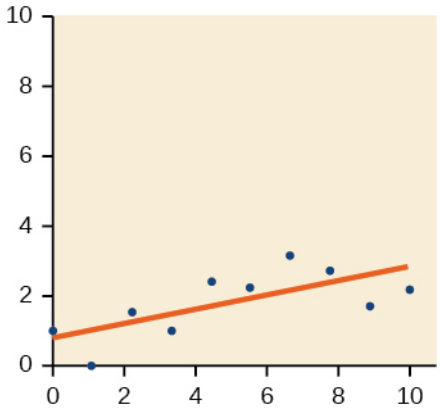
Draw a Best-Fit Line for the Plotted Data

When looking to draw a line of best fit, first make sure a linear model works for the data.

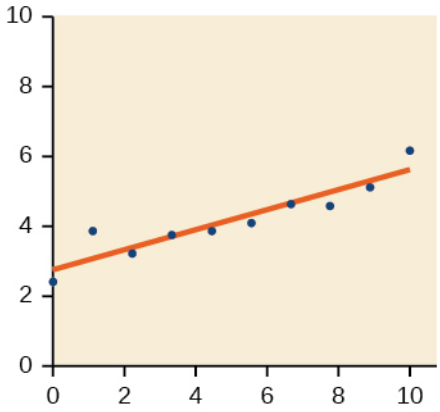
Next, place the line between the data points so that about the same number of points are above and below the line. The aim is to also have points within a close distance to the line of best fit.

Below are two examples of lines of best fit drawn through scatter plots.

1.

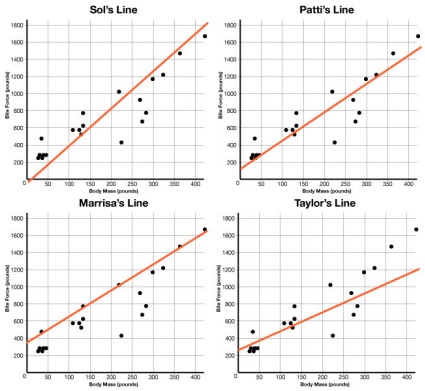


2.



TRY IT **Draw a Best-fit Line for the Plotted Data**

The scatter plots below are the same scatter plot, but each student drew their best fit line differently. Which line do you believe to be the best fit, and why?



3.2.3 Writing Linear Models without Technology

Activity

Linear models can also be found by hand in some situations when the data follows a linear pattern since the slope remains the same between data points.

Linear Models from Tables

For questions 1 – 4, use the scenario and table below.

The table below shows the number of songs Marcus will have in his collection as he adds new songs each month.

Number of months, t	0	1	2	3
Number of songs, N	200	215	230	245

1. What is the initial amount of songs, or the y-intercept of this situation?

2. What is the slope, or rate of change?

3. Write the equation of the linear model that represents the situation in slope-intercept form.

4. How many songs will Marcus have in 8 months?

For questions 5 – 8, use the situation and table below.

A new plant food is introduced to a young tree to test its effect on the height of the tree. The table shows the height of the tree, H, in feet x months since measurements began.

Number of months, x	2	4	6	8
Height in feet, H	13.5	14.5	15.5	16.5

5. What is the initial height of the tree, or the y-intercept, when measurements began? (Hint: Work each row backwards to where $x = 0$.)

6. What is the slope, or rate of change?

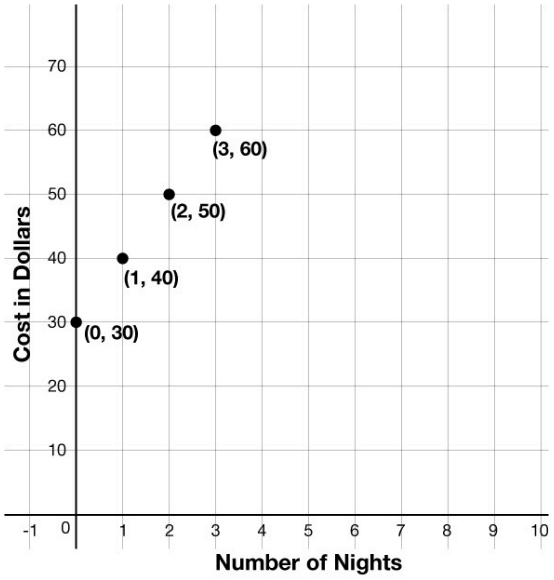
7. Write the equation of the linear model that represents the situation in slope-intercept form.

8. What will the height of the tree be in 14 months?

Linear Models from Graphs

For questions 9 – 12, use the situation and graph below.

The graph models the cost in dollars, C, of renting a tent at a campground for n nights.



9. What is the y-intercept of the graph?

10. What is the slope, or rate of change?

11. Write the equation of the linear model that represents the situation in slope-intercept form.

12. How much does it cost to rent a tent for 5 nights?

Self Check

The table below shows the revenue for the number of pizzas sold at a restaurant.

Number of Pizzas	20	30	40	50
Revenue	110	160	210	260

Which linear model could be used to predict the revenue if 4000 pizzas were sold?

$y = 80(4000)$

$y = 5(4000) + 10$

$y = 4000(10)$

I am not sure.

$y = \frac{1}{5}(4000) + 10$

Additional Resources

Writing Linear Equations From Tables and Graphs

Linear Equations from Tables

The number of texts a teen sends, T, in days, d, is shown in the table below.

Days, d	1	2	3	4
Number of texts, T	65	130	195	260

Write an equation in slope-intercept form to represent the situation then predict the number of texts sent in 8 days.

Step 1 - Find the y-intercept.

Since the value when $d = 0$ is not present, work backwards to $d = 0$.

If $d = 0$ was present, then $T = 0$ since as each x-value increases by 1, the y-values increase by 65.

Step 2 - Find the slope of the situation.

Find the rate of change. The change in y is 65 and the change in x is 1, so the slope is 65

Step 3 - Write the equation in slope-intercept form.

$y = mx + b$

$T = 65d + 0$

$T = 65d$

Step 4 - Make a prediction.

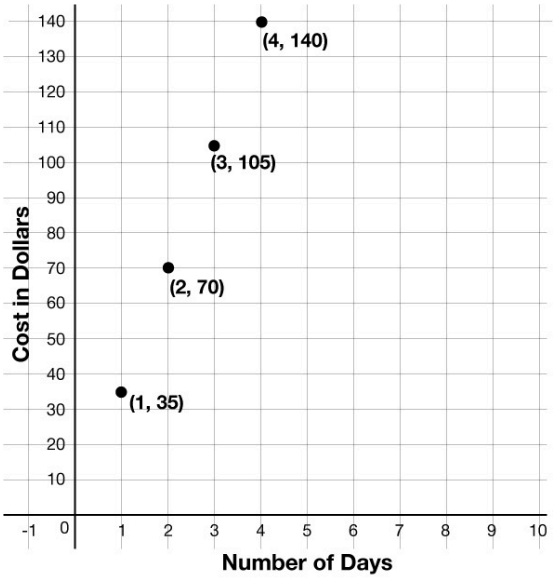
Substitute $d = 8$ into the equation.

$T = 65(8) = 520$

After 8 days, the teen would have sent 520 texts.

Linear Equations from Graphs

The cost, C, for the number of days, d, a dog spends at doggie daycare is shown in the graph below.



Write an equation in slope-intercept form to represent the situation then predict the cost of a dog staying at doggie daycare after 7 days.

Step 1 - Find the y-intercept.

Since the value when $d = 0$ is not present, work backwards to $d = 0$.

If $d = 0$ was present, then $C = 0$ since as each x-value increases by 1, the y-values increase by 35.

Step 2 - Find the slope of the situation.

Find the rate of change. The change in y is 35 and the change in x is 1, so the slope is 35

Step 3 -Write the equation in slope-intercept form.

$y = mx + b$

$C = 35d + 0$

$C = 35d$

Step 4 -Make a prediction.

Substitute $d = 7$ into the equation.

$C = 35(7) = 245$

After 7 days, the cost of doggie daycare would be $245.

TRY IT **Try It-Writing Linear Equations From Tables and Graphs**

Naomi is a professional painter. The table below shows how many square feet, F, she can paint in h hours.

Time in hours, h	1	2	3	4
Number of square feet, F	120	240	360	480

How many square feet can Naomi paint in 8 hours?

3.2.4 Assessing the Fit of a Linear Model

Activity



The weight of ice cream sold at a small store in pounds (x) and the average temperature outside in degrees Celsius (y) are recorded in the table.

x	20	18	21	17	21.5	19.5	21	18
y	6	4.5	6.5	3.5	7.5	6.5	7	5

1. For these data, create a scatter plot by hand on a coordinate grid and sketch a line that fits the data well.

Using Technology to Find the Equation of the Best Fit Line

Previously, you drew the best fit line by hand and found its equation without the use of technology. In this activity, and throughout the remainder of this lesson, you will use technology to find the equation of the best fit line. Click here (https://openstax.org/r/desmos-guide) for instructions on how to use technology to find the equation of the best fit line.

2. Now, use technology to create the scatter plot and compute the best fit line.

3. What is the value of the slope for the line of best fit? Round your answer to the nearest hundredth (two decimal places).

4. What does this slope value mean in this situation?

5. What is the value of the y-intercept for the line of best fit? Round your answer to the nearest hundredth (two decimal places).

6. What does this y-intercept value mean in this situation?

7. Use the best-fit line to predict the y-value when x is 10.

8. Is this a good estimate for the data? Be prepared to share your reasoning.

9. Your teacher will give you a data table for one of the other scatter plots from the previous activity. Use technology and this table of data to create a scatter plot then determine the line of best fit.

10. Interpret the meaning of the value of the slope of this line of best fit.

Are you ready for more?

Extending Your Thinking

Priya uses several different ride services to get around her city. The table shows the distance, in miles, she traveled during her last 10 trips and the price of each trip, in dollars.

distance (miles)	price ($)
3.1	12.5
4.2	14.75
5	16
3.5	13.25
2.5	12
1	9
0.8	8.75
1.6	9.75
4.3	12
3.3	14

1. Priya creates a scatter plot of the data using the distance, x, and the price, y. She determines that a linear model is appropriate to use with the data. Use technology to find the equation of a line of best fit.
2. Interpret the slope and the y-intercept of the equation of the line of best fit in this situation.
3. Use the line of best fit to estimate the cost of a 3.6-mile trip. Will this estimate be close to the actual value? Be prepared to show your reasoning.
4. On her next trip, Priya tries a new ride service and travels 3.6 miles, but she pays only $4.00 because she receives a discount. Include this trip in the table and calculate the equation of the line of best fit for the 11 trips. Did the slope of the equation of the line of best fit increase, decrease, or stay the same? Why? Be prepared to show your reasoning.
5. Priya uses the new ride service for her 12^{th} trip. She travels 4.1 miles and is charged $24.75. How do you think the slope of the equation of the line of best fit will change when this 12^{th} trip is added to the table?

Video: Finding Lines of Best Fit

Watch the following video to learn more about how to find the line of best fit.

Determining Lines of Best Fit (https://openstax.org/r/best-fit)

Self Check

Old Faithful is a geyser in Yellowstone National Park. The following table offers some rough estimates of the length of an eruption (in minutes) and the amount of water (in gallons) in that eruption.

Length (minutes)	1.5	2	3	4.5
Amount of Water (gallons)	3,700	4,100	6,450	8,400

Use technology to find an equation of the line of best fit. Round to the nearest hundredth.

a. $y = -1151.19x + 1640.48$
b. $y = 1640.48x + 1151.19$
c. $y = 800x + 2500$
d. $y = 1151.19x + 1640.48$

Additional Resources

Technology for Best Fitting Lines

Scientists are interested in finding out how different species adapt to finding food sources. One group studied crocodilians to find out how their bite force was related to body mass and diet. The table below displays the information they collected on body mass (in pounds) and bite force (in pounds).

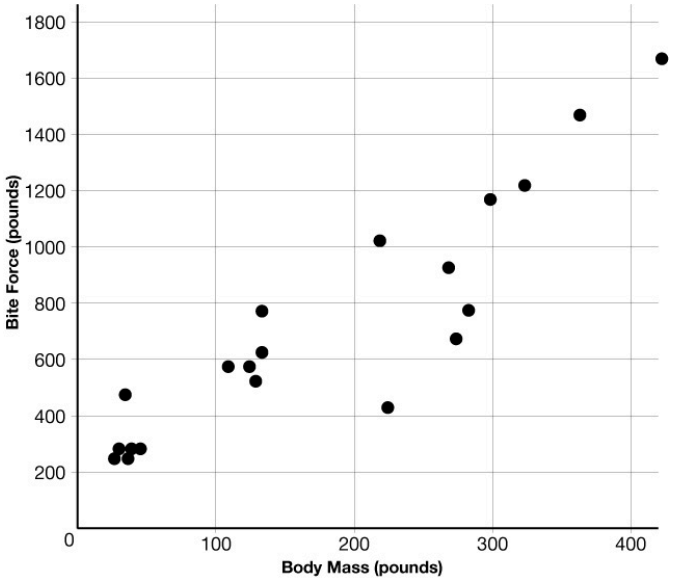
Species	Body mass (pounds)	Bite force (pounds)
Dwarf crocodile	35	450
Crocodile F	40	260

Crocodilian Biting

Species	Body mass (pounds)	Bite force (pounds)
Alligator A	30	250
Caiman A	28	230
Caiman B	37	240
Caiman C	45	255
Crocodile A	110	550
Nile crocodile	275	650
Crocodile B	130	500
Crocodile C	135	600
Crocodile D	135	750
Caiman D	125	550
Indian gharial crocodile	225	400
Crocodile G	220	1,000
American crocodile	270	900
Crocodile E	285	750
Crocodile F	425	1,650
American alligator	300	1,150
Alligator B	325	1,200
Alligator C	365	1,450

Crocodilian Biting

The scatter plot below displays the data on body mass and bite force for the crocodilians in the study.



Using technology, the line of best fit for the data is $y = 3.02x + 154.68$.

The slope of the line of best fit is 3.02. This signifies that for every pound more the crocodile weighs, it has 3.02 more pounds of bite force.

The y-intercept is 154.68, which doesn't quite make sense for a crocodile that weighs 0 pounds, but it does give a minimum bite force to expect from a crocodile.

TRY IT

Try It: Technology for Best Fitting Lines

The following table gives the times of the gold, silver, and bronze medal winners for the men's 100-meter race (in seconds) for the past 10 Olympic Games.

Use technology to write an equation of the line that represents the line of best fit for the years since 1976 and the mean times each year. Round to the nearest hundredth.

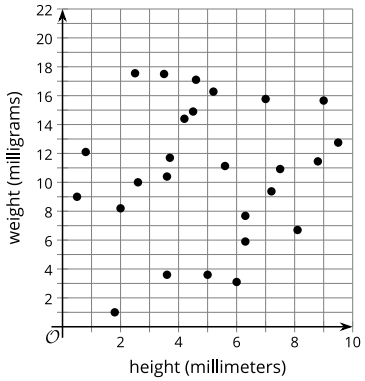
Year	2012	2008	2004	2000	1996	1992	1988	1984	1980	1976
Number of Years (since 1976)	36	32	28	24	20	16	12	8	4	0
Gold	9.63	9.69	9.85	9.87	9.84	9.96	9.92	9.99	10.25	10.06
Silver	9.75	9.89	9.86	9.99	9.89	10.02	9.97	10.19	10.25	10.07
Bronze	9.79	9.91	9.87	10.04	9.90	10.04	9.99	10.22	10.39	10.14
Mean Time	9.72	9.83	9.86	9.97	9.88	10.01	9.96	10.13	10.30	10.09

3.2.5 Working with Fitted Lines

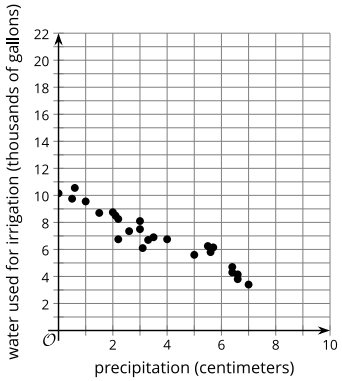
Cool Down

1. Which of these scatter plots shows data that would best be modeled with a linear function? Be prepared to show your reasoning.

Graph A



Graph B



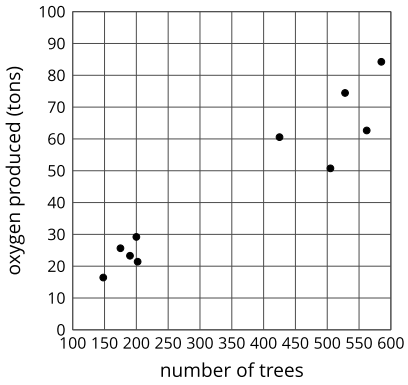
2. Which of the following lines is most likely the line of best fit for the data provided in the graph and table below?

a. $y = -0.51x + 225.12$

b. $y = 0.34x - 34.05$

c. $y = 0.13x - 0.19$

d. $y = 0.98x - 21.13$



number of trees in a forest	tons of oxygen produced by the forest
148	16.43
175	25.64
190	23.28
200	29.2
202	21.41
425	60.56
505	50.75
528	74.45
562	62.66
585	84.24

3.2.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Use graphing technology to create a scatter plot and find the best fit line.

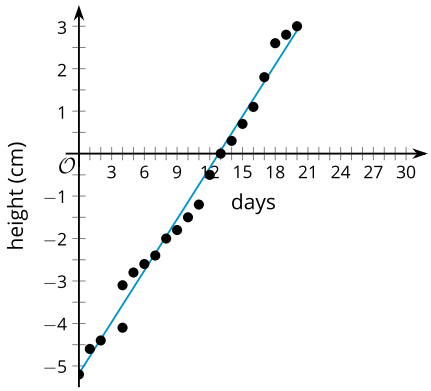
What does the best fit line estimate for the y-value when x is 100?

x	y
83	102
87	115
91	107
93	122
97	125
97	127
101	120
104	127

a. 120
b. 150
c. 125
d. 100

Access multimedia content (http://openstax.org/books/algebra-1/pages/3-2-6-practice)

2. A seed is planted in a glass pot, and its height is measured in centimeters every day.



The best fit line is given by the equation $y = 0.404x - 5.18$, where y represents the height of the plant above ground level, and x represents the number of days since it was planted.

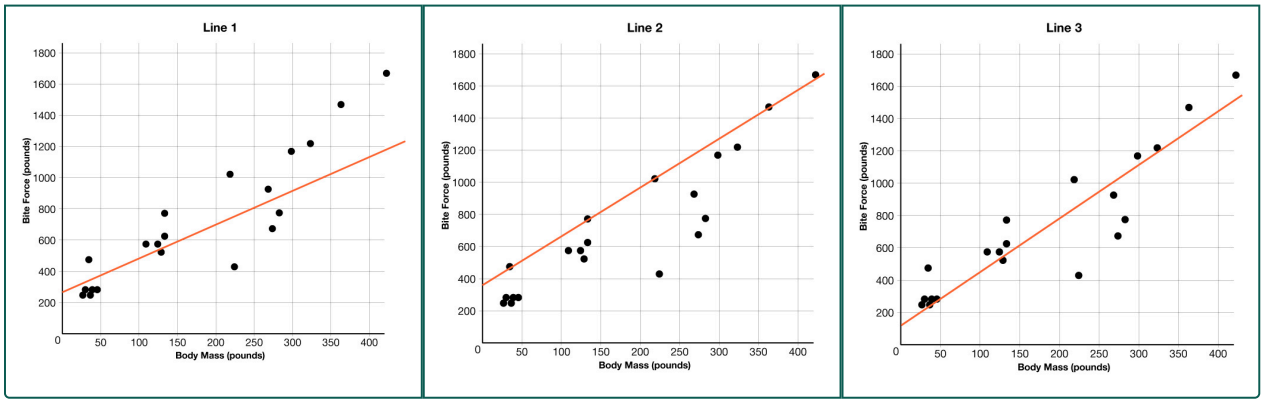
What is the slope of the best fit line? What does the slope of the line mean in this situation?

a. The slope is 0.404, which means the plant grows 1 centimeter every 0.404 days.
b. The slope is -5.18, which means for every day, the plant shrinks -5.18 centimeters.
c. The slope is 0.404, which means for every day, the plant grows 0.404 centimeters.
d. The slope is 5.18, which means for every day, the plant grows 5.18 centimeters.

Use the following information to answer questions 3 - 5.

The scatter plots below show different lines that students used to model the relationship between body mass (in pounds) and bite force (in pounds) for crocodiles.

Match each graph to one of the equations below, and explain your reasoning. Let B represent bite force (in pounds) and W represent body mass (in pounds).



3. Which graph represents data that could be modeled by the equation, $B = 3.28W + 126$?

a. Line 1
b. Line 2
c. Line 3

4. Which graph represents data that could be modeled by the equation, $B = 3.04W + 351$?

a. Line 1
b. Line 2
c. Line 3

5. Which graph represents data that could be modeled by the equation, $B = 2.16W + 267$?

a. Line 1
b. Line 2
c. Line 3

6. Mike owns a grass cutting business. The table below shows how many yards, y, he can cut in x days.

Number of days, x	1	2	3	4
Number of yards cut, y	6	12	18	24

a. $y = \frac{1}{6}x + 6$
b. $y = \frac{1}{6}x$
c. $y = 6x + 6$
d. $y = 6x$

3.2.7 Lesson Summary

In this lesson, you learned how to:

- Write linear models by hand from tables and graphs.
- Describe the rate of change and y-intercept for a linear model in everyday language.
- Use technology to find the line of best fit.

Here are the **activities** that helped you reach those goals:

- 3.2.1: Selecting the Best Line
  - In this activity, you visually assessed the best line fit for data among a set of given choices. You explored bad fit, good fit, and best fit.
- 3.2.2: Determining the Best Line That Fits the Data
  - In this activity, you distinguished between linear and nonlinear relationships in bivariate, numerical data and informally assessed the fit of a linear model. You also compared the slope and the vertical intercepts of different linear models and were able to describe the relationship between
  - 3.2.2: Self Check
  - 3.2.2: Additional Resources
- 3.2.3: Writing Linear Models without Technology
  - In this activity, you were given data in either table or graphic form and then asked to find a linear model that could represent it without using technology. Once you wrote the equation in slope-intercept form, you used it to make a prediction.
  - 3.2.3: Self Check
  - 3.2.3: Additional Resources
- 3.2.4: Assessing the Fit of a Linear Model
  - In this activity, you used technology to compute a line of best fit for data given in a table and to recognize the meaning of the slope and y-intercept.
  - 3.2.4: Self Check
  - 3.2.4: Additional Resources
- 3.2.5: Working with Fitted Lines
  - In this activity, you had an opportunity to work with fitted lines. You identified scatter plots that showed data that would best be modeled with a linear function and the line of best fit for the data provided.

After these activities, you completed the following **practice** :

- 3.2.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

3.3 RESIDUALS

3.3.0 Lesson Overview

In this lesson, you will assess the fit of a function by plotting and analyzing residuals.

When you finish this lesson, you will be able to:

- Plot and calculate residuals for a data set and use the information to judge whether a linear model is a good fit.

Here are the **activities** that will help you reach those goals:

- 3.3.1: Subtracting an Estimated Value from an Actual Value
- 3.3.2: Plotting and Analyzing Residuals
  - 3.3.2: Self Check
  - 3.3.2: Additional Resources
- 3.3.3: Matching Residuals Graphs to Scatter Plots
  - 3.3.3: Self Check
  - 3.3.3: Additional Resources
- 3.3.4: Identifying Best Fit Line for Residuals on a Graph

After that, you'll **practice and review**.

- 3.3.5: Practice
- 3.3.6: Lesson Summary

3.3.1 Subtracting an Estimated Value from an Actual Value

Warm Up

Use the following information to answer questions 1 - 4

> Mentally calculate how close the estimate is to the actual value using the difference:
>
> *actual value − estimated value.*

1. Actual value: 24.8 grams. Estimated value: 19.6 grams

2. Actual value: $112.11. Estimated value: $109.30

3. Actual value: 41.5 centimeters. Estimated value: 45.90 centimeters

4. Actual value: -1.34 degrees Celsius. Estimated value: -2.45 degrees Celsius

3.3.2 Plotting and Analyzing Residuals

Activity



Here is the data table for orange weights from a previous lesson. Use this information to answer the questions.

Number of oranges	Weight in kilograms
3	1.027
4	1.162
5	1.502
6	1.617
7	1.761
8	2.115
9	2.233
10	2.569

1. Use the graphing tool or technology outside the course. Create the scatter plot and calculate the best fit line using the Desmos tool below. Access multimedia content (http://openstax.org/books/algebra-1/pages/3-3-2-plotting-and-analyzing-residuals)
2. What level of accuracy makes sense for the slope and intercept values? Be prepared to show your reasoning.
3. What does the linear model estimate for how much the weight increases for each additional orange that is placed in the box of oranges?
4. Compare the weight of the actual box with 3 oranges in it to the estimated weight of the box with 3 oranges in it. Be prepared to show your reasoning.
5. How many oranges are in the box when the linear model estimates the weight best? Be prepared to show your reasoning.
6. How many oranges are in the box when the linear model estimates the weight least well? Be prepared to show your reasoning.

Use the following information to answer questions 7 and 8.

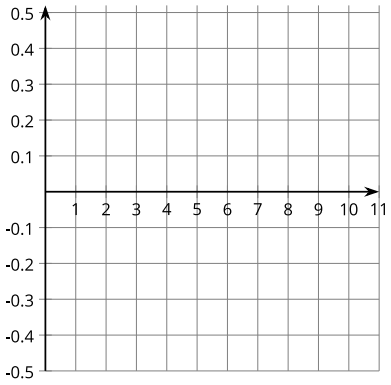
The difference between the actual value and the value estimated by a linear model is called the residual.

- If the actual value is greater than the estimated value, the residual is positive.
- If the actual value is less than the estimated value, the residual is negative.

7. For the orange weight data set, what is the residual for the line of best fit when there are 3 oranges?

8. On the same axes as the scatter plot, plot this residual at the point where $x = 3$ and y has the value of the residual.

9. Find the residuals for each of the other points in the scatter plot and graph them.



10. Which point on the scatter plot has the residual closest to zero? What does this mean about the weight of the box with that many oranges in it?

11. How can you use the residuals to decide how well a line fits the data?

Video: Finding Residuals

Watch the following video to learn more about finding residuals.

Calculating Residuals (https://openstax.org/r/calculate-residuals)

Self Check

The equation for a line of best fit is $y = 2.2x + 3.4$. Find the residual for the point $(3, 9)$.

a. 14.2
b. -1
c. 1
d. 2.4

Additional Resources

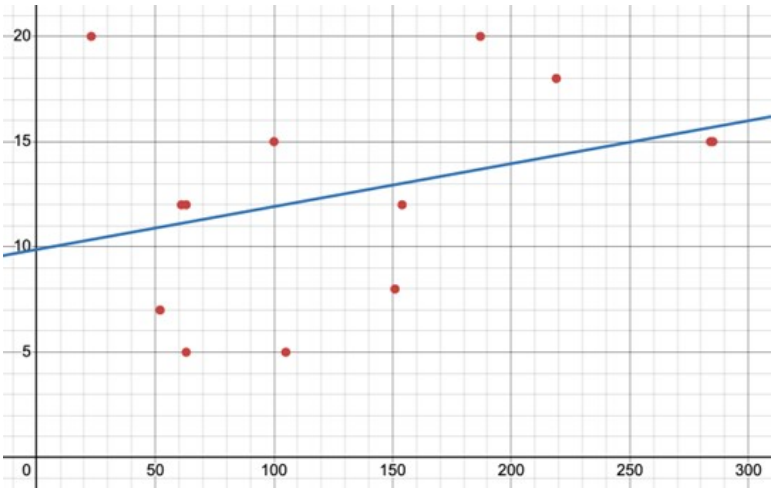
Finding Residuals

The gestation time for an animal is the typical duration between conception and birth. The longevity of an animal is the typical lifespan for that animal. The gestation times, in days, and longevities, in years, for 13 types of animals are shown in the table below.

Animal	Gestation Time (Days)	Longevity (Years)
Baboon	187	20
Black Bear	219	18
Beaver	105	5
Bison	285	15

Animal	Gestation Time (Days)	Longevity (Years)
Cat	63	12
Chimpanzee	23	20
Cow	284	15
Dog	61	12
Fox (Red)	52	7
Goat	151	8
Lion	100	15
Sheep	154	12
Wolf	63	5

The equation of a line of best fit is $y = 9.875 + 0.02039x$, where x represents the gestational time, in days, and y represents longevity, in years.



A lion's gestation time is 100 days, and its longevity is 15 years. What does this line of best fit predict the lion's longevity to be?

Solution

The line of best fit is $y = 9.875 + 0.02039(100) = 11.9$

The residual would be: actual value – estimated value or $15 - 11.9 = 3.1$.

The residual value is 3.1.

TRY IT **Finding Residuals**

In the table above, a dog has a gestation time of 61 days and a longevity of 12 years.

Using the same equation, $y = 9.875 + 0.02039x$, what would the residual value be?

3.3.3 Matching Residuals Graphs to Scatter Plots

Activity

With a partner, you will take turns to match the graphs (https://openstax.org/r/match-graphs-3-3-3) of residuals to the scatter plots that display linear models. The scatter plots and their lines of best fit are located on cards A - F. The graphs of the residuals are located on cards G - L.

1. Match the scatter plots and given linear models to the graph of the residuals.
2. Turn over the scatter plots so that only the residuals are visible. Based on the residuals, which line would produce the most accurate estimates? Which line fits its data worst?

What Do Residual Plots Tell Us?

Residuals tell us how close to accurate a line of best fit really is.

Ideally, the residual from a line of best fit will be scattered randomly above and below the horizontal line $y = 0$ on the residual plot. The residual plot would have points above and below the horizontal line $y = 0$. This shows that the relationship between the two variables in the data set can be approximated by a linear model.

When the residuals are equal to 0, this indicates that the line of best fit goes through those points.

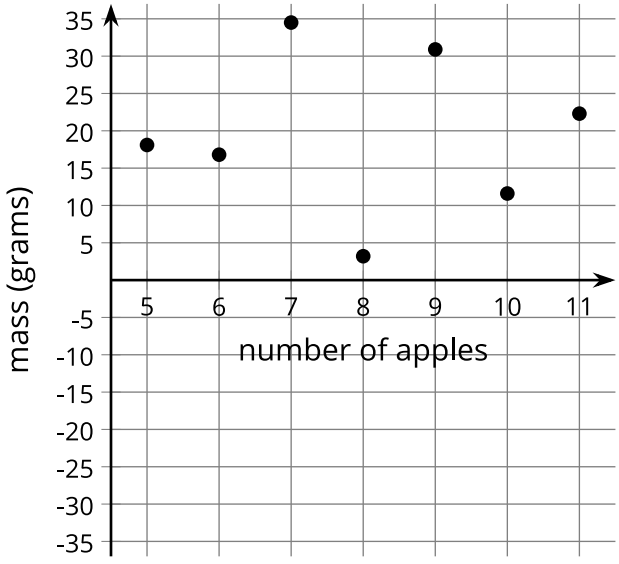
A curved pattern shows that a nonlinear model would be better to describe the relationships between points.

Are you ready for more?

Extending Your Thinking

Use the following information to answer questions 1 – 2.

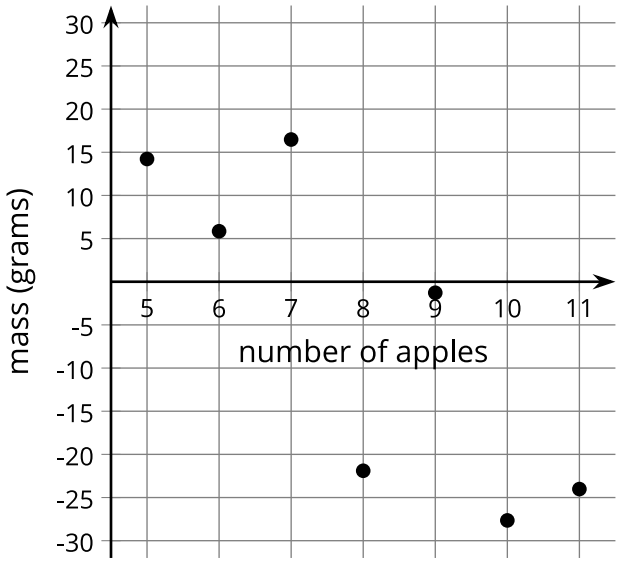
Tyler estimates a line of best fit for some linear data about the mass, in grams, of different numbers of apples. Here is the graph of the residuals.



1. What does Tyler's line of best fit look like according to the graph of the residuals?
2. How well does Tyler's line of best fit model the data? Be prepared to show your reasoning.

Use the following information to answer questions 3 and 4.

Lin estimates a line of best fit for the same data. The graph shows the residuals.

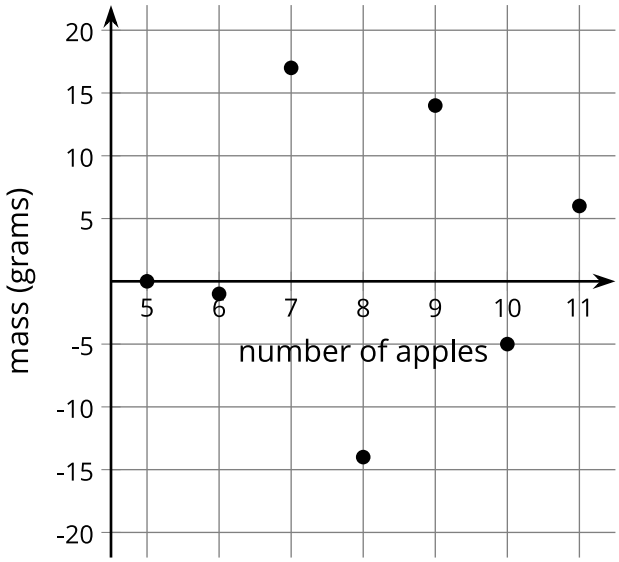


3. What does Lin's line of best fit look like in comparison to the data?

4. How well does Lin's line of best fit model the data? Be prepared to show your reasoning.

Use the following information to answer questions 5 – 6.

Kiran also estimates a line of best fit for the same data. The graph shows the residuals.



5. What does Kiran's line of best fit look like in comparison to the data?

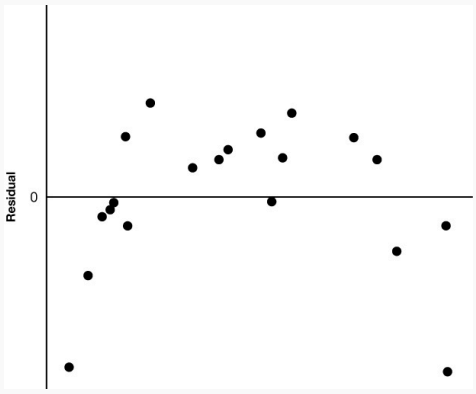
6. How well does Kiran's line of best fit model the data? Be prepared to show your reasoning.

7. Who has the best estimate of the line of best fit: Tyler, Lin, or Kiran? Be prepared to show your reasoning.

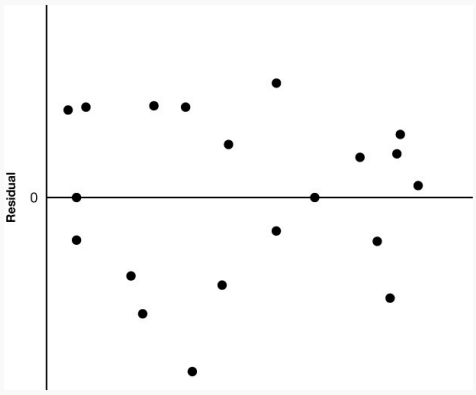
Self Check

Which residual plot shows the most accurate line of best fit for a scatter plot?

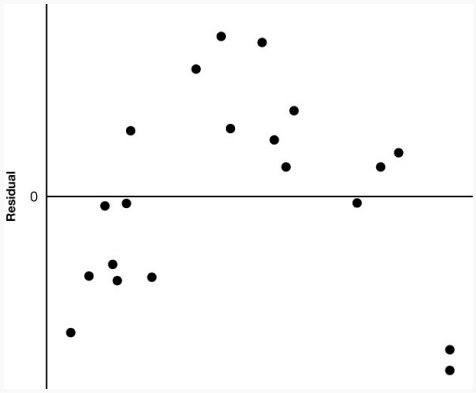
a.



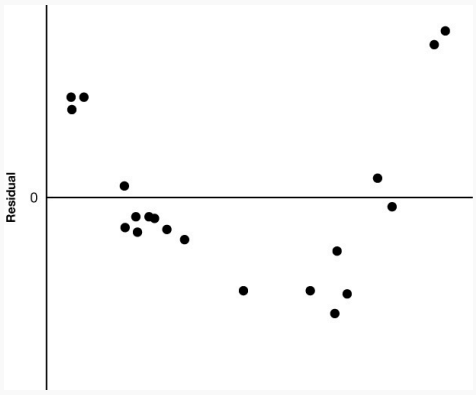
b.



c.



d.

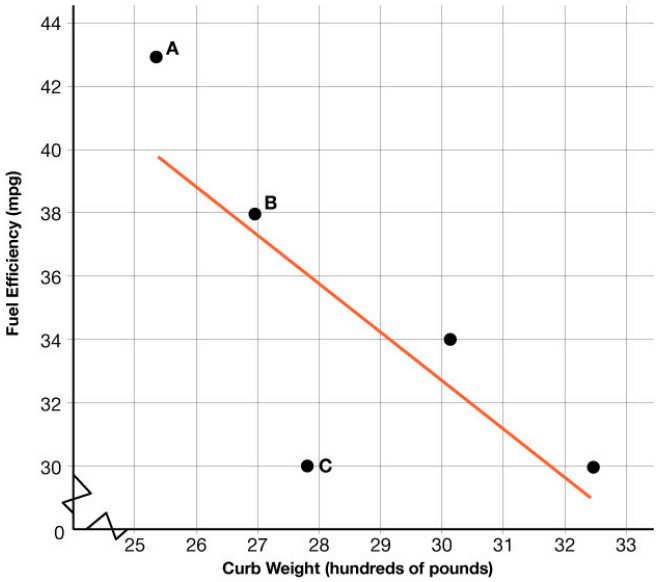


Additional Resources

The Meaning of Residuals

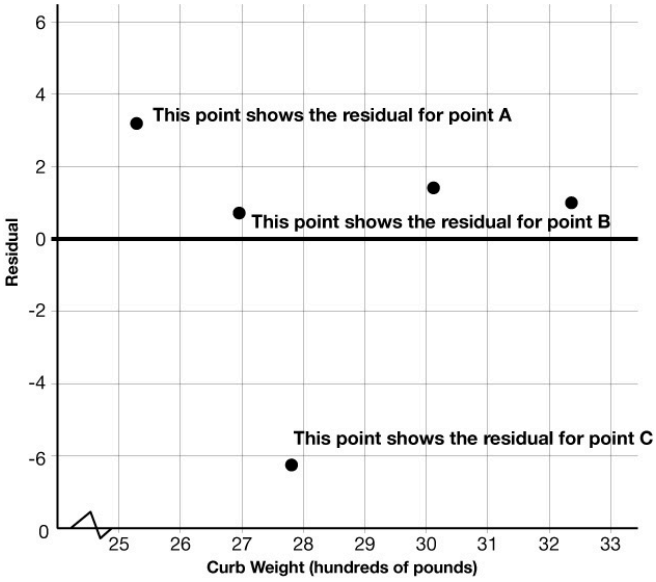
Suppose that you have a scatter plot and that you have drawn the line of best fit on your plot. Remember that the residual for a point in the scatter plot is the vertical distance of that point from the line of best fit. In the previous lesson,

you looked at a scatter plot showing how fuel efficiency was related to curb weight for five compact cars. The scatter plot and line of best fit are shown below.



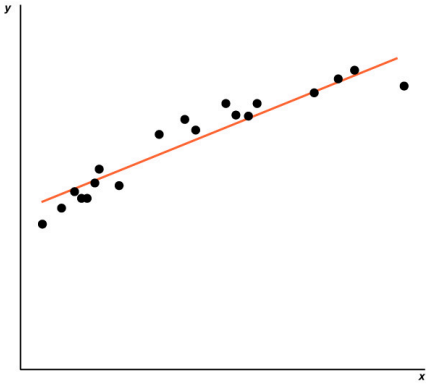
Consider the following questions:

1. What kind of residual does Point A have?
   Point A has a large positive residual.
2. What kind of residual does Point B have?
   Point B has a small positive residual.
3. What kind of residual does Point C have?
   Point C has a very large negative residual.



TRY IT **The Meaning of Residuals**

Suppose you are given a scatter plot and a line of best fit that looks like this:

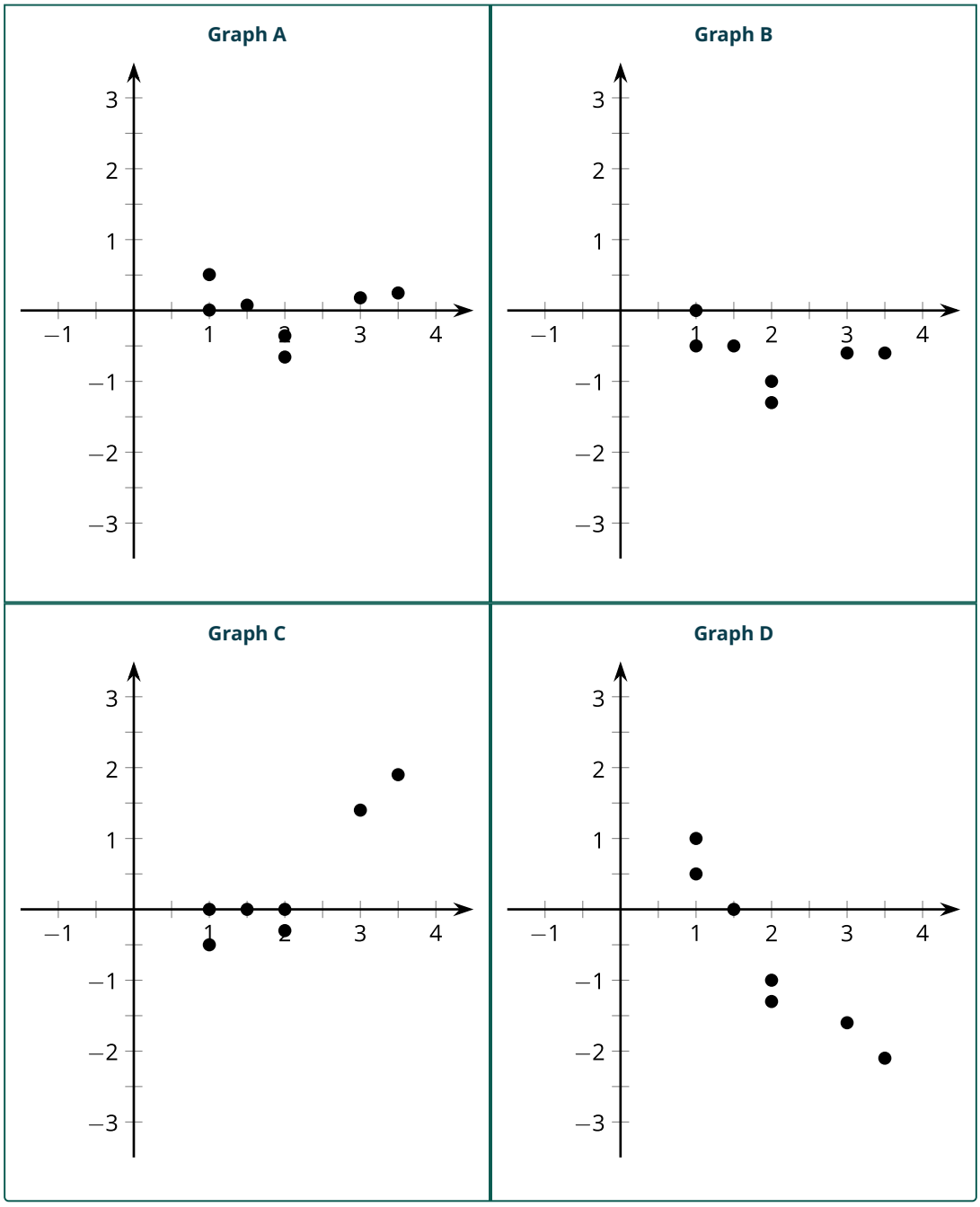


Describe what you think the residual plot would look like.

3.3.4 Identifying Best Fit Line for Residuals on a Graph

Cool Down

Each of these graphs of residuals is from the same set of data using different lines to fit the data. Which graph is most likely to represent the residuals from the best fit line? Be prepared to show your reasoning.


Graph A
Graph B
Graph C
Graph D

3.3.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

Using the following information, find the residual for each of the coordinate pairs, (x, y) in 1 - 6.

A line of best fit for a data set is $y = 1.1x + 3.4$.

1. $(5, 8.8)$

a. -2.76
b. 0
c. 0.32
d. -0.2
e. -0.1

2. $(2.5, 5.95)$

a. -2.76
b. 0
c. 0.32
d. -0.2
e. -0.1

3. $(0, 3.72)$

a. -2.76
b. 0
c. 0.32
d. -0.2
e. -0.1

4. $(1.5, 5.05)$

a. -2.76
b. 0
c. 0.32
d. -0.2
e. -0.1

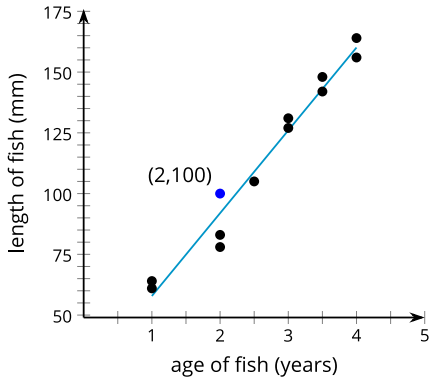
5. $(-3, 0)$

a. -2.76
b. 0
c. 0.32
d. -0.2
e. -0.1

6. $(-5, -4.86)$

a. -2.76
b. 0
c. 0.32
d. -0.2
e. -0.1

7. In the scatter plot below, the length of fish and the age of the fish are shown. The residual for the fish that is 2 years old and 100 mm long is 8.06 mm.



What does this residual mean?

a. The actual fish at 2 years old is about 8 mm shorter than the linear model estimates.
b. The actual fish at 2 years old is about 8 mm longer than the linear model estimates.

Use the following information to answer questions 8 – 11.

Four athletes on a track team are comparing their personal bests in the 100-meter and 200-meter events. A table of their best times is shown below.

Athlete	100 m time (seconds)	200 m time (seconds)
1	12.95	26.68
2	13.81	29.48
3	14.66	28.11
4	14.88	30.93

The line of best fit has an equation of $y = 1.5115x + 7.526$, where x represents the 100-meter time and y represents the 200-meter time.

8. What is the residual for athlete number 1?

a. -1.08
b. -0.91
c. 0.91
d. 1.08
e. 0.42
f. -0.42
g. -1.57
h. 1.57

9. What is the residual for athlete number 2?

a. -1.08
b. -0.91
c. 0.91
d. 1.08
e. 0.42
f. -0.42
g. -1.57

h. 1.57

10. What is the residual for athlete number 3?

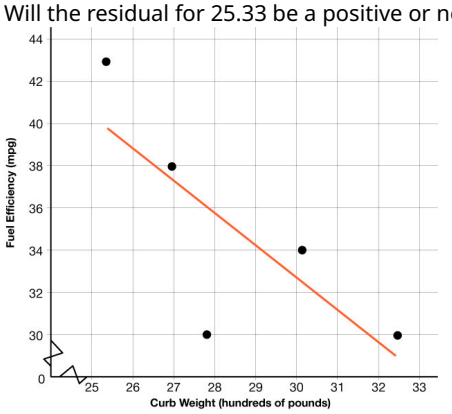
a. -1.08
b. -0.91
c. 0.91
d. 1.08
e. 0.42
f. -0.42
g. -1.57
h. 1.57

11. What is the residual for athlete number 4?

a. -1.08
b. -0.91
c. 0.91
d. 1.08
e. 0.42
f. -0.42
g. -1.57
h. 1.57

12. The curb weight of a car is the weight of the car without luggage or passengers. The scatter plot below shows the curb weight, in pounds, of 5 cars and their fuel efficiency, in miles per gallon.

Will the residual for 25.33 be a positive or negative value?



a. positive
b. negative

3.3.6 Lesson Summary

In this lesson, you learned how to:

- Plot and calculate residuals for a data set and use the information to judge whether a linear model is a good fit.

Here are the **activities** that helped you reach those goals:

- 3.3.1: Subtracting an Estimated Value from an Actual Value
  - In this activity, you discussed strategies and understandings you had for subtracting an estimated value from an actual value. This discussion helped you develop fluency and helped prepare you to compute residuals from a linear model.
- 3.3.2: Plotting and Analyzing Residuals
  - In this activity, you explored the concept of residuals, plotted residuals, and analyzed the residuals to informally assess the fit of a function. In this activity, you also fitted a function to data and used the function to solve problems. You learned that a residual is the difference between the actual y-value for a point and the expected y-value for the point on the linear model with the same associated x-value.
  - 3.3.2: Self Check
  - 3.3.2: Additional Resources
- 3.3.3: Matching Residuals Graphs to Scatter Plots
  - In this activity, you took turns with a partner, matching graphs of residuals to scatter plots that displayed linear models. You explained your thinking and listened in order to explain your reasoning and critique the reasoning of others. You recognized that a plot of the residuals for data that are fit well by a linear model shows residuals that are close to the x-axis and do not show a noticeable pattern.
  - 3.3.3: Self Check
  - 3.3.2: Additional Resources
- 3.3.4: Identifying Best Fit Line for Residuals on a Graph
  - In this activity, you identified a graph most likely to represent the residuals from the best fit line.

After these activities, you completed the following **practice**:

- 3.3.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

3.4 THE CORRELATION COEFFICIENT

3.4.0 Lesson Overview

In this lesson, you will find and interpret the **correlation coefficient** and use it to understand the strength of a linear relationship.

When you finish this lesson, you will be able to:

- Describe the “goodness of fit” of a linear model using the correlation coefficient.
- Match the correlation coefficient with a scatter plot and linear model.

Here are the **activities** that will help you reach those goals:

- 3.4.1: Compare Scatter Plots with Linear and Nonlinear Trends
- 3.4.2: Scatter Plot Differences and Connections to Correlation Coefficient
  - 3.4.2: Self Check
  - 3.4.2: Additional Resources
- 3.4.3 Finding the Correlation Coefficient with Technology
  - 3.4.3: Self Check
  - 3.4.3: Additional Resources
- 3.4.4: Matching Correlation Coefficients
  - 3.4.4: Self Check
  - 3.4.4: Additional Resources
- 3.4.5: Using Correlation Coefficient to Determine Best Fit

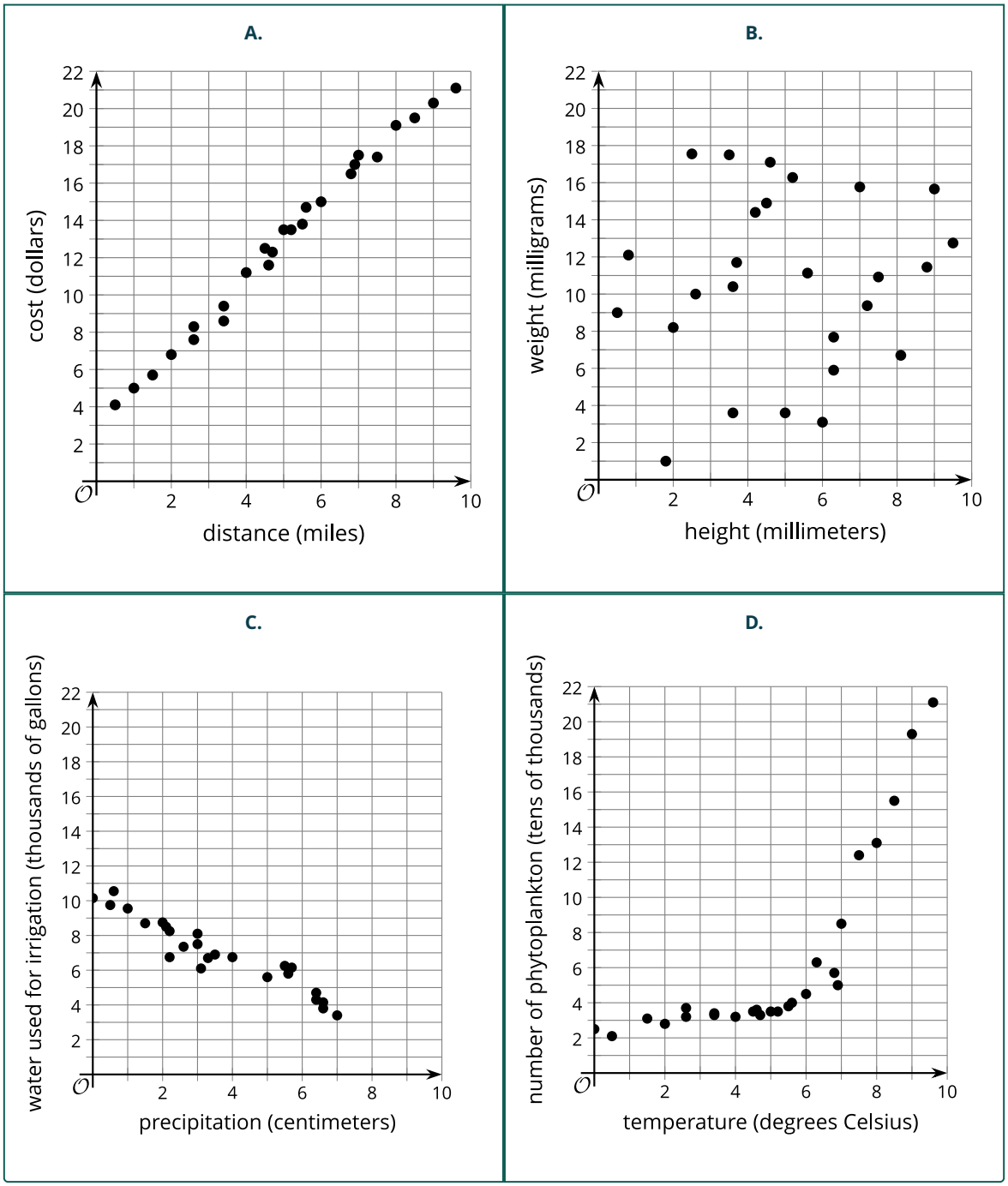
After that, you’ll **practice and review**.

- 3.4.6: Practice
- 3.4.7: Lesson Summary

3.4.1 Comparing Scatter Plots with Linear and Nonlinear Trends

Warm Up

Which one doesn’t belong?



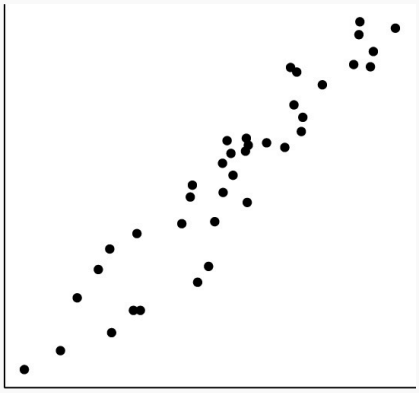
3.4.2 Recognizing Differences in Scatter Plots

Activity

Your teacher will give you a set of cards (https://openstax.org/r/set-cards-3-4-2) that show scatter plots of data. Sort the cards into 2 categories of your choosing. Be prepared to explain the meaning of your categories. Then, sort the cards into 2 categories in a different way. Be prepared to explain the meaning of your new categories.

Self Check

Which correlation coefficient best matches the scatter plot below?

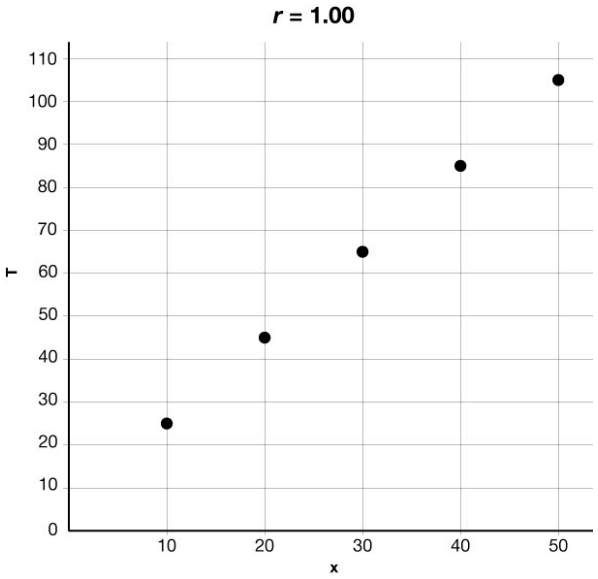


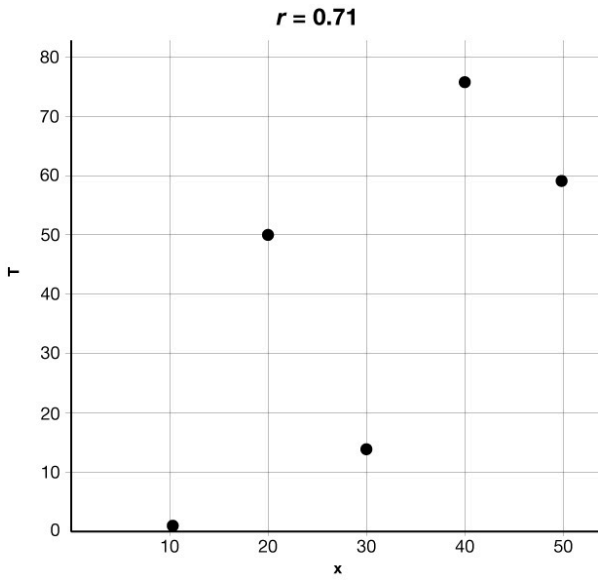
a. $r = -0.39$
b. $r = 0.26$
c. $r = -0.89$
d. $r = 0.95$

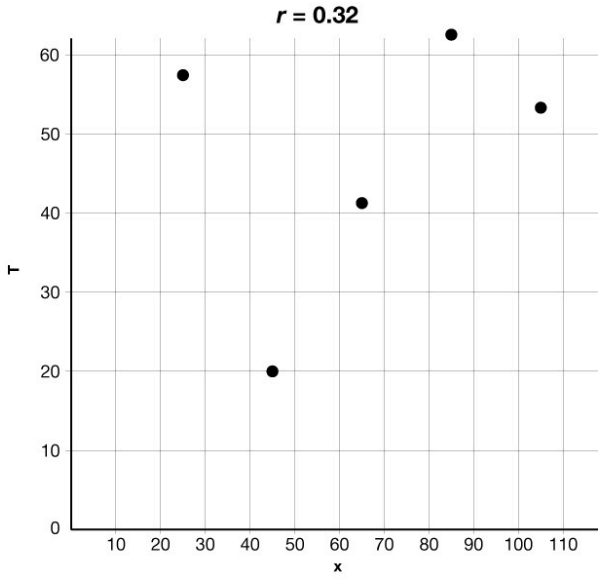
Additional Resources

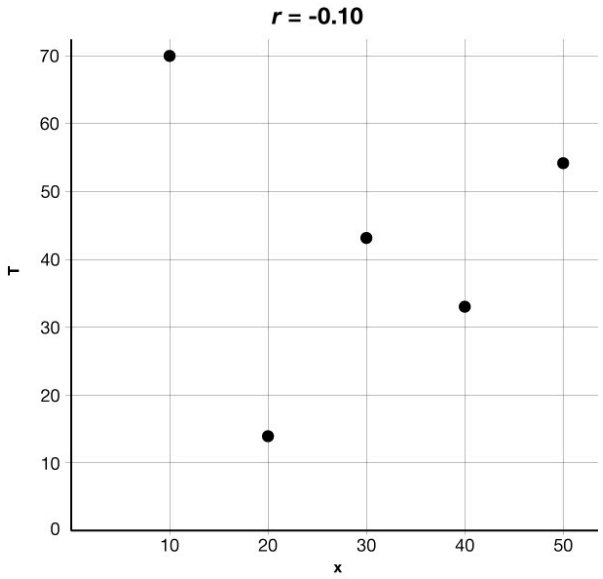
Correlation Coefficient

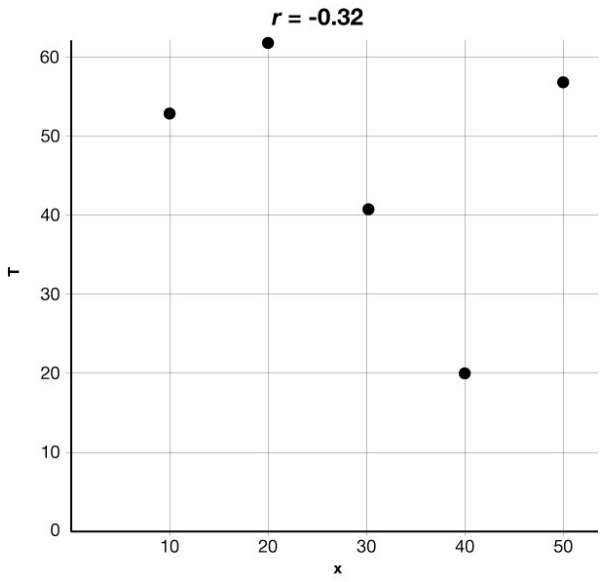
The correlation coefficient is a number between -1 and $+1$ (including -1 and $+1$) that measures the strength and direction of a linear relationship. The correlation coefficient is denoted by the letter r. Several scatter plots are shown below. The value of the correlation coefficient for the data displayed in each plot is also given.

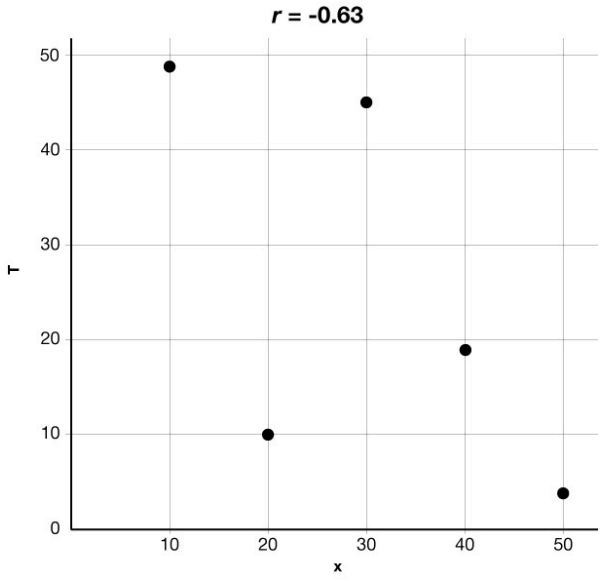


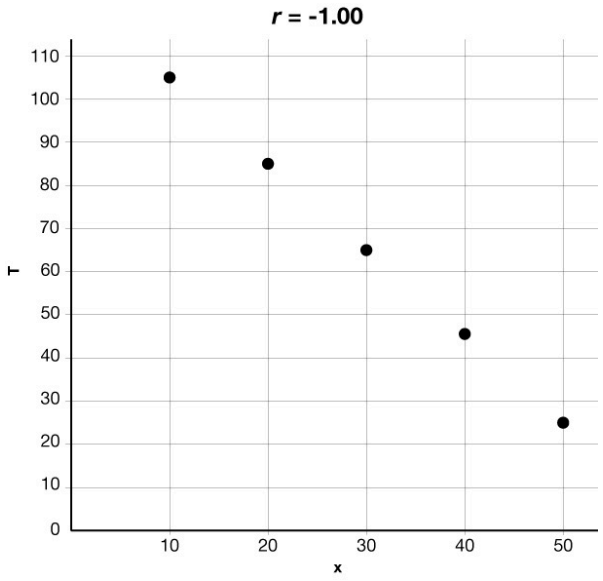

r = 0.71
80
70
60
50
40
30
20
10
0
y
10
20
30
40
50
x


r = 0.32
60
50
40
30
20
10
0
y
10
20
30
40
50
60
70
80
90
100
110
x


r = -0.10
70
60
50
40
30
20
10
0
y
10
20
30
40
50
x


r = -0.32
60
50
40
30
20
10
0
T
10
20
30
40
50
x


r = -0.63
50
40
30
20
10
0
T
10
20
30
40
50
x


r = -1.00
110
100
90
80
70
60
50
40
30
20
10
0
T
10
20
30
40
50
x

When is the value of the correlation coefficient positive?

- The correlation coefficient is positive when as the x-values increase, the y-values also tend to increase.

When is the value of the correlation coefficient negative?

- The correlation coefficient is negative when as the x-values increase, the y-values tend to decrease.

Is the linear relationship stronger when the correlation coefficient is closer to 0 or to 1 (or -1)?

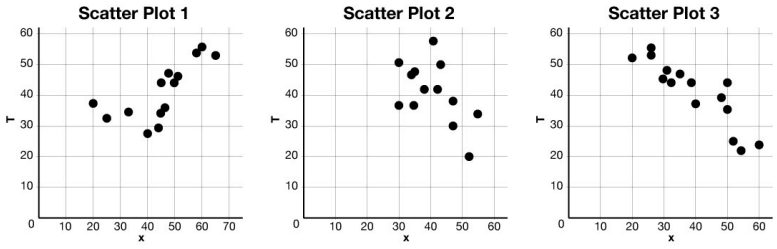
- As the points form a stronger negative or positive linear relationship, the correlation coefficient gets farther from 0. Students note that when all of the points are on a line with a positive slope, the correlation coefficient is $+1$. The correlation coefficient is -1 if all of the points are on a line with a negative slope.

The table below shows how you can informally interpret the value of a correlation coefficient.

If the value of the correlation coefficient is . . .	You can say that . . .
$r = 1.0$	There is a perfect positive linear relationship.
$0.7 \leq r < 1.0$	There is a strong positive linear relationship.
$0.3 \leq r < 0.7$	There is a moderate positive linear relationship.
$0 < r < 0.3$	There is a weak positive linear relationship.
$r = 0$	There is no linear relationship.
$-0.3 < r < 0$	There is a weak negative linear relationship.
$-0.7 < r \leq -0.3$	There is a moderate negative linear relationship.
$-1.0 < r \leq -0.7$	There is a strong negative linear relationship.
$r = -1.0$	There is a perfect negative linear relationship.

TRY IT **Correlation Coefficient**

Which of the three scatter plots below shows the strongest linear relationship? Which shows the weakest linear relationship? Explain your answers.



3.4.3 Finding the Correlation Coefficient with Technology

Activity

1. Click on the "+" sign in the Desmos graphing tool. Choose table from the drop down menu. This will generate a table for entering a data set.

Access multimedia content (http://openstax.org/books/algebra-1/pages/3-4-3-finding-the-correlation-coefficient-with-

technology)

2. Enter the data set shown in the table below into the corresponding x_1 and y_1 columns.

(If preferred, the data can be copied and pasted into Desmos from a spreadsheet using keyboard shortcuts. Windows: Ctrl+C for copy and Ctrl+V for paste. Mac: Command+C for copy and Command+V for paste.)

x_1	y_1
2	12
9	11
15	6
21	3
22	-1

Hint: The window should be appropriately set to display the points in the table, but sometimes it will be necessary to adjust the window of the graph in order to properly view your data set. Use the mouse to drag and change the viewing window. To zoom in or out, use the buttons in the top right or the scrolling wheel on the mouse.

3. In the Desmos window, click in the entry box #2 below the table from Question 1 and enter the command $corr(x_1, y_1)$. (You can just type x_1 and y_1. The subscripts will appear automatically.)

4. A number between -1 and 1 will appear after an equal sign in the bottom right-hand corner of the same box. What is the number rounded to the thousandths place?

5. What does this number tell you about the strength of the linear relationship?

Self Check

Use the Desmos graphing tool below to find the correlation coefficient for this data.

x	8	15	26	31	56
y	23	41	53	72	103

Access multimedia content (http://openstax.org/books/algebra-1/pages/3-4-3-finding-the-correlation-coefficient-with-technology)

Additional Resources

Finding The Correlation Coefficient Using Technology

Let's find the correlation coefficient for a given data set.

Step 1 - Click on the "+" sign in the Desmos graphing tool. Choose table from the drop down menu. This will generate a table for entering a data set.

Access multimedia content (http://openstax.org/books/algebra-1/pages/3-4-3-finding-the-correlation-coefficient-with-technology)

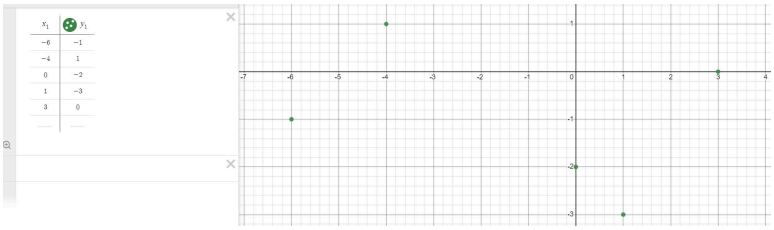
Step 2 - Enter the data into the corresponding x_1 and y_1 columns in the table you created in Step 1. Press Enter or use arrow keys to enter a new data value in the table.

(If preferred, the data can be copied and pasted into Desmos from a spreadsheet using keyboard shortcuts. Windows: Ctrl+C for copy and Ctrl+V for paste. Mac: Command+C for copy and Command+V for paste.)

x_1	y_1
-6	-1
-4	1
0	-2
1	-3
3	0

Hint: The window should be appropriately set to display the points in the table, but sometimes it will be necessary to adjust the window of the graph in order to properly view your data set. Use the mouse to drag and change the viewing window. To zoom in or out, use the buttons in the top right or the scrolling wheel on the mouse.

You should have created the following table and scatter plot:



Step 3 - In the Desmos window, click in the entry box #2 below the table from Question 1 and enter the command $corr(x_1, y_1)$. (You can just type x_1 and y_1. The subscripts will appear automatically.)

As soon as you enter the correlation command, Desmos will display the correlation coefficient after an equal sign in the bottom right corner of the same box.



Step 4 - Round to the nearest thousandth. The correlation coefficient is -0.299.

Step 5 - Determine the strength of the linear relationship. This correlation coefficient shows a weak negative linear relationship.

TRY IT **Try It: Finding the Correlation Coefficient with Technology**

1. Find the correlation coefficient of the data set. Round the answer to the nearest thousandth.
   Access multimedia content (http://openstax.org/books/algebra-1/pages/3-4-3-finding-the-correlation-coefficient-with-technology)

x_1	y_1
-1	4
0	9
3	14
5	-3
6	1

2. What is the strength of the linear relationship?

3.4.4 Matching Correlation Coefficients

Activity

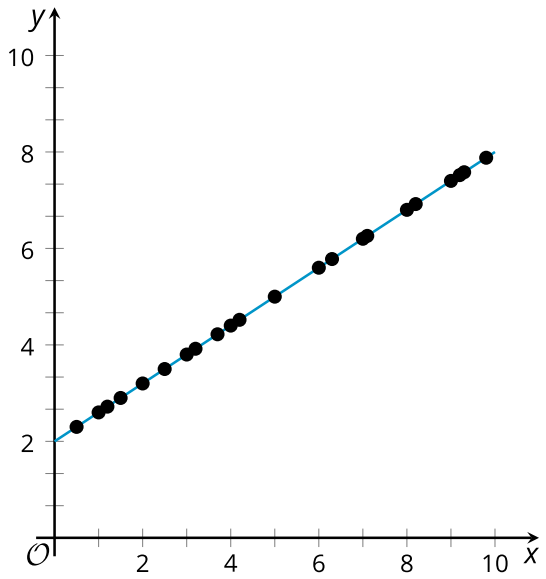
Thinking back to the last activity, you learned about the *r*-value. Write these characteristics about the *r*-value.

The *r*-value is called a correlation coefficient. A **correlation coefficient** is one way to measure the strength of a linear relationship.

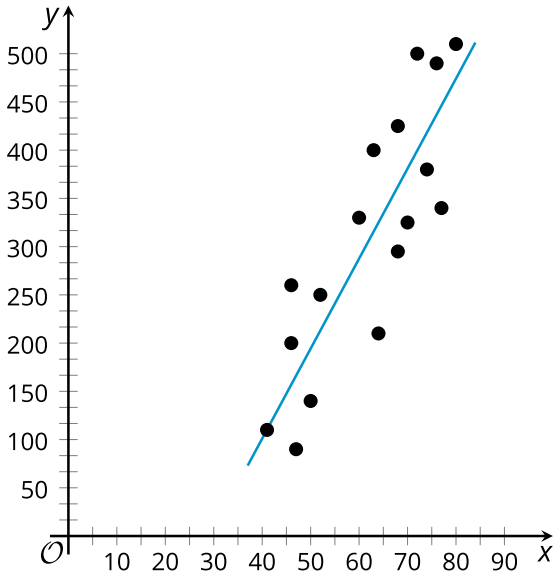
- The sign of *r* is the same as the sign of the slope of the best fit line.
- The values for *r* go from –1 to 1 and include –1 and 1.
- The closer *r* is to 1 or –1, the stronger the linear relationship between the variables.
- The closer *r* is to 0, the weaker the linear relationship between the variables.

Take turns with your partner to match a scatter plot with a correlation coefficient. For each match you find, explain to your partner how you know it's a match. For each match your partner finds, listen carefully to their explanation. If you disagree, discuss your thinking and work to reach an agreement.

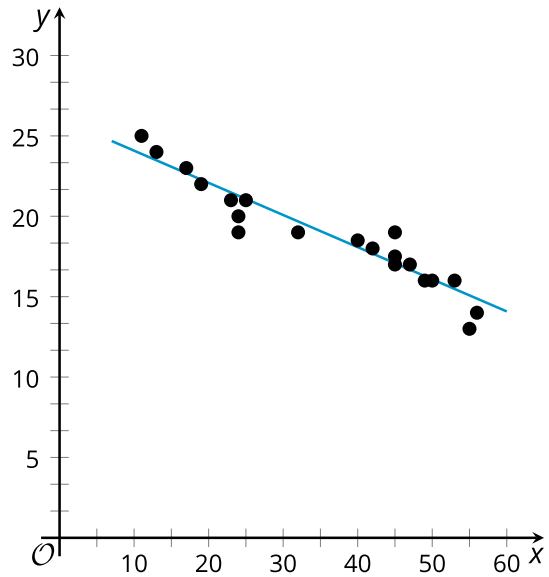
A


y
10
8
6
4
2
O
2
4
6
8
10
x

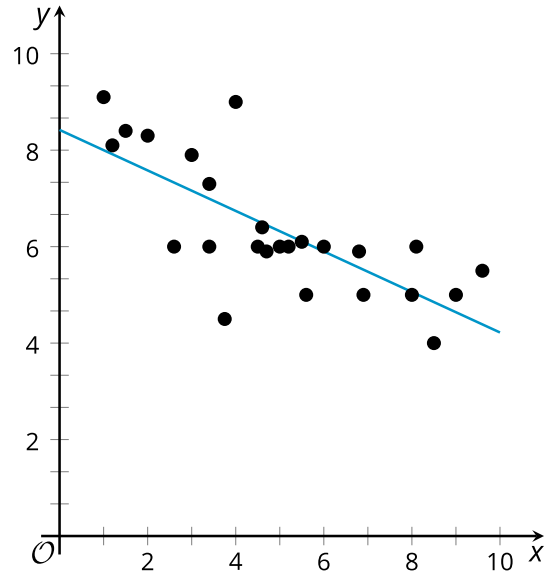
B


y
500
450
400
350
300
250
200
150
100
50
O
10
20
30
40
50
60
70
80
90
x

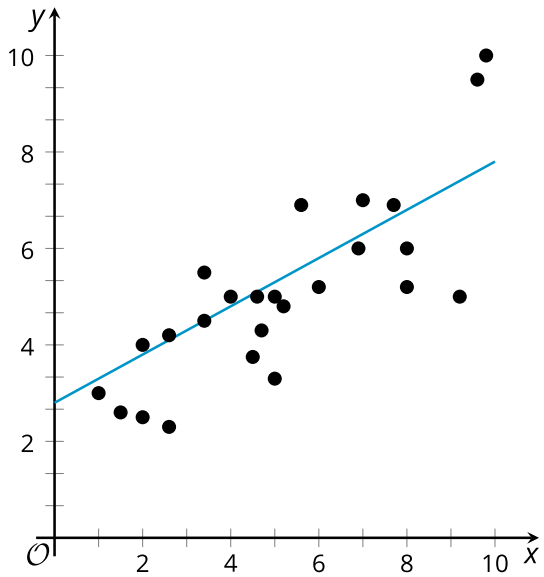
C



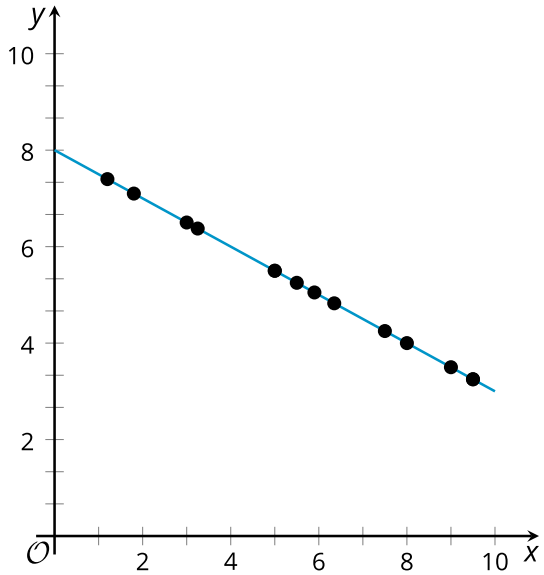
D



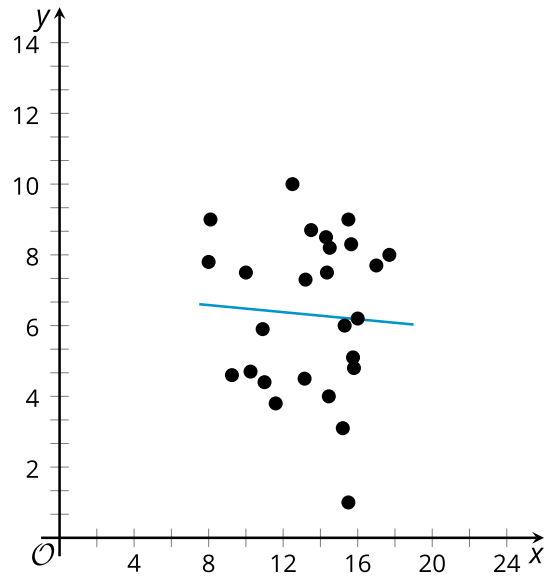
E



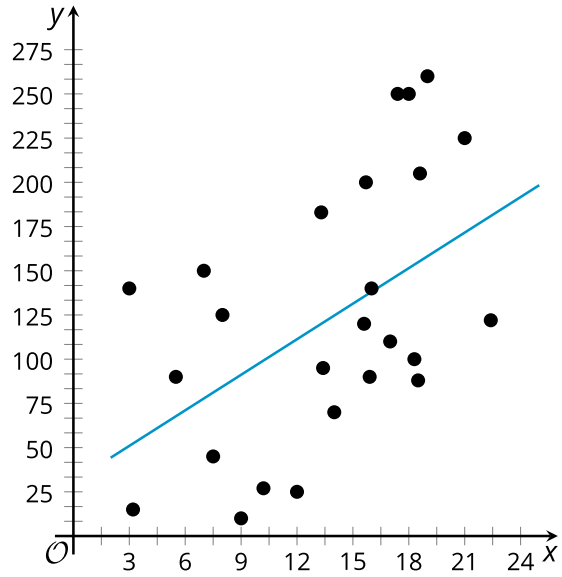
F



G



H

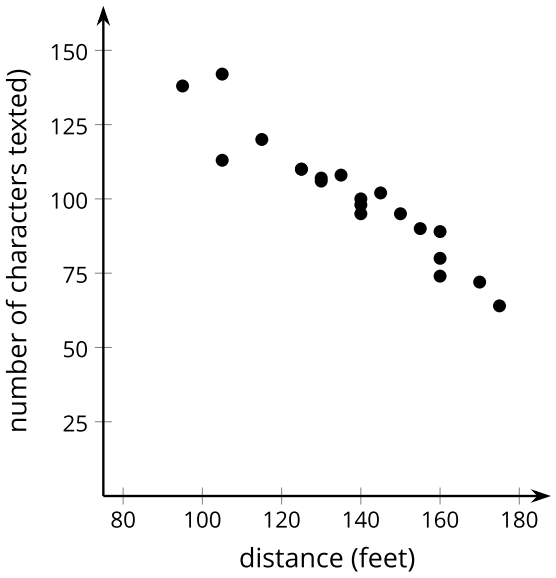


1. $r = -1$
2. $r = -0.95$
3. $r = -0.74$
4. $r = -0.06$
5. $r = 0.48$
6. $r = 0.65$
7. $r = 0.9$
8. $r = 1$

Are you ready for more?

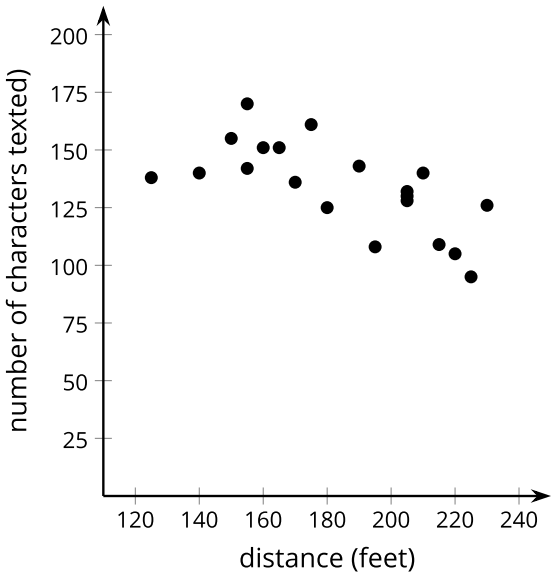
Extending Your Thinking

Jada wants to know if the speed that people walk is correlated with their texting speed. To investigate this, she measured the distance, in feet, that 5 of her friends walked in 30 seconds and the number of characters they texted during that time. Each of the 5 friends took 4 walks for a total of 20 walks. Here are the results of the first 20 walks.



Distance (feet)	Number of characters texted	Distance (feet)	Number of characters texted
105	14	95	138
125	110	125	110
115	120	160	80
140	98	175	64
145	102	130	106
160	89	140	95
170	72	150	95
140	100	155	90
130	107	160	74
105	113	135	108

Over the next few days, the same 5 friends practiced walking and texting to see if they could walk faster and text more characters. They did not record any more data while practicing. After practicing, each of the 5 friends took another 4 walks. Here are the results of the final 20 walks.



Distance (feet)	Number of characters texted	Distance (feet)	Number of characters texted
140	410	165	151
150	155	170	136
160	151	190	143
155	170	205	132
180	125	205	128
205	130	210	140
225	95	215	109
175	161	220	105
195	108	230	126
155	142	225	138

1. What do you notice about the 2 scatter plots?

2. Jada noticed that her friends walked farther and texted faster during the last 20 walks than they did during the first 20 walks. Since both were faster, she predicts that the correlation coefficient of the line of best fit for the last 20 walks will be closer to –1 than the correlation coefficient of the line of best fit for the first 20 walks. Do you agree with Jada? Be prepared to show your reasoning.

3. Use technology to find an equation of the line of best fit for each data set.
4. The correlation coefficient for the first line of best fit is -0.95 and the second line of best fit is -0.68. Was your answer to question 2 correct? Why do you think the correlation coefficients for the two data sets are so different? Be prepared to show your reasoning.

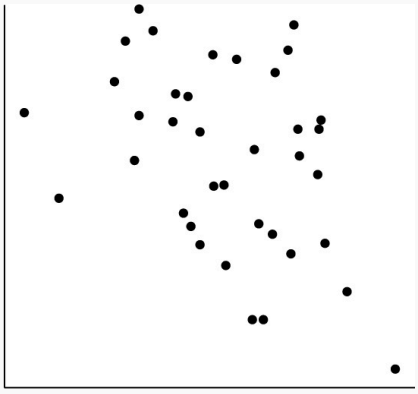
Video: Matching Correlation Coefficients

Watch the following video to learn more about the connections of scatter plots to the correlation coefficient.

Measuring Strength of Fit with Correlation Coefficients (https://openstax.org/r/correlation-coefficients)

Self Check

Which correlation coefficient best matches the scatter plot below?



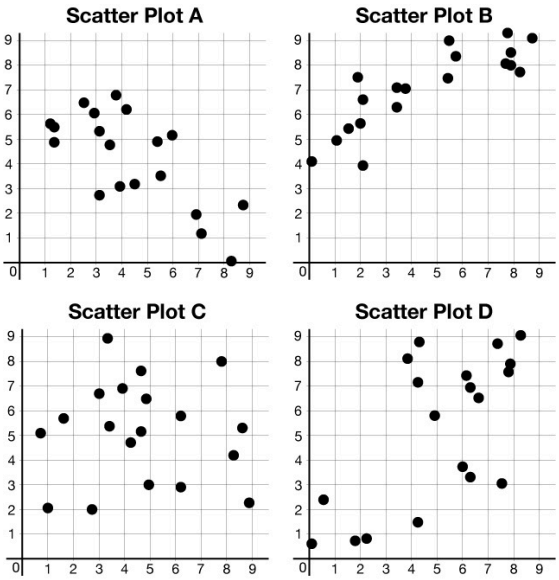
a. $r = -0.39$
b. $r = 0.26$
c. $r = -0.89$
d. $r = 0.95$

Additional Resources

Correlation Coefficient

Try matching each scatter plot to one of the following correlation coefficients:

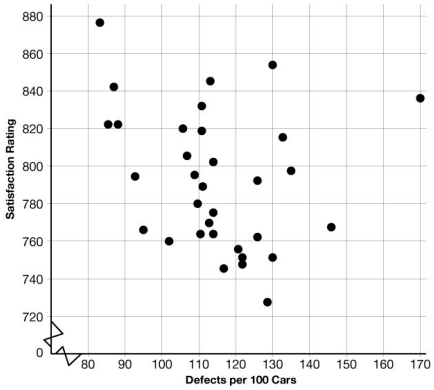
$r = -0.02$, $r = 0.84$, $r = -0.72$, $r = 0.65$



Scatter plot	Correlation coefficient
Scatter plot A	$r = -0.72$
Scatter plot B	$r = 0.84$
Scatter plot C	$r = -0.02$
Scatter plot D	$r = 0.65$

TRY IT **Correlation Coefficient**

The scatter plot below displays data on the number of defects per 100 cars and a measure of customer satisfaction (on a scale from 1 to 1,000, with higher scores indicating greater satisfaction) for the 33 brands of cars sold in the United States in 2009.



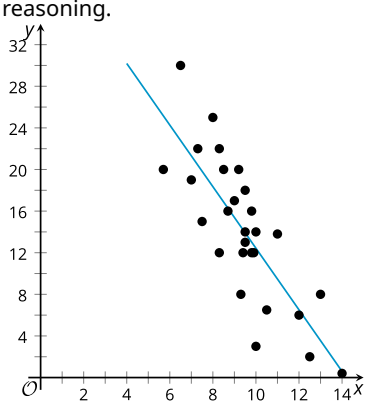
Which of the following is the value of the correlation coefficient for this data set: $r = -0.95$, $r = -0.24$, $r = 0.83$, or $r = 1.00$? Explain why you selected this value.

3.4.5 Using Correlation Coefficient to Determine Best Fit

Cool Down

1. What information does a correlation coefficient tell you about the data in a scatter plot?

2. Which value best estimates the value for the correlation coefficient of the scatter plot: –1, –0.8, –0.2, 0.2, 0.8, or 1? Be prepared to show your reasoning.



3.4.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Select **three** values for r that indicate a positive slope for the line of best fit.

a. 1
b. -1
c. 0.5
d. -0.5
e. 0
f. 0.8
g. -0.8

2. Select **three** values for r that indicate a negative slope for the line of best fit.

a. 0
b. 1
c. -1
d. -0.87
e. 0.32
f. -0.24
g. 0.24

3. The correlation coefficient, r, is given for several different linear models. Order the correlation coefficients as follows: start with the r-value indicating the strongest linear relationship and end with the r-value indicating the weakest linear relationship.

$0.4, -0.85, 0.72, -0.31$

a. $-0.85, 0.72, 0.4, -0.31$
b. $-0.31, 0.4, 0.72, -0.85$
c. $-0.85, -0.31, 0.4, 0.72$
d. $0.72, 0.4, -0.31, -0.85$

4. The correlation coefficient, r, is given for several different linear models. Order the correlation coefficients as follows: start with the r-value indicating the strongest linear relationship and end with the r-value indicating the weakest linear relationship

$-0.65, 0.47, -1, 0.89$

a. $0.89, 0.47, -1, -0.65$
b. $-1, -0.65, 0.47, 0.89$
c. $0.47, -0.65, 0.89, -1$
d. $-1, 0.89, -0.65, 0.47$

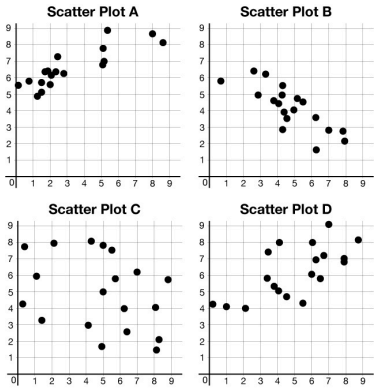
5. Match the following scatter plots to the correct correlation coefficient from the following:

$r = 0.67$

$r = -0.33$

$r = 0.88$

$r = -0.79$



Scatter Plot A

a. r=0.67
b. r=-0.33
c. r=0.88
d. r=-0.79

Scatter Plot B

a. r=0.67
b. r=-0.33
c. r=0.88
d. r=-0.79

Scatter Plot C

a. r=0.67
b. r=-0.33
c. r=0.88
d. r= -0.79

Scatter Plot D

a. r=0.67
b. r=-0.33
c. r=0.88
d. r= -0.79

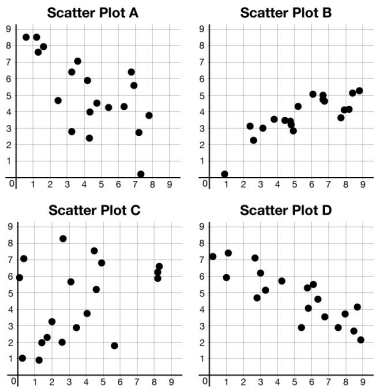
6. Match the following scatter plots to the correct correlation coefficient from the following:

$r = 0.37$

$r = 0.83$

$r = -0.67$

$r = -0.85$



Scatter Plot A

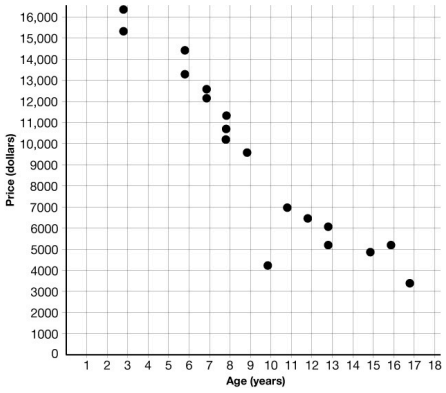
a. r=0.37
b. r=0.83
c. r=-0.67
d. r=-0.85

Scatter Plot B
a. r=0.37
b. r=0.83
c. r=-0.67
d. r=-0.85

Scatter Plot C
a. r=0.37
b. r=0.83
c. r=-0.67
d. r=-0.85

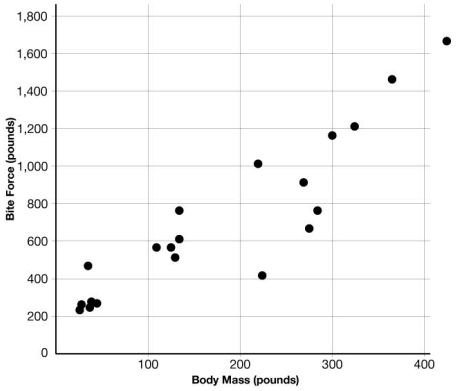
Scatter Plot D
a. r=0.37
b. r=0.83
c. r=-0.67
d. r=-0.85

7. Which of the following correlation coefficients would best describe this scatter plot?



a. 0.32
b. -0.87
c. 0.87
d. -0.32

8. Which of the following correlation coefficients would best describe this scatter plot?



a. 0.62

b. -0.94
c. 0.94
d. -0.62

9. Which of the following has the stronger linear relationship? Explain.

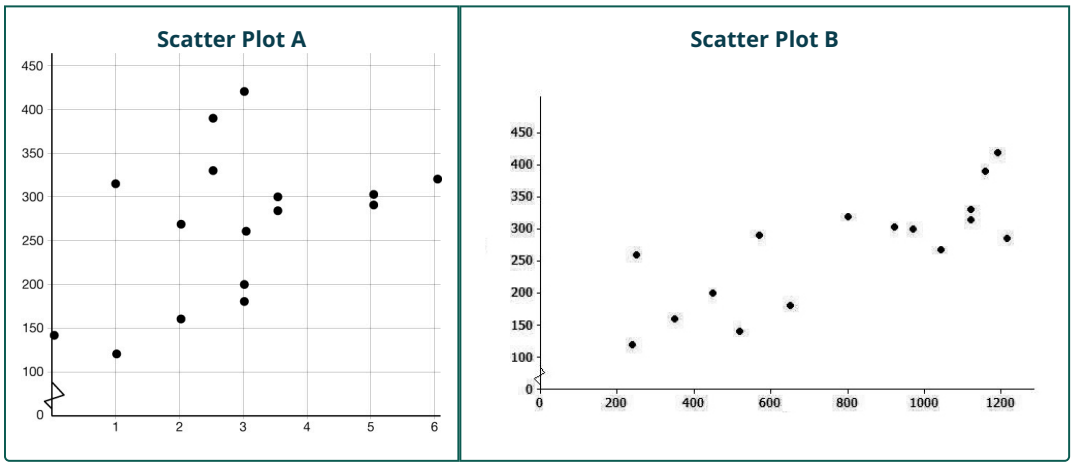
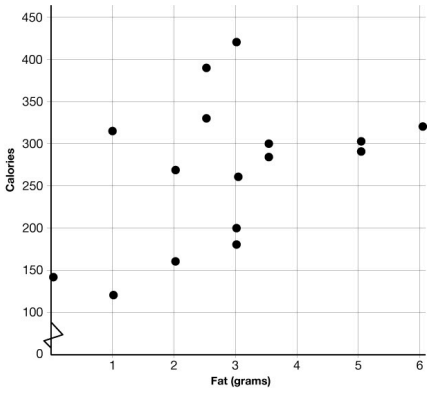


Table 3.4.0

a. Scatter plot B
b. Scatter plot A

10. Which of the following correlation coefficients would best describe the scatter plot?



a. -0.43
b. 0.43
c. 0.89
d. -0.89

3.4.7 Lesson Summary

In this lesson, you learned how to:

- Describe the “goodness of fit” of a linear model using the correlation coefficient.
- Match the correlation coefficient with a scatter plot and linear model.

Here are the **activities** that helped you reach those goals:

- 3.4.1: Comparing Scatter Plots with Linear and Nonlinear Trends
  - In this activity, you compared four scatter plots displaying data with linear and nonlinear trends. During this activity, you had an opportunity to use language precisely and to hear how you use terminology and talk about characteristics of items in comparison to one another.
- 3.4.2: Recognizing Differences in Scatter Plots
  - In this activity, you were given cards displaying scatter plots of data that could be fit by linear models with varying accuracy. As you worked, you began to recognize differences in scatter plots, lines of best fit, and r -values, and the connection to the correlation coefficient.
  - 3.4.2: Self Check
  - 3.4.2: Additional Resources
- 3.4.3: Finding the Correlation Coefficient with Technology
  - In this activity, you used technology to find the correlation coefficient.
  - 3.4.3: Self Check
  - 3.4.3: Additional Resources
- 3.4.4: Matching Correlation Coefficients
  - In this activity, you gained a better understanding of correlation coefficients by taking turns with a partner to match scatter plots and correlation coefficients. You had a chance to explain your thinking and critique the reasoning of others.
  - 3.4.4: Self Check
  - 3.4.4: Additional Resources
- 3.4.5: Using Correlation Coefficient to Determine Best Fit
  - In this activity, you determined the information a correlation coefficient tells you about the data in a scatter plot.

After these activities, you completed the following **practice**:

- 3.4.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

3.5 USING THE CORRELATION COEFFICIENT

3.5.0 Lesson Overview

In this lesson, you will compute and interpret the correlation coefficient for a bivariate, numerical data set.

When you finish this lesson, you will be able to:

- Describe the strength of a relationship between two variables.
- Use technology to find the correlation coefficient and explain what the value tells you about a linear model in everyday language.

Here are the **activities** that will help you reach those goals:

- 3.5.1: Using Bivariate Data in Context
- 3.5.2: Interpreting the Strength of Linear Relationships
  - 3.5.2: Self Check
  - 3.5.2: Additional Resources
- 3.5.3: Describing Relationships between Two Variables
  - 3.5.3: Self Check
  - 3.5.3: Additional Resources
- 3.5.4: Interpreting the Correlation Coefficient

After that, you'll **practice and review**.

- 3.5.5: Practice
- 3.5.6: Lesson Summary

3.5.1 Using Bivariate Data in Context

Warm Up

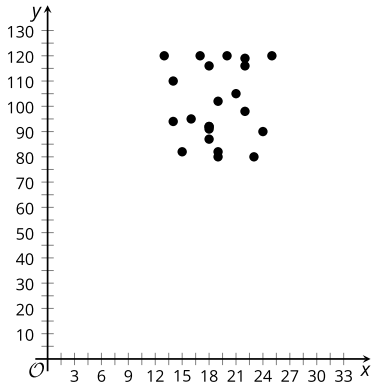
In the following activity, you will complete a series of questions regarding scatter plots and the variables that represent the data.

CLASSROOM CONNECTION Log into student.desmos.com using the information provided by your teacher to complete the activity.

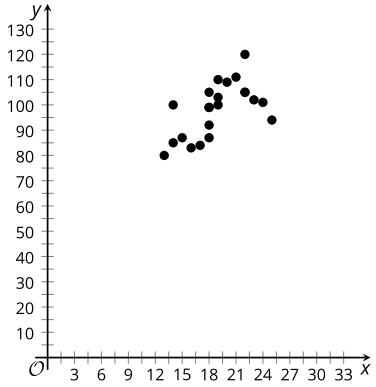
Examine the unlabeled graphs below, then match the variables to the scatter plot you think they best fit. Choose from among the following variable combinations. Be prepared to explain your reasoning.

x variable = average number of free throws shot in a season y variable = basketball team score per game
x variable = daily low temperature in Celsius for Denver, CO y variable = boxes of cereal in stock at a grocery in Miami, FL
x variable = measured student height in feet y variable = measured student height in inches
x variable = average number of minutes spent in a waiting room y variable = hospital satisfaction rating

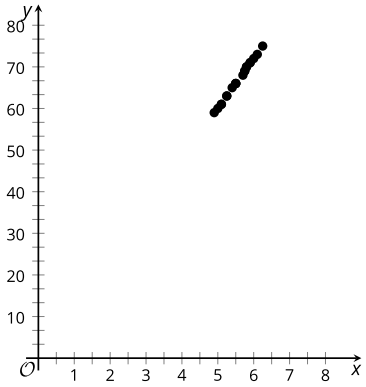
1. Which variables best describe the relationship represented in the graph below?



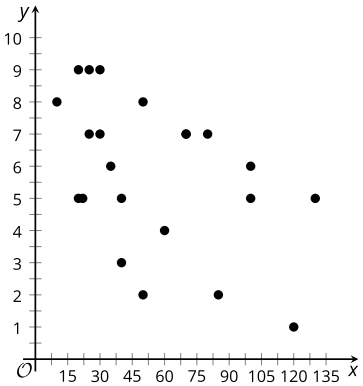
2. Which variables best describe the relationship represented in the graph below?



3. Which variables best describe the relationship represented in the graph below?



4. Which variables best describe the relationship represented in the graph below?



3.5.2 Finding and Using Correlation Coefficient to Interpret the Strength of Linear Relationships

Activity

Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

Priya takes note of the distance the car drives and the time it takes to get to the destination for many trips.

Distance (mi) (x)	Travel time (min) (y)
2	4
5	7
10	11
10	15
12	16
15	22
20	23
25	25
26	28
30	36
32	35
40	37

Distance (mi) (x)	Travel time (min) (y)
50	51
65	70
78	72

1. Distance is one factor that influences the travel time of Priya's car trips. What are some other factors?
2. Which of these factors (including distance) most likely has the most consistent influence for all the car trips? Be prepared to show your reasoning.
3. Use the graphing tool or technology outside the course. Create the scatter plot and calculate the best fit line using the Desmos tool below. Access multimedia content (http://openstax.org/books/algebra-1/pages/3-5-2-finding-and-using-correlation-coefficient-to-interpret-the-strength-of-linear-relationships)
4. What do the slope and y-intercept for the line of best fit mean in this situation?
5. Use technology to find the correlation coefficient for this data. Based on the value, how would you describe the strength of the linear relationship?
6. How long do you think it would take Priya to make a trip of 90 miles if the linear relationship continues?

Self Check

Determine the correlation coefficient of the data in the table below to 3 decimal places and interpret the value.

x	8	15	26	31	56
y	23	41	53	72	103

a. $r = -0.987$; there is a strong negative relationship.

b. $r = 0.987$; there is a strong positive relationship.

c. $r = 0.987$; there is a weak positive relationship.

d. $r = -0.987$; there is a weak negative relationship.

Additional Resources

Calculating and Interpreting Correlation Coefficients

Properties of the correlation coefficient:

- Property 1: The sign of r (positive or negative) corresponds to the direction of the linear relationship.
- Property 2: A value of $r = +1$ indicates a perfect positive linear relationship, with all points in the scatter plot falling exactly on a straight line.
- Property 3: A value of $r = -1$ indicates a perfect negative linear relationship, with all points in the scatter plot falling exactly on a straight line.
- Property 4: The closer the value of r is to $+1$ or -1, the stronger the linear relationship.

Using the data of shoe length in inches and height in inches for 10 men, find the correlation coefficient of the data with graphing technology.

x (Shoe length) inches	y (Height) inches
12.6	74
11.8	65
12.2	71
11.6	67
12.2	69
11.4	68
12.8	70
12.2	69
12.6	72
11.8	71

The correlation coefficient is $r \approx 0.65$.

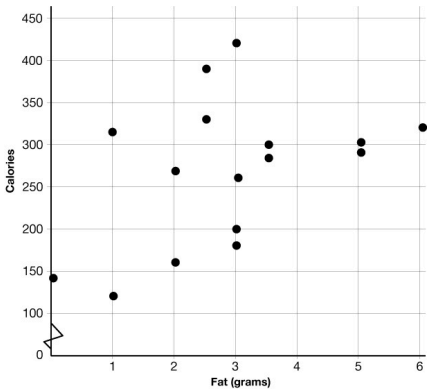
Based on the table, there is a moderate positive linear relationship between shoe length and height.

The table below shows how you can informally interpret the value of a correlation coefficient

If the value of the correlation coefficient is . . .	You can say that . . .
$r = 1.0$	There is a perfect positive linear relationship.
$0.7 \leq r < 1.0$	There is a strong positive linear relationship.
$0.3 \leq r < 0.7$	There is a moderate positive linear relationship.
$0 < r < 0.3$	There is a weak positive linear relationship.
$r = 0$	There is no linear relationship.
$-0.3 < r < 0$	There is a weak negative linear relationship.
$-0.7 < r \leq -0.3$	There is a moderate negative linear relationship.
$-1.0 < r \leq -0.7$	There is a strong negative linear relationship.
$r = -1.0$	There is a perfect negative linear relationship.

TRY IT **Calculating and Interpreting Correlation Coefficients**

Consumer Reports published a study of fast-food items. The table and scatter plot below display the fat content (in grams) and number of calories per serving for 16 fast-food items.



Fat(g)	Calories(kcal)
2	268
5	3,003
3	260
3.5	300
1	315
2	160
3	200
6	320
3	420
5	290
3.5	285
2.5	390
0	140
2.5	330
1	120
3	180

What is the correlation coefficient for this data and what does it tell us about the strength and type of relationship represented?

3.5.3 Using Correlation Coefficient to Describe Relationships between Two Variables

Activity

For each situation, describe the linear relationship between the variables, based on the correlation coefficient. Make sure to mention whether there is a **strong relationship** or not as well as whether it is a **positive relationship** or **negative relationship**.

1. Number of steps taken per day and number of kilometers walked per day. $r = 0.92$
2. Temperature of a rubber band and distance the rubber band can stretch. $r = 0.84$
3. Car weight and distance traveled using a full tank of gas. $r = -0.86$
4. Average fat intake per citizen of a country and average cancer rate of a country. $r = 0.73$
5. Score on science exam and number of words written on the essay question. $r = 0.28$
6. Average time spent listening to music per day and average time spent watching TV per day. $r = -0.17$

Are you ready for more?

Extending Your Thinking

A biologist is trying to determine if a group of dolphins is a new species of dolphin or if it is a new group of individuals within the same species of dolphin. The biologist measures the width (in millimeters) of the largest part of the skull, zygomatic width, and the length (in millimeters) of the snout, rostral length, of 10 dolphins from the same group of individuals.

x, Rostral length (mm)	y, Zygomatic width (mm)
288	147
247	147
268	171
278	177
258	168
272	184
272	161
258	159
273	168
277	166

The data appears to be linear and the equation of the line of best fit is $y = 0.201x + 110.806$ and the r-value is 0.201.

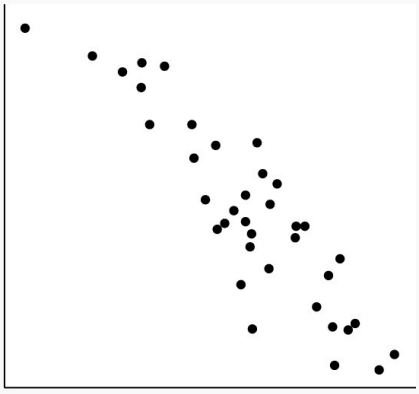
1. After checking the data, the biologist realizes that the first zygomatic width listed as 147 mm is an error. It is supposed to be 180 mm. Use technology to find the equation of a line of best fit and the correlation coefficient for the corrected data. What is the equation of the line of best fit and the correlation coefficient?
2. Compare the new equation of the line of best fit with the original. What impact did changing one data point have on the slope, y-intercept, and correlation coefficient on the line of best fit?
3. Why do you think that weak positive association became a moderately strong association? Be prepared to show your reasoning.

For questions 4—6, use technology to change the y-value for the first and second entries in the table.

4. How does changing each point's y-value impact the correlation coefficient?
5. Can you change two values to get the correlation coefficient closer to 1? Use data to support your answer.
6. By leaving (288,180), can you change a value to get the relationship to change from a positive one to a negative one? Use data to support or refute your answer.

Self Check

In the graph below, which **best** describes the scatter plot of points?



a. $r = 0.24$, positive correlation
b. $r = -0.24$, weak, negative correlation
c. $r = -0.90$, strong, negative correlation
d. $r = 0.90$, positive, weak, correlation

Additional Resources

Determine the Strength of a Linear Relationship

Finding a Correlation Coefficient

1. Calculate the correlation coefficient for cricket-chirp data in the table below.
2. Tell if the linear relationship is positive or negative and if it is strong, weak, or moderate.

Chirps	44	35	20.4	33	31	35	18.5	37	26
Temperature	80.5	70.5	57	66	68	72	52	73.5	53

1. Using graphing technology, enter the chirps in the x-column of the table and the temperature in the y-column of the table. The correlation coefficient is $r = 0.9509$.
2. This value is very close to 1, which suggests a strong, positive, linear relationship.

TRY IT **Determine the Strength of a Linear Relationship**

Calculate the correlation coefficient for data in the table below. Tell if the linear relationship is positive or negative and if it is strong, weak, or moderate.

x	21	25	30	31	40	50
y	17	11	2	-1	-18	-40

3.5.4 Interpreting the Correlation Coefficient

Cool Down

Doctors suspect a strain of bacteria found in the hospital is becoming resistant to antibiotics. They put various concentrations of antibiotic in petri dishes and add some of the bacteria to allow it to grow. The bacteria grow into groups in the dish called colonies. After some time, the doctors return to the petri dishes and count the number of colonies for the different amounts of antibiotic.

The data is plotted with a best fit line. The correlation coefficient was $r = -0.83$.

1. What does the sign of the correlation coefficient tell you about the relationship between the number of bacteria colonies and the concentration of antibiotic in the dish?

2. What does the numerical value of the correlation coefficient tell you about the relationship between the number of bacteria colonies and the concentration of antibiotic in the dish?

3. In a follow-up study, a group of scientists collect data that was fit by a linear model with a correlation coefficient of $r = -0.94$. Which study suggests a stronger relationship between the number of bacteria colonies and the concentration of antibiotics: the doctors' study or the scientists' study? Be prepared to show your reasoning.

3.5.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. The number of hours worked, x, and the total dollars earned, y, have a strong positive relationship.

Explain what it means to have a strong positive relationship in this situation.

a. As the number of hours worked increases, the total dollars earned remains unchanged.
b. As the number of hours worked decreases, the total dollars earned increases.
c. As the number of hours worked increases, the total dollars earned increases.
d. As the number of hours worked increases, the total dollars earned decreases.

2. The number of minutes on the phone and the customer satisfaction rating have a weak negative relationship.

Explain what it means to have a weak negative relationship in this context.

a. As the number of minutes the customer waits on the phone increases, the customer satisfaction rating increases.
b. As the number of minutes the customer waits on the phone decreases, the customer satisfaction rating decreases.
c. As the number of minutes the customer waits on the phone decreases, the customer satisfaction rating remains unchanged.
d. As the number of minutes the customer waits on the phone increases, the customer satisfaction rating decreases.

For questions 3 – 6, use a graphing calculator to answer the questions.

3. Find the equation of the line of best fit. What is the value of the correlation coefficient?

x	y
5	2
6	4.6
7.5	7.2
8	8.4
8.3	8.2
9	9.1
10.2	10.3
11.4	9.9
11.4	11
12	12.5

a. $r \approx -0.04$
b. $r \approx 0.04$
c. $r \approx 0.96$
d. $r \approx -0.96$

For questions 4 – 6, use the following scenario.

The Collegiate Review publishes information about colleges and universities. The data below are for six public 4-year colleges in New York. Graduation rate is the percentage of students who graduate within six years. Student-to-faculty ratio is the number of students per full-time faculty member.

School	Number of Full-Time Students	Student-to-Faculty Ratio	Graduation Rate
CUNY Bernard M. Baruch College	11,477	17	63
CUNY Brooklyn College	9,876	15.3	48
CUNY City College	10,047	13.1	40
SUNY at Albany	14,013	19.5	64
SUNY at Binghamton	13,031	20	77
SUNY College at Buffalo	9,398	14.1	47

4. Use technology to calculate the value of the correlation coefficient between the number of full-time students and graduation rate. Let x represent the number of full-time students and y represent the graduation rate. Round to the nearest hundredth.

a. $r \approx 0.27$
b. $r \approx -0.83$
c. $r \approx 0.83$
d. $r \approx -0.27$

5. Is the linear relationship between graduation rate and number of full-time students weak, moderate, or strong?

a. weak
b. moderate
c. strong

6. Let x represent the student-to-faculty ratio and the y represent the graduation rate. Use technology to calculate the value of the correlation coefficient between the student-to-faculty ratio and the graduation rate. Round to the nearest hundredth.

a. $r \approx -0.05$
b. $r \approx 0.05$
c. $r \approx -0.95$
d. $r \approx 0.95$

3.5.6 Lesson Summary

In this lesson, you learned how to:

- Describe the strength of a relationship between two variables.
- Use technology to find the correlation coefficient and explain what the value tells me about a linear model in everyday language.

Here are the **activities** that helped you reach those goals:

- 3.5.1: Using Bivariate Data in Context
  - In this activity, you matched bivariate data with its context. You thought about whether there would be a strong correlation or not as well as whether the relationship was a positive or negative correlation.
- 3.5.2: Finding and Using Correlation Coefficient to Interpret the Strength of Linear Relationships
  - In this activity, you used technology to find a correlation coefficient given data and used it to interpret the strength of the linear relationship in context. In addition, you interpreted the coefficients of the equation of the line of best fit and used the equation to make predictions.
  - 3.5.2: Self Check
  - 3.5.2: Additional Resources
- 3.5.3: Using Correlation Coefficient to Describe Relationships between Two Variables
  - In this activity, you used the correlation coefficient to describe the relationship between two variables. You examined a pair of variables and a correlation coefficient to describe the relationship between the variables as strong or weak and as positive or negative.
  - 3.5.3: Self Check
  - 3.5.3: Additional Resources
- 3.5.4: Interpreting the Correlation Coefficient
  - In this activity, you described what the sign and numerical value of the correlation coefficient tell you about the strength of the linear relationship.

After these activities, you completed the following **practice**:

- 3.5.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

3.6 CAUSAL RELATIONSHIPS

3.6.0 Lesson Overview

In this lesson, you will compute and interpret the correlation coefficient for a bivariate, numerical data set.

When you finish this lesson, you will be able to:

- Describe the strength of a relationship between two variables.
- Analyze connections between two variables to determine whether or not there is a causal relationship.

Here are the **activities** that will help you reach those goals:

- 3.6.1: Describing the Relationship between Variables
- 3.6.2: Describing How Two Variables Are Related
  - 3.6.2: Self Check
  - 3.6.2: Additional Resources
- 3.6.3: Using the Term Causal Relationship
  - 3.6.3: Self Check
  - 3.6.3: Additional Resources
- 3.6.4: Determining the Type of Relationship

After that, you'll **practice and review**.

- 3.6.5: Practice
- 3.6.6: Lesson Summary

3.6.1 Describe the Relationship between Variables

Warm Up

Describe the strength and sign of the relationship you expect for each pair of variables. Be prepared to show your reasoning.

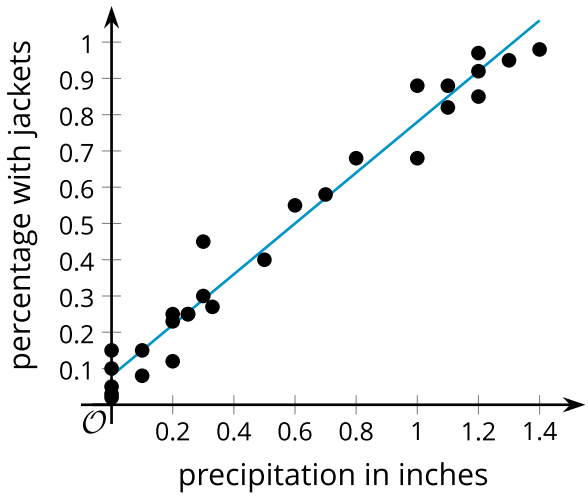
1. Used car price and original sale price of the car
2. Used car price and number of cup holders in the car
3. Used car price and number of oil changes the car has had
4. Used car price and number of miles the car has been driven

3.6.2 Describing How Two Variables Are Related

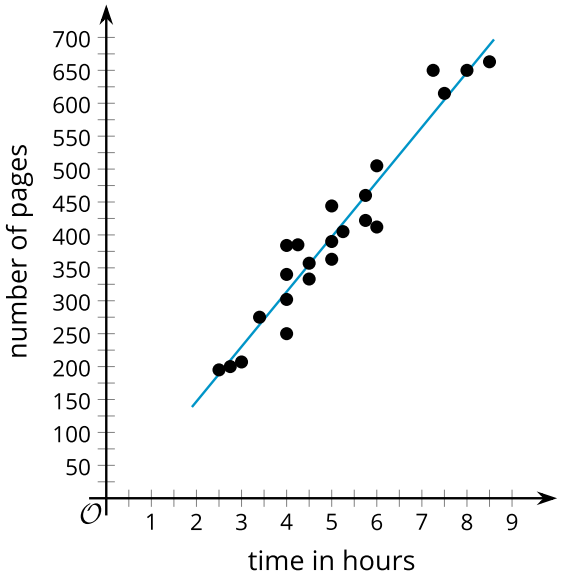
Activity

Each of the scatter plots shows a strong relationship. Write a sentence or two describing how you think the variables are related.

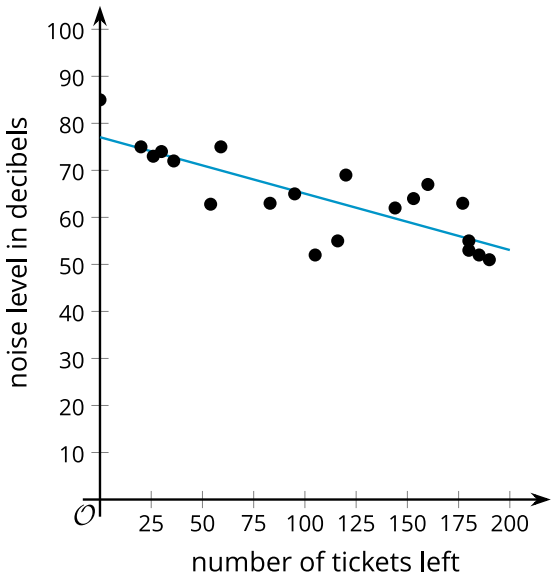
1. During the month of April, Elena keeps track of the number of inches of rain, or precipitation, recorded for the day and the percentage of people who come to school with rain jackets.



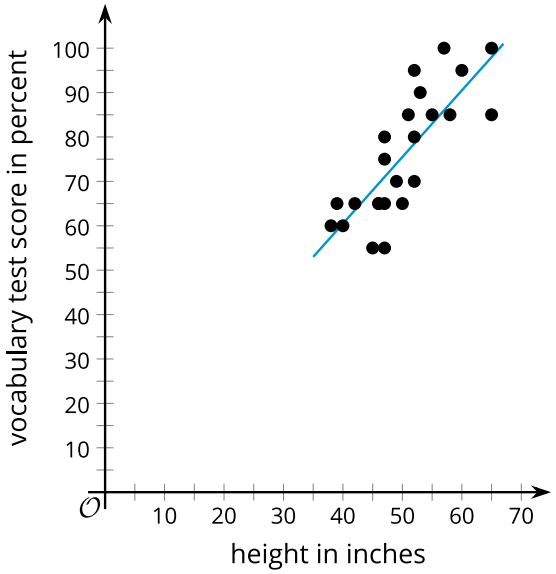
2. A school book club has a list of 100 books for its members to read. They keep track of the number of pages in the books the members read from the list and the amount of time it took to read the book.



3. Number of tickets left for holiday parties at a venue and noise level at the party



4. The height and score on a test of vocabulary for several children ages 6 to 13



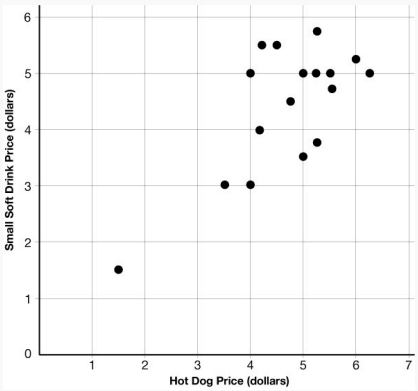
Why Should I Care?



One of the most powerful tools you can give a farmer to help improve their crop is statistics. Statistics can help farmers at every stage. For example, seeds are expensive, and if you plant too many strawberries in a market where there are hundreds of strawberry farmers, you might not sell enough to make a profit. With statistical models, farmers can use demand to precisely calculate the proper amount of crops to plant.

Self Check

In this scatter plot, which sentence best describes how the two variables are related?

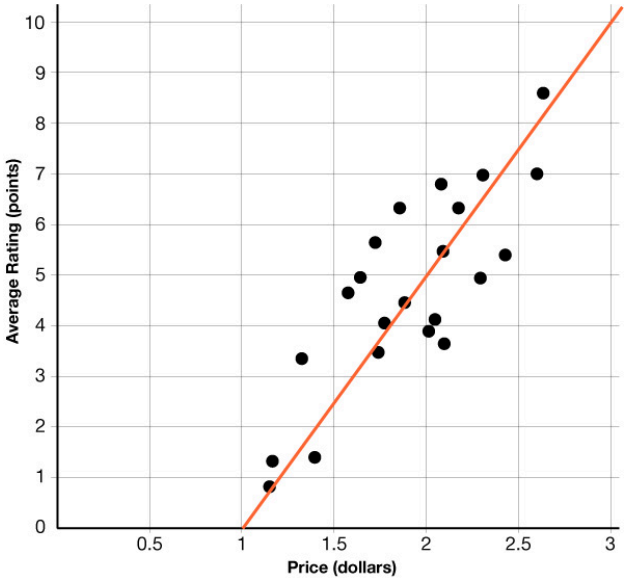


a. As hot dog prices go up, soft drink prices go down.
b. At different locations, as hot dog prices increase, small soda prices also increase.
c. Hot dog prices and small soft drink prices are not related.
d. The more hot dogs that are bought, the more sodas that are bought.

Additional Resources

Causal Relationships

A panel is rating different kinds of potato chips. The scatter plot below shows the relationship between their average rating and the price of the chips, along with the line of best fit. How could the variables be related?

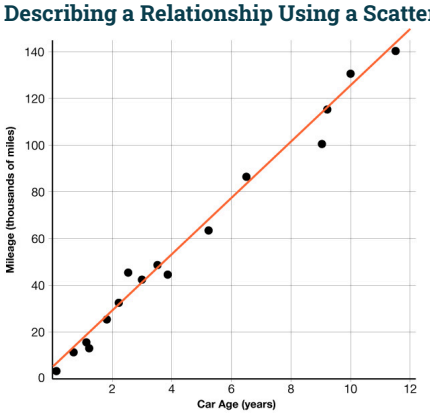


As the price of a bag of chips in dollars increases, the average rating of the bag of chips in points increases.

Does this represent a causal relationship or simply an association?

In this situation, the independent variable, price, does not seem to cause the rating a person gives the chips. While there appears to be a positive association, there are other factors that may cause the ratings for a bag of chips to go up. For instance, advertising on tv or social media might cause a particular kind of chip to increase in ratings without a change in their price.

TRY IT **Describing a Relationship Using a Scatter Plot**



The scatter plot shows a strong relationship. How could the variables be related?

3.6.3 Using the Term Causal Relationship

Activity

The phrases “correlation, not causation” or “association, not causation” are often used to refer to situations in which there is a relationship, but it is not a causal relationship. While **correlation** used to describe the strength of the relationship between variables, a **causal relationship** means that a change in one of the variables actually causes a change in the other variable.

Describe a pair of variables with each condition. Be prepared to show your reasoning.

1. Two variables have a **causal relationship**.
2. Two variables have an **association** in which the variables are strongly related, but a third factor might be the cause for the changes in the variables.
3. The variables are not related.
4. How are associations and causal relationships similar?
5. How are associations and causal relationships different?
6. How are associations and causal relationships different from data sets representing data that are not related?

Are you ready for more?
Extending Your Thinking

For questions 1 – 4, use the following instructions.

Look through news articles or advertisements for claims of causation or correlation. Find 2 or 3 claims and read or watch the articles or the advertisements. Answer these questions for each of the claims.

1. What is the claim?
2. What evidence is provided for the claim?
3. Does there appear to be evidence for causation or correlation? Explain your thinking.
4. Choose the claim with the least or no evidence. Describe an experiment or other way that you could collect data to show correlation or causation.

Self Check

Which of the following is a causal relationship?

a. The lower the temperature, the more people go ice skating.
b. The size of a deck and number of wood boards used.
c. The temperature and number of people buying hot chocolate.
d. The number of kids in a family and the number of pets in a family.

Additional Resources

Is It Causation?

Correlation means there is a relationship or pattern between the values of two variables. A scatter plot displays data about two variables as a set of points in the xy-plane and is a useful tool for determining if there is a correlation between the variables.

Causation means that one event causes another event to occur. Causation can only be determined from an appropriately designed experiment. In such experiments, similar groups receive different treatments, and the outcomes of each group are studied. We can only conclude that a treatment causes an effect if the groups have noticeably different outcomes.

Why doesn't correlation mean causation?

Even if there is a correlation between two variables, we cannot conclude that a change in one variable causes a change in the other. This relationship could be coincidental, or a third factor may be causing both variables to change.

EXAMPLE

Liam collected data on the sales of ice cream cones and air conditioners in his hometown. He found that when ice cream sales were low, air conditioner sales tended to be low, and that when ice cream sales were high, air conditioner sales tended to be high.

Liam can conclude that sales of ice cream cones and air conditioners are positively correlated.

Liam can't conclude that selling more ice cream cones causes more air conditioners to be sold. It is likely that the increases in the sales of both ice cream cones and air conditioners are caused by a third factor, an increase in temperature!

TRY IT **Is It Causation?**

Vivek notices that students in his class with larger shoe sizes tend to have higher grade point averages. Based on this observation, what is the best description of the relationship between shoe size and grade point average?

3.6.4 Determining the Type of Relationship

Cool Down

For each pair of variables, decide whether there is:

- a very weak or no relationship
- a strong relationship that is not a causal relationship
- a causal relationship

Be prepared to show your reasoning.

1. Number of snow plows owned by a city and mitten sales in the city
2. Explain the reasoning for your answer to question 1.
3. Number of text messages sent per day by a person and number of shirts owned by the person.
4. Explain the reasoning for your answer to question 3.
5. Price of a pizza and number of calories in the pizza
6. Explain the reasoning for your answer to question 5.
7. Amount of gas used on a trip and number of miles driven on the trip
8. Explain the reasoning for your answer to question 7.

Video: Determining the Type of Relationship

Watch the following video to learn more about using the term causal relationship.

Determining the Type of Relationship (https://openstax.org/r/relationship-type)

3.6.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Chloe creates a scatter plot showing the relationship between the number of steps she takes and her heart rate. The correlation coefficient of the line of best fit is 0.88.

Are they correlated? Be prepared to show your reasoning.

a. No, the correlation coefficient is more than 0.
b. Yes, the correlation coefficient is more than 0.
c. No, the correlation coefficient is close to 1.
d. Yes, the correlation coefficient is close to 1.

2. Kiran creates a scatter plot showing the relationship between the number of students attending drama club and the number of students attending poetry club each week. The correlation coefficient for the line of best fit is –0.36. Are they strongly correlated? Be prepared to show your reasoning.

a. Yes, the correlation coefficient is close to 0.
b. No, the correlation coefficient is negative.
c. Yes, the correlation coefficient is more than 0.
d. No, the correlation coefficient is closer to 0.

3. Which of the following is a causal relationship?

a. The number of minutes spent running and the number of calories burned.
b. The temperature outside and number of people wearing shorts.
c. The number of people in a family and the amount of the electric bill.
d. The age of a person and the number of words they know.

4. Which of the following is a causal relationship?

a. The number of children in a family and the number of pets in the home.
b. The amount of sugar a person eats and their level of happiness.
c. The number of students in a school and the state test scores.
d. The number of miles driven and the cost of gas for a trip.

3.6.6 Lesson Summary

In this lesson, you learned how to:

- Analyze connections between two variables to determine whether or not there is a causal relationship.

Here are the **activities** that helped you reach those goals:

- 3.6.1: Describing the Relationship between Variables
  - In this activity, you described the relationship between variables using mathematical terminology, such as strong or weak relationship and positive or negative relationship.
- 3.6.2: Describing How Two Variables Are Related
  - In this activity, you described how two variables are related and determined whether or not there is a causal relationship. You began to recognize that some variables may be related, but one does not always cause the other to change. You were able to recognize that at this point, the mathematics of scatter plot analysis cannot determine whether there is a causal relationship. You recognized that the relationship must be thought through carefully to decide, based on the situation, whether the related variables have a causal relationship.
  - 3.6.2: Self Check
  - 3.6.2: Additional Resources
- 3.6.3: Using the Term Causal Relationship
  - In this activity, you practiced using the term causal relationship and thought of situations to which it might apply and whether relationships were causal.
  - 3.6.3: Self Check
  - 3.6.3: Additional Resources
- 3.6.4: Determining the Type of Relationship
  - In this activity, you decided if variable pairs had a very weak or no relationship, a strong relationship that is not a causal relationship, or a causal relationship.

After these activities, you completed the following **practice**:

- 3.6.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

PROJECT 3: TWO-VARIABLE STATISTICS

Project 3 Overview

In this culminating project, you will integrate ideas from the unit as you collect, summarize, interpret, and draw conclusions from bivariate data using scatter plots, best fit lines, and correlation coefficients.

Technology isn't required for this lesson, but there are opportunities for you to choose to use appropriate technology to solve problems.

Project Activities

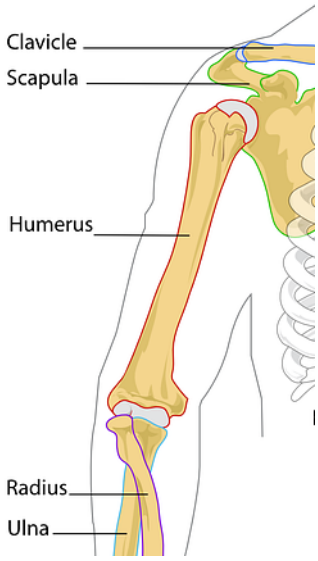
1. Estimating Lengths
2. Analyze Data

Learning Targets

When you finish this project, you will have demonstrated how to:

- Collect data, create a linear model to fit the data, determine if the linear model is a good fit, and use the information from my linear model to answer questions.

Estimate Lengths



An anthropologist finds a fossilized humerus bone of an ancient human ancestor. The humerus is an arm bone running from the shoulder to the elbow. It is 24 centimeters in length. Use data from your classmates to estimate the height of this ancient human.

Analyze Data

1. Before researching data, do you think the number of penalties a team gets in a season is correlated to the number of wins the team gets in that season? Do you think it is a positive or negative relationship? Do you think the relationship is weak or strong? Be prepared to show your reasoning.
2. Is there a relationship between the number of penalties a team gets in a season and the number of wins in that season? Show any mathematical work that leads to your answer.
3. Do penalties cause a change in wins or wins cause a change in penalties or neither? Be prepared to show your reasoning.

Unit 3 Wrap Up

Throughout this unit you learned that farmers, such as the Black and underserved farmers Farms To Grow founders Gordon Reed and Gail Myers work with, can use statistics to help them make decisions about their business. Statistics can help them decide the amount of crops they should grow, and how to package and ship their product. They can study weather and rainfall patterns and investigate how types of soil affects their crop. These decisions help assure them they will have a sustainable farm.

In this unit, you learned how a linear model is used to describe the relationship between two variables quantitatively. You learned that a linear model is an approximation based on data from a scatter plot. The linear model allows you to make predictions in a range near the given data.

You used technology to find the equation of the line of best fit and interpreted the slope and y-intercepts in the context of the situation. You used the linear model to estimate values not given in the data set.

You assessed the fit of the linear model informally using residuals and them formally using the correlation coefficient.

Finally, you learned that if a change in one of the variables directly causes a change in the other variable, then it is a causal relationship. You learned that a strong correlation does not imply a causal relationship, so you learned to analyze the relationship between two variables to determine whether they are causal or not.



Access our Unit 3 Student Self-Assessment (https://openstax.org/r/unit-3-self-assesment) to reflect on the skills you learned during this unit. Congratulations on completing Unit 3!

Building Character: Purpose

Throughout this unit, you learned that having **purpose** is good for both you and the world. It drives you to make positive contributions to the world that are in line with your own interests and strengths, as you saw Gordon and Gail do with their organization. As you continue along your journey in school and beyond, remember to think about what you want to contribute to the world and how you can be helpful to others. This may help you lead a happier, more fulfilling life.

INQUIRY PROJECT: DEFINING FUNCTIONS

Inquiry Project Overview

In this project, you will apply your prior knowledge about plotting coordinate points, evaluating expressions using the correct order of operations, and simplifying algebraic expressions, to generate initial ideas about functions.

First, you will be asked to play with a simulation for a few minutes. This open play time allows for you to learn how to use a simulation for learning and discover how to interact with it. You will have an opportunity to share how you used the simulation during the open play time. Be sure you are aware of the controls and how to use the simulation.

When you engage with the activity sheet, you should be pausing to discuss some of your responses with a partner. Be prepared to stop and share your individual responses and partner discussions with the whole class.

By the end of the lesson, you should have completed the activity sheets using the simulation, and should be able to describe a function as having inputs, outputs, and behaving consistently. You should also be able to evaluate a function using different inputs, and be able to summarize the impact that certain operations have on the algebraic function, the table of values, the appearance of the graph, and the outputs.

Project Activities

1. Simulating Relationships
2. What is a Function?
3. Different Kinds of Functions

Learning Targets

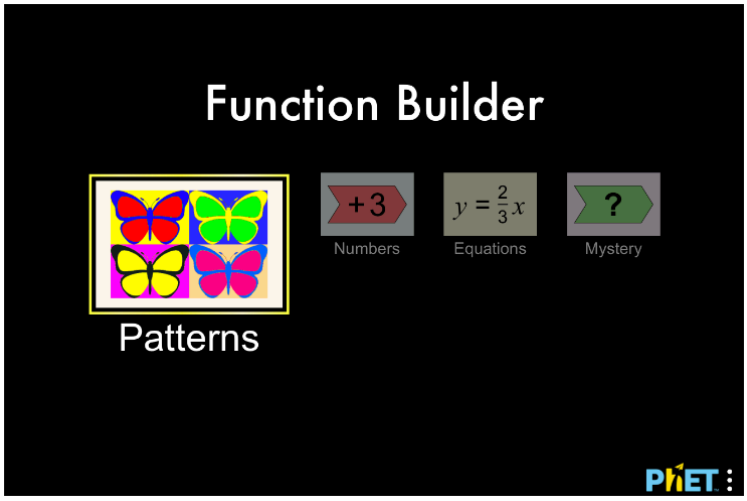
When you finish this project, you will have demonstrated how to:

- Define a function as a rule relating each input to exactly one output and consistently acting on inputs.
- Predict outputs of a function using given inputs.
- Describe a linear function in terms of a table, graph, equation, or series of operations.

Simulating Relationships

CLASSROOM CONNECTION Access the interactive simulator (https://www.openstax.org/r/function-builder) to begin this project.

Today you are going to investigate something called an interactive simulation. It is a tool that will help you visualize how functions work.



Step 1 - Open the interactive simulator, Function Builder (https://www.openstax.org/r/function-builder) in your web browser. You will use it to complete this project. It will look like the image above.

Step 2 - Play with any of the screens on the Function Builder for 5 minutes.

Step 3 - As you discover how the tool works, share your thinking with your group members. You may want to use the following question and sentence stem to guide your discussions:

- What did you try? What happened?
- I tried ... and it caused ... to happen.

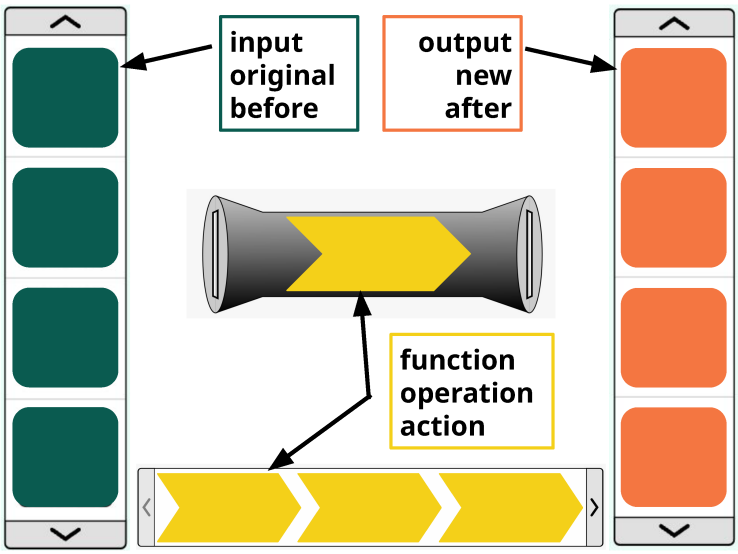
Step 4 - Write down two things you notice or have questions about, and one thing a neighbor noticed that is interesting to you.

What is a function?

Access the interactive simulator (https://www.openstax.org/r/function-builder) to begin this project.

In this activity, whenever you see ◈, stop and share your responses with your partner. If you have different responses, try to come to a consensus.

In the first activity, you may have noticed that all of the simulation screens share a three-part structure. These three parts can be described in a variety of ways, such as the ones listed in the image below.



1. Come up with your own names or labels for the three parts of a function.

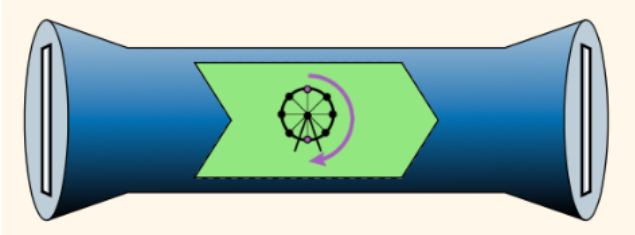
2. Share your labels with your group. Write down any similarities. ⚠

For questions 3 – 4, use the following instructions.

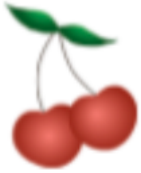
> Use the simulation screen labeled “Patterns”. Note you can switch between simulation screens using the toolbar located below the function builder.

3. The blue “builder” in the middle of the screen is a function. Describe what a function is using the labels you've agreed on as a class. ⚠

4. Suppose you build this function:



a. a. Complete the following table, sketching what happens after going through the function.

Input	Output
	
	
	
	

Input	Output
	
	

b. b. Write a rule that describes what the function does. Compare with your group. ◈

Different Kinds of Functions

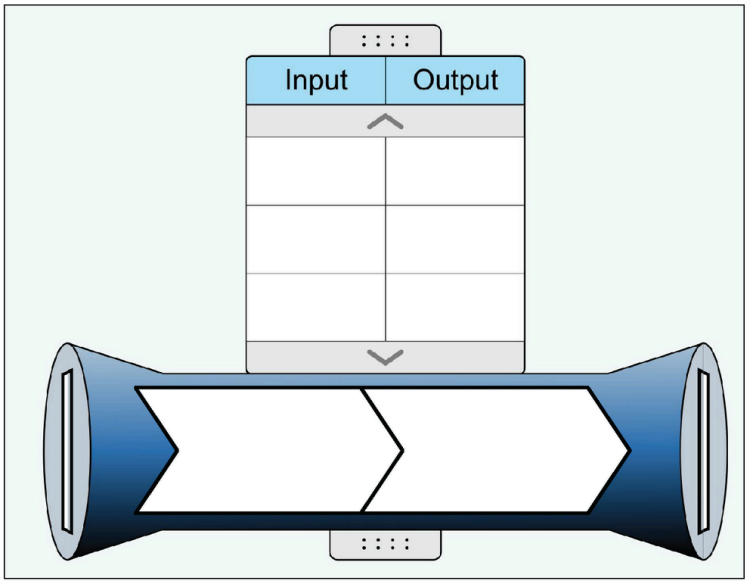
CLASSROOM CONNECTION Access the interactive simulator (https://www.openstax.org/r/function-builder) to begin this project.

In this activity, whenever you see ◈, stop and share your responses with your partner. If you have different responses, try to come to a consensus.

For questions 1 – 3, use the following instructions.

Use the simulation screen labeled “Numbers”. Note you can switch between simulation screens using the toolbar located below the function builder.

1. Build a function. Fill in the function builder and table. Access the input and output information by clicking the gray tabs above and below the function builder.



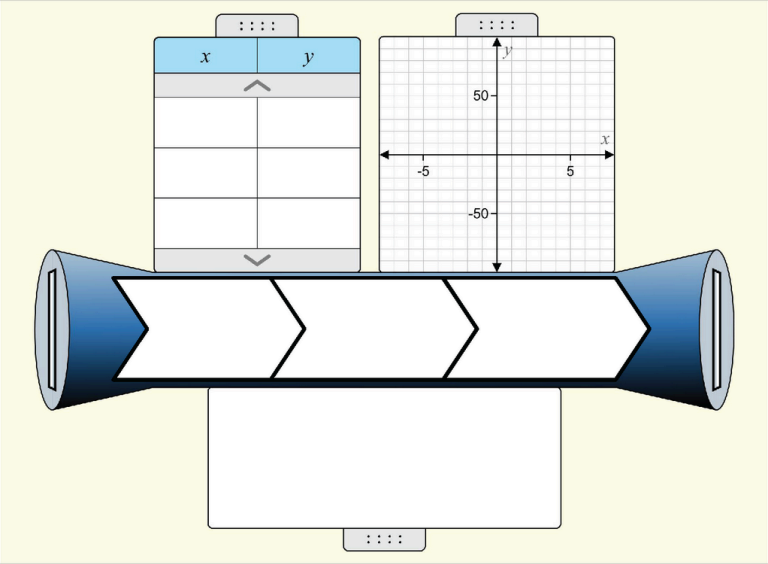
2. What is the output when the input is 10? Switch papers with a teammate and check that you found the correct output. ◈

3. Describe how to find the output for your function if given any input. **Challenge yourself to describe in multiple ways.**

For questions 4 – 8, use the following instructions.

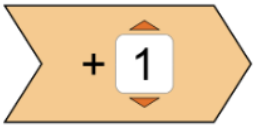
> Use the simulation screen labeled “Equations”. Note you can switch between simulation screens using the toolbar located below the function builder.

4. Build a custom function. Fill in the function builder and representations. Access the table and graph information by clicking the gray tabs above and below the function builder.



5. What is y when x is 100?

6. Manipulate your function in different ways. Describe the effects on the table, graph, and equation that each of your actions has.

Action	Effect on table	Effect on graph	Effect on equation
Click the up arrow on the addition operation 			

7. What does your graph look like? What other graphs can you make?

8. Briefly describe how different operations impact the graph of your function.

a. Addition
b. Subtraction
c. Multiplication
d. Division

Extension

Play with the game! Challenge yourself to figure out the mystery functions. They get tricky when you have two or three operations. Does your rule match the simulation's rule? Can there be more than one answer?

CLASSROOM CONNECTION Access the mystery functions game (https://openstax.org/r/mystery-functions) for an extra challenge

UNIT 4 OVERVIEW AND READINESS

Unit 4 Overview

Why Should I Care?

What do you do when faced with a challenge or problem? Do you brainstorm ideas to solve it? Ask others for help? Watch the video to learn how researcher Supratim Das uses creativity and functions to solve the issue of a longer life for batteries.

Unit 4 Overview (https://openstax.org/r/linear-equations-budget)

In this unit, you will learn about functions. Understanding how to use functions to model relationships can help you, for example, decide which smartphone plan is better and how much you will need to earn each week to pay for the one you choose.

Building Character: Creativity

As a scientist, Supratim Das uses his creativity to solve challenging issues. Similarly, in this course, you will want to use your creativity to solve problems.

Throughout this unit, you will work on developing your **creativity**. Creativity is the ability to think of new solutions. When you approach life with a creative attitude, you are more open to new ideas and possibilities, better able to understand others' perspectives, and more likely to seek out multiple solutions to problems. Despite the stereotype of the mad genius, everyday creativity is just as important. You can use it to help you improve your mood and cope with stress, feel more connected with other people and the world—and even help you find meaning and purpose in life.

Think about your current sense of creativity. How many of the following statements are true for you?

- I often seek out new experiences and ways of doing things.
- I like to think of different ways to reach my goals.
- I often make decisions that take me outside my comfort zone.
- I have a playful attitude toward learning something new.
- I enjoy connecting the dots between seemingly different perspectives and thoughts.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

Am I Ready to Learn This?

Getting Ready for Unit 4

To be ready for this unit, you need to be able to:

- Find the slope of the line that goes through two points on a graph.
- Interpret the meaning of a situation from a graph.
- Use a pattern to find the next term in a sequence.

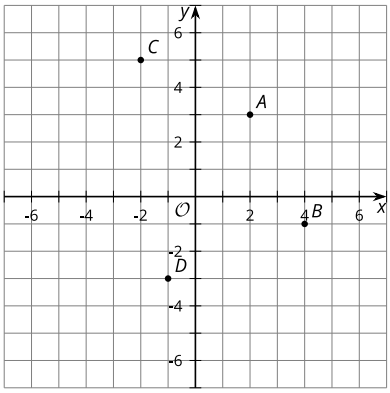
Next, you will answer some questions that will help you see how ready you are for this unit. If you need help, some mini-lessons will help you brush up on anything you need to work on.

Find Slope: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #1: Find Slope

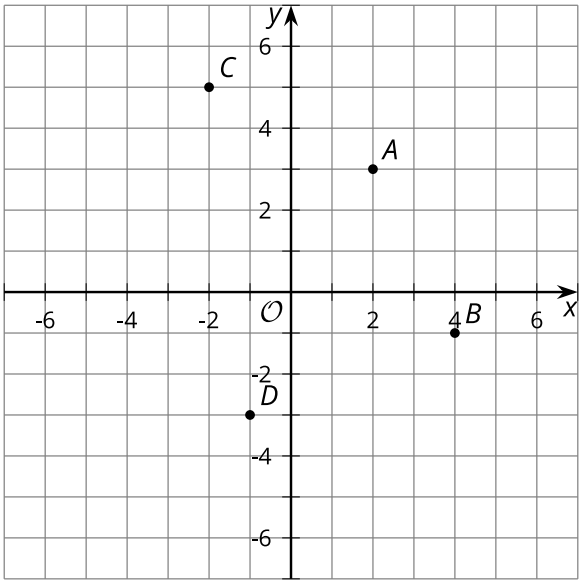
What is the slope of the line that goes through points A and D?



a. $-\frac{1}{2}$
b. $\frac{1}{2}$
c. -2
d. 2

Find the Slope of a Line Using Points on a Graph

The slope of a line is the ratio of the change in y and the change in x. You can use the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$, to find the slope of the line through points (x_1, y_1) and (x_2, y_2).



Find the slope of the line that goes through points

a. A and D
b. A and B
c. C and D

Here's how to find the slope given two points.

a.

Step 1 - First, name $B(-3, -4)$ as point 1 and $D(4, 3)$ as point 2.

(x_1, y_1) and (x_2, y_2)

$(-3, -4)$ and $(4, 3)$

Step 2 - Use the slope formula.

$m = \frac{y_2 - y_1}{x_2 - x_1}$

Step 3 - Substitute the values: $y_2 = -3$, $y_1 = 3$, $x_2 = -1$, $x_1 = 2$.

$m = \frac{-3-3}{-1-2}$

Step 4 - Simplify.

$m = \frac{-6}{-3}$

$m = 2$

b.

Step 1 - First, name $A(2, 3)$ as point 1 and $B(4, -1)$ as point 2.

(x_1, y_1) and (x_2, y_2)

$(2, 3)$ and $(4, -1)$

Step 2 - Use the slope formula.

$m = \frac{y_2 - y_1}{x_2 - x_1}$

Step 3 - Substitute the values: $y_2 = -1$, $y_1 = 3$, $x_2 = 4$, $x_1 = 2$.

$m = \frac{-1-3}{4-2}$

Step 4 - Simplify.

$m = \frac{-4}{2}$

$m = -2$

c.

Step 1 - First, name $C(-2, 5)$ as point 1 and $D(-1, -3)$ as point 2.

(x_1, y_1) and (x_2, y_2)

$(-2, 5)$ and $(-1, -3)$

Step 2 - Use the slope formula.

$m = \frac{y_2 - y_1}{x_2 - x_1}$

Step 3 - Substitute the values: $y_2 = -3$, $y_1 = 5$, $x_2 = -1$, $x_1 = -2$.

$m = \frac{-3-5}{-1-(-2)}$

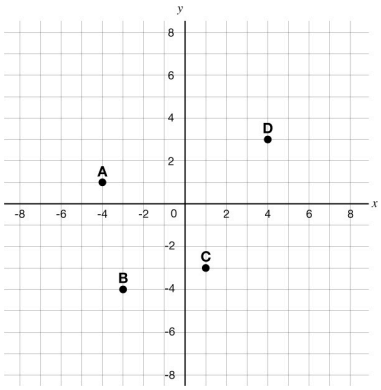
Step 4 - Simplify.

$m = \frac{-8}{1}$

$m = -8$

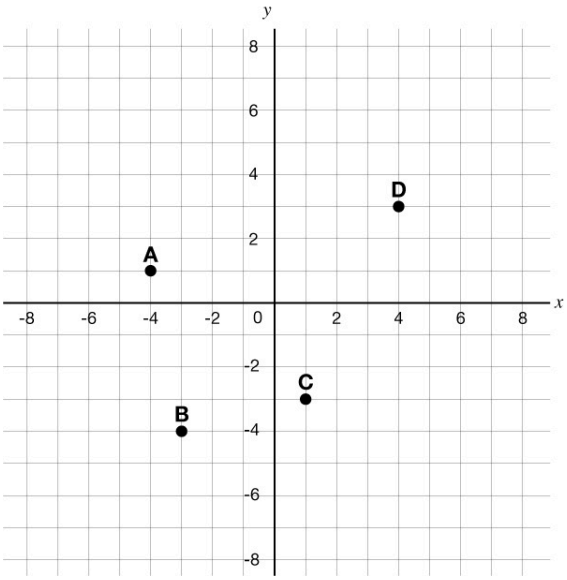
TRY IT **Find the Slope of a Line Using Points on a Graph**

Find the slope of the line that goes through points B and D.



Check Your Understanding

What is the slope of the line that goes through points A and D?



a. $-\frac{1}{4}$

b. $\frac{1}{4}$

c. -3

d. 3

Videos: Finding the Slope of a Line

Khan Academy: Slope from a Graph

Watch this video to see how to find the slope from a graph.

Finding the slope of a line from its graph | Algebra I | Khan Academy (https://www.youtube.com/embed/R948Tsyq4vA)

Khan Academy: Slope from Two Points

Watch this video to see how to find the slope given two points.

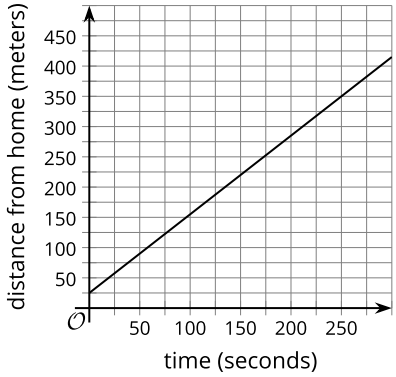
Slope from two ordered pairs example 1 | Algebra I | Khan Academy (https://www.youtube.com/embed/WkspBxrzuZo)

Describe Graphs: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #2: Describe Graphs

Which statement best describes the situation shown in the graph?

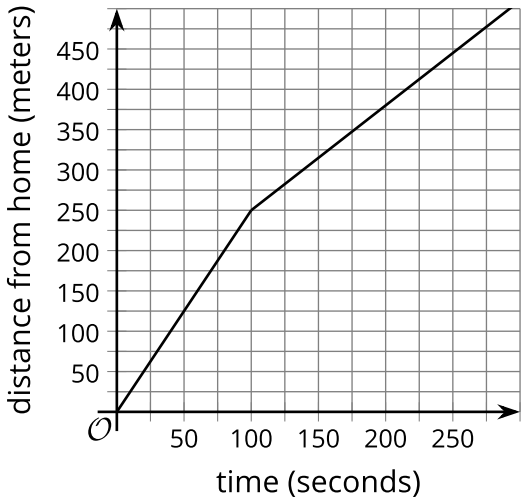


a. Mai begins at a neighbor's house and runs away from home, and then walks for a while.
b. Mai begins at a neighbor's house and walks away from home at a constant rate.
c. Mai begins at home and walks away from her home at a constant rate.
d. Mai begins her walk at school and walks home at a constant rate.

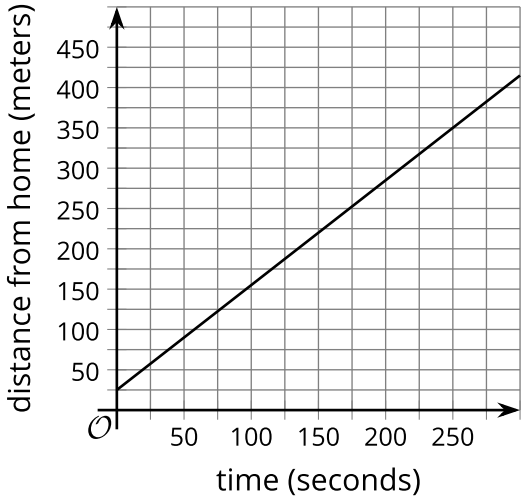
Match a Description with a Graph

A graph can be used to model a situation. You can use what you know about slope and y-intercept to connect the graphs to real situations.

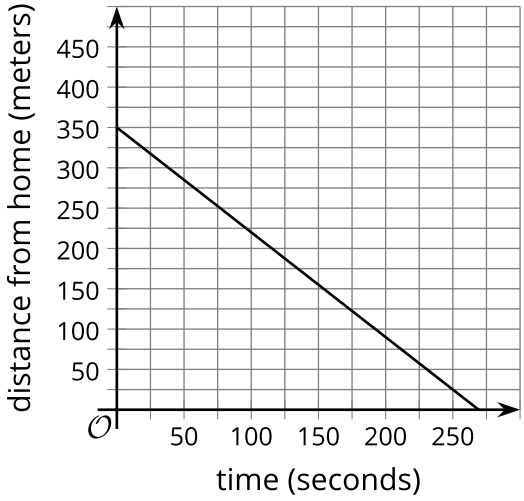
A.



B.



C.



Match each description with one of the graphs above:

1. Elena begins at home and runs away from her home at a constant rate.

Elena starts at home, so the graph must start at the origin. She moves away from home at a constant speed, so the graph must be a line that increases to the right. **Graph B** matches the description.

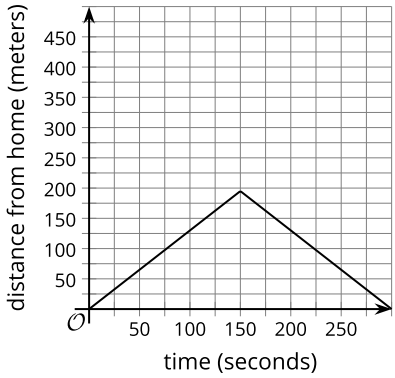
2. Clare begins her walk at school and walks home at a constant rate.

Clare starts at school, so the y-intercept is a point on the y-axis that shows the distance her home is from the school. She moves closer to home at a constant speed, so the graph must be a line that decreases to the right. **Graph C** matches the description.

3. Lin begins at home and runs away from home, and then walks for a while.

Lin starts at home, so the graph must start at the origin. She starts running and then walks, so the graph is a line with a slope that starts off steep and then continues with a less steep slope. **Graph A** matches the description.

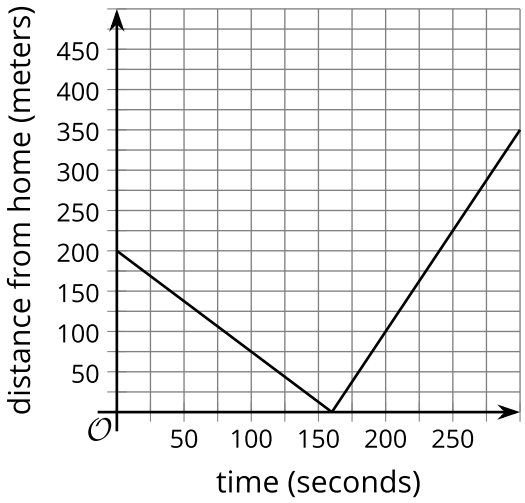
TRY IT **Match a Description with a Graph**



Describe the graph. You can use the wording from the previous problem as a guide.

Check Your Understanding

Which statement best describes the situation shown in the graph?



a. Max begins his walk at the library and walks away from home at a constant rate.

b. Max begins his walk at the library and walks home at a constant rate.

c. Max begins his walk at the library, walks home at a constant rate, and then walks away from home at a constant rate.

d. Max begins at home and walks away from home for a while, then walks back home.

Video: Slope and Intercept in Tables

Watch the following video to find the meaning of slope and intercepts in context.

Khan Academy: Slope and Intercept in Tables

Slope and intercept in tables (https://www.youtube.com/embed/TYbN--4KHIk)

Extend a Pattern: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #3: Extend a Pattern

Which describes the rule and the next two terms in the sequence?

2, 6, 18, 54, . . .

a. Multiply by 4; 216, 864
b. Add 3; 57, 60
c. Add 4; 58, 62
d. Multiply by 3; 162, 486

Use a Rule to Extend a Pattern

Pattern rules can use one or more mathematical operations to describe the relationship between consecutive numbers in a list of numbers, or sequence. Knowing the pattern rule will help you extend the pattern. To extend the pattern means to use the pattern rule to write the numbers that would come next in the sequence.

Try to see the difference between consecutive numbers. It will help you understand the relationship between the numbers.

Find the next two numbers in the following sequence:

3, 6, 9, 12, . . .

First, figure out the pattern rule. The numbers are increasing, so the rule likely involves addition or multiplication.

- Look at the first two terms, 3 and 6. What operation could you perform on 3 to get 6?
  - You could add 3.
  - You could multiply by 2.
  - You could do a combination of two or more operations.
- Look at the second two terms, 6 and 9.
  - If you add 3 to 6, you get 9. So the pattern rule “add 3” seems to work.
- Make sure the pattern rule “add 3” works for every term.
  - $9 + 3 = 12$

The pattern rule works. To extend the pattern, apply the pattern rule of “add 3” to the last term, 12, to get the next term, and then “add 3” to the result to get another term. The answer is that the extended pattern is:

3, 6, 9, 12, 15, 18, . . .

You can find a particular term in the pattern by extending the pattern until you’ve reached the term you are looking for.

What is the seventh number in the following sequence?

64, 32, 16, 8, . . .

First, figure out the pattern rule. The numbers are decreasing, so the rule likely involves subtraction or division.

- Look at the first two terms, 64 and 32. What operation could you perform on 64 to get 32?
  - You could subtract 32.
  - You could divide by 2.
  - You could do a combination of two or more operations.
- Look at the second two terms, 32 and 16.
  - If you subtract 32 from 32, you get 0, not 16. So the pattern rule “subtract 32” does not work.
  - If you divide 32 by 2, you get 16. So the pattern rule “divide by 2” seems to work.
- Make sure the pattern rule “divide by 2” works for every term.
  - $16 \div 2 = 8$

The pattern rule works. To extend the pattern, apply the pattern rule of “divide by 2” to the last term, 8, to get the next term, and continue dividing by 2 until you reach the seventh term.

64, 32, 16, 8, 4, 2, 1

The seventh term of the sequence is 1.

TRY IT **Use a Rule to Extend a Pattern**

Describe the rule and find the next two terms in the sequence.

3, 9, 27, 81, . . .

Check Your Understanding

Which describes the rule and the next two terms in the sequence? 2, 6, 10, 14, ...

a. Add 4; 18, 22

b. Multiply by 3; 42, 126

c. Multiply by 4; 56, 224

d. Add 3; 17, 20

Video: Finding Patterns in Numbers

Watch the following video to learn more identifying patterns in numbers.

Khan Academy: Finding Patterns in Numbers

Practice finding patterns in numbers | 3rd grade | Khan Academy (https://www.youtube.com/embed/l-6uEtTBH7g)

Unit 4 Overview: Wrap Up

- Find Slope
- Describe Graphs
- Extend a Pattern

If you are still struggling, be sure to reach out to your teacher for additional help. Nobody is good at problems like this the first try. Keep at it, and it will get easier.

To continue with the unit, open the course index and select “Section A Overview: Functions and Their Representations.” Otherwise, select the previous button to repeat this lesson.

4.1 DESCRIBING AND GRAPHING SITUATIONS

4.1.0 Lesson Overview

In this lesson, you will learn about functions. You will examine examples of functions and their representations in context.

When you finish this lesson, you will be able to:

- Explain when a relationship between two quantities is a function.
- Identify independent and dependent variables in a function, and use words and graphs to represent the function.
- Explain what descriptions and graphs of functions tell us about situations.

Here are the **activities** that will help you reach those goals:

- 4.1.1: Contrasting Two Relationships by Reasoning
- 4.1.2: Reasoning Graphically About the Relationship between the Two Quantities
  - 4.1.2: Self Check
  - 4.1.2: Additional Resources
- 4.1.3 Examining Relations and Functions
  - 4.1.3: Self Check
  - 4.1.3: Additional Resources
- 4.1.4: Describing Functional Relationships
  - 4.1.4: Self Check
  - 4.1.4: Additional Resources
- 4.1.5: Modeling Relationships between Two Variables

After that, you'll **practice and review**.

- 4.1.6: Practice
- 4.1.7: Lesson Summary

4.1.1 Contrasting Two Relationships by Reasoning

Warm Up





A customer at a bagel shop is buying 13 bagels. The store worker says, "That would be $16.25." Jaime, Pedro, and Hannah, who are in the shop, all think it is a mistake.

- Jaime says, "Shouldn't the total be $13.25?"
- Pedro says, "I think it should be $13.00."
- Hannah says, "No, I think it should be $11.25."

Consider how the store worker, Jaime, Pedro, and Hannah could all be right.

In the following activity, you will be asked to use the pricing information displayed at the bagel shop to determine who is correct about the pricing for 13 bagels.

CLASSROOM CONNECTION Log into student.desmos.com using the information provided by your teacher to complete the activity.

1. Create a table of values using the pricing information from the bagel shop's sign to identify the different possibilities for calculating the price of 13 bagels.

Number of bagels	Store worker's price calculation (in dollars)	Jaime's price calculation (in dollars)	Pedro's price calculation (in dollars)	Hannah's price calculation (in dollars)
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

Number of bagels	Store worker's price calculation (in dollars)	Jaime's price calculation (in dollars)	Pedro's price calculation (in dollars)	Hannah's price calculation (in dollars)
11				
12				
13				

2. Use the table you created in the previous question to explain how each person in the scenario is correct.

a. Using the table you created in question 1, explain why the store worker thinks the total for 13 bagels should cost $16.25.

Use the sentence stem: The store worker thinks the price for 13 bagels should be $16.25 because ...

b. Using the table you created in question 1, explain why Jaime thinks the total for 13 bagels should cost $13.25.

Use the sentence stem: Jaime thinks the price for 13 bagels should be $13.25 because ...

c. Using the table you created in question 1, explain why Pedro thinks the total for 13 bagels should cost $13.00.

Use the sentence stem: Pedro thinks the price for 13 bagels should be $13.00 because ...

d. Using the table you created in question 1, explain why Hannah thinks the total for 13 bagels should cost $11.25.

Use the sentence stem: Hannah thinks the price for 13 bagels should be $11.25 because ...

3. Create a table of values using the pricing information from the bagel shop's sign to identify the best price for each number of bagels bought.

Number of bagels	Best price (in dollars)
1	
2	
3	
4	
5	
6	
7	

Number of bagels	Best price (in dollars)
8	
9	
10	
11	
12	
13	

4. How might a price list like this benefit the owner of the bagel shop?

5. As a customer, do you like items to be priced like this or not? Explain your reasoning.

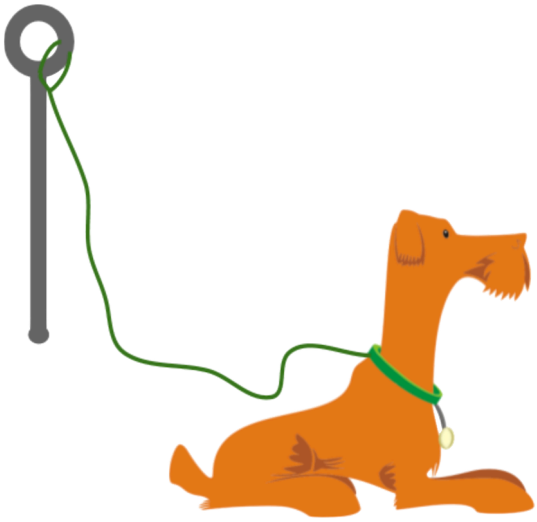
In this activity, because there was only one **best price** for a particular number of bagels, the best price is a **function** of the number of bagels bought.

A **function** assigns one output to each input.

4.1.2 Reasoning Graphically about the Relationship between the Two Quantities

Activity

Three days in a row, a dog owner tied his dog's 5-foot-long leash to a post outside a store while he ran into the store to get a drink. Each time, the owner returned within minutes.



The dog's movement each day is described here.

- Day 1: The dog walked around the entire time while waiting for its owner.

- Day 2: The dog walked around for the first minute and then laid down until its owner returned.
- Day 3: The dog tried to follow its owner into the store but was stopped by the leash. Then, it started walking around the post in one direction. It kept walking until its leash was completely wound up around the post. The dog stayed there until its owner returned.
- Each day, the dog was 1.5 feet away from the post when the owner left.
- Each day, 60 seconds after the owner left, the dog was 4 feet from the post.

Your teacher will assign one of the days for you to analyze. You will sketch a graph that could represent the dog's distance from the post, in feet, as a function of time, in seconds, since the owner left.

Then, meet with someone in your group who had Day 2's movement and someone who had Day 3's movement. Work together to explain your graphs and make changes where needed before checking your work. Be sure to sketch each other's graphs.

Day _____

Sketch the graph.

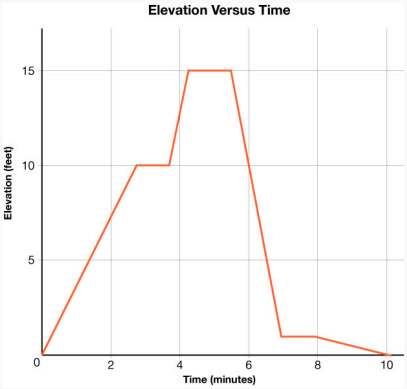
Are you ready for more?

Extending Your Thinking

From the graph, is it possible to tell how many times the dog changed directions while walking around? Be prepared to show your reasoning.

Self Check

The graph below shows a person's elevation for a period of 10 minutes. Which statement below is true based on the graph?



a. The person never stopped climbing.
b. During the first 3 minutes, the person was hiking up a hill.
c. The person slowed down for the second half of the hike.
d. The person sped up in the first 3 minutes.

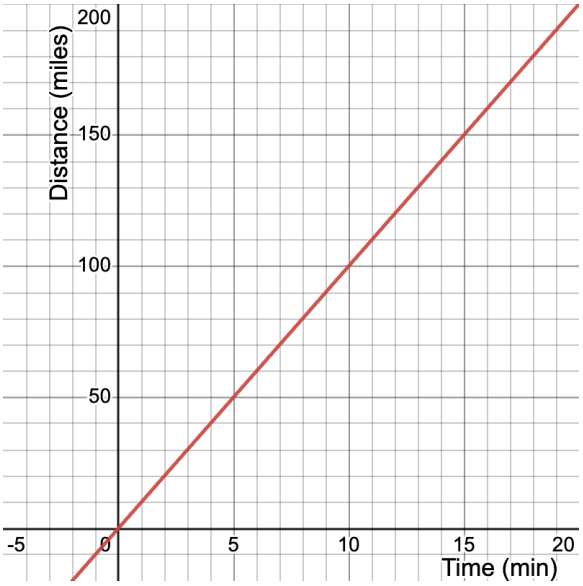
Additional Resources

Representing Everyday Situations with Graphs

Graphing functions to represent everyday situations can be useful to provide a visual for what is happening in a situation.

EXAMPLE 1

The graph shows the distance y, in miles, for the time of each of Mia's runs x, in minutes.



Notice that at 0 minutes, Mia has run 0 miles. At 30 minutes, Mia has run 3 miles. At 180 minutes, Mia has run 18 miles.

TRY IT **Represent Everyday Situations with Graphs**

Sky spends 5 hours at the library. She is on their internet for an hour. Then she takes an hour break to read a book. She then is back online for 2 hours before her friend comes to talk for an hour. She spends her last hour at the library on the internet.

Draw this situation where the time spent on the internet, in hours, is a function of time spent at the library, in hours.

4.1.3 Examining Relations and Functions

Activity

A **relation** is any set of ordered pairs, (x, y).

A **function** is a special type of relation that maps every input, x, to exactly one output, y.

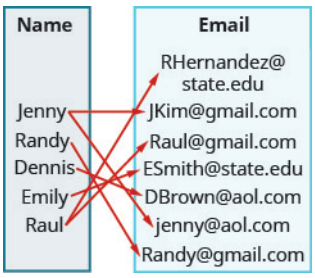
Tell if each relation is a function and explain.

1.

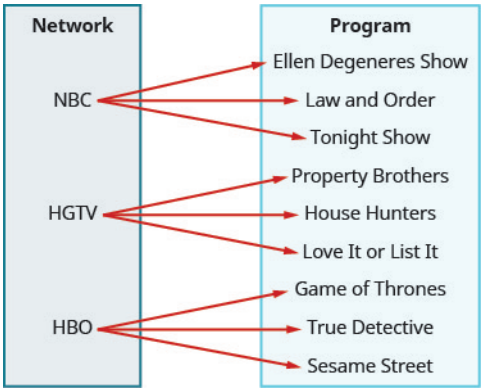
x	y
-3	3
-2	2
-1	1
0	0
1	1

x	y
2	2
3	3

2.



3.



4. $(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)$

5. $(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)$

Self Check

Consider the two relations below. Which statement is true about both relations?

A. $\{(-3, -6), (-2, -4), (-1, -2), (0, 0), (1, 2), (2, 4), (3, 6)\}$
B. $\{(8, -4), (4, -2), (2, -1), (0, 0), (2, 1), (4, 2), (8, 4)\}$

a. Neither relation is a function.
b. Relation b is a function, but relation a is not a function.
c. Relation a is a function, but relation b is not a function.
d. Both relations are functions.

Additional Resources

Is the Relation a Function?

A **relation** is a set of ordered pairs. The set of the first components of each ordered pair are the inputs and the set of the second components of each ordered pair are the outputs. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.

$(1, 2), (2, 4), (3, 6), (4, 8), (5, 10)$

The set of inputs is $\{1, 2, 3, 4, 5\}$. The set of outputs is $\{2, 4, 6, 8, 10\}$.

Each **input** value is the **independent variable**, and is often labeled with the lowercase letter x. Each **output** value, is the **dependent variable**, and is often labeled lowercase letter y.

A function f is a relation that assigns a single output to each input. In other words, no x-values are repeated. For our example that relates the first five natural numbers to numbers double their values, this relation is a function because each input, {1,2,3,4,5}, is paired with exactly one output, $\{2, 4, 6, 8, 10\}$.

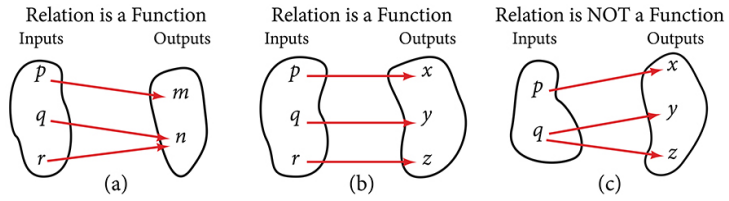
Now let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

{ (odd,1),(even,2),(odd,3),(even,4),(odd,5) }

Notice that each input, {even,odd} is not paired with exactly one output, $\{1, 2, 3, 4, 5\}$. For example, the term "odd" corresponds to three output values, {1,3,5} and the term "even" corresponds to two output values, $\{2, 4\}$. This violates the definition of a function, so this relation is not a function.

A **mapping** is sometimes used to show a relation. The arrows show the pairing of the inputs with the outputs.

Here are some relations that are functions and not functions.



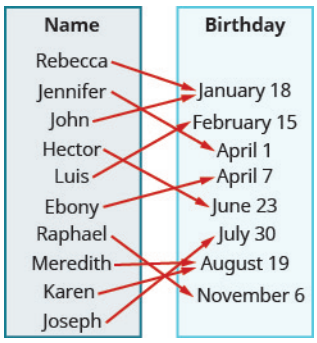
a. This relationship is a function because each input is associated with a single output. Note that inputs q and r both give output n.
b. This relationship is also a function. In this case, each input is associated with a single output.
c. This relationship is not a function because input q is associated with two different outputs.

How to determine if a relationship is a function.

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

TRY IT **Is the Relation a Function?**

Is the following relation a function? Explain your answer.



4.1.4 Describing Functional Relationships

Activity

In a function, there are two quantities called variables. The **independent variable** is the input, while the **dependent variable** is the output.

Here are two pairs of quantities from a situation you've seen in this lesson. Each pair has a relationship that can be defined as a function.

- Time, in seconds, since the dog owner left and the total number of times the dog has barked
- Time, in seconds, since the owner left and the total distance, in feet, that the dog has walked while waiting

Choose one pair of quantities and express their relationship as a function.

1. In that function, which variable is independent? Which one is dependent?

2. Write a sentence of the form "____ is a function of ____."

3. On a piece of graph paper, sketch a possible graph that represents your chosen scenario.

Video: Determining Independent and Dependent Variables

Watch the following video to learn more about independent and dependent variables.

Determining Independent and Dependent Variables (https://openstax.org/r/variables-identification)

Self Check

The number of unread emails in Bryan's account is 100. This number grows by 15 unread emails a day. Which of the following is true about this relationship?

a. The number of unread emails is a function of the number of new unread emails.
b. There is not a functional relationship between time and the number of emails.
c. The number of emails is a function of time in days.
d. Time, in days, is a function of the number of emails.

Additional Resources

Identifying if a Relationship Is a Function

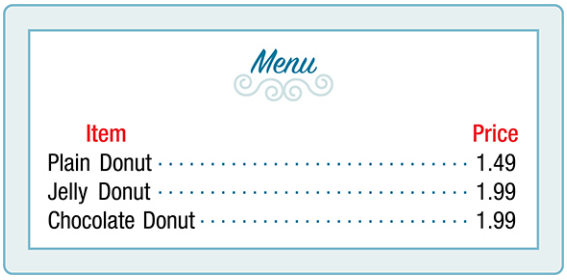
A function is a relation in which each possible input value leads to exactly one output value. We say "the output is a function of the input."

How to determine if a relationship is a function.

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

EXAMPLE 1

The coffee shop menu, shown below, consists of items and their prices.



a. Is price a function of the item?

b. Is the item a function of the price?

EXAMPLE 2

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? The table below shows a possible rule for assigning grade points.

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92 – 100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades, ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

TRY IT **Identifying if a Relationship Is a Function**

The table below lists the five greatest baseball players of all time in order of rank.

Player	Rank
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?

b. Is the player name a function of the rank?

4.1.5 Modeling Relationships between Two Variables

Cool Down

A parent is using a garden hose to fill up a small inflatable pool for her young child. The pool has a capacity of 90 gallons. She turns the water off after 5 minutes, but she leaves the hose in the pool for another 3 minutes before putting it away.

In this situation, the relationship between the gallons of water in the pool and the time, in minutes, since the parent started filling the pool can be seen as a function.

In that function, which variable is independent?

In this function, which variable is dependent?

Write a sentence of the form "____ is a function of ____."

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-1-5-modeling-relationships-between-two-variables)
Use the graphing tool above or technology outside the course. Sketch a possible graph of the relationship on the coordinate plane. Be sure to label and indicate a scale on each axis.

4.1.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 - 4, use the following scenario.

The relationship between the amount of time a car is parked, in hours, and the cost of parking, in dollars, can be described with a function.

1. Identify the independent variable in this function.

a. Cost of parking
b. Amount of time car is parked

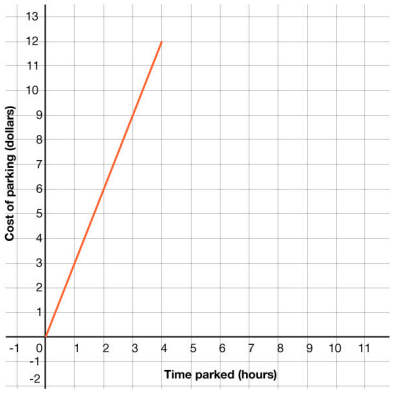
2. Identify the dependent variable in this function.

a. Cost of parking
b. Amount of time car is parked

3. Correctly describe the function with a sentence of the form "___ is a function of ___."

a. The cost of parking is a function of the amount of time a car is parked.
b. The amount of time a car is parked is a function of the cost of parking.

4. Suppose it costs \$3 per hour to park, with a maximum cost of \$12. Here is a graph of the situation.



What does the point $(2, 6)$ mean in this situation?

a. 6 hours of parking costs a total of \$2
b. 2 hours of parking costs a total of \$6
c. 6 hours of parking costs \$2 per hour
d. 2 hours of parking costs \$6 per hour

5. The distance a person walks, d, in kilometers, is a function of time, t, in minutes, since the walk began.

Select the **two statements** about the input variable of this function that are true.

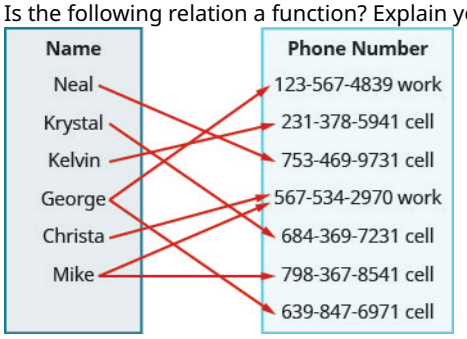
a. Distance is the input.
b. Time of day is the input.
c. Time since the person starts walking is the input.
d. t represents the input.
e. d represents the input.
f. The input is not measured in any particular unit.
g. The input is measured in hours.
h. For each input, there are sometimes two outputs.

6. Determine if the following relation is a function.

$\{(27,-3),(8,-2),(1,-1),(0,0),(1,1),(8,2),(27,3)\}$

a. Yes
b. No

7. Is the following relation a function? Explain your answer.



a. No
b. Yes

8. Determine if the following relation is a function.
$\{(7,-3),(-5,-4),(8,0),(0,0),(-6,4),(-2,2),(-1,3)\}$;

a. Yes
b. No

4.1.7 Lesson Summary

In this lesson, you learned how to:

- Explain when a relationship between two quantities is a function.
- Identify independent and dependent variables in a function, and use words and graphs to represent the function.
- Explain what descriptions and graphs of functions tell us about situations.

Here are the **activities** that helped you reach those goals:

- 4.1.1: Contrasting Two Relationships by Reasoning
  - In this activity, you looked at which relationships represent a function and learned which quantities are functions of the other.
- 4.1.2: Reasoning Graphically About the Relationship between the Two Quantities
  - In this activity, you sketched graphs about a dog's movement as distance was a function of time.
  - 4.1.2: Self Check
  - 4.1.2: Additional Resources
- 4.1.3: Examining Relations and Functions
  - In this activity, you determined if relations are functions by looking at the given inputs and outputs.
  - 4.1.3: Self Check
  - 4.1.3: Additional Resources
- 4.1.4: Describing Functional Relationships
  - In this activity, you identified the independent and dependent variables in functional relationships. You also determined which variable was a function of the other to make statements about the relationship.
  - 4.1.4: Self Check
  - 4.1.4: Additional Resources
- 4.1.5: Modeling Relationships between Two Variables
  - In this activity, you summarized everything you have learned in this lesson by identifying independent and dependent variables of a situation, determining which variable is a function of the other, and sketching a graph of the situation.

After these activities, you completed the following **practice**:

- 4.1.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

4.2 FUNCTION NOTATION

4.2.0 Lesson Overview

In this lesson, you will be introduced to function notation as a more concise and precise way to represent functions. Further, you will learn about the relationship between input and output.

When you finish this lesson, you will be able to:

- Use function notation to express functions that have specific inputs and outputs.
- Explain what function notation is and why it exists.
- Explain a statement written in function notation in terms of a situation.

Here are the **activities** that will help you reach those goals:

- 4.2.1: Interpreting Graphs to Answer Questions
- 4.2.2: Interpreting Statements Written in Function Notation
  - 4.2.2: Self Check
  - 4.2.2: Additional Resources
- 4.2.3: Finding a Unique Output for Each Input
  - 4.2.3: Self Check
  - 4.2.3: Additional Resources
- 4.2.4: Understanding Function Notation Using Application Problems

After that, you'll **practice and review**.

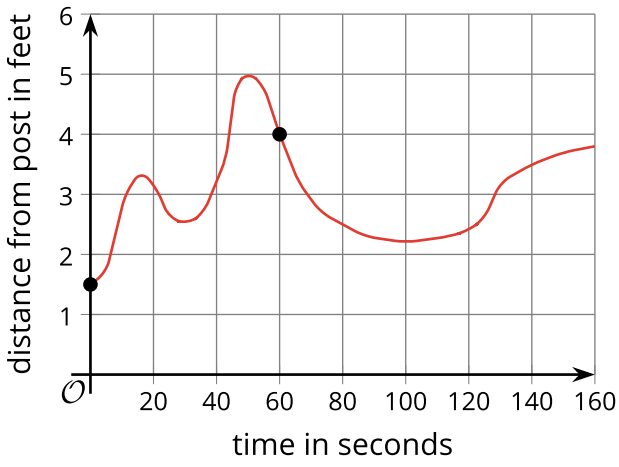
- 4.2.5: Practice
- 4.2.6: Lesson Summary

4.2.1 Interpreting Graphs to Answer Questions

Warm Up

Here are the graphs of some situations you saw before. Each graph represents the distance of a dog from a post as a function of time since the dog owner left to purchase something from a store. Distance is measured in feet, and time is measured in seconds. Use the given graphs to answer questions about each of the three days.

Day 1



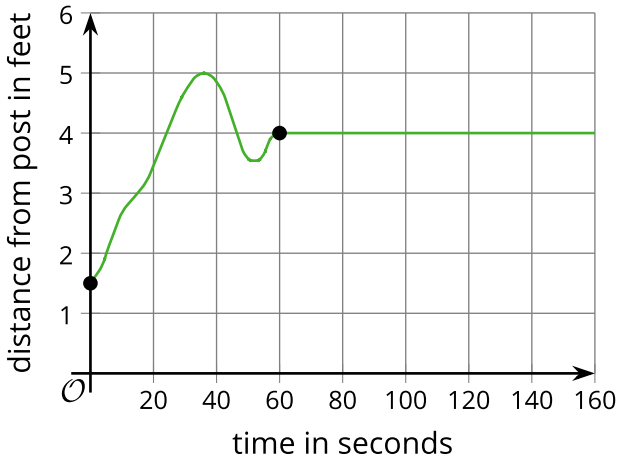
1. How far away was the dog from the post 60 seconds after the owner left?

2. How far away was the dog from the post when the owner left?

3. How far away was the dog from the post 160 seconds after the owner left?

4. How many seconds passed before the dog reached the farthest point it could reach from the post?

Day 2



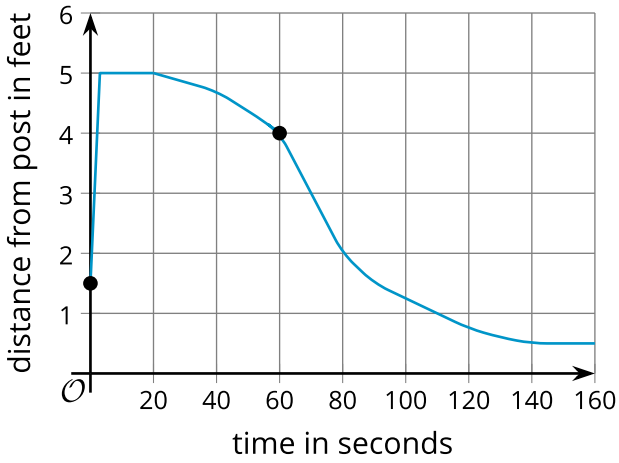
5. Consider the statement, "The dog was 2 feet away from the post after 80 seconds." Do you agree with the statement?

6. How far away was the dog from the post when the owner left?

7. The owner returned 160 seconds after he left. How far away was the dog from the post at that time?

8. How many seconds passed before the dog reached the farthest point it could reach from the post?

Day 3



9. How far away was the dog from the post 60 seconds after the owner left?

10. How far away was the dog from the post when the owner left?

11. The owner returned 160 seconds after he left. How far away was the dog from the post at that time?

12. How many seconds passed before the dog reached the farthest point it could reach from the post?

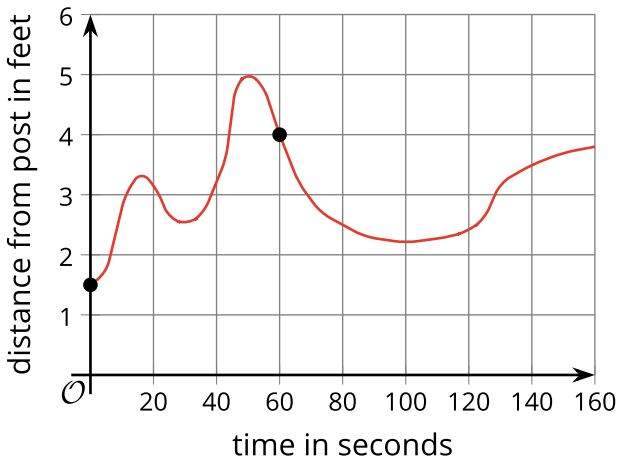
13. Consider the statement, "The dog was 2 feet away from the post after 80 seconds." Do you agree with the statement?

4.2.2 Interpreting Statements Written in Function Notation

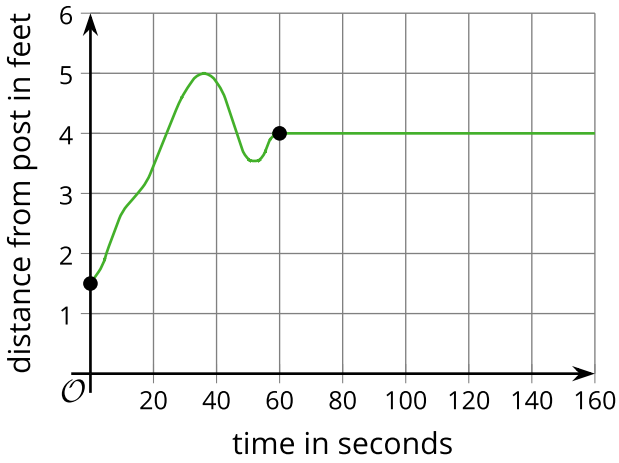
Activity

Let's name the functions that relate the dog's distance from the post and the time since its owner left: function f for Day 1, function g for Day 2, function h for Day 3. The input of each function is time in seconds, t.

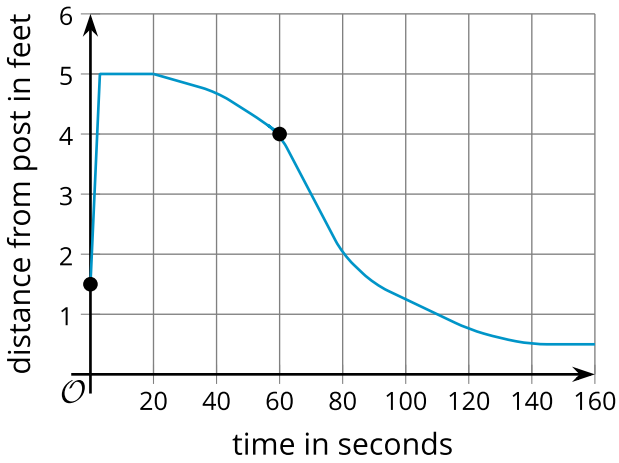
Day 1



Day 2



Day 3



Use **function notation** to complete the table.

	Day 1	Day 2	Day 3
a. distance from post 60 seconds after the owner left			
b. distance from post when the owner left			
c. distance from post 150 seconds after the owner left			

1. Use function notation to represent the dog's distance from the post 60 seconds after the owner left.

a. Enter your answer for Day 1:

b. Enter your answer for Day 2:

c. Enter your answer for Day 3:

2. Use function notation to represent the dog's distance from the post when the owner left.

a. Enter your answer for Day 1:

b. Enter your answer for Day 2:

c. Enter your answer for Day 3:

3. Use function notation to represent the dog's distance from the post 150 seconds after the owner left.

a. Enter your answer for Day 1:

b. Enter your answer for Day 2:

c. Enter your answer for Day 3:

4. Describe what each expression represents in this context:

a. $f(15)$

b. $g(48)$

c. $h(t)$

5. The equation $g(120) = 4$ can be interpreted to mean: "On Day 2, 120 seconds after the dog owner left, the dog was 4 feet from the post." Describe what each equation represents in this context.

a. $h(40) = 4.6$

b. $f(t) = 5$

c. $g(t) = d$

Video: Interpreting Function Notation

Watch the following video to learn more about function notation.

Interpreting Function Notation (https://openstax.org/r/function-notation)

Self Check

A function $f(y) = N$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ represent?

a. In the year 2005, there were 300 more police officers than in 2004.
b. In the year 2005, there were 300 police officers.
c. At year 300, there were 2005 police officers.
d. After 2005 years, there were 300 police officers.

Additional Resources

Understanding Function Notation

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables h for height and a for age. The letters f, g, and h are often used to represent functions, just as we use x, y, and z to represent numbers and A, B, and C to represent sets.

- h is f of a: We name the function f; height is a function of age.
- $h = f(a)$: We use parentheses to indicate the function input.
- $f(a)$: We name the function f; the expression is read as "f of a."

Remember, we can use any letter to name the function; the notation $h(a)$ shows us that h depends on a. The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

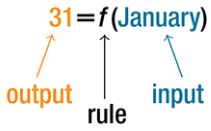
We can also give an algebraic expression as the input to a function. For example, $f(a + b)$ means "first add a and b, and the result is the input for the function f." The operations must be performed in this order to obtain the correct result.

Function Notation

The notation $y = f(x)$ defines a function named f. This is read as "y is a function of x." The letter x represents the input value, or independent variable. The letter y, or $f(x)$, represents the output value, or dependent variable.

EXAMPLE: USING FUNCTION NOTATION FOR DAYS IN A MONTH

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month (don't include leap years).



For example, f(March)$= 31$, because March has 31 days. The notation $d = f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Analysis

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this course will have numbers as inputs and outputs.

TRY IT **Understanding Function Notation**

Use function notation to express the weight of a pig, W, in pounds as a function of its age in days, d.

4.2.3 Finding a Unique Output for Each Input

Activity

Rule G takes a person's name as its input and gives their high school graduation date as the output.

Input	Output
Taylor Swift	2008
Zendaya	2015
Storm Reid	2020
Johnny Orlando	2021

Rule P takes a high school graduation date and gives a person with that graduation date as the output.

Input	Output
2020	Lil Tecca
2021	Olivia Rodrigo
2022	Forrest Wheeler
2023	Miles Brown

1. Complete the table for Rule G with three more examples of input-output pairs.

Your second input-output pair example:

Your third input-output pair example:

2. Complete the table for Rule P with three more examples of input-output pairs

Your second input-output pair example:

Your third input-output pair example:

3. If you use your own name as the input to G, explain how many outputs are possible and how you know.

4. If you use your graduation date as the input to P, explain how many outputs are possible and how you know.

5. Only one of these two relationships, G or P, is a function. The other is not a function. Which one is the function?

6. How do you know the relationship you chose is a function?

7. For the relationship that is a function, write two input-output pairs from the table using function notation.

a. What is the first input-output pair example?

b. What is the second input-output pair example?

Self Check

The price for pizzas, in dollars, is a function of the number of pizzas bought, n.
Identify the input, output, and meaning of the statement: $p(11) = 55$

a. Input: 11
   Output: 55
   55 pizzas cost $11
b. Input: 55
   Output: 11
   55 pizzas cost $11
c. Input: 11
   Output: 55
   11 pizzas cost $55
d. Input: 55
   Output: 11
   11 pizzas cost $55

Additional Resources

Meanings in Functions

Review the following examples to help you gain a better understanding of meanings in functions.

EXAMPLE 1

The number of cubic yards of dirt, D, needed to cover a garden with an area of a square feet is given by $D = g(a)$.

1. A garden with area 5000 ft^2 requires 50 yd^3 of dirt. Express this information in terms of the function g.

2. Explain the meaning of the statement $g(100) = 1$.

EXAMPLE 2

Let $f(t)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:

1. $f(5) = 30$

2. $f(10) = 40$

TRY IT **Meanings in Functions**

The amount of garbage, G, produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people.

1. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.

2. Explain the meaning of the statement $f(5) = 2$.

Are you ready for more?

Extending Your Thinking

1. Write a rule that describes these input-output pairs:

$F(\text{ONE}) = 3$

$F(\text{TWO}) = 3$

$F(\text{THREE}) = 5$

$F(\text{FOUR}) = 4$

What rule could define function F?

2. Here are some input-output pairs with the same inputs but different outputs:

$v(\text{ONE}) = 2$

$v(\text{TWO}) = 1$

$v(\text{THREE}) = 2$

$v(\text{FOUR}) = 2$

What rule could define function v?

4.2.4 Understanding Function Notation Using Application Problems

Cool Down

Function Q gives a puppy's weight, in pounds, as a function of its age, in months.

1. What does each expression or equation represent in this situation?
a. $Q(18)$
b. $Q(30) = 27.5$

2. Use function notation to represent each statement.
a. When the puppy turned 12 months old, it weighed 19.6 pounds.
b. When the puppy was m months old, it weighed w pounds.

4.2.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. The height of water in a bathtub, w, is a function of time, t. Let P represent this function. Height is measured in inches and time in minutes.

What does each equation mean in this situation?

a. $P(0) = 0$

a. The height of the water is 10 inches after 4 minutes.
b. After 10 minutes, the height of the water is 4 inches.
c. The bathtub starts out with no water.
d. After 20 minutes, the bathtub is empty.

b. $P(4) = 10$

a. The height of the water is 10 inches after 4 minutes.
b. After 10 minutes, the height of the water is 4 inches.
c. The bathtub starts out with no water.
d. After 20 minutes, the bathtub is empty.

c. $P(10) = 4$

a. The height of the water is 10 inches after 4 minutes.
b. After 10 minutes, the height of the water is 4 inches.
c. The bathtub starts out with no water.
d. After 20 minutes, the bathtub is empty.

d. $P(20) = 0$

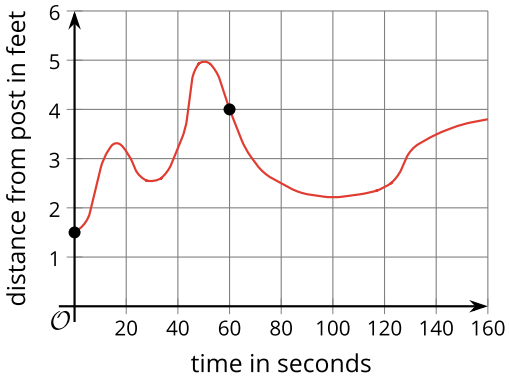
a. The height of the water is 10 inches after 4 minutes.
b. After 10 minutes, the height of the water is 4 inches.
c. The bathtub starts out with no water.
d. After 20 minutes, the bathtub is empty.

2. Function C takes time for its input and gives a student's Monday class for its output.

Use function notation to represent: A student has English at 10:00.

a. C(10:00)=Monday
b. C(Monday)=English at 10:00
c. C(10:00)=English
d. C(English)=10:00

3. Function f gives the distance of a dog from a post, in feet, as a function of time, in seconds, since its owner left.



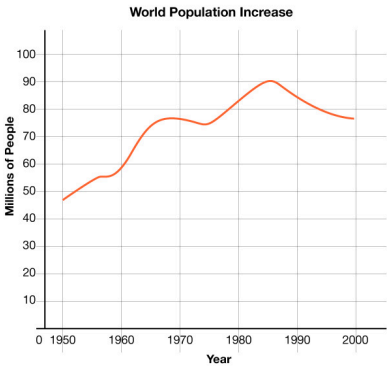
a. Find the value of $f(20)$.

a. 3.5
b. 3.4
c. 3.3
d. 3.2
e. 3.1
f. 3.0

b. Find the value of $f(140)$.

a. 3.5
b. 3.4
c. 3.3
d. 3.2
e. 3.1
f. 3.0

4. Looking at the graph below, what does the point $(1980, 83)$ mean?



a. There were 83 million more people in 1980 than 1979.
b. In 1980, the world population increased by 83 million people.

5. Function R gives the number of inches of rain after h hours. What does the statement $R(6) = 5$ mean?

a. After 6 hours, there was 5 inches of rain.
b. After 5 hours, there was 6 inches of rain.
c. At 6 hours, the rate is 5 inches per hour of rain.

6. Function R gives the number of inches, T, of rain after h, hours. Choose which function notation represents this situation.

a. $T = R(h)$
b. $h = T(R)$
c. $h = R(T)$
d. $R = T(h)$

7. Use function notation to represent that the number of miles, m, a car travels as a function of time, t.

a. $t = f(m)$
b. $t = m(f)$
c. $m = f(t)$
d. $f = m(t)$

4.2.6 Lesson Summary

In this lesson, you learned how to:

- Use function notation to express functions that have specific inputs and outputs.
- Explain what function notation is and why it exists.
- Explain a statement written in function notation in terms of a situation.

Here are the **activities** that helped you reach those goals:

- 4.2.1: Interpreting Graphs to Answer Questions
  - In this activity, you used graphs to answer questions and also interpreted points on graphs to answer questions.
- 4.2.2: Interpreting Statements Written in Function Notation
  - In this activity, you used function notation to represent different situations. You identified the inputs for the functions as they changed.
  - 4.2.2: Self Check
  - 4.2.2: Additional Resources
- 4.2.3: Finding a Unique Output for Each Input
  - In this activity, you identified inputs and outputs given rules. You determined if the inputs and outputs created a function.
  - 4.2.3: Self Check
  - 4.2.3: Additional Resources
- 4.2.4: Understanding Function Notation Using Application Problems
  - In this activity, you read function notation in terms of the situation. You also wrote function notation to represent the given situation.

After these activities, you completed the following **practice**:

- 4.2.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

4.3 INTERPRETING USING FUNCTION NOTATION

4.3.0 Lesson Overview

In this lesson, you will continue to develop your ability to interpret statements in function notation in terms of a situation, including reasoning about inequalities.

When you finish this lesson, you will be able to:

- Describe the connections between a statement in function notation and the graph of the function.
- Use function notation to efficiently represent a relationship between two quantities in a situation.
- Use statements in function notation to sketch a graph of a function.

Here are the **activities** that will help you reach those goals:

- 4.3.1: Comparing Function Values
- 4.3.2: Writing Statements from Function Notation
  - 4.3.2: Self Check
  - 4.3.2: Additional Resources
- 4.3.3: Comparing Statements in Function Notation
  - 4.3.3: Self Check
  - 4.3.3: Additional Resources
- 4.3.4: Using Function Notation with a Real-World Example

After that, you'll **practice and review**.

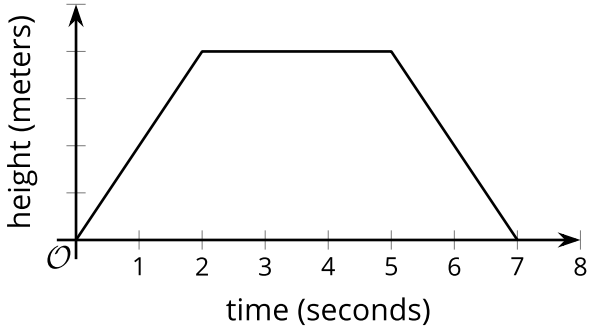
- 4.3.5: Practice
- 4.3.6: Lesson Summary

4.3.1 Comparing Function Values

Warm Up



Here is a graph that represents function f, which gives the height of a drone, in meters, t seconds after it leaves the ground.



1. Decide which function value is greater: $f(0)$ or $f(4)$
2. Explain your thinking as to which is greater, $f(0)$ or $f(4)$, or if they are equal.
3. Decide which function value is greater: $f(2)$ or $f(5)$, or if they are equal.
4. Explain your thinking as to which is greater, $f(2)$ or $f(5)$, or if they are equal.
5. Decide which function value is greater, $f(3)$ or $f(7)$, or if they are equal.
6. Explain your thinking as to which is greater, $f(3)$ or $f(7)$, or if they are equal.
7. Decide which function value is greater, $f(t)$ or $f(t+1)$, or if they are equal.
8. Explain your thinking as to which is greater, $f(t)$ or $f(t+1)$, or if they are equal.

4.3.2 Writing Statements from Function Notation

Activity



The function P gives the number of people, in millions, who own a smartphone, t years after the year 2000. So, any value for which t is negative would represent the number of years before 2000.

1. What does this equation tell us about smartphone ownership?

$P(17) = 2320$

2. What does this equation tell us about smartphone ownership?

$P(-10) = 0$

3. Use function notation to represent this statement.

In 2010, the number of people who owned a smartphone was 296,600,000.

4. Use function notation to represent this statement.

In 2015, about 1.86 billion people owned a smartphone.

5. Mai is curious about the value of t in $P(t) = 1000$.

What would the value of t tell Mai about the situation?

6. Mai is curious about the value of t in $P(t) = 1000$.

Is 4 a possible value of t here?

7. Use your own graph paper to sketch a graph of the function. Your graph should go through these four points: $(-10, 0)$, $(10, 296.6)$, $(15, 1860)$, and $(17, 2320)$.

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-3-2-writing-statements-from-function-notation)

Are you ready for more?

Extending Your Thinking

What can you say about the value or values of t when $P(t) = 1000$?

Video: Understanding Equations in Function Notation

Watch the following video to learn more about reading equations in function notation.

Understanding Equations in Function Notation (https://openstax.org/r/equations-function-notation)

Self Check

The function Q gives the number of people, in millions, who owned a smart watch after 2015. What does the equation $Q(6) = 7.25$ mean?

a. In 2021, 7.25 million more people owned smart watches than in 2015.
b. In 2021, 7.25 million people owned smart watches.
c. In the beginning of 2023, 6 million people owned smart watches.
d. In 2006, 7.25 million people owned smart watches.

Additional Resources

Interpreting Equations in Function Notation

What does a statement like $p(3) = 12$ mean?

On its own, $p(3) = 12$ only tells us that when p takes 3 as its input, its output is 12.

If we know what quantities the **input** and **output** represent, however, we can learn much more about the situation that the function represents.

If function p gives the perimeter of a square whose side length is x and both measurements are in inches, then we can interpret $p(3) = 12$ to mean "a square whose side length is 3 inches has a perimeter of 12 inches."

We can also interpret statements like $p(x) = 32$ to mean "a square with side length x has a perimeter of 32 inches," which then allows us to reason that x must be 8 inches and to write $p(8) = 32$.

If function p gives the number of blog subscribers, in thousands, x months after a blogger started publishing online, then $p(3) = 12$ means "3 months after a blogger started publishing online, the blog has 12,000 subscribers."

It is important to pay attention to the units of measurement when analyzing a function. Otherwise, we might mistake what is happening in the situation. If we miss that $p(x)$ is measured in thousands, we might misinterpret $p(36)$ to mean "there are 36 blog subscribers after x months," while it actually means "after 36 months, there are x thousands of blog subscribers."

TRY IT **Interpret Equations in Function Notation**

If a function A gives the area of a square, with a side length x centimeters, then what does the

function $A(8) = 64$ mean?

4.3.3 Comparing Statements in Function Notation

Activity

The function W gives the temperature, in degrees Fahrenheit, of a pot of water on a stove, t minutes after the stove is turned on.

Take turns with your partner to explain the meaning of each statement in this situation. When it's your partner's turn, listen carefully to their interpretation. If you disagree, discuss your thinking and work to reach an agreement.

1. What did you and your partner agree is the meaning of $W(0) = 72$
2. What did you and your partner agree is the meaning of $W(5) > W(2)$
3. What did you and your partner agree is the meaning of $W(10) = 212$
4. What did you and your partner agree is the meaning of $W(12) = W(10)$
5. What did you and your partner agree is the meaning of $W(15) > W(30)$
6. What did you and your partner agree is the meaning of $W(0) < W(30)$
7. Sketch a possible graph of function W using the informaiton in the previous questions. If all statements in the previous question represent the situation, sketch a possible graph of function W.

Building Character: Creativity



Creativity is the ability to think of new solutions. When you approach life with a creative attitude, you are more open to new ideas and possibilities, better able to understand others' perspectives, and more likely to seek out multiple solutions to problems.

Think about your current sense of creativity. Are the following statements true for you?

- I often seek out new experiences and ways of doing things.
- I like to think of different ways to reach my goals.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

Self Check

The function C gives the temperature, in degrees Fahrenheit, of a pot of coffee x minutes after it is made.

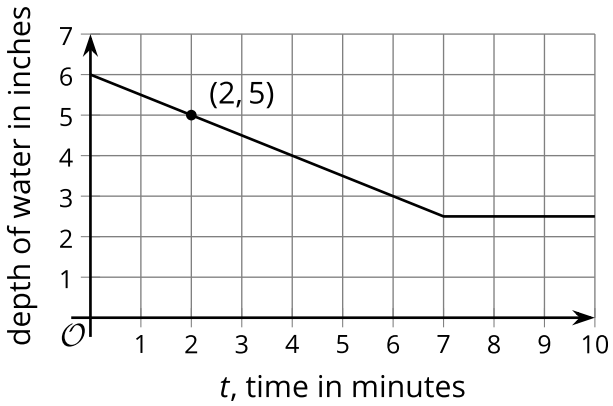
What does the statement $C(10) < C(5)$ mean in the context of the problem?

a. The temperature of coffee after 10 hours was cooler than after 5 hours.
b. The temperature of the coffee after 10 minutes was warmer than after 5 minutes.
c. The temperature of the coffee after 10 minutes was cooler than after 5 minutes.
d. The coffee pot has less coffee in it after 10 minutes than after 5 minutes.

Additional Resources

Comparing Outputs

Function f gives the depth, in inches, of water in a tub as a function of time, t, in minutes, since the tub started being drained.



Each point on the graph has the coordinates $(t, f(t))$, where the first value is the input of the function and the second value is the output.

- $f(2)$ represents the depth of water 2 minutes after the tub started being drained. The graph passes through $(2, 5)$, so the depth of water is 5 inches when $t = 2$. The equation $f(2) = 5$ captures this information.
- $f(0)$ gives the depth of the water when the draining began, at $t = 0$. The graph shows the depth of water to be 6 inches at that time, so we can write $f(0) = 6$.
- $f(t) = 3$ tells us that t minutes after the tub started draining, the depth of the water is 3 inches. The graph shows that this happens when t is 6.
- In the graph, $f(2) > f(7)$.
- $f(7) = f(10)$
- $f(4) > f(6)$
- $f(0) = 6$

TRY IT **Comparing Outputs**

Function P gives the depth of the pool water after t hours of it being refilled.

What does the statement $P(12) < P(4)$ mean in the context of the situation?

4.3.4 Using Function Notation with a Real-World Example

Cool Down

An art museum opens at 9 a.m. and closes at 5 p.m. The function V gives the number of visitors in the museum h hours

after it opens.

1. Explain what this statement tells us about the situation: $V(1.25) = 28$.

2. Use function notation to represent each statement:
   a. At 1 p.m., there were 257 visitors in the museum.
   b. At the time of closing, there were no visitors in the museum.

3. Use the previous statements about the visitors in the museum to sketch a graph that could represent the function.

4.3.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Function f gives the temperature, in degrees Celsius, t hours after midnight.

Choose which equation or expression represents the statement in function notation.

a. The temperature was the same at 9 a.m. and at 4 p.m.

a. $f(9) = f(16)$
b. $f(9) = f(4)$

b. It was warmer at 9 a.m. than at 6 a.m.

a. $f(9) < f(6)$
b. $f(9) > f(6)$

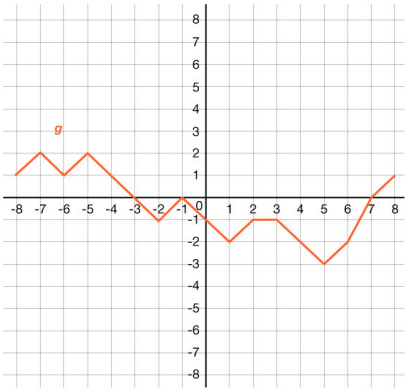
c. Some time after midnight, the temperature was 24 degrees Celsius.

a. $f(12) = 24$
b. $f(t) = 24$
c. $f(0) < 24$
d. $f(24) = t$

2. Select the **two** points that are on the graph of f if we know that $f(2) = -4$ and $f(5) = 3.4$.

a. (–4, 2)
b. (2, –4)
c. (3.4, 5)
d. (5, 3.4)
e. (2, 5)

3. Looking at the graph of the function g, below, which is greater, $g(-5)$ or $g(5)$?



a. $g(-5)$
b. $g(5)$

4. The function T tells how many people are living on floor n of an apartment building. What does $T(4) = 82$ mean?

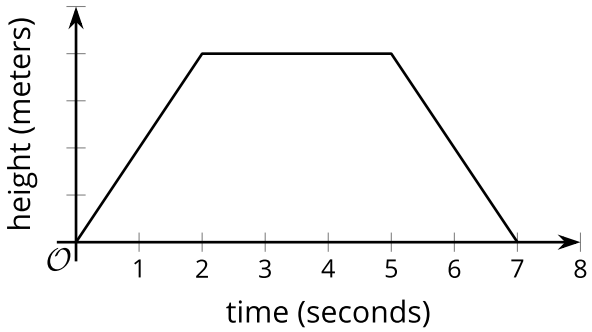
a. There are 4 people living on floor 82.
b. There are 82 people living on floor 4.

5. The function, T, gives the temperature of a house, in degrees Fahrenheit, h hours after midnight. Choose which function notation represents the temperature at 6 a.m. was 70 degrees.

a. $70(T) = 6$
b. $6(T) = 70$
c. $T(6) = 70$

d. $T(70) = 6$

6. The graph below represents a function, f, which gives the height of a drone, in meters, t seconds after it leaves the ground. Which statement describes the point $f(2) = 4$?



a. The height of the drone after 4 seconds is 2 meters off the ground.
b. The height of the drone after 2 seconds is 4 meters off the ground.

4.3.6 Lesson Summary

In this lesson, you learned how to:

- Describe the connections between a statement in function notation and the graph of the function.
- Use function notation to efficiently represent a relationship between two quantities in a situation.
- Use statements in function notation to sketch a graph of a function.

Here are the **activities** that helped you reach those goals:

- 4.3.1: Comparing Function Values
  - In this activity, you compared function values given function notation and a graph.
- 4.3.2: Writing Statements from Function Notation
  - In this activity, you interpreted the meaning of function notation in a given situation and wrote a statement describing the situation.
  - 4.3.2: Self Check
  - 4.3.2: Additional Resources
- 4.3.3: Comparing Statements in Function Notation
  - In this activity, you interpreted equations and inequalities involving function notation to compare which values were equal, larger, or smaller.
  - 4.3.3: Self Check
  - 4.3.3: Additional Resources
- 4.3.4: Using Function Notation with a Real-World Example
  - In this activity, you looked at a real-world example of function notation where time was the input and the number of visitors to a website was the output.

After these activities, you completed the following **practice**:

- 4.3.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

4.4 USING FUNCTION NOTATION TO DESCRIBE RULES, PART 1

4.4.0 Lesson Overview

In this lesson, you will use function notation to describe the rule of a function or how a function behaves at any value of input. In addition, you will represent situations symbolically to solve problems.

When you finish this lesson, you will be able to:

- Explain rules of functions when they are written in function notation and create tables and graphs to represent the functions.
- Write equations that represent the rules of functions.

Here are the **activities** that will help you reach those goals:

- 4.4.1: Using Function Notation in Tables of Values
- 4.4.2: Defining Functions by a Rule
  - 4.4.2: Self Check
  - 4.4.2: Additional Resources
- 4.4.3: Using Tables to Find Function Rules
  - 4.4.3: Self Check
  - 4.4.3: Additional Resources
- 4.4.4: Writing a Rule Using Function Notation

After that, you'll **practice and review**.

- 4.4.5: Practice
- 4.4.6: Lesson Summary

4.4.1 Using Function Notation in Tables of Values

Warm Up

Use the tables to answer each question.

x	$f(x) = 10 - 2x$
1	8
1.5	7
5	0
–2	14

x	$g(x) = x^3$
–2	–8
0	0

x	$g(x) = x^3$
1	1
3	27

1. What do you notice?
2. What do you wonder?

4.4.2 Defining Functions by a Rule

Activity

For questions 1 – 4, match each equation with a verbal description that represents the same function.

1. $f(x) = 3x - 7$

a. Subtract 7 from the input, then divide the result by 3.

b. Subtract 7 from the input, then multiply the result by 3.

c. Multiply the input by 3, then subtract 7 from the result.

d. Divide the input by 3, and then subtract 7 from the result.

2. $g(x) = 3(x - 7)$

a. Subtract 7 from the input, then divide the result by 3.

b. Subtract 7 from the input, then multiply the result by 3.

c. Multiply the input by 3, then subtract 7 from the result.

d. Divide the input by 3, and then subtract 7 from the result.

3. $h(x) = \frac{x}{3} - 7$

a. Subtract 7 from the input, then divide the result by 3.

b. Subtract 7 from the input, then multiply the result by 3.

c. Multiply the input by 3, then subtract 7 from the result.

d. Divide the input by 3, and then subtract 7 from the result.

4. $h(x) = \frac{x-7}{3}$

a. Subtract 7 from the input, then divide the result by 3.

b. Subtract 7 from the input, then multiply the result by 3.

c. Multiply the input by 3, then subtract 7 from the result.

d. Divide the input by 3, and then subtract 7 from the result.

5. For one of the functions, when the input is 6, the output is –3. Which is that function: f, g, h, or k?

6. Explain how you knew the answer to question 5.

7. Which function value, f, g, h, or k is the greatest when the input is 0.

8. Which function value, f, g, h, or k is the greatest when the input is 10.

Are you ready for more?

Extending Your Thinking

Mai says $f(x)$ is always greater than $g(x)$ for the same value of x. Is this true? Explain how you know.

Self Check

Which function has the least value when $x = 0$?

a. $f(x) = -4x + 5$
b. $f(x) = 2(x - 1)$
c. $f(x) = 3x$
d. $f(x) = 2x + 1$

Additional Resources

Evaluating Functions with Function Notation

Find the Value of a Function

It is very convenient to name a function, and most often we name it f, g, or h. In any function, for each x-value from the domain we get a corresponding y-value in the range. For the function f, we write this range value y as $f(x)$. This is called function notation, and is read f of x or the value of f at x. In this case, the parentheses do not indicate multiplication.

Function Notation

For the function $y = f(x)$:

- f is the name of the function.
- x is the input value.
- $f(x)$ is the output value, y, corresponding to the value of x.

We read $f(x)$ as f of x or the value of f at x.

We call x the independent variable as it can be any value in the domain. We call y the dependent variable as its value depends on x.

Independent and Dependent Variables

For the function $y = f(x)$,

x is the independent variable as it can be any value.

y is the dependent variable as its value depends on x.

Much as when you first encountered the variable x, function notation may be rather unsettling. It seems strange because it is new. You will feel more comfortable with the notation as you use it.

Let's look at the equation $y = 4x - 5$. To find the value of y when $x = 2$, we know to substitute $x = 2$ into the equation and then simplify.

Original Equation	$y = 4x - 5$
Substitute $x = 2$ in for x.	$y = 4(2) - 5$
Simplify.	$y = 3$

The value of the function at $x = 2$ is 3.

We can do the same thing using function notation. The equation $y = 4x - 5$ can be written as $f(x) = 4x - 5$. To find the value when $x = 2$, we write:

Original Equation	$f(x) = 4x - 5$
Substitute $x = 2$ in for x.	$f(2) = 4(2) - 5$
Simplify.	$f(2) = 3$

The value of the function at $x = 2$ is 3.

This process of finding the value of $f(x)$ for a given value of x is called evaluating the function.

TRY IT **Evaluating Functions with Function Notation**
Find the value of $f(x) = 5x + 2$ when $x = 3$.

4.4.3 Using Tables to Find Function Rules

Activity

1. A square that has a side length of 9 cm has an area of 81 cm^2. The relationship between the side length and the area of the square is a function.

For questions a-e use the table by answering each question.

Side Length (cm)	**Area** (cm^2)
1	a. _____
2	b. _____
4	c. _____
6	d. _____
s	e. _____

a. What is the area in cm^2 when the side length is 1cm?

b. What is the area in cm^2 when the side length is 2cm?

c. What is the area in cm^2 when the side length is 4cm?

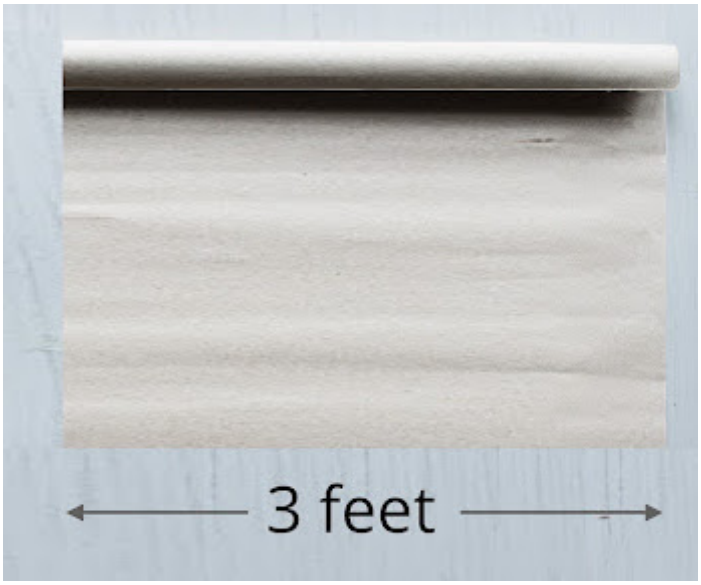
d. What is the area in cm^2 when the side length is 6cm?

e. Write a rule for a function, A, that gives the area of the square in cm^2 when the side length is s cm.

f. What does $A(2)$ represent in this situation? What is its value? Write your answers, then click **solution** to compare your answer.

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-4-3-using-tables-to-find-function-rules)
g. Sketch a graph of this function. Use the desmos graphing tool or technology outside the course.

2. A roll of paper that is 3 feet wide can be cut to any length.



a. If we cut a length of 2.5 feet, what is the perimeter of the paper? Enter your answer in feet.**Use integers only.**

For questions b-f, keep in mind that the roll of paper is 3 feet wide. Complete the table by answering each question.

side length (feet)	perimeter (feet)
1	b. _____
2	c. _____
6.3	d. _____
11	f. _____

b. If we cut a length of 1 foot, what is the perimeter of the paper to the nearest whole foot?

c. To the nearest whole foot, what is the perimeter when the side length is 2 feet?

d. What is the exact perimeter, in feet, when the side length is 6.3 feet?

e. What is the perimeter, to the nearest whole foot,when the side length is 11 feet?

f. Write a rule for a function, P, that gives the perimeter of the paper in feet when the side length in feet is **l**.

g. What does P(11) represent in this situation? What is its value?

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-4-3-using-tables-to-find-function-rules)
h. Sketch a graph of this function. Use the desmos graphing tool or technology outside the course.

Video: Writing Function Rules

Watch the following video to learn more about writing rules with function notation.

Writing Function Rules (https://openstax.org/r/function-rules)

Self Check

Which function rule can describe the table below?

x	y
1	3
2	5
5	11

a. $f(x) = x + 2$
b. $f(x) = 2x + 2$
c. $f(x) = 2x + 1$
d. $f(x) = x + 3$

Additional Resources

Write the Function Rule from a Table

Kelly can tune 6 cars in 3 hours. The table below is completed to show this relationship. The last column has the output for any x. This is the rule of the function.

Time spent tuning cars (x)	2	3	6	7	x
Number of cars tuned up (y)	4	6	12	14	$2x$

Kelly can tune 6 cars in 3 hours or 2 cars in 1 hour, so the function rule is $f(x) = 2x$.

TRY IT **Write the Function Rule from a Table**

Determine the rule in function notation for the table below:

x (input)	1	2	3	4	x
y (output)	5	8	11	14	?

4.4.4 Writing a Rule Using Function Notation

Cool Down

1. Complete the table with the perimeter of a square for each given side length.

side length (inches)	perimeter (inches)
0.5	a. _____
7	b. _____
20	c. _____

a. _____
b. _____
c. _____

2. Write a rule for a function, P, that gives the perimeter of a square in inches when the side length is x inches.

3. What is the value of $P(9.1)$? What does it tell us about the side length and perimeter of the square?

4.4.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Match each equation with a verbal description that represents the same function.

a. $f(x) = 2x + 4$

a. To get the output, multiply the input by 4, then add 2 to the result.
b. To get the output, multiply the input by 2, then add 4 to the result.
c. To get the output, add 2 to the input, then multiply the result by 4.
d. To get the output, add 4 to the input, then multiply the result by 2.

b. $g(x) = 2(x + 4)$

a. To get the output, multiply the input by 4, then add 2 to the result.
b. To get the output, multiply the input by 2, then add 4 to the result.
c. To get the output, add 2 to the input, then multiply the result by 4.
d. To get the output, add 4 to the input, then multiply the result by 2.

c. $h(x) = 4x + 2$

a. To get the output, multiply the input by 4, then add 2 to the result.
b. To get the output, multiply the input by 2, then add 4 to the result.
c. To get the output, add 2 to the input, then multiply the result by 4.
d. To get the output, add 4 to the input, then multiply the result by 2.

d. $k(x) = 4(x + 2)$

a. To get the output, multiply the input by 4, then add 2 to the result.
b. To get the output, multiply the input by 2, then add 4 to the result.
c. To get the output, add 2 to the input, then multiply the result by 4.
d. To get the output, add 2 to the input, then multiply the result by 2.

2. Functions A and B are defined by these equations.

$A(x) = 25 + 10x$
$B(x) = 80 - 15x$

Which function has a greater value when x is 2.5?

a. $A(x) = 25 + 10x$
b. $B(x) = 80 - 15x$

3. Which function is greater when $x = 4$?

a. $f(x) = x - 10$
b. $f(x) = -5x + 2$
c. $f(x) = -3x + 1$
d. $f(x) = 2x - 7$

4. Which function rule describes the table below:

x	2	3	4	5	6
y	3	5	7	9	11

a. $f(x) = x + 2$
b. $f(x) = 2x - 1$
c. $f(x) = 3x - 3$
d. $f(x) = x + 1$

5. Write a function $g(x)$ for the following statement: to get the output, subtract 4 from the input, then multiply the result by 2.

a. $g(x) = 2(x - 4)$
b. $g(x) = x - 8$
c. $g(x) = -4 - 2x$
d. $g(x) = 2x - 4$

6. Write a function $h(x)$ for the following statement: to get the output, multiply the input by 5 and then add 2.

a. $h(x) = x + 2$
b. $h(x) = x + 2(5)$
c. $h(x) = 5x + 2$
d. $h(x) = 5(x + 2)$

7. Find the value of $f(x) = -2x - 3$ when $x = 2$.

a. –7
b. –1
c. 1
d. 7

8. Examine the table below. Find $g(2.5)$ using the rule for this table.

x	1	2	3	4
$g(x)$	2	6	10	14

4.4.6 Lesson Summary

In this lesson, you learned how to:

- Explain rules of functions when they are written in function notation and create tables and graphs to represent the functions.
- Write equations that represent the rules of functions.

Here are the **activities** that helped you reach those goals:

- 4.4.1: Using Function Notation in Tables of Values
  - In this activity, you made observations about tables of values that incorporated function notation.
- 4.4.2: Defining Functions by a Rule
  - In this activity, you matched equations that used function notation to their descriptions. You also compared output values when the same input was entered into different equations.
  - 4.4.2: Self Check
  - 4.4.2: Additional Resources
- 4.4.3: Using Tables to Find Function Rules
  - In this activity, you worked to find function rules for a table of values. You also determined the meaning of values given function notation in the context of the problem. You then sketched a graph of the function.
  - 4.4.3: Self Check
  - 4.4.3: Additional Resources
- 4.4.4: Writing a Rule Using Function Notation
  - In this activity, you wrote a rule from a table of values in function notation. You then used that rule to find other values of the function.

After these activities, you completed the following **practice**:

- 4.4.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

4.5 USING FUNCTION NOTATION TO DESCRIBE RULES, PART 2

4.5.0 Lesson Overview

In this lesson, you will use rules of functions to find the output when the input is given (or to evaluate functions) and to find the input when the output is known (or to solve equations that define functions). To make connections between different representations of functions, you will interpret rules of functions by comparing cell phone data plans to choose the most cost-effective plan.

When you finish this lesson, you will be able to:

- Use technology to graph a function given in function notation, and use the graph to find the values of the function.
- Be able to identify if a graph represents a function through the vertical line test.
- Explain different ways to find the value of a function and to solve equations written in function notation.
- Explain what makes a function a linear function.

Here are the **activities** that will help you reach those goals:

- 4.5.1: Finding Solutions of Equations
- 4.5.2: Using the Vertical Line Test
  - 4.5.2: Self Check
  - 4.5.2: Additional Resources
- 4.5.3: Comparing Functions
  - 4.5.3: Self Check
  - 4.5.3: Additional Resources
- 4.5.4: Using Function Notation and Graphing Technology
  - 4.5.4: Self Check
  - 4.5.4: Additional Resources
- 4.5.5: Using Graphical and Algebraic Approaches for Finding Unknowns of Linear Functions

After that, you'll **practice and review**.

- 4.5.6: Practice
- 4.5.7: Lesson Summary

4.5.1 Finding Solutions of Equations

Warm Up

Consider the equation $q = 4 + 0.8p$.

1. What value of q would make the equation true when p is 7?
2. What value of q would make the equation true when p is 100?
3. What value of p would make the equation true when q is 12?
4. What value of p would make the equation true when q is 60?

4.5.2 Using the Vertical Line Test

Activity

A table for the relation f is below.

x	1	2	3	1	4	5
$f(x)$	-1	5	-2	1	3	-2

1. Is this relation a function? Explain your answer.

2. Use the Desmos graphing tool or technology outside the course. Graph the points in the relation.

Interact with the graph. Graph the points in the relation on a coordinate grid.

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-5-2-using-the-vertical-line-test)

3. What do you notice about the points $(1, -1)$ and $(1, 1)$?

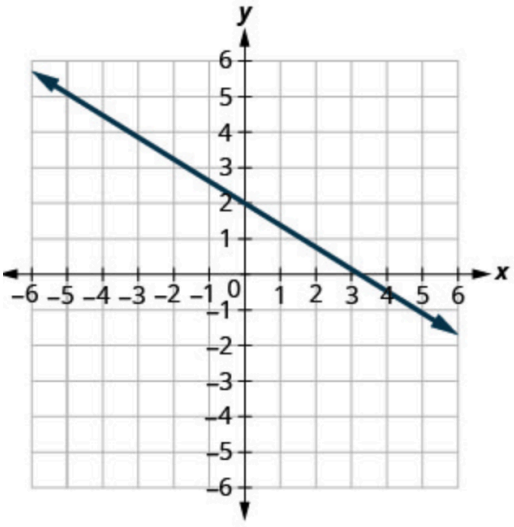
VERTICAL LINE TEST

A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point.

If any vertical line intersects the graph in more than one point, the graph does not represent a function.

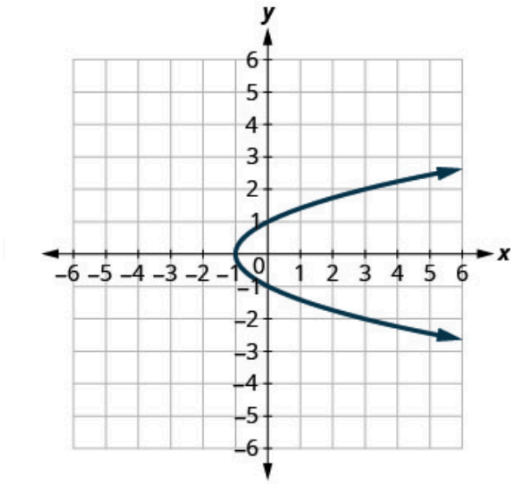
If the graph does represent a function, we say it "passes the vertical line test."

4. For the function graphed below, determine if the graph passes the vertical line test and then explain how this identifies the graph as a function or not.



(a)

5. For the function graphed below, determine if the graph passes the vertical line test and then explain how this identifies the graph as a function or not.



(b)

Notice that the graph of a non-vertical line will always be a function. This is called a **linear function**.

Video: Interpreting Function Notation

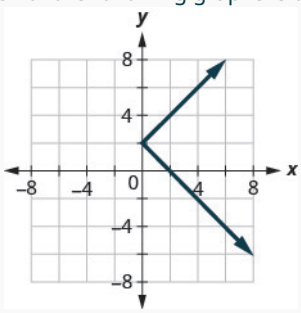
Watch the following video to learn more about function notation.

Using the Vertical Line Test (https://openstax.org/r/vertical-line-test)

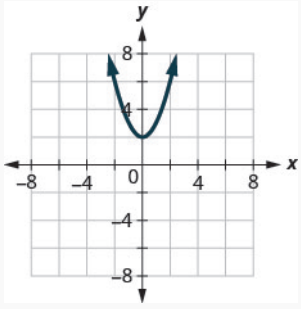
Self Check

Which of the following graphs is the graph of a function?

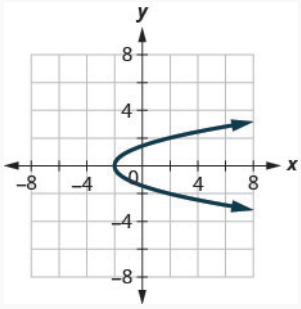
a.



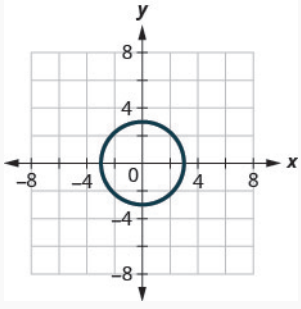
b.



c.

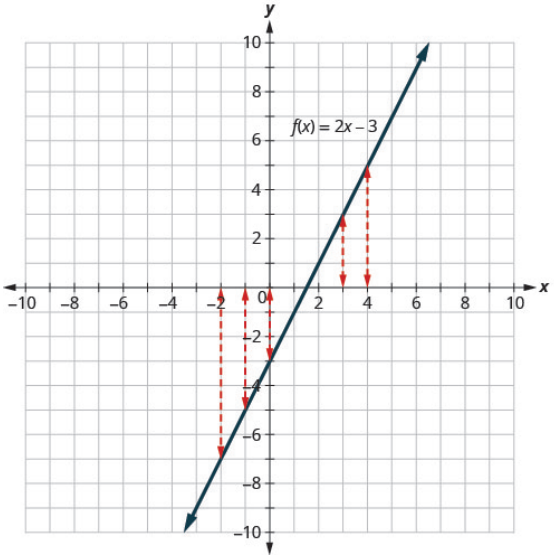


d.



Additional Resources

Determine If a Graph Is a Function



$f(x) = 2x - 3$		
x	$f(x)$	$(x, f(x))$
–2	–7	(–2, –7)
–1	–5	(–1, –5)
0	–3	(0, –3)
3	3	(3, 3)
4	5	(4, 5)

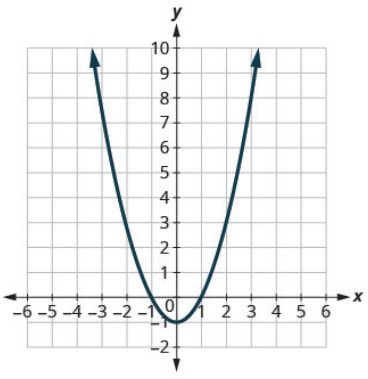
A relation is a function if every input has exactly one output value. So the relation defined by the equation $y = 2x - 3$ is a function. Notice that a line will always be a function, so it is called a linear function.

If we look at the graph, each vertical dashed line intersects the line at only one point. This makes sense because in a function, for every x-value there is only one y-value.

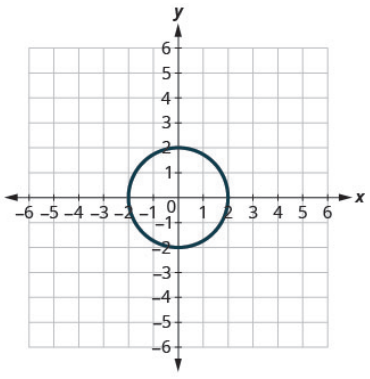
If the vertical line hit the graph twice, the x-value would be mapped to two y-values, so the graph would not represent a function.

This leads us to the vertical line test. A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. If any vertical line intersects the graph in more than one point, the graph does not represent a function.

Look at the graphs below:



(a)



(b)

Graph a passes the vertical line test because if an imaginary vertical line were drawn anywhere on the graph, it would only touch one time.

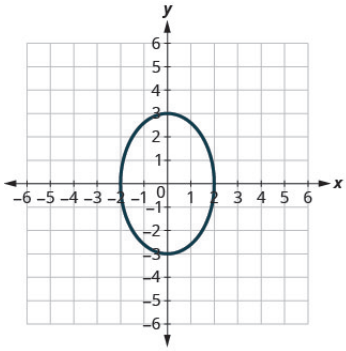
Graph a is a function.

Graph b does not pass the vertical line test because when a vertical line is drawn down the graph, there is at least one place where the vertical line touches twice.

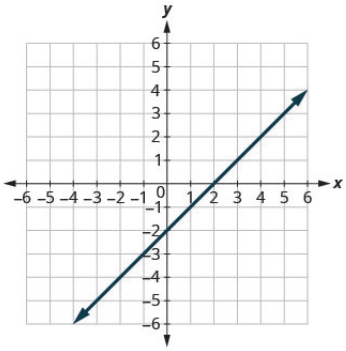
Graph b is not a function.

TRY IT **Determine if a Graph Is a Function**

Determine if each graph is a function. Use the vertical line test to explain.



(a)



(b)

4.5.3 Comparing Functions

Activity

A college student is choosing between two data plans for her new cell phone. Both plans include an allowance of 2 gigabytes of data per month. The monthly cost of each option can be seen as a function and represented with an equation:

- Option A: $A(x) = 60$
- Option B: $B(x) = 10x + 25$

In each function, the input, x, represents the gigabytes of data used **over** the monthly allowance.

The student decides to find the cost of each option when she exceeds the 2 gigabytes of data allowed each month. Use the function rules to help her compare the data plans.

1. What is the value of A(1)

2. What is the value of B(1)

3. Explain which data plan seems to be the best option for when 1 additional gigabyte of data is used.

4. What is the value of $A(7.5)$?

5. What is the value of $B(7.5)$?

6. Explain which data plan seems to be the best option for when 7.5 additional gigabytes of data are used.

7. Describe option A, $A(x) = 60$, in words.

8. Describe option B, $B(x) = 10x + 25$, in words.

9. Use the desmos graphing tool or technology outside the course. Graph each function on the same coordinate plane.

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-5-3-comparing-functions)

10. Explain which plan you think she should choose.

11. The student only budgeted $50 a month for her cell phone. She thought, "I wonder how many gigabytes of data I would have for $50 if I go with Option B?" and wrote $B(x) = 50$. How many additional gigabytes can the student use beyond the 2 allotted by the plan?

12. Explain how you knew how many gigabytes of data she would have for $50.

Are you ready for more?

Extending Your Thinking

Describe a different data plan that, for any amount of data used, would cost no more than one of the given plans and no less than the other given plan. Explain or show how you know this data plan would meet these requirements.

Self Check

Rental car company A charges \$50 a day plus \$0.50 a mile, m, and is modeled by function A.

a. Company A is always more than Company B.
b. $A(100) < B(100)$
c. $A(100) = B(100)$
d. $A(100) > B(100)$

Additional Resources

Comparing Function Values

Functions are used in a variety of real-life situations. They are useful in making comparisons, especially when comparing the cost.

EXAMPLE

The daily cost to printing company A to print a book is modeled by a function $C(x) = 3.25x + 1500$, where C is the total daily cost, in dollars, and x is the number of books printed.

Their competitor, company B, can print a book at a cost modeled by $C(x) = 2.75x + 2500$.

Find $C(1000)$ for each company and make a statement about the result.

Company A: $C(1000) = 3.25(1000) + 1500$

$C = 4750$

Company B: $C(1000) = 2.75(1000) + 2500$

$C = 5250$

When 1,000 books are printed, Company A has the better price.

TRY IT **Comparing Function Values**

The daily cost to manufacturing company A is modeled by the function$C(x) = 7.25x + 2500$, where $C(x)$ is the total daily cost and x is the number of items manufactured.

The daily cost to manufacturing company B is modeled by the function $C(x) = 8.5x + 1500$.

Which company is less at $C(1000)$? Explain the meaning of the result.

4.5.4 Using Function Notation and Graphing Technology

Activity

The function B is defined by the equation $B(x) = 10x + 25$. Use the graphing tool here or technology outside the course. Graph the function.

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-5-4-using-function-notation-and-graphing-technology)

If needed, remember -

- You can collapse the equation entry tab by selecting the button.
- To change the scale along either the x- or y-axis, click on the wrench button and then enter minimum and maximum values for each axis and set the step value.
- To find a point on the line, click on the line and drag the mouse along the line to display the ordered pairs for each location.

1. What is the value of $B(6)$?

2. What were the window values you used for the x-axis? (Write the values as an inequality.)

3. What were the window values you used for the y-axis? (Write the values as an inequality.)

4. What is the value of $B(2.75)$?

5. What is the value of $B(1.482)$?

6. What is the value of x if $B(x) = 93$?

7. What were the window values you used for the x-axis? (Write the values as an inequality.)

8. What were the window values you used for the y-axis? (Write the values as an inequality.)

9. What is the value of x if $B(x) = 42.1$?

10 What is the value of x if $B(x) = 116.25$?

Self Check

The function D is defined by the equation $D(x) = 2.5x + 12$.

Which describes the window that can be used to find the value of x when $D(x) = 23.25$ using graphing technology and gives the correct value of x?

a. Graph $y = 2.5x + 12$, $y = 23.25$; window: x from 20 to 25, y from 0 to 10; $x = 4.5$
b. Graph $y = 2.5x$, $y = 23.25$; window: x from 0 to 10, y from 20 to 25; $x = 9.3$
c. Graph $y = 2.5x + 12$, $y = 23.25$; window: x from 0 to 10, y from 20 to 25; $x = 4.5$
d. Graph $y = 2.5x + 12$, $x = 23.25$; window: x from 0 to 30, y from 0 to 100; $x = 70.125$

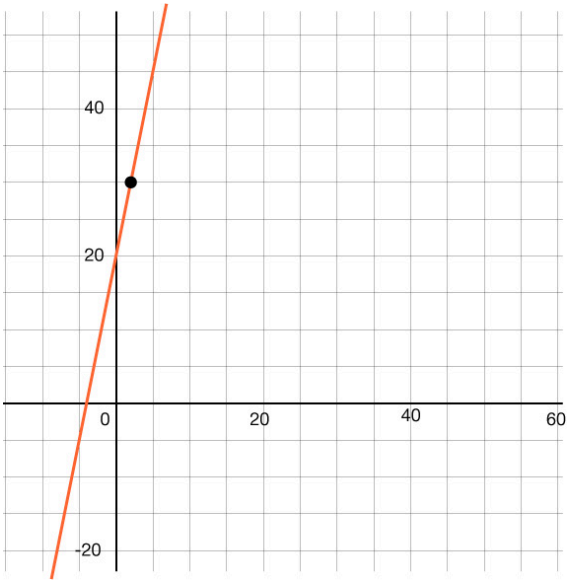
Additional Resources

Finding Missing Values

Function G is defined by $G(x) = 5x + 20$. Find the following values using graphing technology.

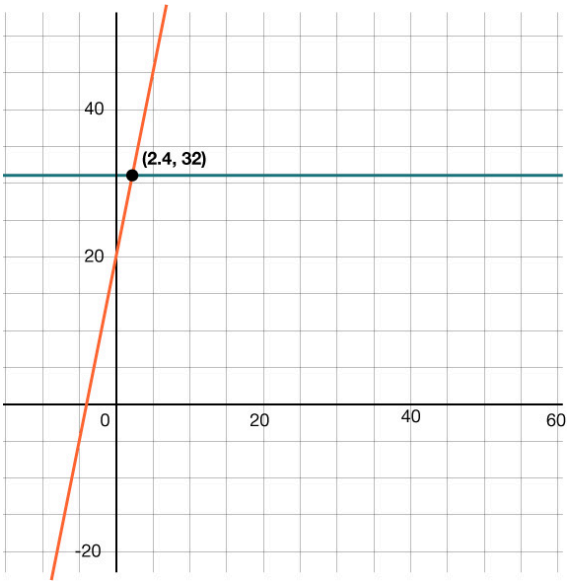
a. $G(2)$

To find $G(2)$, look at $x = 2$ on the x-axis on the graph of $G(x)$. Use proper steps with your graphing technology to find the y-value on the line: $G(2) = 30$.



b. $G(x) = 32$

To find the value of x where $G(x) = 32$, look at where the graph crosses the y-value of 32. Use proper steps with your graphing technology to find the x-value where the graph meets $y = 32$.



$G(2.4) = 32$

TRY IT **Finding Missing Values**

Function H is defined by $H(x) = 2x + 4$. Find $H(x) = 10.4$ using graphing technology.

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-5-4-using-function-notation-and-graphing-technology)

Use the graphing tool here or technology outside the course.

4.5.5 Using Graphical and Algebraic Approaches for Finding Unknowns of Linear Functions

Cool Down

The college student who is looking for a data plan for her cell phone found a third option, which also offers a 2-gigabyte monthly allowance for data.

The monthly cost of this option can be represented by function C, defined by $C(x) = 30x + 5$, where x is the gigabytes of data used over the monthly allowance.

1. Find $C(0.7)$ and explain what it means in this situation.
2. Her budget is still $50 a month. To find out how many gigabytes of data she could use if she chose this plan, she writes $C(x) = 50$ and solves for x. What value of x makes this equation true? Be prepared to show your reasoning.

4.5.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. The total monthly cost of a cell phone plan from Company C costs \$10, plus \$15 per gigabyte for data used. The total monthly cost from Company D costs \$80 with unlimited data.

Rule C gives the monthly cost, in dollars, of using g gigabytes of data on Company C's plan. Rule D gives the monthly cost, in dollars, of using g gigabytes of data on Company D's plan.

a. Which sentence describes the meaning of the statement $C(2) = 40$?

a. It costs \$2 to use 40 gigabytes of data using plan D.
b. It costs \$2 to use 40 gigabytes of data using plan C.
c. It costs \$40 to use 2 gigabytes of data using plan D.
d. It costs \$40 to use 2 gigabytes of data using plan C.

b. Which is less, $C(4)$ or $D(4)$?

a. $C(4)$
b. $D(4)$

c. Which is less, $C(5)$ or $D(5)$?

a. $C(5)$
b. $D(5)$

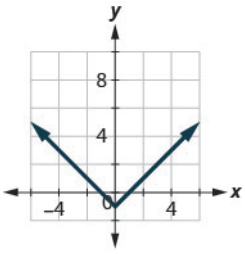
d. For what number g is $C(g) = 130$?

2. Function G takes a student's first name for its input and gives the number of letters in the first name for its output.

a. Find the value of $G(\text{Jada})$.

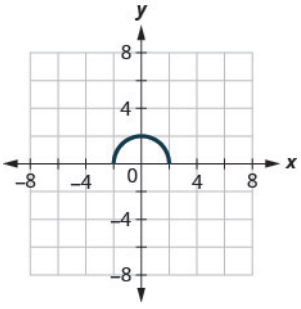
b. Find the value of $G(\text{Diego})$.

3. Is the following graph a function?



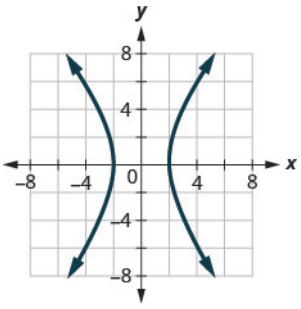
a. Yes
b. No

4. Is the following graph a function?



a. Yes
b. No

5. Is the following graph a function?



a. Yes
b. No

4.5.7 Lesson Summary

In this lesson, you learned how to:

- Use technology to graph a function given in function notation, and use the graph to find the values of the function.
- Be able to identify if a graph represents a function through the vertical line test.
- Explain different ways to find the value of a function and to solve equations written in function notation.
- Explain what makes a function a linear function.

Here are the **activities** that helped you reach those goals:

- 4.5.1: Finding Solutions of Equations
  - In this activity, you evaluated functions at a specific input value to find the output.
- 4.5.2: Using the Vertical Line Test
  - In this activity, you determined which graphs were functions based on the vertical line test.
  - 4.5.2: Self Check
  - 4.5.2: Additional Resources
- 4.5.3: Comparing Functions
  - In this activity, you compared the values of two functions representing cell phone plans. You explained the function notation in words and graphed the functions.
  - 4.5.3: Self Check
  - 4.5.3: Additional Resources
- 4.5.4: Using Function Notation and Graphing Technology
  - In this activity, you used graphing technology to find the output value of a function given an input value. You also worked backward to find the input value given an output value of a function.
  - 4.5.4: Self Check
  - 4.5.4: Additional Resources
- 4.5.5: Using Graphical and Algebraic Approaches for Finding Unknowns of Linear Functions
  - In this activity, you used graphical or algebraic approaches to find missing inputs or outputs. You explained the meaning of function notation in the given context.

After these activities, you completed the following **practice**:

- 4.5.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

4.6 FEATURES OF GRAPHS

4.6.0 Lesson Overview

In this lesson, you will further develop your ability to describe the characteristics of graphs. You will use mathematical terms such as intercept, maximum, and minimum in your graphical analyses, and you will relate features of graphs to features of the functions represented.

When you finish this lesson, you will be able to:

- Identify important features of graphs of functions and explain what they mean in the situations represented.
- Explain and apply the terms "horizontal intercept," "vertical intercept," "maximum," and "minimum" when talking about functions and their graphs.
- Identify important features of linear functions.

Here are the **activities** that will help you reach those goals:

- 4.6.1: Interpreting Statements in Function Notation
- 4.6.2: Analyzing Graphs of Functions
  - 4.6.2: Self Check
  - 4.6.2: Additional Resources
- 4.6.3: Connecting Graphical and Verbal Representations of a Function
  - 4.6.3: Self Check
  - 4.6.3: Additional Resources
- 4.6.4: Key Features of Linear Functions
  - 4.6.4: Self Check
  - 4.6.4: Additional Resources
- 4.6.5: Interpreting Graphs

After that, you'll **practice and review**.

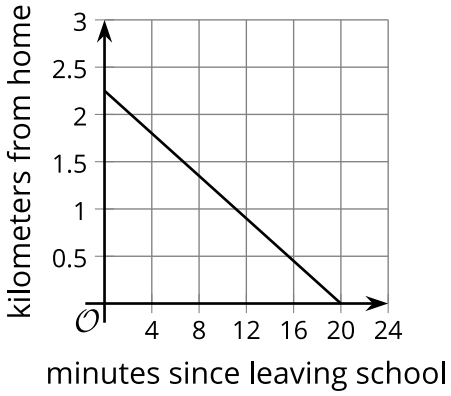
- 4.6.6: Practice
- 4.6.7: Lesson Summary

4.6.1 Interpreting Statements in Function Notation

Warm Up

Diego is walking home from school at a constant rate. This graph represents function d, which gives his distance from home, in kilometers, m minutes since leaving the school.

Use the graph to find or estimate:



1. What is $d(0)$, and what does it represent in the situation?
2. What is $d(12) =$
3. What does $d(m) = 1$ mean in the context of the problem, and what is m?
4. What does $d(m) = 0$ mean in the context of the problem, and what is m?

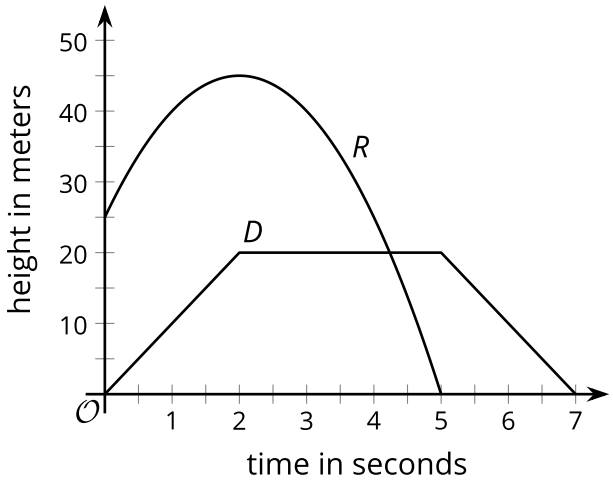
4.6.2 Analyzing Graphs of Functions

Activity

A toy rocket and a drone were launched at the same time.

Here are the graphs that represent the heights of two objects as a function of time since they were launched.

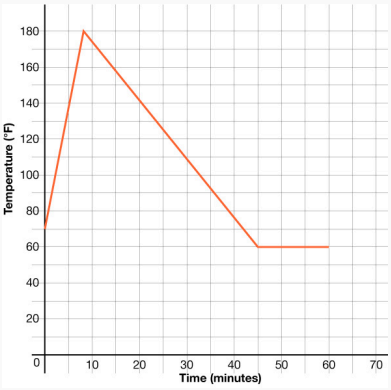
Height is measured in meters above the ground, and time is measured in seconds since launch.



1. Analyze the graph of the toy rocket and describe as precisely as you can what was happening. Your description should be complete and precise enough that someone who is not looking at the graph could visualize how the object was behaving. Include where the toy rocket starts and when it stops.
2. Analyze the graph of the drone and describe as precisely as you can what was happening. Your description should be complete and precise enough that someone who is not looking at the graph could visualize how the object was behaving. Include where the drone starts and when it stops.
3. At what point were the drone and the toy rocket at the same height?
4. Which parts or features of the graphs show important information about each object's movement? List the features or mark them on the graphs.

Self Check

The graph below represents the temperature, $t(m)$, in degrees Fahrenheit of water used for a cup of hot tea over time in minutes, m. What does $t(0)$ represent?

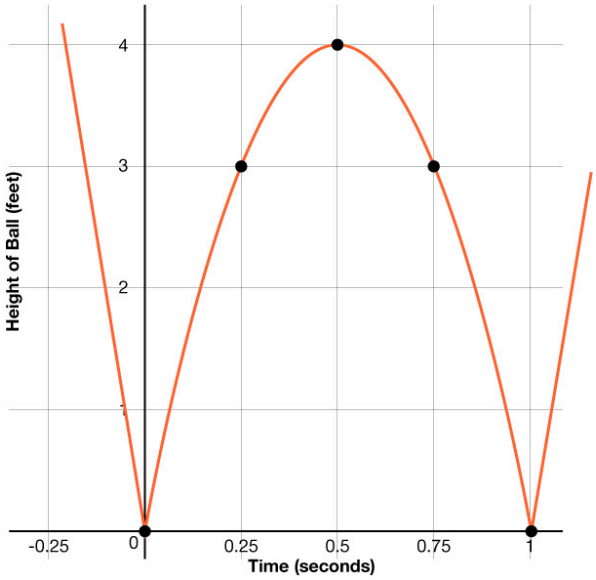


a. $t(0) = 180$; After 180 minutes, the temperature is 0 degrees.
b. $t(0) = 70$; After 0 minutes, the temperature is 70 degrees.
c. $t(0) = 180$; After 0 minutes, the temperature is 180 degrees.
d. $t(0) = 70$; After 70 minutes, the temperature is 0 degrees.

Additional Resources

Real-Life Graphs of Functions

A ball is bouncing across the school yard. The vertical or y-axis represents the height of the ball in feet, $h(t)$, while the horizontal or x-axis represents the time, t, that has passed in seconds.



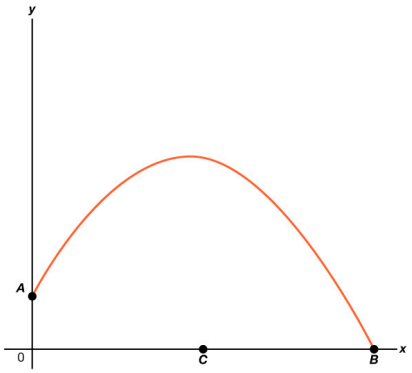
On the graph, $t = 0$ represents the first time the ball touches the ground.

What is $h(1)$, and what does it represent in the context of the problem?

$h(1) = 0$, and it represents the height of the ball after 1 second, or the ball has a height of zero after 1 second. The value inside of the parentheses is the value from the x-axis or horizontal axis. The solution is the value of the graph at that point.

TRY IT **Real-Life Graphs of Functions**

A ball is thrown across the field from point A to point B. It hits the ground at point B. The path of the ball is shown in the diagram below. The x-axis shows the horizontal distance the ball travels in feet, and the y-axis shows the height of the ball in feet.

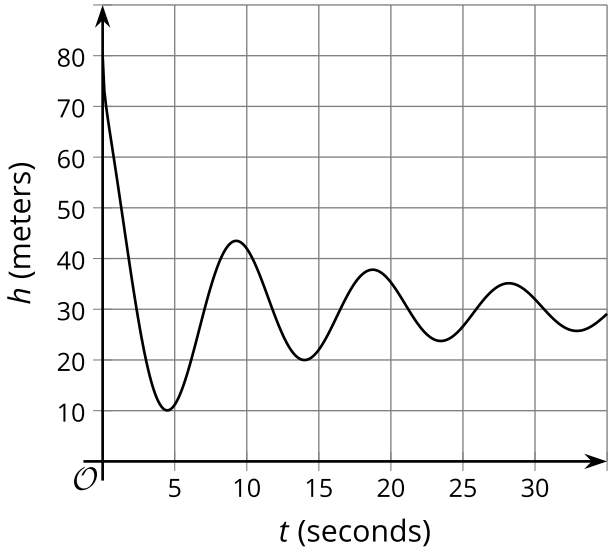


Explain what $f(C)$ means in this graph.

4.6.3 Connecting Graphical and Verbal Representations of a Function

Activity

In a bungee jump, the height of the jumper is a function of time since the jump begins.



Expressions and Equations:

- $h(0)$
- $h(t) = 0$
- $h(4)$
- $h(t) = 80$
- $h(t) = 45$

Features:

- first dip in the graph
- vertical intercept
- first peak in the graph
- horizontal intercept
- maximum

1. Match each description about the jump to a corresponding expression or equation and to a feature on the graph.

One expression or equation does not have a matching verbal description. Its corresponding graphical feature is also not

shown on the graph. Interpret that expression or equation in terms of the jump and in terms of the graph of the function. Record your interpretation in the last row of the table.

Description of Jump	Expression or Equation	Feature of Graph
A. the greatest height that the jumper is from the river		
B. the height from which the jumper was jumping		
C. the time at which the jumper reached the highest point after the first bounce		
D. the lowest point that the jumper reached in the entire jump		
E. the time at which the jumper hits the surface of the river		

A.Expression or Equation

Feature of Graph

B. Expression or Equation

Feature of Graph

C. Expression or Equation

Feature of Graph

D. Expression or Equation

Feature of Graph

E. Description of Jump

Expression or Equation

Feature of Graph

2. Use the graph to:

A. estimate $h(0)$ and $h(4)$.

2. If $t = 4$, then $h(4)$ equals what value?

3. If $h(t) = 45$, then t equals what value?

4. If $h(t) = 0$, then t equals what value?

Are you ready for more?

Extending Your Thinking

Based on the information available, how long do you think the bungee cord is? Make an estimate and explain your reasoning.

Video: The Meaning of Function Notation

Watch the following video to learn more about connecting graphical and verbal representations of a function.

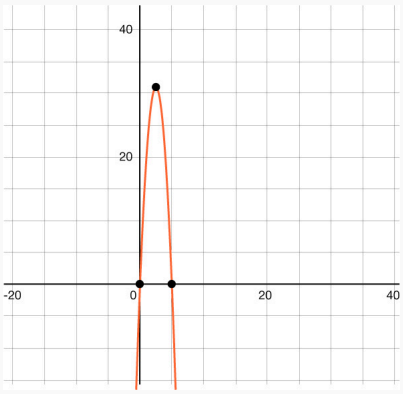
The Meaning of Function Notation (https://openstax.org/r/function-notation-meaning)

The horizontal intercept is also called the x-intercept, or zero .

The vertical intercept is also called the y-intercept.

Self Check

The graph models the height, in feet, of a kickball over time. When is the ball at its maximum height?

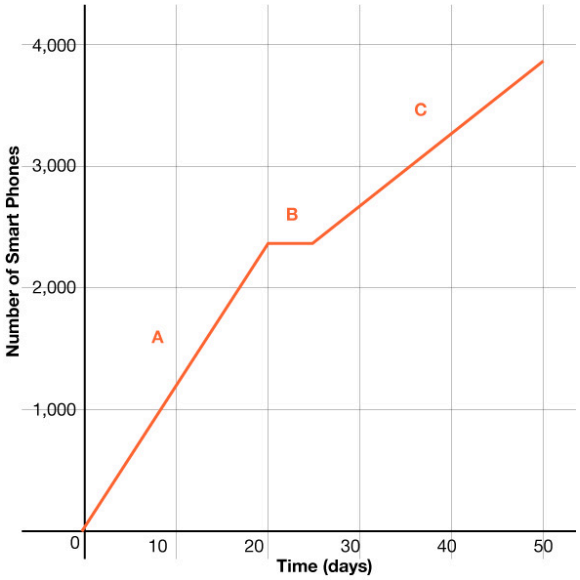


a. $h(2.5) = 31$; The maximum height is about 2.5 feet. It occurred about 31 seconds after the ball was kicked.
b. $h(1) = 20$; The maximum height occurred 1 second after the ball was kicked.
c. $h(5) = 0$; The maximum height occurs about 6 seconds after the ball was kicked.
d. $h(2.5) = 31$; The maximum height is about 31 feet. It occurred about 2.5 seconds after the ball was kicked.

Additional Resources

More Real-Life Graphs

The graph below represents the total number of smartphones that are shipped to a retail store over the course of 50 days.

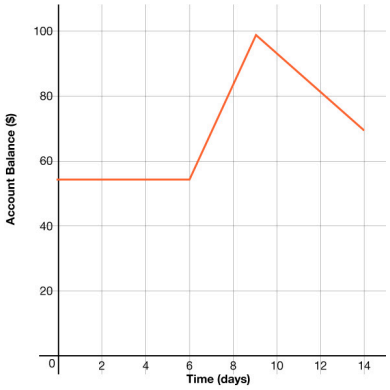


Why does $f(20) = f(22)$?

Since section B is not changing, it represents the period of time where the store did not receive new shipments, so $f(20) = f(22)$.

TRY IT **More Real-Life Graphs**

The relationship between Jameson's account balance B, in dollars, and time t, in days, is modeled by the graph below.



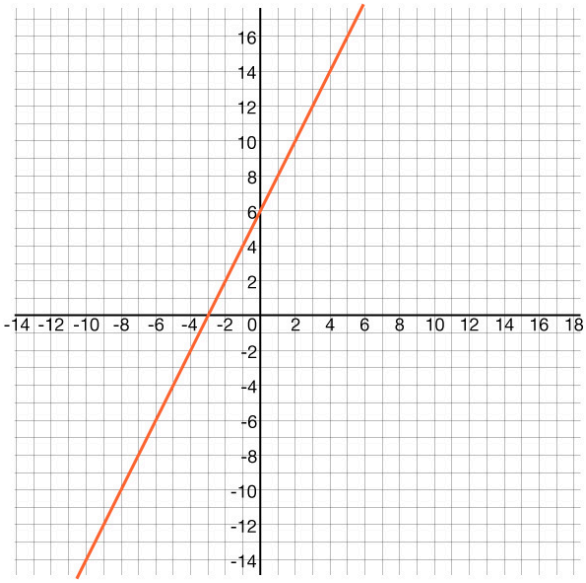
What does the point $B(9) = 98$ represent in the context of the problem?

4.6.4 Key Features of Linear Functions

Activity

Use the following information to answer questions 1 – 4.

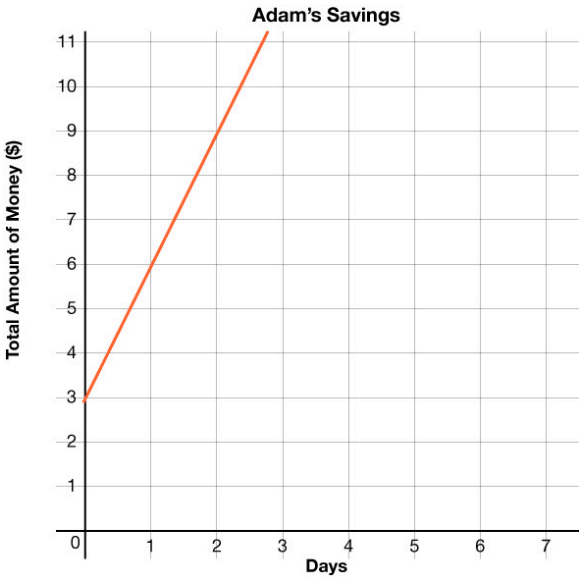
The graph of $f(x) = 2x + 6$ is graphed below. Answer the questions about the linear function and its key features.



1. What is the value of $f(-3)$?
2. What key feature of the linear function is the point $(-3, 0)$?
3. What is the slope of the linear function?
4. What part of the linear function is the point $(0, 6)$?

Use the following information to answer questions 5 – 8.

The amount of money Adam saves in dollars is shown in the graph below.



5. What is the y-intercept of the graph?

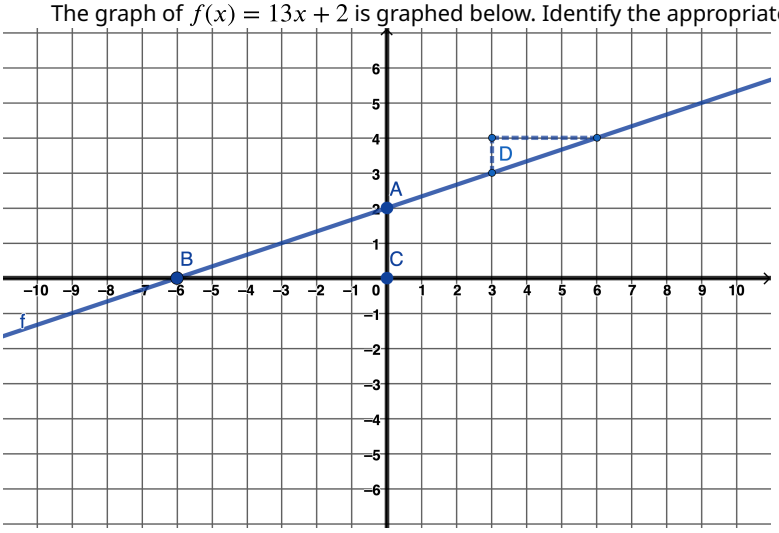
6. What does the y-intercepts mean in the context of the problem

7. What is the slope?

8. What does the slope mean in the context of the problem

Use the following information to answer questions 9 – 12.

The graph of $f(x) = 13x + 2$ is graphed below. Identify the appropriate label for each key feature of the graph.



9. What is the key feature labeled A on the graph?

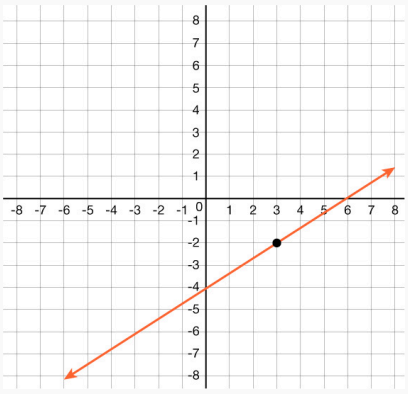
10. What is the key feature labeled B on the graph?

11. What is the key feature labeled C on the graph?

12. What is the key feature labeled D on the graph?

Self Check

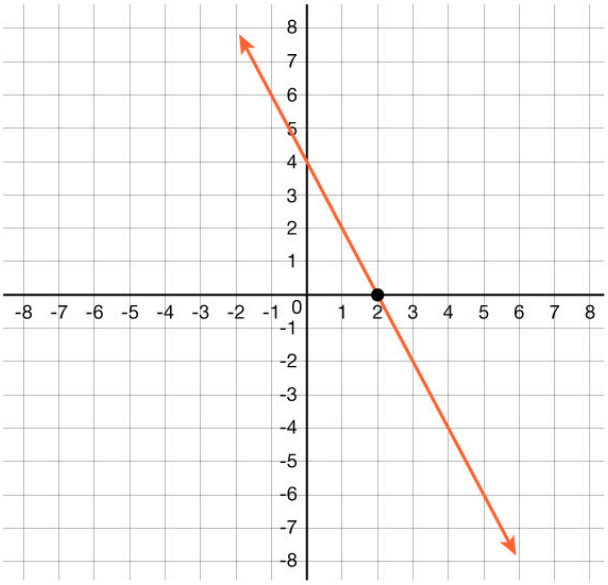
What is the zero of the linear function graphed below?



a. $(0, 0)$
b. $(0, 6)$
c. $(0, -4)$
d. $(6, 0)$

Additional Resources

Features of Linear Functions



The graph of a linear function forms a line that goes on forever in both directions. Since the ends go to infinity and negative infinity, linear functions do not have maximums or minimums, unless their domain is restricted to a specific part of the graph.

You can still pick out key features of linear functions.

The x-intercept, or zero, is where the line crosses the x-axis. This is where the y-coordinate is 0.

In this linear function, the x-intercept, or zero, is $x = 2$ and written as the point $(2, 0)$.

The y-intercept is where the line crosses the y-axis and the x-coordinate is 0.

In this linear function, the y-intercept is $y = 4$ and written as the point $(0, 4)$.

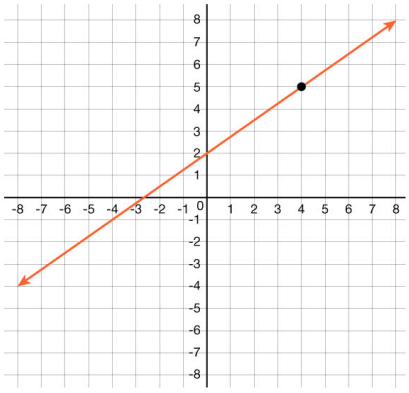
The slope of the line can be found as in previous units as the change in y over the change in x.

$$\text{m} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in y}}{\text{change in x}}$$

The slope of this linear function is $m = \frac{4}{-2} = \frac{-4}{2} = -2$.

TRY IT **Features of Linear Functions**

Find the x-intercept and y-intercept of the linear function graphed below. Write each answer as a point.

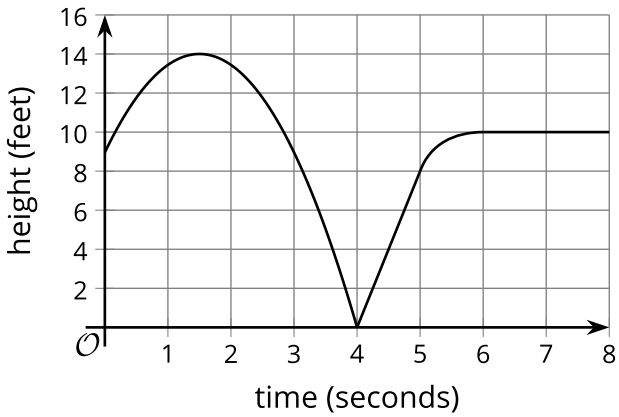


4.6.5 Interpreting Graphs

Cool Down

A squirrel runs up and down a tree.

The graph shows the height of the squirrel, h (feet), as a function of time, t (seconds).

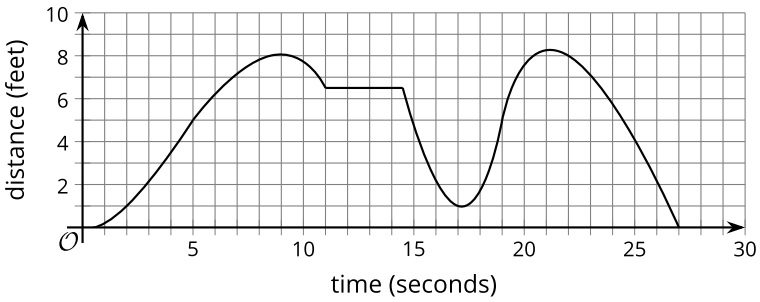


1. What is the highest point the squirrel reaches?
2. Solve $h(t) = 0$. What does this solution tell you about the squirrel?
3. Find the vertical intercept of the graph. What does it tell you about the squirrel?

4.6.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. This graph represents Andre's distance from his bicycle as he walks in a park.



Decide whether the following statements are true or false.

a. The graph has multiple horizontal intercepts.

   a. true
   b. false

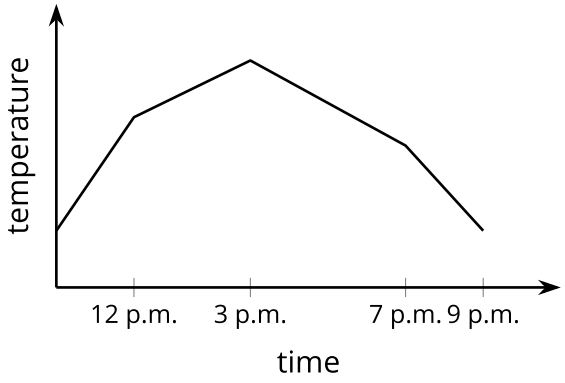
b. A horizontal intercept of the graph represents the time when Andre was with his bike.

   a. true
   b. false

c. A maximum of the graph is the point $(17, 1)$.

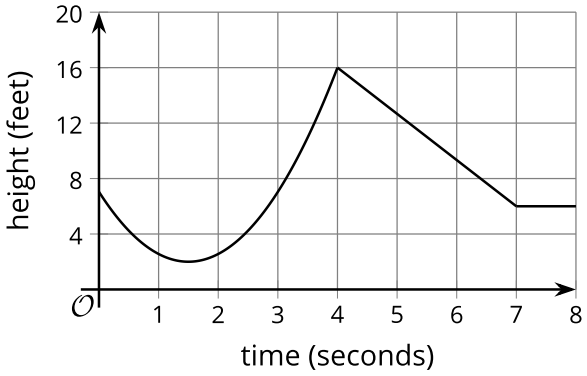
   a. true
   b. false

2. The graph represents the temperature, in degrees Fahrenheit, as a function of time.



At what hour in the afternoon does the maximum of the graph occur?

3. Match each feature of the situation with a corresponding statement in function notation.



a. $h(0) = 7$

a. Starting height
b. Height staying the same
c. Minimum height
d. Maximum height

b. $h(1.5)$

a. Starting height
b. Height staying the same
c. Minimum height
d. Maximum height

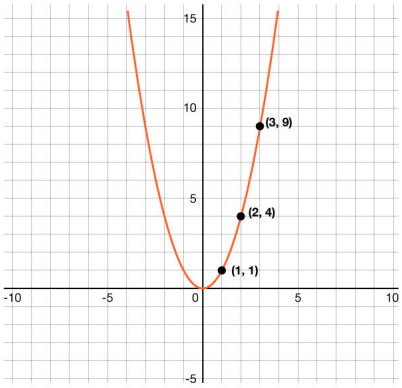
c. $h(4)$

a. Starting height
b. Height staying the same
c. Minimum height
d. Maximum height

d. $h(t) = 6$ for $7 \leq t \leq 8$

a. Starting height
b. Height staying the same
c. Minimum height
d. Maximum height

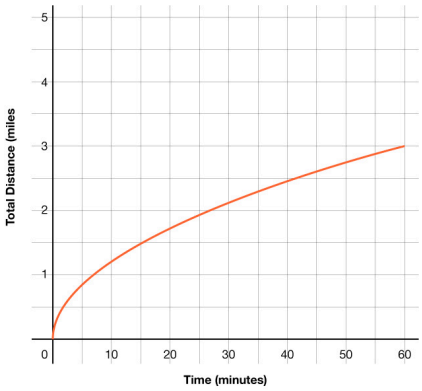
4. The graph of $f(x) = x^2$ is provided.



a. What is $f(2)$?

b. What is the value of $f(1)$?

5. Shannon is running on a flat, rocky trail that eventually rises up a steep mountain. The graph below represents the total distance she covers with respect to time.



How many miles does Shannon run in 60 minutes?

4.6.7 Lesson Summary

In this lesson, you learned how to:

- Identify important features of graphs of functions and explain what they mean in the situations represented.
- Explain and apply the terms “horizontal intercept,” “vertical intercept,” “maximum,” and “minimum” when talking about functions and their graphs.
- Identify important features of linear functions.

Here are the **activities** that helped you reach those goals:

- 4.6.1: Interpreting Statements in Function Notatio
  - In this activity, you interpreted function notation using a graph. You completed mathematical statements and found the value of variables by reading the information from a graph.
- 4.6.2: Analyzing Graphs of Functions
  - In this activity, you looked at the graphs of two different real-life functions. You identified key features such as minimum and maximum, x-intercepts and y-intercepts, and what they mean in a real-life situation.
  - 4.6.2: Self Check
  - 4.6.2: Additional Resources
- 4.6.3: Connecting Graphical and Verbal Representations of a Function
  - In this activity, you continued working with graphs, function notation, and real-life situations. You interpreted statements about a real-life graph and connected them to the function notation that represented them.
  - 4.6.3: Self Check
  - 4.6.3: Additional Resources
- 4.6.4: Key Features of Linear Functions
  - In this activity, you recognized the x-intercepts, zeros, slope, and y-intercepts of linear functions.
  - 4.6.4: Self Check
  - 4.6.4: Additional Resources
- 4.6.5: Interpreting Graphs
  - In this activity, you examined specific points and key features on the graph and interpreted their meaning verbally and algebraically.

After these activities, you completed the following **practice**:

- 4.6.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

4.7 FINDING SLOPE

4.7.0 Lesson Overview

In this lesson, you will work to find the slope of a line from two points, a graph, and a table. You will then use this knowledge to write equations of lines in three different forms: slope-intercept form, point-slope form, and standard form.

When you finish this lesson, you will be able to:

- Write the slope of a line.
- Write the equation of a line in different forms.

Here are the **activities** that will help you reach those goals:

- 4.7.1: Evaluating Fractions
- 4.7.2: Finding Slope From Tables, Graphs, and Points
  - 4.7.2: Self Check
  - 4.7.2: Additional Resources
- 4.7.3: Writing Linear Equations
  - 4.7.3: Self Check
  - 4.7.3: Additional Resources
- 4.7.4: Writing Equations From Two Points

After that, you'll **practice and review**.

- 4.7.5: Practice
- 4.7.6: Lesson Summary

4.7.1 Evaluating Fractions

Warm Up

Evaluate Mentally.

1. $\frac{102-96}{45-42}$

2. $\frac{-8-4}{6-2}$

3. $\frac{31-18}{5-10}$

4. $\frac{4-9}{12-18}$

4.7.2 Finding Slope From Tables, Graphs, and Points

Activity

Recall that the **slope** of a line is the rate of change of the linear function. On the graph, it is the steepness of the line.

1. Find the slope of the line that connects $(0, 0)$ and $(3, 2)$

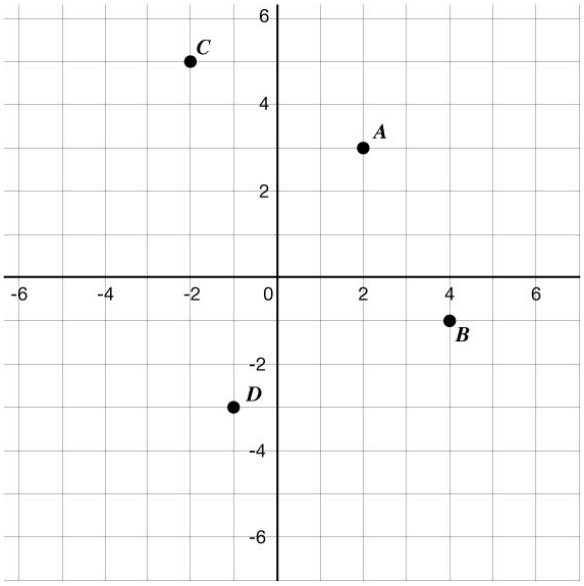
2. Find the slope of the line that connects $(4, 2)$ and $(10, 7)$

3. Find the slope of the line that connects $(1, -2)$ and $(2, 5)$

4. Find the slope of the line that connects $(-3, 4)$ and $(-5, -2)$

5. Find the slope of the line that connects $(8, 3)$ and $(10, -9)$

For questions 6 – 9, use the graph to find the slope of the line that goes through the two points.



6. A and B

7. A and D

8. B and C

9. C and D

10. Determine the slope of the equation whose values are in the given table.

x	1	3	5	7
y	5	10	15	20

11. Determine the slope of the equation whose values are in the given table. Simplify your answer.

x	2	5	8	11
y	4	1	-3	-6

Self Check

Find the slope of the line that goes through $(-1, 5)$ and $(6, 2)$.

a. $-\frac{3}{7}$
b. $\frac{3}{7}$
c. $-\frac{7}{3}$
d. $-\frac{3}{5}$

Additional Resources

Slope of a Line

Here's how to find the slope of a line from its graph using $m = \frac{\text{rise}}{\text{run}}$.

Finding Slope from a Graph

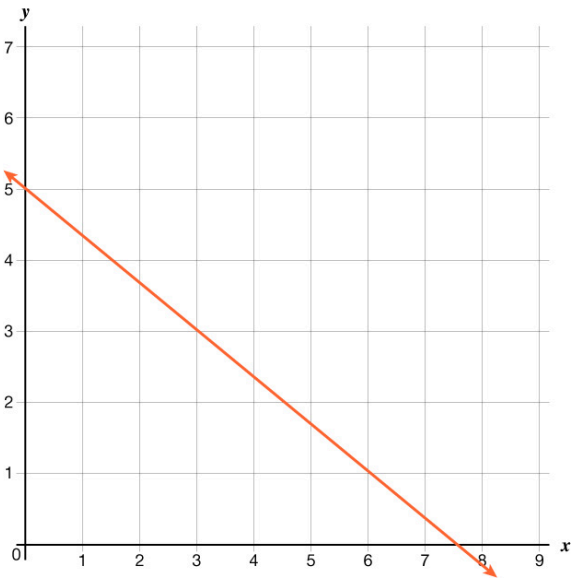
Step 1 - Locate two points on the line whose coordinates are integers.

Step 2 - Starting with one point, sketch a right triangle, going from the first point to the second point.

Step 3 - Count the rise and the run on the legs of the triangle.

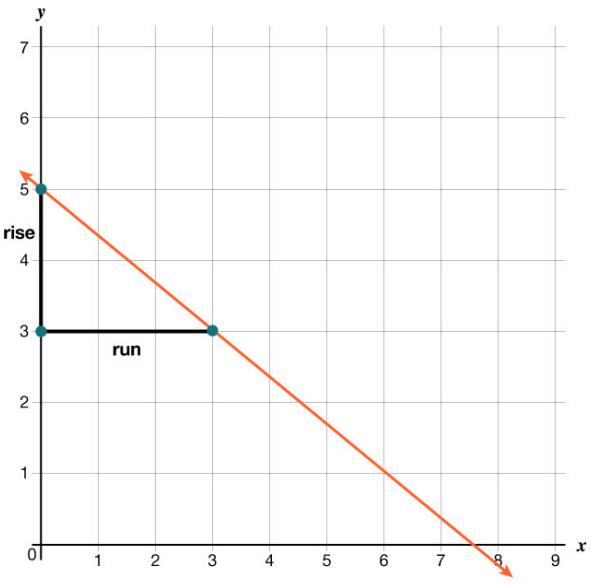
Step 4 - Take the ratio of rise to run to find the slope: $m = \frac{\text{rise}}{\text{run}}$.

EXAMPLE 1



Step 1 - Locate two points on the line whose coordinates are integers. $(0, 5)$ and $(3, 3)$

Step 2 - Starting with one point, sketch a right triangle, going from the first point to the second point.



Step 3 - Count the rise and the run on the legs of the triangle.

The rise is -2.

The run is 3.

Step 4 - Take the ratio of rise to run to find the slope: $m = \frac{\text{rise}}{\text{run}}$.

$m = -\frac{2}{3}$

Here's how to find the slope of a line when given two points on the line.

Finding Slope from Two Points

Step 1 - Label the points (x_1, y_1) and (x_2, y_2).

Step 2 - Substitute the values into the slope formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$

Step 3 - Simplify the slope expression.

EXAMPLE 2

Use the slope formula to find the slope of the line through the points $(-2, -3)$ and $(-7, 4)$.

Step 1 - Label the points (x_1, y_1) and (x_2, y_2).

$(-2, -3)$ $(-7, 4)$

(x_1, y_1) (x_2, y_2)

Step 2 - Substitute into the slope formula.

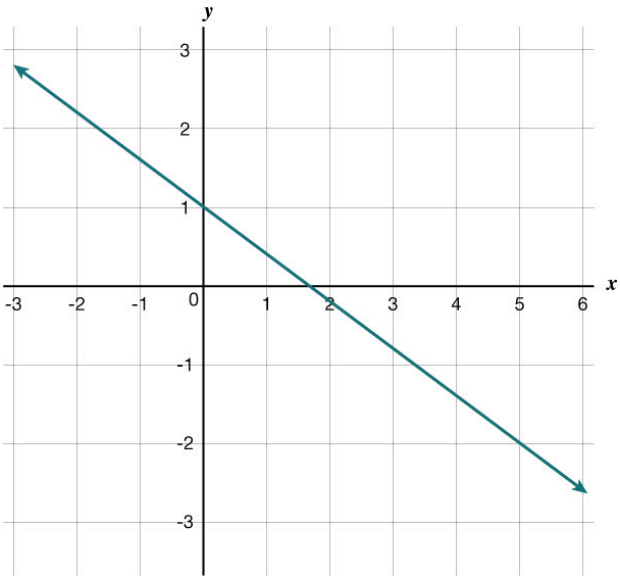
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-(-3)}{-7-(-2)}$

Step 3 - Simplify

$m = \frac{4-(-3)}{-7-(-2)} = \frac{4+3}{-7+2} = \frac{7}{-5}$

You Try: Finding Slope from a Graph or Two Points

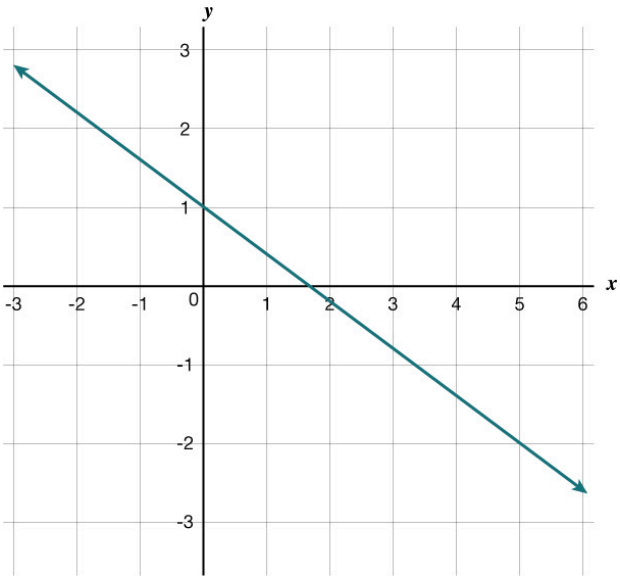
Find the slope of the line using the graph, then confirm your answer using the slope formula.



Step 1 - Locate two points on the line whose coordinates are integers.

$(0, 1)$ and $(5, -2)$

Step 2 - Starting with one point, sketch a right triangle, going from the first point to the second point.



Step 3 - Count the rise and the run on the legs of the triangle.

The rise is 3.

The run is -5.

Step 4 - Take the ratio of rise to run to find the slope: $m = \frac{\text{rise}}{\text{run}}$.

$m = -\frac{3}{5}$

Now, using the slope formula:

Step 1 - Label the points (x_1, y_1) and (x_2, y_2).

$(0, 1)\ (5, -2)$

$(x_1, y_1)\ (x_2, y_2)$

Step 2 - Substitute into the slope formula.

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-1}{5-0}$

Step 3 - Simplify.

$m = \frac{-2-1}{5-0} = \frac{-3}{5}$

The slope of the line is $-\frac{3}{5}$.

Here's how to find the slope of a line, given a table of values.

EXAMPLE 3

Find the slope of the line whose table of values is shown:

x	3	5	7	9	11
y	4	7	10	13	16

Step 1 - Locate two points on the line whose coordinates are integers.

$(3, 4)\ (5, 7)$

Step 2 - Label the points (x_1, y_1) and (x_2, y_2).

$(3, 4)$ $(5, 7)$

(x_1, y_1) (x_2, y_2)

Step 3 - Substitute into the slope formula.

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-4}{5-3}$

Step 4 - Simplify.

$m = \frac{7-4}{5-3} = \frac{3}{2}$

TRY IT **Finding Slope from a Table**

Find the slope of the line whose table of values is shown:

x	-3	-1	1	3
y	2	4	6	8

4.7.3 Writing Linear Equations

Activity

In Algebra, equations are commonly written in three different ways: slope-intercept form, point-slope form, and standard form. They are equivalent ways to write the same equation, but each one is useful for different reasons.

Slope-intercept form	Point-slope form	Standard form
$y = mx + b$ Given: $m =$ slope $b = y$-intercept	$y - y_1 = m(x - x_1)$ Given: a point (x_1, y_1) $m =$ slope	$Ax + By = C$ $m = -\frac{A}{B} =$ slope Where: There are no fractions and A cannot be negative.

For **slope-intercept form**, the slope and y-intercept are needed:

1. Write an equation in **slope-intercept form** for a line whose slope is -2 and y-intercept is $(0, 4)$.

For **point-slope form**, the slope and one point are needed:

2. Write an equation in point-slope form for a line whose slope is $\frac{1}{2}$ and goes through the point $(3, -5)$

Standard form of a linear equation is often used to find x- and y-intercepts of graphs.

Often, an equation starts in another form and gets put into standard form.

Standard form does not have fractions or decimals and cannot lead with a negative number.

3. What is the slope in the equation $y = 2x + 3$?

4. Write the equation $y = 2x + 3$ in standard form.

Work with a partner to write equations in different forms.

5. What is the slope of $y - 3 = 2(x + 4)$

6. Write the equation $y - 3 = 2(x + 4)$ in slope-intercept form.

7. Write the equation $y + 5 = -3(x - 2)$ into standard form.

8. Write the equation $y = \frac{1}{3}x - 4$ in standard form.

9. What is the slope of $3x + 12y = 24$?

Video: Writing Equations in Different Forms

Watch the following video to learn more about writing equations in different forms.

Worked Solution 4.7 (https://openstax.org/r/worked-solution-47)

Self Check

Write the equation of the line in point-slope form that has a slope of 7 and goes through $(-1, 2)$.

a. $2 - y = 7(-1 - x)$
b. $y - 2 = 7(x - 1)$
c. $y - 2 = 7(x + 1)$
d. $y = 7x + 9$

Additional Resources

Writing Linear Equations in Different Forms

EXAMPLE 1

When you look at equations in these forms, it's easy to identify the slope of m.

Equation	Equation formula	Slope formula	Slope value
$y = 5x - 15$	$y = mx + b$	m	5
$y = 6x + 8$	$y = mx + b$	m	6
$y - 7 = 9(x + 1)$	$y - y_1 = m(x - x_1)$	m	9
$y - 20 = 3(x + 16)$	$y - y_1 = m(x - x_1)$	m	3
$18x - 6y = 24$	$Ax + By = C$	$m = -\frac{A}{B}$	$-(\frac{18}{-6}) = 3$
$30x + 15y = 24$	$Ax + By = C$	$m = -\frac{A}{B}$	$-\frac{30}{15} = -2$

EXAMPLE 2

Find the equation of a line with slope −9 and y-intercept $(0, -4)$ in slope-intercept form.

Step 1 - Identify the m and b.

$m = -9$, $b = -4$

Step 2 - Substitute into the formula.

$y = -9x - 4$

EXAMPLE 3

Find the equation of a line in point-slope form given a slope of $-\frac{1}{3}$ and the point $(6, -4)$

Step 1 - Identify the m and (x_1, y_1).

$m = -\frac{1}{3} x_1 = 6 y_1 = -4$

Step 2 - Substitute into the formula.

$y - (-4) = -\frac{1}{3}(x - 6)$

Step 3 - Simplify

$y + 4 = -\frac{1}{3}(x - 6)$

EXAMPLE 4

Write the equation $y + 4 = -\frac{1}{3}(x - 6)$ in standard form.

Step 1 - Multiply to get rid of the fraction.

Remember, standard form cannot have decimals or fractions.

Multiply every term by 3.

$3(y + 4) = 3(-\frac{1}{3}(x - 6))$

$3y + 12 = -(x - 6)$

Step 2 - Distribute.

$3y + 12 = -x + 6$

Step 3 - Rearrange terms so the x term and the y term are both on the left side of the equation.

Add x to both sides.

$3y + 12 + x = -x + x + 6$

Step 4 - Simplify.

$3y + 12 + x = 6$

Step 5 - Bring the constants to the right side.

Subtract 12 from both sides.

$3y + 12 + x - 12 = 6 - 12$

Step 6 - Simplify.

$3y + x = -6$

Step 7 - Write the x term before the y term.

$3x + y = -6$

TRY IT **Writing Linear Equations in Different Forms**

Write the equation $y - 3 = 2(x + 5)$ in standard form.

4.7.4 Writing Equations From Two Points

Cool Down

With a partner, use what you have learned to write an equation in each form from two points.

1. Find the slope of the line that goes through $(2, 3)$ and $(6, -4)$.
2. Write the equation of the line in point-slope form that goes through $(2, 3)$ and $(6, -4)$.

3. Write the equation of the line in slope-intercept form that goes through $(2, 3)$ and $(6, -4)$. (Hint: use the equation from number 2)

4. Write the equation of the line in standard form that goes through $(2, 3)$ and $(6, -4)$.
(Hint: use the equation from number 3.)

4.7.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Evaluate $\frac{140-20}{9-12}$ mentally.

2. Find the slope of the line that goes through the points $(1, -3)$ and $(2, 4)$.

3. Write the equation of the line in slope-intercept form that goes through $(0, 3)$ and has a slope of $\frac{1}{4}$.

a. $4x - y = -3$
b. $y = 3x + \frac{1}{4}$
c. $y = \frac{1}{4}x + 3$
d. $y = -\frac{1}{4}x + 3$

4. Write the equation of the line that passes through $(2, -1)$ and has a slope of $-\frac{1}{2}$ in point-slope form.

a. $2x + y = 0$
b. $y = -\frac{1}{2}x$
c. $y + 1 = -\frac{1}{2}(x - 2)$
d. $y = -\frac{1}{2}x - 1$

5. **Select two** equations that are correctly in standard form.

a. $6x - 3y = 3$
b. $-4x + 5y = 9$
c. $\frac{1}{2}x + 3y = 8$
d. $5x + 2y = 9$
e. $5y + 2x = 10$

6. Write $y = 4x - 2$ in standard form.

7. Write the equation $y - 4 = -2(x - 5)$ in slope-intercept form.

8. Find the slope of the line between the points $(3, -1)$ and $(2, -4)$.

9. Write the equation $y - 2 = 3(x + 5)$ in standard form.

10. Evaluate $\frac{36-48}{10-14}$ mentally.

4.7.6 Lesson Summary

In this lesson, you learned how to:

- Write the slope of a line.
- Write the equation of a line in different forms.

Here are the **activities** that helped you reach those goals:

- 4.7.1: Evaluating Fractions
  - In this activity, you simplified fractions that involved adding and subtracting in both the numerator and denominator.
- 4.7.2: Finding Slope From Tables, Graphs, and Points
  - In this activity, you learned how to find the slope using the slope formula, given two coordinates, a graph of a line, and a table of values.
  - 4.7.2: Self Check
  - 4.7.2: Additional Resources
- 4.7.3: Writing Linear Equations
  - In this activity, you learned how to write equations in three different forms and also created equivalent equations by writing the equations into the different forms.
  - 4.7.3: Self Check
  - 4.7.3: Additional Resources
- 4.7.4: Writing Equations From Two Points
  - In this activity, you put everything together to write equations in all three forms, given two points which you used to find the slope of the line.

After these activities, you completed the following **practice**:

- 4.7.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

4.8 USING GRAPHS TO FIND AVERAGE RATE OF CHANGE

4.8.0 Lesson Overview

In this lesson, you will learn to characterize changes in functions quantitatively, by using average rates of change. You will also learn that the average rate of change can be used to measure how fast a function changes over a given interval.

When you finish this lesson, you will be able to:

- Explain the meaning of the term “average rate of change.”
- Estimate or calculate the average rate of change between two points from the graph of a function.

Here are the **activities** that will help you reach those goals:

- 4.8.1: Comparing Changes in Output Based on Input
- 4.8.2: Average Rate of Change and Slope
  - 4.8.2: Self Check
  - 4.8.2: Additional Resources
- 4.8.3: Finding and Interpreting Average Rates of Change
  - 4.8.3: Self Check
  - 4.8.3: Additional Resources
- 4.8.4: Rate of Change of Linear Functions
  - 4.8.4: Self Check
  - 4.8.4: Additional Resources
- 4.8.5: Interpreting Average Rates of Change with Real-World Examples

After that, you’ll **practice and review**.

- 4.8.6: Practice
- 4.8.7: Lesson Summary

4.8.1 Comparing Changes in Output Based on Input

Warm Up

Here are the recorded temperatures at three different times on a winter evening.

time	4 p.m.	6 p.m.	10 p.m.
temperature	25°F	17°F	8°F

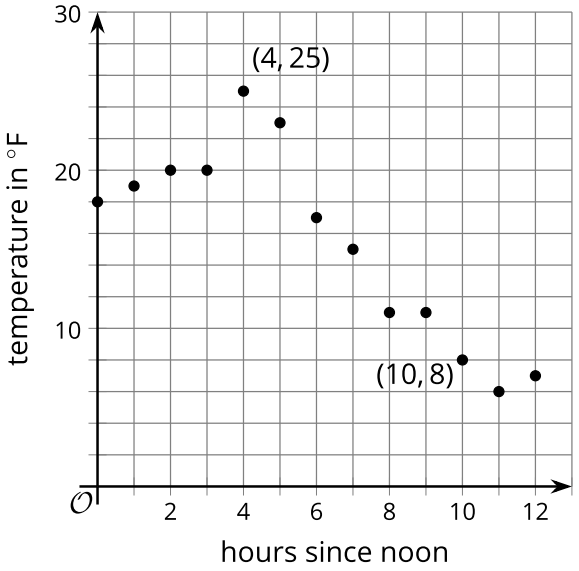
- Tyler says the temperature dropped faster between 4 p.m. and 6 p.m.
- Mai says the temperature dropped faster between 6 p.m. and 10 p.m.

Who do you agree with? Be prepared to show your reasoning.

4.8.2 Average Rate of Change and Slope

Activity

The table and graph show a more complete picture of the temperature changes on the same winter day. The function T gives the temperature, in degrees Fahrenheit, h hours since noon.



h	0	1	2	3	4	5	6	7	8	9	10	11	12
$T(h)$	18	19	20	20	25	23	17	15	11	11	8	6	7

In questions 1 – 3, find the average rate of change for the following intervals. Be prepared to show your reasoning.

1. between noon and 1 p.m.

2. between noon and 4 p.m.

3. between noon and midnight

In questions 4 – 6, use the slope formula to find the slope of the line that connects the following points.

4. $(0, 18)$ and $(1, 19)$

5. $(0, 18)$ and $(4, 25)$

6. $(0, 18)$ and $(12, 7)$

7. Compare your answers to questions 1 – 3 with your answers to questions 4 – 6. Explain what this tells you about the average rate of change and the slope of a line.

For questions 8 and 9, think back to Mai and Tyler's disagreement about temperature to solve these problems.

8. Use the **average rate of change** to show which time period—4 p.m. to 6 p.m. or 6 p.m. to 10 p.m.—experienced a faster temperature drop.

9. Use the slope formula to show which line has the greater slope—the line connecting the points that represent the temperature at 4 p.m. and 6 p.m. or the line connecting the points that represent the temperature at 6 p.m. and 10 p.m.

10. Compare your answers to question 8 and question 9.

Are you ready for more?

Extending Your Thinking

Refer to graph and table of the previous function T to answer the questions:

1. Over what interval did the temperature decrease the most rapidly?

2. Over what interval did the temperature increase the most rapidly?

Self Check

The temperature t, in degrees Fahrenheit, of a cup of coffee that was poured and then forgotten for m minutes is represented in the table below

Time in minutes	0	15	25	40
Temperature	185°F	155°F	140°F	120°F

What was the average rate of change from when the cup was poured to 40 minutes later?

a. -1.625 degrees per minute
b. -1.333 degrees per minute
c. -0.615 degrees per minute
d. -2 degrees per minute

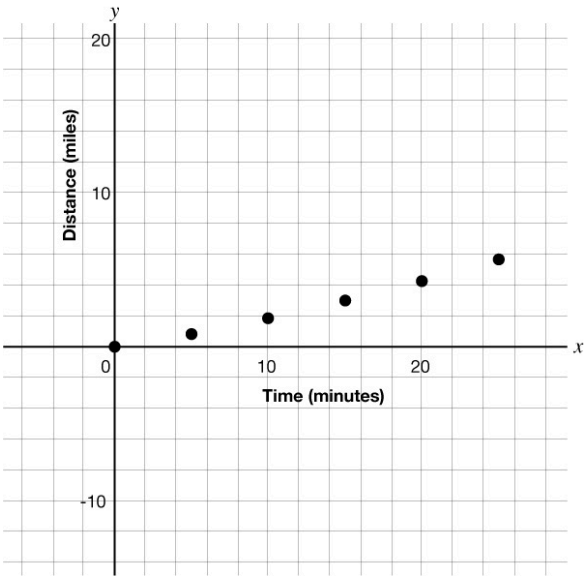
Additional Resources

Finding Rates of Change

Jay was racing bikes with a friend on the way to the park. A table and a graph of the times are shown below.

Time (minutes)	Distance (miles)
0	0
5	0.84
10	1.86
15	3.00
20	4.27
25	5.67

Jay



What is the average rate of change of the first 10 minutes of the race?

To find the average rate of change of the first 10 minutes of the race, you would look at the first data value on the table and the third data value on the table, since it corresponds to 10 minutes.

Remember, the average rate of change formula is $\frac{f(b)-f(a)}{b-a}$, where $(b, f(b))$ typically represents your second data point and $(a, f(a))$ represents your first data point.

For this situation, it would be $(0, 0)$ or $a = 0$ and $f(a) = 0$ and $(10, 1.86)$ or $b = 10$ and $f(b) = 1.86$.

Substituting the points into the average rate of change formula results in the following:

$\frac{f(b)-f(a)}{b-a}$, which is $\frac{1.86}{10}$ or 0.186 miles per minute.

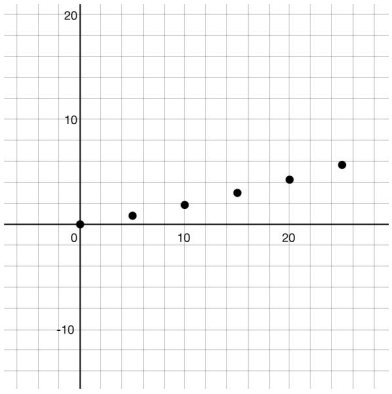
Notice that the answer is positive and the points on the graph are higher each time.

TRY IT **Finding Rates of Change**

Jay was racing bikes with a friend on the way to the park. A table and a graph of the times are shown below.

Time (minutes)	Distance (miles)
0	0
5	0.84
10	1.86
15	3.00
20	4.27
25	5.67

Jay

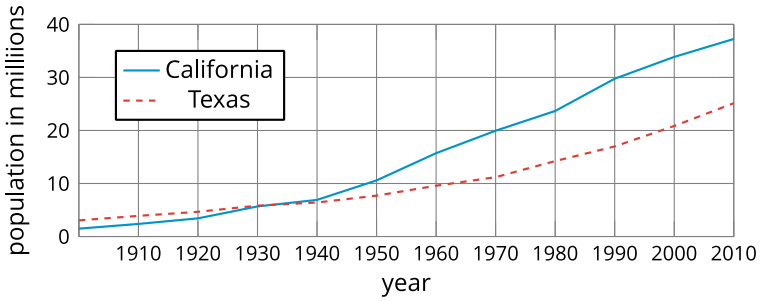


What was Jay's average rate of change from the beginning to 25 minutes later?

4.8.3 Finding and Interpreting Average Rates of Change

Activity

The graphs show the populations of California and Texas over time.



1. Estimate the average rate of change in the population in California between 1970 and 2010. Be prepared to show your reasoning.
2. In this situation, what does the rate of change mean for California?
3. Estimate the average rate of change in the population in Texas between 1970 and 2010. Be prepared to show your reasoning.
4. In this situation, what does the rate of change mean for Texas?
5. Which state's population grew more quickly between 1900 and 2000? Be prepared to show your reasoning.

Video: Interpreting Average Rate of Change

Watch the following video to learn more about interpreting the average rate of change.

Interpreting Average Rate of Change (https://openstax.org/r/rate-of-change)

Self Check

Brooke was racing bikes with a friend on the way to the park. A table of her distance traveled over time is shown.

Time (minutes)	Distance (miles)
0	0
5	1.2
10	2.4
15	3.6
20	4.8
25	6.0

Brooke

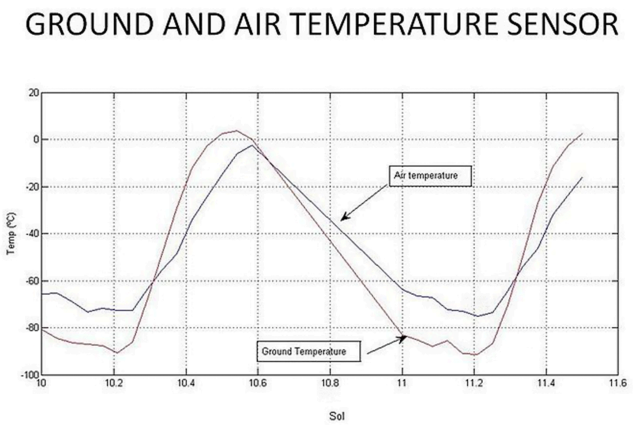
What was Brooke's average rate of change between 5 minutes and 20 minutes?

a. The average rate of change is 4.17 minutes per mile.
b. The average rate of change is 0.24 mile per minute.
c. The average rate of change is 15 minutes per mile.
d. The average rate of change is 3.6 miles per minute.

Additional Resources

Interpreting Slope

A Mars rover collected the following temperature data over 1.6 Martian days. A Martian day is called a Sol. Use the graph to answer the following questions.



Which had a greater average rate of change between 10.2 and 10.4 Sol, ground temperature or air temperature? What is the meaning of each rate change?

Since ground temperature has a greater slant or slope, it has a greater rate of change. Slope is another word for the rate of change of a graph.

To find the average rate of change, first you need to find Sol on the graph (which is what a day is called on Mars). Find the values of 10.2 and 10.4 on the x-axis. Find the corresponding y-values of the points for both the ground temperature

and the air temperature graphs. By finding the average rate of change, you will be able to determine which one was greater.

Ground Temperature: $(10.2, -90)$ $(10.4, -20)$

Air Temperature: $(10.2, -70)$ $(10.4, -40)$

Substitute each set of points into the average rate of change formula:

Ground Temperature = $\frac{-20-(-90)}{10.4-10.2} = \frac{70}{0.2} = 350$

Air Temperature = $\frac{-40-(-70)}{10.4-10.2} = \frac{30}{0.2} = 150$

Remember that to interpret the meaning of the rate of change, you need to compare the axes. It is the change in the y-axis units divided by the change in the x-axis units, or $\frac{\text{change in temperature in degrees celsius}}{\text{change in Sol}}$. That means that the average rate of change of each can be interpreted as:

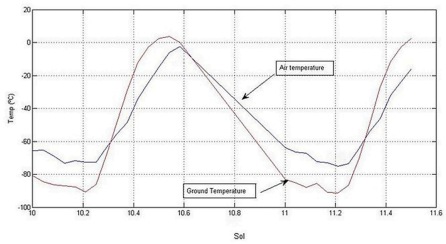
Ground Temperature's average rate of change is 350 degrees celsius per Sol (or day on Mars).

Air Temperature's average rate of change is 150 degrees celsius per Sol (or day on Mars).

TRY IT **Interpreting Slope**

A Mars rover collected the following temperature data over 1.6 Martian days. A Martian day is called a Sol. Use the graph to answer the following questions.

GROUND AND AIR TEMPERATURE SENSOR

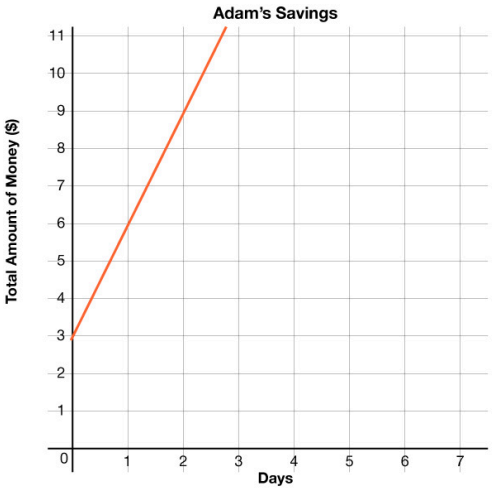


Which had a greater average rate of change between 10.6 and 11 Sol, ground temperature or air temperature? What is the meaning of each rate change?

4.8.4 Rate of Change of Linear Functions

Activity

Three people, Adam, Bianca, and Carmen, are competing to see who can save the most money in one month. Use the table, graph, and equation below to answer the questions.



Bianca's Savings

Input (Number of Days)	Output (Total amount of money in dollars)
5	17
8	26
12	38
20	62

Carmen's Savings

$f(x) = 2x + 30$; where $f(x)$ represents the total amount of money for x number of days

1. Use the graph to find the average rate of change of the function that represents Adam's savings for the following intervals:

a. between 0 and 1 days

b. between 1 and 2 days

c. between 1 and 3 days

2. Use the table to find the average rate of change for the function that represents Bianca's savings for the following intervals:

a. between 5 and 8 days

b. between 12 and 20 days

c. between 5 and 20 days

3. Use the equation to find the average rate of change for the function that represents Carmen's savings for the following intervals:

a. between 1 and 2 days

b. between 2 and 4 days

c. between 1 and 4 days

4. Which statements are true?

a. All three functions are linear functions.
b. All three functions have the same rate of change.
c. All three functions have a constant rate of change.

5. Who has the most money after 30 days? Explain how you know.

Self Check

What is the average rate of change for the linear function?

Input	Output
1	2
2	-1

Input	Output
4	-7
6	-13

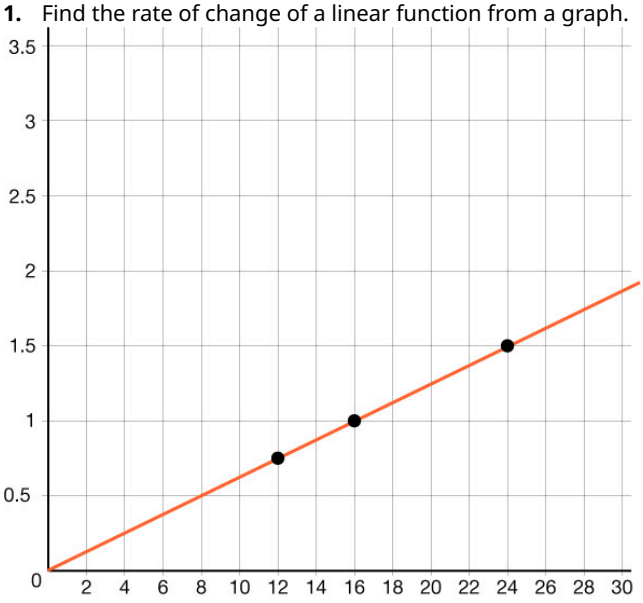
a. $\frac{1}{3}$
b. $-\frac{1}{3}$
c. -3
d. 3

Additional Resources

Rate of Change of Linear Functions.

The graph of a linear function is identical to the graph of the linear equation that describes it. The rate of change of a linear equation is constant. The rate of change is the slope, $m = \frac{y_2 - y_1}{x_2 - x_1}$

1. Find the rate of change of a linear function from a graph.



The rate of change of a linear function is constant, so you can choose any 2 points to find the slope.

Use $(16, 1)$ and $(24, 1.5)$.

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1.5 - 1}{24 - 16} = \frac{0.5}{8} = \frac{1}{16}$

The rate of change is $\frac{1}{16}$

2. Find the rate of change of a linear function from a table.

Input	Output
-2	3
8	-2

Input	Output
10	-3
20	-8

The rate of change of a linear function is constant, so you can choose any 2 points to find the slope.

Use $(-2, 3)$ and $(8, -2)$.

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-3}{8-(-2)} = \frac{-5}{10} = -\frac{1}{2}$

The rate of change is $-\frac{1}{2}$.

3. Find the rate of change of a linear function from an equation.

$F(x) = 7x + 6$

Use the slope intercept form, $y = mx + b$, where m is the slope and b is the y-intercepts.

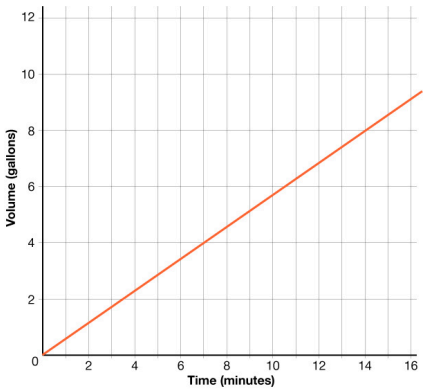
$y = 7x + 6$

The slope is 7. The rate of change is 7.

TRY IT **Try It: Rate of Change of Linear Functions**

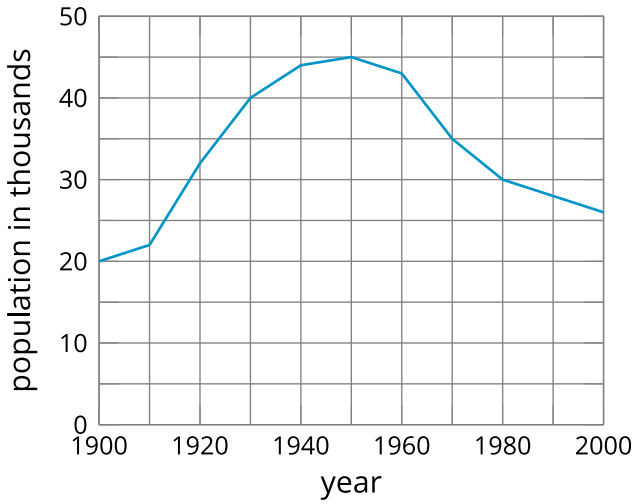
The function $V(x)$ gives the volume of water that flows from a faucet in gallons during ◈ minutes.

$V(x)$ is a linear function with the graph shown.What is the rate of change of the volume?



4.8.5 Interpreting Average Rates of Change with Real-World Examples

Cool Down



The graph above shows the population of a city from 1900 to 2000. Use the graph to answer questions 1 – 5.

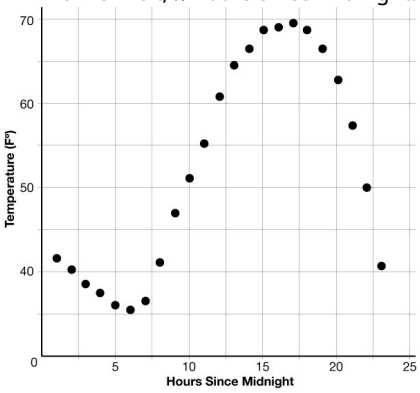
1. What is the average rate of change of the population between 1930 and 1950? Be prepared to show your reasoning.
2. Decide if the average rate of change is positive or negative for the time period from 1930 to 1940.
3. Decide if the average rate of change is positive or negative for the time period from 1950 to 1970.
4. Decide if the average rate of change is positive or negative for the time period from 1930 to 1970.
5. In which decade (10-year interval) did the population grow the fastest? Explain how you know.

4.8.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

Use the following information to answer questions 1 - 5.

The temperature was recorded at several times during the day. Function T gives the temperature in degrees Fahrenheit, n hours since midnight. A graph for this function is provided.



1. For the interval from $n = 1$ to $n = 5$, determine if the average rate of change of the temperature is positive, negative, or zero.

   a. positive
   b. negative
   c. zero

2. For the interval from $n = 5$ to $n = 7$, determine if the average rate of change of the temperature is positive, negative, or zero.

   a. positive
   b. negative
   c. zero

3. For the interval from $n = 10$ to $n = 20$, determine if the average rate of change of the temperature is positive, negative, or zero.

   a. positive
   b. negative
   c. zero

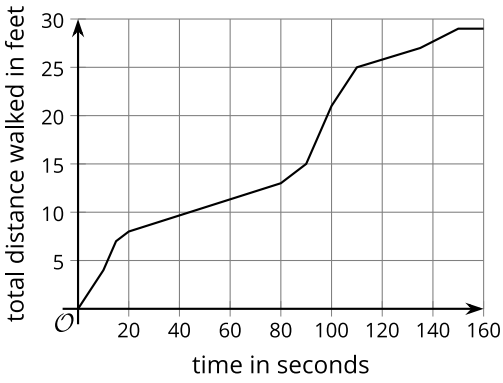
4. For the interval from $n = 15$ to $n = 18$, determine if the average rate of change of the temperature is positive, negative, or zero.

   a. positive
   b. negative
   c. zero

5. For the interval from $n = 20$ to $n = 24$, determine if the average rate of change of the temperature is positive, negative, or zero.

   a. positive
   b. negative
   c. zero

The graph below shows the total distance, in feet, walked by a person as a function of time, in seconds. Use this information to answer questions 6 - 7.



6. Given the graph of total distance walked as a function of time, was the person walking faster between 20 and 40 seconds or between 80 and 100 seconds?

a. Between 80 and 100 seconds
b. Between 20 and 40 seconds
c. Between 0 and 40 seconds

7. Was the person walking faster between 0 and 40 seconds or between 40 and 100 seconds, based on the given graph?

a. Between 40 and 100 seconds
b. Between 40 and 80 seconds
c. Between 0 and 40 seconds

For questions 8 - 11, use the following scenario.

The height, in feet, of a squirrel running up and down a tree is a function of time, in seconds.

Match each description with a statement about the average rate of change of the function for that interval.

8. The squirrel runs up the tree very fast. The average rate of change is -

a. Large and positive
b. Small and positive
c. Zero
d. Negative

9. The squirrel starts and ends at the same height. The average rate of change is -

a. Large and positive
b. Small and positive
c. Zero
d. Negative

10. The squirrel runs down the tree. The average rate of change is _____.

a. Large and positive
b. Small and positive
c. Zero
d. Negative

11. The squirrel runs up the tree slowly. The average rate of change is _____.

a. Large and positive
b. Small and positive
c. Zero
d. Negative

12. The percentage of voters ages 18 to 29 who participated in each United States presidential election between the years 1988 and 2016 is shown in the table.

Year	1988	1992	1996	2000	2004	2008	2012	2016
Percentage of voters ages 18–29	35.7	42.7	33.1	34.5	45.0	48.4	40.9	43.4

The function P gives the percent of voters ages 18 to 29 years old who participated in the election in year t.

Determine the average rate of change for P between 1992 and 2000.

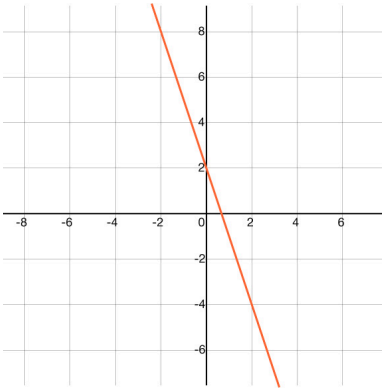
a. –2.4 percent per year
b. 0.35 percent per year
c. –1.025 percent per year
d. 1.025 percent per year

13. Some points from a linear function, $f(x)$, are listed in the table below. What is the rate of change of this linear function?

x	0	5	10	15
$f(x)$	4	6	8	10

a. 6
b. $-\frac{2}{5}$
c. $\frac{2}{5}$
d. 2

14. Find the rate of change of the function, $g(x)$, graphed below.



15. What is the rate of change of the function $h(x) = 4x - 7$?

4.8.7 Lesson Summary

In this lesson, you learned how to:

- Explain the meaning of the term “average rate of change.”
- Estimate or calculate the average rate of change between two points from the graph of a function.

Here are the **activities** that helped you reach those goals:

- 4.8.1: Comparing Changes in Output Based on Input
  - In this activity, you compared the change in output over different periods of the input while looking at a table. You made generalizations about the change over time, without finding the specific rate of change.
- 4.8.2: Average Rate of Change and Slope
  - In this activity, you learned the average rate of change formula and how to apply it to data given in a table or a graph. You were then able to apply the formula to the generalizations you made in activity 4.8.1 to be more precise in your analysis.
  - 4.8.2: Self Check
  - 4.8.2: Additional Resources
- 4.8.3: Finding and Interpreting Average Rates of Change
  - In this activity, you used the average rate of change formula to find the average rate of change from a graph. Once you found the average rate of change, you interpreted the average rate of change in the context of the problem.
  - 4.8.3: Self Check
  - 4.8.3: Additional Resources
- 4.8.4 Rate of Change of Linear Functions
  - In this activity, you found the rate of change of linear functions from a graph, a table, and an equation.
  - 4.8.4: Self Check
  - 4.8.4: Additional Resources
- 4.8.5: Interpreting Average Rates of Change with Real-World Examples
  - In this activity, you determined if the rate of change was positive, negative, or zero. Then you looked at the rate of change more precisely and compared larger intervals of time on the graph.

After these activities, you completed the following **practice**:

- 4.8.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

4.9 INTERPRETING AND CREATING GRAPHS

4.9.0 Lesson Overview

In this lesson, you will apply your understanding of average rate of change to interpret or create graphs of functions that are less well defined and that model real-life situations that are more complex.

When you finish this lesson, you will be able to:

- Explain the average rate of change of a function in terms of a situation.
- Describe important features of a graph and explain what they mean in a situation.
- Sketch a graph that shows important features of the situation, given a description or a visual representation of a situation.

Here are the **activities** that will help you reach those goals:

- 4.9.1: Analyzing and Comparing the Properties of Graphs
- 4.9.2: Interpreting Graphs Without Units
  - 4.9.2: Self Check
  - 4.9.2: Additional Resources
- 4.9.3: Sketching Graphs of Functions
  - 4.9.3: Self Check
  - 4.9.3: Additional Resources
- 4.9.4: Using Verbal Descriptions to Create Graphs of Functions
- 4.9.5: Representing Quantities in a Situation

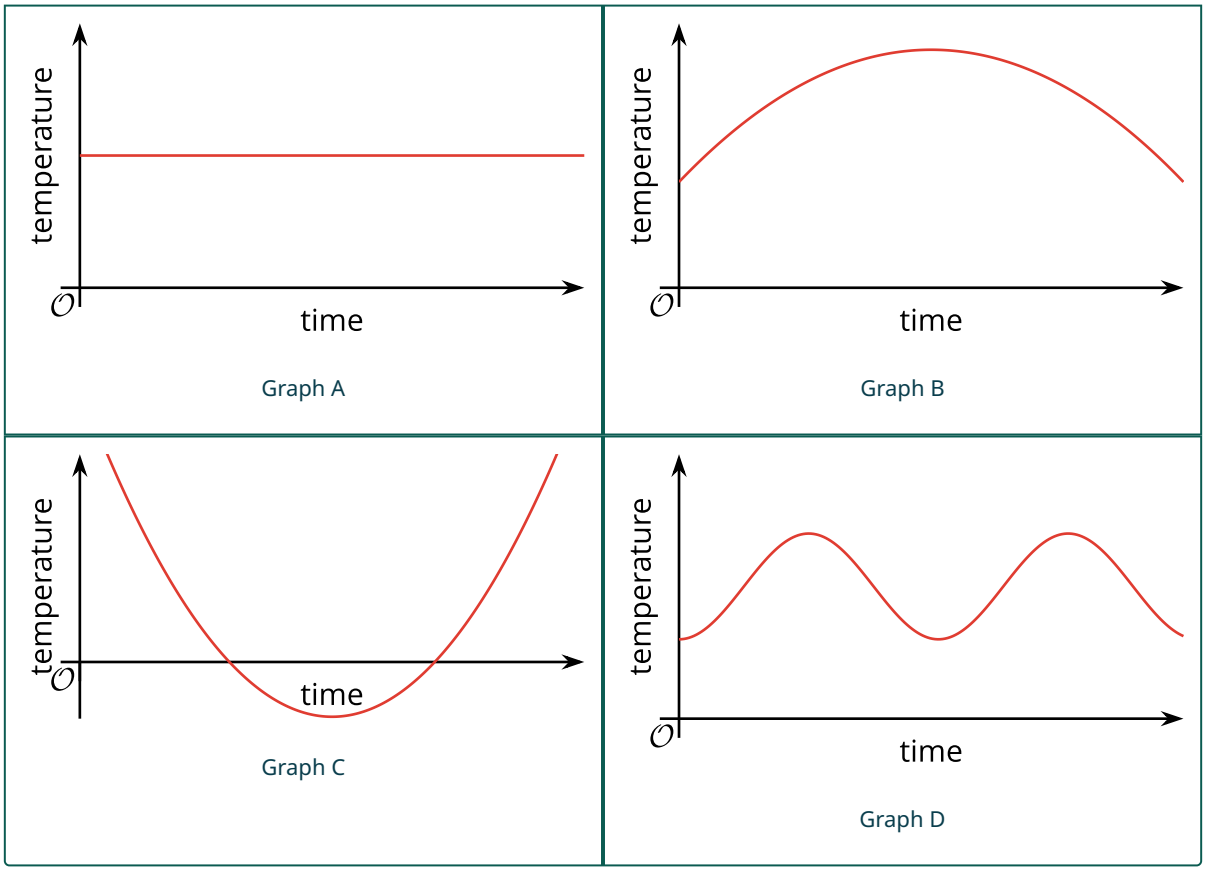
After that, you'll **practice and review**.

- 4.9.6: Practice
- 4.9.7: Lesson Summary

4.9.1 Analyzing and Comparing the Properties of Graphs

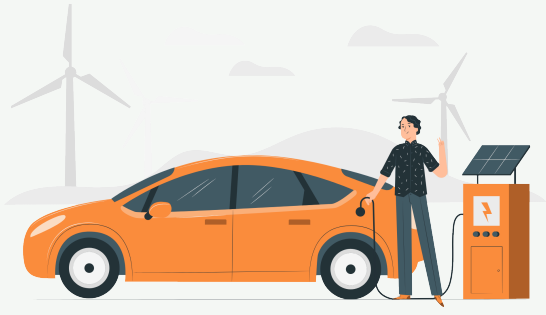
Warm Up

Examine the following 4 graphs and think about why 1 of the graphs does not belong in the same group as the other 3.



1. Which graph do you think does not belong in a group with the other 3 graphs?
2. Explain why you think that graph does not belong.

Why Should I Care?



Supratim Das is a PhD candidate at MIT who is interested in how batteries work and how to make them last longer. He believes that if we produce longer lasting batteries for our mobile devices and cars, we can reduce our overall energy

consumption.

One of the most important mathemtical tools he has used is functions. Functions help him calculate the energy inputs and outputs and the changes that make a battery lose power.

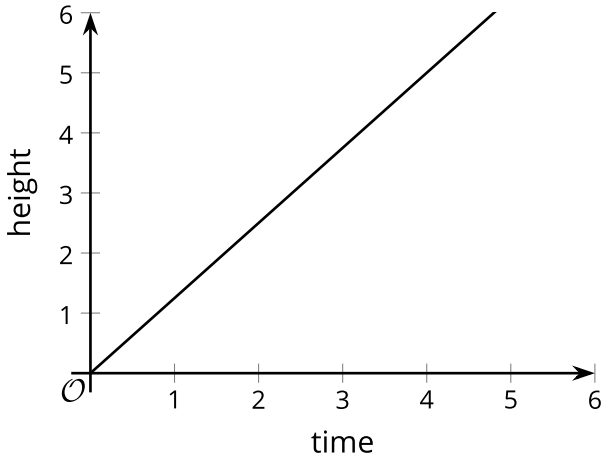
4.9.2 Interpreting Graphs Without Units

Activity

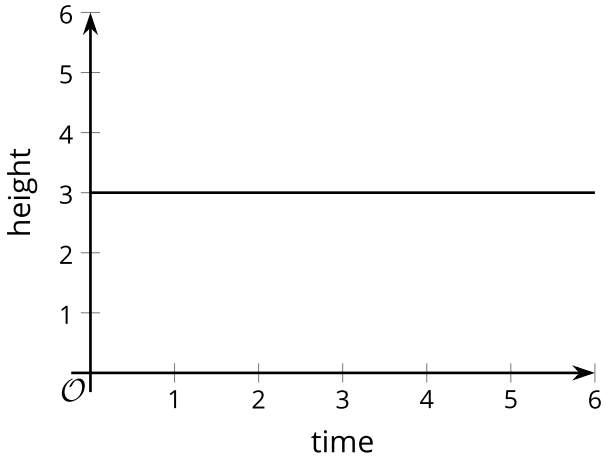
A flag ceremony is held at a Fourth of July event. The height of the flag is a function of time.

Here are some graphs that could each be a possible representation of the function.

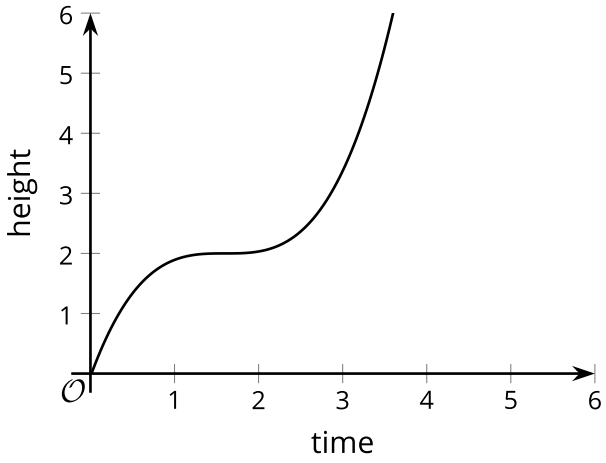
a.



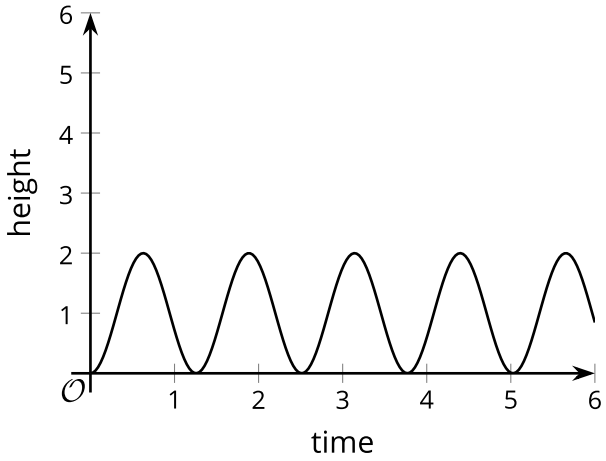
b.



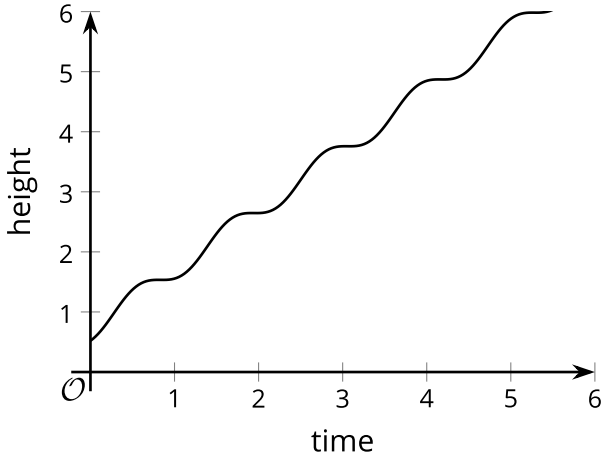
c.


height
time
O
1
2
3
4
5
6

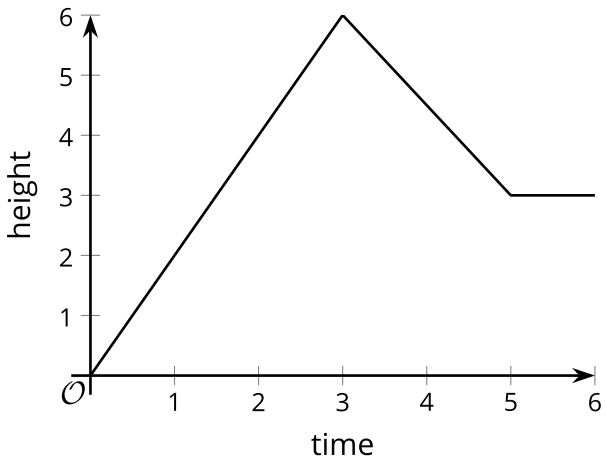
d.


height
time
O
1
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4
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e.


height
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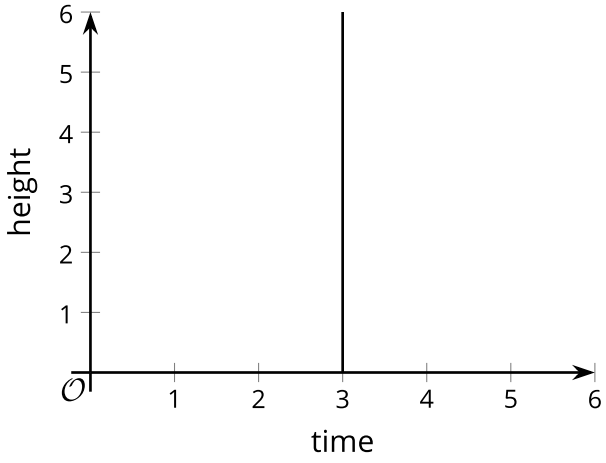
f.



After you examine the graphs, discuss with a partner how each graph could be used to describe the movement of the flag as it is raised on the flagpole.

1. Either select one of the graphs above or use the one assigned to you by your teacher. Explain what the graph tells us about the flag and its height over time.
2. Which graph seems the most realistic and likely to happen at a flag ceremony? Be prepared to explain your reasoning.
3. Which graphs seem most unrealistic or least likely to happen at a flag ceremony? Be prepared to explain your reasoning.
4. Which graphs might be possible, but seem unlikely? Be prepared to explain your reasoning.

Use the following graph to answer questions 5 – 6. Note that the graph also relates time and height.



5. Can this graph represent the time and height of the flag? Be prepared to show your reasoning.
6. Is this a graph of a function? Be prepared to show your reasoning.

Are you ready for more?

Extending Your Thinking

Suppose an ant is moving at a rate of 1 millimeter per second and keeps going at that rate for a long time. If time, x, is measured in seconds, then the distance the ant has traveled in millimeters, y, is $y = 1x$. If time, x, is measured in minutes, the distance in millimeters is $y = 60x$.

1. Explain why the equation $y = (365 \cdot 24 \cdot 3600)x$ gives the distance the ant has traveled, in millimeters, as a function of time, x, in years.

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-9-2-interpreting-graphs-without-units)

2. Use the Desmos graphing tool or technology outside the course to graph the equation.

3. Does the graph look like that of a function? Why do you think it looks this way?
4. Using graphing technology, adjust the graphing window until the graph no longer looks this way. If you manage to do so, describe the graphing window that you use.
5. Do you think the last graph in the flag activity, Graph F, could represent a function relating time and height of a flag? Explain your reasoning.

Video: Interpreting Graphs Without Units

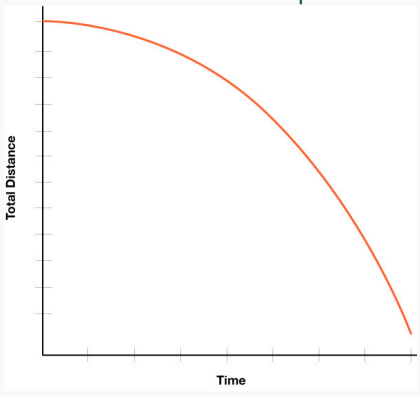
Watch the following video to learn more about how to determine if a graph is realistic when it doesn't have units.

Interpreting Graphs Without Units (https://openstax.org/r/unitless-graphs)

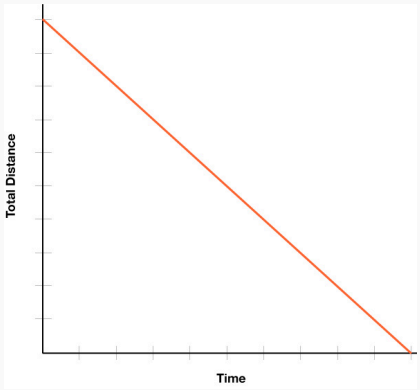
Self Check

Shannon is running on a flat, rocky trail that eventually rises up a steep mountain. Which graph below represents the total distance she covers with respect to time?

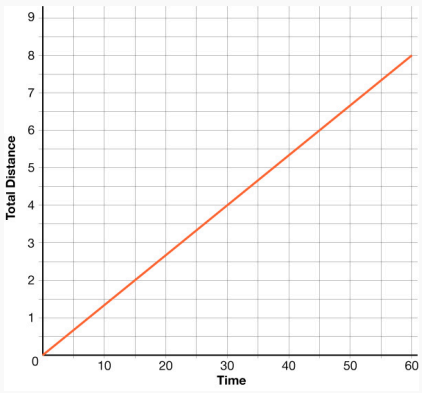
a.



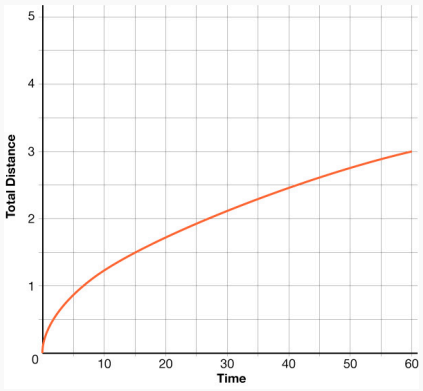
b.



c.



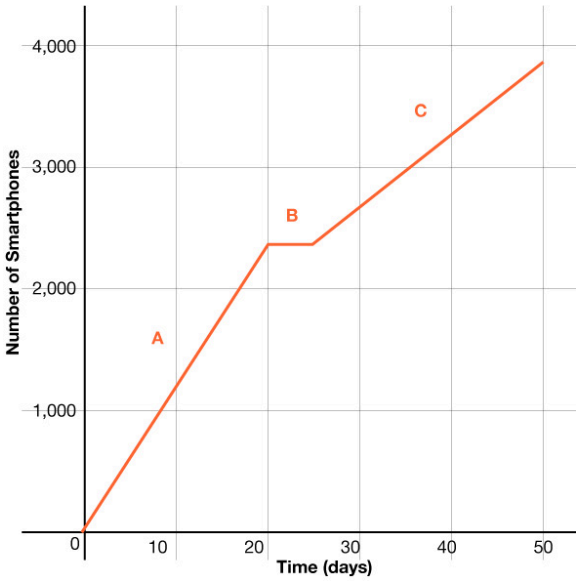
d.



Additional Resources

Interpreting Graphs of Functions

The graph below represents the total number of smartphones that are shipped to a retail store over the course of days.



Match each part of the graph (A, B, and C) to its verbal description. Be prepared to explain the reasoning behind your choice.

Description	Details
Description 1	Half of the factory workers went on strike, and not enough smartphones were produced for normal shipments.
Description 2	The production schedule was normal, and smartphones were shipped to the retail store at a constant rate.
Description 3	A defective electronic chip was found, and the factory had to shut down, so no smartphones were shipped.

Description 1 best matches part C of the graph. If half of the workers went on strike, then the number of smartphones produced would be less than normal. The rate of change for C is less than the rate of change for A.

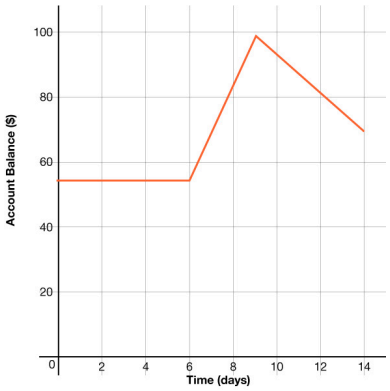
Description 3 best matches part B of the graph. If no smartphones are shipped to the sore, the total number remains

constant during that time.

Part A of the graph best matches description 2. If the production schedule is normal, the rate of change of interval A is greater than the rate of change of interval C.

TRY IT **Interpreting Graphs of Functions**

The relationship between Jameson's account balance and time is modeled by the graph below.



Write a story that models the situation represented by the graph.

4.9.3 Sketching Graphs of Functions

Activity

Your teacher will show a video of a flag being raised. Function H gives the height of the flag over time. Height is measured in feet. Time is measured in seconds since the flag is fully secured to the string, which is when the video clip begins.

1. Use your own graph paper to sketch a graph that could represent the function H. Be sure to include a label and a scale for each axis.
   Hint: Suggestion for how to set up graph (https://openstax.org/r/set-up-graph)

Self Check

Marcus is going to the library with his friend Monica.

- Monica lives between the library and Marcus's house.
- Marcus lives 0.5 mile from Monica, and she lives 0.25 mile from the library.
- Marcus walks to Monica's house in 5 minutes, and he stays there for 15 minutes before they walk together to the library in 3 minutes.

If you were sketching a graph of Marcus's distance from home in miles with respect to time, in minutes, how would you represent the 15 minutes he was at Monica's house graphically?

a. A horizontal line connecting $(5, 0.5)$ and $(20, 0.5)$
b. A line with a constant rate connecting $(0, 0)$ and $(20, 0.5)$
c. A line with a constant rate connecting $(0, 0)$ and $(5, 0.5)$
d. A vertical line connecting $(5, 0.5)$ and $(5, 20)$

Additional Resources

Graphing Situations as Functions

EXAMPLE 1

Draw a graph representing a function that describes Sam's distance from home over an hour.

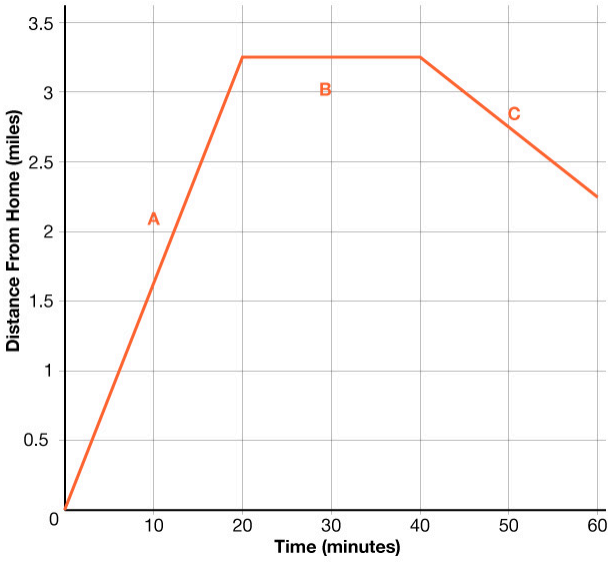
A: Sam rides his bike to his friend's house at a constant rate for 20 minutes.

B: Sam plays for 20 minutes at his friend's house.

C: Sam and his friend bike together to an ice cream shop that is between their houses for 20 minutes.

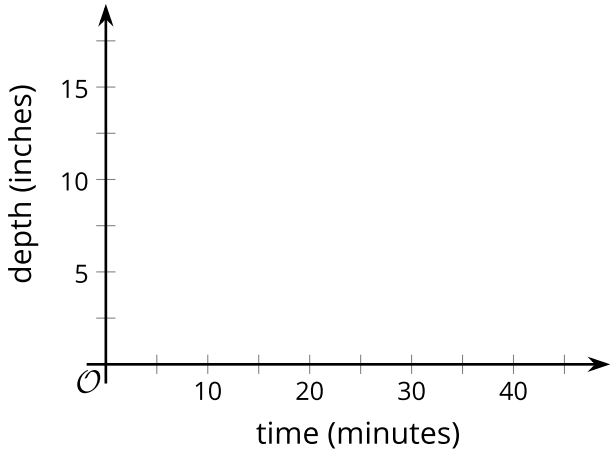
Here is a sample graph. Notice that the x-axis is labeled "time in minutes" and the y-axis is labeled "distance from home (miles)."

The distance from home is increasing as Sam rides to his friend's house. While playing at his friend's house, the distance remained constant and unchanged. When Sam left his friend's house for the ice cream shop, the distance from home decreased because the shop is in between their houses, so Sam was getting closer to home.



EXAMPLE 2

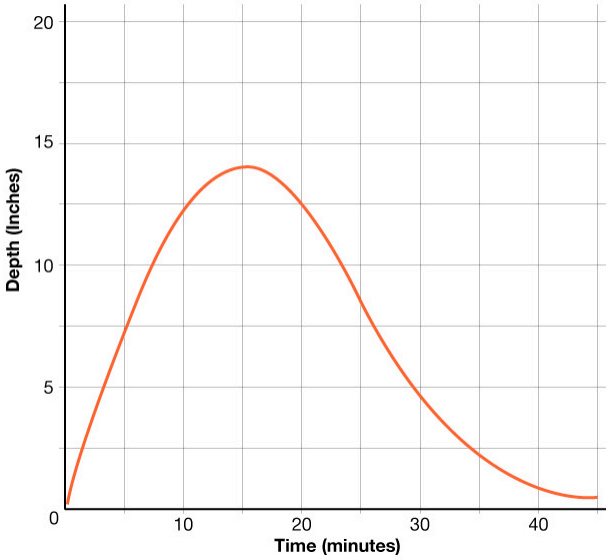
Tyler filled up his bathtub, took a bath, and then drained the tub. The function gives the depth of the water B, in inches, t minutes after Tyler began to fill the bathtub.



These statements describe how the water level in the tub was changing over time. Use the statements to sketch an approximate graph of the function.

- $B(0) = 0$
- $B(1) < B(7)$
- $B(9) = 11$
- $B(10) = B(20)$
- $B(20) > B(40)$

A sample graph below meets all of the statements above.



TRY IT **Graphing Situations as Functions**

A rock climber begins her descent from a height of 50 feet. She slowly descends at a constant rate for 4 minutes. She takes a break for 1 minute; she then realizes she left some of her gear on top of the rock and climbs more quickly back to the top at a constant rate.

Create a graph representing this situation.

4.9.4 Representing Quantities in a Situation

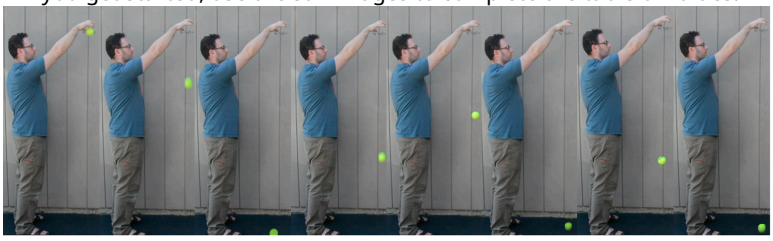
Activity

Watch the video of the tennis ball being dropped.

Height of Tennis Ball Dropping (full speed) (https://openstax.org/r/tennis-ball-full)

The height of the ball is a **function** of time. Suppose the height is h feet, t seconds after the ball is dropped.

1. Create a table of values that displays the height of the tennis ball, in feet, at key moments during the video. To help you get started, use the still images to complete the table of values.



Time (seconds)	Height (feet)
0	A. _____
0.28	B. _____
0.54	C. _____
0.74	D. _____
1.03	E. _____
1.48	F. _____
1.88	G. _____
2.25	H. _____

2. Use a blank coordinate plane to sketch a graph of the height of the tennis ball as a function of time. Use the table of values you created to help guide your graph.

Hint: Suggestion for how to set up graph here (https://openstax.org/r/set-up-graph-4-9-4).

3. Identify the horizontal intercepts (x-intercepts) of the graph. Explain what the coordinates tell us about the tennis ball.
4. Identify the vertical intercept (y-intercept) of the graph. Explain what the coordinates tell us about the tennis ball.
5. Find the maximum value of the function. Explain what it tells us about the tennis ball.
6. Find the minimum value of the function. Explain what it tells us about the tennis ball.

Are you ready for more?

Extending Your Thinking

If you only see the still images of the ball and not the video of the ball bouncing, can you accurately graph the height of the ball as a **function** of time? Explain your reasoning.

4.9.5 Using Verbal Descriptions to Create Graphs of Functions

Cool Down

To prepare for a backyard party, a parent uses two identical hoses to fill a small pool that is 15 inches deep and a large pool that is 27 inches deep.

The height of the water in each pool is a function of time since the water is turned on.



Here are descriptions of three situations. For each situation, sketch the graphs of the two functions on the same coordinate plane, so that $S(t)$ is the height of the water in the small pool after t minutes, and $L(t)$ is the height of the water in the large pool after t minutes.

In both functions, the height of the water is measured in inches.

Situation 1: Each hose fills one pool at a constant rate. When the small pool is full, the water for that hose is shut off. The other hose keeps filling the larger pool until it is full.

1. Sketch your interpretation of the graph of situation 1.

Situation 2: Each hose fills one pool at a constant rate. When the small pool is full, both hoses are shut off.

2. Sketch your interpretation of the graph of situation 2.

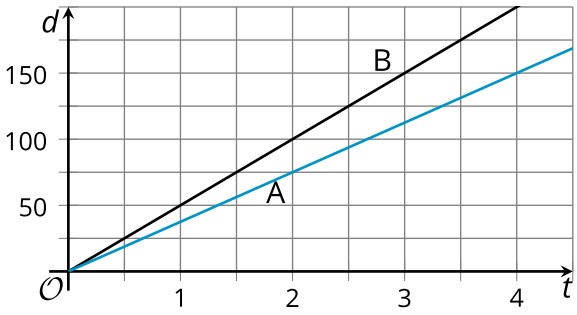
Situation 3: Each hose fills one pool at a constant rate. When the small pool is full, both hoses are used to fill the large pool until it is full.

3. Sketch your interpretation of the graph of situation 3.

4.9.6 Practice

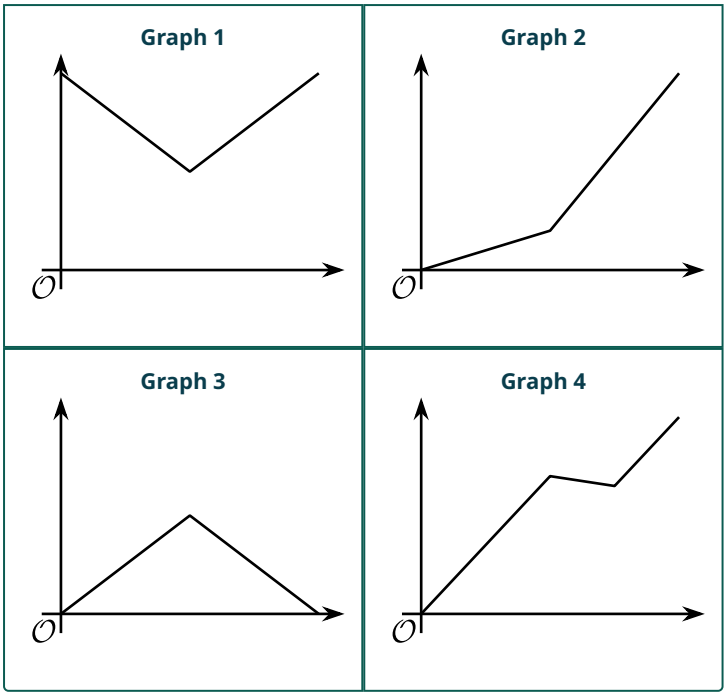
Complete the following questions to practice the skills you have learned in this lesson.

1. The graphs show the distance, d, traveled by two cars, A and B, over time, t. Distance is measured in miles, and time is measured in hours. Which car traveled slower?



a. Car A
b. Car B

For questions 2 – 5, use the graphs below to match each description of a situation in which the volume of water in a tank is a function of time to its corresponding graph.



2. An empty 20-gallon water tank is filled at a constant rate for 3 minutes until it is half full. Then, it is emptied at a constant rate for 3 minutes.

a. Graph 1
b. Graph 2
c. Graph 3
d. Graph 4

3. A full 10-gallon water tank is drained for 30 seconds, until it is half full. Afterward, it gets refilled.

a. Graph 1

b. Graph 2
c. Graph 3
d. Graph 4

4. A 2,000-gallon water tank starts out empty. It is being filled for 5 hours, slowly at first, and faster later.

a. Graph 1
b. Graph 2
c. Graph 3
d. Graph 4

5. An empty 100-gallon water tank is filled in 50 minutes. Then, a dog jumps in and splashes around for 10 minutes, letting 7 gallons of water out. The tank is refilled afterward.

a. Graph 1
b. Graph 2
c. Graph 3
d. Graph 4

For questions 6 – 7, use this situation and function:

The daily cost to the manufacturing company is modeled by the function $C(x) = 7.25x + 2500$, where $C(x)$ is the total daily cost and x is the number of items manufactured.

6. What value equals $C(0)$?

7. What does $C(0)$ mean in the context of the situation?

a. Maximum cost
b. Cost before any items are manufactured
c. Horizontal intercept
d. When the cost is 0

4.9.7 Lesson Summary

In this lesson, you learned how to:

- Explain the average rate of change of a function in terms of a situation.
- Describe important features of a graph and explain what they mean in a situation.
- Sketch a graph that shows important features of the situation, given a description or a visual representation of a situation.

Here are the **activities** that helped you reach those goals:

- 4.9.1: Analyzing and Comparing the Properties of Graphs
  - In this activity, you analyzed and compared graphs that did not have units of measure on the axes. Vocabulary words from prior lessons, such as “maximum,” “minimum,” “increasing,“ and “decreasing,” were used to review and to describe the graphs.
- 4.9.2: Interpreting Graphs Without Units
  - In this activity, you described different graphs that all represented the same situation. You determined which graph best matched the scenario, and which graphs were the least logical considering the scenario. The importance of an appropriate viewing window for technology was also discovered in this activity. You also reviewed the graph of a vertical line, including how it is not a function and why it would not work in the real-world scenario.
  - 4.9.2: Self Check
  - 4.9.2: Additional Reources
- 4.9.3: Sketching Graphs of Functions
  - In this activity, you had to create the graph of function after watching a video of a situation. You had to share and discuss vocabulary, including the different key features of your graph. There was an opportunity to discuss differences in graphs and determine which best reflected the situation from the video.
  - 4.9.3: Self Check
  - 4.9.3 Additional Resources
- 4.9.4: Using Verbal Descriptions to Create Graphs of Functions
  - In this activity, you used several different written or verbal descriptions to create graphs. You had to consider the context of the situation to know when one or both of the graphs became constant.
- 4.9.5: Representing Quantities in a Situation
  - In this activity, you watched a video of a tennis ball being dropped and used the video and still pictures from it to approximate values to sketch a graph. You were then asked specific questions about key features of your sketch.

After these activities, you completed the following **practice**:

- 4.9.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

4.10 COMPARING GRAPHS

4.10.0 Lesson Overview

In this lesson, you will deepen your understanding of functions by comparing representations of several functions relating the same pair of quantities. You will also analyze two or more graphs simultaneously, interpreting their relative features and their average rates of change in context.

When you finish this lesson, you will be able to:

- Compare the features of graphs of functions and explain what they mean in the situations represented.
- Make sense of an equation of the form $f(x) = g(x)$ in terms of a situation and a graph, and know how to find the solutions.
- Explain statements about two or more functions when they are written in function notation.

Here are the **activities** that will help you reach those goals:

- 4.10.1: Analyzing Graphs and Statements in Function Notation
- 4.10.2: Interpreting Graphs and Statements in Terms of a Situation
  - 4.10.2: Self Check
  - 4.10.2: Additional Resources
- 4.10.3: Comparing Functions Represented in Separate Graphs
  - 4.10.3: Self Check
  - 4.10.3: Additional Resources
- 4.10.4: Compare Graphs and Statements that Represent Functions without a Context
- ◦ 4.10.4: Self Check
  - 4.10.4: Additional Resources
- 4.10.5: Analyzing Functions Using Real-World Context

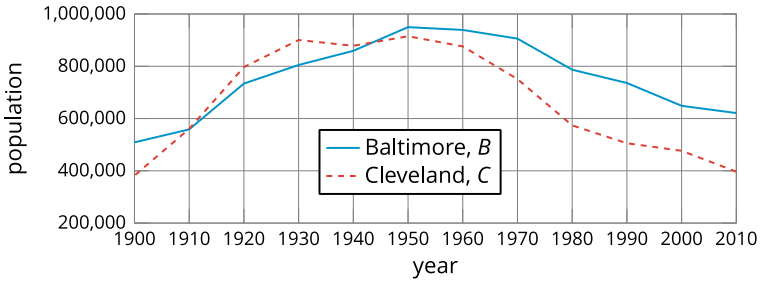
After that, you'll **practice and review**.

- 4.10.6: Practice
- 4.10.7: Lesson Summary

4.10.1 Analyzing Graphs and Statements in Function Notation

Warm Up

This graph shows the populations of Baltimore and Cleveland in the 20^{th} century. $B(t)$ is the population of Baltimore in year t. $C(t)$ is the population of Cleveland in year t.



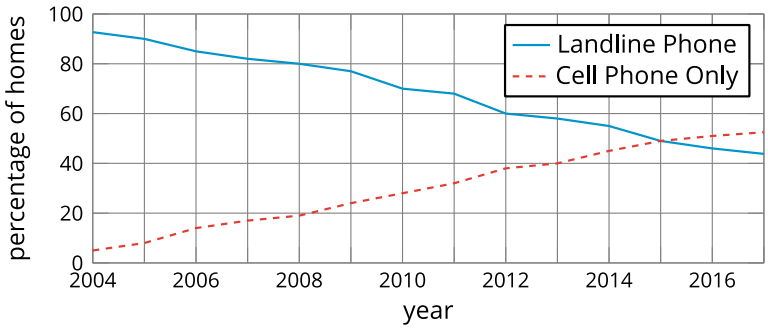
1. Estimate $B(1930)$ and explain what it means in this situation.
2. Which statement is true: $B(2000) \geq C(2000)$ or $B(2000) \leq C(2000)$?
3. Explain the reasoning for your response to question 2.
4. Which statement is true: $B(1900) = C(1900)$ or $B(1900) > C(1900)$?

5. Explain the reasoning for your response to question 4.
6. Were the two cities' populations ever the same? If so, when?

4.10.2 Interpreting Graphs and Statements in Terms of a Situation

Activity

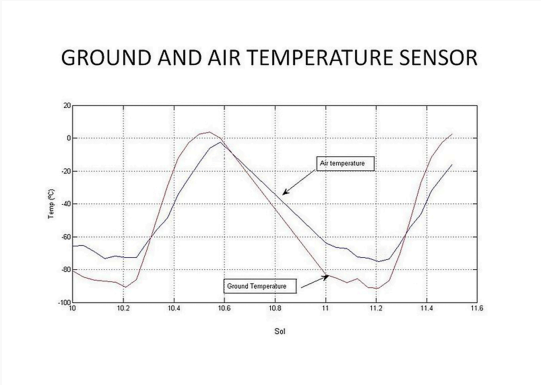
$H(t)$ is the percentage of homes in the United States that have a landline phone in year t. $C(t)$ is the percentage of homes with only a cell phone. Here are the graphs of H and C.



1. Estimate $H(2006)$.
2. Explain what this value tells us about the phones.
3. Estimate $C(2006)$.
4. Explain what this value tells us about the phones.
5. What is the approximate solution to $C(t) = 20$?
6. Explain what the solution to $C(t) = 20$ means in this situation.
7. Is this equation true: $C(2011) = H(2011)$?
8. Explain how you know if the equation is true or not.
9. Is this equation true: $C(2015) = H(2015)$?
10. Explain how you know if the equation is true or not.
11. Between 2004 and 2015, did the percentage of homes with landlines decrease at the same rate at which the percentage of cell-phones-only homes increased?
12. Explain the reasoning for your response to question 11.

Self Check

A Mars rover collected the following temperature data over 1.6 Martian days. A Martian day is called a Sol. The graph below displays the air and ground temperatures in Celsius (y-axis) for specific times measured in Sols (x-axis). Which of the following is true about the comparison between ground temperature and air temperature?

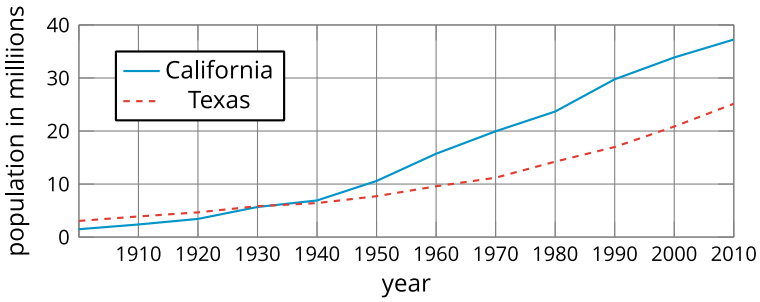


a. The air temperature is always warmer than the ground temperature.
b. At 11 Sols, the air temperature is warmer than the ground temperature.
c. The ground temperature is always warmer than the air temperature.
d. At 11 Sols, the ground temperature is warmer than the air temperature.

Additional Resources

Comparing Populations

Graphs are very useful for comparing two or more functions. Here are graphs of functions C and T, which give the populations (in millions) of California and Texas in year x.



What can we tell about the populations?	How can we tell?	How can we convey this with function notation?
In the early 1900s, California had a smaller population than Texas.	The graph of C is below the graph of T when $x$is 1900.	$C(1900) < T(1900)$
Around 1935, the two states had the same population of about 5 million people.	The graphs intersect at about $(1935, 5)$.	$C(1935) = 5$ and $T(1935) = 5$, and $C(1935) = T(1935)$
After 1935, California has had more people than Texas.	When x is greater than 1935, the graph of $C(x)$ is above that of $T(x)$.	$C(x) > T(x)$ for $x > 1935$
Both populations have increased over time, with no periods of decline.	Both graphs slant upward from left to right.	
From 1900 to 2010, the population of California has risen faster than that of Texas. California had a greater average rate of change.	If we draw a line to connect the points for 1900 and 2010 on each graph, the line for C has a greater slope than that for T.	$\frac{C(2010)-C(1900)}{201-1900} > \frac{T(2010)-T(1900)}{2010-1900}$

TRY IT **Comparing Populations**

Examine the graph given above that displays the populations of California and Texas over time.

1. Which state had a greater population in 1920?

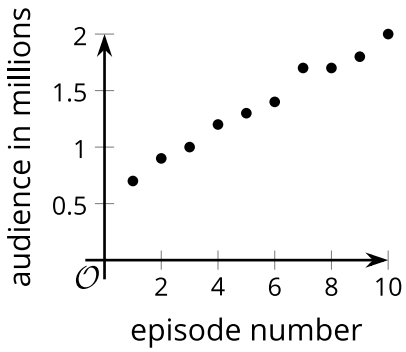
2. Write a true mathematical statement using function notation to describe which state has the greater population in 1920.

4.10.3 Comparing Functions Represented in Separate Graphs

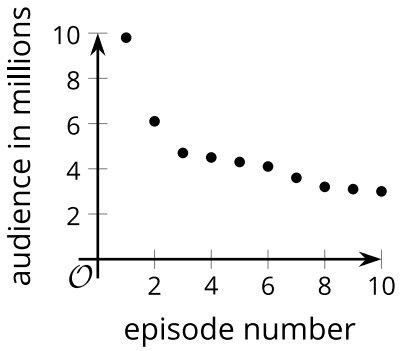
Activity

The number of people who watched a TV episode is a function of that show's episode number. Here are three graphs of three functions—A, B, and C—representing three different TV shows.

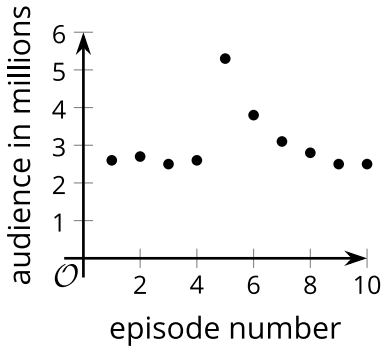
Show A



Show B



Show C



Match each description with a graph that could represent the situation described. One of the descriptions has no corresponding graph.

1. This show has a good core audience. It had a guest star in the fifth episode that brought in some new viewers, but most of them stopped watching after that.
2. This show is one of the most popular shows, and its audience keeps increasing.
3. This show has a small audience, but it's improving, so more people are noticing.
4. This show started out huge. Even though it feels like it crashed, it still has more viewers than another show.
5. Which is greatest, $A(7)$, $B(7)$, $C(7)$
6. Explain what the answer to question 5 tells us about the shows.
7. Use the Desmos graphing tool or technology outside the course to sketch a graph of the viewership of the fourth TV show that did not have a matching graph.

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-10-3-comparing-functions-represented-in-separate-graphs)

Video: Comparing Functions Represented in Separate Graphs

Watch the following video to learn more about how to compare functions that are represented in separate graphs.

Comparing Functions Represented in Separate Graphs (https://openstax.org/r/compare-functions)

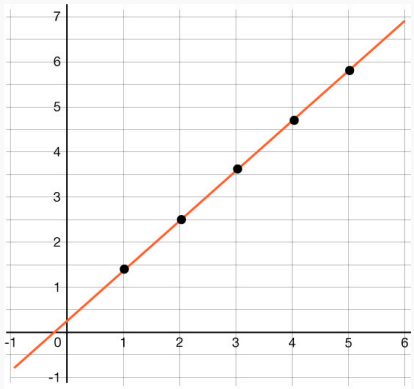
Self Check

For which of the following functions is $f(3)$ the greatest value?

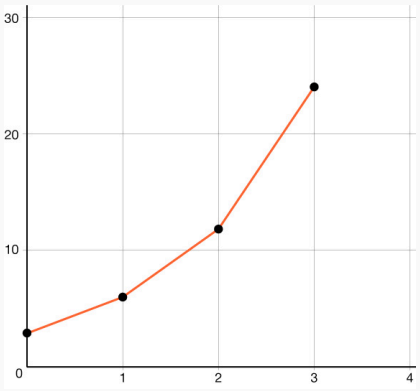
a.



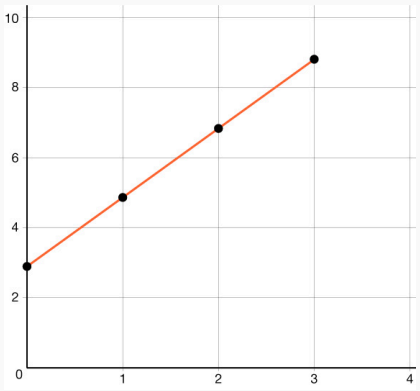
b.



c.



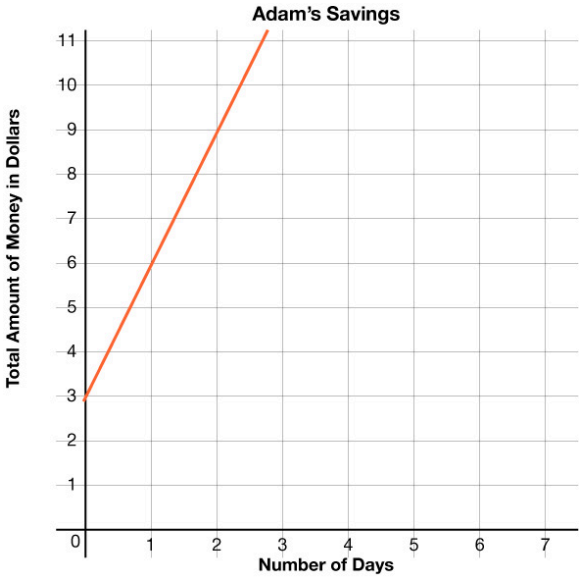
d.

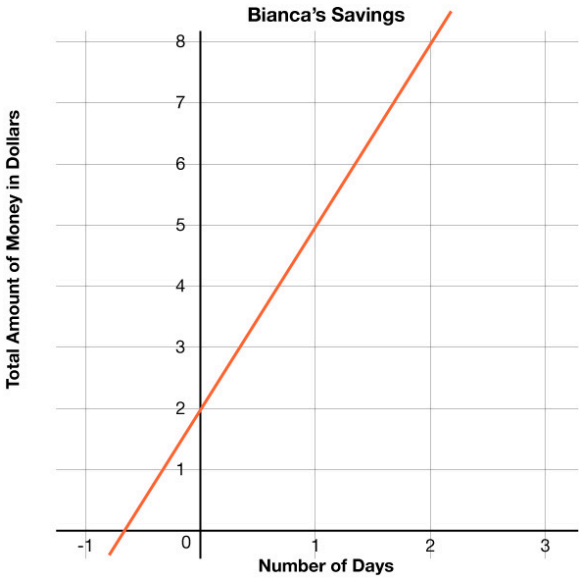


Additional Resources

Comparing Real-Life Functions

Two people, Adam and Bianca, are competing to see who can save more money in one month. Use the table and the graph below to determine who will save more money at the end of the month. State how much money each person had at the start of the competition. (Assume each is following a linear function in his or her saving habit.) Let Adam's function be represented by $A(t)$, where t is days, and Bianca's function be represented by $B(t)$.



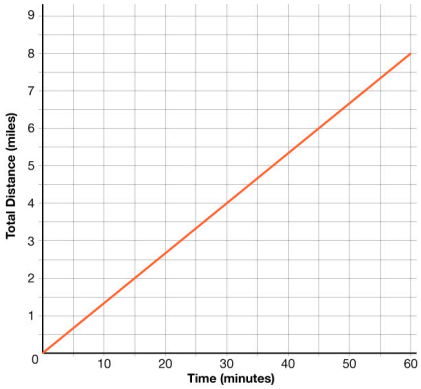


Which is greater, $A(2)$ or $B(2)$? Explain what this answer means.

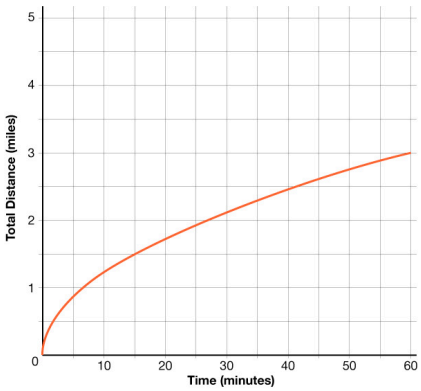
$A(2) = 9$ and $B(2) = 8$, so after 2 days, Adam has saved more money.

TRY IT **Comparing Real-Life Functions**

Aleph is running at a constant rate on a flat, paved road. The graph below, $A(t)$, represents the total distance, A, he covers with respect to time, t.



Shannon is running on a flat, rocky trail that eventually rises up a steep mountain. The graph below, $S(t)$, represents the total distance, S, she covers with respect to time, t.

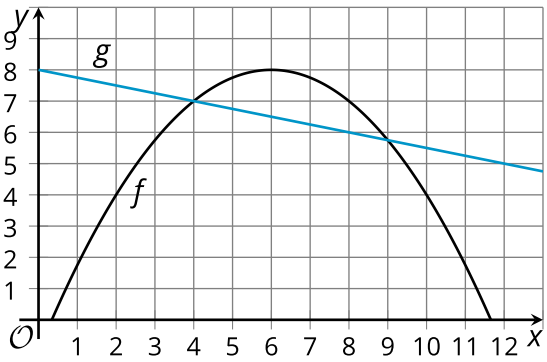


Who has traveled farther after 40 minutes? Explain your answer.

4.10.4 Comparing Graphs and Statements that Represent Functions without a Context

Activity

Here are graphs that represent two **functions**, f and g.

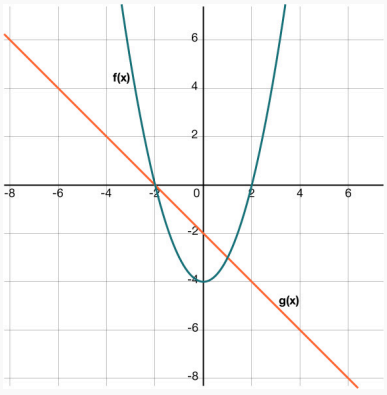


Decide which function value is greater for each given input. Be prepared to explain your reasoning.

1. Which is greater, $f(2)$ or $g(2)$
2. Which is greater, $f(4)$ or $g(4)$
3. Which is greater, $f(6)$ or $g(6)$
4. Which is greater, $f(8)$ or $g(8)$
5. Is there a value of x at which the **equation** $f(x) = g(x)$ is true? Be prepared to show your reasoning.
6. Identify at least two values of x at which the inequality $f(x) < g(x)$ is true.

Self Check

Which of the following is true about the graphs of $f(x)$ and $g(x)$ below?

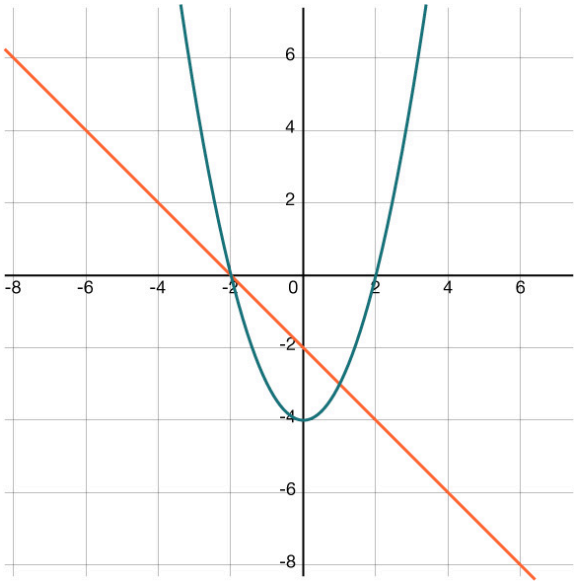


a. $f(-1) = g(-1)$
b. $f(-3) < g(-3)$
c. $f(1) = g(1)$
d. $f(0) > g(0)$

Additional Resources

Using Graphs to Find Values

Two functions, $f(x)$ and $g(x)$, are graphed below.



What is the value of $f(-1)$?

Look at where $x = -1$ touches the graph of f. The y-value is –3.

$f(-1) = -3$

What is the value of $g(-1)$?

Look where $x = -1$ touches the graph of g. The y-value is –1.

$g(-1) = -1$

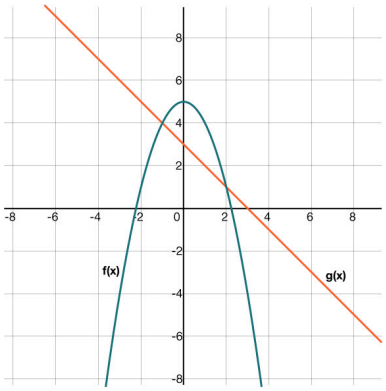
Which value is greater?

$g(-1) > f(-1)$

TRY IT

Using Graphs to Find Values

Two functions, $f(x)$ and $g(x)$, are graphed below.

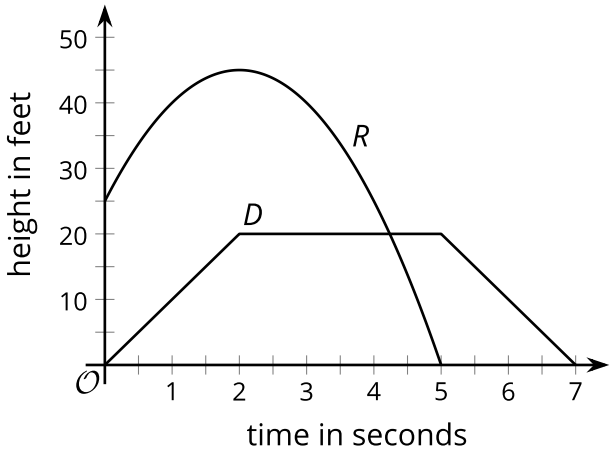


Which is greater, $f(1)$ or $g(1)$?

4.10.5 Analyzing Functions Using Real-World Context

Cool Down

Functions R and D give the height, in feet, of a toy rocket and a drone, t seconds after they are released. Here are the graphs of R (for the rocket) and D (for the drone).



1. Which of the inequalities is true: $R(2) > D(2)$ or $D(2) > R(2)$?
2. What was the height of the drone when the toy rocket hit the ground?
3. For what value of t is $R(t) = D(t)$ true?
4. What does this tell you about the drone and the toy rocket?

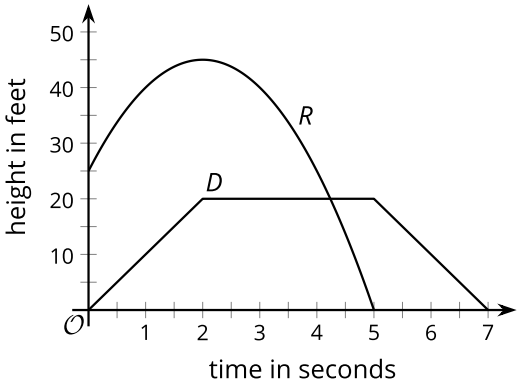
4.10.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

Use the following scenario and graph to answer question 1.

Functions R and D give the height, in feet, of a toy rocket and a drone, t seconds after they are released.

Here are the graphs of R (for the rocket) and D (for the drone).



1. Decide whether the following statements are true or false.

a. $R(3) > D(3)$

a. true
b. false

b. $R(0) = D(0)$

a. true
b. false

c. $R(4.5) = D(4.5)$

a. true
b. false

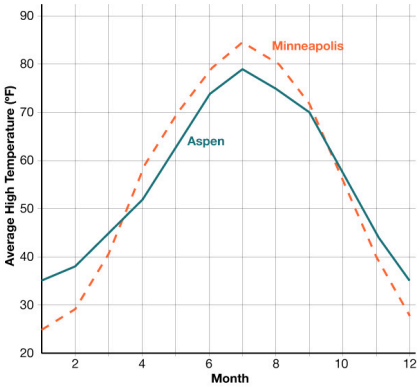
d. $R(0) = D(0) + 25$

a. true
b. false

e. $D(7) > R(5)$

a. true
b. false

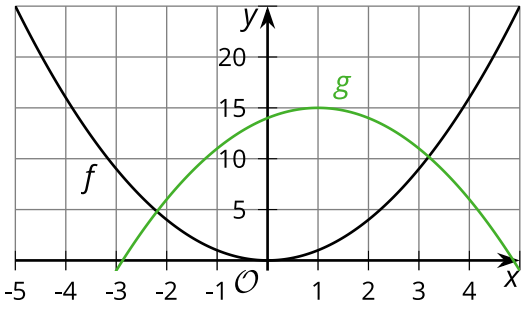
2. $A(t)$ is the average high temperature in Aspen, Colorado, t months after the start of the year. $M(t)$ is the average high temperature in Minneapolis, Minnesota, t months after the start of the year. Temperature is measured in degrees Fahrenheit.



Which function had the higher average rate of change between the beginning of January and middle of March?

a. Aspen
b. Minneapolis

3. Here are two graphs representing functions f and g.

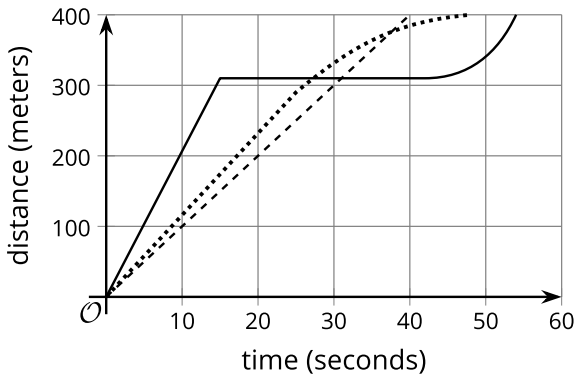


Select the three statements that are true about functions f and g.

a. $f(0) > g(0)$
b. There are values of x where $f(x) = g(x)$
c. $f(-1) < g(-1)$
d. $f(-3) > g(4)$

For questions 4 – 6, use the three graphs below:

The three graphs represent the progress of three runners in a 400-meter race.
The solid line represents Runner A. The dotted line represents Runner B. The dashed line represents Runner C.



4. Which runner ran at a constant rate throughout the race?

a. Runner A
b. Runner B
c. Runner C

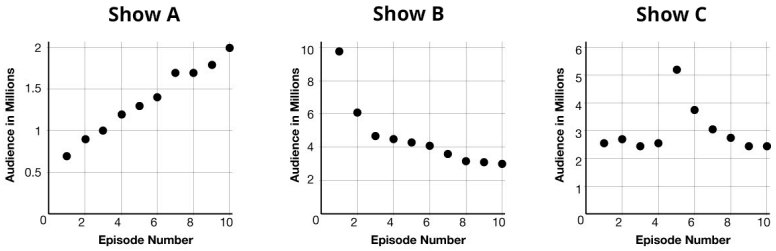
5. Which runner stopped running for a while?

a. Runner A
b. Runner B
c. Runner C

6. Which runner won the race?

a. Runner A
b. Runner B
c. Runner C

Use the graphs below for questions 7 – 8.



7. Which show had the largest audience for episode 4?

a. Show A
b. Show B
c. Show C

8. Which show had the largest audience for their first episode?

a. Show A
b. Show B
c. Show C

4.10.7 Lesson Summary

In this lesson, you learned how to:

- Compare the features of graphs of functions and explain what they mean in the situations represented.
- Make sense of an equation of the form $f(x) = g(x)$ in terms of a situation and a graph, and know how to find the solutions.
- Explain statements about two or more functions when they are written in function notation.

Here are the activities that helped you reach those goals:

- 4.10.1: Analyzing Graphs and Statements in Function Notation
  - In this activity, you used function notation to find specific values on the graph and compare the two functions. You had to estimate the values of functions and determine where the graphs were the same in the context of the problem.
- 4.10.2: Interpreting Graphs and Statements in Terms of a Situation
  - In this activity, more emphasis was placed on the meaning of function notation and what its solution represented. You also used function notation to do calculations.
  - 4.10.2: Self Check
  - 4.10.2: Additional Resources
- 4.10.3: Comparing Functions Represented in Separate Graphs
  - In this activity, you compared the size of the audience per episode of three different TV shows. It was important to notice and analyze the units of each axis before you could compare the three different graphs. You realized that just looking at the shape of the graph can be deceiving, and exact (or at least approximate) data points are better for comparisons.
  - 4.10.3: Self Check
  - 4.10.3: Additional Resources
- 4.10.4: Compare Graphs and Statements that Represent Functions without a Context
  - In this activity, you used a graph with two functions to compare values.
  - 4.10.4: Self Check
  - 4.10.4: Additional Resources
- 4.10.5: Analyzing Functions Using Real-World Context
  - In this activity, you compared the heights of a toy rocket and a drone in a real-life context.

After these activities, you completed the following **practice**:

- 4.10.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

4.11 GRAPHING A FUNCTION USING TRANSFORMATIONS

4.11.0 Lesson Overview

In this lesson, you will work to graph linear functions using different types of transformations, including vertical shifts, vertical compressions, vertical stretches, and vertical reflections.

When you finish this lesson, you will be able to:

- Use slope-intercept form and transformations to graph linear functions.
- Recognize the shifts made to a parent linear function to form another linear function.

Here are the **activities** that will help you reach those goals:

- 4.11.1: Graphing Linear Functions
- 4.11.2: Vertical Shifts
  - 4.11.2: Self-Check
  - 4.11.2: Additional Resources
- 4.11.3: Horizonntal Shifts
  - 4.11.3: Self-Check
  - 4.11.3: Additional Resources
- 4.11.4: Vertical Stretches and Compressions
  - 4.11.4: Self-Check
  - 4.11.4: Additional Resources
- 4.11.5: Horizontal Stretches and Compressions
  - 4.11.5: Self-Check
  - 4.11.5: Additional Resources
- 4.11.6: Graphing Using Transformations

After that, you'll **practice and review**.

- 4.11.7: Practice
- 4.11.8: Lesson Summary

4.11.1 Graphing Linear Functions

Warm Up

1. Use the Desmos graphing tool or technology outside the course. Graph the following lines given in slope-intercept form.
- $y = 3x + 4$
- $y = 3x - 2$

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-11-1-graphing-linear-functions)

2. What is the relationship between the two lines you graphed in question 1?

3. When you examine the equations of the lines, what do you notice is the same about the equations?

4. When you examine the equations of the lines, what do you notice is different about the equations?

5. When you examine the graph of the lines, how many vertical spaces are between the two lines?

6. Use the graphing tool above or technology outside the course, on your coordinate grid, use the slope and y-intercept of each equation to graph each equation. Use a different color for each equation.
- $y = 3x - 2$
- $y = -\frac{1}{3}x - 3$

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-11-1-graphing-linear-functions)

7. When you examine the graph of the lines in question 6, explain if the lines appear to be perpendicular.

4.11.2 Vertical Shifts

Activity

Vertical Shifts

1. Use the Desmos graphing tool or technology outside the course. Graph the following linear functions.

- $f(x) = x + 4$
- $f(x) = x + 2$
- $f(x) = x$
- $f(x) = x - 2$
- $f(x) = x - 4$

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-11-2-vertical-shifts)

2. What are the slopes of each line?

The parent function of a **linear function** is $f(x) = x$.

- The y-intercept is (0,0).
- The slope is 1.

3. How can the parent function, $f(x) = x$ be shifted to overlap $f(x) = x + 2$?

4. How can the parent function, $f(x) = x$, be shifted to overlap $f(x) = x - 4$?

In $f(x) = mx + b$, the b acts as the vertical shift. This is a type of transformation to the graph of $f(x) = x$.

Vertical Shift of a Function

A vertical shift "transforms" the parent function into another function by moving the graph up or down d units.

Vertical Shift → $f(x) + d$

If the d value is positive, the graph of the function shifts up. If the d value is negative, the graph of the function shifts down.

Self Check

What transformation takes place from the graph of $y = 4x - 3$ to $y = 4x + 5$?

a. Left 8
b. Up 8
c. Right 8
d. Down 8

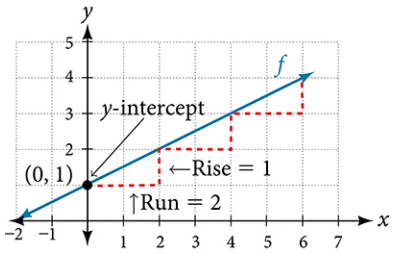
Additional Resources

Vertical Shifts and y-Intercepts

In the equation $f(x) = mx + b$:

- b is the y-intercept of the graph and indicates the point $(0, b)$ at which the graph crosses the y-axis.

- m is the slope of the line and indicates the vertical displacement (rise) and horizontal displacement (run) between each successive pair of points.



The y-intercept tells where the parent function, $f(x) = x$, has shifted up or down.

EXAMPLE 1

Tell how $f(x) = x + 7$ has shifted from the parent function $f(x) = x$.

Solution Since $b = 7$, the y-intercept is +7, so the graph shifts up 7 from the parent function.

EXAMPLE 2

Tell how $f(x) = x - 9$ has shifted from the parent function $f(x) = x$.

Solution Since $b = -9$, the y-intercept is –9, so the graph shifts down 9 from the parent function.

EXAMPLE 3

Tell how $f(x) = 2x + 1$ has shifted from $f(x) = 2x + 5$.

Solution Since $b = 1$, the y-intercept is 1. This point is 4 vertical units below the y-intercepts of the original line that had a y-intercept of 5. So, the graph shifts down 4 units from $f(x) = 2x + 5$ to arrive at $f(x) = 2x + 1$.

Vertical Shift of a Function

A vertical shift "transforms" the parent function into another function by moving the graph up or down d units.

Vertical Shift → $f(x) + d$

If the d value is positive, the graph of the function shifts up. If the d value is negative, the graph of the function shifts down.

4.11.3 Horizontal Shifts

Activity

Horizontal Shifts

1. Use the Desmos graphing tool or technology outside the course. Graph the following linear functions.

- $f(x) = 2(x + 4)$
- $f(x) = 2(x + 2)$
- $f(x) = 2x$
- $f(x) = 2(x - 2)$
- $f(x) = 2(x - 4)$

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-11-3-horizontal-shifts)

2. What is the direction and horizontal distance between the x-intercepts of $f(x) = 2(x)$ and $f(x) = 2(x - 2)$?

3. How can the function, $f(x) = 2(x)$ be shifted to overlap $f(x) = 2(x - 4)$?

4. What is the direction and horizontal distance between the x-intercepts of $f(x) = 2(x)$ and $f(x) = 2(x + 2)$?

5. How can the function, $f(x) = 2(x)$, be shifted to overlap $f(x) = 2(x + 4)$?

In linear functions, when a value is added to or subtracted from the input value, or the x-values, the graph acts as if a horizontal shift is applied. This is another type of transformation to the graph.

Horizontal Shift of a Function

A horizontal shift "transforms" the parent function into another function by moving the graph to the left or right c units.

Horizontal Shift → $f(x - c)$

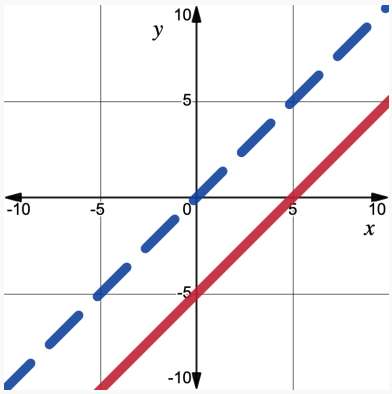
If the c value is positive, the graph of the function shifts right. If the c value is negative, the graph of the function shifts left.

But, be careful! The sign of the c value is preceded by subtraction in the equation. When the equation is simplified, the sign of the c value is changed.

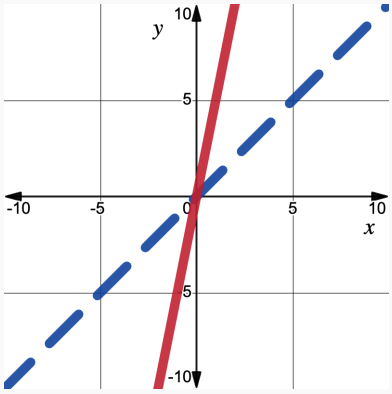
Self Check

Which graph represents $f(x - 5)$ when $f(x) = x$, the linear parent function? The parent function is represented by the dashed, blue line.

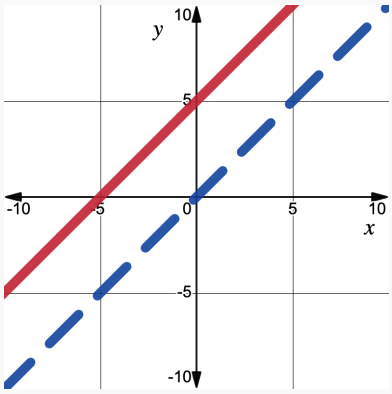
a.



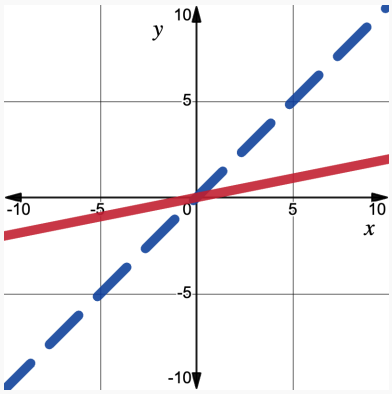
b.



c.



d.



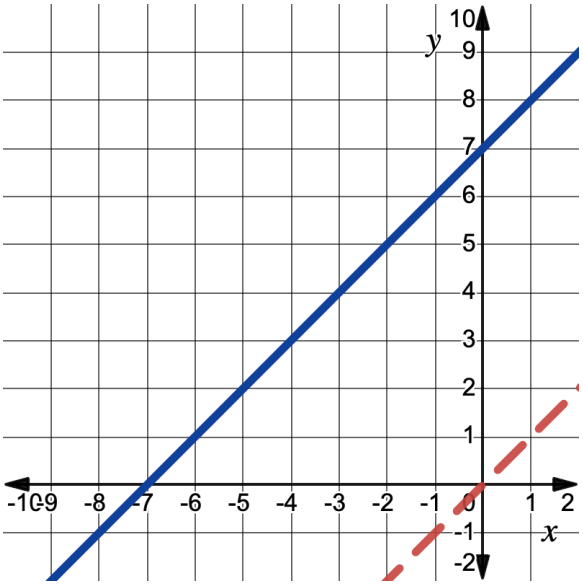
Additional Resources

Horizontal Shifts and x-intercepts

In addition to being shifted vertically, graphs can also be shifted to the right or left. For linear functions, this is sometimes hard to identify because a line that is shifted horizontally may appear to have a vertical shift.

Let's shift the parent function, $f(x) = x$, horizontally to the left 7 units.

If the line shifts horizontally 7 units to the left, the x-intercept is shifted 7 units to the left, as well.



The graph of the parent function, $f(x) = x$, is represented by the dashed orange line. The graph of the solid blue line represents the line that we horizontally shift 7 units to the left.

To determine the equation of the shifted line, we use the slope of 1 and $(-7, 0)$ as a point on the line.

Substituting these values into point-slope form, we get:

$$(y - y_1) = m(x - x_1)$$
$$(y - 0) = 1(x - (-7))$$
$$y = 1(x + 7)$$

So, the equation of a line that has been shifted 7 units to the left from the parent function, $f(x) = x$, is $y = x + 7$

But, doesn't it look like the function $f(x) = x + 7$ was shifted up 7 units, too? This is because $f(x) = x + 7$ is equivalent to $f(x) = 1(x + 7)$. And this is what makes horizontal shifts difficult to see.

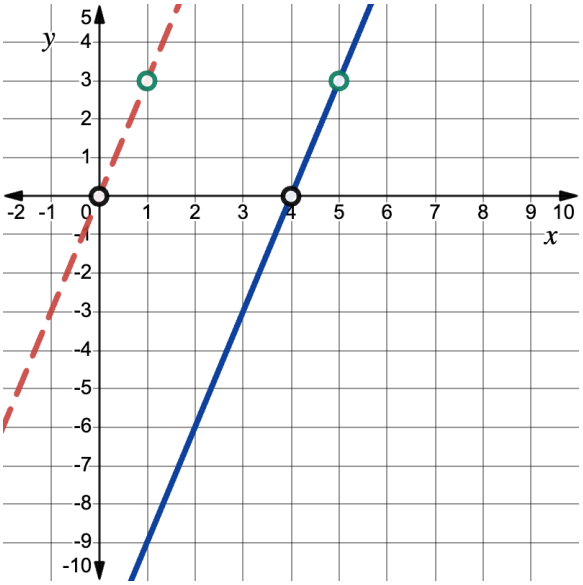
EXAMPLE 1

How has the function $f(x) = 3(x - 4)$ been transformed from $f(x) = 3x$?

Solution

To answer this question, first consider how the x-values, including the x-intercept, are being changed. The function says to subtract 4 from each x-value. This means the function is being shifted 4 units.

Next, we need to determine which direction the x-values are being shifted. Let's look at the graphs.



The solid blue line represents the graph of $f(x) = 3(x - 4)$. The dashed orange line represents the graph of $f(x) = x$.

On the graph, we can look at specific points on the parent function like the x-intercept (0, 0) and (1, 3). The transformed line is to the right of our parent function. To determine how far to the right the line has been shifted, we count the number of horizontal units between the lines.

- We see that (0, 0) corresponds to the point (4, 0) on the transformed line.
- The point (1, 3) corresponds to the point (5, 3) on the transformed line.
- Each point on the parent function has been shifted 4 units to the right.

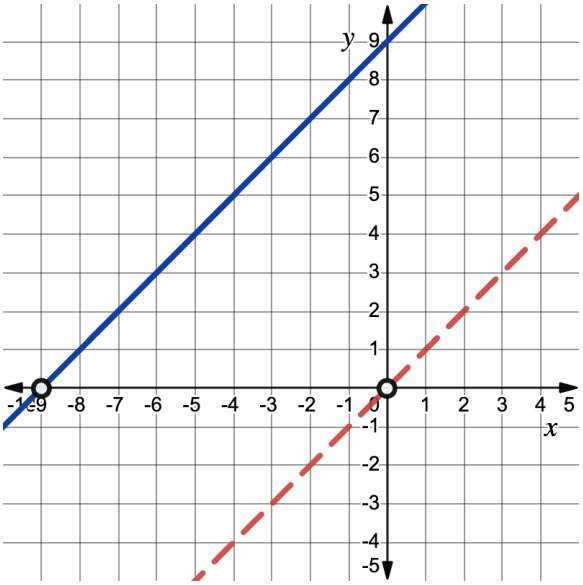
Therefore, we conclude the line representing $f(x) = 3(x - 4)$ appears to have been shifted 4 units to the right from the function $f(x) = 3x$.

EXAMPLE 2

Use the x-intercept to describe how the function $f(x) = x + 9$ has been transformed from the parent function $f(x) = x$.

Solution

We know the shift is 9 units, but we need to consider which direction the x-intercept has been moved.



The orange dashed line represents the parent function, $f(x) = x$ and the solid blue line represents the shifted function.

Carefully looking at the x-intercepts, we see the line representing $f(x) = x + 9$ has been shifted 9 units to the left from the parent function, $f(x) = x$.

Often the horizontal shift is denoted using parentheses to explicitly identify how the values of the independent variable, or domain, are being changed.

EXAMPLE 3

Describe how the function $f(x) = ½(x + 4)$ is horizontally transformed to $g(x) = ½(x - 2)$.

Solution

Let's rewrite the equations in point-slope form, $(y - y_1) = m(x - x_1)$:

$$f(x) = \frac{1}{2}(x + 4)$$
$$(y - 0) = \frac{1}{2}(x + 4)$$
$$(y - 0) = \frac{1}{2}(x - (-4))$$

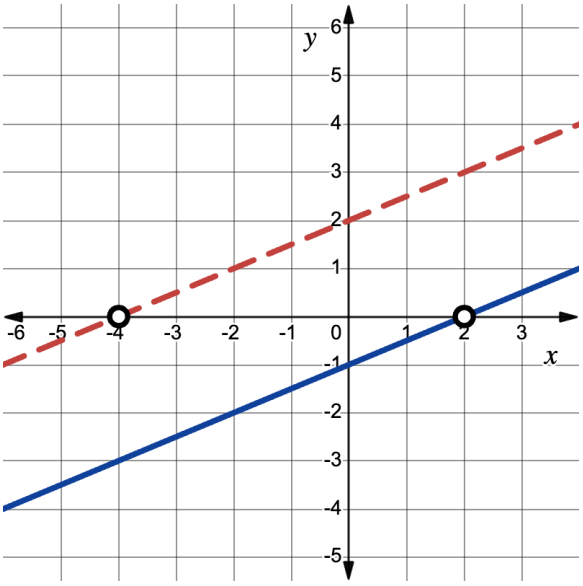
So, the x-intercept is $(-4, 0)$.

$$g(x) = \frac{1}{2}(x - 2)$$
$$(y - 0) = \frac{1}{2}(x - 2)$$

So, the x-intercept is $(2, 0)$.

The x-intercept of $f(x)$ starts at $(-4, 0)$ and moves to $(2, 0)$, so this represents a shift of 6 units to the right.

We can also check this using graphs.



Examining the x-intercept for $f(x) = ½(x + 4)$, on the orange dashed line, we see that $(-4, 0)$ gets shifted 6 units to the right and lands at $(2, 0)$ on $g(x) = ½(x - 2)$, represented by the blue solid line.

So, we can identify horizontal shifts using either graphs or algebraic equations.

Horizontal Shift of a Function

A horizontal shift "transforms" the parent function into another function by moving the graph to the left or right c units.

Horizontal Shift → $f(x - c)$

If the c value is positive, the graph of the function shifts right. If the c value is negative, the graph of the function shifts left. But, be careful! The sign of the c value is preceded by subtraction in the equation. When the equation is simplified, the sign of the c value is changed.

TRY IT **Horizontal Shifts and x-intercepts**

Tell the shift made to $f(x) = x$ to arrive at $f(x) = x - 5$ using the x-intercepts to describe the change.

4.11.4 Vertical Stretches and Compressions

Activity

Vertical Stretches and Compressions

1. On the same coordinate grid graph the following linear functions. You do not need to label the lines in the graphing tool.

Line Label	Function
Line a	$f(x) = 3x$
Line b	$f(x) = 2x$
Line c	$f(x) = x$

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-11-4-vertical-stretches-and-compressions)

2. What do you notice about the slopes of lines c, b, and a as the number of the coefficient gets larger?

3. Now, graph lines d and e on the same graph as $a - c$.

Line Label	Function
Line d	$f(x) = \frac{1}{2}x$
Line e	$f(x) = \frac{1}{3}x$

4. How does Line d, representing $f(x) = \frac{1}{2}x$, compare to Line b, representing $f(x) = 2x$?

5. Examine lines d and e. What can you say about the lines when $0 < m < 1$?

6. Now graph lines f, g, and h on the same coordinate grid.

Line Label	Function
Line f	$f(x) = -2x$
Line g	$f(x) = -x$
Line h	$f(x) = -\frac{1}{2}x$

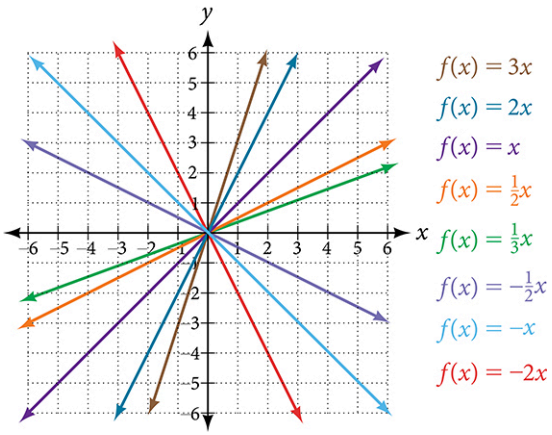
Access multimedia content (http://openstax.org/books/algebra-1/pages/4-11-4-vertical-stretches-and-compressions)

7. Compare the equations of lines $f - h$ to the graphs of lines $b - d$. What changed with the equations of these functions?

8. Compare the graphs of lines $f - h$ to the graphs of lines $b - d$. How did the change in their equations affect their graphs?

In the equation $f(x) = ax$, the a is acting as the **vertical stretch** or **vertical compression** of the parent function. It dilates the graph. When a is negative, there is also a **vertical reflection** of the graph.

- Multiplying the equation of $f(x) = x$ by a vertically stretches or expands the graph of f by a factor of a units if $a > 1$.
- The a vertically compresses the graph of f by a factor of a units if $0 < a < 1$.
- This means the larger the absolute value of a, the steeper the slope.



In $f(x) = mx + b$, the m acts as the **vertical dilation** (stretch or compression factor). This is a type of **transformation** to the graph of $f(x) = x$.

Vertical Dilations of a Function

A vertical stretch or compression "transforms" the parent function into another function by vertically dilating the graph by "a" units.

Vertical Dilation → $af(x)$

If the a value is negative, the graph is reflected over the x-axis. If the a value is greater than 1, the graph is stretched by a scale factor of a. If the a value is between 0 and 1, the graph is compressed by a scale factor of a.

Video: Identifying Vertical Stretches and Compressions

Watch the following video to learn more about vertical stretches and compressions.

Identifying Vertical Stretches and Compressions (https://openstax.org/r/vertical-transformations)

Self Check

How does the graph of $f(x) = \frac{1}{4}x - 3$ change from the parent function, $f(x) = x$?

a. Vertical stretch of $\frac{1}{4}$, left 3
b. Vertical compression of $\frac{1}{4}$, up 3
c. Vertical compression of $\frac{1}{4}$, down 3
d. Vertical stretch of $\frac{1}{4}$, down 3

Additional Resources

Vertical Transformations of Linear Functions

Given the equation of a linear function, use transformations to graph the linear function in the form $f(x) = mx + b$.

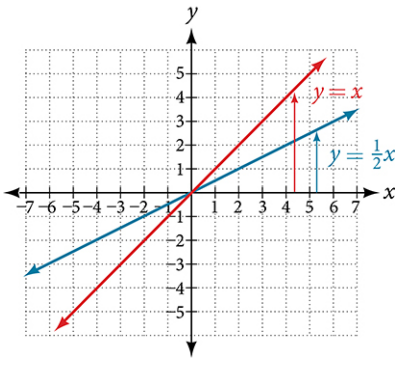
EXAMPLE

Graph $f(x) = \frac{1}{2}x - 3$ using transformations.

Solution

Step 1 - Graph the parent function, $f(x) = x$.

Step 2 - Vertically stretch or compress the graph by a factor m. The equation for the function shows that $m = \frac{1}{2}$, so the identity function is vertically compressed by $\frac{1}{2}$.

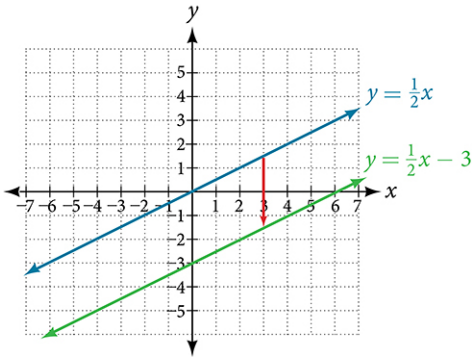


This graph shows two functions on an (x, y) coordinate plane. One shows an increasing function of $y = x$ divided by 2 that runs through the points $(0, 0)$ and $(2, 1)$. The second shows an increasing function of $y = x$ and runs through the points $(0, 0)$ and $(1, 1)$.

Step 3 - Shift the graph up or down b units.

The equation for the function also shows that $b = -3$, so the parent function is vertically shifted down 3 units.

Now, show the vertical shift:



This graph shows two functions on an (x, y) coordinate plane. The first is an increasing function of $y = x$ divided by 2 and runs through the points $(0, 0)$ and $(2, 1)$. The second shows an increasing function of $y = x$ divided by 2 minus 3 and passes through the points $(0, 3)$ and $(2, -2)$. An arrow pointing downward from the first function to the second function reveals the vertical shift.

TRY IT **Vertical Transformations of Linear Functions**

Name the transformations to go from the parent function, $f(x) = x$, to $f(x) = 4 + 2x$.

4.11.5 Horizontal Stretches and Compressions

Horizontal Stretches and Compressions

Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

1. On the same coordinate grid, graph the following linear functions. You do not need to label the lines in the graphing tool.

Line Label	Function
Line i	$f(x) = x$
Line j	$f(2x)$
Line k	$f(\frac{1}{2}x)$

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-11-5-horizontal-stretches-and-compressions)

2. What is the difference between what you entered for the functions of lines j and k in comparison to the function for line j?

3. Now, graph lines m and n on the same graph as $i - k$.

Line Label	Function
Line m	$m(x) = 2x$
Line n	$n(x) = \frac{1}{2}x$

4. How does line j, representing $f(2x)$, compare to line m, representing $m(x) = 2x$?

5. How does line k, representing $f(\frac{1}{2}x)$, compare to line n, representing $n(x) = \frac{1}{2}x$?

Horizontal dilations that stretch and compress **linear** functions are difficult to identify from graphs because they can appear as if they are vertical dilations. However, they are different because:

- The changes to the function are applied to the input values of a function when the transformation is a **horizontal dilation**.
- The changes to the function are applied to the output values of a function when the transformation is a **vertical dilation**.

6. Now graph the lines given below on the same coordinate grid.

Line Label	Function
Line i	$f(x) = x$
Line j	$f(2x)$
Line p	$f(3x)$
Line q	$f(4x)$

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-11-5-horizontal-stretches-and-compressions)

7. As the coefficient on the input increases, what is happening to the graph?

8. Graph the lines given below on the same coordinate grid.

Line Label	Function
Line i	$f(x) = x$
Line k	$f(\frac{1}{2}x)$
Line r	$f(\frac{1}{4}x)$
Line s	$f(\frac{1}{10}x)$

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-11-5-horizontal-stretches-and-compressions)

9. As the coefficient on the input approaches 0, what is happening to the graph?

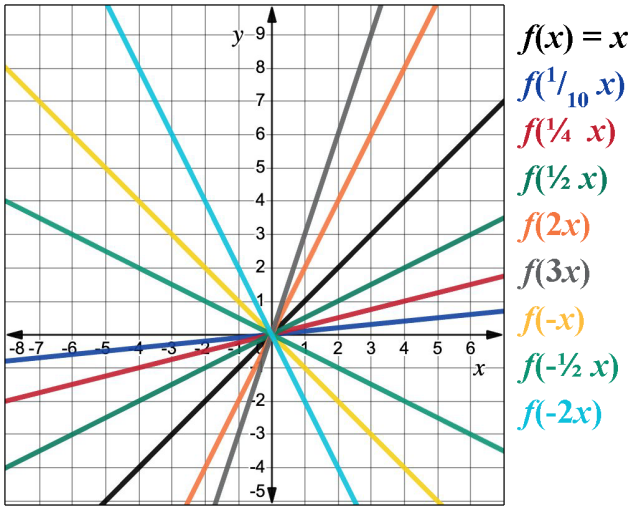
10. Graph the lines given below on the same coordinate grid.

Line Label	Function
Line t	$f(\frac{1}{2}x)$
Line u	$f(-\frac{1}{2}x)$

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-11-5-horizontal-stretches-and-compressions)

In the function $f(bx)$, the b is acting as the **horizontal stretch** or **horizontal compression** of the parent function. It dilates the graph.

- When b is negative, there is also a **horizontal reflection** of the graph over the y-axis.
- Multiplying the function's input by b horizontally compresses the graph of f by a factor of b units if $b > 1$.
- The b horizontally expands the graph of f by a factor of b units if $0 < b < 1$.



Horizontal Dilations of a Function

A horizontal stretch or compression "transforms" the parent function into another function by horizontally dilating the graph by "b" units.

Horizontal Dilation → $f(bx)$

If the b value is negative, the graph is reflected over the y-axis. If the b value is greater than 1, the graph is horizontally compressed. If the b value is between 0 and 1, the graph is horizontally stretched.

Self Check

Which function represents $g(x)$, a function transformed from $f(x)$ by a horizontal compression?

a. $g(x) = f(x + 5)$
b. $g(x) = f(x) + 5$
c. $g(x) = f(5x)$
d. $g(x) = 5f(x)$

Additional Resources

Horizontal Transformations of Linear Functions

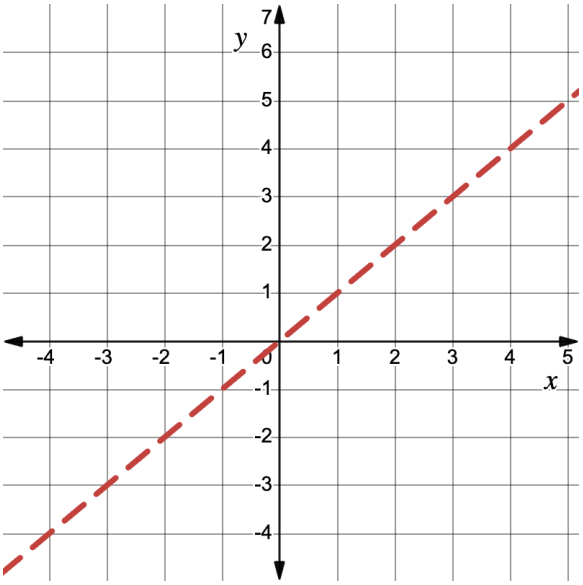
Horizontal transformations are the result of altering the input values before applying the function rule. Vertical transformations result from altering the output values after the function rule has been applied.

EXAMPLE 1

Describe how $f(x)$ is transformed into $f(x-3)$.

Solution

First, graph the given function, $f(x)$, the parent linear function.



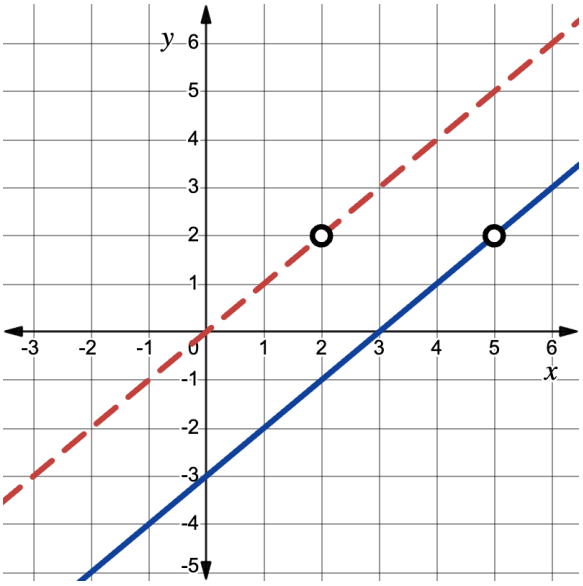
Now, let's examine an input value of $x=2$ using $f(x)$.

- For the parent function $f(x)$, all points on the line are represented by $(x, f(x))$.
- So, when $x=2$, then $f(x)$ becomes $f(2)$.
- That means this point is located at $(2, f(2))$ or $(2, 2)$.

In order to get the same output value for $f(x-3)$ as we had for $f(x)$, we need to figure out how to change the input value.

- Because our transformed equation is subtracting three from what we input, we have to increase the prior input value by 3 units to yield the same output as $f(x)$.
- The input value that is 3 units larger than $x=2$ is 5. In the transformed equation, we must use $x=5$.

So, in the original equation where $x=2$ resulted in $f(2)=2$, in the transformed equation, we have to use $x=5$ in order for $f(x-3)=f(5-3)=f(2)$.



On the graph, $x = 5$ is located 3 units to the right of $x = 2$. So, the transformed equation of $f(x - 3)$ is 3 units to the right of $f(x)$.

Horizontal transformations are represented by the rule $f(x - c)$ where c indicates the number of units to the right or left that the graph is shifted.

EXAMPLE 2

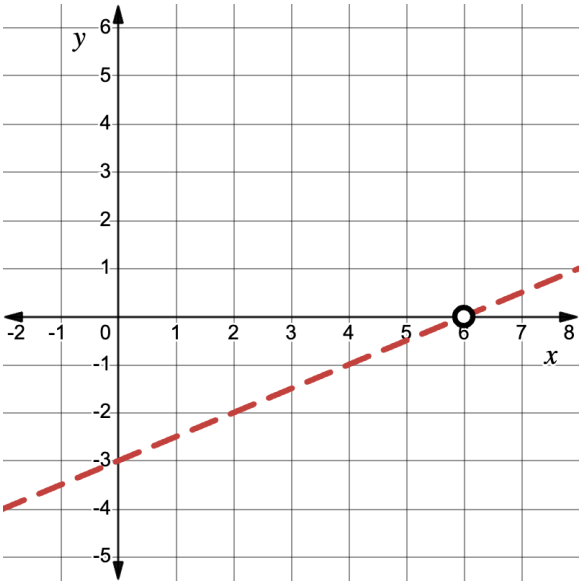
Graph $f(x + 2)$ when $f(x) = \frac{1}{2}x - 3$. Then, determine the equation of the transformed function.

Solution

Step 1 - Graph the given function, $f(x) = \frac{1}{2}x - 3$.

The y-intercept is (0, -3) and the slope is $\frac{1}{2}$.

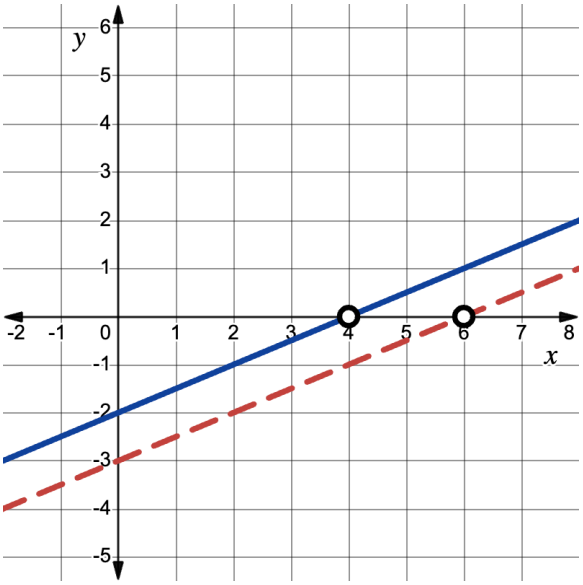
For horizontal shifts, it is helpful to identify the x-intercept. On our graph, the x-intercept or zero is located at (6,0).



Step 2 - Apply the designated transformation. Remember that the rule for a horizontal shift is $f(x - c)$. If we rewrite $f(x + 2)$ in an equivalent form, we can make it is easier to see $x - c$.

$f(x+2) = f(x-(-2))$ That means $c = -2$.

So, $f(x+2)$ represents a horizontal shift of two units to the left. On our graph, we move the x-intercept two units left from (6, 0) to (4, 0).



The orange dashed line is the graph of the original function and the blue solid line is the transformed function.

Step 3 - Determine the equation of the transformed function.

The y-intercept of the transformed line is (0, -2) and the slope of the line is $\frac{2}{4}$ or $\frac{1}{2}$.

That means the equation $f(x+2) = \frac{1}{2}x - 2$

Want to check your work? We can check our equation algebraically by substituting $x+2$ into $f(x)$.

$f(x) = \frac{1}{2}x - 3$

$f(x+2) = \frac{1}{2}(x+2) - 3 = \frac{1}{2}x + 1 - 3 = \frac{1}{2}x - 2$

EXAMPLE 3

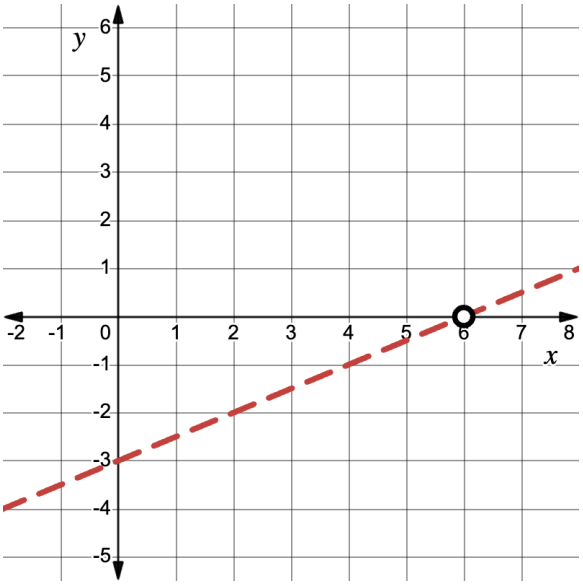
Graph $f(4x)$ when $f(x) = \frac{1}{2}x - 3$. Then, determine the equation of the transformed function.

Solution

Step 1 - Graph the given function, $f(x) = \frac{1}{2}x - 3$.

The y-intercept is (0, -3) and the slope is $\frac{1}{2}$.

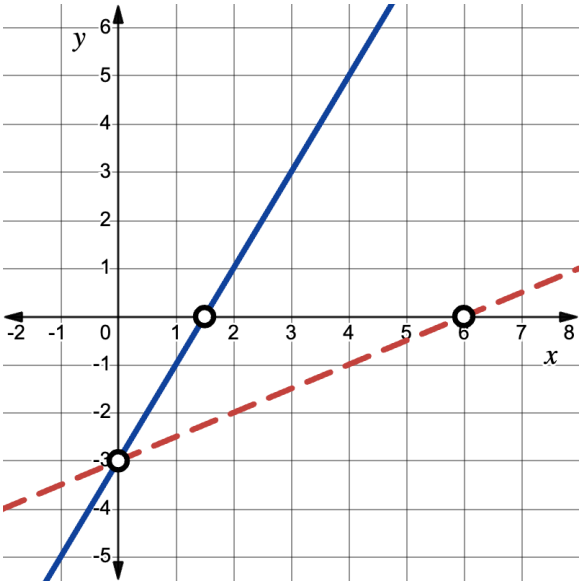
For horizontal shifts, it is helpful to identify the x-intercept. On our graph, the x-intercept or zero is located at (6,0).



Step 2 - Apply the designated transformation. Remember that the rule for a horizontal shift is $f(bx)$. And, when the value of b is greater than 1, the function experiences a horizontal compression.

As a result of the horizontal compression, the linear function should look steeper because it is being compressed by a scale factor of 4.

On our graph, this means the x-intercept is compressed from (6, 0) to (1.5, 0).



The orange dashed line is the graph of the original function and the blue solid line is the transformed function.

Step 3 - Determine the equation of the transformed function.

The y-intercept of the transformed line is (0, -3) and the slope of the line is 3/1.5 or 2.

That means the equation $f(4x) = 2x - 3$.

Want to check your work? We can check our equation algebraically by substituting 4x into $f(x)$.

$f(x) = \frac{1}{2}x - 3$

$f(4x) = \frac{1}{2}(4x) - 3 = 2x - 3$✔

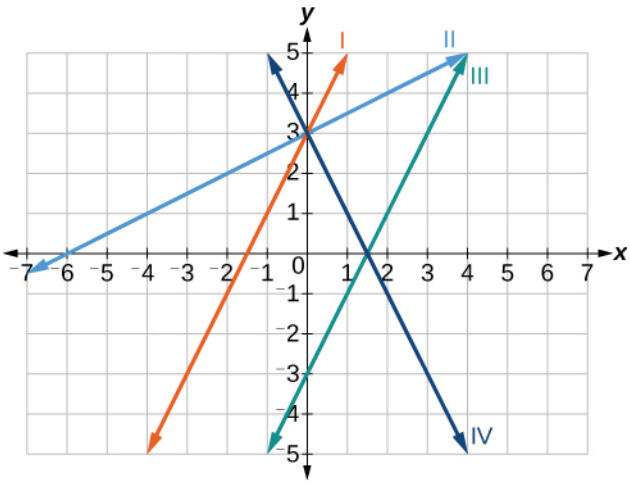
TRY IT **Horizontal Transformations of Linear Functions**

Name the transformation to go from the parent function, $f(x) = x$, to $f(x - 1)$.

4.11.6 Graphing Using Transformations

Cool Down

For questions 1 – 4, match each equation of the linear functions with one of the lines in the given graph.

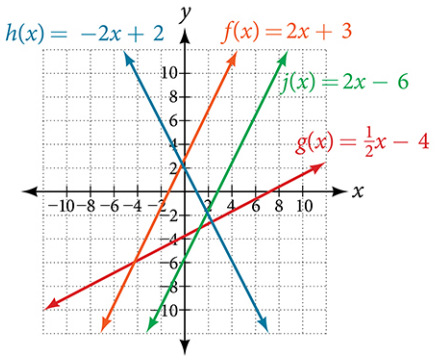


1. $f(x) = 2x + 3$
2. $g(x) = 2x - 3$
3. $h(x) = -2x + 3$
4. $j(x) = \frac{1}{2}x + 3$
5. How will the graph of $k(x) = x$ be affected if the transformation $k(x - 3)$ is applied?

4.11.7 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 - 3, use the graph below.



1. Tell the shift from $j(x) = 2x - 6$ to $f(x) = 2x + 3$.

a. Up 8
b. Up 9
c. Down 8
d. Down 3

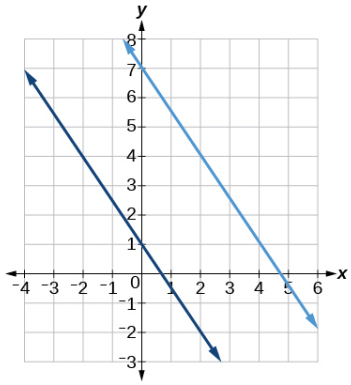
2. Tell the transformations from $f(x) = 2x + 3$ to $h(x) = -2x + 2$.

a. Vertical stretch, left 1
b. Vertical stretch, right 1
c. Vertical reflection, down 1
d. Vertical reflection, up 1

3. Describe the transformations made from the parent linear function to achieve $g(x) = \frac{1}{2}x - 4$. **Select two** that apply:

a. Vertical stretch of $\frac{1}{2}$
b. Vertical compression of $\frac{1}{2}$
c. Up 4
d. Down 4
e. Right 4
f. Left 4

4. What are the equations of the two lines shown?



Select two that apply:

a. $y = x + 4$
b. $y = x - 4$

c. $y = -\frac{3}{2}x + 1$
d. $y = \frac{3}{2}x - 7$
e. $y = \frac{3}{2}x + 1$
f. $y = -\frac{3}{2}x + 7$

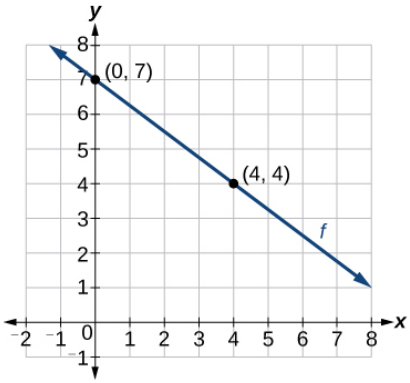
5. The linear function $f(x) = x$ has a vertical stretch of 3, a vertical reflection, and a vertical shift down 4. Which of the following is the new function?

a. $f(x) = -3x - 4$
b. $f(x) = 3x - 4$
c. $f(x) = -3x + 4$
d. $f(x) = 3x + 4$

6. The linear function $f(x) = x$ has a vertical compression of $\frac{1}{3}$ and a vertical shift up 2. Which of the following is the new function?

a. $f(x) = 2x - \frac{1}{3}$
b. $f(x) = 2x + \frac{1}{3}$
c. $f(x) = -\frac{1}{3}x + 2$
d. $f(x) = \frac{1}{3}x + 2$

7. What is the equation of the function graphed?



a. $y = -\frac{4}{3}x + 7$
b. $y = -\frac{3}{4}x + 7$
c. $y = \frac{3}{4}x + 7$
d. $y = 3x + 7$

8. The linear function $f(x) = x$ has a vertical compression of $\frac{1}{3}$ and a vertical shift down 2. Which of the following is the new function?

a. $f(x) = 2x - \frac{1}{3}$
b. $f(x) = 2x + \frac{1}{3}$
c. $f(x) = -\frac{1}{3}x - 2$
d. $f(x) = \frac{1}{3}x - 2$

9. Which two transformations can be used to obtain the graph of $g(x) = -(x - c)$ from the graph of $g(x) = x$?

a. A translation to the left c units followed by a reflection across the y-axis
b. A translation to the right c units followed by a reflection across the y-axis
c. A translation to the left c units followed by a reflection across the x-axis
d. A translation to the right c units followed by a reflection across the x-axis

10. For the functions h and g, which statement is true if $h(x) = g(x + 14) - 12$?

a. The graph of h is the result of the graph of g being translated left 14 units and up 12 units.
b. The graph of h is the result of the graph of g being translated right 14 units and up 12 units.
c. The graph of h is the result of the graph of g being translated left 14 units and down 12 units.
d. The graph of h is the result of the graph of g being translated right 14 units and down 12 units.

11. Which type of transformation can be used to obtain the graph of $g(x) = 4(2 + x)$ from the graph of $f(x) = 2 + x$?

a. Vertical stretch
b. Vertical shift up
c. Vertical shift down
d. Vertical shrink or compression

4.11.8 Lesson Summary

In this lesson, you learned how to:

- Use slope-intercept form and transformations to graph linear functions.
- Recognize the shifts made to a parent linear function to form another linear function.

Here are the **activities** that helped you reach those goals:

- 4.11.1: Graphing Linear Functions
  - In this activity, you reviewed how to graph linear functions using slope-intercept form. You looked at how different slopes and y-intercepts changed the graphs.
- 4.11.2: Vertical Shifts
  - In this activity, you explored how changing the y-intercept causes a vertical shift of a linear function.
  - 4.11.2: Self Check
  - 4.11.2: Additional Resources
- 4.11.3: Horizontal Shifts
  - In this activity, you explored how horizontal shifts impact the equation of a linear function and the x-intercepts on a graph.
  - 4.11.3: Self-Check
  - 4.11.3: Additional Resources
- 4.11.4: Vertical Stretches and Compressions
  - In this activity, you explored how changing the slope causes vertical stretches, compressions, and reflections.
  - 4.11.4: Self-Check
  - 4.11.4: Additional Resources
- 4.11.5: Horizontal Stretches and Compressions
  - In this activity, you explored how changes to the magnitude of the input values for a function causes horizontal compressions, stretches, and reflections.
  - 4.11.5: Self-Check
  - 4.11.5: Additional Resources
- 4.11.6: Graphing Using Transformations
  - In this activity, you matched linear functions in slope-intercept form with their corresponding lines on a graph.
  -

After these activities, you completed the following **practice**:

- 4.11.7: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

4.12 DOMAIN AND RANGE, PART 1

4.12.0 Lesson Overview

In this lesson, you will learn to identify the inputs and outputs as the domain and range of a function. You will calculate real-world examples using money, area, and perimeter to understand how the use of the domain and range of a function can be applied in everyday life.

When you finish this lesson, you will be able to:

- Describe the “domain” and “range” of a function.
- Determine a reasonable domain and range for the function when given a description of a function in a situation.

Here are the **activities** that will help you reach those goals:

- 4.12.1: Determining Reasonable Inputs and Outputs
- 4.12.2: Domain: The Input of a Function
  - 4.12.2: Self Check
  - 4.12.2: Additional Resources
- 4.12.3: Range: The Output of a Function
  - 4.12.3: Self Check
  - 4.12.3: Additional Resources
- 4.12.4: Using Graphs to Determine the Domain of a Function
- 4.12.5: Using Functions to Answer Real-World Questions

After that, you’ll **practice and review**.

- 4.12.6: Practice
- 4.12.7: Lesson Summary

4.12.1 Determining Reasonable Inputs and Outputs

Warm Up

Earlier, you saw a situation where the total number of times a dog has barked was a function of the time, in seconds, after its owner tied its leash to a post and left. Less than 3 minutes after he left, the owner returned, untied the leash, and walked away with the dog.

1. Could each value be an input of the function? Be prepared to explain your reasoning.
a. 15
b. $84\frac{1}{2}$
c. 300

2. Could each value be an output of the function? Be prepared to explain your reasoning.
a. 15
b. $84\frac{1}{2}$
c. 300

Building Character: Creativity



Despite the stereotype of the mad genius, everyday creativity is just as important. You can use it to help you improve your mood and cope with stress, feel more connected with other people and the world—and even help you find meaning and purpose in life.

Think about your current sense of creativity. Are the following statements true for you?

- I often make decisions that take me outside my comfort zone.
- I have a playful attitude toward learning something new.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

4.12.2 Domain: The Input of a Function

Activity

Your teacher will give you a set of cards (https://openstax.org/r/set-cards-4-12-2) that each contain a number. Decide whether each number is a possible input for the functions described here. Sort the cards into two groups: possible inputs and impossible inputs. Record your sorting decisions.

1. The area of a square, in square centimeters, is a function of its side length, s, in centimeters. The equation $A(s) = s^2$ defines this function.

a. Possible inputs:

b. Impossible inputs:

2. A tennis camp charges $40 per student for a full-day camp. The camp only runs if at least 5 students sign up, and it limits the enrollment to 16 campers a day. The amount of revenue, in dollars, that the tennis camp collects is a function of the number of students who enroll. The equation $R(n) = 40n$ defines this function.

a. Possible inputs:

b. Impossible inputs:

3. The relationship between temperature in Celsius and the temperature in Kelvin can be represented by a function k. The equation $k(c) = c + 273.15$ defines this function, where c is the temperature in Celsius and $k(c)$ is the temperature in Kelvin.

a. Possible inputs:

b. Impossible inputs:

Write this definition in your math notebook: The **domain** of a function is the set of all of its possible input values.

Self Check

Jose created a function $W(h)$ for the weight, W, of his family members given their height, h, in inches.

Which of the following provides a group of possible inputs for the function?

a. $62, -68, 70\frac{3}{4}$
b. $47, 50.2, 60\frac{1}{2}$
c. $5, 40, 62$
d. $36, 52, -70$

Additional Resources

Finding Possible Input Values

1. Micah mows lawns every week where the function M represents the number of lawns mowed as a function of d, days, modeled by the equation $M(d) = 4d$. Select **three** of the following that are inputs for this function.

a. 0
b. -2
c. 2.5
d. 4

2. Elena runs at a constant speed. Her distance, D, in miles is a function of the time, t, in minutes. Her running distance is represented by the function $D(t) = 9t$. Select **three** of the following that are inputs for this function.

a. -7
b. $\frac{2}{3}$
c. 45
d. 60.5

Using Inequalities to Determine Domain

The domain and range of a function describe the set of possible input values (domain) and output values (range) for that function. The domain represents the values that can be plugged into the function, Inequalities can be used to represent the domain. Remember to consider the nature of the function, its properties, and any specific restrictions when determining the domain using inequalities. Graphing the function can often provide visual clues and help confirm your analysis.

Use the following information to determine the domain of a function using inequalities:

- Identify any restrictions or limitations on the input variable, usually denoted as x.
- Look for values of x that could result in undefined or non-existent outputs.
- Common restrictions include square roots of negative numbers, division by zero, or values excluded by the function's definition.

Use the following information to express the domain of a function using inequalities:

- Use interval notation or set notation to represent the valid input values for the function.
- Use inequalities to express any restrictions.

For example, if x cannot be negative, you would write $x \geq 0$ or $[0, +\infty)$ in interval notation.

Special cases

Consider any specific exclusions or additional conditions mentioned in the function's definition.

For instance, if a function has a denominator and x cannot equal certain values, denote those exclusions in the

domain using the notation "$x \neq$ value."

EXAMPLE

Determine the domain of this function: $p(x) = 3, x \geq -2$.

The inequality $x \geq -2$ indicates that the function is defined for all values of x that are greater than or equal to $x \geq -2$.

Therefore, the domain of $p(x) = 3$, $x \geq -2$ is $x \geq -2$, or in interval notation, $[-2, +\infty)$.

TRY IT **Determine the Domain of a Function Using Inequalities**

1. The distance a turtle has walked, in feet, after his race with the hare begins is represented by the function $f(x) = 0.5x$ where x is the time in seconds. For this scenario, what is the domain of the function $f(x) = 0.5x$ using inequalities?

2. The profit a dance team makes from a fundraising raffle is represented by the function $f(x) = 3x$ where x represents the number of tickets sold. For this scenario, what is the domain of the function $f(x) = 3x$ using inequalities?

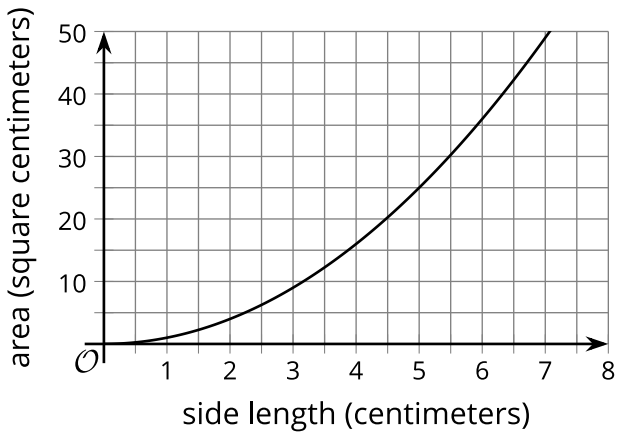
3. What is the domain of the function $g(x) = \frac{1}{(x-2)}$, $x \neq 2$ using inequalities?

4.12.3 Range: The Output of a Function

Activity

In an earlier activity, you saw a function representing the area of a square (function A) and another representing the revenue of a tennis camp (function R). Refer to the descriptions of those functions to answer these questions.

1. Here is a graph that represents function A, defined by $A(s) = s^2$, where s is the side length of the square in centimeters.



a. Name three possible input-output pairs of this function. Enter your answers as sets of ordered pairs.

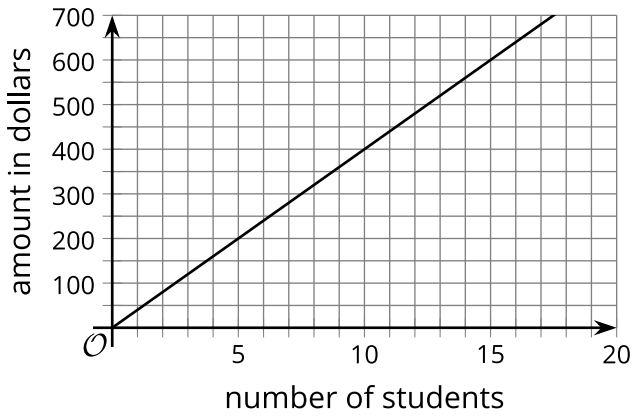
b. Earlier, we described the set of all possible input values of A as “any number greater than or equal to 0.” How would you describe the set of all possible output values of A?

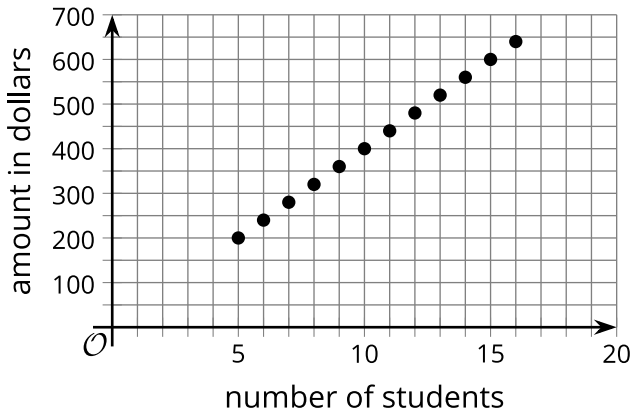
2. Function R is defined by $R(n) = 40n$, where n is the number of campers.

a. Is 20 a possible output value in this situation? What about 100? Be prepared to show your reasoning.

b. Here are two graphs that relate number of students and camp revenue in dollars. Which graph could represent

function R? Explain why the other one could not represent the function.





c. Describe the set of all possible output values of R.

Write the following definition in your math notebook: The set of output values form the **range** of the function.

Are you ready for more?

Extending Your Thinking

Making a Real-World Connection

If the camp wishes to collect at least $500 from the participants, how many students can they have? Explain how this information is shown on the graph.

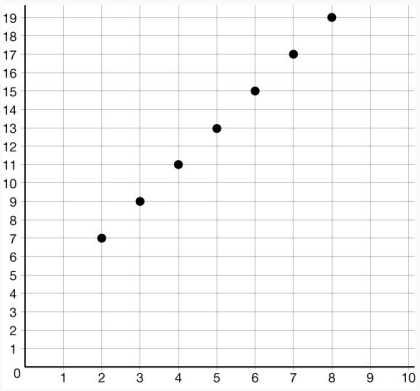
Video: Identifying Outputs of a Function

Watch the following video to learn more about identifying outputs of a function.

Identifying Outputs of a Function (https://openstax.org/r/function-outputs)

Self Check

Looking at the graph below, which statement about the values graphed is true?



a. All odd numbers are outputs.
b. The largest input value is 19.
c. The largest output value is 19.
d. The largest output value is 8.

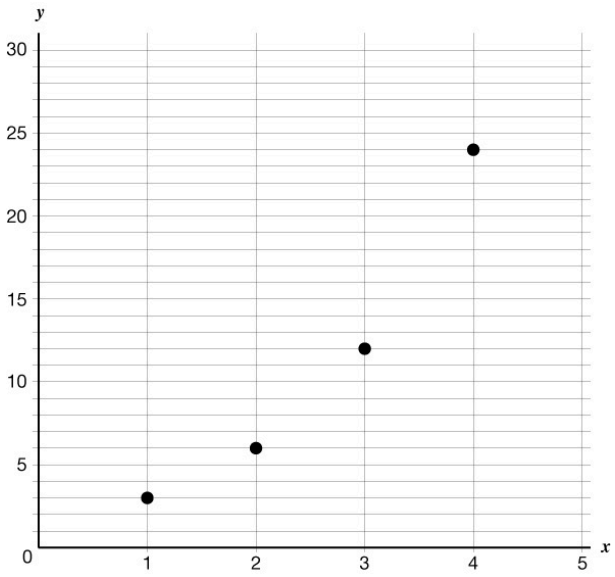
Additional Resources

Naming Input-Output Pairs

A graph is yet another way that a relation can be represented. The set of ordered pairs of all the points plotted is the relation. The set of all x-coordinates, or the input values, is the **domain** of the relation, and the set of all y-coordinates, or outputs, is the **range**.

Recall a coordinate is written (x, y).

Name the input-output pairs for the graph below:

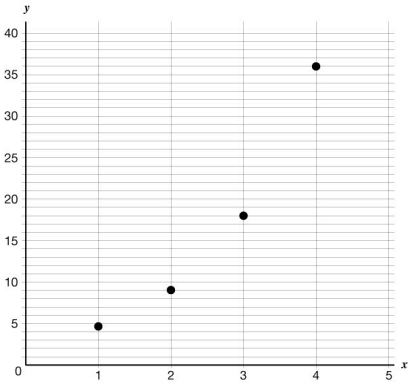


The input values are the x-values, and the output values are y-values: (1,3), (2,6), (3,12), (4,24).

This means the domain for the graphed relation is {1, 2, 3, 4} and the range is {3, 6, 12, 24}

TRY IT **Naming Input-Output Pairs**

Name the input-output pairs of the graph below:



Using Inequalities to Determine Range

The domain and range of a function describe the set of possible input values (domain) and output values (range) for that function. The range represents the values that the function can output. Inequalities can be used to represent the range of a function. Remember to consider the nature of the function, its properties, and any specific restrictions when determining the range using inequalities. Graphing the function can often provide visual clues to help confirm your analysis.

Use the following information to determine the range of a function using inequalities:

- Analyze the behavior and possible output values of the function, usually denoted as $f(x)$.
- Consider the range of the function based on its graph or algebraic properties.
- Identify the minimum and maximum possible values that the function can reach.

Use the following information to express the range of a function using inequalities:

- Use interval notation or set notation to represent the valid output values for the function.
- Use inequalities to express the minimum and maximum possible values.

For example, if the function's graph never goes below −3 and can reach any positive value, you would write $-3 \leq y < +\infty$ as an inequality.

Special cases

Consider any specific exclusions or additional conditions mentioned in the function's definition.

For instance, if a function is taking an absolute value, then the y-values will not be negative. So, denote those exclusions in the range using the notation "$y \neq 0$," or other value as appropriate.

EXAMPLE

Determine the range of this function: $p(x) = 3, x \geq -2$.

Since the function $p(x)$ is constant and always equal to 3, the range of the function is a single value, which is 3.

Therefore, the range of $p(x) = 3, x \geq -2$ is the set {3}.

TRY IT **Determine the Range of a Function Using Inequalities**

1. What is the range of the function $f(x) = 2x - 3$ using inequalities?
2. What is the range of the function $g(x) = \frac{1}{(x-2)}$, $x \neq 2$ using inequalities?

4.12.4 Using Graphs to Determine the Domain of a Function

Activity

Consider the function $f(x) = \frac{6}{x-2}$.

To find out the sets of possible input and output values of the function, Clare created a table and evaluated f at some values of x. Along the way, she ran into some trouble.

1. Find $f(x)$ for each x-value Clare listed. Describe what Clare's trouble might be.

x	-10	0	$\frac{1}{2}$	2	8
$f(x)$					

2. Use graphing technology to graph function f. What do you notice about the graph?

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-12-4-using-graphs-to-determine-the-domain-of-a-function)

3. Use a calculator to compute the value you and Clare had trouble computing. What do you notice about the computation?

4. How would you describe the domain of function f?

Are you ready for more?

Extending Your Thinking

Why do you think the graph of function f looks the way it does? Why are there two parts that split at $x = 2$, with one curving down as it approaches $x = 2$ from the left and the other curving up as it approaches $x = 2$ from the right?

Evaluate function f at different x-values that approach 2 but are not exactly 2, such as 1.8, 1.9, 1.95, 1.999, 2.2, 2.1, 2.05, 2.001, and so on. What do you notice about the values of $f(x)$ as the x-values get closer and closer to 2?

4.12.5 Using Functions to Answer Real-World Questions

Cool Down

Diego's club earns money for charity when members of the club perform community service after school. For each student who does community service, the club earns $5. There are 12 students in the club.

The total dollar amount earned, E, is a function of the number of members who perform community service, n.

1. Is 5 a possible input value? Why or why not?

2. Is 24 a possible output value? Why or why not?

3. Describe the domain of this function.

4. Describe the range of this function.

4.12.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. The cost for an upcoming field trip is $15 per student. The cost of the field trip C, in dollars, is a function of the number of students x. **Select two** possible outputs for the function defined by $C(x) = 15x$.

a. 20
b. 30
c. 50
d. 90
e. 100

2. A rectangle has an area of $24cm^2$. Function f gives the length of the rectangle, in centimeters, when the width is w cm. Determine which values, in centimeters, are possible inputs of the function. **Select three** that apply.

a. 3
b. 0.5
c. 48
d. –6
e. 0

3. **Select three** possible input-output pairs for the function $y = x^3$.

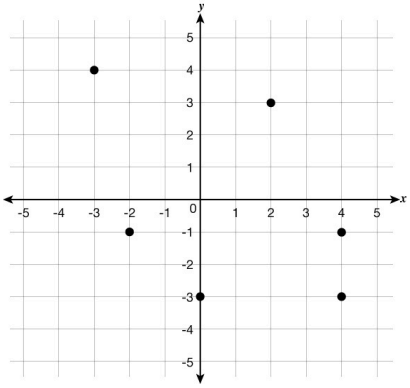
a. $(-1, -1)$
b. $(-2, 8)$
c. $(3, 9)$
d. $\left(\frac{1}{2}, \frac{1}{8}\right)$
e. $(4, 64)$
f. $(1, -1)$

4. Two functions are defined by the equations $f(x) = 5 - 0.2x$ and $g(x) = 0.2(x + 5)$.

Select two possible statements that are true about the functions.

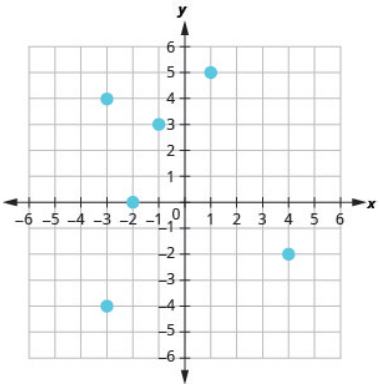
a. $f(3) > 0$
b. $f(3) > 5$
c. $g(-1) = 0.8$
d. $f(0) = g(0)$

5. Is the relation graphed below a function?



a. Yes
b. No

6. Is the relation graphed below a function?



a. Yes
b. No

Use the following scenario for questions 7 - 9.

For a fundraiser, Chi and her class are having a talent show and charging $7 a ticket. There are 100 seats in the theater. The function T, gives the total for selling n tickets and is represented by $T(n) = 7n$.

7. Enter the minimum domain value.

8. Enter the maximum domain value.

9. Is 30 a possible output in the range?

a. Yes
b. No

4.12.7 Lesson Summary

In this lesson, you learned how to:

- Describe the “domain” and “range” of a function.
- Determine a reasonable domain and range for the function when given a description of a function in a situation.

Here are the **activities** that helped you reach those goals:

- 4.12.1: Determining Reasonable Inputs and Outputs
  - In this activity, you worked to decide if inputs and outputs were reasonable in functions given the context of a situation.
- 4.12.2: Domain: The Input of a Function
  - In this activity, you determined which inputs are possible and which are impossible through a card sort activity. You noticed the reasonableness of an input depends on the situation’s context.
  - 4.12.2: Self Check
  - 4.12.2: Additional Resources
- 4.12.3: Range: The Output of a Function
  - In this activity, you identified possible outputs of a function to make the range. You also identified input-output pairs from graphs.
  - 4.12.3: Self Check
  - 4.12.3: Additional Resources
- 4.12.4: Using Graphs to Determine the Domain of a Function
  - In this activity, you explored what happens when an input gives an undefined output.
- 4.12.5: Using Functions to Answer-Real World Questions
  - In this activity, you answered real-world questions by describing the elements of the domain and range of a given situation.

After these activities, you completed the following **practice**:

- 4.12.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

4.13 DOMAIN AND RANGE, PART 2

4.13.0 Lesson Overview

In this lesson, you will use graphs to learn about the domain of functions. You will examine and interpret different types of graphs that represent real-world situations to determine the domain and range.

When you finish this lesson, you will be able to:

- Determine a reasonable domain and range for the function when given a description of a function in a situation.

Here are the **activities** that will help you reach those goals:

- 4.13.1: Different Features of a Graph
- 4.13.2: Graphs of Functions
  - 4.13.2: Self Check
  - 4.13.2: Additional Resources
- 4.13.3: Finding Domain and Range Using a Graph
  - 4.13.3: Self Check
  - 4.13.3: Additional Resources
- 4.13.4: Real-World Domain and Range
  - 4.13.4: Self Check
  - 4.13.4: Additional Resources
- 4.13.5: Graph of a Function and the Domain and Range Connection

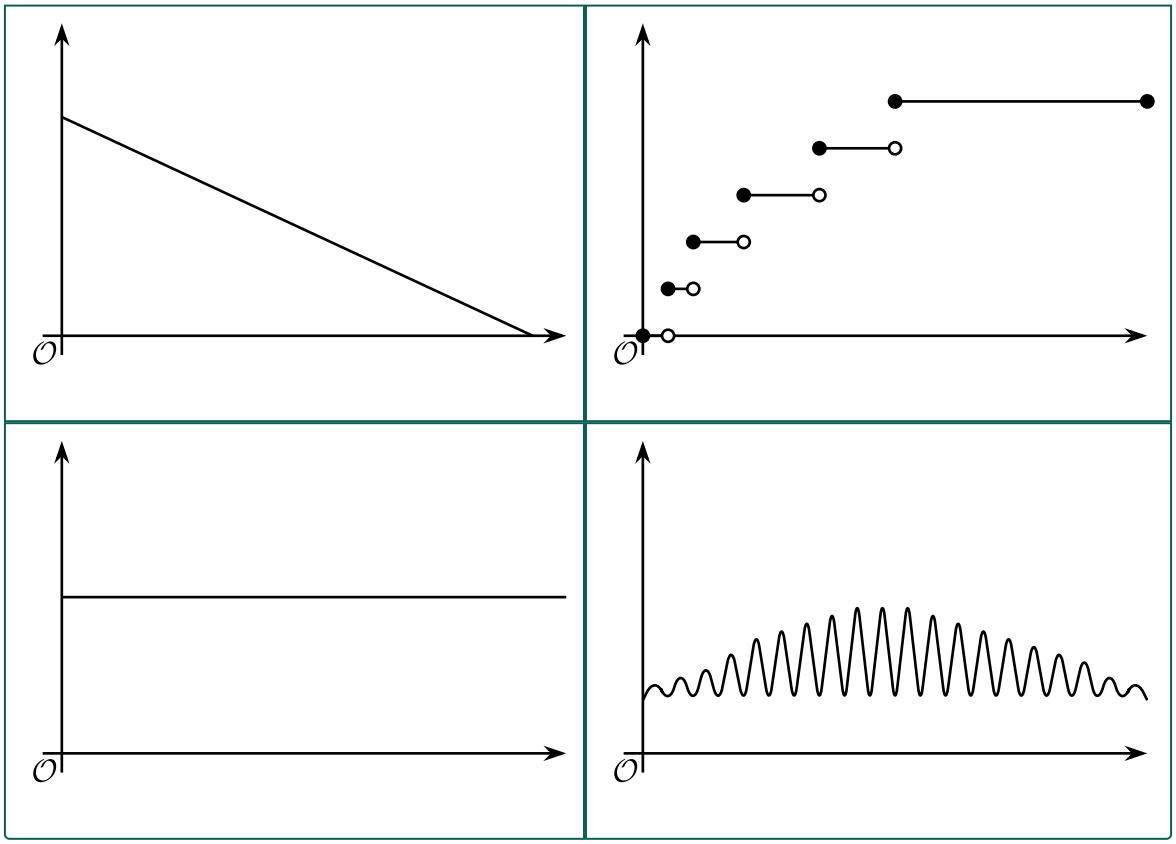
After that, you'll **practice and review**.

- 4.13.6: Practice
- 4.13.7: Lesson Summary

4.13.1 Different Features of a Graph

Warm Up

Examine the following 4 graphs and think about why 1 of the graphs does not belong in the same group as the other 3.



4.13.2 Graphs of Functions

Activity

For questions 1 – 4, you will be provided with descriptions of four functions and four graphs representing the situation described below.

A child gets on a swing in a playground, swings for 30 seconds, and then gets off the swing.

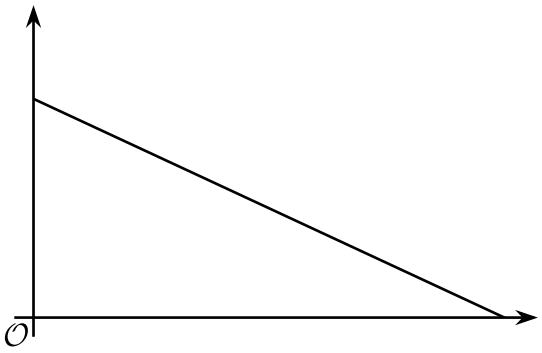


The independent variable in each function is time, measured in seconds.

CLASSROOM CONNECTION Log into student.desmos.com using the information provided by your teacher to

complete the activity.

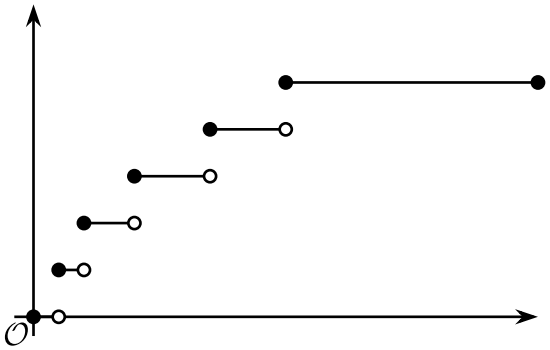
1. Examine Graph A below.



Which function description best matches the representation provided in the graph? Be prepared to explain how you made your match.

a. Function h: The height of the swing, in feet, as a function of time since the child gets on the swing
b. Function r: The amount of time left on the swing as a function of time since the child gets on the swing
c. Function d: The distance, in feet, of the swing from the top beam (from which the swing is suspended) as a function of time since the child gets on the swing
d. Function s: The total number of times an adult pushes the swing as a function of time since the child gets on the swing

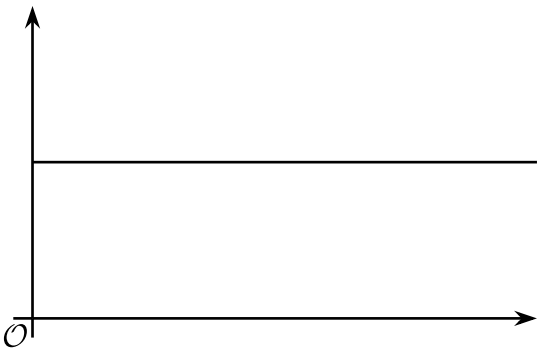
2. Examine Graph B below.



Which function description best matches the representation provided in the graph? Be prepared to explain how you made your match.

a. Function h: The height of the swing, in feet, as a function of time since the child gets on the swing
b. Function r: The amount of time left on the swing as a function of time since the child gets on the swing
c. Function d: The distance, in feet, of the swing from the top beam (from which the swing is suspended) as a function of time since the child gets on the swing
d. Function s: The total number of times an adult pushes the swing as a function of time since the child gets on the swing

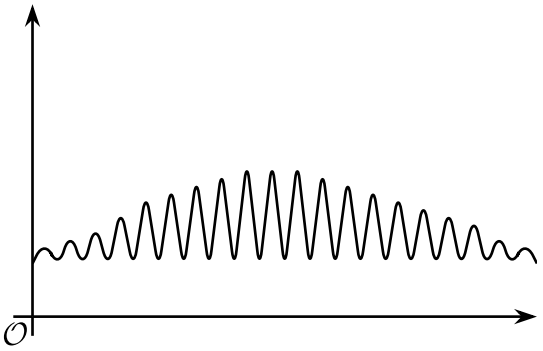
3. Examine Graph C.



Which function description best matches the representation provided in the graph? Be prepared to explain how you made your match.

a. Function h: The height of the swing, in feet, as a function of time since the child gets on the swing
b. Function r: The amount of time left on the swing as a function of time since the child gets on the swing
c. Function d: The distance, in feet, of the swing from the top beam (from which the swing is suspended) as a function of time since the child gets on the swing
d. Function s: The total number of times an adult pushes the swing as a function of time since the child gets on the swing

4. Examine Graph D.



Which function description best matches the representation provided in the graph? Be prepared to explain how you made your match.

a. Function h: The height of the swing, in feet, as a function of time since the child gets on the swing
b. Function r: The amount of time left on the swing as a function of time since the child gets on the swing
c. Function d: The distance, in feet, of the swing from the top beam (from which the swing is suspended) as a function of time since the child gets on the swing
d. Function s: The total number of times an adult pushes the swing as a function of time since the child gets on the swing

For questions 5 – 8, continue to use the scenario from above and include the following information to determine the domain and range for each function.

- The child is given 30 seconds on the swing.
- While the child is on the swing, an adult pushes the swing a total of 5 times.
- The swing is 1.5 feet (18 inches) above ground.
- The chains that hold the seat and suspend it from the top beam are 7 feet long.
- The highest point that the child swings up to is 4 feet above the ground.

[Hint: It may be helpful to revisit the scenario, the matched graph, and the additional list of information to identify specific values for the domain and range.]

5. Describe the domain and range for function h: The height of the swing, in feet, as a function of time since the child gets on the swing.

6. Describe the domain and range for function r: The amount of time left on the swing as a function of time since the

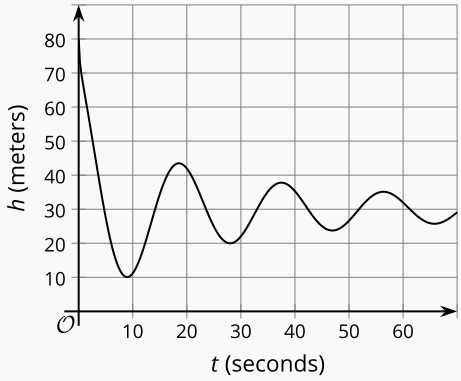
child gets on the swing.

7. Describe the domain and range for function d: The distance, in feet, of the swing from the top beam (from which the swing is suspended) as a function of time since the child gets on the swing.

8. Describe the domain and range for function s: The total number of times an adult pushes the swing as a function of time since the child gets on the swing.

Self Check

Which of the following is true about the range of the graph below?

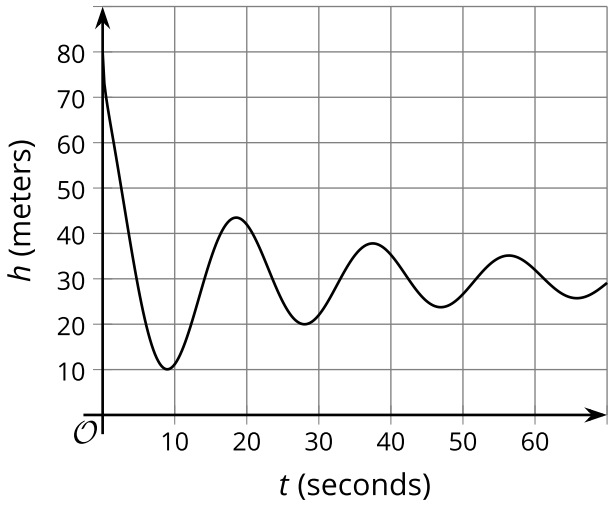


a. The range of this function only includes 10 and 75.
b. The range of this function includes all values that are at least 10 and at most 75.
c. The range of this function includes all values that are at least 0 and no more than 90.
d. The range of this function includes all values that are at least 0 and no more than 70.

Additional Resources

Describing Domain and Range Graphs

In a previous lesson, you explored the height, h, of a bungee jumper at t seconds.



Assuming the jump ended at 35 seconds, what is the domain?

For domain, consider the x-values, so look from left to right.

The leftmost x-value is 0, and the rightmost x-value is 35.

The domain includes all the values from at least 0 up to and including 35 meters.

What are the values in the range?

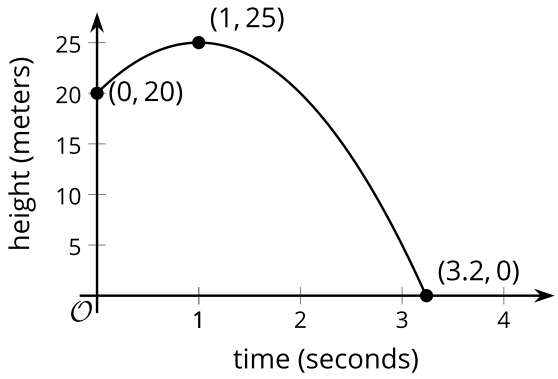
For the range, consider the y-values, so look from bottom to top.

The lowest y-value is 10, and the highest y-value is about 75.

So, the range includes all of the values from at least 10 up to about 75.

TRY IT **Try It: Describing Domain and Range Graphs**

Describe the values of the domain and range using the graph below.



4.13.3 Finding Domain and Range Using a Graph

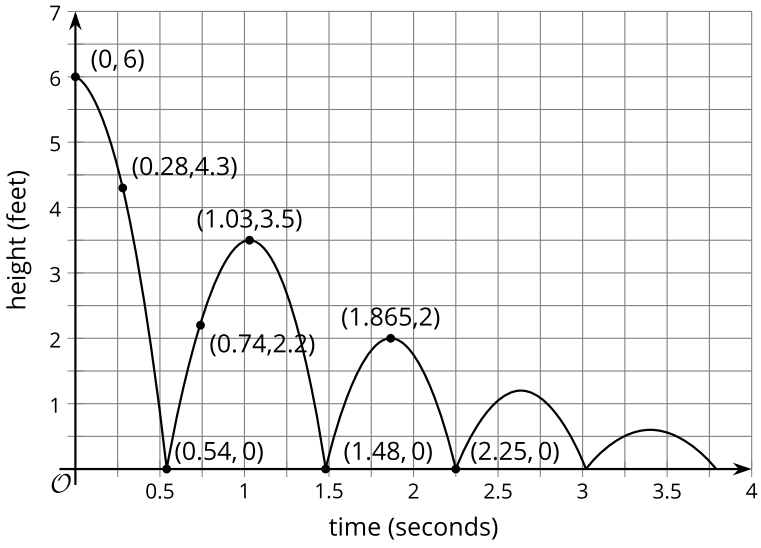
Activity

A tennis ball was dropped from a certain height. It bounced several times, rolled along for a short period, and then stopped. Function H gives its height over time.

Here is a partial graph of H. Height is measured in feet. Time is measured in seconds.

Use the graph to help you answer the questions.

Be prepared to explain what each value or set of values means in this situation.



1. Find $H(0)$. What point is this on the graph? What does it represent?
2. Solve $H(x) = 0$.

3. Describe the domain of the function.
4. Describe the range of the function.

Are you ready for more?

Extending Your Thinking

In function H, the input was time in seconds, and the output was height in feet.

Think about some other quantities that could be inputs or outputs in this situation.

1. Describe a function whose domain includes only integers. Be sure to specify the units.
2. Sketch a graph of this function.



3. Describe a function whose range includes only integers. Be sure to specify the units.
4. Sketch a graph of this function.



Video: Interpreting Function Notation

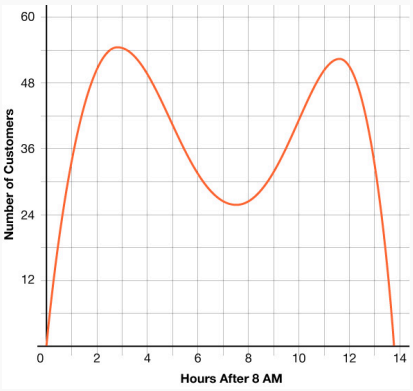
Watch the following video to learn more about function notation.

4.13.3 (https://openstax.org/r/function-notation-2)

Self Check

The graph below shows the number of customers, C, in a restaurant x hours after 8 a.m.

What does $C(14)$ equal, and what does it mean?

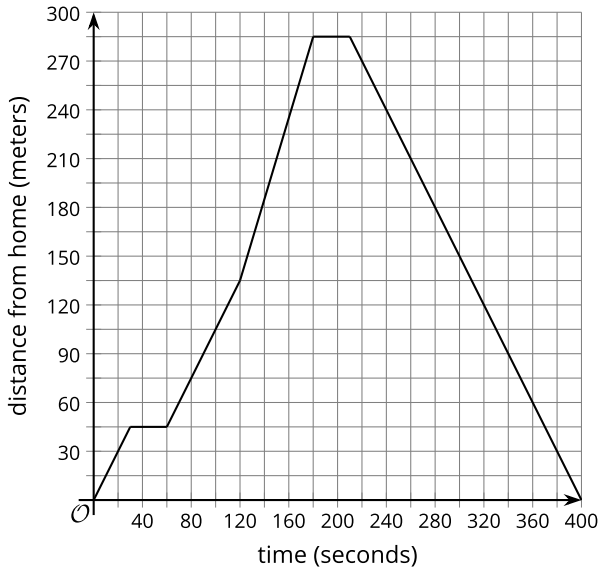


a. $C(14) = 0$ means there have not been any customers over 14 hours.
b. $C(14) = 0$ and means there are 14 customers at 8 a.m.
c. $C(14) = 0$ and is when the restaurant closes and has no more customers around 10 p.m.
d. $C(14) = 0.5$ and means there are 14 customers around 8:30.

Additional Resources

Domain and Range from Graphs

The graph below shows the distance Carl drives from home, $D(t)$, when t is time in seconds.



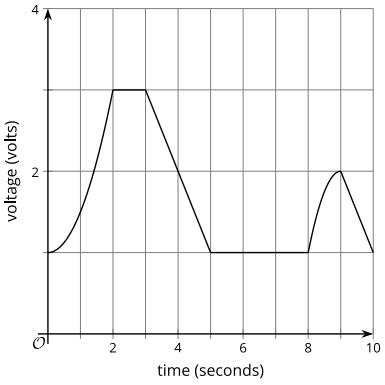
1. What is the domain of the graph?

2. What is the range of the graph?

3. What are the values of x where $D(x) = 0$, and what do they mean?

TRY IT **Domain and Range from Graphs**

What are the values for the domain and range of the graph below?

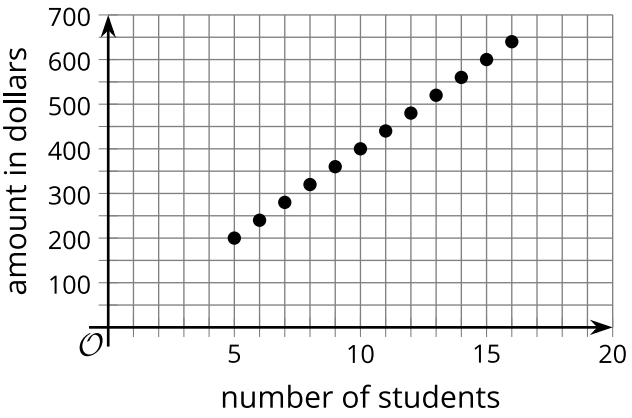


4.13.4 Real-World Domain and Range

Activity

A **discrete function** has distinct separate values in the domain and range, rather than an interval of values.

1. Below is a graph from the previous lesson that graphs the number of students attending a camp and the amount, in dollars, the camp earns. There must be 5 students to hold the camp.



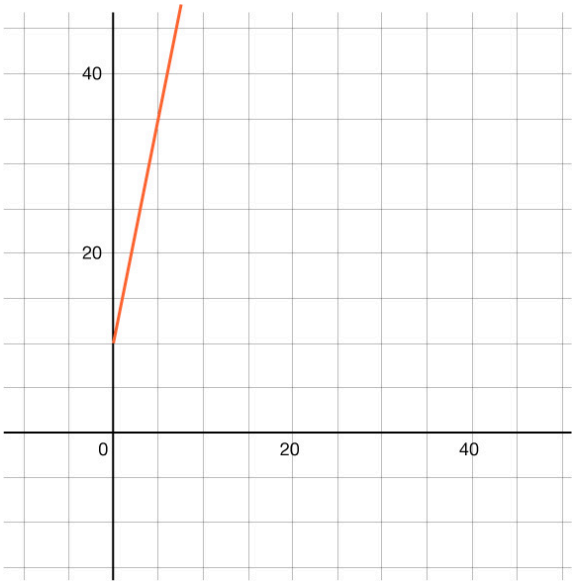
a. What are the x-values of this function that make up the domain?

b. Why can't we say the domain is all values from 5 to 16?

c. What are the y-values of this function that make up the range?

A **continuous function** refers to a function that does not have any breaks in its graph. The domain and range of a continuous function are often represented by an interval that can be written as an inequality.

2. The graph below represents a continuous function. The function, P, is the cost of parking in a garage for t hours. It is represented by the function $P(t) = 5h + 10$.



a. What does $P(0)$ represent?

b. In words, what are the values of the domain?

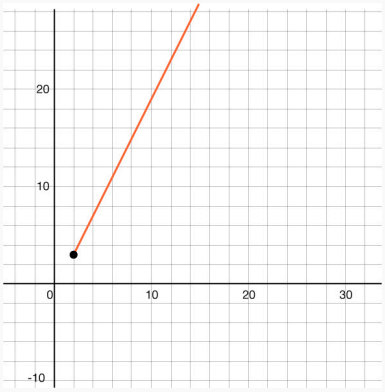
c. How can you write the domain values as an inequality?

d. In words, what are the values of the range?

e. How can you write the range values as an inequality?

Self Check

Which of the following correctly describes the domain and range of the graph below?



a. Domain: $x > 2$
Range: $y > 3$

b. Domain: $x \leq 2$
Range: $y \leq 3$

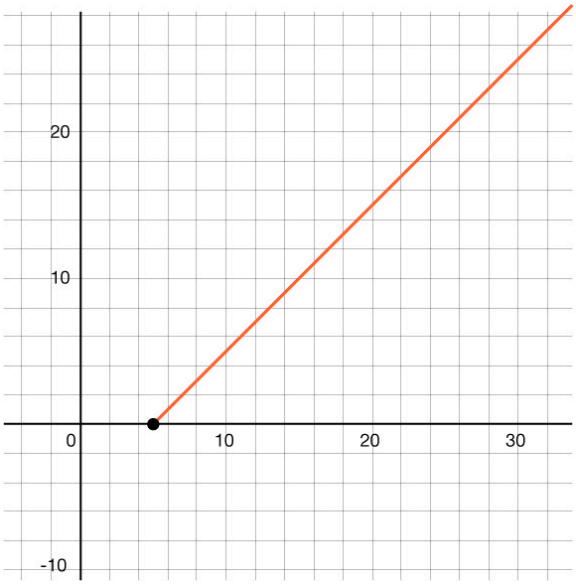
c. Domain: $x \geq 2$
Range: $y \geq 3$

d. Domain: $x \geq 3$
Range: $y \geq 2$

Additional Resources

Domain and Range as Inequalities

Looking at the linear function below, use inequalities to represent the domain and range.



Domain:

Since the domain is all x-values, look left to right for the least to greatest values.

The smallest x-value on this linear function is 5, and then the line continues to the right forever.

So, the domain is all values that are at least 5.

As an inequality, write this as $x \geq 5$.

Range:

Since the range is all y-values, look from the bottom to the top for the least to greatest values.

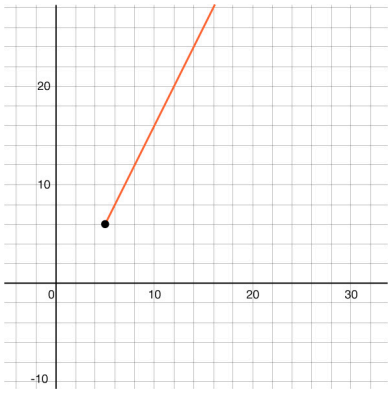
The smallest y-value is 0, and then the line continues up forever.

So, the range is all values that are at least 0.

An an inequality, the range is $y \geq 0$.

TRY IT **Domain and Range as Inequalities**

Looking at the linear function below, use inequalities to represent the domain and range.

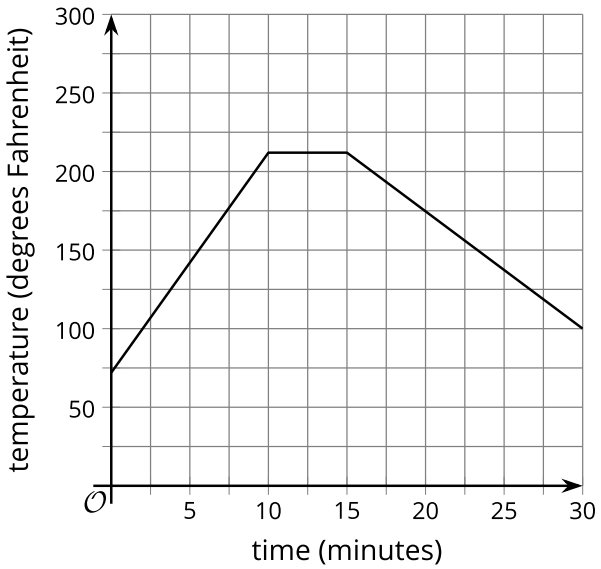


4.13.5 Graph of a Function and the Domain and Range Connection

Cool Down

The function W gives the temperature, in degrees Fahrenheit, of a pot of water on a stove t minutes after the stove is turned on. After 30 minutes, the pot is taken off the stove.

The graph of the function is shown.

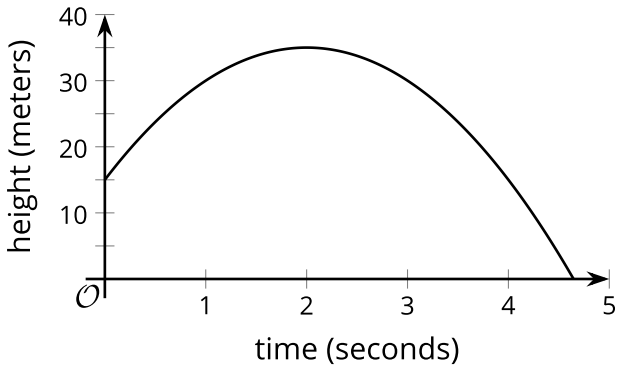


1. Is 250 in the range of function W? Explain how you know.
2. Describe the range of the function.
3. Does $W(t) = 0$ have a solution? Explain how you know.

4.13.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. The graph of function H shows the height, in meters, of a rocket t seconds after it was launched.



a. Find $H(0)$ in meters.

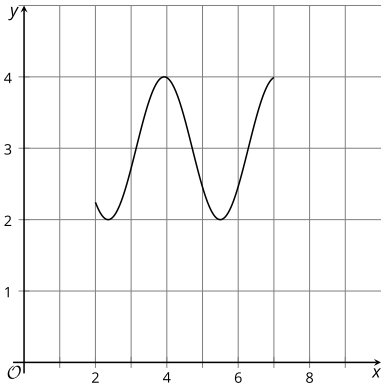
b. Which of the following describes the domain of this function?

a. $15 \leq x \leq 35$
b. $0 \leq x \leq 2$
c. $0 \leq x \leq 4.5$
d. $0 \leq x \leq 35$

c. Which of the following describes the range of this function?

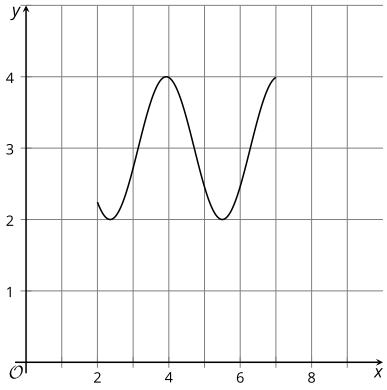
a. $15 \leq y \leq 35$
b. $0 \leq y \leq 2$
c. $0 \leq y \leq 4.5$
d. $0 \leq y \leq 35$

2. Which of the following describes the domain of the function graphed below?



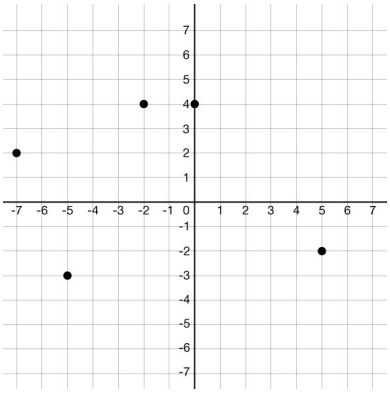
a. $2 \leq x \leq 4$
b. $2 \leq x \leq 7$
c. $0 \leq x \leq 4$
d. $0 \leq x \leq 9$

3. Which of the following describes the range of the graph below?



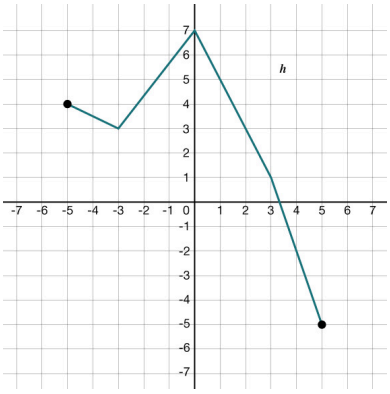
a. $2 \leq x \leq 7$
b. $2 \leq y \leq 7$
c. $2 \leq x \leq 4$
d. $2 \leq y \leq 4$

4. **Select four** of the values of the range for the function h, graphed below.



a. –5
b. –3
c. –2
d. 0
e. 2
f. 4
g. 5

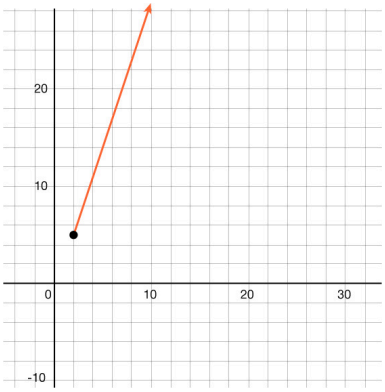
5. Which of the following is the domain of the function graphed below?



a. $-5 \leq y \leq 7$
b. $-5 \leq y \leq 5$

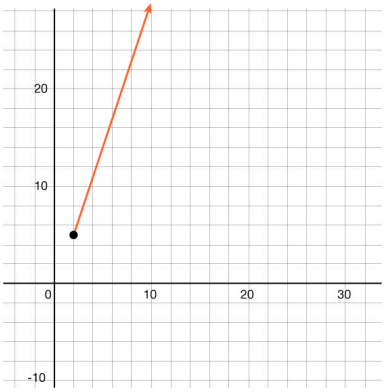
c. $-5 \leq x \leq 7$
d. $-5 \leq x \leq 5$

6. Which of the following represents the domain of the function graphed below?



a. $x \geq 2$
b. $y \geq 2$
c. $x \geq 5$
d. $y \geq 5$

7. Which of the following represents the range of the function graphed below?



a. $x \geq 2$
b. $y \geq 2$
c. $x \geq 5$
d. $y \geq 5$

4.13.7 Lesson Summary

In this lesson, you learned how to:

- Determine a reasonable domain and range for the function when given a description of a function in a situation.

Here are the **activities** that helped you reach those goals:

- 4.13.1: Different Features of a Graph
  - In this activity, you compared features of 4 different graphs to determine which did not belong.
- 4.13.2: Graphs of Functions
  - In this activity, you matched graphs to situations they could represent. You then determined the domain and range when you were given more information about the graphs.
  - 4.13.2: Self Check
  - 4.13.2: Additional Resources
- 4.13.3: Finding Domain and Range Using a Graph
  - In this activity, you used a graph to determine the domain and range using maximum and minimum values. You also determined the value of the x- and y-intercepts.
  - 4.13.3: Self Check
  - 4.13.3: Additional Resources
- 4.13.4: Real-World Domain and Range
  - In this activity, you looked at graphs representing real-world situations that were both discrete and continuous functions. You then determined the domain and range of each type of function. For continuous functions, you wrote the domain and range as inequalities.
  - 4.13.4: Self Check
  - 4.13.4: Additional Resources
- 4.13.5: Graph of a Function and the Domain and Range Connection
  - In this activity, you examined a graph and determined if a value was in the range. You then identified the range and also determined if there is a solution to the graph.

After these activities, you completed the following **practice**:

- 4.13.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

4.14 SEQUENCES

4.14.0 Lesson Overview

In this lesson, you will learn how to work with sequences and describe them recursively in an informal way. You will see how sequences are used to accomplish recurring tasks.

When you finish this lesson, you will be able to:

- Give examples of a sequence.
- Explain the meaning of the term “sequence.”
- Explain the meaning of the term “term,” as it is used in a mathematical sequence.

Here are the **activities** that will help you reach those goals:

- 4.14.1: Describe the Pattern
- 4.14.2: What Is a Sequence?
  - 4.14.2: Self Check
  - 4.14.2: Additional Resources
- 4.14.3: The Connection Between Sequences and Terms
  - 4.14.3: Self Check
  - 4.14.3: Additional Resources
- 4.14.4: Rules for a Sequence

After that, you’ll **practice and review**.

- 4.14.5: Practice
- 4.14.6: Lesson Summary

4.14.1 Describe the Pattern

Warm Up

Here is a rule for making a list of numbers: **Each number is 1 less than twice the previous number.**

Pick a number to start with. Then follow the rule to build a list of 5 numbers.

4.14.2 What Is a Sequence?

Activity

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-14-2-what-is-a-sequence)
In the Tower of Hanoi puzzle, a set of discs sits on a peg, while there are 2 other empty pegs.

A move in the Tower of Hanoi puzzle involves taking a disc and moving it to another peg. There are two rules:

- Only move 1 disc at a time.
- Never put a larger disc on top of a smaller one.

You complete the puzzle by building the complete tower on a different peg from the starting peg.

1. What is the smallest number of moves in which you are able to complete the puzzle with 3 discs?

2. What is the smallest number of moves in which you are able to complete the puzzle with 4 discs?

3. Jada says she used the solution for 3 discs to help her solve the puzzle for 4 discs. Describe how this might happen.

4. How many moves do you think it will take to complete a puzzle with 5 discs? Explain.

5. How many moves do you think it will take to complete a puzzle with 7 discs?

In this activity, you have started to find a **sequence** of **terms**.

Write these definitions in your math notebook.

- A sequence is a list of numbers, possibly going on forever, such as all the odd positive integers arranged in order: 1, 3, 5, 7, ...
- The term (of a sequence) is one of the numbers in a sequence.

Self Check

The table below shows the number of chips in a game and the number of moves that can be made.

Number of chips	1	2	3	4
Number of possible moves	3	5	7	9

How many moves would be possible with 7 chips?

a. 13
b. 15
c. 14
d. 11

Additional Resources

Using a Table to Find the Pattern

Sometimes it is helpful to keep a table to find the pattern needed to answer a question.

For the Tower of Hanoi game, enter the number of discs used and the smallest number of moves to solve the puzzle in a table like the one below. Use the answers from the activity.

Number of Discs	3	4	5	6	7	8
Number of Moves to Solve Puzzle	7	15	31	?	127	?

The pattern was to double the previous answer and then add 1.

Using this pattern, complete the table for 6 discs.

$2(31) + 1 = 63$

Then use this rule to find how many moves are needed to complete the puzzle with 8 discs.

There were 127 moves for 7 discs.

So, double 127 and add 1: $2(127) + 1 = 255$.

TRY IT **Using a Table to Find the Pattern**

Now it's your turn. For the same game, how many moves would it take to solve the puzzle with 9 discs?

4.14.3 The Connection between Sequences and Terms

Activity

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-14-3-the-connection-between-sequences-and-terms)

Some checkers are lined up, with blue on one side, red on the other, and 1 empty space between them. A **move** in this checker game pushes any checker forward 1 space, or jumps over any 1 checker of the other color. Jumping the same color is not allowed, moving backward is not allowed, and 2 checkers cannot occupy the same space.



You complete the puzzle by switching the colors completely: ending up with black on the right, red on the left, and 1 empty space between them.

1. Using 1 checker on each side, complete the puzzle. What is the smallest number of moves needed?
2. Using 3 checkers on each side, complete the puzzle. What is the smallest number of moves needed?
3. Estimate the number of moves needed if there are 2 or 4 checkers on each side. Then test your guesses.
4. Noah says he used the solution for 3 checkers on each side to help him solve the puzzle for 4 checkers. Describe how this might happen.
5. How many moves do you think it will take to complete a puzzle with 7 checkers on each side?

Self Check

The table below shows the number of checkers and the number of moves needed to win a skipping game.

Number of Checkers	1	2	3	4
Number of Skips Needed	2	8	18	32

How many skips are needed with 6 checkers on each side?

a. 48
b. 128
c. 72
d. 50

Additional Resources

Using a Table to Find Missing Values

Number of Checkers	1	2	3	4	7	8
Number of Moves Needed	3	8	15	24	63	?

What is the pattern that you used in the previous activity to find the number of moves depending on the number of checkers?

Each row is multiplied by 1 more, so:

$1 \cdot 3 = 3$

$2 \cdot 4 = 8$

$3 \cdot 5 = 15$

$4 \cdot 6 = 24$

How many moves are needed with 5 checkers?

$5 \cdot 7 = 35$ moves

What about 6 checkers?

$6 \cdot 8 = 48$ moves

TRY IT **Using a Table to Find Missing Values**
Using the table and the pattern, how many moves are needed with 8 checkers?

4.14.4 Rules for a Sequence

Cool Down

A sequence starts 3, 6, . . .

1. Give a rule the sequence could follow, and list the next 3 terms.
2. Give a different rule the sequence could follow, and list the next 3 terms.

Video: Finding Rules of Sequences

Watch the following video to learn more about function notation.

Finding Rules of Sequences (https://openstax.org/r/sequence-rules)

4.14.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Here is a rule to make a list of numbers: Each number is the sum of the previous two numbers. Start with the numbers 0 and 1, and then follow the rule to build a sequence of 10 numbers.

Enter the 10^{th} number in the sequence.

2. A sequence of numbers follows the rule: Multiply the previous number by –2 and add 3. The 4^{th} term in the sequence is –7.

a. Give the 8^{th} term in the sequence.

b. Give the term that came before –7 in the sequence.

3. Find the 4^{th} term if the first term is 2 and each term is 3 more than the previous number

4. Find the 8^{th} term of the sequence: 4, 7, 12, 19, 28, . . .

5. Find the 8^{th} term of the sequence: –8, –6, –3, 1, 6, . . .

6. Find the next term in the sequence: –3, 6, –12, 24, . . .

7. Starting with 2, each number is three times the previous number.

What is the 4^{th} term in the sequence?

4.14.6 Lesson Summary

In this lesson, you learned how to:

- Give examples of a sequence.
- Explain the meaning of the term “sequence.”
- Explain the meaning of the term “term,” as it is used in a mathematical sequence.

Here are the **activities** that helped you reach those goals:

- 4.14.1: Describe the Pattern
  - In this activity, you were given a rule and then made up a list of numbers using that rule.
- 4.14.2: What Is a Sequence?
  - In this activity, you explored the Tower of Hanoi puzzle and started to make sense of the patterns to solve the puzzle, depending on how many discs there were and how many moves were needed.
  - 4.14.2: Self Check
  - 4.14.2: Additional Resources
- 4.14.3: The Connection Between Sequences and Terms
  - In this activity, you explored a game with checkers and worked to find a pattern with the number of checkers on each side and the number of moves needed to complete the game.
  - 4.14.3: Self Check
  - 4.14.3: Additional Resources
- 4.14.4: Rules for a Sequence
  - In this activity, you gave a rule for a sequence and listed 3 terms. You then thought of another rule for the sequence and gave a different list of the next 3 terms.

After these activities, you completed the following **practice**:

- 4.14.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

4.15 INTRODUCING GEOMETRIC SEQUENCES

4.15.0 Lesson Overview

In this lesson, you will begin to understand what makes a sequence a geometric sequence and how that connects to exponential functions. Using graphs, you will explore how growing and shrinking patterns represent geometric sequences.

When you finish this lesson, you will be able to:

- Find missing terms in a geometric sequence.

Here are the **activities** that will help you reach those goals:

- 4.15.1: What Are the Patterns?
- 4.15.2: Using Tables and Graphs to Represent Geometric Sequences
  - 4.15.2: Self Check
  - 4.15.2: Additional Resources
- 4.15.3: Complete the Sequence
  - 4.15.3: Self Check
  - 4.15.3: Additional Resources
- 4.15.4: Connecting the Common Ratio and the Term in a Sequence

After that, you'll **practice and review**.

- 4.15.5: Practice
- 4.15.6: Lesson Summary

4.15.1 What Are the Patterns?

Warm Up

Examine the patterns of numbers given below.

- 40, 120, 360, 1,080, 3,240
- 2, 8, 32, 128, 512
- 1,000, 500, 250, 125, 62.5
- 256, 192, 144, 108, 81

1. What do you notice?

2. What do you wonder?

4.15.2 Using Tables and Graphs to Represent Geometric Sequences

Activity

Clare takes a piece of paper, cuts it in half, and then stacks the pieces. She takes the stack of two pieces and cuts it in half again to form four pieces, stacking them. She keeps repeating the process.

1. The original piece of paper has length 8 inches and width 10 inches. Complete the table.

number of cuts	number of pieces	area in square inches of each piece
0		
1		
2		

number of cuts	number of pieces	area in square inches of each piece
3		
4		
5		

2. Describe in words how you can use the results after 5 cuts to find the results after 6 cuts.

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-15-2-using-tables-and-graphs-to-represent-geometric-sequences)

3.

a. Use the graphing tool above or technology outside the course, on the given axes, sketch a graph of the number of pieces as a function of the number of cuts.

b. How can you see on the graph how the number of pieces is changing with each cut?

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-15-2-using-tables-and-graphs-to-represent-geometric-sequences)

4.

a. Use the graphing tool above or technology outside the course, on the given axes, sketch a graph of the area of each piece as a function of the number of cuts.

b. How can you see how the area of each piece is changing with each cut?

Are you ready for more?

Extending Your Thinking

1. Clare has a piece of paper that is 8 inches by 10 inches.

a. How many pieces of paper will Clare have if she cuts the paper in half n times?

b. What will the area of each piece be?

2. Why is the product of the number of pieces and the area of each piece always the same? Explain how you know.

Self Check

Given the table below, what would be the next term in the sequence?

x	1	2	3	4	5
y	270	90	30	10	$\frac{10}{3}$

a. $\frac{10}{27}$
b. $\frac{10}{9}$
c. $\frac{11}{3}$
d. $\frac{10}{6}$

Additional Resources

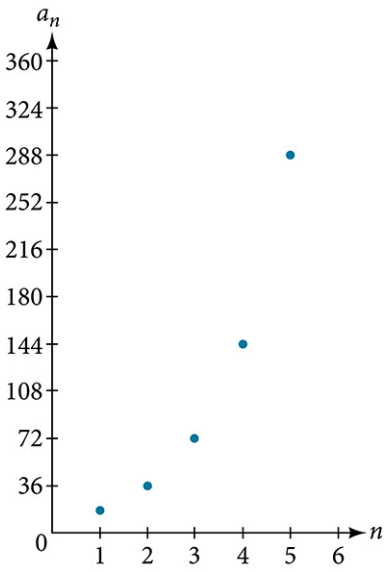
Graphs of Sequences

Graph the sequence {18, 36, 72, 144, 288, . . .}.

First, make a table:

Term	1	2	3	4	5
Value	18	36	72	144	288

Now, plot the points:



Notice that each height is double the previous height.

Each term is also double the previous term.

TRY IT **Graphs of Sequences**

Graph the following sequence and explain the pattern.

1, 3, 9, 27, . . .

4.15.3 Complete the Sequence

Activity

Write the following terms and definitions in your math notebook:

A **geometric sequence** is a sequence in which each term is found by multiplying the previous term by the same constant.

The number that is multiplied by each term is called the **common ratio**.

1. Complete each geometric sequence.

1.5, 3, 6, _a__, 24, _b__

Blank “a”:

Blank “b”:

b. 40, 120, 360, __a__, __b__

Blank “b”:

c. 200, 20, 2, __a__, 0.02, __b__

Blank “b”:

d. $\frac{1}{7}$, __a__, $\frac{9}{7}$, $\frac{27}{7}$, __b__

Blank “b”:

e. 24, 12, 6, __a__, __b__

Blank “b”:

2. For each sequence, find its common ratio.

a. 1.5, 3, 6, ___, 24, ___

b. 40, 120, 360, ___, ___

c. 200, 20, 2, ___, 0.02, ___

d. $\frac{1}{7}$, ___, $\frac{9}{7}$

$\frac{27}{7}$

e. 24, 12, 6, ___, ___

Video: Completing Geometric Sequences

Watch the following video to learn more about geometric sequences.

Completing Geometric Sequences (https://openstax.org/r/geometric-sequences)

Self Check

Which of the following sequences is geometric?

a. 10, 7, 4, 1, . . .
b. 3, 5, 8, 10, 13, . . .
c. 48, 24, 12, 6, . . .
d. 5, 10, 15, 20 . . .

Additional Resources

Finding a Common Ratio

Definition of a Geometric Sequence

A geometric sequence is one in which any term divided by the previous term is a constant. This constant is called the common ratio of the sequence. The common ratio can be found by dividing any term in the sequence by the previous term. If a_1 is the initial term of a geometric sequence and r is the common ratio, the sequence will be

$\{a_1, a_1 \cdot r, a_1 \cdot r^2 , a_1 \cdot r^3 \ldots\}$

How to:

Given a set of numbers, determine if they represent a geometric sequence.

1. Divide each term by the previous term.
2. Compare the quotients. If they are the same, a common ratio exists and the sequence is geometric.

EXAMPLE

Is the sequence geometric? If so, find the common ratio.

a. 1, 2, 4, 8, 16, . . .
b. 48, 12, 4, 2, . . .

a. Is a geometric sequence. Each term is multiplied by 2, so 2 is the common ratio.
b. This is not geometric because each term is not multiplied by the same constant. There is not a common ratio.

TRY IT **Finding a Common Ratio**

Is the sequence geometric? If so, find the common ratio.

100, 20, 4, $\frac{4}{5}$, . . .

4.15.4 Connecting the Common Ratio and the Term in a Sequence

Cool Down

Here is a sequence: 500, 100, 20, . . .

1. Explain why this sequence could be a geometric sequence.
2. What is the next term?
3. Explain how you know the next term.

4.15.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Here are the first two terms of a geometric sequence: 2, 4. What is the 5^{th} term in the sequence?

2. What are the common ratios of each geometric sequence?

a. 1, 1, 1, 1

b. 256, 128, 64

c. 18, 54, 162

d. 0.8, 0.08, 0.008

e. 0.008, 0.08, 0.8

3. A person owes $1,000 on a credit card that charges an interest rate of 2% per month.

Complete this table showing the credit card balance each month if they do not make any payments. Enter each answer to the hundredths place.

Month	Total Bill In Dollars
1	1,000
2	1,020
3	1,040.40
4	a.
5	b.
6	c.
7	d.
8	e.

Table 4.15.0

a. Month 4
Round to the hundredths place.

b. Month 5
Round to the hundredths place.

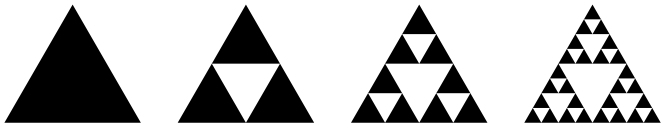
c. Month 6
Round to the hundredths place.

d. Month 7
Round to the hundredths place.

e. Month 8
Round to the hundredths place.

4. A Sierpinski triangle can be created by starting with an equilateral triangle, breaking the triangle into 4 congruent equilateral triangles, and then removing the middle triangle. Starting from a single black equilateral triangle with

an area of 256 square inches, here are the first four steps:



Complete this table showing the number of shaded triangles in each step and the area of each triangle. Enter each answer as an integer.

Step Number	Number Of Shaded Triangles	Area Of One Shaded Triangle In Square Inches
0	1	256
1	3	a.
2	b.	c.
3	d.	e.
4	f.	g.

Table 4.15.1

a. In Step 1, what is the area of each shaded triangle in square inches?

b. In Step 2, what is the number of shaded triangles?

c. In Step 2, what is the area of each shaded triangle in square inches?

d. In Step 3, what is the number of shaded triangles?

e. In Step 3, what is the area of each shaded triangle in square inches?

f. In Step 4, what is the number of shaded triangles?

g. In Step 4, what is the area of each shaded triangle in square inches?

5. For each geometric sequence with missing terms, find the common ratio that might help determine the missing terms.

a. ___, 5, 25, ___, 625

b. –1, ___, –36, 216, ___

c. 10, 5, ___, ___, 0.625

d. ___, ___, 36, –108, ___

e. ___, 12, 18, 27, ___

4.15.6 Lesson Summary

In this lesson, you learned how to:

- Find missing terms in a geometric sequence.

Here are the **activities** that helped you reach those goals:

- 4.15.1: What Are the Patterns?
  - In this activity, you noticed and wondered about the patterns and characteristics of different sequences.
- 4.15.2: Using Tables and Graphs to Represent Geometric Sequences
  - In this activity, you used tables and graphs to recognize a sequence and find the next term. You learned that geometric sequences can get larger or smaller.
  - 4.15.2: Self Check
  - 4.15.2: Additional Resources
- 4.15.3: Complete the Sequence
  - In this activity, you worked to complete the geometric sequences by filling in the blanks. You used a common ratio to find the pattern to complete the sequence.
  - 4.15.3: Self Check
  - 4.15.3: Additional Resources
- 4.15.4: Connecting the Common Ratio and the Term in a Sequence
  - In this activity, you summarized what you learned in the lesson by explaining why a sequence is geometric and how you are able to find the next term.

After these activities, you completed the following **practice**:

- 4.15.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

4.16 INTRODUCING ARITHMETIC SEQUENCES

4.16.0 Lesson Overview

In this lesson, you will learn how to explain the difference between an arithmetic and a geometric sequence. You will determine arithmetic sequences that will help you solve real-world problems, such as how much fuel is left in your vehicle or how fast the money in your bank account is increasing.

When you finish this lesson, you will be able to:

- Explain what it means for a sequence to be arithmetic or geometric.

Here are the **activities** that will help you reach those goals:

- 4.16.1: Interpreting Function Notation
- 4.16.2: What Is an Arithmetic Sequence?
  - 4.16.2: Self Check
  - 4.16.2: Additional Resources
- 4.16.3: A Sequence Is a Type of Function
  - 4.16.3: Self Check
  - 4.16.3: Additional Resources
- 4.16.4: Identify and Generate Sequences

After that, you'll **practice and review**.

- 4.16.5: Practice
- 4.16.6: Lesson Summary

4.16.1 Interpreting Function Notation

Warm Up

Consider the function f given by $f(n) = 3n - 7$. This function takes an input, multiplies it by 3, and then subtracts 7.

Evaluate.

1. $f(10)$

2. $f(10) - 1$

3. $f(10 - 1)$

4. $f(5) - f(4)$

4.16.2 What Is an Arithmetic Sequence?

Activity

Here are the values of the first 5 terms of 3 sequences:

1. 30, 40, 50, 60, 70, . . .
2. 0, 5, 15, 30, 50, . . .
3. 1, 2, 4, 8, 16, . . .

1. For each sequence, describe a way to produce a new term from the previous term.

a. 30, 40, 50, 60, 70, . . .

b. 0, 5, 15, 30, 50, . . .

c. 1, 2, 4, 8, 16, . . .

2. If the patterns you described continue, which sequence has the second-greatest value for the 10^{th} term?

3. Which of these could be geometric sequences? Explain how you know.

An **arithmetic sequence** is a sequence in which each term is the previous term plus a constant. This constant is called the **common difference**.

4. Which of these could be arithmetic sequences? Explain how you know.

Are you ready for more?

Extending Your Thinking

Elena says that it's not possible to have a sequence of numbers that is both arithmetic and geometric. Do you agree with Elena? Be prepared to show your reasoning.

Video: Comparing Sequences

Watch the following video to learn more about comparing arithmetic and geometric sequences.

Comparing Sequences (https://openstax.org/r/compare-sequences)

Self Check

Which of the following sequences is arithmetic?

a. 25, 5, 1, $\frac{1}{5}$, . . .
b. 3, 7, 11, 15, . . .
c. 1, 4, 16, 64, . . .
d. 3, 4, 6, 9, 13, . . .

Additional Resources

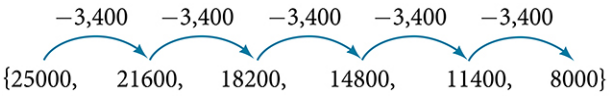
Arithmetic Sequences

Finding Common Differences

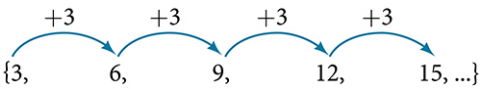
Companies often make large purchases, such as computers and vehicles, for business use.

- A woman who starts a small contracting business purchases a new truck for $25,000.
- After five years, she estimates that she will be able to sell the truck for $8,000. The loss in value of the truck will therefore be $17,000, which is $3,400 per year for five years.
- The truck will be worth $21,600 after the first year; $18,200 after two years; $14,800 after three years; $11,400 after four years; and $8,000 at the end of five years.

The values of the truck in the example are said to form an arithmetic sequence because they change by a constant amount each year. Each term increases or decreases by the same constant value called the common difference of the sequence. For this sequence, the common difference is –3,400.



The sequence below is another example of an arithmetic sequence. In this case, the common difference is 3. You can choose any term of the sequence and add 3 to find the subsequent term.



ARITHMETIC SEQUENCE

An arithmetic sequence is a sequence that has the property that the difference between any two consecutive terms is a constant. This constant is called the common difference. If a_1 is the first term of an arithmetic sequence and d is the common difference, the sequence will be:

$$\{a_n\} = \{a_1, a_1 + d, a_1 + 2d, a_1 + 3d \ldots\}$$

Example

Is each sequence arithmetic? If so, find the common difference.

a. {1, 2, 4, 8, 16, . . .}
b. {3, 1, 5, 9, 13, . . .}

Subtract each term from the subsequent term to determine whether a common difference exists.

a. The sequence is not arithmetic because there is no common difference.

$$2-1=1 \quad 4-2=2 \quad 8-4=4 \quad 16-8=8$$

b. The sequence is arithmetic because there is a common difference. The common difference is 4.

$$1-(-3)=4 \quad 5-1=4 \quad 9-5=4 \quad 13-9=4$$

TRY IT **Arithmetic Sequences**

Is the given sequence arithmetic? If so, find the common difference.

{18, 16, 14, 12, 10, . . .}

4.16.3 A Sequence Is a Type of Function

Activity

Jada and Mai are trying to decide what type of sequence this could be:

term number	value
1	2
2	6
5	18

Jada says: “I think this sequence is geometric because in the value column, each row is 3 times the previous row.”

Mai says: “I don’t think it is geometric. I graphed it, and it doesn’t look geometric.”

Do you agree with Jada or Mai? Be prepared to show your reasoning using a graph.

Self Check

Which two terms could be next in the sequence to make it arithmetic?

5, 8, ____, ____

a. 13, 21
b. 13, 20
c. 11, 14
d. 12.8, 7.68

Additional Resources

Creating a Sequence

For a sequence: 1, 3, . . .

Write the next two terms that would make the sequence

1. geometric
2. arithmetic

Solution

1. To make the sequence geometric, there must be a common ratio. $1 \cdot 3 = 3$, so use a common ratio of 3 to multiply each term by. The sequence becomes: 1, 3, 9, 27 . . .
2. To make the sequence arithmetic, there must be a common difference. $3 - 1 = 2$, so use a common difference of 2 to add to each term. The sequence becomes: 1, 3, 5, 7 . . .

TRY IT **Creating a Sequence**

Write the next two terms to create an arithmetic sequence.

2, 6, . . .

4.16.4 Identify and Generate Sequences

Cool Down

Many sequences start with the terms 2 and 8.

1. Find the next two terms of the arithmetic sequence 2, 8, ____, ____.

2. Find the next two terms of the geometric sequence 2, 8, ____, ____.

3. Find two possible next terms of a sequence 2, 8, ____, ____ that is neither geometric nor arithmetic.

4.16.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Here are the first two terms of some different arithmetic sequences. What is the 5^{th} term of each sequence?

a. –2, 4 ...

b. 11, 111 ...

c. 5, 7.5 ...

d. 5, –4 ...

2. For each sequence, decide whether it could be arithmetic, geometric, or neither.

a. 200, 40, 8, . . .

   a. Neither
   b. Arithmetic
   c. Geometric

b. 2, 4, 16, . . .

   a. Neither
   b. Arithmetic
   c. Geometric

c. 10, 20, 30, . . .

   a. Neither
   b. Arithmetic
   c. Geometric

d. 100, 20, 4, . . .

   a. Neither
   b. Arithmetic
   c. Geometric

e. 6, 12, 18, . . .

   a. Neither
   b. Arithmetic
   c. Geometric

3. For each arithmetic sequence with missing terms, find the common difference that might help determine the missing terms.

a. –3, –2, ___, ___, 1

b. ___, 13, 25, ___, ___

c. 1, 0.25, ___, –1.25, ___

d. 92, ___, ___ ,___, 80

4. Find the common difference for each of the following arithmetic sequences.
HINT: It may be helpful to complete the sequence with the missing terms.

a. ___, 6, 10, ___, 18

b. –1, ___, –11, –16, ___

c. 10, 5, ___, ___, –10

d. ___, ___, 4.5, 6, ___

e. ___, 12, 6, 0, ___

4.16.6 Lesson Summary

In this lesson, you learned how to:

- Explain what it means for a sequence to be arithmetic or geometric.

Here are the **activities** that helped you reach those goals:

- 4.16.1: Interpreting Function Notation
  - In this activity, you mentally evaluated functions given a function in function notation.
- 4.16.2: What Is an Arithmetic Sequence?
  - In this activity, you learned what makes a sequence arithmetic.
  - 4.16.2: Self Check
  - 4.16.2: Additional Resources
- 4.16.3: A Sequence Is a Type of Function
  - In this activity, you represented a sequence in a graph and discussed if it could be geometric or arithmetic.
  - 4.16.3: Self Check
  - 4.16.3: Additional Resources
- 4.16.4: Identify and Generate Sequences
  - In this activity, you completed the same sequence with different terms to make it arithmetic, then geometric, and then neither.

After these activities, you completed the following **practice**:

- 4.16.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

4.17 REPRESENTING SEQUENCES

4.17.0 Lesson Overview

In this lesson, you will use your knowledge about arithmetic and geometric sequences to learn how to use recursive definitions to generate sequences.

When you finish this lesson, you will be able to:

- Define arithmetic and geometric sequences recursively using function notation.
- Represent a sequence in different ways.
- Ask questions to get the information needed to represent a sequence in different ways.

Here are the **activities** that will help you reach those goals:

- 4.17.1: Define a Sequence
- 4.17.2: What Is a Recursive Definition?
  - 4.17.2: Self Check
  - 4.17.2: Additional Resources
- 4.17.3: Ways to Represent a Sequence
  - 4.17.3 Self Check
  - 4.17.3: Additional Resources
- 4.17.4: Geometric Recursive Definitions

4.17.1 Define a Sequence

Warm Up

Recall that:

An **arithmetic sequence** is a sequence in which each term is the previous term plus a constant.

A **geometric sequence** is a sequence in which each term is a constant times the previous term.

Use the first 5 terms of each sequence to state if the sequence is arithmetic, geometric, or neither.

A. 30, 40, 50, 60, 70, . . .

B. 80, 40, 20, 10, 5, 2.5, . . .

C. 1, 2, 4, 8, 16, 32, . . .

D. 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, . . .

E. 20, 13, 6, –1, –8, . . .

F. 1, 3, 7, 15, 31, . . .

4.17.2 What Is a Recursive Definition?

Activity

The **recursive definition** is a function that provides a repeated, or recurring, process to find terms in a sequence.

Start with the recursive definition: $R(n) = 3R(n-1) + 1$, $n \geq 2$ and given $R(1) = -1$.

$$R(n) = 3R(n\text{-}1) + 1, \quad n \geq 2 \quad \text{given } R(1) = \text{-}1$$

Rule Domain Basis

- The first part of the recursive definition provides the instructions for how to generate the sequence. It provides the function rule or formula for the n^{th} term.
- The second part states the domain of the sequence. It describes where we start and the values that are substituted into the function.
- The third part is the basis for the sequence or the smallest term value.

See the table below for how to substitute to find more terms in the sequence using the recursive definition. Substitute the term number, n, in for n in the recursive definition. Complete the table with a partner.

Term number (n)	Substitute in $R(n)$	Value
1	given	-1
2	$3R(2-1)+1 = 3R(1)+1 = 3(-1)+1 = -2$	-2
3	$3R(3-1)+1 = 3R(2)+1 = 3(-2)+1 = -5$	-5
4	$3R(4-1)+1 = 3R(3)+1 = 3(-5)+1$	a. ___
5	a. ___	b. ___
6	b. ___	c. ___

1. Using the recursive definition:

a. Substitute $n = 5$ in $R(n)$

b. Substitute $n = 6$ in $R(n)$

2. Determine the value for each term:

a. Value when $n = 4$

b. Value when $n = 5$

c. Value when $n = 6$

3. Write the full sequence out to 6 terms.

4. Determine if the following sequences are arithmetic or geometric.

a. 3, 6, 12, 24

b. 18, 36, 72, 144

c. 3, 8, 13, 18

d. 18, 13, 8, 3

e. 18, 9, 4.5, 2.25

f. 18, 20, 22, 24

g. 3, 15, 75, 375

CLASSROOM CONNECTION **Desmos Activity:**
Log into student.desmos.com using the information provided by your teacher to complete the activity.

5. Examine the following recursive definitions.

$F(1) = 18$, $F(n) = F(n-1) + 2$, $n \geq 2$

$G(1) = 18$, $G(n) = \frac{1}{2} \cdot G(n-1)$, $n \geq 2$

$H(1) = 3$, $H(n) = 5 \cdot H(n-1)$, $n \geq 2$

$J(1) = 3$, $J(n) = J(n-1) + 5$, $n \geq 2$

$K(1) = 18$, $K(n) = K(n-1) - 5$, $n \geq 2$

$L(1) = 18$, $L(n) = 2 \cdot L(n-1)$, $n \geq 2$

$M(1) = 3$, $M(n) = 2 \cdot M(n-1)$, $n \geq 2$

Match each definition with one of the sequences from the previous question. Be prepared to explain to a partner how you know it's a match. For each match that your partner finds, listen carefully to their explanation. If you disagree, discuss your thinking and work to reach an agreement.

a. 3, 6, 12, 24

b. 18, 36, 72, 144

c. 3, 8, 13, 18

d. 18, 13, 8, 3

e. 18, 9, 4.5, 2.25

f. 18, 20, 22, 24

g. 3, 15, 75, 375

Video: Learning About Recursive Definitions and Sequences

Watch the following video to learn more about matching sequences to their recursive definitions.

Learning About Recursive Definitions and Sequences (https://openstax.org/r/recursive-sequences)

Self Check

Which of the following sequences matches the recursive definition for $G(n)$ where $G(n) = G(n-1) + 4$ for $n \geq 2$ when $G(1) = 3$?

a. 3, 12, 48, 144
b. 3, 7, 11, 15
c. 1, 3, 7, 11, 15
d. 4, 8, 12, 16, 20

Additional Resources

Writing Recursive Definitions

Sometimes we can define a sequence recursively. That is, we can describe how to calculate the next term in a sequence if we know the previous term.

Here's a sequence: 6, 10, 14, 18, 22, . . . This is an arithmetic sequence, where each term is 4 more than the previous term. Since sequences are functions, let's call this sequence f. Then we can use function notation to write $f(n) = f(n-1) + 4$. Here, $f(n)$ is the term, $f(n-1)$ is the previous term, and + 4 represents the common difference since f is an arithmetic sequence.

When we define a function recursively, we must also say what the first term is. Without that, there would be no way of knowing if the sequence defined by $f(n) = f(n-1) + 4$ started with 6 or 81 or any other number. Here, one possible initial condition is $f(1) = 6$. It could also make sense to number the terms starting with 0, using $f(0) = 6$.

Combining this information gives the recursive definition: $f(1) = 6$ and $f(n) = f(n-1) + 4$ for $n \geq 2$, where n is an integer. We include the $n \geq 2$ at the end since the value of f at 1 is already given, and the other terms in the sequence are generated by inputting integers larger than 1 into the definition.

Let's look at some examples from the first activity and write the recursive definition for each type of sequence:

EXAMPLE 1

Sequence A: $30, 40, 50, 60, 70, \ldots$

Step 1 - Determine the type of sequence

This sequence is arithmetic.

Step 2 - Determine the common difference or common ratio

The terms increase by adding a value of 10, so the common difference is 10.

Step 3 - Determine the formula or rule

Since one term's value comes from adding 10 to the term value before it, that means $n = (n-1) + 10$. Now write this rule using function notation.

$A(n) = A(n-1) + 10$

Step 4 - Determine the initial condition or basis

$A(1) = 30$

Step 5 - Determine the domain of the formula

We use the first term to find the second term, so that is where we start at $n = 2$. So, our domain is $n \geq 2$.

The recursive definition for 30, 40, 50, 60, 70, ... is $A(1) = 30$, $A(n) = A(n-1) + 10$, $n \geq 2$.

EXAMPLE 2

Sequence B: $80, 40, 20, 10, 5, 2.5, \ldots$

Step 1 - Determine the type of sequence

This sequence is geometric.

Step 2 - Determine the common difference or common ratio

The terms are divided by 2 each time, so the common ratio is $\frac{1}{2}$ (since it is divided by 2).

Step 3 - Determine the formula or rule

Since one term's value comes from multiplying $\frac{1}{2}$ to the term value before it, that means $n = (n-1) \times \frac{1}{2}$. Now write this rule using function notation.

$B(n) = B(n-1) \times \frac{1}{2}$

Step 4 - Determine the initial condition or basis

$B(1) = 80$

Step 5 - Determine the domain of the formula

We use the first term to find the second term, so that is where we start at $n = 2$. So, our domain is $n \geq 2$.

The recursive definition for 80, 40, 20, 10, 5, 2.5, ... is $B(1) = 80$, $B(n) = B(n-1) \times \frac{1}{2}$, $n \geq 2$.

EXAMPLE 3

Sequence F: $1, 3, 7, 15, 31, \ldots$

Step 1 - Determine the type of sequence

This sequence is neither arithmetic nor geometric.

TRY IT **Writing Recursive Definitions**

Write $E(1)$ and a recursive definition for $E(n)$:

E: 20, 13, 6, –1, –8, . . .

4.17.3 Ways to Represent a Sequence

Activity

Your teacher will give you either a problem card or a data card. Do not show or read your card to your partner.

If your teacher gives you the data card:

1. Silently read the information on your card.
2. Ask your partner, "What specific information do you need?" and wait for your partner to ask for information. Only give information that is on your card. (Do not figure out anything for your partner!)
3. Before telling your partner the information, ask, "Why do you need to know (that piece of information)?"
4. Read the problem card and solve the problem independently.
5. Share the data card and discuss your reasoning.

If your teacher gives you the problem card:

1. Silently read your card and think about what information you need to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain to your partner how you are using the information to solve the problem.
4. When you have enough information, share the problem card with your partner and solve the problem independently.
5. Read the data card and discuss your reasoning.

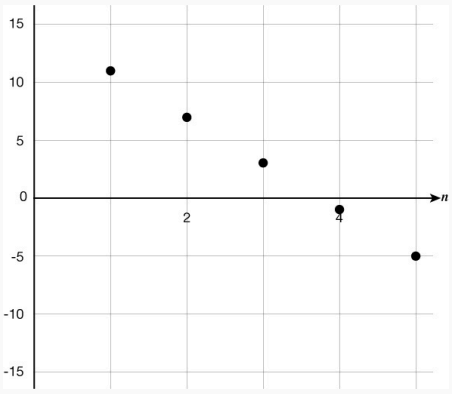
Are you ready for more?

Extending Your Thinking

Make a visual pattern (for example, using dots or boxes), starting with Step 0, so the pattern for Step n contains $n^2 + 3n + 3$ dots.

Self Check

Which of the following recursive definitions is represented by the graph below?



a. $a(1) = 11$ and $a(n) = a(n) - 4$ for $n \geq 2$
b. $a(11) = 1$ and $a(n) = a(n-1) - 4$ for $n \geq 2$
c. $a(1) = 11$ and $a(n) = a(n-1) - 4$ for $n \geq 2$
d. $a(1) = 11$ and $a(n) = 4a(n-1)$ for $n \geq 2$

Additional Resources

Representing Sequences

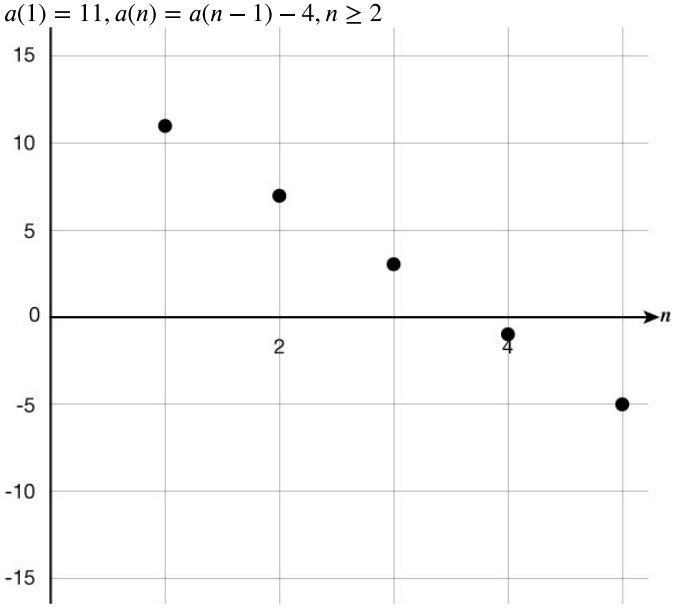
Let's look at some more examples for finding the recursive definition of arithmetic and geometric sequences and ways to represent them.

EXAMPLE 1

An arithmetic sequence, a, begins 11, 7, . . .

1. Write a recursive definition for this sequence using function notation.
2. Sketch a graph of the first 5 terms of this sequence.
3. Explain how to use the recursive definition to find $a(100)$. (Don't actually determine the value.)

Solutions:

1. $a(1) = 11, a(n) = a(n-1) - 4, n \geq 2$
2. 

3. Starting with 11, keep subtracting 4 until you do that 99 times.

EXAMPLE 2

A geometric sequence, g, starts 80, 40, . . .

1. Write a recursive definition for this sequence using function notation.
2. Use your definition to make a table of values for $g(n)$ for the first 6 terms.
3. Explain how to use the recursive definition to find $g(100)$. (Don't actually determine the value.)

Solutions:

1. $g(1) = 80, g(n) = \frac{1}{2} \cdot g(n-1), n \geq 2$
2.

n	1	2	3	4	5	6
$g(n)$	80	40	20	10	5	2.5

3. Starting with 80, keep dividing by 2 until you do that 99 times.

TRY IT **Representing Sequences**

Write a recursive definition for $P(n)$, the sequence below:

1, 2, 4, 8, 16

4.17.4 Geometric Recursive Definitions

Cool Down

The geometric sequence G starts 12, 6, . . .

1. Write a recursive definition for sequence G.

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-17-4-geometric-recursive-definitions)

2. Use the graphing tool above or technology outside the course. Sketch a graph representing the first 5 terms of sequence G.

4.17.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 3, match each sequence with one of the definitions. Note that only the part of the definition showing the relationship between the current term and the previous term is given so as not to give away the solutions.

1. 6, 12, 18, 24

a. $d(n) = d(n-1) + 20$
b. $c(n) = c(n-1) + 6$
c. $b(n) = \frac{1}{2}b(n-1)$
d. $a(n) = 7 \cdot a(n-1)$

2. 2, 14, 98, 686

a. $d(n) = d(n-1) + 20$
b. $c(n) = c(n-1) + 6$
c. $b(n) = \frac{1}{2}b(n-1)$
d. $a(\mathrm{n}) = 7 \cdot a(n-1)$

3. 160, 80, 40, 20

a. $d(n) = d(n-1) + 20$
b. $c(n) = c(n-1) + 6$
c. $b(n) = \frac{1}{2}b(n-1)$
d. $a(\mathrm{n}) = 7 \cdot a(n-1)$

For questions 4 – 5, select the first five terms of each sequence. Determine whether each sequence is arithmetic, geometric, or neither.

4. $a(1) = 7$, $a(n) = a(n-1) - 3$ for $n \geq 2$.

a. 7, 4, 1, –2, –5, arithmetic
b. 7, –21, 63, –189, arithmetic
c. 7, –21, 63, –189, geometric
d. 7, 4, 1, –2, –5, geometric

5. $c(1) = 3$, $c(n) = 10 \cdot c(n-1)$ for $n \geq 2$.

a. 3, 13, 23, 33, 43, arithmetic
b. 3, 13, 23, 33, 43, geometric
c. 3, 30, 300, 3000, 30000, geometric
d. 3, 30, 300, 3000, 30000, arithmetic

For questions 6 – 8, match each sequence with one of the recursive definitions. Note that only the part of the definition showing the relationship between the current term and the previous term is given so as not to give away the solutions.

6. 3, 15, 75, 375

a. $d(n) = d(n-1) + n - 1$
b. $c(n) = 5 \cdot c(n-1)$
c. $b(n) = b(n-1) - 4$
d. $a(n) = \frac{1}{3} \cdot a(n-1)$

7. 18, 6, 2, $\frac{2}{3}$

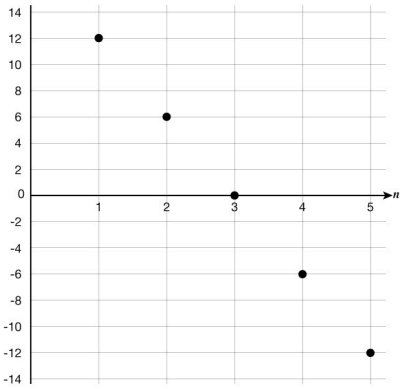
a. $d(n) = d(n-1) + n - 1$
b. $c(n) = 5 \cdot c(n-1)$
c. $b(n) = b(n-1) - 4$
d. $a(n) = \frac{1}{3} \cdot a(n-1)$

8. 17, 13, 9, 5

a. $d(n) = d(n-1) + n - 1$

b. $c(n) = 5 \cdot c(n-1)$
c. $b(n) = b(n-1) - 4$
d. $a(n) = \frac{1}{3} \cdot a(n-1)$

9. An arithmetic sequence k starts 12, 6, . . . and is represented by the graph below.



Which of the following is a recursive definition for sequence k when $n \geq 2$?

a. $k(1) = 12, 6 \cdot k(n) = k(n-1)$
b. $k(1) = 12, k(n) = k(n-1) + 6$
c. $k(1) = 12, k(n) = k(n-1) - 6$
d. $k(1) = 12, k(n) = k(n) - 6$

4.17.6 Lesson Summary

In this lesson, you learned how to:

- Define arithmetic and geometric sequences recursively using function notation.
- Represent a sequence in different ways.
- Ask questions to get the information needed to represent a sequence in different ways.

Here are the **activities** that helped you reach those goals:

- 4.17.1: Define a Sequence
  - In this activity, you reviewed identifying sequences as geometric, arithmetic, or neither.
- 4.17.2: What Is a Recursive Definition?
  - In this activity, you matched sequences with their recursive definitions.
  - 4.17.2: Self Check
  - 4.17.2: Additional Resources
- 4.17.3: Ways to Represent a Sequence
  - In this activity, you matched sequences with their recursive definitions and also represented the sequences on graphs.
  - 4.17.3: Self Check
  - 4.17.3: Additional Resources
- 4.17.4: Geometric Recursive Definitions
  - In this activity, you wrote a recursive definition given a geometric sequence. You then graphed the terms of the sequence.

After these activities, you completed the following **practice**:

- 4.17.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

4.18 THE Nth TERM OF A SEQUENCE

4.18.0 Lesson Overview

In this lesson, you will learn how a function can be represented by two different equations. You will find any term of an arithmetic sequence directly, by using the n^{th} term formula to find any term number n.

When you finish this lesson, you will be able to:

- Describe the n^{th} term.
- Find terms of sequences directly.
- Explain why different equations can represent the same sequence.

Here are the **activities** that will help you reach those goals:

- 4.18.1 Repeated Operations
- 4.18.2 Identifying Domain for a Function
- 4.18.3 Define an Arithmetic Sequence by the n^{th} Term
  - 4.18.3: Self Check
  - 4.18.3: Additional Resources
- 4.18.4 Define a Geometric Sequence by the n^{th} Term
  - 4.18.4: Self Check
  - 4.18.4: Additional Resources
- 4.18.5 Interchanging between Formulas
  - 4.18.5: Self Check
  - 4.18.5: Additional Resources
- 4.18.6 Different Types of Equations

After that, you'll **practice and review**.

- 4.18.7 Practice
- 4.18.8 Lesson Summary

4.18.1 Repeated Operations

Warm Up

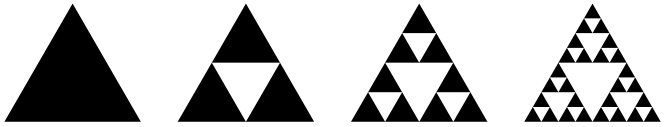
Which one doesn't belong?

1. $5+2+2+2+2+2+2$
2. $5+6 \cdot 2$
3. $5 \cdot 2^6$
4. $5 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

4.18.2 Identifying Domain for a Function

Activity

A Sierpinski triangle can be created by starting with an equilateral triangle, breaking the triangle into 4 congruent equilateral triangles, and then removing the middle triangle. Starting from a single black equilateral triangle:



1. Let S be the number of black triangles in Step n. Define $S(n)$ recursively.

2. Andre and Lin are asked to write an equation for S that isn't recursive. Andre writes $S(n) = 3^n$ for $n \geq 0$, while Lin writes $S(n) = 3^n - 1$ for $n \geq 1$. Whose equation do you think is correct? Be prepared to show your reasoning.

Are you ready for more?

Extending Your Thinking

Here is a geometric sequence. Find the missing terms.

3, ____, 6, ____, 12, ____, 24

Video: Learning About Sierpinski's Triangle

Watch the following video to learn more about how to write a pattern as an n^{th} term.

Learning About SierpinskiÕs Triangle (https://openstax.org/r/sierpinski-triangle)

4.18.3 Define an Arithmetic Sequence by the nth Term

Activity

1. Clare takes a piece of paper with length 8 inches and width 10 inches and cuts it in half. Then she cuts it in half again, and again . . .

a. Instead of writing a recursive definition, Clare writes $C(n) = 80 \cdot (\frac{1}{2})^n$, where C is the area, in square inches, of the paper after n cuts. Explain where the different terms in her expression came from.

b. Approximately what is the area of the paper after 10 cuts?

2. Kiran takes a piece of paper with length 8 inches and width 10 inches and cuts away 1 inch of the width. Then he does it again, and again . . .

a. Complete the table for the area of Kiran's paper $K(n)$, in square inches, after n cuts.

n	$K(n)$
0	80
1	
2	$80 - 8 - 8 = 80 - 8(2) = 64$
3	
4	
5	

b. Kiran says the area after 6 cuts, in square inches, is $80 - 8 \cdot 6$. Explain where the different terms in his expression came from.

c. Write a definition for $K(n)$ that is not recursive.

3. Which is larger, $K(6)$ or $C(6)$?

4. Is $K(n)$ arithmetic or geometric?

5. Is $C(n)$ arithmetic or geometric?

A recursive sequence is a sequence in which terms are defined using one or more of the previous terms. If you know the n^{th} term of an arithmetic or geometric sequence and the common difference or factor, you can find the $(n+1)^{\text{th}}$ term by using the recursive formula.

Different definitions can often create the same sequence. For arithmetic and geometric sequences, there are general rules, called explicit rules, that can be followed to help find any term in the sequences.

These are called the nth **term** or the general term of the sequence.

Arithmetic Sequence Formulas

Recursive Formula

$a_n = a_{n-1} + d$, a_1 = first term

where a_1 is the first term, n is the term you want, and d is the common difference.

Explicit General Formula

$a_n = a_1 + (n-1)d$

Where a_1 is the first term, n is the term you want, and d is the common difference.

Self Check

What is the 12th term in an arithmetic sequence where the first term is 22 and the common difference is –2?

a. -45056
b. 0
c. -22
d. -2

Additional Resources

The n^{th} Term of Arithmetic Sequences
GENERAL TERM (n^{th} TERM) OF AN ARITHMETIC SEQUENCE

The general term of an arithmetic sequence with first term a_1 and common difference d is: $a_n = a_1 + (n-1)d$.

EXAMPLE 1

Write the explicit formula for the sequence given by the terms 18, 21, 24, 27, ...

Solution

Step 1- Write the general formula.

$a_n = a_1 + (n-1)d$

Step 2 - Substitute values for the first term, common difference/ratio, term number.

$a_1 = 18, d = 3$

$a_n = 18 + (n-1)3$

So, the explicit formula for the sequence 18, 21, 24, 27... is $a_n = 18 + (n-1)3$.

Now any term can more easily be determined - even the 1000^{th} term!

EXAMPLE 2

Find the 15^{th} term of a sequence where the first term is 3 and the common difference is 6.

Step 1 - Write the general formula.

$a_n = a_1 + (n-1)d$

Step 2 - Substitute values for the first term, common difference/ratio, term number.

$a_1 = 3, d = 6, n = 15$

$a_{15} = 3 + (15 - 1)6$

Step 3 - Simplify the expression.

$a_{15} = 3 + (14)6$

$a_{15} = 3 + 84$

$a_{15} = 87$

If the explicit formula for this question was needed, we would not have substituted $n = 15$ and the nth term formula would have been $a_n = 3 + (n - 1)6$. This formula is also equivalent to

$a_n = 3 + 6(n - 1)$

TRY IT **Try It: The n^{th} Term of Arithmetic Sequences**

Find the formula for the nth term of a sequence where the first term is 7 and the common difference is 9.

4.18.4 Define a Geometric Sequence by the nth Term

Activity

In 2015, the number of wildcats in a national park was 284. It was estimated that the wildcat population increased by 4% each year.

1. Instead of writing a recursive definition, a researcher for the park writes $W_n = 284(1.04)^n$, where W is the projected wildcat population n years after 2015. Explain where the different factors in her expression came from.

Work with a partner to discuss and create a table of values to estimate the projected wildcat population over the next 7 years.

2. Complete the table when $n = -2$ on your paper and check your answers.

Year	n	Wildcat Population
2015	0	$W_n = 284(1.04)^0 = 284$
2016	1	$W_n = 284(1.04)^1 = 295$
2017	2	$W_n = 284(1.04)^2 =$
2018	3	
2019	4	
2020	5	
2021	6	
2022	7	

3. Use the graphing tool or technology outside the course. Graph the data that represents the population projections for this scenario.

Access multimedia content (http://openstax.org/books/algebra-1/pages/4-18-4-define-a-geometric-sequence-by-the-nth-

term)

4. Calculate and enter the year when the wildcat population will exceed 500 members.

5. Is the data that represents this scenario an arithmetic or geometric sequence? Explain your reasoning.

Remember that a recursive formula defines terms using one or more of the previous terms. If you need to calculate the 100^{th} term or the 500^{th} term in a sequence, the recursive formula becomes difficult to use if you do not know the values of the terms near $n = 100$ or $n = 500$.

Different definitions can often create the same sequence but are more generalizable. These are called general rules or explicit rules. These formulas can be used to find any term in the sequence and may also be referred to as the rule to find the nth term.

Geometric Sequence Formulas

Recursive Formula

$a_n = a_{n-1} \cdot r$,

Where a_1 is the first term, n is the term you want, and r is the common ratio.

Explicit General Formula

$a_n = a_1(r)^{n-1}$

Where a_1 is the first term, n is the term you want, and r is the common ratio.

Self Check

For the function $J(n) = 180(\frac{1}{3})^{n-1}$, what is the value of the function when $n = 5$?

a. $\frac{1}{81}$
b. $\frac{180}{81}$
c. 20
d. 45

Additional Resources

The n^{th} Term of a Geometric Sequence

The n^{th} Term of a Geometric Sequence
General Term (nth Term) of a Geometric Sequence

The general term of a geometric sequence with first term a_1 and the common ratio r is: $a_n = a_1 r^{n-1}$.

EXAMPLE 1

Write the explicit formula for the sequence given by the terms 3, 2, 43, 89 ...

Solution

Step 1- Write the general formula.

$a_n = a_1(r^{n-1})$

Step 2 - Substitute values for the first term, common difference/ratio, term number.

$a_1 = 3, r = \frac{2}{3}$

$a_n = 3(\frac{2}{3})^{n-1}$

So, the explicit formula for the sequence 3, 2, 43, 89 ... is $a_n = 3(\frac{2}{3})^{n-1}$.

Just like it is with arithmetic sequences, the explicit formula makes finding any term easier to determine - especially term numbers that are very large!

EXAMPLE 2

Find the 14^{th} term of a sequence where the first term is 64 and the common ratio, $r = \frac{1}{2}$.

Solution

Step 1 - Write the general formula.

$a_n = a_1(r^{n-1})$

Step 2 - Substitute values for the first term, common difference/ratio, term number.

$a_1 = 64, r = \frac{1}{2}, n = 14$

$a_{14} = 64(\frac{1}{2})^{14-1}$

Step 3 - Simplify the expression.

$a_{14} = 64(\frac{1}{2})^{13}$

$a_{14} = 64(\frac{1}{2})^{13}$

$a_{14} = 64(\frac{1}{8192})$

$a_{14} = \frac{1}{128}$

If the explicit formula for this question was needed, we would not have substituted $n = 14$ and the n^{th} term formula would have been $a_n = 64(\frac{1}{2})^{n-1}$.

TRY IT **The n^{th} Term of a Geometric Sequence**

Find the formula for the n^{th} term of a sequence where the first term is 81 and the common ratio is $r = \frac{1}{3}$.

4.18.5 Interchanging between Formulas

Activity

Converting from a Recursive Formula to an Explicit Formula for Arithmetic Sequences

An arithmetic sequence has the following recursive formula:

$a_1 = 3$

$a_n = a_{n-1} + 2, n \geq 2$

Recall that this formula gives us the following two pieces of information:

- The first term is 3.
- To get any term from its previous term, **add** 2. In other words, the common difference is 2.

Let's find an explicit formula for the sequence.

We can represent a sequence whose first term is A and common difference is B with the standard explicit form $A + B(n - 1)$.

Therefore, an explicit formula of the sequence is $a_n = 3 + 2(n - 1)$.

1. Write an explicit (nth term) formula for an arithmetic sequence where $b_1 = 2$ and $b_n = b_{n-1} + 7, n \geq 2$.

Converting from a Explicit Formula to an Recursive Formula for Arithmetic Sequences

Now try to take the n^{th} term formula and use it to write the recursive formula of an arithmetic sequence.

We are given the following n^{th} term of an arithmetic sequence:

$d_n = 5 + (n-1)\cdot 16$

This formula is given in the standard explicit form $A + B(n-1)$, where A is the first term and B is the common difference. Therefore, the first term of the sequence is 5, and the common difference is 16.

Let's find a recursive formula for the sequence. Recall that the recursive formula gives us two pieces of information:

- The first term, which we know is 5.
- The pattern rule to get any term from the term that comes before it, which we know is "add 16."

Therefore, this is a recursive formula for the sequence: $d_1 = 5$, $d_n = d_{n-1} + 16$, $n \geq 2$.

2. Write the recursive formula of $e_n = 12 + (n-1)2$.

Converting from a Recursive Formula to an Explicit Formula for Geometric Sequences

Now change the recursive formula of a geometric sequence into the nth term.

Here is the recursive formula of a geometric sequence, $g(n)$:

$g_1 = 9$, $g_n = g_{n-1}\cdot(8)$, $n \geq 2$

9 is the first term of the sequence, and 8 is the common ratio.

An explicit formula is structured as: $g(x) =$ (1st term of sequence) $\cdot$ (common ratio)$^{(x-1)}$.

Substitute the values, and you get the explicit formula: $g_n = 9\cdot 8^{(n-1)}$.

3. Write the n^{th} term of the geometric sequence where $g_1 = 5$ and $g_n = g_{n-1}\cdot 4$, $n \geq 2$.

Converting from an Explicit Formula to a Recursive Formula for Geometric Sequences

Finally, geometric sequences represented by the n^{th} term can be represented with the recursive definition.

Start with the n^{th} term formula for g_n where $g_n = 6\cdot 3^{n-1}$.

Working backward, when $n = 1$, $g_1 = 6$. This is the initial term.

The formula for geometric sequences recursively is $g_n = g_{n-1}\cdot r$, where r is the common ratio.

The common ratio of the given sequence is 3.

So, $g_1 = 6$ and $g_n = g_{n-1}\cdot 3$, $n \geq 2$.

4. Write the geometric sequence represented by $g_n = 14\cdot 2^{n-1}$ using the recursive definition.

Why Should I Care?



One of the tools to fight climate change is renewable energy. Wind farms and solar power can produce can produce clean energy. But what happens when there is no wind or sunshine? Energy companies rely on battery storage systems to save energy that is created on sunny or windy days, so it can be used when the sun and wind aren't cooperating.

It is important that the batteries can store as much energy as possible, for as long as possible. Scientists who create

these batteries use functions to help them calculate energy inputs and outputs and the changes that make a battery lose power.

Self Check

Write the nth term formula of the geometric sequence represented by $g_1 = -3$ and $g_n = g_{n-1} \cdot 6$, $n \geq 2$.

a. $g_n = 6(n-1) - 3$
b. $g_n = -3 \cdot 6^{n-1}$
c. $g_n = -3 \cdot 6^n$
d. $g_n = 6n + -3$

Additional Resources

Switching Between Definitions of Sequences

A recursive formula defines each term of a sequence using the previous terms.

Sequence	Recursive Formula	Explicit or n^{th} Term Formula
Arithmetic	$F_1 = a_1$, $F_n = F_{n-1} + d$, where a_1 is the first term and d is the common difference.	$a_n = a_1 + (n-1)d$, where a_1 is the first term, n is the term you want, and d is the common difference.
Geometric	$H_1 = a_1$, $H_n = H_{n-1} \cdot r$, Where a_1 is the first term and r is the common ratio.	$a_n = a_1 r^{n-1}$, Where a_1 is the first term, n is the term you want, and r is the common ratio.

EXAMPLE 1

Find the explicit general formula for the arithmetic sequence given in recursive form.

$a_1 = 5, a_n = a_{n-1} + 6, n \geq 2$

Step 1 - Write the recursive formula. (given)

$a_1 = 5, a_n = a_{n-1} + 6, n \geq 2$

Step 2 - Identify the first term.

$a_1 = 5$

Step 3 - Identify the common difference/ratio.

$d = 6$

Step 4 - Write the explicit formula for the n^{th} term.

$a_n = a_1 + (n-1)d$

Step 5 - Substitute the first term and common difference/ratio into the formula.

$a_n = 5 + (n-1)6$

Check by finding the first 3 terms using each formula:

$a_1 = 5, a(n) = a_n = a_{n-1} + 6, n \geq 2$	$a_1 = 5$	$a_2 = 5 + 6 = 11$	$a_3 = 11 + 6 = 17$
$a_n = 5 + (n-1)6$	$a_1 = 5$	$a_2 = 5 + (2-1)6 = 11$	$a_3 = 5 + (3-1)6 = 17$

Both formulas produce the same first 3 terms: 5, 11, 17.

EXAMPLE 2

Find the explicit general formula for the geometric sequence given in recursive form.

$b_1 = 4, b_n = b_{n-1} \cdot 5, n \geq 2$

Step 1 - Write the recursive formula (given).

$b_1 = 4, b_n = b_{n-1} \cdot 5, n \geq 2$

Step 2 - Identify the first term.

$a_1 = 4$

Step 3 - Identify the common difference/ratio.

$r = 5$

Step 4 - Write the explicit formula for the n^{th} term.

$a_n = a_1 r^{n-1}$

Step 5 - Substitute the first term and the common difference/ratio into the formula.

$a_n = 4(5)^{n-1}$

Check by finding the first 3 terms using each formula:

$b_1 = 4,$ $b_n = b_{n-1} \cdot 5,$ $n \geq 2$	$b_1 = 4$	$b_2 = b_{2-1} \cdot 5 =$ $4 \cdot 5 = 20$	$b_3 = b_{3-1} \cdot 5 =$ $20 \cdot 5 = 100$
$a_n = 4(5)^{n-1}$	$a_1 = 4(5)^{1-1} = 4$	$a_2 = 4(5)^{2-1} = 20$	$a_3 = 4(5)^{3-1} = 100$

Both formulas produce the same first 3 terms: 4, 20, 100.

EXAMPLE 3

Find the recursive formula for the arithmetic sequence given in explicit general form.

$a_n = 21 + (n - 1)(-2)$

Step 1 - Write the explicit formula (given).

$a_n = 21 + (n - 1)(-2)$

Step 2 - Identify the first term.

$a_1 = 21$

Step 3 - Identify the common difference/ratio.

$d = -2$

Step 4 - Write the explicit formula for the n^{th} term.

$F_1 = a_1, F_n = F_{n-1} + d, n \geq 2$

Step 5 - Substitute the first term and the common difference/ratio into the formula.

$F_1 = 21, F_n = F_{n-1} - 2, n \geq 2$

Check your reasoning by finding the first three terms using each formula. Both formulas produce the same first 3 terms: 21, 19, 17.

EXAMPLE 4

Find the recursive formula for the arithmetic sequence given in explicit general form.

$a_n = 3 \cdot (\frac{1}{4})^{n-1}$

Step 1 - Write the explicit formula.

$a_n = 3 \cdot (\frac{1}{4})^{n-1}$

Step 2 - Identify the first term.

$a_1 = 3$

Step 3 - Identify the common difference/ratio.

$r = \frac{1}{4}$

Step 4 - Write the recursive formula.

$H_1 = a_1, H_n = H_{n-1} \cdot r, n \geq 2$

Step 5 - Substitute the first term and the common difference/ratio into the formula.

$H_1 = 3, H_n = H_{n-1} \cdot \frac{1}{4}, n \geq 2$

Check your reasoning by finding the first three terms using each formula. Both formulas produce the same first 3 terms: $3, \frac{3}{4}, \frac{3}{16}$.

TRY IT **Switching Between Definitions of Sequences**

Write the recursive formula for the sequence below that is given in explicit form:

$a_n = 14 + (n - 1)(3)$

4.18.6 Different Types of Equations

Cool Down

1. A sequence is defined by $f_1 = -3, f_n = f_{n-1} + 0.5$ for $n \geq 2$. Diego writes an equation for the n^{th} term of the sequence as $a_n = -3 + (n - 1)(0.5)$. Is he correct? Explain how you know.
2. How would you describe the representation $f_1 = -3, f_n = f_{n-1} + 0.5$ for $n \geq 2$ for Diego's sequence?
3. How would you describe the representation of Diego's sequence, $a_n = -3 + (n - 1)(0.5)$?

4.18.7 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Find the value of each specified term of the arithmetic sequences below.

a. Find the 21^{st} term of a sequence where the first term is 3 and the common difference is 8.

b. Find the 30^{th} term of a sequence where the first term is -14 and the common difference is 5.

c. Find the 16^{th} term of a sequence where the first term is 11 and the common difference is -6.

2. Find the value of each specified term of the geometric sequences below.

a. Find the 11^{th} term given the first term is 8 and the common ratio is 3.

b. Find the 10^{th} term given the first term is -6 and the common ratio is -2.

3. A sequence is defined recursively by $f_1 = -6$, $f_n = f_{n-1} - 2$, for $n \geq 2$. Which of the following gives the n^{th} term for $n \geq 0$?

a. $a_n = -6 + (n)(-2)$
b. $a_n = -6 + (n-1) + (-2)$
c. $a_n = -6 + (n-1)(-2)$
d. $a_n = -2 + (n-1)(-6)$

4. A sequence is defined recursively by $f_1 = -20$, $f_n = f_{(n-1)} + 5$ for $n \geq 2$. Which of the following is the n^{th} term?

a. $a_n = 5 + (n-1)(-20)$
b. $a_n = -20 + (n-1)(5)$
c. $a_n = -20 + (n-1)(-2)$
d. $a_n = -20 + (n-1) + 5$

5. A sequence is defined recursively by $f_1 = 3$, $f_n = 2 \cdot f_{n-1}$, $n \geq 2$. Which of the following gives the n^{th} term for $n \geq 0$?

a. $f_n = 3 + 2^n$
b. $f_n = 3 + 2n$
c. $f_n = 3^n$
d. $f_n = 3 \cdot 2^{n-1}$

6. A sequence of numbers is given by $a_2 = 1$, $a_3 = 5$, $a_4 = 9$, $a_5 = 13$, $a_6 = 17$. Which expression can be used to find the n^{th} term of the sequence?

a. $3n - 4$
b. $7n - 4$
c. $4n - 7$
d. $4n - 3$

7. A sequence of numbers is given by $\frac{1}{27}$, $\frac{1}{9}$, $\frac{1}{3}$, 1, ... What is the equation that can be used to find the n^{th} term of the sequence?

a. $a_n = \frac{1}{27} \div 3(n-1)$
b. $a_n = \frac{1}{27}(3)^{n-1}$
c. $a_n = \frac{1}{27}(\frac{1}{3})^{n-1}$
d. $a_n = \frac{1}{27} - \frac{1}{3}(n-1)$

8. A sequence is given by the set of numbers $5, 1, -3, -7$ ... What is the equation that can be used to find the n^{th} term of the sequence?

a. $a_n = 5 - (n-1)(-4)$
b. $a_n = 5(4)^{n-1}$
c. $a_n = 5(-4)^{n-1}$
d. $a_n = 5 + (n-1)(-4)$

9. A sequence consists of the following terms: 1280, 320, 80, ... What is the explicit formula for this sequence?

a. $a_n = 1280 \div 4(n-1)$
b. $a_n = 1280(\frac{1}{4})^{n-1}$
c. $a_n = 1280(4)^{n-1}$
d. $a_n = 1280 - 960(n-1)$

10. An arithmetic sequence consists of the terms 1, 6, 11, 16 .... If you were asked to find the 100^{th} term, what is the general formula you would use to find that value efficiently?

a. $a_n = 1 + 5(a_{n-1})$
b. $a_n = 5 + a_{n-1}$
c. $a_n = 1(a_{n-1})^5$
d. $a_n = 1 + 5(n-1)$

4.18.8 Lesson Summary

In this lesson, you learned how to:

- Explain why different equations can represent the same sequence.

Here are the **activities** that helped you reach those goals:

- 4.18.1 Repeated Operations
  - In this activity, you looked at different ways to represent repeated addition with multiplication and exponents.
- 4.18.2 Identifying Domain for a Function
- 4.18.3 Define an Arithmetic Sequence by the n^{th} Term
  - In this activity, you learned how to represent arithmetic sequences using an explicit general rule.
  - 4.18.3: Self Check
  - 4.18.3: Additional Resources
- 4.18.4 Define a Geometric Sequence by the n^{th} Term
  - In this activity, you learned how to represent geometric sequences using an explicit general rule.
  - 4.18.4: Self Check
  - 4.18.4: Additional Resources
- 4.18.5 Interchanging between Formulas
  - In this activity, you learned that arithmetic and geometric sequences can be written using the recursive definition and the n^{th} term. It is possible to use them interchangeably.
  - 4.18.5: Self Check
  - 4.18.5: Additional Resources
- 4.18.6 Different Types of Equations
  - In this activity, you looked at how the same arithmetic sequence was represented recursively and with the n^{th} term.

After these activities, you completed the following **practice**:

- 4.18.7 Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

PROJECT 4: USING FUNCTIONS TO MODEL BATTERY POWER

Project 4 Overview

In this culminating project, you will use functions to model real-life situations. Both activities in the lesson use the context of a cell phone's battery power varying over time—either increasing from being charged or decreasing from being used. The relationships in each activity can be appropriately modeled with linear functions.

In the first activity, you will be given a table of values showing times and percentages of battery power. Study the rates of change in the table, which are fairly constant, and produce a simple model to make a prediction.

In the second activity, you will analyze images of battery-usage status from a cell phone. To gather the necessary information, you will need to make sense of the numerical values and the graph on each image. You will also need to determine how to organize and represent the information before you can use it to create a model.

During this project, you will engage in many aspects of the modeling cycle. You will identify important quantities in a situation, reason about relationships using representations that you consider useful, analyze the relationships mathematically to draw conclusions, and reflect on your results in context. Upon gaining new information, you can work to improve your model.

You might want to collect data from your own electronic devices, or to create presentations to explain your models.

Technology isn't required for this lesson, but there are opportunities for you to choose to use appropriate technology to solve problems.

Project Activities

1. Estimate Time to Charge a Battery
2. Analyze Percent Charge Over Time
3. Write an Equation to Model Data

Learning Targets

When you finish this project, you will have demonstrated how to:

1. Use functions to model real-life situations and make predictions.

Estimate Time to Charge a Battery

For this project, think about an electronic device with a battery that you have to charge on a regular basis.

1. What device is it?
2. When you are using the device, approximately how long does it take the battery to go from 100% charged until the time you need to plug it in again to recharge?
3. Approximately how long does it take to charge to 100% starting from 0% or nearly 0%?
4. Approximately how long does it take to charge if you plugged in your device when the battery was 50% charged?
5. How long do you think it would take to recharge the device to 100% from 50% compared to the time it would take if the device was at 0%? Would it be exactly half the time, more than half the time, or less than half the time it would take if starting from 0%?

Analyze Percent Charge Over Time

A cell phone is plugged in to be charged. The table below shows the percent of battery power at a few times after it was plugged in. Use the data to find out. Be prepared to show your reasoning.

Time	Percentage Charged
11:00 a.m.	6%
11:10 a.m.	15%

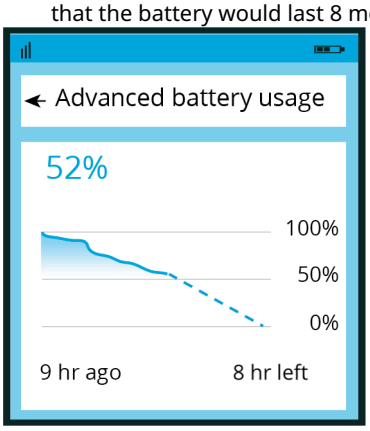
Time	Percentage Charged
11:30 a.m.	35%
11:40 a.m.	43%

1. At what time will the battery be 100% charged?

Write an Equation to Model Data

For questions 1 – 2 , use the following prompt and image.

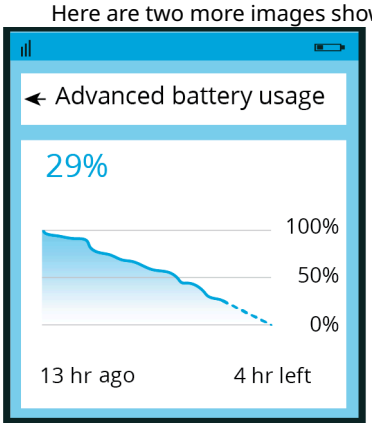
The image shows the battery usage of a cell phone 9 hours after it was fully charged. It also shows a prediction that the battery would last 8 more hours.

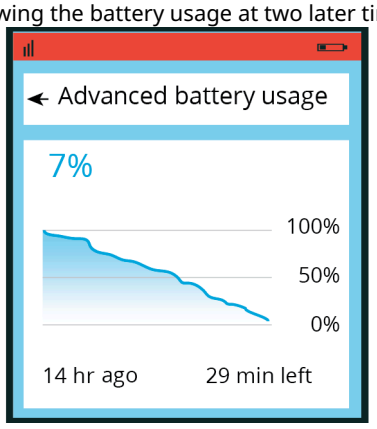


1. Write an equation for a model that fits the data in the image and gives the percent of battery power as a function of time since the phone was fully charged. Be prepared to show your reasoning.
2. Based on your function, what percentage of power would the battery have 4 hours after this image was taken? What about 5 hours after the image was taken? Be prepared to show your reasoning.

For questions 3 – 5, use the following prompt and images.

Here are two more images showing the battery usage at two later times, before the battery was charged again.





3. How well did the function you wrote predict the battery power 4 and 5 hours after the first image was taken (that is, 13 and 14 hours after the battery was fully charged)? Be prepared to show your reasoning.
4. What do you notice about the change in the prediction between $t = 13$ and $t = 14$?
5. Write a new equation for a function that would better fit the data shown in the last image.

If you get stuck, consider creating a table of values or a scatter plot of the data.

Unit 4 Wrap Up

Throughout this unit, you learned that Supratim Das can use functions to help him produce longer-lasting batteries that can be used in mobile devices and electric cars. He can analyze energy inputs and outputs using functions and their graphs.

In this unit, you learned how to identify functions and how to use function notation to name a function and to specify its input and output. You learned that each point on the graph of a function represents an input-output pair of the function. You learned that function notation can be used to find the rule of a function and that the rule can be used to find the output for any input. You learned that a function is linear if the output changes by a constant rate relative to its input.

You related features of graphs of functions using terms such as “intercept,” “maximum,” “minimum,” and “slope.” You used statements in function notation to talk about key features of a graph. You learned that the average rate of change can be used to measure how fast a function changes over an interval. You compared several functions relating the same pair of quantities and analyzed two or more graphs, interpreting their relative features. You compared key features of graphs and interpreted them in context.

You related possible input and output values to domain and range. You determined a reasonable domain and range for a function based on the context.

Finally, you learned that a sequence is defined as a list of numbers, and the term (of a sequence) is one of the numbers in the list. You learned that the terms of a geometric sequence change by the same factor, called the common ratio. An arithmetic sequence is characterized by adding a constant value to get from one term to the other. One formula for a sequence, called a recursive formula, identifies the first term and the common difference or common ratio to identify the terms of the sequence. Another formula, called an n^{th} term formula or explicit formula, is one where the value of the n^{th} term is determined from just the term number.



Access our Unit 4 Student Self-Assessment (https://openstax.org/r/unit-4-self-assesment) to reflect on the skills you learned during this unit. Congratulations on completing Unit 4!

Building Character: Creativity

Throughout this unit, you learned that **creativity** is important for helping you explore different possibilities and seek multiple ways to solve problems. Creativity can help you cope with stress, connect with others, and find meaning in life.

UNIT 5 OVERVIEW AND READINESS

Unit 5 Overview

Overview: Why Should I Care?

Do you know what financial inclusion and dignity is? Watch the video to learn a bit about how Operation Hope uses exponential functions to help underserved populations take part in the economy and achieve financial security.

Unit 5 Overview (https://openstax.org/r/operation-hope)

In this unit, you will learn about exponential functions and models. Understanding exponential functions and models can help you with financial planning, whether you want to know how much interest you will pay on a credit card balance, how much interest you will earn on your savings account, or how much a stock fund has grown in value over the past 10 years.

Building Character: Grit

Through coaching, John Hope Bryan, owner of Operation Hope, equips young people and adults with the financial tools and education to secure a better future. He coaches them through their personal aspirations and life's challenges and facilitates their journey to financial independence. Similarly, you will need to present passion and perseverance to work through this course to ultimately achieve your long-term goals, just as John's organization helps others achieve their long-term financial goals. This unit is the midpoint of the course. Through grit, you will be able to keep working toward your goal, and you will uncover that exponential functions and models can help with long-term planning.

Throughout this unit, you will work on developing your grit. Grit is passion and perseverance to reach long-term goals. Excellence sometimes seems like the result of natural talent. But no matter how gifted you are—no matter how easily you climb up the learning curve—you do need to do that climbing. There are no shortcuts. Grit predicts accomplishing challenging goals of personal significance. For example, grittier students are more likely to graduate from high school, and grittier cadets are more likely to complete their training at West Point. Notably, in most research studies, grit and measures of talent and IQ are unrelated, suggesting that talent puts no limits on the capacity for passion and perseverance.

Think about your current sense of grit. How many of the following statements are true for you?

- I enjoy projects that take years to complete.
- I am working toward a very long-term goal.
- What I do each day is connected to my deepest personal values.
- There is at least one subject or activity that I never get bored of thinking about.
- Setbacks don't discourage me for long.
- I am a hard worker.
- I finish whatever I begin.
- I never stop working to improve.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

Am I Ready to Learn This?

Getting Ready for Unit 5

To be ready for this unit, you need to be able to:

- Write linear functions from a graph.
- Interpret linear functions.
- Evaluate exponential expressions.

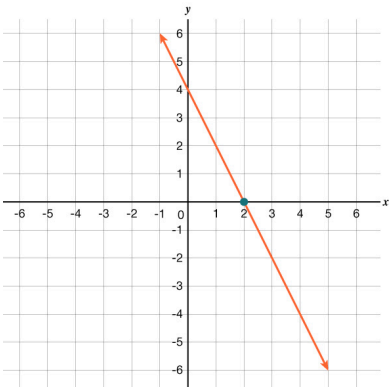
Next, you will answer some questions that will help you see how ready you are for this unit. If you need help, some mini-lessons will help you brush up on anything you need to work on.

Write Linear Functions from a Graph: Mini-Lesson Review

MINI LESSON QUESTION

Question #1: Write Linear Functions from a Graph

Which function could represent the graph?

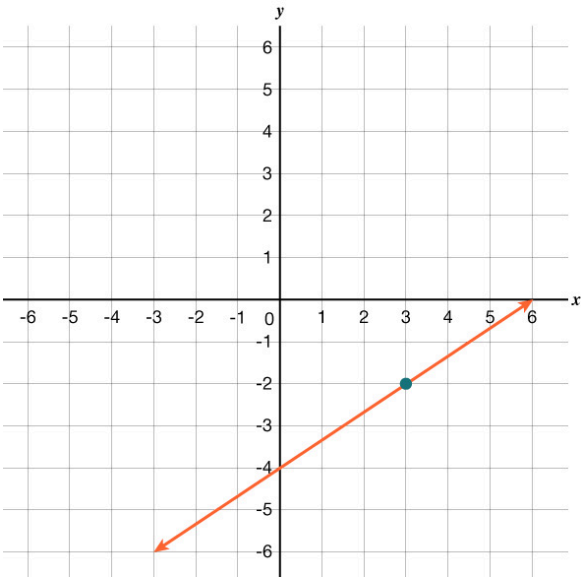


a. $f(x) = -2x + 2$
b. $f(x) = 2x + 4$
c. $f(x) = -\frac{1}{2}x + 4$
d. $f(x) = -2x + 4$

Write Linear Functions from a Graph

You can determine the slope and intercept of a line if the equation is written in slope-intercept form, $y = mx + b$. You can also do the reverse—identify the slope and y-intercept and use them to find the equation of the line.

Write the linear function shown in the graph.



To find the slope, choose two points on the graph. The y-intercept is $(0, -4)$, and the graph passes through $(3, -2)$.

Step 1 - Find the slope by counting the rise and run.

$m = \frac{rise}{run}$

Step 2 - $m = \frac{2}{3}$

Step 3 - Find the y-intercept.

y-intercept $(0, -4)$

Step 4 - Substitute the values into $y = mx + b$.

$y = mx + b$

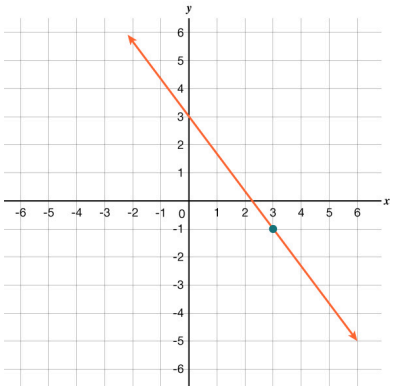
Step 5 - $y = \frac{2}{3}x = 4$

Step 6 - Rewrite the equation using function notation.

$f(x) = \frac{2}{3}x - 4$

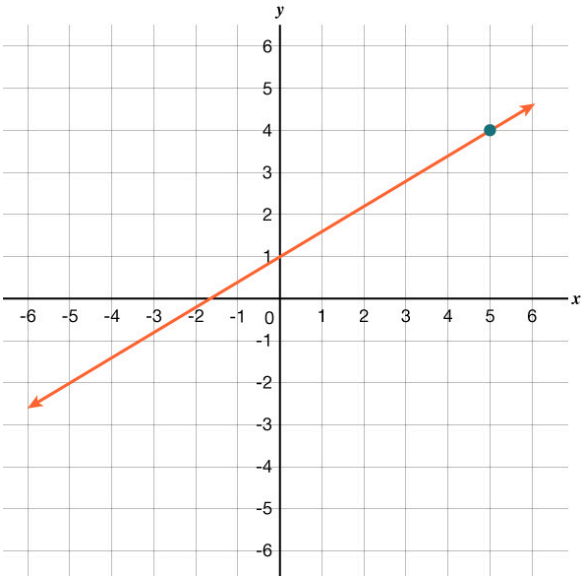
TRY IT **Write Linear Functions from a Graph**

Write the linear function shown in the graph.



Check Your Understanding

Which linear function could represent the graph?



a. $f(x) = \frac{3}{5}x + 1$

b. $f(x) = \frac{5}{3}x + 1$

c. $f(x) = -\frac{3}{5}x + 1$

d. $f(x) = \frac{3}{5}x - \frac{5}{3}$

Video: Write Linear Equations from a Graph

Watch the following video to learn more about writing linear functions from a graph..

Khan Academy: Write Linear Equations

Slope-intercept equation from a graph examples | Algebra I | Khan Academy (https://www.youtube.com/embed/9wOalujeZf4)

Interpret Linear Functions: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #2: Interpret Linear Functions

When Quinn returned from vacation, he turned the heat back on in his home. He set the temperature as high as it could go. The function $Q = 16 + 0.4t$ represents the temperature in Quinn's home (in degrees Celsius) after t minutes.

How much will the temperature increase after 5 minutes?

a. 0.4 degrees Celsius
b. 20 degrees Celsius
c. 18 degrees Celsius
d. 2 degrees Celsius

Interpret Linear Functions

A linear function is a function that represents a line on the coordinate plane. A line can be written in slope-intercept form, $y = mx + b$, where m is the slope and b is the y-intercept. The dependent variable is y, and the independent variable is x. The y can be replaced by $f(x)$, so the function can be written as $f(x) = mx + b$.

Andrei wants to fill a glass tank with marbles, each with the same volume, and then fill the remaining space with water. The function $V(n) = 32 - 0.05n$ represents the volume of water, in liters, Andrei uses if he uses n marbles.

You can rewrite the function in slope-intercept form: $V(n) = -0.05n + 32$. The slope is –0.05, and the y-intercept is 32.

1. What is the volume of the tank?

   $V(n) = -0.05n + 32$ represents the volume left in the tank after n marbles were added. The volume of the tank is the initial volume before any of the marbles have been added, or $V(0)$. $V(0) = -0.05(0) + 32 = 32$. The volume of the tank is 32 liters.

2. What is the volume of each marble?

   The volume of water needed to fill the tank decreases by 0.05 liters each time Andrei puts in a marble, n. This means that the volume of each marble is 0.05 liters.

› **TRY IT** **Interpret Linear Functions**

Rachel is a stunt driver, and she is escaping from a building that is about to explode. The function $D(t) = -38t + 220$ represents the remaining distance, D, in meters after t seconds.

How long is the distance Rachel has to drive?

Check Your Understanding

The amount of money that farmers in Massachusetts paid to maintain their crops between 1991 and 2008 is modeled by the function $P = 3.53t + 100$, where P is the amount of money the farmers paid, in millions of dollars, and t is the year (with 1991 as $t = 0$). What does the 3.53 mean in the equation?

a. The costs for maintaining crops increased by $3.53 million each year between 1991 and 2008.

b. The costs for maintaining crops increased a total of $3.53 million between 1991 and 2008.

c. The costs for maintaining crops was $3.53 million in 1991.

d. The costs for maintaining crops was $3.53 million in 2008.

Videos: Interpreting Linear Functions

Khan Academy: Interpret Linear Functions

Watch the videos to see how to interpret a linear function.

Linear equation word problems (https://www.youtube.com/embed/w6R8rywmgek)

Interpreting linear functions — Basic example | Math | SAT | Khan Academy (https://www.youtube.com/embed/hOsBsHA6okM)

Evaluate Exponential Expressions: Mini-Lesson Review

? **MINI LESSON QUESTION** **Question #3: Evaluate Exponential Expressions**

Evaluate $2(4)^3$.

a. 24

b. 512
c. 64
d. 128

Evaluating Exponential Expressions

A power is repeated multiplication of the same factor. The factor is the base, and the number of times the base is used as a factor is the exponent. The expression is called a power.

$Base \rightarrow \underbrace{5}^{3} \leftarrow Exponent$

The expression 5^3 is read:

- five to the power of three
- five to the third power
- five cubed

5^3 means 5 is used as a factor 3 times: $5^3 = 5 \cdot 5 \cdot 5 = 125$.

Evaluate each expression.

1. $(\frac{1}{2})^5$

   Use $\frac{1}{2}$ as a factor 5 times.

   $(\frac{1}{2})^5 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32}$

   $(\frac{1}{2})^5 = \frac{1}{32}$

2. $7(3)^4$

   Use order of operations. Evaluate the exponent first.

   $7(3)^4 = 7(3 \cdot 3 \cdot 3 \cdot 3) = 7(81)$

   Then multiply.

   $7(81) = 567$

   $7(3)^4 = 567$

3. $20(0.8)^2$

   Use order of operations. Evaluate the exponent first.

   $20(0.8)^2 = 20(0.8 \cdot 0.8) = 20(0.64)$

   Then multiply.

   $20(0.64) = 12.8$

   $20(0.8)^2 = 12.8$

TRY IT **Try It: Evaluating Exponential Expressions**

Evaluate $50(0.2)^2$.

Check Your Understanding

Evaluate $2(3)^3$.

a. 27

b. 216

c. 18

d. 54

Unit 5 Overview: Wrap Up

- Write linear functions from a graph.
- Interpret linear functions.
- Evaluate exponential expressions.

If you are still struggling, be sure to reach out to your teacher for additional help. Nobody is good at problems like this the first try. Keep at it and it will get easier.

5.1 PROPERTIES OF EXPONENTS

5.1.0 Lesson Overview

In this lesson, you will learn about properties of exponents.

When you finish this lesson, you will be able to:

- Simplify expressions using different properties for exponents.
- Simplify expressions containing exponents of zero.
- Use the definition of a negative exponent to simplify expressions.

Here are the **activities** that will help you reach those goals:

- 5.1.1: Understanding Exponents
- 5.1.2: Using Product and Quotient Properties for Exponents
  - 5.1.2: Self Check
  - 5.1.2: Additional Resources
- 5.1.3: Using Zero Exponent Property and Negative Exponents
  - 5.1.3: Self Check
  - 5.1.3: Additional Resources
- 5.1.4: Using Power Properties for Exponents
  - 5.1.4: Self Check
  - 5.1.4: Additional Resources
- 5.1.5: Simplifying Exponents

After that, you'll practice and review.

- 5.1.6: Practice
- 5.1.7: Lesson Summary

5.1.1 Understanding Exponents

Warm Up

An exponent indicates repeated multiplication of the same quantity.

$$\underbrace{a^m = a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ factors}} \qquad \underbrace{(-9)^5 = (-9)(-9)(-9)(-9)(-9)}_{5 \text{ factors}}$$

Expand each exponential expression to show repeated multiplication.

1. 4^3

2. x^5

3. $(-8)^6$

4. $(-z)^3$

5.1.2 Using Product and Quotient Properties for Exponents

Activity

Product Property for Exponents

If a is a real number and m and n are integers, then

$a^m \cdot a^n = a^{m+n}$

To multiply with like bases, add the exponents.

Complete problems 1 - 6 using the product property for exponents.

1. $d^2 \cdot d^3$
2. $c^4 \cdot c^6$
3. $k^3 \cdot k^5$
4. $(-v)^2 \cdot (-v)^1$
5. $(m)^4 \cdot (m)^4$
6. $(-p)^6 \cdot (-p)^8$

Product Quotient for Exponents

If a is a real number, $a \neq 0$, and m and n are integers, then

$\frac{a^m}{a^n} = a^{m-n}$, $m > n$ and $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$, $n > m$

To divide with like bases, subtract the exponents.

Complete problems 7 - 12 using the quotient property for exponents.

7. $\frac{y^5}{y^3}$
8. $\frac{r^4}{r}$
9. $\frac{q^9}{q^2}$
10. $\frac{(-u)^5}{(-u)^2}$
11. $\frac{(j)^6}{(j)^5}$
12. $\frac{(-z)^8}{(-z)^6}$

Self Check

Which of the following equations is correct?

a. $\frac{(-u)^8}{(-u)^2} = 2(-u)^4$
b. $\frac{y^6}{y^3} = y^2$
c. $x^2 \cdot x^4 = x^8$
d. $y^5 \cdot y^3 = y^8$

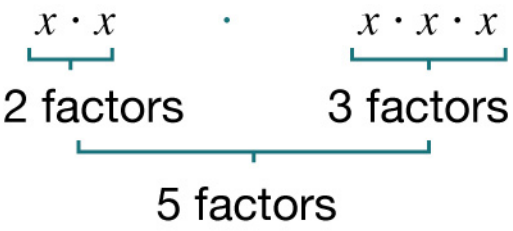
Additional Resources

Using Product and Quotient Properties for Exponents

When we combine like terms by adding and subtracting, we need to have the same base with the same exponent. But when you multiply and divide, the exponents may be different, and sometimes the bases may be different, too.

First, we will look at an example that leads to the Product Property.

What does $x^2 \cdot x^3$ mean?



Notice that 5 is the sum of the exponents, 2 and 3. We see $x^2 \cdot x^3$ is x^{2+3} or x^5.

The base stayed the same, and we added the exponents. This leads to the **Product Property for Exponents**.

Product Property for Exponents

If a is a real number and m and n are integers, then

$a^m \cdot a^n = a^{m+n}$

To multiply with like bases, add the exponents.

EXAMPLE 1

$4^3 \cdot 4^4 = 4^{3+4} = 4^7$

EXAMPLE 2

$12^2 \cdot 12^6 = 12^{2+6} = 12^8$

EXAMPLE 3

$y^7 \cdot 4^8 = y^{7+8} = y^{15}$

EXAMPLE 4

$(2^b)^{10} \cdot (2b)^3 = (2b)^{10+3} = (2b)^{13}$

Now we will look at an exponent property for division. As before, we'll try to discover a property by looking at some examples.

1. Consider $\frac{x^5}{x^2}$

Step 1 - What does it mean?

$\frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}$

Step 2 - Use the Equivalent Fractions Property.

$\frac{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x}}$

Step 3 - Simplify.

x^3

2. Consider $\frac{x^2}{x^3}$

Step 1 - What does it mean?

$\frac{x \cdot x}{x \cdot x \cdot x}$

Step 2 - Use the Equivalent Fractions Property.

$\frac{\cancel{x} \cdot \cancel{x} \cdot 1}{\cancel{x} \cdot \cancel{x} \cdot x}$

Step 3 - Simplify.

$\frac{1}{x}$

Notice that in each case, the bases were the same and we subtracted exponents. We see $\frac{x^5}{x^2}$ is x^{5-2} or x^3. We see $\frac{x^2}{x^3}$ is $\frac{1}{x}$. When the larger exponent was in the numerator, we were left with factors in the numerator. When the larger exponent was in the denominator, we were left with factors in the denominator—notice the numerator of 1. When all the factors in the numerator have been removed, remember this is really dividing the factors to one, so we need a 1 in the numerator. $\frac{\cancel{x}}{\cancel{x}} = 1$. This leads to the **Quotient Property for Exponents**.

Quotient Property for Exponents

If a is a real number, $a \neq 0$, and m and n are integers, then

$\frac{a^m}{a^n} = a^{m-n}$, $m > n$ and $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$, $n > m$

EXAMPLE 5

$7^5 \div 7^3 = 7^{5-3} = 7^2$

EXAMPLE 6

$k^9 \div k^2 = k^{9-2} = k^7$

EXAMPLE 7

$\frac{j^{10}}{j^7} = j^{10-7} = k^3$

TRY IT **Using Product and Quotient Properties for Exponents**

Simplify each expression.

1. $12^2 \cdot 12^5$
2. $20^{14} \div 20^4$
3. $h^4 \cdot h^7$
4. $(-s)^2 \cdot (-s)^2$
5. $\frac{b^7}{b^5}$
6. $(-y)^6 (-y)^3$
7. $\frac{x^3}{x^5}$

Video: Understanding Product and Quotient Properties for Exponents

Watch the following video to learn more about applying the Product and Quotient Properties of Exponents.

Understanding Product and Quotient Properties for Exponents (https://openstax.org/r/exponent-properties)

5.1.3 Using Zero Exponent Property and Negative Exponents

Activity

Zero Exponent Property

If a is a non-zero number, then $a^0 = 1$.

If a is a non-zero number, then a to the power of zero equals 1.

Any non-zero number raised to the zero power is 1.

Complete problems 1 - 6 using the product property for exponents. Double click on mathematical expressions/equations to enlarge.

1. d^0
2. 7^0
3. $(-8)^0$
4. $(-v)^0$
5. $r^0 \cdot r^0$

Definition of a Negative Exponent

If n is an integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$.

Complete problems 6 - 10 using the product property for exponents.

6. g^{-4}
7. 2^{-3}
8. $\frac{1}{c^{-2}}$
9. $\frac{1}{(-h)^{-6}}$
10. $\frac{1}{4^{-3}}$

Self Check

Which of the following equations is correct?

a. $d^{-4} = \frac{1}{d^4}$
b. $7^{-3} = -343$
c. $w^{-6} = \frac{-6}{w^6}$
d. $8^0 = 0$

Additional Resources

Using the Zero Exponent Property

A special case of the Quotient Property is when the exponents of the numerator and denominator are equal, such as an expression like $\frac{a^m}{a^m}$. We know $\frac{x}{x} = 1$ for any x ($x \neq 0$) since any number divided by itself is 1. The Quotient Property for Exponents shows us how to simplify $\frac{a^m}{a^m}$ when $m > n$ and when $n < m$ by subtracting exponents. What if $m = n$? We will

simplify $\frac{a^m}{a^m}$ in two ways to lead us to the definition of the **Zero Exponent Property**. In general, for $a \neq 0$:

$\frac{a^m}{a^m}$	$\frac{a^m}{a^m}$
a^{m-m}	m factors $\overbrace{\text{Ø} \cdot \text{Ø} \cdot \text{Ø} \cdot \ldots \cdot \text{Ø}}$ / $\underbrace{\text{Ø} \cdot \text{Ø} \cdot \text{Ø} \cdot \ldots \cdot \text{Ø}}$ m factors
a^0	1

Zero Exponent Property

If a is a non-zero number, then $a^0 = 1$.

If a is a non-zero number, then a to the power of zero equals 1.

Any non-zero number raised to the zero power is 1.

Using the Definition of a Negative Exponent

We saw that the Quotient Property for Exponents has two forms depending on whether the exponent is larger in the numerator or the denominator. What if we just subtract exponents regardless of which is larger?

Let's consider $\frac{x^2}{x^5}$. We subtract the exponent in the denominator from the exponent in the numerator. We see $\frac{x^2}{x^5}$ is x^{2-5} or x^{-3} .

We can also simplify $\frac{x^2}{x^5}$ by dividing out common factors:

$\frac{x^2}{x^5}$

$\frac{\cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}$

$\frac{1}{x^3}$

This implies that $x^{-3} = \frac{1}{x^3}$, and it leads us to the definition of a *negative exponent*. If n is an integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$.

Let's now look at what happens to a fraction whose numerator is one and whose denominator is an integer raised to a negative exponent.

$\frac{1}{a^{-n}}$

Step 1 - Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.

$\frac{1}{\frac{1}{a^n}}$

Step 2 - Simplify the complex fraction.

$1 \cdot \frac{a^n}{1}$

Step 3 - Multiply.

a^n

This implies $\frac{1}{a^{-n}} = a^n$ and is another form of the definition of **Properties of Negative Exponents**.

Properties of Negative Exponents

If n is an integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$ or $\frac{1}{a^{-n}} = a^n$.

The negative exponent tells us we can rewrite the expression by taking the reciprocal of the base and then changing the sign of the exponent.

Any expression that has negative exponents is not considered to be in simplest form. We will use the definition of a negative exponent and other properties for exponents to write the expression with only positive exponents.

For example, if after simplifying an expression we end up with the expression x^{-3}, we will take one more step and write $\frac{1}{x^3}$. The answer is considered to be in simplest form when it has only positive exponents.

TRY IT **Using the Zero Exponent Property and Negative Exponents**

Simplify each expression.

1. h^0
2. 7^0
3. $(-g)^0$
4. a^{-4}
5. 9^{-3}
6. $\frac{1}{n^{-5}}$

5.1.4 Using Power Properties for Exponents

Activity

Work with a partner to complete the activity. Discuss any differences in the simplified results you find. Make sure you both understand each step you take to simplify.

Quotient to a Negative Power Property

If a and b are real numbers, $a \neq 0$, $b \neq 0$, and n is an integer, then

$(\frac{a}{b})^{-n} = (\frac{b}{a})^n$

Complete problems 1 - 3 using the product property for exponents. Double click on mathematical expressions/equations to enlarge.

1. $(\frac{3}{7})^{-5}$

2. $(-\frac{y}{z})^{-7}$

3. $(\frac{g}{h})^{-6}$

Power Property for Exponents

If a is a real number and m and n are integers, then

$(a^m)^n = a^{m \cdot n}$

Complete problems 4 - 6 using the power property for exponents. Double click on mathematical expressions/equations to enlarge.

4. $(4^2)^5$

5. $(n^3)^7$

6. $(2^3)^2(d^4)^3$

Product to a Power Property for Exponents

If a and b are real numbers and m is a whole number, then

$(ab)^m = a^m b^m$

Complete problems 7 - 9 using the product to a power property for exponents.

7. $(-2tv)^7$

8. $(4pr^2)^5$

9. $(3h^3)^{-2}$

Choose one example above and explain each step to your partner. Listen as your partner does the same for you.

Self Check

Simplify the following expression.

$\left(\frac{3^3x^{-2}}{q^3}\right)^3$

a. $\frac{19{,}683x^6}{q^9}$

b. $\frac{729}{x^6q^9}$

c. $\frac{19{,}683}{x^6q^9}$

d. $\frac{27}{x^2q^3}$

Additional Resources

Quotient to a Negative Power Property

Suppose now we have a fraction raised to a negative exponent. Let's use our definition of negative exponents to lead us to a new property.

$\left(\frac{3}{4}\right)^{-2}$

Step 1 - Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.

$\frac{1}{\left(\frac{3}{4}\right)^2}$

Step 2 - Simplify the denominator.

$\frac{1}{\frac{9}{16}}$

Step 3 - Simplify the complex fraction.

$\frac{16}{9}$

But we know that $\frac{16}{9}$ is $(\frac{4}{3})^2$.

Step 4 - This tells us that

$(\frac{3}{4})^{-2} = (\frac{4}{3})^2$

To get from the original fraction raised to a negative exponent to the final result, we took the reciprocal of the base—the fraction—and changed the sign of the exponent.

This leads us to the **Quotient to a Negative Power Property**.

Quotient to a Negative Power Property

If a and b are real numbers, $a \neq 0$, $b \neq 0$, and n is an integer, then

$(\frac{a}{b})^{-n} = (\frac{b}{a})^n$

Now that we have negative exponents, we can use the Product Property we learned in Activity 5.1.2 with expressions that have negative exponents, too.

Power Property for Exponents

Now let's look at an exponential expression that contains a power raised to a power. See if you can discover a general property.

What does this mean?	$(x^2)^3$ $x^2 \cdot x^2 \cdot x^2$
How mant factors altogether?	$\underbrace{x \cdot x}_{2 \text{ factors}} \cdot \underbrace{x \cdot x}_{2 \text{ factors}} \cdot \underbrace{x \cdot x}_{2 \text{ factors}}$ $\underbrace{\qquad\qquad\qquad}_{6 \text{ factors}}$
So we have	x^6

Notice the 6 is the *product* of the exponents, 2 and 3. We see that $(x^2)^3$ is $x^{2 \cdot 3}$ or x^6.

We multiplied the exponents. This leads to the **Power Property for Exponents**.

Power Property for Exponents

If a is a real number and m and n are integers, then

$(a^m)^n = a^{m \cdot n}$

To raise a power to a power, multiply the exponents.

Product to a Power Property for Exponents

We will now look at an expression containing a product that is raised to a power. Can you find this pattern?

$(2x)^3$

Step 1 - What does this mean?

$2x \cdot 2x \cdot 2x$

Step 2 - We group the like factors together.

$2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x$

Step 3 - How many factors of 2 and of x?

$2^3 \cdot x^3$

Notice that each factor was raised to the power and $(2x)^3$ is $2^3 \cdot x^3$.

The exponent applies to each of the factors. This leads to the **Product to a Power Property for Exponents** .

Product to a Power Property for Exponents

If a and b are real numbers and m is a whole number, then

$(ab)^m = a^m b^m$

To raise a product to a power, raise each factor to that power.

Quotient to a Power Property for Exponents

Now we will look at an example that will lead us to the Quotient to a Power Property.

$(\frac{x}{y})^3$

Step 1 - This means

$\frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y}$

Step 2 - Multiply the fractions.

$\frac{x \cdot x \cdot x}{y \cdot y \cdot y}$

Step 3 - Write with exponents.

$\frac{x^3}{y^3}$

Notice that the exponent applies to both the numerator and the denominator.

We see that $(\frac{x}{y})^3$ is $\frac{x^3}{y^3}$.

This leads to the **Quotient to a Power Property for Exponents**.

Quotient to a Power Property for Exponents

If a and b are real numbers, $b \neq 0$, and m is an integer, then

$(\frac{a}{b})^m = \frac{a^m}{b^m}$

To raise a fraction to a power, raise the numerator and denominator to that power.

TRY IT **Using Power Properties for Exponents**

Simplify each expression using the property identified.

Use the Quotient to a Negative Power Property for Exponents.

1. $(\frac{3}{7})^{-5}$

2. $(-\frac{r}{s})^{-2}$

Use the Power Property for Exponents.

3. $(x^3)^5$

4. $(2^2)^3$

5. $(h^4)^5(h^2)^2$

Use the Product to a Power Property for Exponents.

6. $(-2cd)^6$

7. $(4z)^{-3}$

Use the Quotient to a Power Property for Exponents.

8. $(\frac{2xy^2}{z})^3$

9. $(\frac{4p^{-3}}{q^2})^2$

5.1.5 Simplifying Exponents

Cool Down

Take some time to review each of the exponential properties you learned in the lesson. Look at each property and be sure you know when to use it.

Summary of Exponent Properties

If a and b are real numbers and m and n are integers, then

Property	Description
Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
Zero Exponent Property	$a^0 = 1, a \neq 0$
Quotient to a Power Property	$(\frac{a}{b})^m = \frac{a^m}{b^m}, b \neq 0$
Properties of Negative Exponents	$a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$
Quotient to a Negative Exponent	$(\frac{a}{b})^{-n} = (\frac{b}{a})^n$

Complete the table using all of the exponent properties you learned in the lesson.

1.

Expression	Simplified Result
$j^5 \cdot j^3$	
$\frac{h^7}{h^3}$	
$(-ab)^0$	
$\frac{1}{s^{-4}}$	
$(\frac{m}{n})^{-4}$	
$(p^5)^2$	
$(3s^3t^2)^3$	
$(\frac{b}{d})^{-2}$	

5.1.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

Simplify each expression using the appropriate property from the lesson.

1. $f^2 \cdot f^6$

a. f^8
b. f^{12}
c. f^4
d. f^{36}

2. $\frac{(-y)^6}{(-y)^3}$

a. $(-y)^{18}$
b. $(-y)^9$
c. $(-y)^2$
d. $(-y)^3$

3. $(-4)^0$

a. -4
b. 1
c. 4
d. 0

4. 2^{-3}

a. $-\frac{1}{8}$
b. $\frac{1}{8}$
c. -6
d. 8

5. $\frac{1}{(-t)^{-5}}$

a. $(-t)^5$
b. $\frac{1}{t^5}$
c. t^5
d. t^4

6. $(-\frac{m}{n})^{-3}$

a. $-\frac{m^3}{n^3}$
b. $-\frac{n^3}{m^3}$
c. $\frac{m^3}{n^3}$
d. $\frac{n^3}{m^3}$

7. $(d^3)^5$

a. d^{-2}
b. d^5
c. d^8
d. d^{15}

8. $(5x^{-2})^3$

a. $125x$
b. $\frac{125}{x^6}$
c. $5x$
d. $\frac{125}{x^5}$

9. $(\frac{7b^3c}{d^2})^2$

a. $\frac{49b^5c^2}{d^4}$
b. $\frac{49b^6c^2}{d^4}$
c. $\frac{7b^6c^2}{d^4}$
d. $\frac{d^4}{49b^6c^2}$

10. $(\frac{3z^{-2}}{w^2})^{-2}$

a. $\frac{z^4w^4}{9}$
b. $\frac{w^4}{9z^4}$
c. $\frac{9w^4}{z^4}$
d. $9z^4w^4$

5.1.7 Lesson Summary

In this lesson, you learned how to:

- Simplify expressions using different properties for exponents.
- Simplify expressions containing exponents of zero.
- Use the definition of a negative exponent to simplify expressions.

Here are the **activities** that helped you reach those goals:

- 5.1.1: Understanding Exponents
  - In this activity, you reviewed the concept of exponents. You converted exponential expressions into repeated multiplication expressions.
- 5.1.2: Using Product and Quotient Properties for Exponents
  - In this activity, you learned how to use the Product Property for Exponents to rewrite multiplication problems involving exponential terms with the same base. To simplify, the exponential terms are added and the same base is kept. You also learned how to use the Quotient Property for Exponents to rewrite division problems involving exponential terms with the same base. To simplify those, the exponential terms are subtracted and the same base is kept.
  - 5.1.2: Self Check
  - 5.1.2: Addition Resources
- 5.1.3: Using Zero Exponent Property and Negative Exponents
  - In this activity, you used the Zero Exponent Property to simplify any non-zero number raised to the zero power. The solution is always 1. You also learned how to use the definition of a negative exponent to simplify exponential expressions. To simplify an expression with a negative exponent, the reciprocal of the base is used and the sign of the exponent is flipped.
  - 5.1.3: Self Check
  - 5.1.3: Addition Resources
- 5.1.4: Using Power Properties for Exponents
  - In this activity, you learned various power properties for exponents. The Quotient to a Negative Power Property is used to simplify values being divided while all raised to a negative exponent. You learned that the Power Property for Exponents is used when an exponential expression is raised to another power. The Product to a Power Property for Exponents states that when you raise products to a power, all factors being multiplied are raised to that power. Finally, you learned the Quotient to a Power Property for Exponents. This is used to simplify a division expression that is raised to a power.
  - 5.1.4: Self Check
  - 5.1.4: Addition Resources
- 5.1.5: Simplifying Exponents
  - In this activity, you simplified expressions using all the properties you practiced in the lesson. You also reviewed each of the properties you learned.

After these activities, you completed the following **practice**:

- 5.1.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

5.2 RATIONAL EXPONENTS

5.2.0 Lesson Overview

In this lesson, you will learn how to write terms that include rational exponents as radicals and vice versa.

When you finish this lesson, you will be able to:

- Simplify radical expressions with even and odd indexes.
- Rewrite expressions with rational exponents as roots.
- Use properties for exponents with rational exponents.

Here are the **activities** that will help you reach those goals:

- 5.2.1: Perfect Squares Revisited
- 5.2.2: The n^{th} Root
  - 5.2.2: Self Check
  - 5.2.2: Additional Resources
- 5.2.3: Radicals and Rational Exponents
  - 5.2.3: Self Check
  - 5.2.3: Additional Resources
- 5.2.4: Properties of Exponents with Rationals

After that, you'll **practice and review**.

- 5.2.5: Practice
- 5.2.6: Lesson Summary

5.2.1 Perfect Squares Revisited

Warm Up

1. What is the value of 4×4?

2. What is the value of 4^2?

3. Using what you know about roots, what is the value of $\sqrt{16}$?

4. Talk to a partner. How can you prove $\sqrt[2]{4^2} = 4$?

5. Talk to a partner. What do you think the answer to $\sqrt[5]{4^5}$ would be?

5.2.2 The nth Root

Activity

Work with a partner to discuss and answer the following questions.

1. Complete the table when $n = -2$ on your paper and check your answers.

Expression	Process	Simplified Result
n	(-2)	-2
n^2	$(-2)^2 = (-2)(-2)$	4
n^3		

Expression	Process	Simplified Result
n^4		
n^5		

2. What do you notice about the answers when the powers are even?

3. What do you notice about the answers when the powers are odd?

The $\sqrt{a}$, or square root of a, is the same as $\sqrt[2]{a}$. What about when we want to take roots that are higher than the square root? These are called **radicals**.

The n^{th} root, or radical, is written $\sqrt[n]{a}$. If $b^n = a$, then b is an n^{th} root of a. The n is called the **index** of the radical. Just like $\sqrt[2]{n}$ is called the square root, $\sqrt[3]{n}$ is the cubed root, $\sqrt[4]{n}$ is the fourth root, and so on. The number inside the radical is called the **radicand**.

4. Using what you did with the powers in number 1, complete the table to find the roots of each.

$x^y = z$	$\sqrt[y]{z} = x$
$4^3 = 64$	$\sqrt[3]{64} = 4$
$3^4 = 81$	
	$\sqrt[5]{-32} = -2$

Could we have an even root of a negative number? We know that the square root of a negative number is not a real number. The same is true for any even root. **Even** roots of negative numbers are not real numbers. **Odd** roots of negative numbers are real numbers.

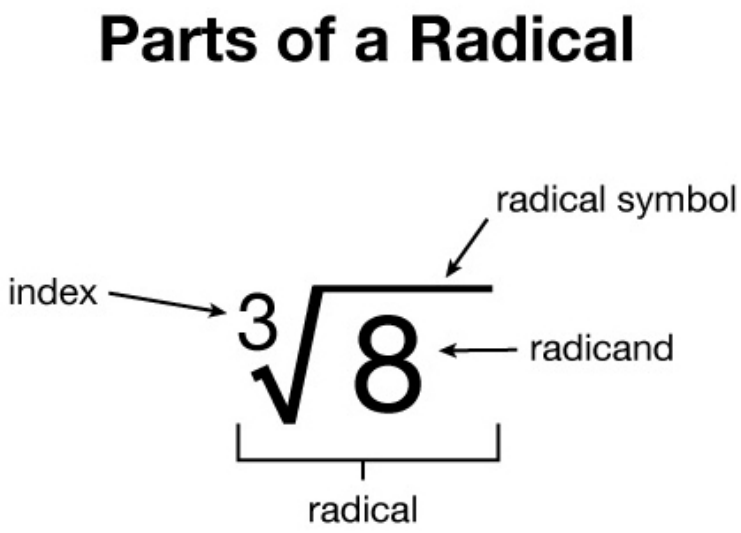
Self Check

Simplify $\sqrt[3]{-27}$.

a. -9
b. -3
c. 9
d. 3

Additional Resources

Simplifying Radicals



When simplifying **radicals**, think which number will be used as a factor the number of times indicated by the **index** to get the number under the radical.

In the example above,

$2 \times 2 \times 2 = 8$, so $\sqrt[3]{8} = 2$.

Remember, when the index is odd, the root is allowed to be negative. When the index is even, the root cannot be negative.

TRY IT **Simplifying Radicals**

Simplify.

$\sqrt[5]{-32}$

5.2.3 Radicals and Rational Exponents

Activity

If $\sqrt[n]{a^m}$ is a real number and $n \geq 2$, then $\sqrt[n]{a^m} = a^{\frac{m}{n}}$.

So radicals can be rewritten as rational exponents.

With a partner, write each radical as a term with rational exponents on your own paper. Then, turn and talk to your partner.

1. $\sqrt[8]{x}$
2. $\sqrt[3]{x^2}$
3. $\sqrt[11]{x^5}$

Now, with a partner, rewrite each term as an equivalent radical.

4. $x^{\frac{1}{2}}$
5. $x^{\frac{1}{7}}$

6. $x^{\frac{4}{5}}$

Video: Learning About Rational Exponents

Watch the following video to learn more about rational exponents.

Matching Verbal Situations to Expressions and Tables (https://openstax.org/r/verbal-expressions)

Self Check

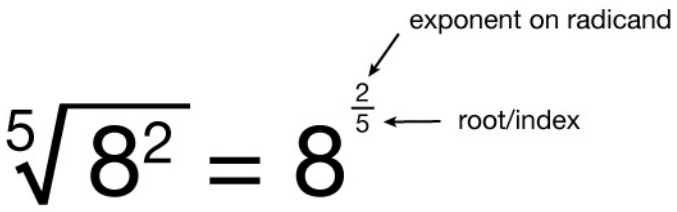
Rewrite $\sqrt[10]{x^7}$ as a term with a rational exponent.

a. $\frac{7}{10}x$
b. $10x^7$
c. $x^{\frac{7}{10}}$
d. $x^{\frac{10}{7}}$

Additional Resources

Rewriting Radicals

EXAMPLE 1



When writing radicals as a term with a rational exponent, the exponent becomes the numerator of the fraction in the exponent, the root or index.

In the example above, the 2 is the exponent and becomes the top of the fraction. The 5 is the index and becomes the bottom of the fraction.

Step 1 - Identify the index.

5

Step 2 - Identify the exponent (inside the radical).

2

Step 3 - Write the power over the root.

$8^{\frac{2}{5}}$

Let's simplify a rational exponent with a variable.

EXAMPLE 2

$\sqrt[4]{5x^3}$

Step 1 - Identify the index.

4

Step 2 - Identify the exponent (inside the radical).

3

Step 3 - Write the power over the root.

$5x^{\frac{3}{4}}$

TRY IT **Rewriting Radicals**

Write $\sqrt[3]{12^2}$ as a term with a rational exponent.

Write $\sqrt[6]{x^5}$ as a term with a rational exponent.

5.2.4 Properties of Exponents with Rationals

Cool Down

The activities you learned in a prior lesson taught you the properties of exponents. These same properties work with rational exponents.

Simplify each expression, and then check your answers.

1. $(x^2)^{\frac{1}{3}}$

2. $x^{\frac{2}{3}} \cdot x^{\frac{2}{3}}$

3. $(x^{-3})^{\frac{2}{5}}$

4. $(x^{\frac{1}{3}})^{-1}$

5.2.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Simplify $\sqrt[4]{81}$.

2. Simplify $\sqrt[5]{-32}$.

3. Which of the following is equivalent to $\sqrt[4]{x^3}$?

a. $\frac{3}{4}x$
b. $4x^3$
c. $x^{\frac{4}{3}}$
d. $x^{\frac{3}{4}}$

4. Which of the following is equivalent to $\sqrt[9]{2^4}$?

a. $2^{\frac{4}{9}}$
b. $\frac{2^4}{9}$
c. $\frac{4}{9}(2)$
d. $2^{\frac{9}{4}}$

5. Simplify $k^3 \cdot k^{\frac{1}{3}}$.

a. $k^{\frac{10}{3}}$
b. $3k^{\frac{1}{3}}$
c. $\frac{1}{3}k^3$
d. k

6. Simplify $(m^{\frac{3}{4}})^{-2}$.

a. $-2m^{\frac{3}{4}}$
b. $m^{\frac{3}{2}}$
c. $\frac{1}{m^{\frac{3}{2}}}$
d. $m^{\frac{6}{8}}$

7. True or false:
$\sqrt[4]{-16} = -2$

a. true
b. false

8. True or false:
$\sqrt[3]{-64} = -4$

a. true
b. false

9. Which of the following expressions are equivalent to perfect squares? **Select the three** that apply.

a. 25

b. $\sqrt{16}$
c. $3 + 3$
d. $\sqrt[5]{9^5}$
e. $3 \cdot 2$

10. Which of the following is equivalent to $(x^{\frac{1}{5}})^4$?

a. x^{20}
b. $x^{\frac{4}{20}}$
c. $x^{\frac{4}{5}}$
d. x^4

5.2.6 Lesson Summary

In this lesson, you learned how to:

- Simplify radical expressions with even and odd indexes.
- Rewrite expressions with rational exponents as roots.
- Use properties of exponents with rational exponents.

Here are the **activities** that helped you reach those goals:

- 5.2.1: Perfect Squares Revisited
  - In this activity, you were reminded of perfect squares and how they are related to repeated multiplication and square roots. You looked at which operations were equivalent and which ones were inverses of each other.
- 5.2.2: The n^{th} Root
  - In this activity, you analyzed what happens to negative numbers when they are raised to an odd and even power. You were also introduced to new vocabulary such as radical, index, and radicand and discovered how they are related to perfect squares and square roots.
  - 5.2.2: Self Check
  - 5.2.2: Addition Resources
- 5.2.3: Radicals and Rational Exponents
  - In this activity, you extended radicals and root indices even further by looking at rational exponents. You rewrote expressions from radical form to rational exponent form.
  - 5.2.3: Self Check
  - 5.2.3: Addition Resources
- 5.2.4: Properties of Exponents with Rationals
  - In this activity, you applied the properties of exponents to expressions with rational exponents. You were able to practice using the same properties of exponents from an earlier lesson.

After these activities, you completed the following **practice** :

- 5.2.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

5.3 PATTERNS OF GROWTH

5.3.0 Lesson Overview

In this lesson, you will encounter two different growth patterns. You will think about and compare the patterns by performing calculations and using graphs. This lesson contains many opportunities for you to notice and make use of structure, for example, noticing that $1,000 + 200 + 200 + 200 + 200$ can be expressed as $1,000 + 200 \cdot 4$.

You will examine two types of patterns by looking at tables, noting that one type is characterized by common differences and the other by common ratios. Then you will match different representations with two contexts. This is an opportunity for you to notice structure in expressions.

When you finish this lesson, you will be able to:

- Compare growth patterns using calculations and graphs.
- Use words and expressions to describe patterns in tables of values.
- Write expressions and create tables of values to represent them using descriptions of linear and exponential relationships.
- Begin making connections between linear and exponential growth and the real world (especially financially).

Here are the **activities** that will help you reach those goals:

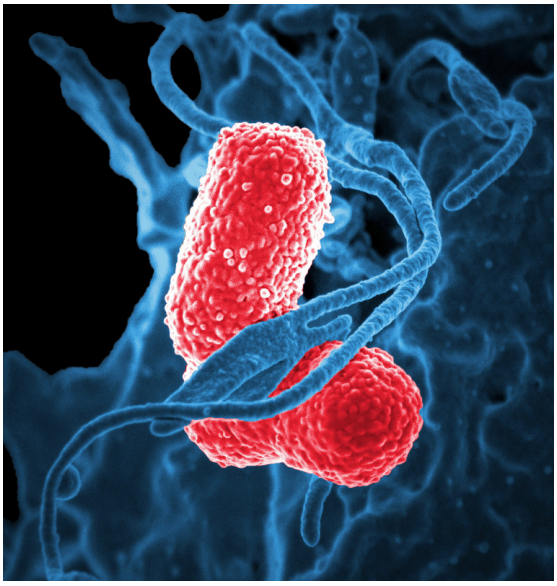
- 5.3.1: Creating a Pictorial Representation of Exponential Growth
- 5.3.2: Exploring Different Growth Patterns
  - 5.3.2: Self Check
  - 5.3.2: Additional Resources
- 5.3.3: Compare and Contrast Two Patterns
  - 5.3.3: Self Check
  - 5.3.3: Additional Resources
- 5.3.4: Match Expressions and Tables with Situations
  - 5.3.4: Self Check
  - 5.3.4: Additional Resources
- 5.3.5: Describe Mathematically Changes in Growth

After that, you'll **practice and review**.

- 5.3.6: Practice
- 5.3.7: Lesson Summary

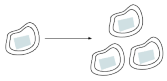
5.3.1 Creating a Pictorial Representation of Exponential Growth

Warm Up



There are some bacteria in a dish. Every hour, each bacterium splits into 3 bacteria.

1. This diagram shows a bacterium in hour 0 and then hour 1. Draw what happens in hours 2 and 3.



2. How many bacteria are there in hour 2?

3. How many bacteria are there in hour 3?

5.3.2 Exploring Different Growth Patterns

Activity

You are walking along a beach, and your toe hits something hard. You reach down, grab onto a handle, and pull out a lamp! It is sandy. You start to brush it off with your towel. Poof! A genie appears.

He tells you, “Thank you for freeing me from that bottle! I was getting claustrophobic. You can choose one of these purses as a reward.”

- Purse A, which contains $1,000 today. If you leave it alone, it will contain $1,200 tomorrow (by magic). The next day, it will be $1,400. This pattern of an additional $200 per day will continue.
- Purse B, which contains 1 penny today. Leave that penny in there because tomorrow it will (magically) turn into 2 pennies. The next day, there will be 4 pennies. The amount in the purse will continue to double each day.

1. How much money will be in Purse A after a 7-day week? Round your answer to the nearest cent.

2. How much money will be in Purse B after a 7-day week? Round your answer to the nearest cent.

3. How much money will be in Purse A after two weeks? Round your answer to the nearest cent.

4. How much money will be in Purse B after two weeks? Round your answer to the nearest cent.

5. The genie later adds that he will let the money in each purse grow for three weeks.

a. How much money will be in Purse A in three weeks? Round your answer to the nearest cent.

b. How much money will be in Purse B in three weeks? Round your answer to the nearest cent.

6. Which purse contains more money after 30 days? Round your answer to the nearest cent.

Building Character: Grit



Grit is passion and perseverance to reach long-term goals.

Think about your current sense of grit. Are the following statements true for you?

- Setbacks don't discourage me for long.
- I never stop working to improve.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

Self Check

Olivia has two different job offers for babysitting next Saturday.

- Job 1: She will receive $5 for accepting the job, then $10 for every hour she works.
- Job 2: She will receive $1.50 for accepting the job, $3 after the first hour, $6 after the second hour, and for each additional hour her pay per hour will double.

Olivia can only work 3 hours. How much would she make at each job?

a. Olivia would make $30 at Job 1 and $21 at Job 2.
b. Olivia would make $25 at Job 1 and $10.50 at Job 2.
c. Olivia would make $35 at Job 1 and $22.50 at Job 2.
d. Olivia would make $35 at Job 1 and $12 at Job 2.

Additional Resources

Exploring Linear and Exponential Growth

The tables listed below are modeling four different functions, all showing growth. Which one does not have the same growth pattern as the others?

x	y
1	8
2	16
3	24
4	32
8	64

Table A

x	y
0	0
2	16
4	32
6	48
8	64

Table B

x	y
0	1
1	4
2	16
3	64
4	256

Table C

x	y
0	4
1	8
2	12

Table D

x	y
3	16
4	20

Table D

When determining growth, there are a few ways to check. When given a table, you could graph each, or you could check the rate of change. Linear functions have a constant rate of change, where consecutive terms have a **common difference**. The common difference is a value that is added to or subtracted from consecutive terms to find the next term. Exponential functions have a **constant ratio** or common multiplier. The constant ratio is a value that is multiplied or divided from consecutive terms to find the next term.

- Table A is linear and has a slope of 8 since the change in y is 8 and the change in x is 1.
- Table B is also linear because the change in y is 16 and the change in x is 2 for a slope of 8.
- Table C is an example of an exponential function. There is a common ratio instead of a common difference. Notice that each y-term is 4 times the previous term. Therefore, the common ratio is 4.
- Table D is also linear with a common difference of 4.

Table C is the only exponential function. Therefore, it has a different growth pattern since Tables A, B, and D are linear.

TRY IT **Exploring Linear and Exponential Growth**

The table shows the height, in centimeters, of the water in a swimming pool at different times since the pool started to be filled.

minutes	height
0	150
1	150.5
2	151
3	151.5

Does the height of the water have a common difference or a common ratio?

Explain how you know.

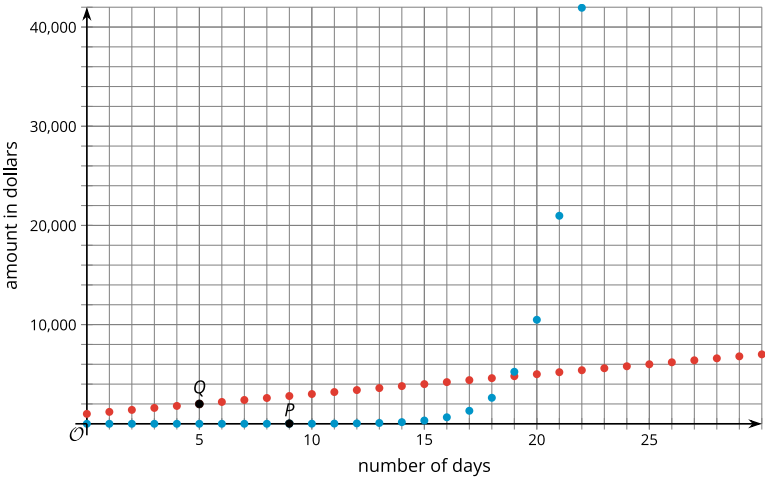
What type of function is represented by the table?

Explain how you know.

5.3.3 Compare and Contrast Two Patterns

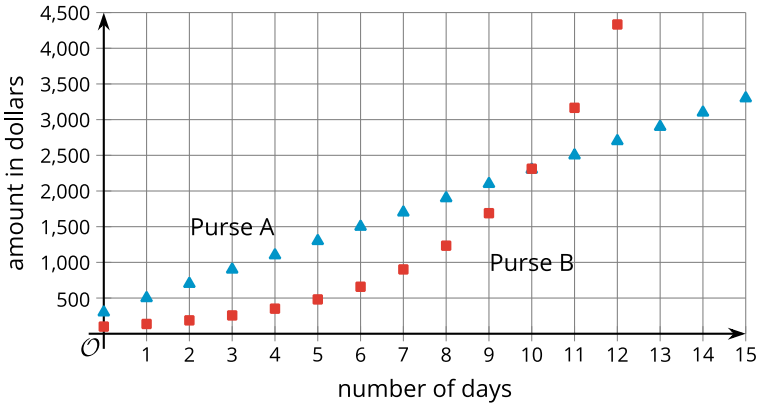
Activity

Here are graphs showing how the amount of money in each of the purses change. Remember, Purse A starts with $1,000 and grows by $200 each day. Purse B starts with $0.01 and doubles each day.



1. Which data on the graph shows the amount of money in Purse A? Which data on the graph shows the amount of money in Purse B? Explain how you know.
2. Points P and Q are labeled on the graph. Explain what they mean in terms of the genie's offer.
3. What are the coordinates of the vertical intercept for each graph? Explain how you know.
4. When does Purse B become a better choice than Purse A? Be prepared to show your reasoning.
5. Knowing what you know now, which purse would you choose? Be prepared to show your reasoning.

While on a beach, your friend discovers a different genie. This genie also offers two purses to choose from, and he gives you the following graph to show how the money in each purse will grow. The set of triangular marks that lie on a line represent values in Purse A, and the set of square marks that make a curve represent those in Purse B.



6. The genie is still deciding how many days he will let the money in the purses grow. Help your friend plan a strategy for picking the purse with more money while the genie thinks.

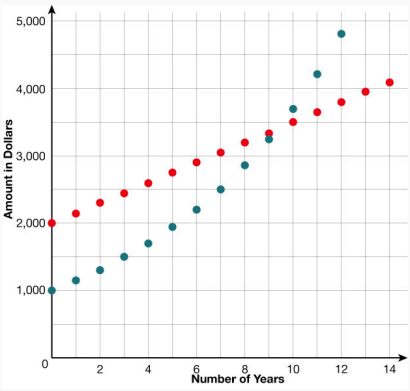
Are you ready for more?

Extending Your Thinking

"Okay, okay," the genie smiles, disappointed. "I will give you an even more enticing deal." He explains that Purse B stays the same, but Purse A now increases by $250,000 every day. Which purse should you choose?

Self Check

The graph shows the yearly balance, in dollars, in two investment accounts. Account A starts with $2,000. Account B starts with $1,000.



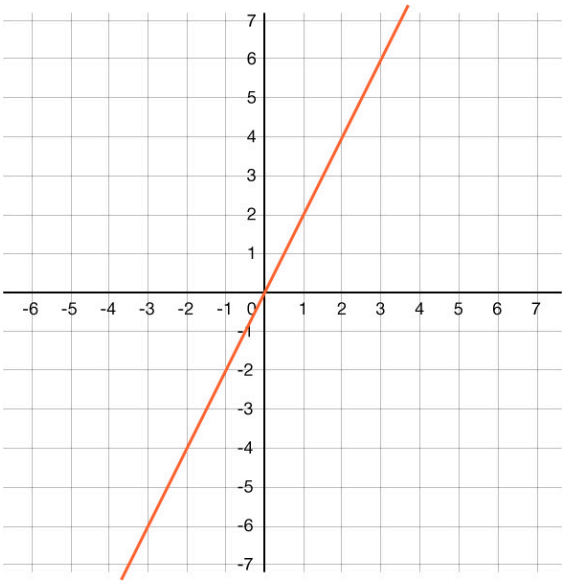
a. Both accounts grow by the same amount each year.
b. Both accounts grow more quickly as time goes on.
c. Account A grows by the same amount each year. Account B grows more quickly as time goes on
d. Account A grows more quickly as time goes on. Account B grows by the same amount each year.

Additional Resources

Compare and Contrast Graphs

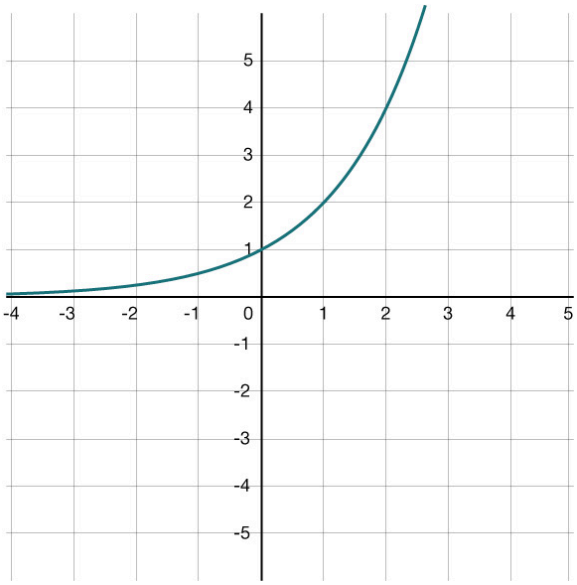
Compare and contrast the graphs of $y = 2x$ and $y = 2^x$.

To compare and contrast the graphs of these functions, you would first need to graph them.



x	$y = 2x$
–1	–2
0	0
1	2
2	4

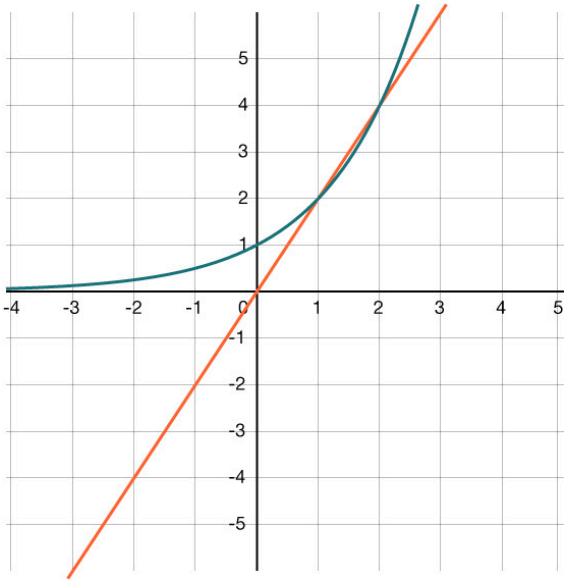
Graph of $(y = 2x)$



x	$y = 2^x$
−1	$\frac{1}{2}$
0	1
1	2
2	4

Graph of $(y = 2^x)$

Both equations have a 2 and an x in them, but in $y = 2x$, x is multiplied by 2, and in $y = 2^x$, 2 is the base and x is the exponent. If you graph the equations together on the same grid, it makes it easier to discuss the key features and compare the graphs.



- The graphs intersect at $(1, 2)$ and $(2, 4)$.

- $y = 2x$ has a y-intercept of $(0, 0)$. $y = 2^x$ has a y-intercept of $(0, 1)$.
- When $1 < x < 2$, $y = 2x$ is greater than $y = 2^x$.

TRY IT **Compare and Contrast Graphs**

Compare and contrast the graphs of $y = 3x + 1$ and $y = 3^x$.

5.3.4 Match Expressions and Tables with Situations

Activity

Here are verbal descriptions of two situations, followed by tables and expressions that could help answer one of the questions in the situations.

- Situation 1: A person has 80 followers on social media. The number of followers triples each year. How many followers will she have after 4 years?
- Situation 2: A tank contains 80 gallons of water and is getting filled at a rate of 3 gallons per minute. How many gallons of water will be in the tank after 4 minutes?

Use the descriptions of Situation 1 and Situation 2 to help you answer questions 1 – 6. Be prepared to explain how the table or expression answers the question.

1. Which situation best describes $80 \cdot 3 \cdot 3 \cdot 3 \cdot 3$?

2. Which situation best describes the following table?

x	0	1	2	3	4
y	80	240	720	2,160	6,480

3. Which situation best describes $80 + 3 + 3 + 3 + 3$?

4. Which situation best describes $80 + 4 \cdot 3$?

5. Which situation best describes the following table?

x	0	1	2	3	4
y	80	83	86	89	92

6. Which situation best describes $80 \cdot 81$?

Video: Matching Verbal Situations to Expressions and Tables

Watch the following video to learn more about matching verbal representations of a function with tables and expressions.

Matching Verbal Situations to Expressions and Tables (https://openstax.org/r/verbal-expressions)

Self Check

Which of the following is **NOT** an example of a way to represent a chain of stores that currently has 4 stores but plans to double the number of stores they have each year for the next 5 years?

a.

x	0	1	2	3	4	5
y	4	8	16	32	64	128

b. $4 \cdot 32$
c. $4 + 5 \cdot 2$
d. $4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

Additional Resources

Multiple Representations Verbal Scenario

Here are verbal descriptions of two situations, followed by tables and expressions that could help answer one of the questions in the situations.

A food company currently has 5 convenience stores. It is considering two plans for expanding its chain of stores.

Plan A: Open 20 new stores each year.

Plan B: Double the number of stores each year.

How many stores will there be after 5 years under each plan?

Match each representation (a table or an expression) with one situation. Be prepared to explain how the table or expression answers the question.

1. $5 + 20 + 20 + 20 + 20 + 20$
2. $5 \cdot 32$
3. $5 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
4.

years	0	1	2	3	4	5
number of stores	5	10	20	40	80	160

5. $5 + 100$
6.

years	0	1	2	3	4	5
number of stores	5	25	45	65	85	105

To determine which options go with which scenario, you first need to analyze each scenario.

Let's look at Plan A.

- Plan A is to open 20 new stores each year for a food company that currently has 5 convenience stores. In both scenarios, the initial value or starting point is 5 stores, which is why in each table the first value is $(0, 5)$. If they are opening 20 additional stores each year, they are adding 20 stores to the total from the previous year. So in year 1 there would be 25 total stores, in year 2 there would be 45 total stores, and so on. These values are represented in option 6, so option 6 would represent Plan A.
- The mathematical expressions contain either multiplication or addition. Plan A is ADDING 20 stores a year, so year 5 could be represented as option 1 because it would be the initial number of stores plus 20 per year for 5 years, or $5 + 20 + 20 + 20 + 20 + 20$.
- Option 5 is also added, and represents Plan A since $5 + 20 + 20 + 20 + 20 + 20$ is the same as $5 + 100$.

Let's look at Plan B.

- Plan B would also start at 5, but it says to double the number of stores each year. Double means to MULTIPLY by 2, so it would be $5 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, which is option 3.
- Since $5 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ can be simplified to $5 \cdot 32$, option 2 also represents Plan B.
- Option 4 is the other table option, and as mentioned before, each table should have an initial value of 5 stores. Since Plan B is to double the number of stores, at the end of year 1 there should be 10 stores. When you check option 4, you can see that each term is double the term before.

Therefore, Plan A is options 1, 5, and 6, and Plan B is options 2, 3, and 4.

Try It: Multiple Representations Verbal Scenario

Represent the following scenario as a table and as two different mathematical expressions:

A tank contains 80 gallons of water and is getting filled at a rate of 3 gallons per minute. How many gallons of water will be in the tank after 4 minutes?

5.3.5 Describe Mathematically Changes in Growth

Cool Down

The tables show the cat population on two islands over several years. Describe mathematically, as precisely as you can, how the cat population on each island is changing.

year	0	1	2	3	4
number of cats on Meow Island	2	6	18	54	162

year	0	1	2	3	4
number of cats on Purr Island	2	6	10	14	18

5.3.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

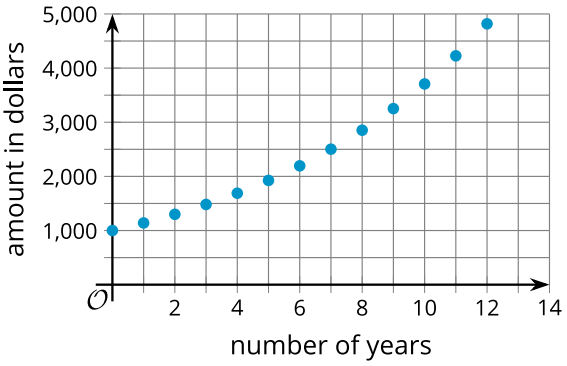
1. Which expression equals 2^7?

a. $2+7$
b. $2 \cdot 7$
c. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
d. $2+2+2+2+2+2+2$

For questions 2 – 3, use the following information :

The graph shows the yearly balance, in dollars, of investment account A. The table shows the yearly balance, in dollars, of investment account B.

Investment account A



Investment account B

Number of years	Amount in dollars
0	$2,000
1	$2,150
2	$2,300
3	$2,450
4	$2,600

2. What type of functions do the graph and the table represent?

a. Investment A is a/n:

a. linear function
b. exponential function

b. Investment B is a/n:

a. linear function
b. exponential function

3. Which of the following statements are true? **Select two** that apply.

a. Investment A has a higher initial balance than investment B.
b. Investment A has a common ratio.
c. Investment B has a common ratio.
d. Investment A is growing by the same number of dollars each year.
e. Investment B is growing by the same number of dollars each year.
f. After 12 years, investment B will result in more money.

4. Investment account 1 starts with a balance of $200 and doubles every year. Investment account 2 starts with $1,000 and increases by $100 each year. How long does it take for each account to double?

a. How many years will it take investment account 1 to double?

a. 15
b. 10

c. 5
d. 1

b. How many years will it take investment account 2 to double?

a. 15
b. 10
c. 5
d. 1

For questions 5 – 7, use the following information:
A population of ants is 10,000 at the start of April. Since then, it triples each month.

5. At the start of April, there were 10,000 ants in the colony. How many ants will there be 1 month later?

a. 90,000
b. 30,000
c. 10,300
d. 10,003

6. Which values complete the table?

months since April	number of ants
0	10,000
1	30,000
2	
3	
4	

a. 33,000, 33,300, 33,330
b. 40,000, 70,000, 110,000
c. 90,000, 270,000, 810,000
d. 50,000, 70,000, 90,000

7. If there are n ants in one month, which expression represents how many ants there will be one month later?

a. $n + 3$
b. $3n$
c. n^3
d. 3^n

8. A swimming pool contains 500 gallons of water. A hose is turned on, and it fills the pool at a rate of 24 gallons per minute. Which expression represents the amount of water in the pool, in gallons, after 8 minutes?

a. $500 \cdot 248$
b. $500 + 24 \cdot 8$
c. $500 + 24 + 8$
d. $500 \cdot 24 \cdot 8$

9. The population of a city is 100,000. It doubles each decade for 5 decades. **Select the two** expressions that represent the population of the city after 5 decades.

a. 32,000

b. 320,000
c. $100,000 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
d. $100,000 \cdot 5^2$
e. $100,000 \cdot 2^5$

10. Bank account C starts with $10 and doubles each week. Bank account D starts with $1,000 and grows by $500 each week. When will account C contain more money than account D?

a. Account C will contain more money than account D after 10 weeks.
b. Account C will contain more money than account D after 7 weeks.
c. Account C will never contain more money than account D.
d. Account C will contain more money than account D after 2 weeks.

5.3.7 Lesson Summary

In this lesson, you learned how to:

- Compare growth patterns using calculations and graphs.
- Use words and expressions to describe patterns in tables of values.
- Write expressions and create tables of values to represent them using descriptions of linear and exponential relationships.
- Begin making connections between linear and exponential growth and the real world (especially financially).

Here are the **activities** that helped you reach those goals:

- 5.3.1: Creating a Pictorial Representation of Exponential Growth
  - In this activity, you create a pictorial representation of exponential growth.
- 5.3.2: Exploring Different Growth Patterns
  - In this activity, you explored two growth patterns and learned that a doubling pattern eventually grows very, very large, even with a small starting value.
  - 5.3.2: Self Check
  - 5.3.2: Addition Resources
- 5.3.3: Compare and Contrast Two Pattern
  - In this activity, you compared and contrasted the two patterns visually. You did this by making sense of the provided graphs and used them to answer questions in the context of the scenario. You recalled that a quantity that grows by adding the same amount at each step forms a line when plotted. Also, after viewing an example, you learned that a quantity that grows by the same ratio or factor at each step forms a curve when plotted.
  - 5.3.3: Self Check
  - 5.3.3: Addition Resources
- 5.3.4: Match Expressions and Tables with Situations
  - In this activity, you studied two stories characterized by different growth patterns and then matched expressions and tables with the two situations.
  - 5.3.4: Self Check
  - 5.3.4: Addition Resources
- 5.3.5: Describe Mathematically Changes in Growth
  - In this activity, you described mathematically, as precisely as you could, how a population changed over time.

After these activities, you completed the following **practice**:

- 5.3.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

5.4 REPRESENTING EXPONENTIAL GROWTH

5.4.0 Lesson Overview

In this lesson, you will learn about equations that represent exponential growth and the **growth factor** or **constant ratio** in an exponential function. You will apply this knowledge in a real-world activity about the growth of bacteria.

When you finish this lesson, you will be able to:

- Explain the connections between an equation and a graph that represents exponential growth.
- Write and interpret an equation that represents exponential growth.

Here are the **activities** that will help you reach those goals:

- 5.4.1: Exponent Rules
- 5.4.2: Zero Exponent Rule
  - 5.4.2: Self Check
  - 5.4.2: Additional Resources
- 5.4.3: Exponential Change: The Growth Factor
  - 5.4.3: Self Check
  - 5.4.3: Additional Resources
- 5.4.4: Graphing Exponential Expressions
  - 5.4.4: Self Check
  - 5.4.4: Additional Resources
- 5.4.5: Interpret the Growth Factor of the Equation

After that, you'll **practice and review**.

- 5.4.6: Practice
- 5.4.7: Lesson Summary

5.4.1 Exponent Rules

Warm Up

Find the value of x that makes the equation true.

1. $2^3 \cdot 2^4 = 2^x$

2. $2^5 \cdot 2 = 2^x$

3. $2^{10} \div 2^7 = 2^x$

4. $2^9 \div 2 = 2^x$

5.4.2 Zero Exponent Rule

Activity

1. Complete the table by entering the values for a, b, and c below the table. Take advantage of any patterns you notice.

x	4	3	2	1	0
3^x	81	27	a. _____	b. _____	c. _____

a. _____

b. _____

c. _____

2. Here are some equations. Find the solution to each equation using what you know about exponent rules. Be

prepared to explain your reasoning.

a. $9^x \cdot 9^7 = 9^7$

b. $\frac{9^{12}}{9^x} = 9^{12}$

3. Answer the following questions.

a. What is the value of 5^0 ?

b. What about 2^0 ?

Are you ready for more?

Extending Your Thinking

We know, for example, that $(2+3)+5 = 2+(3+5)$ and $2\cdot(3\cdot5) = (2\cdot3)\cdot5$. The grouping with parentheses does not affect the value of the expression.

Is this true for exponents? That is, are the numbers $2^{\left(3^5\right)}$ and $\left(2^3\right)^5$ equal? If not, which is bigger?

Which of the two would you choose as the meaning of the expression 2^{3^5} written without parentheses?

Self Check

Santos and Sara are checking their work on the following problem: $4(2 \cdot 3 \cdot 5)^0$.

Santos says the answer is 1. Sara says the answer is 4. Who is correct, and why?

a. Sara is correct because you first would simplify the parenthesis, and since it's raised to the zero power, it would be 1. Then you would multiply that by 4, for a final answer of 4.
b. Both Santos and Sara are correct since there are two methods to solve the problem.
c. Neither Santos nor Sara did the problem correctly.
d. Santos is correct because anything raised to the zero power is 1.

Additional Resources

Zero as an Exponent

Why is $2^0 = 1$ and $5^0 = 1$, but 2^3 is not the same as 5^3?

Answering this question will help us understand the **zero exponent rule**.

Exponents in general represent repeated multiplication, much like multiplication represents repeated addition. Let's use that understanding to evaluate 2^3 and 5^3:

$2^3 = 2\cdot2\cdot2 = 4\cdot2 = 8$

$5^3 = 5\cdot5\cdot5 = 25\cdot5 = 125$

8 and 125 are obviously very different, but the method to get to each answer was the same. If we rewrite it as a table, we can work backward to understand how 2^0 and 5^0 are both 1.

Using $y = 2^x$ and filling in the parts we have found results in Table 1:

x	0	1	2	3
y		2	4	8

Repeating that process for $y = 5^x$ results in Table 2:

x	0	1	2	3
y		5	25	125

Since exponents represent repeated multiplication in both tables, additional terms are found by multiplying again by the base. What if we worked backward? If you multiply to move to the right in the table, you can divide by the common ratio to work backward. Start with the 8, and work backward to fill in the table:

x	0	1	2	3
y	$2 \div 2 = 1$	$4 \div 2 = 2$	$8 \div 2 = 4$	8

Repeat the process for Table 2:

x	0	1	2	3
y	$5 \div 5 = 1$	$25 \div 5 = 5$	$125 \div 5 = 25$	125

After you have worked backward in each table, you will see that 2^0 and 5^0 both equal 1. When 1 is the exponent, your answer is the same as the common ratio, so when you divide by the common ratio, you end up with 1 as your answer every time. This is an important rule that can save a lot of time.

TRY IT **Zero as an Exponent**

Find the value of each of the following:

- 3^0
- $1,246^0$
- $(2 \cdot 3 \cdot 7)^0$

5.4.3 Exponential Change: The Growth Factor

Activity

In an exponential function, the output is multiplied by the same factor every time the input increases by one. The multiplier is called the **growth factor**.

1. In a biology lab, 500 bacteria reproduce by splitting. Every hour, on the hour, each bacterium splits into two bacteria. Fill in the table by writing an expression to show the number of bacteria after each hour.

hour	number of bacteria
0	500
1	a. _____
2	b. _____
3	c. _____

hour	number of bacteria
6	d. ____
t	e. ____

a. ____

$500 + 2$

$500 \cdot 2$

b. ____

$500 + 2 + 2$

$500 \cdot 2 \cdot 2$

c. ____

$500 + 2 + 2 + 2$

$500 \cdot 2 \cdot 2 \cdot 2$

d. ____

$500 + 2 + 2 + 2 + 2$

$500 + 2 + 2 + 2 + 2 + 2 + 2$

$500 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

$500 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

e. ____

$500 + 2t$

$500 + 2^t$

$500 \cdot 2^t$

$500 \cdot 2 \cdot t$

2. Which equation relates n, the number of bacteria, to t, the number of hours?

a. $n = 500 + 2t$

b. $n = 500 + 2^t$

c. $n = 500 \cdot 2^t$

d. $n = 500 \cdot 2 \cdot t$

3. Use your equation to find n when t is 0. What does this value of n mean in this situation?

4. In a different biology lab, a population of single-cell parasites also reproduces hourly. An equation that gives the number of parasites, p, after t hours is $p = 100 \cdot 3^t$. Explain what the numbers 100 and 3 mean in this situation.

Self Check

The population of a city is 100,000. It doubles each decade. Write an equation p that relates the population to t, time in decades.

a. $p = 2^t$
b. $p = 100,000 \cdot t^2$
c. $p = 100,000 \cdot 2^t$
d. $p = 100,000 \cdot 2t$

Additional Resources

Identifying the Constant Ratio of the Exponential Function

What is the constant ratio (also called the growth factor) of the exponential function shown in the table? Rewrite each term to show the initial value and repeated use of the constant ratio.

x	0	1	2	3	4
$f(x)$	4	12	36	108	324

Defining Exponential Growth

Because the output of exponential functions increases very rapidly, the term "exponential growth" is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

EXPONENTIAL GROWTH

A function that models **exponential growth** grows by a rate proportional to the amount present. For any real number x and any positive real numbers a and b such that $b \neq 1$, an exponential growth function has the form

$$f(x) = ab^x$$

where

- a is the initial or starting value of the function.
- b is the growth factor or growth multiplier per unit x.

The initial or starting value of the function is 4, because that is when the value of x is 0.

The constant ratio, b, would be 3, since every value in the table is 3 times the previous term.

x	0	1	2	3	4
$f(x)$	4	$4 \cdot 3^1$	$4 \cdot 3^2$	$4 \cdot 3^3$	$4 \cdot 3^4$

TRY IT **Identifying the Constant Ratio of the Exponential Function**

What is the constant ratio of the following exponential function?

x	1	2	3	4
$f(x)$	10	20	40	80

5.4.4 Graphing Exponential Expressions

Activity

Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

In an exponential function, the output is multiplied by the same factor every time the input increases by one. The multiplier is called the growth factor.

Use the graphing tool or technology outside the course. Refer back to your work in the table in activity 5.4.3: Exponential Change: The Growth Factor, where $n = 500 \cdot 2^t$ and $p = 100 \cdot 3^t$. Use that information and the given coordinate planes to graph the following using the Desmos tool below.

Use the equation $n = 500 \cdot 2^t$ to answer questions 1 – 7

Access multimedia content (http://openstax.org/books/algebra-1/pages/5-4-4-graphing-exponential-expressions)

1. What is the domain?
2. What is the range?
3. The scenario dictates that the domain and range be restricted in comparison to the graph of the function that models it. Write a reasonable domain and range based on the scenario using inequalities.
4. Graph (t, n) when t is 0, 1, 2, 3, and 4.

Access multimedia content (http://openstax.org/books/algebra-1/pages/5-4-4-graphing-exponential-expressions)

5. On the graph of n, where can you see each number that appears in the equation?
6. Graph (t, p) when t is 0, 1, 2, 3, and 4. (If you get stuck, you can create a table.)

Access multimedia content (http://openstax.org/books/algebra-1/pages/5-4-4-graphing-exponential-expressions)

7. On the graph of p, where can you see each number that appears in the equation?

Video: Graphing Exponential Functions

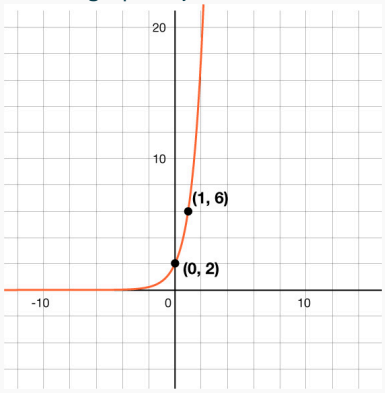
Watch the following video to learn more about graphing exponential functions.

Graphing Exponential Functions (https://openstax.org/r/graph-exponential)

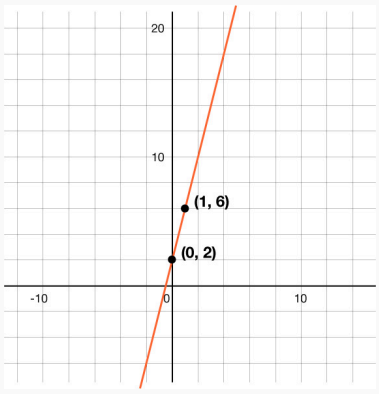
Self Check

Which is the graph of $y = 2 \cdot 3^x$?

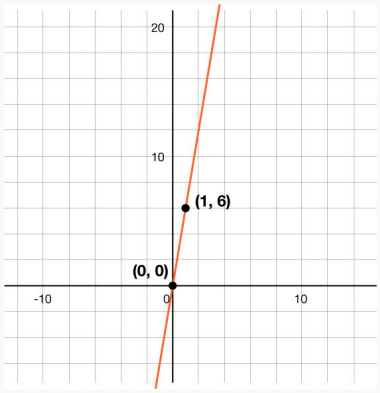
a.



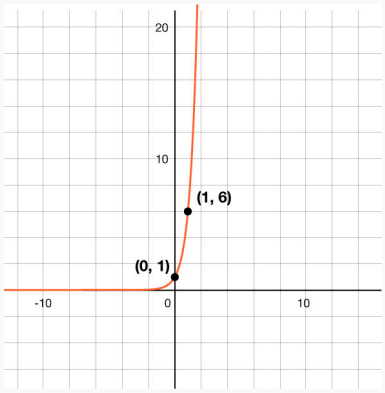
b.



c.



d.



Additional Resources

Graphing Exponential Functions

Graph $y = 4^x$.

When graphing any function, it often helps to start with a table. The table should always contain a minimum of 3 to 5 entries. The more information you have, the easier it is to sketch the graph. The information below is generic and applies to the graph of most exponential growth functions.

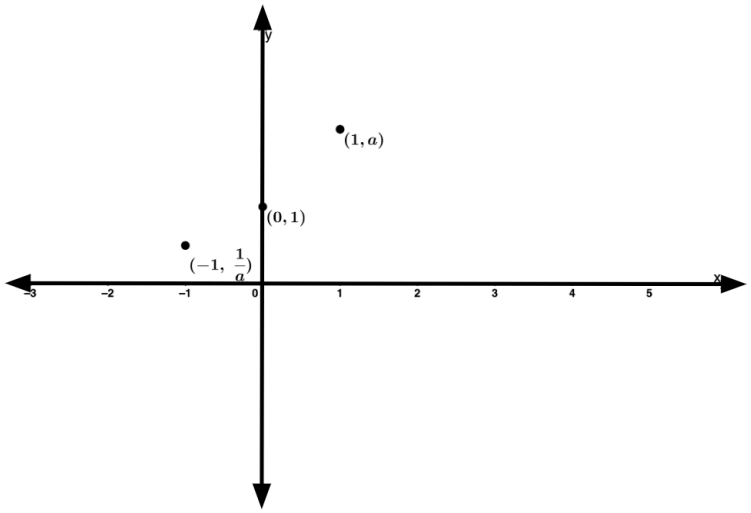
Domain	$(-\infty, \infty)$
Range	$(0, \infty)$
x -intercept	None

Properties of the graph of $f(x) = a^x$ when $a > 1$

y -intercept	$(0, 1)$
asymptote	none
contains	$(1, a)$ and $(-1, \frac{1}{a})$

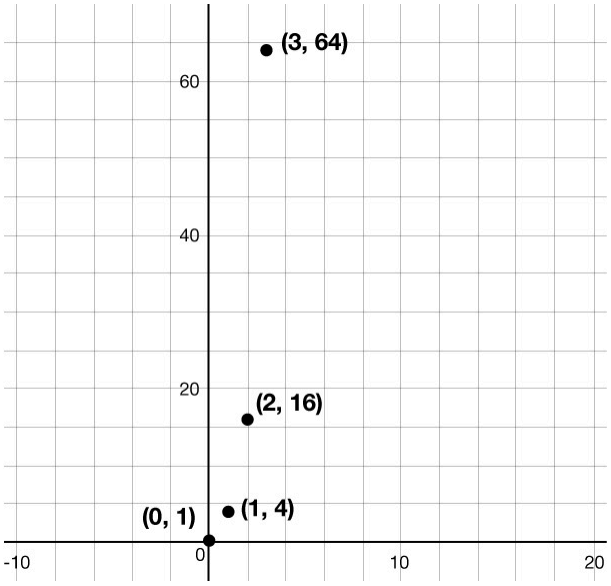
Properties of the graph of $f(x) = a^x$ when $a > 1$

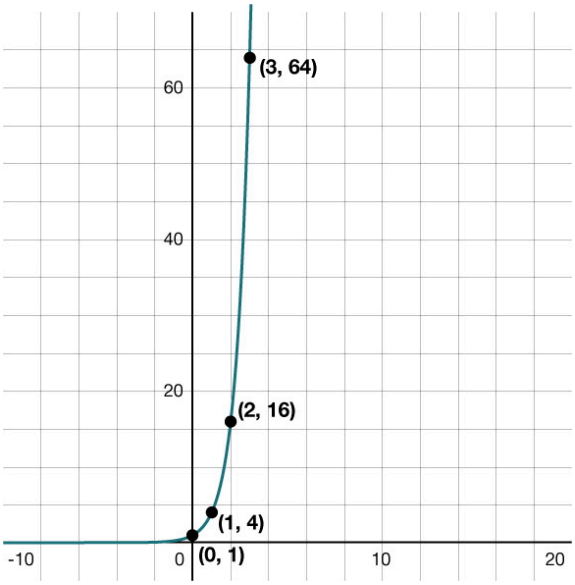
This information can be used to create a graph and connect the points with a smooth curve.



Once a table of values is created, plot each of the points on a coordinate plane, remembering that exponential functions are more of a smooth curve. It is possible that your points don't fit on the graph you have created, but they can be useful to "see" your graph continue off the coordinate plane.

x	y	(x, y)
0	$4^0 = 1$	$(0, 1)$
1	$4^1 = 4$	$(1, 4)$
2	$4^2 = 16$	$(2, 16)$
3	$4^3 = 64$	$(3, 64)$
4	$4^4 = 256$	$(4, 256)$
5	$4^5 = 1,024$	$(5, 1024)$





An asymptote is a line or curve that a function approaches but does not intersect or touch. In other words, as the input values of a function approach a certain value or tend towards infinity or negative infinity, the function values get arbitrarily close to the asymptote without actually crossing it. Asymptotes can provide valuable information about the behavior of a function and can help in understanding its limits and range.

Since equation $y = 4^x$ represents an exponential function, we determine the asymptote of this function by considering the behavior of the function as x approaches positive or negative infinity.

As x approaches positive infinity, the values of 4^x will grow exponentially larger. There is no upper bound, and the function increases without bound as x increases. Therefore, there is no horizontal asymptote in the positive direction.

Similarly, as x approaches negative infinity, the values of 4^x will become very small but never reach zero. In this case, there is also no horizontal asymptote in the negative direction.

Therefore, the function $y = 4^x$ does not have a horizontal asymptote in either direction. The graph of the function will continue to increase, **continue off the coordinate plane**, indefinitely as x increases or decreases.

TRY IT **Graphing Exponential Functions**

Graph $f(x) = 3^x$.

5.4.5 Interpret the Growth Factor of the Equation

Cool Down

A group of biologists is surveying the mice population in a forest. The equation $n = 75 \cdot 3^t$ gives the total number of mice, n, t years since the survey began. Explain what the numbers 75 and 3 mean in this situation.

5.4.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Which expression is equal to $4^0 \cdot 4^2$?

a. 64
b. 16
c. 1
d. 0

2. **Select three** expressions that are equivalent to 3^8.

a. 8^3
b. $\frac{3^{10}}{3^2}$
c. $3 \cdot 8$
d. $\left(3^4\right)^2$
e. $(3 \cdot 3)^4$
f. $\frac{1}{3^8}$

For 3 – 8 use the following information and table:

An investment bond is initially bought for $250. It doubles in value every decade. For each of the missing spaces in the table, complete the table.

decades since bond is bought	dollar value of bond
0	3. _____
1	4. _____
2	5. _____
3	6. _____

3. The dollar value of the bond, 0 decades

a. $750
b. $500
c. $250
d. $0

4. The dollar value of the bond, 1 decade

a. $1000
b. $750
c. $500
d. $250
e. $0

5. The dollar value of the bond, 2 decades

a. $1500

b. $1250
c. $1000
d. $750
e. $500

6. The dollar value of the bond, 3 decades

a. $2250
b. $2000
c. $1750
d. $1500
e. $1250
f. $1000

7. How many decades does it take before the bond is worth more than $10,000?

8. Choose an equation that relates V, the value of the bond, to d, the number of decades since the bond was bought.

a. $V = 250 \cdot 2^d$
b. $V = 250 \cdot 2 \cdot d$
c. $V = 500^d$
d. $V = 250 \cdot d^2$

For questions 9 – 10, use the following information:
A sea turtle population p is modeled by the equation $p = 400 \cdot (1.25)^y$ where y is the number of years since the population was first measured.

9. How many turtles are in the population when it is first measured?

10. After how many years will the turtle population reach 700? Round to the nearest full year.

Use the following information to answer questions 11 and 12.

- Bank Account A starts with $5,000 and grows by $1,000 each week.
- Bank Account B starts with $1 and doubles each week

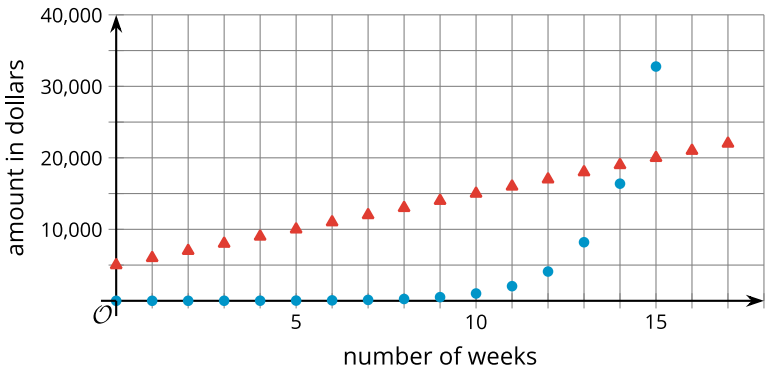
11. Which account has more money after one week?

a. Bank Account A
b. Bank Account B

12. Which account has more money after fifteen weeks?

a. Bank Account A
b. Bank Account B

For 13 - 15, use this graph showing the two account balances.



13. The graph of the red triangles corresponds to which bank account?

a. Bank Account A
b. Bank Account B

14. The graph of the blue circles corresponds to which bank account?

a. Bank Account A
b. Bank Account B

15. Which of the two accounts would you choose for long-term saving?

a. Bank Account A
b. Bank Account B

5.4.7 Lesson Summary

In this lesson, you learned how to:

- Explain the connections between an equation and a graph that represents exponential growth.
- Write and interpret an equation that represents exponential growth.

Here are the **activities** that helped you reach those goals:

- 5.4.1: Exponent Rules
  - In this activity, you reviewed the basic properties of simplifying exponents and practiced their use.
- 5.4.2: Zero Exponent Rule
  - In this activity, you focused on zero as an exponent. You looked at patterns in exponents to determine the value of zero as an exponent and applied the basic rules of exponents with zero.
  - 5.4.2: Self Check
  - 5.4.2: Additional Resources
- 5.4.3: Exponential Change: The Growth Factor
  - In this activity, you learned to continue the exponential growth tables to the equations of the form $y = a \cdot b^x$.
  - 5.4.3: Self Check
  - 5.4.3: Additional Resources
- 5.4.4: Graphing Exponential Expressions
  - In this activity, you continued working with the multiple representations of exponential functions. During this activity, you expanded upon the table that was created in 5.4.3 and represented it in a graph.
  - 5.4.4: Self Check
  - 5.4.4: Additional Resources
- 5.4.5: Interpret the Growth Factor of the Equation
  - In this activity, you analyzed all of the parts of an exponential equation and what they meant in the context of a problem.

After these activities, you completed the following **practice**:

- 5.4.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

5.5 REPRESENTING EXPONENTIAL DECAY

5.5.0 Lesson Overview

In this lesson, you will work more with growth factors. You will continue to write equations and interpret graphs as you learn about exponential decay.

When you finish this lesson, you will be able to:

- Use only multiplication to represent "decreasing a quantity by a fraction of itself."
- Write an expression or equation to represent a quantity that decays exponentially.
- Know the meanings of "exponential growth" and "exponential decay."
- Explain the meanings of a and b in an equation that represents exponential decay and is written as $y = a \cdot b^x$.
- Find a growth factor from a graph and write an equation to represent exponential decay.
- Graph equations that represent quantities that change by a growth factor between 0 and 1.

Here are the **activities** that will help you reach those goals:

- 5.5.1: Identifying Linear and Exponential Quantities
- 5.5.2: Exponential Decay
  - 5.5.2: Self Check
  - 5.5.2: Additional Resources
- 5.5.3: Using Graphs to Represent Exponential Decay
  - 5.5.3: Self Check
  - 5.5.3: Additional Resources
- 5.5.4: Justifying Exponential Models
  - 5.5.4: Self Check
  - 5.5.4: Additional Resources
- 5.5.5: Writing Equations to Represent Exponential Decay

After that, you'll **practice and review**.

- 5.5.6: Practice
- 5.5.7: Lesson Summary

5.5.1 Identifying Linear and Exponential Quantities

Warm Up

Use the patterns you notice to complete the tables. Show your reasoning. Complete your work on paper.

1.

x	0	1	2	3	4	25
y	2.5	10	17.5	25		

Table A

2.

x	0	1	2	3	4	25
y	2.5	10	40	160		

Table B

5.5.2 Exponential Decay

Activity

Every year after a new car is purchased, it loses $\frac{1}{3}$ of its value. Another way to think about this is that the car retains $\frac{2}{3}$ of its value each year. Let's say that a new car's value is $18,000.

1. A buyer worries that the car will be worth nothing in three years. Do you agree? Be prepared to show your reasoning.

2. Write an expression to show how to find the value of the car for each year listed in the table.

Year	Value of Car (Dollars)
0	18,000
1	
2	
3	
6	
t	

3. Write an equation relating the value of the car in dollars, v, to the number of years, t.

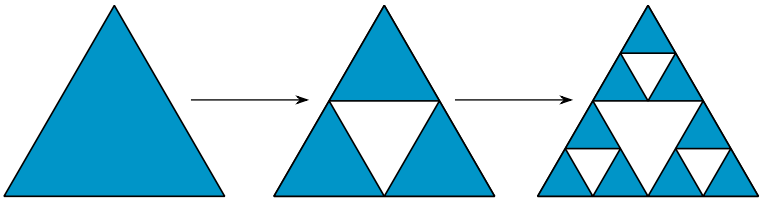
4. Use your equation to find v when t is 0. What does this value of v mean in this situation?

5. A different car loses value at a different rate. The value of this different car in dollars, d, after t years can be represented by the equation $d = 10,000 \cdot (\frac{4}{5})^t$. Explain what the numbers 10,000 and $\frac{4}{5}$ mean in this situation.

Are you ready for more?

Extending Your Thinking

Start with an equilateral triangle with area 1 square unit, divide it into 4 congruent pieces as in the figure, and remove the middle one. Then, repeat this process with each of the remaining pieces. Repeat this process over and over for the remaining pieces. The figure shows the first two steps of this construction.



What fraction of the area is removed each time? How much area is removed after the n^{th} step? Use a calculator to find out how much area remains in the triangle after 50 such steps have been taken.

Self Check

Which of the following could represent a population of bacteria, b, that decreases at a rate of $\frac{1}{8}$ every hour, h? Consider an initial amount of 100 bacteria.

a. $b = 100 \cdot (\frac{1}{8})h$
b. $b = 100 \cdot (\frac{7}{8})h$
c. $b = 100 \cdot (\frac{1}{8})^h$
d. $b = 100 \cdot (\frac{7}{8})^h$

Additional Resources

Analyzing Exponential Decay

At the beginning of April, a colony of ants had a population of 5,000.

1. The colony decreases by $\frac{1}{5}$ during April. Write an expression that represents the ant population at the end of April.

2. During May, the colony decreases again by $\frac{1}{5}$ of its size. Write an expression that represents the ant population at the end of May.
3. The colony continues to decrease by $\frac{1}{5}$ of its size each month. Write an expression that represents the ant population after 6 months.
4. Write an equation to represent the population of ants, p, after m months.

When you are analyzing exponential decay and are given the fraction or percent by which it decreases, you need to remember that one minus that fraction or percent is the growth rate. In the case of the ant colony, because it is decreasing by $\frac{1}{5}$, that means that the growth factor is $1-\frac{1}{5}$ or $\frac{4}{5}$.

Part 1 asks for an expression to represent the ant population at the end of April. There are several options:

- $5000-(\frac{1}{5}\cdot 5000)$
- $5000\cdot\frac{4}{5}$
- 4000

Each line is the simplification of the line above it. Part 2 has a few more options.

Part 2 is asking you to find $\frac{4}{5}$ of April's total. There are several expressions to represent that:

- $5000\cdot\frac{4}{5}\cdot\frac{4}{5}$
- $4000\cdot\frac{4}{5}$
- $4000-(\frac{1}{5}\cdot 4000)$
- 3200

Basically, you take the answer from April and multiply by $\frac{4}{5}$ again since the growth factor is $\frac{4}{5}$. You could also look at it as taking the initial value times $\frac{4}{5}$ twice.

Part 3 asks you to extend the pattern to 6 months after the initial population was measured. After the 2^{nd} month, you multiplied the initial value by the growth factor twice, so for this situation you would need to multiply the initial value by the growth factor 6 times, or raise it to the sixth power. The possible answers would be $5000\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}$ or $5000\cdot(\frac{4}{5})^6$, which means that after 6 months the population would be about 1,311.

Part 4 wants the general equation to represent the situation. If you use the pattern from part 3, you can see that it can be written in the exponential form $y=ab^x$, where a is the initial value and b is the growth factor. Using the variables suggested, the equation would be $p=5000(\frac{4}{5})^m$.

TRY IT **Analyzing Exponential Decay**

A doctor prescribes 125 milligrams of a therapeutic drug that decreases in effectiveness by $\frac{3}{10}$ each hour.

a. Write an expression to represent how much is left after 1 hour.

b. Write an expression to represent how much is left after 3 hours.

c. Write an equation that represents the amount of drug, d, left after h hours.

5.5.3 Using Graphs to Represent Exponential Decay

Activity

To control an algae bloom in a lake, scientists introduce some treatment products.

Once the treatment begins, the area covered by algae in square yards, A, is given by the equation $A=240\cdot(\frac{1}{3})^t$. Time, t, is measured in weeks.



1. In the equation, what does 240 tell us about the algae?
2. What does the $\frac{1}{3}$ tell us?
3. Create a graph to represent $A = 240 \cdot (\frac{1}{3})^t$ when t is 0, 1, 2, 3, and 4. Think carefully about how you choose the scale for the axes. If you get stuck, consider creating a table of values. Use the graphing tool or technology outside the course. Graph the equation that represents this scenario using the Desmos tool below.

Access multimedia content (http://openstax.org/books/algebra-1/pages/5-5-3-using-graphs-to-represent-exponential-decay)

4. Approximately how many square yards will the algae cover after 2.5 weeks? Be prepared to show your reasoning.

Are you ready for more?

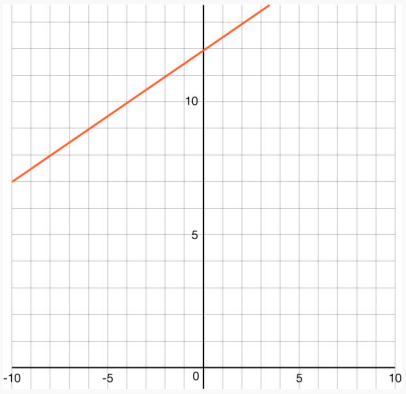
Extending Your Thinking

The scientists estimate that to keep the algae bloom from spreading after the treatment concludes, they will need to get the area covered under one square foot. How many weeks should they run the treatment to achieve this?

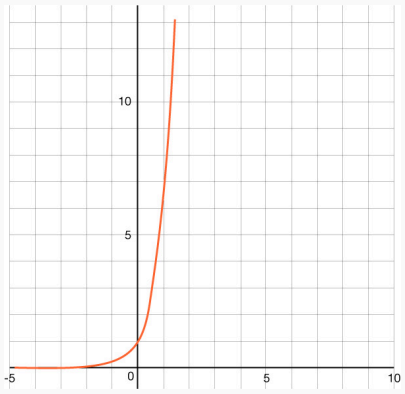
Self Check

Which of the following is the graph of $y = 12(\frac{1}{2})^x$?

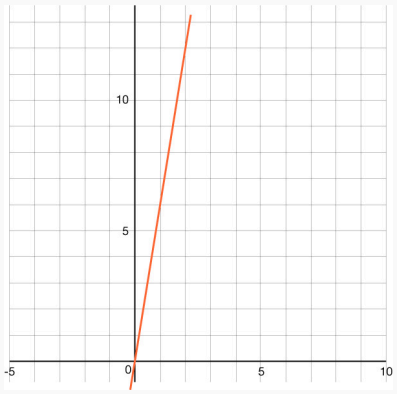
a.



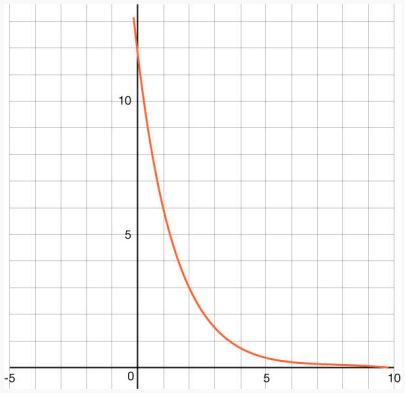
b.



c.



d.



Additional Resources

Connecting Tables and Graphs in Exponential Decay Functions

A population p of migrating butterflies satisfies the equation $p = 100,000 \cdot (\frac{4}{5})^w$, where w is the number of weeks since they began their migration.

Answer the following questions that would complete the table and create a graph to show the population p of migrating butterflies at the number of weeks since they began their migration.

w	0	1	2	3	4
p					

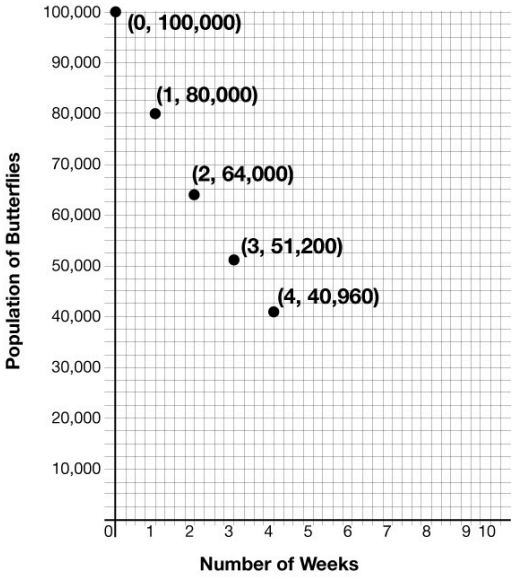
1. The number of butterflies when they began their migration.
2. The number of butterflies 1 week after they began their migration.
3. The number of butterflies 3 weeks after they began their migration.
4. Create a graph that best represents the butterfly population.

5. What is the vertical intercept, or y-intercept, of the graph? What does it tell you about the butterfly population?
6. When does the butterfly population reach approximately 50,000?
7. What is the asymptote of the graph? What does it tell you about the butterfly population over time?

When you have the equation that represents the situation, you can substitute values in and use the order of operations to find the values for the table.

- Part 1 is asking for the initial value. Since the equation $p = 100,000 \cdot (\frac{4}{5})^w$ is in the form $y = ab^x$, you know the initial value is a or 100,000.
- Part 2 is 1 week after migration, so replace the w in the equation with 1 to get 80,000.
- In part 3, the w can be replaced with 3 to get 51,200.
- The rest of the values of the table can be found the same way, and once you have the completed table, you can plot the points. Once you have plotted the points, you can connect them with a smooth curve.

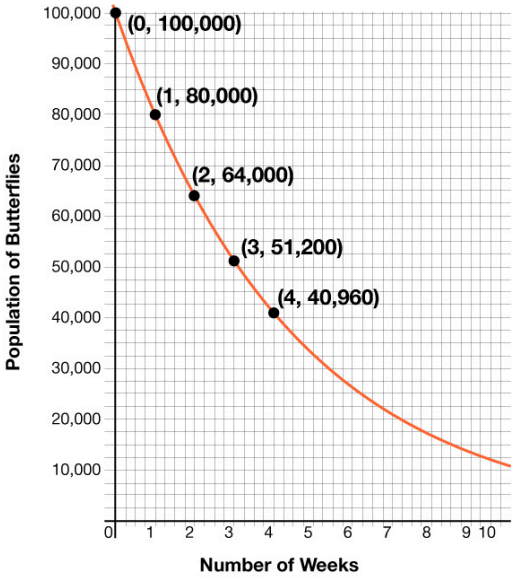
w	0	1	2	3	4
p	100,000	80,000	64,000	51,200	40,960



- Part 5 asks about the vertical intercept and what it means. The vertical intercept is 100,000, which is the initial amount of the butterfly population. This means that when the butterflies start their migration (at $w = 0$ weeks), the population is initially 100,000.

  The y-intercept is the value of p when $w = 0$, which corresponds to the initial condition or starting point of the migration. Substituting $w = 0$ into the equation, we have $p = 100000 \cdot (\frac{4}{5})^0 = 100000 \cdot 1 = 100000$.

- For part 6, you could continue the table or use the graph to see where it is 50,000, which is around 3 weeks.



- For part 7, an asymptote is a line or curve that a function approaches but does not intersect or touch. In other words, as the input values of a function approach a certain value or tend towards infinity or negative infinity, the function values get arbitrarily close to the asymptote without actually crossing it. Asymptotes can provide valuable information about the behavior of a function and can help in understanding its limits and range.

  To determine the asymptote, we examine the behavior of the function as w approaches infinity. In this case, as w becomes larger and larger, the term $(\frac{4}{5})^w$ will grow exponentially smaller. Consequently, the population p will also grow infinitely smaller as it gets closer and closer to zero. Thus, the horizontal asymptote in this equation is the x-axis. The graph of the function will continue to decrease towards the x-axis as w increases, approaching $y = 0$. The butterfly population will decrease over time due to the migration.

TRY IT **Connecting Tables and Graphs in Exponential Decay Functions**

The number of copies of a book sold the year it was released was 600,000. Each year after that, the number of copies sold decreased by $\frac{1}{2}$.

1. Complete the table showing the number of copies of the book sold each year.

Years Since Published	Number of Copies Sold
0	
1	
2	
3	
y	

2. Write an equation representing the number of copies, c, sold y years after the book was released.

3. Use the graphing tool or technology outside the course. Graph the equation that represents the situation, using the Desmos tool below.

Access multimedia content (http://openstax.org/books/algebra-1/pages/5-5-3-using-graphs-to-represent-exponential-decay)

4. Use your equation or graph to find c when $y = 6$. What does this mean in terms of the book sales?

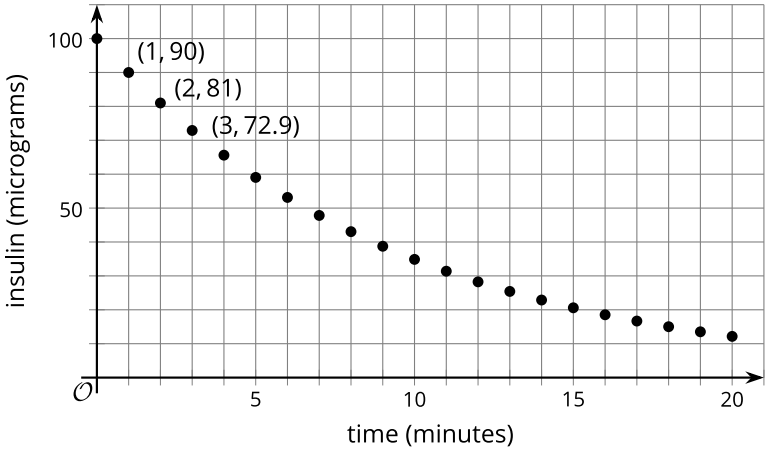
5. What is the vertical intercept, or y-intercept of the graph? What does it tell you about the number of copies sold?

6. What is the asymptote of the graph? What does it tell you about the number of copies sold over time?

5.5.4 Justifying Exponential Models

Activity

A patient who is diabetic receives 100 micrograms of insulin. The graph shows the amount of insulin, in micrograms, remaining in his bloodstream over time, in minutes.



1. Scientists have found that the amount of insulin in a patient's body changes exponentially. How can you check if the graph supports the scientists' claim?

2. Answer the following questions:

a. How much insulin broke down in the first minute?

b. What fraction of the original insulin is that?

3. Answer the following questions.

a. How much insulin broke down in the second minute?

b. What fraction is that of the amount one minute earlier?

4. What fraction of insulin remains in the bloodstream for each minute that passes? Be prepared to show your reasoning.

5. Complete the table to show the predicted amount of insulin 4 and 5 minutes after injection.

Time after Injection (Minutes)	0	1	2	3	4	5
Insulin in the Bloodstream (Micrograms)	100	90	81	72.9		

6. Answer the following questions.

a. Describe how you would find how many micrograms of insulin remain in his bloodstream after 10 minutes.

b. After m minutes?

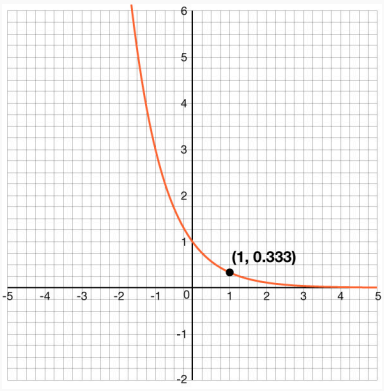
Video: Analyzing Exponential Graphs and Writing Equations

Watch the following video to learn more about analyzing exponential graphs and using the graphs to write an equation.

Analyzing Exponential Graphs and Writing Equations (https://openstax.org/r/exponential-graph-equations)

Self Check

Which of the following is the equation of the graph provided?



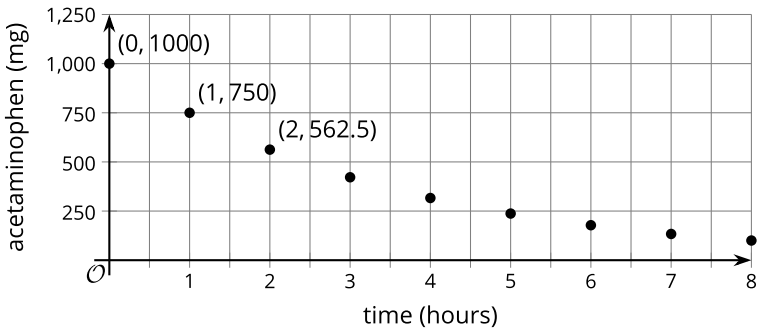
a. $y = (\frac{1}{3})^x$
b. $y = -\frac{1}{3}x$
c. $y = -3x + 1$
d. $y = 3^x$

Additional Resources

More Exponential Models

The graph shows the amount of a chemical in a patient's body at different times, measured in hours since the levels were first checked.

Could the amount of this chemical in the patient be decaying exponentially? Explain how you know.



The graph does seem to have the shape of exponential decay. To confirm if the chemical in the patient is decaying exponentially, you can find the growth factor. Since terms are usually multiplied by the growth factor, you can divide consecutive terms to see if there is the same growth factor each time.

Since $\frac{600}{1000} = \frac{360}{600} = \frac{6}{10}$, the growth factor is $\frac{6}{10}$. Therefore, it is decaying exponentially, and the equation is $y = 1000(\frac{6}{10})^x$, where y is the amount of chemical in mg in the patient and x is the time in hours.

As the time continues, the amount of chemical in the patient is approaching 0 mg. We can see this in the graph as the data points approach the x-axis where $y = 0$. The equation of the line that the data points approach is called the asymptote. In this example it is $y = 0$.

TRY IT **More Exponential Models**

The following table represents the number of people who have read a book. Does the table represent

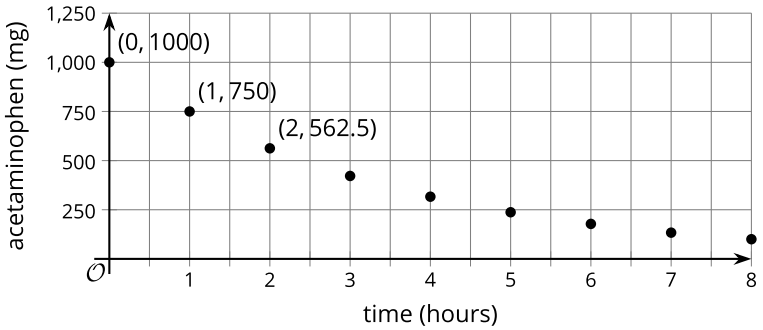
exponential growth? Be prepared to show your reasoning.

Number of Months	Number of People who have Read a Book
0	300
1	600
2	1200
3	2400

5.5.5 Writing Equations to Represent Exponential Decay

Cool Down

Acetaminophen is a common pain reliever and fever reducer. Here is a graph showing the amount of acetaminophen, in milligrams, in an adult's body at different times after they take a normal dose.



1. Answer the following questions.
   a. What is the vertical intercept?
   b. What does it mean in this situation?
2. What fraction of the medicine remained after one hour?
3. Write an equation that represents the amount of medicine, a, after t hours.

5.5.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. A new bicycle sells for \$300. It is on sale for $\frac{1}{4}$ off the regular price. **Select three** expressions that represent the sale price of the bicycle in dollars.

a. $300 \cdot \frac{1}{4}$
b. $300 \cdot \frac{3}{4}$
c. $300 \cdot (1 - \frac{1}{4})$
d. $300 - \frac{1}{4}$
e. $300 - \frac{1}{4} \cdot 300$

For questions 2–4, use the following information:

A computer costs \$800. It loses $\frac{1}{4}$ of its value every year after it is purchased.

2. Answer the following questions that would complete the table to show the value of the computer at the listed times.

Time (Years)	Value of Computer (Dollars)
0	
1	
2	
3	
t	

a. The value of the computer, in dollars, when it is purchased.

b. The value of the computer, in dollars, after 1 year.

c. The value of the computer, in dollars, after 2 years.

3. Which of the following expressions would represent the value of the computer, in dollars, after t years?

a. $v = 800 \cdot (\frac{3}{4})^t$
b. $v = 800 \cdot (\frac{1}{4})^t$
c. $v = 800 \cdot \frac{3}{4} \cdot t$
d. $v = 800 \cdot \frac{1}{4} \cdot t$

4. Use your equation from question 3 to find v when t is 5. What does this value of v mean?

a. 253.13; After 5 years, the computer's value is \$253.13.
b. 3,000; After 5 years, the computer's value is \$3,000.
c. 142.38; After 5 years, the computer's value is \$142.38.
d. 189.84; After 5 years, the computer's value is \$189.84.

5. A piece of paper is folded into thirds multiple times. The area, A, of the piece of paper in square inches, after n folds, is $A = 90 \cdot (\frac{1}{3})^n$.

a. What is the value of A when $n = 0$?

b. How many folds are needed before the area is less than 1 square inch?

For questions 6 – 7, use the following information:

The area of another piece of paper in square inches, after n folds, is given by $A = 100 \cdot (\frac{1}{2})^n$.

6. What does the number 100 mean in this situation?

a. The area after the paper is folded
b. The area before the paper is folded
c. The number of folds to have an area of 0
d. The growth factor of the situation

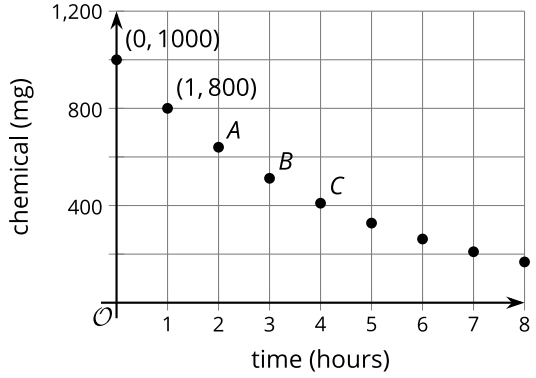
7. What does the number $\frac{1}{2}$ mean in this situation?

a. The area of the paper before it is folded
b. The area of the paper after 1 fold
c. The growth factor of the situation
d. The initial value of the situation

8. Lin has 13 mystery novels. Each month, she gets 2 more. **Select three** expressions that represent the total number of mystery novels Lin has after 3 months.

a. $13 + 2 + 2 + 2$
b. $13 \cdot 2 \cdot 2 \cdot 2$
c. $13 \cdot 8$
d. $13 + 6$
e. 19

9. The graph shows the amount of a chemical in a water sample. It is decreasing exponentially.



a. What are the coordinates of point A?

a. $(1, 640)$
b. $(\frac{4}{5}, 1000)$
c. $(2, 640)$
d. $(0, 1000)$

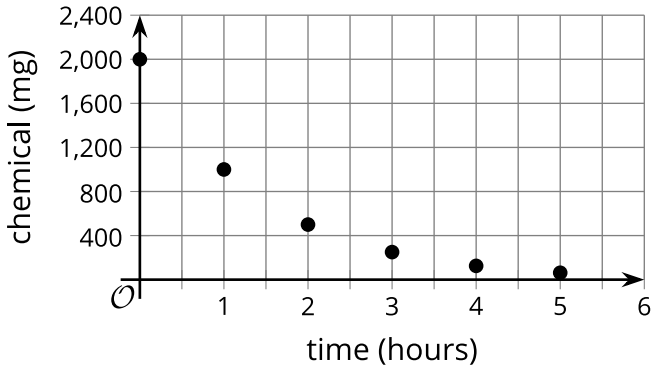
b. What are the coordinates of point B?

a. $(\frac{4}{5}, 512)$
b. $(4, 410)$
c. $(3, 512)$
d. $(2, 640)$

c. Which of the following equations would represent the chemical, C, after time, x, from the graph?

a. $C = 1000 + \frac{4}{5} + x$
b. $C = 1000(\frac{4}{5})^x$
c. $C = 1000 \cdot \frac{4}{5} \cdot x$
d. $C = 1000 + \frac{4}{5}x$

10. The graph shows the amount of a chemical in a water sample at different times after it was first measured. **Select two** statements that are true.



a. The amount of the chemical in the water sample is decreasing exponentially.
b. The amount of the chemical in the water sample is not decreasing exponentially.
c. It is not possible to tell for certain whether the amount of the chemical is decreasing exponentially.
d. When it was first measured, there were 2,000 mg of the chemical in the water sample.
e. After 4 hours, there were 200 mg of the chemical in the water.

11. The height of a plant in mm is 7. It doubles each week. **Select two** expressions that represent the height of the plant, in mm, after 4 weeks.

a. $7 + 4 \cdot 2$
b. $7 \cdot 2^4$
c. $2 + 7^4$
d. $7 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
e. $7 \cdot 2 \cdot 4$

5.5.7 Lesson Summary

In this lesson, you learned how to:

- Use only multiplication to represent "decreasing a quantity by a fraction of itself."
- Write an expression or equation to represent a quantity that decays exponentially.
- Know the meanings of "exponential growth" and "exponential decay."
- Explain the meanings of a and b in an equation that represents exponential decay and is written as $y = a \cdot b^x$.
- Find a growth factor from a graph and write an equation to represent exponential decay.
- Graph equations that represent quantities that change by a growth factor between 0 and 1.

Here are the **activities** that helped you reach those goals:

- 5.5.1: Identifying Linear and Exponential Quantities
  - In this activity, you analyzed the difference between linear and exponential tables. You also extended the pattern of the equation of each.
- 5.5.2: Exponential Decay
  - In this activity, you learned how the value of a used car can be represented using exponential decay. You also learned how to use the values from the problem to write the equation.
  - 5.5.2: Self Check
  - 5.5.2: Additional Resources
- 5.5.3: Using Graphs to Represent Exponential Decay
  - In this activity, you analyzed how an algae bloom shrinks after being treated. The values of the equations were analyzed in the context of the situation, and each of the observations was graphed.
  - 5.5.3: Self Check
  - 5.5.3: Additional Resources
- 5.5.4: Justifying Exponential Models
  - In this activity, you analyzed how much insulin is left in a diabetic's bloodstream over time. This scenario was used to justify the fact that the insulin decreases exponentially over time.
  - 5.5.4: Self Check
  - 5.5.4: Additional Resources
- 5.5.5: Writing Equations to Represent Exponential Decay
  - In this activity, you used the graph of the breakdown of acetaminophen to create an exponential equation to represent the situation.

After these activities, you completed the following **practice** :

- 5.5.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

5.6 NEGATIVE EXPONENTS AND SCIENTIFIC NOTATION

5.6.0 Lesson Overview

In this lesson, you will examine the meaning of negative exponents in the context of exponential growth and decay. Using data in a table, you will write equations and answer questions, some of which require working with negative values representing time. In one case, you will also produce a graph. As with many other activities in this unit, the mathematical work is grounded and interpreted in a context. These equations with negative exponents are frequently used in solving real-world problems where estimates are taken, including population sizes, financial calculations, and many other scientific measurements.

You will learn to convert numbers fluently between decimal form and scientific notation. This will allow you to easily perform operations on and represent large and small values in any context. Scientific notation is an extremely valuable tool for representing very large and small numbers that frequently occur as scientists take real-world measurements.

When you finish this lesson, you will be able to:

- Describe the meaning of a negative exponent in equations that represent exponential growth and decay.
- Write equations that represent exponential growth and decay to solve problems.
- Graph an equation that represents exponential decay to solve a problem.
- Convert numbers between decimal form and scientific notation.
- Use scientific notation to solve problems arising from real-world situations.

Here are the **activities** that will help you reach those goals:

- 5.6.1: Checking Exponent Rules
- 5.6.2: Interpreting Negative Exponents in Exponential Growth
  - 5.6.2: Self Check
  - 5.6.2: Additional Resources
- 5.6.3: Interpreting Negative Exponents in Exponential Decay
  - 5.6.3: Self Check
  - 5.6.3: Additional Resources
- 5.6.4: Using Scientific Notation
  - 5.6.4: Self Check
  - 5.6.4: Additional Resources
- 5.6.5: Applying Scientific Notation
  - 5.6.5: Self Check
  - 5.6.5: Additional Resources
- 5.6.6: Interpreting Negative Exponents and Scientific Notation

After that, you'll **practice and review**.

- 5.6.7: Practice
- 5.6.8: Lesson Summary

5.6.1 Checking Exponent Rules

Warm Up

How would you rewrite each of the following as an equivalent expression with a single exponent?

1. $2^4 \cdot 2^0$

2. $2^4 \cdot 2^{-1}$

3. $2^4 \cdot 2^{-3}$

4. $2^4 \cdot 2^{-4}$

5.6.2 Interpreting Negative Exponents in Exponential Growth

Activity

A marine biologist estimates that a structure of coral has a volume of 1,200 cubic centimeters and that its volume doubles each year.



1. Write an equation of the form $y = a \cdot b^t$ representing the relationship, where t is time in years since the coral was measured and y is volume of coral in cubic centimeters. (You need to figure out what numbers a and b are in this situation.)
2. Find the volume of the coral when t is 5, 1, 0, –1, and –2. Note that when $t = -1$, this represents one year before the marine biologist made the original estimate.

Time (years)	5	1	0	-1	-2
Volume of Coral (cubic centimeters)	a.____	b.____	c.____	d.____	e.____

a. Enter the volume of coral when t = 5.

b. Enter the volume of coral when t = 1.

c. Enter the volume of coral when t = 0.

d. Enter the volume of coral when t = -1.

e. Enter the volume of coral when t = -2.

3. What does it mean, in this situation, when t is –2?
4. In a certain year, the volume of the coral is 37.5 cubic centimeters. Which year is this? Be prepared to show your reasoning.

Self Check

Using the equation $p = 10,000 \cdot 2^d$, find the value of p when $d = -2$.

a. 40,000
b. 5,000
c. 20,000
d. 2,500

Additional Resources

Interpreting Negative Exponents in Exponential Growth

Let's look at a problem involving an exponential growth equation and negative exponents.

A forest fire has been burning for several days. The burned area, in acres, is given by the equation $y = 4800 \cdot 2^d$, where d is the number of days since the area of the fire was first measured.

1. Let's complete the table.

d, days since first measurement	Process column	y, acres burned since fire started
0	$= 4800 \cdot 2^0 = 4800 \cdot 1$	4800
–1	$= 4800 \cdot 2^1 = 4800 \cdot \frac{1}{2}$	2400
–2	0	c. _____
–3	0	d. _____
–5	0	e. _____

To determine the value of y, given d, follow these steps:

EXAMPLE 1

Determine the value for y when $d = -2$

Step 1 -Substitute the given domain value for d into the expression.

$4800 \cdot 2^{-2}$

Step 2 -Evaluate the exponential value. Recall that a negative exponent represents how many times the reciprocal of the base should be multiplied.

$= 4800 \cdot (\frac{1}{2})^2 = 4800 \cdot \frac{1}{4}$

Step 3 -Evaluate the remaining expression to determine y.

$= 1200$

EXAMPLE 2

Determine the value for y when $d = -3$

Step 1 -Substitute the given domain value for d into the expression.

$4800 \cdot 2^{-3}$

Step 2 -Evaluate the exponential value. Recall that a negative exponent represents how many times the reciprocal of the base should be multiplied.

$= 4800 \cdot (\frac{1}{2})^3 = 4800 \cdot \frac{1}{8}$

Step 3 -Evaluate the remaining expression to determine y.

$= 600$

EXAMPLE 3

Determine the value for y when $d = -5$

Step 1 -Substitute the given domain value for d into the expression.

$4800 \cdot 2^{-5}$

Step 2 -Evaluate the exponential value. Recall that a negative exponent represents how many times the reciprocal of the base should be multiplied.

$= 4800 \cdot (\frac{1}{2})^5 = 4800 \cdot \frac{1}{32}$

Step 3 -Evaluate the remaining expression to determine y.

$= 150$

EXAMPLE 4

What does the value of y tell you about the area burned in the fire when $d = -1$?

The y value at $d = -1$ represents the fact that 2,400 acres had burned 1 day before the first measurement was taken.

EXAMPLE 5

How much area had the fire burned a week before it measured 4,800 acres? Explain your reasoning.

Since there are 7 days in a week, to find the area burned a week before the fire was measured, we need to determine the value of y when $d = -7$.

Step 1 -Substitute the given domain value for d into the expression.

$4800 \cdot 2^{-7}$

Step 2 -Evaluate the exponential value. Recall that a negative exponent represents how many times the reciprocal of the base should be multiplied.

$= 4800 \cdot \frac{1}{2}^7 = 4800 \cdot \frac{1}{128}$

Step 3 -Evaluate the remaining expression to determine y.

$= 37.5$

37.5 acres had burned a week before the fire measured $4,800$ acres. The value is found by substituting -7 for d since that represents 7 days before the first measurement of $4,800$ acres.

TRY IT **Interpreting Negative Exponents in Exponential Growth**

The value of a home in 2015 was $400,000. Its value has doubled each decade.

1. If v is the value of the home, in dollars, write an equation for v in terms of d, the number of decades since 2015.
2. What is v when $d = -1$?
3. What does this value mean?
4. What does this value mean?
5. What is v when $d = -3$?

5.6.3 Interpreting Negative Exponents in Exponential Decay

Activity

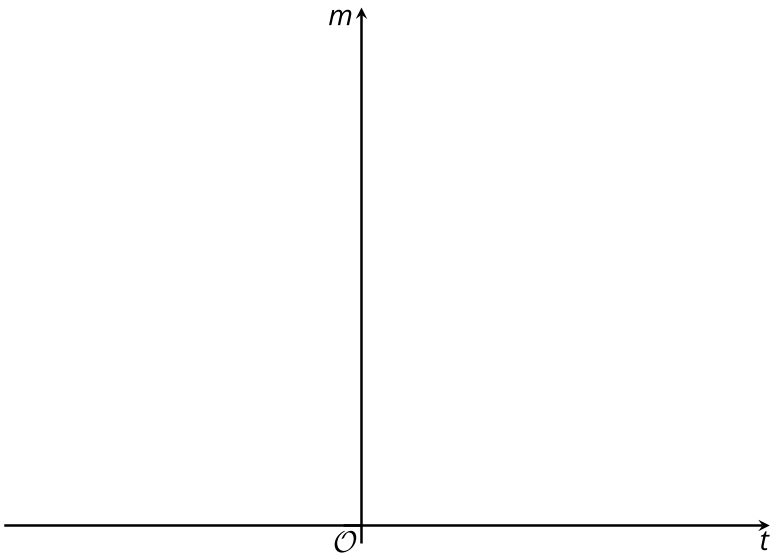
A person took some medicine but does not remember how much. Concerned that she took too much, she has a blood test every hour for several hours.

t, time (hours)	m, medicine (mg)
–3	
–1	

t, time (hours)	m, medicine (mg)
0	100
1	50
2	25

Answer the following questions.

1. Time t is measured in hours since the blood test, and amount of medicine in her body, m, is measured in milligrams. What is the decay factor? That is, what is b in an equation of the form $m = a \cdot b^t$?
2. What is a?
3. Find the amounts of medicine in the patient's body when t is –1.
4. Find the amounts of medicine in the patient's body when t is –3.
5. What does $t = 0$ mean in this context?
6. What does $t = -3$ mean in this context?
7. The medicine was taken when t is –5. Assuming the person did not have any of the medication in her body beforehand, how much medicine did the patient take?
8. Create a graph by plotting the points whose coordinates are in the table from question 1 above. Make sure to draw and label tick marks on the axes.



9. Based on your graph, when do you think the patient will have: 500 mg of medicine remaining in her body
10. Based on your graph, when do you think the patient will have: No medicine remaining in her body

Are you ready for more?

Extending Your Thinking

Without evaluating them, describe each of the following quantities as close to 0, close to 1, or much larger than 1.

1. $\frac{1}{1-2^{-10}}$

Without evaluating them, describe each of the following quantities as close to 0, close to 1, or much larger than 1.

2. $\frac{2^{10}}{2^{10}+1}$

Without evaluating them, describe each of the following quantities as close to 0, close to 1, or much larger than 1.

3. $\frac{2^{-10}}{2^{10}+1}$

Without evaluating them, describe each of the following quantities as close to 0, close to 1, or much larger than 1.

4. $\frac{1-2^{-10}}{2^{10}}$

Without evaluating them, describe each of the following quantities as close to 0, close to 1, or much larger than 1.

5. $\frac{1+2^{10}}{2^{-10}}$

Self Check

A census of hemlock trees is taken in a national park. The rangers have found that only $\frac{4}{5}$ of the population

y, years since census	h, number of hemlocks
–2	
0	64,000
1	51,200
3	32,768

a. 100,000
b. 80,000
c. 125,000
d. 156,250

Additional Resources

Interpreting Negative Exponents in Exponential Decay

Let's look at a problem involving an exponential decay equation and negative exponents.

A scientist records the population of a bacterial population in a petri dish. The data are shown in the table.

d, days since first test	b, number of bacteria
–3	
–1	
0	40,000
1	20,000
3	5,000

The number of days d since the first test corresponds with the number of bacteria, b, estimated to be alive in the petri dish.

1. What is the decay factor? That is, what is f in an equation of the form $b = n \cdot f^d$? What is the starting population, n, on

the first day of testing?

Since the number of bacteria found on day 0 is 40,000, then $n = 40,000$. Since after each day, the number of bacteria is cut in half, then the decay factor, f, is $\frac{1}{2}$.

The equation to represent this situation is $b = 40,000 \cdot \left(\frac{1}{2}\right)^d$.

2. What do $d = -1$ and $d = -3$ mean in this context?

Since the day the first test is taken is defined as $d = 0$, then days with a negative value occured before the first test. $d = -1$ means 1 day before the scientist did the first test. $d = -3$ means 3 days before the scientist did the first test.

3. How many bacteria were present 1 day before the first test?

Substitute –1 for d in the equation $b = 40,000 \cdot \left(\frac{1}{2}\right)^d$.

$b = 40,000 \cdot \left(\frac{1}{2}\right)^d$

$b = 40,000 \cdot \left(\frac{1}{2}\right)^{-1}$

$b = 40,000 \cdot 2$

$b = 80,000$

There were 80,000 bacteria in the petri dish 1 day before the test.

4. Assume the peak of the bacterial population was 3 days before the first test. How many bacteria were present at the peak of the population?

Substitute –3 for d in the equation $b = 40,000 \cdot \left(\frac{1}{2}\right)^d$.

$b = 40,000 \cdot \left(\frac{1}{2}\right)^d$

$b = 40,000 \cdot \left(\frac{1}{2}\right)^{-3}$

$b = 40,000 \cdot 8$

$b = 320,000$

There were 320,000 bacteria in the petri dish 3 days before the test at the peak of the population.

TRY IT **Interpreting Negative Exponents in Exponential Decay**

An investor has tracked the value of a particular investment since 2016 that has depreciated over time. The data are shown in the table.

y, years since 2016	v, value of investment ($)
–2	
–1	
0	36,000
1	12,000
2	4,000

1. What is the equation that represents this situation?
2. What do $y = -1$ and $y = -2$ mean in this context?
3. If the value of the investment was at its highest in 2014, what was the maximum value of the

investment?

5.6.4 Using Scientific Notation

Activity

1. How would you write each of the following numbers in scientific notation?

Decimal Form	Scientific Notation
0.00000654	
0.0000947	
683,000	
0.022	
5,200	
0.000352	

2. How would you write each of the following scientific notation values in decimal form?

Scientific Notation	Decimal Form
7.26×10^{-4}	
3.32×10^{-2}	
9.4×10^{6}	
8.87×10^{-1}	
6.395×10^{4}	
4.0×10^{-3}	

Video: Converting between Scientific Notation and Decimal Form

Watch the following video to learn more about rewriting scientific notation in decimal form and decimal form in scientific notation.

Converting between Scientific Notation and Decimal Form (https://openstax.org/r/scientific-notation)

Self Check

Which of the following equations is true?

a. $2.36 \times 10^{-5} = 0.00000236$
b. $8.5 \times 10^{-4} = 0.00085$
c. $9.11 \times 10^{5} = 91,100$
d. $7.64 \times 10^{3} = 76,400$

Additional Resources

Using Scientific Notation

Working with very large or very small numbers can be awkward. Since our number system is base ten, we can use powers of ten to rewrite very large or very small numbers to make them easier to work with. Consider the numbers 4,000 and 0.004.

Using place value, we can rewrite the numbers 4,000 and 0.004. We know that 4,000 means $4 \times 1,000$ and 0.004 means $4 \times \frac{1}{1000}$.

If we write the 1,000 as a power of ten in exponential form, we can rewrite these numbers in this way:

4,000

$4 \times 1,000$

4×10^3

0.004

$4 \times \frac{1}{1000}$

$4 \times \frac{1}{10^3}$

4×10^{-3}

When a number is written as a product of two numbers, where the first factor is a number greater than or equal to 1 but less than 10, and the second factor is a power of 10 written in exponential form, it is said to be in **scientific notation**.

A number is expressed in scientific notation when it is in this form :

$a \times 10^n$, where $1 \leq \left| a \right| < 10$ and n is an integer.

It is customary in scientific notation to use × as the multiplication sign, even though we avoid using this sign elsewhere in algebra.

To convert a decimal to scientific notation:

Step 1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.

Step 2. Count the number of decimal places, n, that the decimal point was moved.

Step 3. Write the number as a product with a power of 10. If the original number is

- greater than 1, the power of 10 will be 10^n .
- between 0 and 1, the power of 10 will be 10^{-n} .

Step 4. Check.

EXAMPLE

Write in scientific notation:

1. 37,000
   Solution

The original number, 37,000, is greater than 1 so we will have a positive power of 10.	37,000
Move the decimal point to get 3.7, a number between 1 and 10.	37,000
Count the number of decimal places the point was moved.	4 places
Write as a product with a power of 10.	3.7×10^4
Check. 3.7×10^4 $3.7 \times 10{,}000$ $37{,}000$	
	$37{,}000 = 3.7 \times 10^4$

2. 0.0052

Solution

The original number, 0.0052, is between 0 and 1 so we will have a negative power of 10	0.0052
Move the decimal point to get 5.2, a number between 1 and 10.	0.0052
Count the number of decimal places the point was moved.	3 places
Write as a product with a power of 10.	5.2×10^{-3}
Check. 5.2×10^{-3} $5.2 \times \frac{1}{10^3}$ $5.2 \times \frac{1}{1000}$ 5.2×0.001 0.0052	
	$0.0052 = 5.2 \times 10^{-3}$

To convert scientific notation to decimal form:

Step 1. Determine the exponent, n, on the factor of 10.

Step 2. Move the decimal n places, adding zeros if needed.

- If the exponent is positive, move the decimal point n places to the right.
- If the exponent is negative, move the decimal point n places to the left.

Step 3. Check.

EXAMPLE

Write in decimal form:

1. 6.2×10^3

Solution

	6.2×10^3
Determine the exponent, n, on the factor 10.	The exponent is 3.
	6.200
Since the exponent is positive, move the decimal point 3 places to the right.	6,000
Add zeros as needed for placeholders.	
	$6.2 \times 10^3 = 6{,}200$

2. -8.9×10^{-2}

Solution

	-8.9×10^{-2}
Determine the exponent, n, on the factor 10.	The exponent is –2.
Since the exponent is negative, move the decimal point 2 places to the left.	–.8.9
Add zeros as needed for placeholders.	–0.089
	$-8.9 \times 10^{-2} = -0.089$

TRY IT **Using Scientific Notation**

Write in scientific notation.

1. 65,000,000

2. 0.00034

Write in decimal form.

3. 5.3×10^5

4. 4.25×10^{-3}

5.6.5 Applying Scientific Notation

Activity

1. A biologist receives bird population estimates for different parks. The estimates are shown in the table. Work with your partner to arrange these estimates in order from least to greatest.

Park	Bird Population
Shady Elm	7.5×10^3
Westcott Park	105,000
Brighton Park	4.5×10^2
Red Elk	22,300
Bear Park	1.1×10^5
Daniels Park	3,300
Blue Spruce	1.25×10^4
Yellow Jay Park	2.25×10^4

2. Enrico weighs the small objects shown for a science project. Work with your partner to arrange these weights in order from heaviest to lightest.

Object	Weight (lb)
grain of sand	1.4×10^{-4}
grain of pollen	0.00001014
one drop of water	0.00011
a feather	1.8×10^{-5}
one kernel of corn	1.1×10^{-3}
house fly	3.08×10^{-5}

3. What do you notice when comparing numbers in scientific notation with different negative exponents?

Self Check

Which of the numbers shown is the greatest?

a. 9.6×10^{-5}
b. 0.00084
c. 7.2×10^{-3}
d. 0.0065

Additional Resources

Applying Scientific Notation

Let's look at how to solve a real-world problem involving scientific notation.

The specimens below are measured lengths of different tiny animals. Arrange them in order from shortest to longest.

Specimen	Length (m)
A	7.0×10^{-3}
B	0.0000847
C	0.00022
D	1.8×10^{-4}
E	8.92×10^{-3}
F	6.03×10^{-2}

To solve this problem, we can convert all of the measurements to decimal form or standard form.

Let's convert all measurements to decimal form from scientific notation.

Specimen	Length (m)
A	$7.0 \times 10^{-3} = 0.007$
B	0.0000847
C	0.00022
D	$1.8 \times 10^{-4} = 0.00018$
E	$8.92 \times 10^{-3} = 0.00892$
F	$6.03 \times 10^{-2} = 0.0603$

We can now arrange the decimal values in order from shortest to longest:

B, D, C, A, E, F

Let's convert each length into scientific notation and check our answer.

Specimen	Length (m)
A	7.0×10^{-3}
B	$0.0000847 = 8.47 \times 10^{-5}$
C	$0.00022 = 2.2 \times 10^{-4}$
D	1.8×10^{-4}
E	8.92×10^{-3}
F	6.03×10^{-2}

The negative exponent with the largest absolute value represents the smallest number, so B is the shortest length.

The negative exponent with the smallest absolute value represents the largest number, so F is the longest length.

When the exponents are equal, compare the decimal part of the number.

The order B, D, C, A, E, F is correct from shortest to longest.

TRY IT **Applying Scientific Notation**

The measurements in the table are sample wavelengths for the given types of waves. Arrange them in order from shortest to longest.

Color	Wavelength (m)
red light	7.0×10^{-7}
blue light	0.00000045
infrared light	5.0×10^{-4}
ultraviolet light	2.0×10^{-9}
microwaves	1.5×10^{-1}
radio waves	2.1

5.6.6 Interpreting Negative Exponents and Scientific Notation

Cool Down

The equation $p = 5,000 \cdot 2^t$ represents the population of an invasive fish species in a large lake, t years since 2005, when the fish population in the lake was first surveyed.

1. What was the population in 2005?
2. What is this value of the population in 2005 in scientific notation?
3. For this model, what does it mean when t is –2?
4. For $t = -2$, is the fish population more or less than 1,000? How do you know?
5. Write the value of the fish population when $t = -2$ in scientific notation.

5.6.7 Practice

Complete the following questions to practice the skills you have learned in this lesson.

Use the following scenario to answer questions 1 – 3.

- In a newly formed swamp, it is estimated that there are 500 mosquitoes. The population is estimated to double each day.

1. Let m represent the total population of mosquitoes and d represent the number of days that have passed since the first estimate. What is the equation that represents the growth of the mosquito population?

a. $m = 500 \cdot d^2$
b. $m = 500 \cdot 2^d$
c. $m = 2 \cdot 500^d$
d. $m = d \cdot 500^2$

2. What does $d = -2$ represent in this context?

a. This represents 2 days after the first estimate was taken.
b. This represents the day the first estimate was taken.
c. This represents the day before the first estimate was taken.
d. This represents 2 days before the first estimate was taken.

3. What is the estimated number of mosquitoes that were in the new swamp 1 day before the estimate was taken?

a. 125
b. 1000
c. 250
d. 500

Use the following scenario to answer questions 4 – 6.

A desert pond is slowly drying up. The average volume of the pond is shown in the table. The first year it was measured, the average volume was determined to be 40,000 liters.

y, years	v, volume (liters)
–4	
–2	
0	40,000
1	20,000
3	5,000

4. Which equation represents the pond's water loss?

a. $v = \frac{1}{2} \cdot (40,000)^y$
b. $v = 20,000 \cdot \left(\frac{1}{4}\right)^y$
c. $v = 40,000 \cdot \left(\frac{1}{2}\right)^y$
d. $y = 40,000 \cdot \left(\frac{1}{2}\right)^v$

5. What does $y = -2$ represent in this context?

a. the time 2 years before the first time the volume was measured
b. the time 2 years after the first time the volume was measured
c. the first time the volume was measured

d. the time it took for the pond to shrink by a factor of 2

6. What was the average volume of the pond 4 years before the first time the volume was measured?

a. 640,000 liters
b. 80,000 liters
c. 160,000 liters
d. 320,000 liters

7. Which of the following is equivalent to 6,350,000 in scientific notation?

a. 6.35×10^{7}
b. 6.35×10^{6}
c. 6.35×10^{5}
d. 6.35×10^{4}

8. Which of the following is equivalent to 0.00032 in scientific notation?

a. 3.2×10^{-6}
b. 3.2×10^{-5}
c. 3.2×10^{-4}
d. 3.2×10^{-3}

9. Which of the following is equivalent to 7.24×10^{3} in decimal form?

a. 724,000
b. 72,400
c. 724
d. 7,240

10. Which of the following is equivalent to 5.47×10^{-5} in decimal form?

a. 0.000547
b. 0.00000547
c. 0.0000547
d. 0.00547

11. Which of the following correctly orders these values from greatest to least?

Value R	0.00161
Value S	6.35×10^{-3}
Value T	0.000541
Value U	7.23×10^{-4}

a. Value R, Value S, Value U, Value T
b. Value S, Value R, Value U, Value T
c. Value R, Value S, Value T, Value U
d. Value S, Value R, Value T, Value U

5.6.8 Lesson Summary

In this lesson, you learned how to:

- Describe the meaning of a negative exponent in equations that represent exponential growth and decay.
- Write equations that represent exponential growth and decay to solve problems.
- Graph an equation that represents exponential decay to solve a problem.
- Convert numbers between decimal form and scientific notation.
- Use scientific notation to solve problems arising from real-world situations.

Here are the **activities** that helped you reach those goals:

- 5.6.1: Checking Exponent Rules
  - In this activity, you reviewed performing operations with exponents, including negative exponents. You used the Product Property of Exponents to simplify the expressions.
- 5.6.2: Interpreting Negative Exponents in Exponential Growth
  - In this activity, you wrote an equation of exponential growth to represent a real-life situation. You interpreted the meaning of negative exponents in this equation. You used negative exponents in exponential growth equations to solve real-world problems.
  - 5.6.2: Self Check
  - 5.6.2: Additional Resources
- 5.6.3: Interpreting Negative Exponents in Exponential Decay
  - In this activity, you wrote an equation of exponential decay to represent a real-life situation. You also created a graph to represent the situation. You interpreted the meaning of negative exponents in this equation. You used negative exponents in exponential decay equations to solve real-world problems.
  - 5.6.3: Self Check
  - 5.6.3: Additional Resources
- 5.6.4: Applying Scientific Notation
  - In this activity, you converted values between decimal form and scientific notation.
  - 5.6.4: Self Check
  - 5.6.4: Additional Resources
- 5.6.5: Applying Scientific Notation
  - In this activity, you used what you know about scientific notation to solve real-world problems.
  - 5.6.5: Self Check
  - 5.6.5: Additional Resources
- 5.6.6: Interpreting Negative Exponents and Scientific Notation
  - In this activity, you interpreted negative exponents in a scenario about the population of fish. Then you changed between the standard from and the scientific notation at specific points.

After these activities, you completed the following **practice**:

- 5.6.7: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

5.7 ANALYZING GRAPHS

5.7.0 Lesson Overview

In this lesson, you will determine whether situations are characterized by exponential growth or by exponential decay given real-world scenarios of descriptions and graphs.

When you finish this lesson, you will be able to:

- Use graphs to compare and contrast situations that involve exponential decay.
- Use information from a graph to write an equation that represents exponential decay.

Here are the **activities** that will help you reach those goals:

- 5.7.1: Patterns in Real Numbers
- 5.7.2: Comparing Graphs
  - 5.7.2: Self Check
  - 5.7.2: Additional Resources
- 5.7.3: Describing Graphs
  - 5.7.3: Self Check
  - 5.7.3: Additional Resources
- 5.7.4: Quantities that Change Exponentially

After that, you'll **practice and review**.

- 5.7.5: Practice
- 5.7.6: Lesson Summary

5.7.1 Patterns in Real Numbers

Warm Up

In the table, find as many patterns as you can. Use one or more patterns to help you complete the table. Be prepared to explain your reasoning.

fraction	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
decimal	0.5	0.25	0.125	a.____	b.____

1. What is the value for a?

2. What is the value for b?

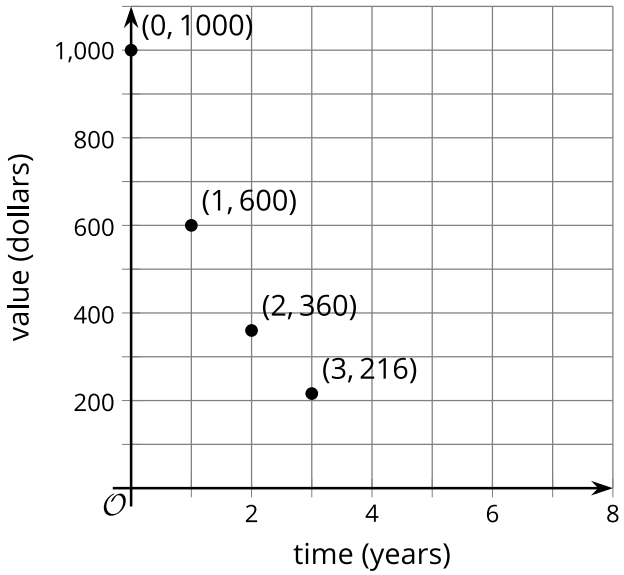
3. Describe the pattern you used to find the values that completed the table.

5.7.2 Comparing Graphs

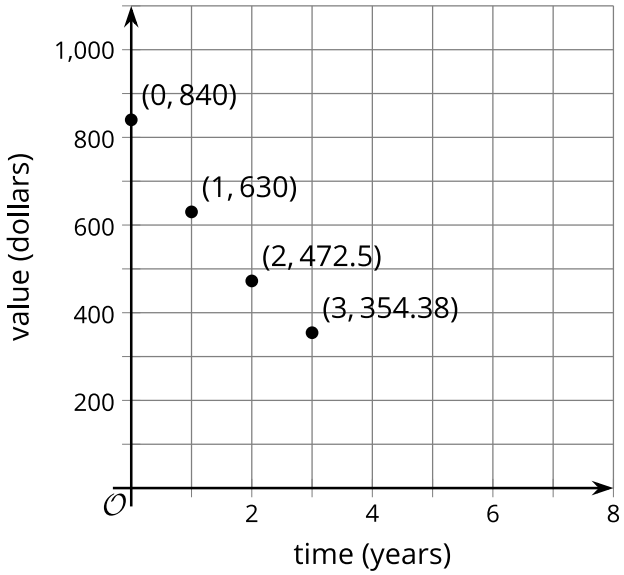
Activity

The value of some cell phones changes exponentially after initial release. Here are graphs showing the depreciation of two phones 1, 2, and 3 years after they were released.

Phone A



Phone B



1. Which phone is more expensive to buy when it is first released?

2. Which graph depicts the value of the cell phone falling in value more quickly? Explain how you know.

3. How does the value of phone A change with every passing year?

4. How does the value of phone B change with every passing year?

5. If the phones continue to depreciate by the same factor each year, what will the value of each phone be 4 years after its initial release?

a. Enter your answer for Phone A.

b. Enter your answer for Phone B.

6. For each cell phone, write an equation that relates the value of the phone in dollars to the years since release, t. Use v for the value of Phone A and w for the value of Phone B.

a. Enter your answer for Phone A.

b. Enter your answer for Phone B.

Are you ready for more?

Extending Your Thinking

When given data, it is not always clear how to best model it. In this case, we were told the value of the cell phones was changing exponentially. Suppose, however, we were instead just given the initial values of the cell phones when released and the values after each of the first three years.

1. Use technology to compute the best fit line for each cell phone. Round any numbers to the nearest dollar.

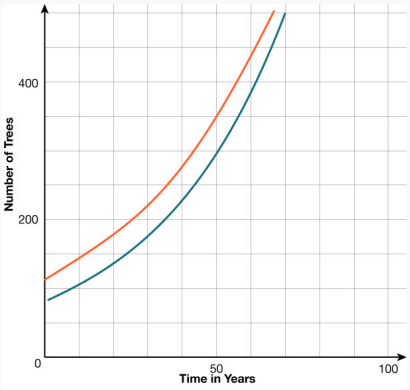
a. Enter your answer for Phone A.

b. Enter your answer for Phone B.

2. Explain why, in this situation, an exponential model might be more appropriate than the linear model you just created.

Self Check

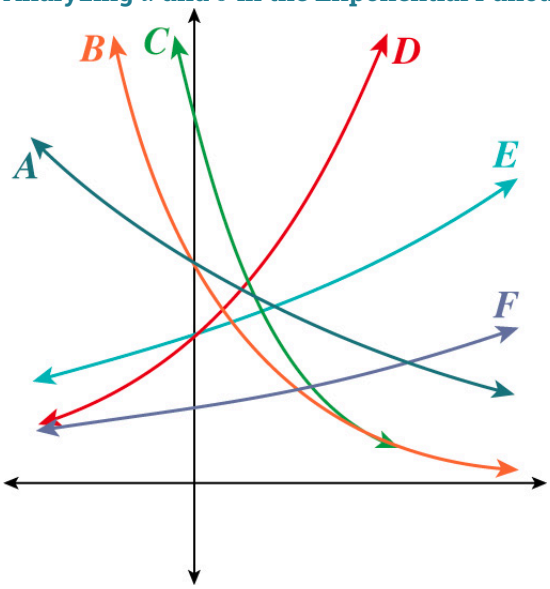
For the following exercise, consider this scenario: For each year t, the population of trees in Forest A is represented by the function $A(t) = 115(1.025)^t$, (red curve). In a neighboring forest, Forest B, the population of the same type of tree is represented by the function $B(t) = 82(1.029)^t$, (blue curve). Looking at the graph provided, which forest had a greater number of trees initially?



a. Forest B, because it has a greater growth rate.
b. Forest A, because it has a greater growth rate.
c. Forest A, because it starts higher on the y-axis.
d. Both forests started with the same number of trees.

Additional Resources

Analyzing a and b in the Exponential Function $y = ab^x$



1. Which graph has the largest value for b?
2. Which graph has the smallest value for b?
3. Which graph has the largest value for a?
4. Which graph has the smallest value for a?

When using the form $y = ab^x$, it is important to remember that a is the initial value, or the y-intercept, and b is the growth factor. The growth factor reflects an exponential growth function when the value is greater than one. The growth factor reflects an exponential decay function when the value is between zero and one.

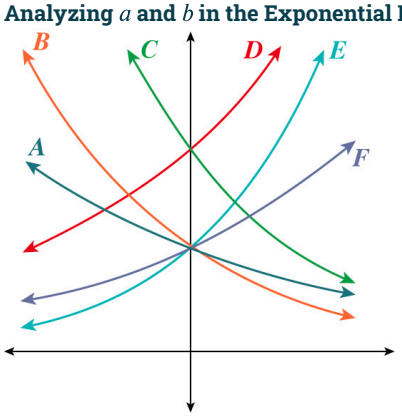
For question 1, you are looking for the largest growth factor, which would mean an exponential growth function that is increasing the most rapidly. First, identify the graphs that reflect exponential growth, which would be D, E, and F, and then decide which is the steepest. Graph D increases much faster than both graphs E and F.

For question 2, you are looking for the steepest rate of exponential decay, so you need the graph that has the smallest b-value. The graphs that reflect exponential decay are graphs A, B, and C. The graph that has a b-value that is closest to 0 is graph C. Graph C has the steepest exponential decay because its b-value is the closest to 0 and, therefore, the smallest.

For question 3, you are looking for the largest initial value or y-intercept. This could be with a growth or decay function; it just needs to cross the y-axis higher than the other graphs. Graph C has the highest a value.

For question 4, you are looking for the smallest initial value or y-intercept. This could be a growth or decay function; it just needs to cross the y-axis lower than the other graphs. Graph F has the lowest a value.

TRY IT **Analyzing a and b in the Exponential Function $y = ab^x$**



1. Which graphs represent exponential growth?
2. Which graphs represent exponential decay?
3. Which graph has the highest value for b?
4. Which graph has the highest value for a?

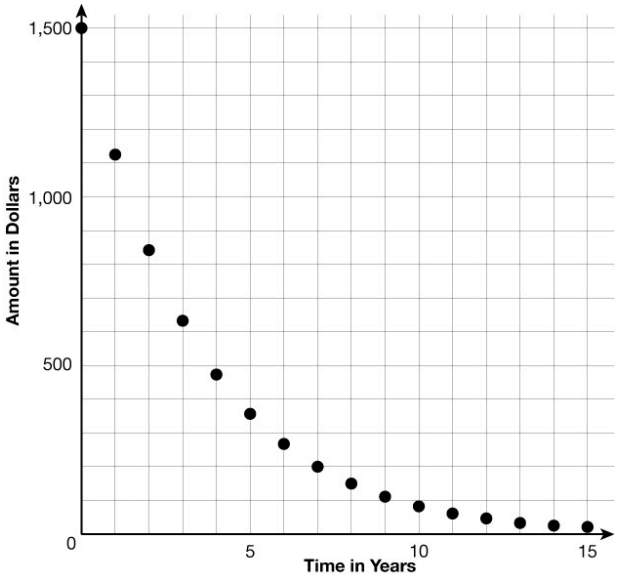
5.7.3 Describing Graphs

Activity

Your teacher will give you a set of cards (https://openstax.org/r/set-cards-5-7-3) containing descriptions of situations and graphs. Match each situation with a graph that represents it. Record your matches and be prepared to explain your reasoning.

Use the cards to answer questions 1 – 4. Match each graph representation to the description of the situation characterized by exponential change. Then choose the relationship between the number of years since purchase and the value.

1. Choose the description of exponential change shown in the graph.



a. The value of a stock doubles approximately every 4 years.

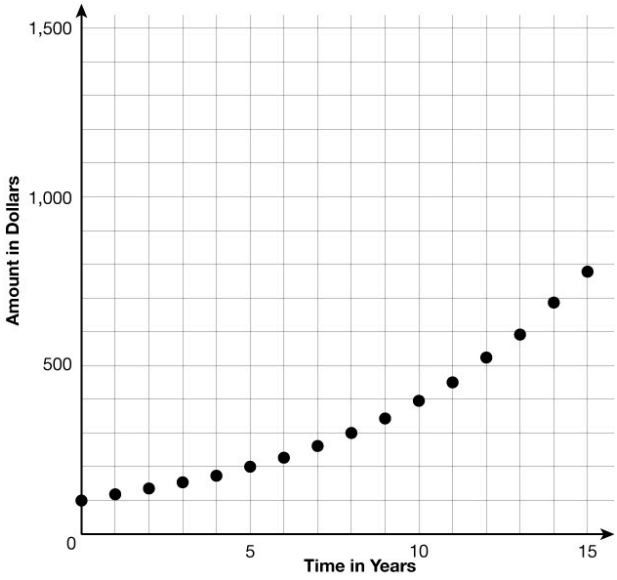
b. A car loses $\frac{1}{4}$ of its value every year after purchase.

c. The value of a stock triples roughly every 8 years.

d. A laptop loses $\frac{2}{5}$ of its value every year after purchase.

2. What is the equation for this graph?

3. Choose the description of exponential change shown in the graph.



a. The value of a stock doubles approximately every 4 years.

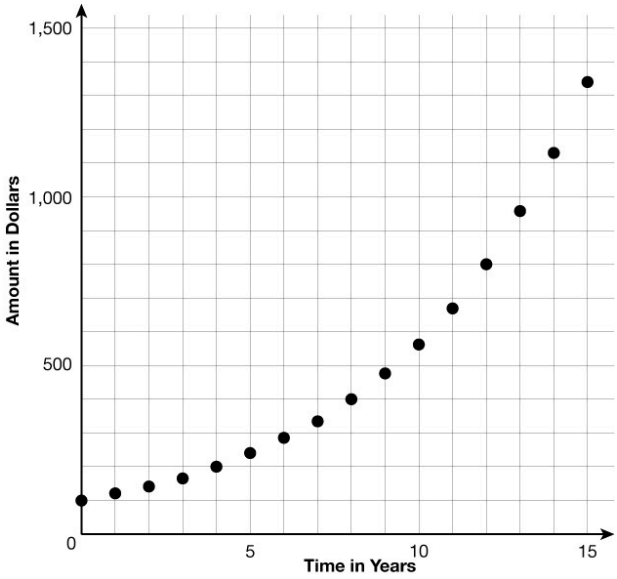
b. A car loses $\frac{1}{4}$ of its value every year after purchase.

c. The value of a stock triples roughly every 8 years.

d. A laptop loses $\frac{2}{5}$ of its value every year after purchase.

4. What is the equation for this graph?

5. Choose the description of exponential change shown in the graph.



a. The value of a stock doubles approximately every 4 years.

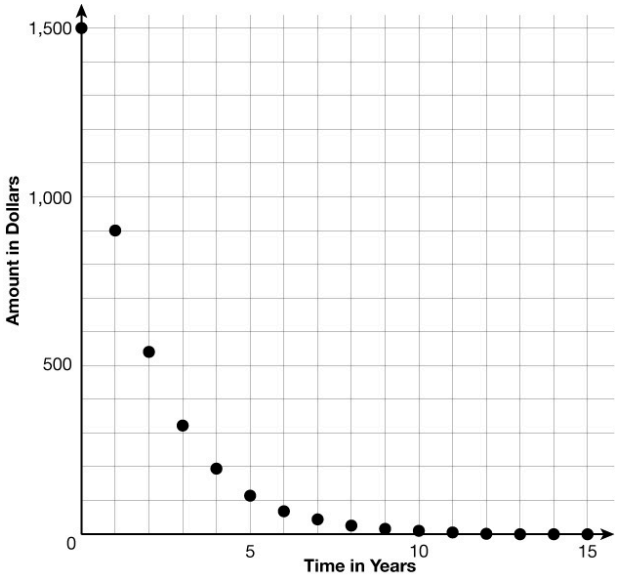
b. A car loses $\frac{1}{4}$ of its value every year after purchase.

c. The value of a stock triples roughly every 8 years.

d. A laptop loses $\frac{2}{5}$ of its value every year after purchase.

6. What is the equation for this graph?

7. Choose the description of exponential change shown in the graph.



a. The value of a stock doubles approximately every 4 years.

b. A car loses $\frac{1}{4}$ of its value every year after purchase.

c. The value of a stock triples roughly every 8 years.

d. A laptop loses $\frac{2}{5}$ of its value every year after purchase.

8. What is the equation for this graph?

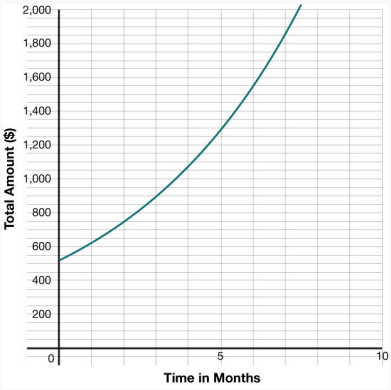
Video: Analyzing Graphs of Exponential Scenarios

Watch the following video to learn more about how to analyze a graph to determine which exponential scenario it represents.

6.7.3 (https://openstax.org/r/which-exponential)

Self Check

Which scenario could be represented by the following graph?



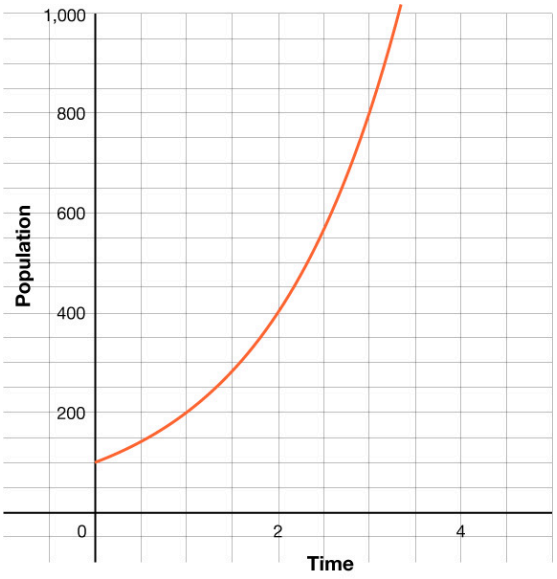
a. A recent graduate's credit card balance started around $500 and decreases at a rate of approximately 20% each month.
b. The population of sea birds started at 518 and is increasing exponentially each month.
c. The population of sea birds started at 518 and is decreasing exponentially each month.
d. A recent graduate's credit card balance started around $500 and increases at a rate of approximately 20% each month.

Additional Resources

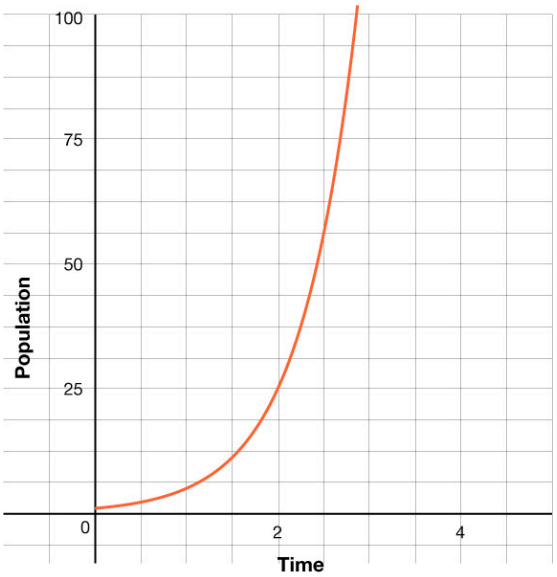
Connecting Exponential Situations and Graphs

Match each graph representation to the description of the situation characterized by exponential change.

Graph 1



Graph 2

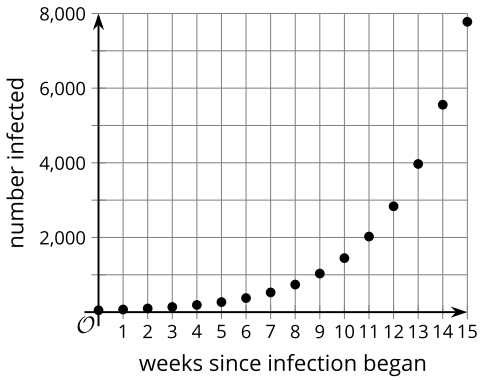


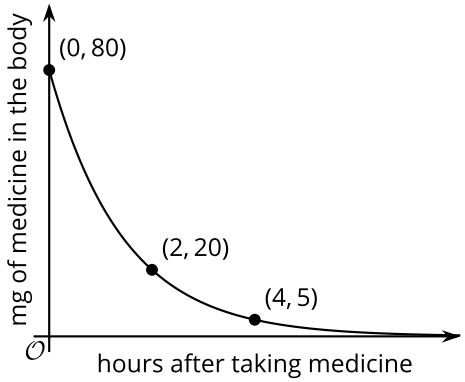
Situation 1: A dangerous bacterial compound forms in a closed environment but is immediately detected. This bacteria is known to double in concentration in a closed environment every hour and can be modeled by the function $P(t) = 100 \cdot 2^t$, where t is measured in hours.

Situation 2: Loggerhead turtles reproduce every 2 to 4 years, laying approximately 120 eggs in a clutch. Using this information, we can derive an approximate equation to model the turtle population. As is often the case in biological studies, we will count only the female turtles. If we start with a population of one female turtle in a protected area and assume that all turtles survive, we can roughly approximate the population of female turtles by $P(t) = 5^t$.

Since each graph represents exponential growth, and they both have the same labels, you have to look more closely at each. The biggest difference is the y-intercept or initial value. Graph 1 has an initial value of 100, and Graph 2 has a very small initial value. When you reread the situations, you can see that Situation 1 starts with 100 bacteria, and Situation 2 starts with 1 female turtle. This helps you see that Situation 1 belongs to Graph 1 and Situation 2 belongs to Graph 2.

TRY IT **Connecting Exponential Situations and Graphs**

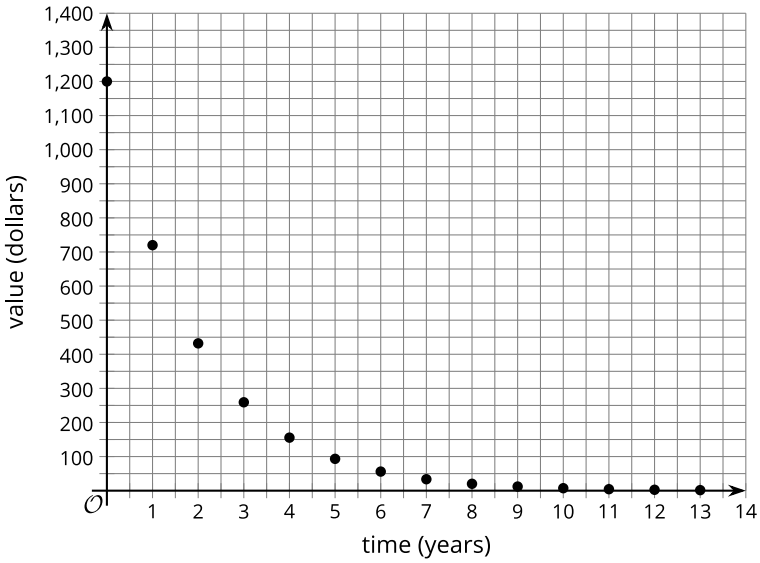




Write a possible situation for each graph. Make sure to include vocabulary about exponential functions.

5.7.4 Quantities that Change Exponentially

Cool Down



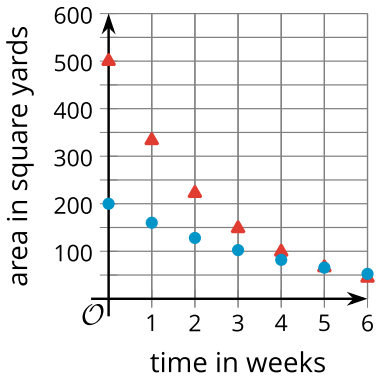
1. This graph represents one of the following descriptions. Which one?
   a. A phone loses $\frac{4}{5}$ of its value every year after purchase; the relationship between the number of years since purchasing the phone and the value of the phone.
   b. The number of stores a company has tripled approximately every 5 years; the relationship between the number of years and the number of stores.
   c. A camera loses $\frac{2}{5}$ of its value every year after purchase; the relationship between the number of years since purchasing the camera and the value of the camera.
2. Explain how you know the graph represents the description you chose.

5.7.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 2, use the following scenario:

The two graphs show models characterized by exponential decay representing the area covered by two different algae blooms, in square yards, w weeks after different chemicals were applied.



1. Which algae bloom covered a larger area when the chemicals were applied?

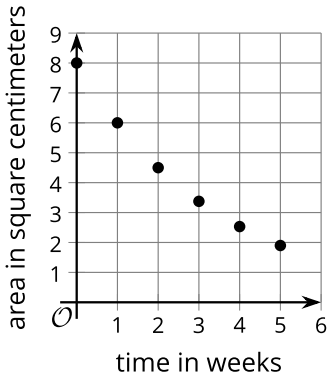
a. The algae bloom represented by the circles because the curve is always below the other one on the graph.
b. Both algae blooms covered the same area after 5 weeks.
c. The algae bloom represented by the circles because the curve has a lower y-intercept.
d. The algae bloom represented by the triangles because the curve has a higher y-intercept.

2. Which algae population is decreasing more rapidly? Explain how you know.

a. The algae bloom represented by the triangles because its graph is steeper.
b. The algae bloom represented by the circles because the curve is always below the other one on the graph.
c. Both algae blooms are decreasing at the same rate.
d. The algae bloom represented by the triangles because the curve has a higher y-intercept.

For questions 3 – 5, use the following scenario:

A medicine is applied to a burn on a patient's arm. The area of the burn in square centimeters decreases exponentially and is shown in the graph.



3. What fraction of the burn area remains each week?

a. $\frac{2}{3}$ or 0.66
b. $\frac{1}{4}$ or 0.25
c. $\frac{3}{4}$ or 0.75

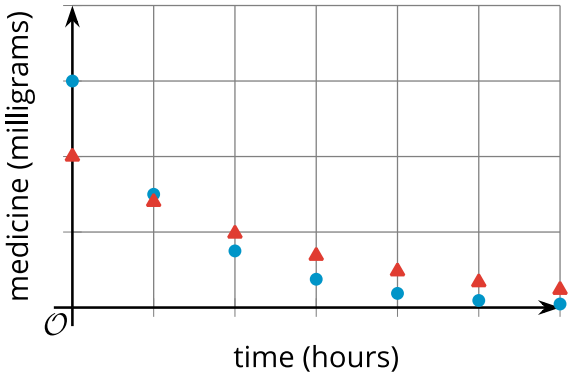
d. $\frac{1}{2}$ or 0.5

4. Which of the following is an equation representing the area of the burn, a, after t weeks?

a. $a = \frac{-6}{5}t + 8$
b. $a = -2t + 8$
c. $a = 8(\frac{3}{4})^t$
d. $a = 8(\frac{1}{4})^t$

5. What is the area of the burn after 7 weeks? Round to three decimal places.

6. The graphs show the amounts of medicine in two patients after receiving injections. The circles show the medicine in patient A, and the triangles show the medicine in patient B.

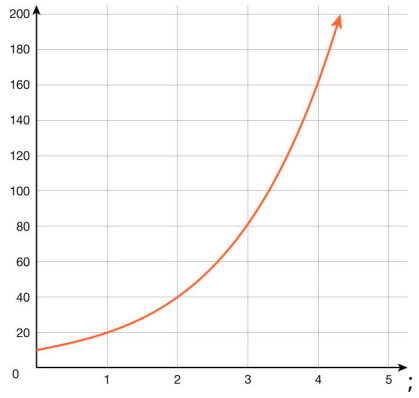


One equation that gives the amount of medicine in milligrams, m, in patient A, h hours after an injection, is $m = 300(\frac{1}{2})^h$.

What could be an equation for the amount of medicine in patient B?

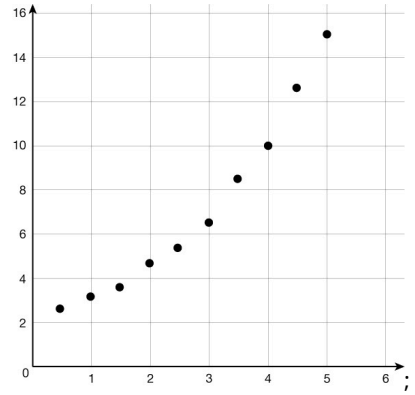
a. $m = 200(\frac{7}{10})^h$
b. $m = 200(\frac{3}{10})^h$
c. $m = 500(\frac{7}{10})^h$
d. $m = 500(\frac{3}{10})^h$

7. Which scenario could describe the graph?



a. A population of bacteria starts with 200 and decays by a factor of $\frac{1}{2}$ every hour.
b. A population of bacteria increases by a factor of 10 every hour and starts with 2 bacteria.
c. A savings account starts with $200 and decays by a factor of $\frac{1}{2}$ each month.
d. A population of bacteria doubles every hour and starts with 10 bacteria.

8. Which equation could represent the graph?



a. $y = 0.479(2.104)^x$
b. $y = 2.104(0.479)^x$
c. $y = 2.5x + 2$
d. $y = 2.104(1.479)^x$

9. Given a formula for an exponential function, is it possible to determine whether the function grows or decays exponentially just by looking at the formula? Explain.

a. No, it is not possible to determine growth and decay from the formula.
b. Yes, if the growth rate is greater than one it's growth; if the growth rate is, less than zero it's decay.
c. Yes, if the growth rate is between zero and one it's decay; if the growth rate is greater than one it is growth.
d. Yes, if the initial value is greater than one it's growth; if the initial value is less than one it's decay.

5.7.6 Lesson Summary

In this lesson, you learned how to:

- Use graphs to compare and contrast situations that involve exponential decay.
- Use information from a graph to write an equation that represents exponential decay.

Here are the **activities** that helped you reach those goals:

- 5.7.1: Patterns in Real Numbers
  - In this activity, you analyzed all the possible patterns connecting a set of numbers in a table. The numbers were written in both decimal and fraction form, and you may have noticed different patterns in each format.
- 5.7.2: Comparing Graphs
  - In this activity, you looked at the graphs of the values of different cell phones over time. You were able to analyze each cell phone and compare them to one another by looking at key features of the graphs.
  - 5.7.2: Self Check
  - 5.7.2: Additional Resources
- 5.7.3: Describing Graphs
  - In this activity, you connected the verbal description of a graph to the matching graph. Each description was a real-life scenario, and all of the graphs were exponential. You had to consider words that represent different growth rates and whether the situation was growth or decay.
  - 5.7.3: Self Check
  - 5.7.3: Additional Resources
- 5.7.4: Quantities that Change Exponentially
  - In this activity, you analyzed a graph and determined which of the scenarios provided it matches. It was a wrap up of all of the other skills from this lesson.

After these activities, you completed the following **practice**:

- 5.7.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

5.8 EXPONENTIAL SITUATIONS AS FUNCTIONS

5.8.0 Lesson Overview

In this lesson, you start to view situations characterized by exponential change using descriptions, tables, and graphs as exponential functions. This means choosing an independent and dependent variable and expressing the relationships using function language and, in some cases, function notation. You use variables to represent one real-world quantity as a function of another, which is an example of decontextualizing and reasoning abstractly.

When you finish this lesson, you will be able to:

- Use function notation to write equations that represent exponential relationships.
- Determine whether the relationships in descriptions, tables, equations, or graphs are functions.

Here are the **activities** that will help you reach those goals:

- 5.8.1: Meaning of a Function Presented Graphically
- 5.8.2: Represent a Function as a Table of Values, Graph, and Equation
  - 5.8.2: Self Check
  - 5.8.2: Additional Resources
- 5.8.3: Using Function Language and Notation
  - 5.8.3: Self Check
  - 5.8.3: Additional Resources
- 5.8.4: Function Notation

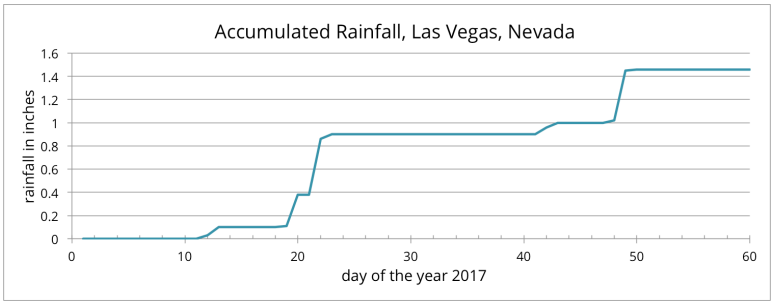
After that, you'll **practice and review**.

- 5.8.5: Practice
- 5.8.6: Lesson Summary

5.8.1 Meaning of a Function Presented Graphically

Warm Up

Here is a graph of the accumulated rainfall in Las Vegas, Nevada, in the first 60 days of 2017.



Use the graph to support your answers to the following questions.

1. Is the accumulated amount of rainfall a function of time?

2. Is time a function of accumulated rainfall?

5.8.2 Represent a Function as a Table of Values, Graph, and Equation

Activity

Clare noticed mold on the last slice of bread in a plastic bag. The area covered by the mold was about 1 square millimeter. She left the bread alone to see how the mold would grow. The next day, the area covered by the mold had

doubled, and it doubled again the day after that.

1. If the doubling pattern continues, how many square millimeters will the mold cover 4 days after she noticed the mold? Be prepared to show your reasoning.
2. Represent the relationship between the area A, in square millimeters, covered by the mold and the number of days d since the mold was spotted using:

a. A table of values, showing the values from the day the mold was spotted through 5 days later.

b. An equation

c. A graph

3. Discuss with your partner: Is the relationship between the area covered by mold and the number of days a function? If so, write ____ is a function of ____. If not, explain why it is not.

Are you ready for more?

Extending Your Thinking

What do you think is an appropriate domain for the mold area function A? Be prepared to show your reasoning.

Self Check

Which of the following equations represents the values in the table?

x	0	1	2	3
y	1	3	9	27

a. $y = 3x + 1$
b. $y = 1 \cdot 3^x$
c. $y = 3 \cdot 1^x$
d. $y = 3x$

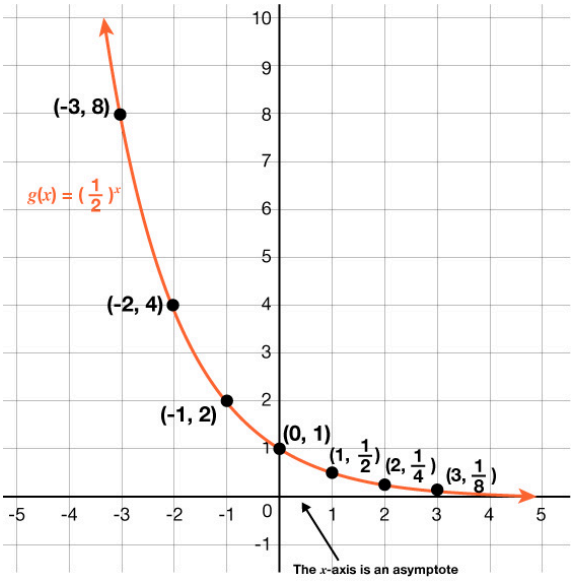
Additional Resources

Different Exponential Representations

A table of values is shown below for a function $g(x) = (\frac{1}{2})^x$, which represents exponential decay.

x	**–3**	**–2**	**–1**	**0**	**1**	**2**	**3**
$g(x) = (\frac{1}{2})^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

When these points are graphed, they show the decay because the function graphed decreases from left to right:



TRY IT **Try It: Different Exponential Representations**

Make a table for the function $f(x) = 4^x$. Then graph the function.

5.8.3 Using Function Language and Notation

Activity

An exponential function is written in the form $f(x) = ab^x$, where a is the initial value and b is the growth factor.

1. In a biology lab, a population of 50 bacteria reproduce by splitting. Every hour, on the hour, each bacterium splits into two bacteria.

a. Write a sentence of the form "_____ is a function of _____."

b. Indicate which is the independent and which is the dependent variable.

c. Write an equation that represents the situation using function notation.

2. Every year after a new car is purchased, it loses $\frac{1}{3}$ of its value. Let's say that the new car costs $18,000.

a. Write a sentence of the form "_____ is a function of _____."

b. Indicate which is the independent and which is the dependent variable.

c. Write an equation that represents the situation using function notation.

3. To control an algae bloom in a lake, scientists introduce some treatment products. The day they begin treatment, the area covered by algae is 240 square yards. Each day since the treatment began, a third of the previous day's area (in square yards) remains covered by algae. Time t is measured in days.

a. Write a sentence of the form "_____ is a function of _____."

b. Indicate which is the independent and which is the dependent variable.

c. Write an equation that represents the situation using function notation.

Video: Writing an Exponential Function from a Situation

Watch the following video to learn more about exponential functions:

Writing an Exponential Function from a Situation (https://openstax.org/r/exponential-function)

Self Check

A new computer is purchased for $2,500. Each year after its purchase, it loses $\frac{1}{5}$ of its value. Which equation represents this situation?

a. $f(t) = \frac{4}{5} \cdot (2500)^t$
b. $f(t) = 2500 \cdot (\frac{4}{5})^t$
c. $f(t) = 2500 \cdot (\frac{4}{5})t$
d. $f(t) = 2500 + (\frac{4}{5})^t$

Additional Resources

Writing Functions from Exponential Situations

EXPONENTIAL GROWTH

A function that models exponential growth grows by a rate proportional to the amount present. For any real number x and any positive real numbers a and b such that $b \neq 1$, an exponential growth function has the form

$f(x) = ab^x$

where

- a is the initial or starting value of the function.
- b is the growth factor or growth multiplier per unit x.

EXAMPLE 1

The starting value of the function is 10. This function doubles every year.

Growth factor function	$f(x) = ab^x$
Identify the initial value.	$a = 10$
Identify the growth factor.	$b = 2$
Identify the unit.	$x = t$ for time in years.
Write the exponential equation.	$f(y) = 10 \cdot 2^t$

EXAMPLE 2

An acre of land is worth $40,000 at the time of purchase. It doubles in value every decade. Write an equation for this situation where t is time in decades.

Identify the initial value.	$a = 40,000$
Identify the growth factor.	$b = 2$
Write the exponential equation.	$f(t) = 40,000 \cdot 2^t$

TRY IT **Writing Functions from Exponential Situations**

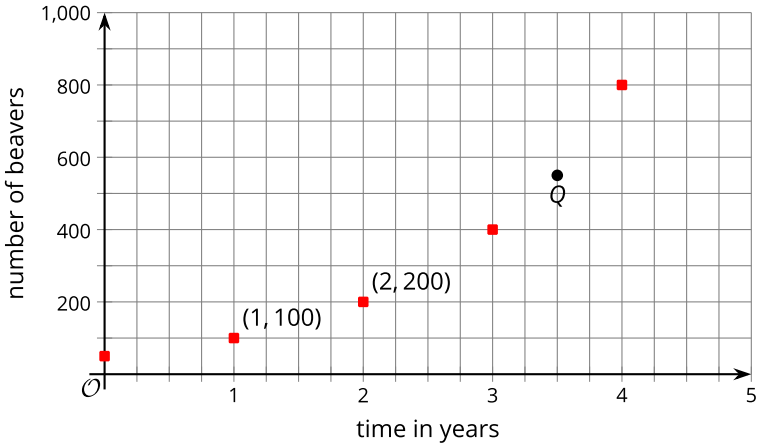
1. The starting value of the function is 150. This function triples every decade

2. When first measured, the population of deer in a forest is 40. Over time, the population triples every year. Write an exponential equation where t is time in years.

5.8.4 Function Notation

Cool Down

For questions 1 - 3, the graph shows the population of beavers in a forest for different numbers of years after 1995. The beaver population is growing exponentially.



1. Explain why we can think of the beaver population as a function of time in years.

2. What is the meaning of the point labeled Q in this context?

3. Write an equation using function notation to represent this situation.

4. Which equation is most appropriate for modeling this data?

x	1	2	3	4	5
y	14	26	74	266	1034

a. $y = 10 + (4x)^2$

b. $y = 10 + 4^x$

c. $y = 10 + 4x^2$

d. $y = 10 + 4^{\frac{1}{x}}$

5.8.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. For an experiment, a scientist designs a can, 20 cm in height, that holds water. A tube is installed at the bottom of the can allowing water to drain out. At the beginning of the experiment, the can is full. Every minute after the start of the experiment, $\frac{2}{3}$ of the water is drained.

The height of the water, h, in cm, is a function f of time t in minutes since the beginning of the experiment, $h = f(t)$. Which is an expression for $f(t)$?

a. $f(t) = 20 + \frac{2}{3}t$
b. $f(t) = 20(\frac{2}{3})^t$
c. $f(t) = 20(\frac{1}{3})^t$
d. $f(t) = 20 + \frac{1}{3}t$

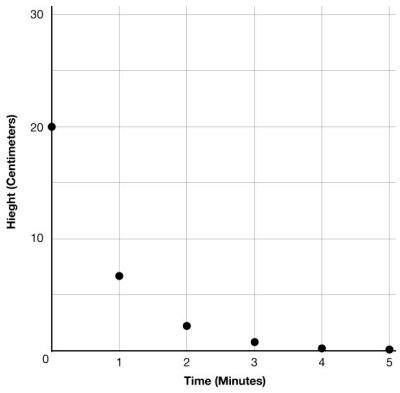
2. A bacteria population is 10,000. It triples each day.
The bacteria population, b, is a function of the number of days, d. Which equation relates b and d?

a. $b = 10000 + 3^d$
b. $b = 10000(\frac{1}{3})^d$
c. $b = 10000 \cdot 3d$
d. $b = 10000 \cdot 3^d$

3. The area, a, covered by a city is 20 square miles. The area grows by a factor of 1.1 each year, t, since it was 20 square miles. Which equation expresses a in terms of t?

a. $a = 20 - 1.1^t$
b. $a = 20 \cdot 1.1^t$
c. $a = 20 \cdot 1.1t$
d. $a = 20 + 1.1^t$

4. The graph below models the water draining out of a tube during an experiment represented by the function, $f(t) = 20(\frac{1}{3})^t$. What does the value $(2, \frac{20}{9})$ mean in the context of the problem?



a. After $\frac{20}{9}$ minutes, the height of the water is 2 centimeters less than at the start.
b. After 2 minutes, the height of the water is $\frac{20}{9}$ centimeters.
c. After $\frac{20}{9}$ minutes, the height of the water is 2 centimeters.
d. After 2 minutes, the height of the water is $\frac{20}{9}$ centimeters less than at the start.

5. A scientist measures the height, h, of a tree each month, and m is the number of months since the scientist first measured the height of the tree. Is the height, h, a function of the month, m?

a. Yes
b. No

6. Cary started with 3 goldfish in her tank, and the population doubled every month. Which is an equation that represents the population, P, of goldfish after m months?

a. $P = 2 \cdot 3^m$
b. $P = 3 \cdot 2^m$
c. $P = 3(2m)$
d. $P = 3 + 2m$

Use this equation for problems 7 – 10: $f(t) = 50 \cdot 2^t$

7. What is the initial value?

8. What is the growth factor?

9. What is the independent variable?

a. $f(t)$
b. t

10. What is the dependent variable?

a. $f(t)$
b. t

11. Which equation is most appropriate for modeling this data?

x	1	2	3	4	5	6
y	50	100	200	400	800	1600

a. $y = 50 \cdot 2^{\frac{1}{x}}$
b. $y = 25 \cdot x^2$
c. $y = 25 \cdot 2^x$
d. $y = 25 \cdot (2x)^2$

5.8.6 Lesson Summary

In this lesson, you learned how to:

- Use function notation to write equations that represent exponential relationships.
- Determine whether the relationships in descriptions, tables, equations, or graphs are functions.

Here are the **activities** that helped you reach those goals:

- 5.8.1: Meaning of a Function Presented Graphically
  - In this activity, you reviewed the meaning of a function presented graphically. Interpreting the graph in terms of the context helped prepare you for your work with functions in the rest of the unit.
- 5.8.2: Represent a Function as a Table of Values, Graph, and Equation
  - In this activity, you represented a situation using a table of values, a graph, and an equation.
  - 5.8.2: Self Check
  - 5.8.2: Additional Resources
- 5.8.3: Using Function Language and Notation
  - In this activity, you revisited describing and analyzing situations involving exponential change, using graphs, tables, and equations, viewing them as functions and expressing them using function language and notation.
  - 5.8.3: Self Check
  - 5.8.3: Additional Resources
- 5.8.4: Function Notation
  - In this activity, you considered a situation as a function of time in years, determined the meaning of a point in this context, and wrote an equation using function notation to represent the situation.

After these activities, you completed the following **practice**:

- 5.8.5: Practice
- 5.8.6: Lesson Summary

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

5.9 INTERPRETING EXPONENTIAL FUNCTIONS

5.9.0 Lesson Overview

In this lesson, you will study exponential functions and their graphs in context. Given a graph or a description of a relationship, you write one quantity as a function of another and then use the function to answer questions about the context. You will also produce graphs of exponential functions, paying close attention to the appropriate domain and range.

When you finish this lesson, you will be able to:

- Analyze a situation and determine whether it makes sense to connect the points on the graph that represents the situation.
- Describe the relationship in a graph of an exponential function using function notation.

Here are the **activities** that will help you reach those goals:

- 5.9.1: Evaluating Expressions for Different Values of x
- 5.9.2: Analyze Underlying Relationship in a Graph of a Function
  - 5.9.2: Self Check
  - 5.9.2: Additional Resources
- 5.9.3: Graph Equations to Solve Problem
  - 5.9.3: Self Check
  - 5.9.3: Additional Resources
- 5.9.4: Deciding on Graphing Window
- 5.9.5: Interpret Graphs to Find Approximate Values

After that, you'll **practice and review**.

- 5.9.6: Practice
- 5.9.7: Lesson Summary

5.9.1 Evaluating Expressions for Different Values of x

Warm Up

Lin and Diego are discussing two expressions: x^2 and 2^x.

- Lin says, "I think the two expressions are equivalent."
- Diego says, "I think the two expressions are only equal for some values of x."

1. Do you agree with either of them? Explain or show your reasoning.

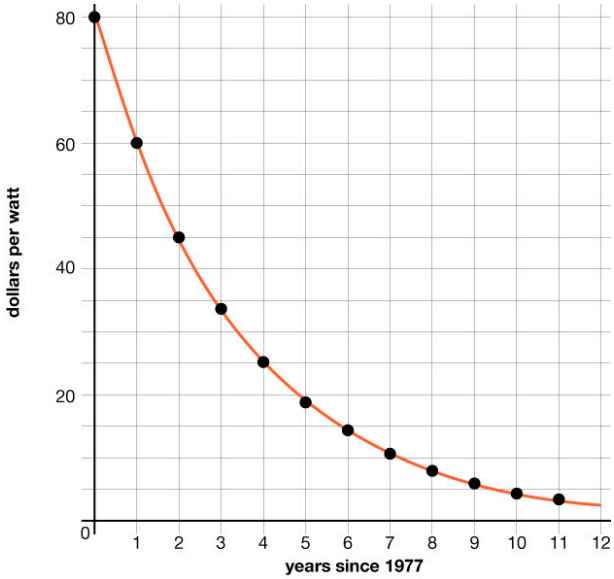
5.9.2 Analyze Underlying Relationship in a Graph of a Function

Activity

The cost, in dollars, to produce 1 watt of solar energy is a function of the number of years since 1977, t.

From 1977 to 1987, the cost could be modeled by an exponential function f. Here is the graph of the function.





1. What is the statement $f(9) \approx 6$ saying about this situation?
2. What is $f(4)$? What about $f(3.5)$? What do these values represent in this context?
3. When $f(t) = 45$, what is t? What does that value of t represent in this context?
4. By what factor did the cost of 1 watt of solar energy change each year? (If you get stuck, consider creating a table.)

Video: Understanding Characteristics of Graphs

Watch the following video to learn more about the characteristics of graphs:

Understanding Characteristics of Graphs (https://openstax.org/r/graph-characteristics)

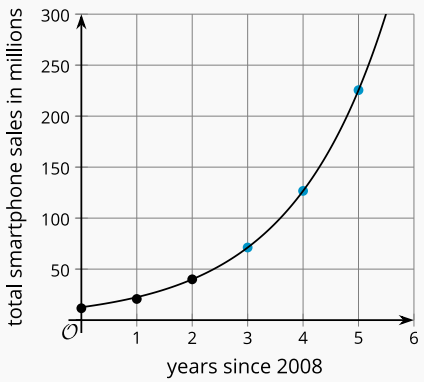
An imaginary line that a function value will not touch or cross is called an **asymptote**.

The value of the range that the function will never touch is called the **horizontal asymptote**.

In the function graphed, the horizontal asymptote is $y = 0$.

Self Check

The graph below shows the total smartphones, s, in millions that were sold in the years, t, since 2008. What does the point $(4, 125)$ represent in this context?

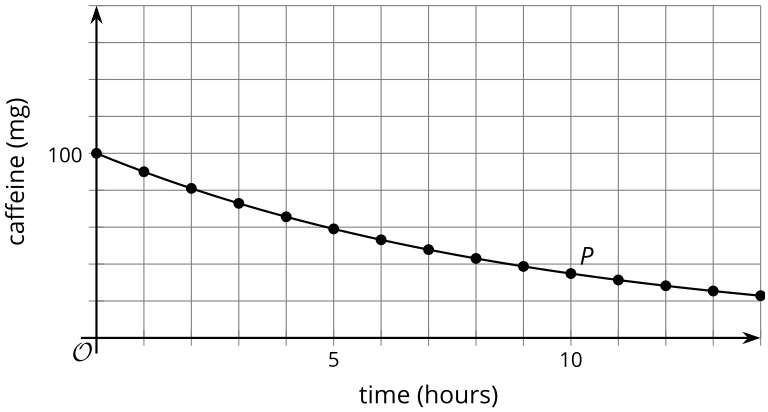


a. In 2004, there were 125 million more smartphones than in 2000.
b. In 2012, there were 125 million smartphones sold.
c. In 2012, there were 125 million more smartphones than in 2008.
d. In 2004, there were 125 million smartphones sold.

Additional Resources

Reading Exponential Graphs

The graph below describes the amount of caffeine, c, in a person's body t hours after an initial measurement of 100 mg.



What does the point $c(10) = 35$ mean?

After 10 hours, there is 35 mg of caffeine in a person's body.

What is the value of t when $c(t) = 80$?

When the caffeine concentration is 80 mg, time is 2 hours, so $t = 2$.

TRY IT **Reading Exponential Graphs**

Using the graph above, what is the value of $c(5)$, and what does it mean in the context of the situation?

5.9.3 Graph Equations to Solve Problems

Activity

1. The thickness t in millimeters of a folded sheet of paper after it is folded n times is given by the equation $t = (0.05) \cdot 2^n$.

a. What does the number 0.05 represent in the equation?

b. What does the number 2 represent in the equation?

c. Use graphing technology to graph the equation $t = (0.05) \cdot 2^n$.

Access multimedia content (http://openstax.org/books/algebra-1/pages/5-9-3-graph-equations-to-solve-problems)
d. How many folds does it take before the folded sheet of paper is more than 1 mm thick? How many folds before it is more than 1 cm thick? Explain how you know.

2. The area of a sheet of paper is 93.5 square inches.

a. Find the remaining visible area of the sheet of paper after it is folded in half once, twice, and three times.

b. Write an equation expressing the visible area a of the sheet of paper in terms of the number of times it has been folded, n.

c. Use graphing technology to graph the equation.

Access multimedia content (http://openstax.org/books/algebra-1/pages/5-9-3-graph-equations-to-solve-problems)
d. In this context, can n take negative values? Be prepared to show your reasoning.

e. Can a take negative values? Be prepared to show your reasoning.

f. Identify the domain of the graph using an inequality.

g. Identify the range of the graph using an inequality.

h. What equation represents the horizontal asymptote?

3. Use the equation to make a prediction of the thickness with 7 folds.

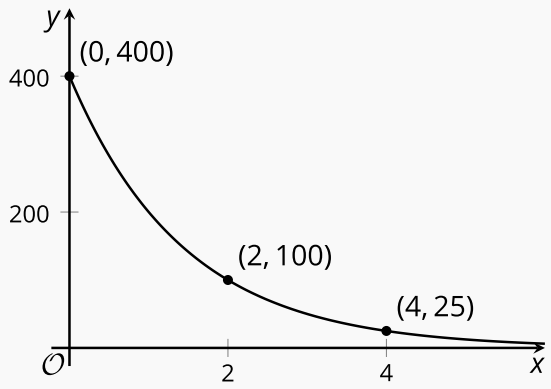
4. The y-intercept represents the initial thickness of the paper t before folding (when n is zero). Calculate the value of the y-intercept.

5. The asymptote represents the possible limit of thickness of the paper as it is folded. What does this mean?

Self Check

The graph below represents the function that gives the value of a computer in dollars, y, as a function of time, x, measured in years since the time of purchase.

What is the range for this graph and situation?



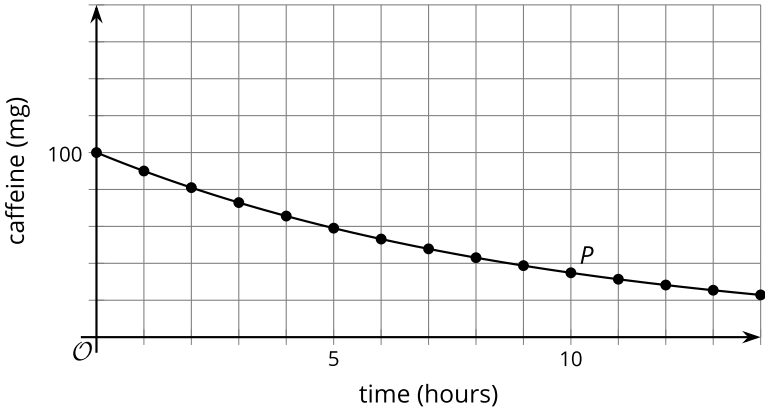
a. $y \geq 0$
b. \(0<y 400\)<br="" \leq=""></y>
c. $x \geq 0$
d. $y \leq 400$

Additional Resources

Domain and Range of Exponential Graphs

The graph below describes the amount of caffeine, c, in a person's body t hours after an initial measurement of 100 mg.

The equation of this curve is $y = 100(1 - 0.1)^t$.



What is the smallest value c can reach?

Since c is the amount of caffeine, it can get close to 0.

Can t ever take on negative values?

No, time will not be negative in this relationship because time is moving forward.

What are the lowest and highest values for t?

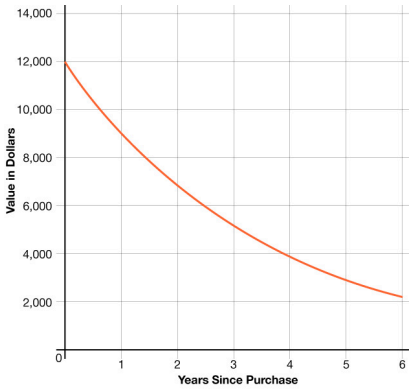
At the beginning, $t = 0$, and t can go as long as it takes for c to get so close to 0 it is insignificant.

The domain (or values of t) written as an inequality is $t \geq 0$.

The range (or values of c) written as an inequality is $0 < c \leq 100$.

TRY IT **Try It: Domain and Range of Exponential Graphs**

The dollar value of a car is a function, f, of the number of years, t, since the car was purchased. The car was purchased at $12,000. Tell the domain and range. The equation of this curve is $y = 12,000(1 - 0.25)^t$.



5.9.4 Deciding on Graphing Window

Activity

The equation $m = 20 \cdot (0.8)^h$ models the amount of medicine m (in milligrams) in a patient's body as a function of hours, h, after injection.

1. Without using a graphing tool, decide if the following horizontal and vertical boundaries are suitable for graphing this function.

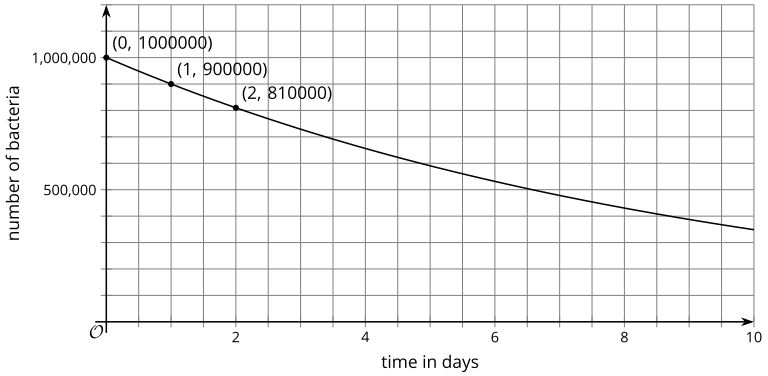
Be prepared to show your reasoning. $-10 < h < 100$ $-100 < m < 1,000$

2. Verify your answer by graphing the equation using graphing technology and using the given graphing window. What do you see? Sketch or describe the graph.
3. If your graph in the previous question is unhelpful, modify the window settings so that the graph is more useful. Record the window settings here. Convince a partner why the horizontal and vertical boundaries that you set are better.

5.9.5 Interpret Graphs to Find Approximate Values

Cool Down

The graph shows the bacteria population on a petri dish as a function f of the days d since an antibiotic is introduced.



1. What is the approximate value of $f(4.5)$?
2. Approximately what is d when $f(d) = 400,000$?
3. Explain what you would do, using your usual graphing technology, to be able to see $f(15)$ on the graph.

Why Should I Care?



Exponential functions and models can help you understand the power of investment.

Let’s say you invest $1000 in a stock that grows 10% every year. By the end of the first year, that money will grow and you will have $1100. The next year, the investment will grow again, based on the new amount. Your investment has compounded.

Now, imagine that money after 20 years: $7330!

Can you think of any other areas of your life where exponential functions and models can help you?

5.9.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

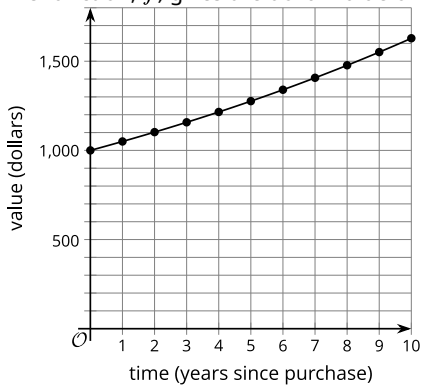
1. The number of people with the flu during an epidemic is a function, f, of the number of days, d, since the epidemic began. The equation $f(d) = 50 \cdot (\frac{3}{2})^d$ defines f.

a. How many people had the flu at the beginning of the epidemic?

b. What does $f(1)$ mean in this situation?

a. The number of people infected after 1 year.
b. The number of people infected grows by a factor of 1.
c. The number of people infected 1 day after the epidemic began.
d. There was 1 year of the epidemic.

2. The function, f, gives the dollar value of a bond t years after the bond was purchased. The graph of f is shown.



a. What is $f(0)$?

b. When is $f(t) = 1500$? Estimate to the nearest whole number.

3. A function f gives the number of stray cats in a town t years since the town started an animal control program. The program includes both sterilizing stray cats and finding homes to adopt them. An equation representing f is $f(t) = 243(\frac{1}{3})^t$.

a. What is the value of $f(t)$ when t is 0?

b. What is the value of $f(t)$ when t is 1?

4. Function g gives the amount of a chemical in a person's body, in milligrams, t hours since the patient took the drug. The equation $g(t) = 600 \cdot (\frac{3}{5})^t$ defines this function.

a. Which is the best choice for the domain of g?

a. $0 \leq t \leq 600$
b. $t \leq 600$
c. $t \geq 600$
d. $t \geq 0$

b. Which is the best choice for the range of g?

a. $0 < g \leq 600$
b. $g \leq 600$
c. $g \geq 600$
d. $g > 0$

5.9.7 Lesson Summary

In this lesson, you learned how to:

- Analyze a situation and determine whether it makes sense to connect the points on the graph that represents the situation.
- Describe the relationship in a graph of an exponential function using function notation.

Here are the **activities** that helped you reach those goals:

- 5.9.1: Evaluating Expressions for Different Values of x
  - In this activity, you addressed a common confusion of mistaking the expression 2^x for the expression x^2.
- 5.9.2: Analyze Underlying Relationship in a Graph of a Function
  - In this activity, you were given a graph of a function, analyzed the underlying relationship, and described it using function notation.
  - 5.9.2: Self Check
  - 5.9.2: Additional Resources
- 5.9.3: Graph Equations to Solve Problems
  - In this activity, you graphed equations to solve problems and thought about appropriate values of their variables.
  - 5.9.3: Self Check
  - 5.9.3: Additional Resources
- 5.9.4: Deciding on Graphing Window
  - In this activity, you were given an additional opportunity to apply what you know about key characteristics of an exponential function and used it to model real-world data.
- 5.9.5: Interpret Graphs to Find Approximate Values
  - In this activity, you interpreted graphs to determine an approximate value of a function.

After these activities, you completed the following **practice**:

- 5.9.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

5.10 LOOKING AT RATES OF CHANGE

5.10.0 Lesson Overview

In this lesson, you will revisit the idea of average rate of change and apply it to **exponential functions**. While rate of change has an unambiguous meaning for linear functions, for nonlinear functions rates of change are not constant, so an interval must be specified.

In this lesson, you are first invited to recall how to calculate an average rate of change for a specific interval for a function represented by a data table. You then consider the average rate of change for an exponential function describing the increase in the number of coffee shops for a business. By comparing the average rate of change of different intervals that all start at the same point, you will observe that while the average rate of change can describe how the data are changing with reasonable accuracy over some intervals, it is not a good predictor over larger intervals since exponential functions do not have a constant rate of change. In the last activity, you use the average rate of change to describe how the cost of producing 1 watt from a solar cell decreases in two separate intervals, which illuminates the nature of exponential decay to decrease at a slower rate. Contexts are used throughout this lesson to give you an opportunity to reason abstractly and quantitatively.

When you finish this lesson, you will be able to:

- Calculate the average rate of change of a function over a specified period of time.
- Recognize how the average rate of change of an exponential function differs from that of a linear function.

Here are the activities that will help you reach those goals:

- 5.10.1: Calculate an Average Rate of Change from Two Points
- 5.10.2: Explore Average Rates of Change in an Exponential Growth Function
  - 5.10.2: Self Check
  - 5.10.2: Additional Resources
- 5.10.3: Explore Average Rates of Change in an Exponential Decay Context
  - 5.10.3: Self Check
  - 5.10.3: Additional Resources
- 5.10.4: Average Rate of Change

After that, you'll **practice and review**.

- 5.10.5: Practice
- 5.10.6: Lesson Summary

5.10.1 Calculate an Average Rate of Change from Two Points

Warm Up

Let p be the function that gives the cost $p(t)$, in dollars, of producing 1 watt of solar energy t years after 1977. Here is a table showing the values of p from 1977 to 1987.

t	$p(t)$
0	80
1	60
2	45
3	33.75
4	25.31

t	$p(t)$
5	18.98
6	14.24
7	10.68
8	8.01
9	6.01
10	4.51

Which expression best represents the average rate of change in solar cost between 1977 and 1987?

a. $p(10) - p(0)$

b. $p(10)$

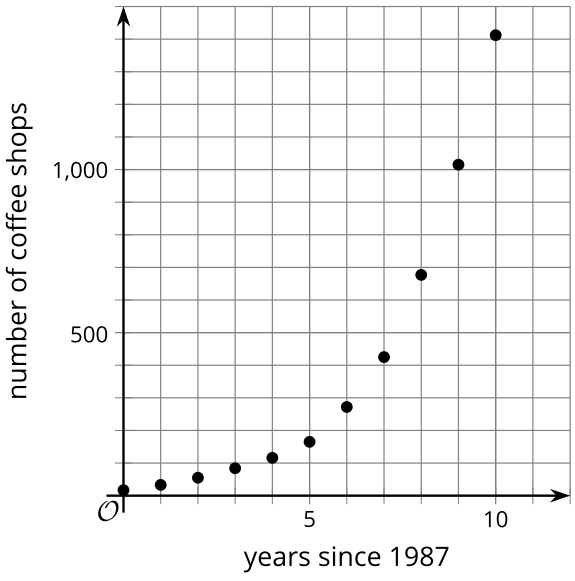
c. $\frac{p(10)-p(0)}{10-0}$

d. $\frac{p(10)}{p(0)}$

5.10.2 Explore Average Rates of Change in an Exponential Growth Function

Activity

Here is a table and a graph that show the number of coffee shops worldwide that a company had in its first 10 years, between 1987 and 1997. The growth in the number of shops was roughly exponential.



Year	Number Of Shops
1987	17
1988	33
1989	55
1990	84
1991	116
1992	165
1993	272
1994	425
1995	677
1996	1,015
1997	1,412

1. Find the average rate of change for each period of time. Show your reasoning.

a. 1987 to 1990

b. 1987 to 1993

c. 1987 to 1997

2. Make some observations about the rates of change you calculated. What do these average rates tell us about how the company was growing during this time period?

3. Use the graph to support your answers to these questions. How well do the average rates of change describe the growth of the company in:

a. the first 3 years?

b. the first 6 years?

c. the entire 10 years?

4. Let f be the function so that $f(t)$ represents the number of stores t years since 1987. The value of $f(20)$ is 15,011. Find $\frac{f(20)-f(10)}{20-10}$ and say what it tells us about the change in the number of stores.

Video: Analyzing Average Rate of Change of an Exponential Function

Watch the following video to learn more about the average rate of change of exponential functions:

Analyzing Average Rate of Change of an Exponential Function (https://openstax.org/r/exponential-rate-change)

Self Check

Suppose a population of cells starts at 500 and triples every day. The number of cells each day can be calculated as follows:

Number Of Days	Number Of Cells
0	500
1	1,500
2	4,500
3	13,500

Find the average rate of change from day 0 to day 2.

a. -2,000
b. 2,000
c. 4,500
d. 1,000

Additional Resources

Average Rate of Change from a Table

Let's look at an exponential function we studied earlier. Let $A(t)$ be the function that models the area A, in square yards, of algae covering a pond t weeks after beginning treatment to control the algae bloom. Here is a table showing approximately how many square yards of algae remain during the first 5 weeks of treatment.

t	$A(t)$
0	240
1	80
2	27
3	9
4	3

The average rate of change of A from the start of treatment to week 2 is about –107 square yards per week since $\frac{A(2)-A(0)}{2-0} \approx -107$. The average rate of change of A from week 2 to week 4, however, is only about –12 square yards per week since $\frac{A(4)-A(2)}{4-2} \approx -12$.

TRY IT **Try It: Average Rate of Change from a Table**

Using the same table from above, what is the average rate of change of A from week 1 to week 3?

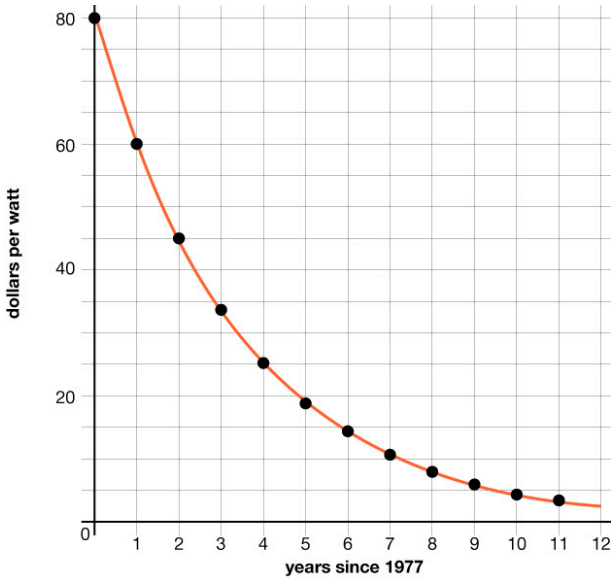
t	$A(t)$
0	240
1	80

t	$A(t)$
2	27
3	9
4	3

5.10.3 Explore Average Rates of Change in an Exponential Decay Context

Activity

Here is a graph you saw in an earlier lesson. It represents the exponential function p, which models the cost $p(t)$, in dollars, of producing 1 watt of solar energy from 1977 to 1988, where t is years since 1977.



1. Clare said, "In the first five years, between 1977 and 1982, the cost fell by about \$12 per year. But in the second five years, between 1983 and 1988, the cost fell only by about \$2 a year." Show that Clare is correct.
2. If the trend continues, will the average decrease in price be more or less than \$2 per year between 1987 and 1992? Be prepared to show your reasoning.

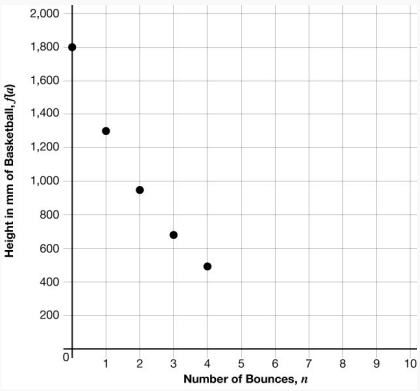
Are you ready for more?

Extending Your Thinking

Suppose the cost of producing 1 watt of solar energy had instead decreased by \$12.20 each year between 1977 and 1982. Compute what the costs would be each year and plot them on a graph. Compare your graph to the graph from the beginning of the 5.10.3 activity (top of page). How do these alternate costs compare to the actual costs shown?

Self Check

The graph below shows the bounce height of a basketball at each bounce. What is the approximate average rate of change from the first bounce to the third bounce?

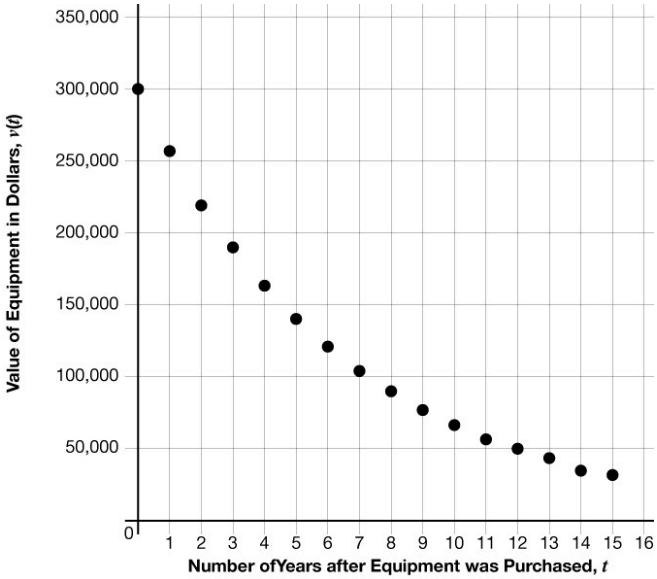


a. -600
b. 600
c. -300
d. 300

Additional Resources

Trends in Average Rates of Change

A construction company purchased some equipment costing $300,000. The equipment depreciates in value over time.



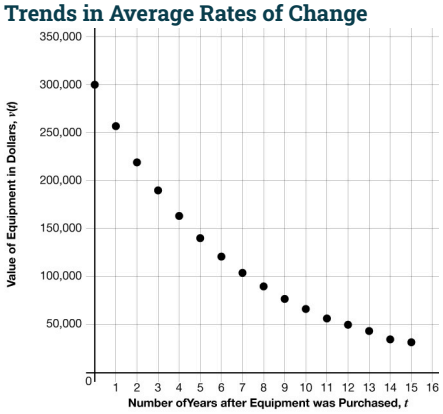
Approximately how much did the value drop from years 0 to 5 compared to years 6 to 10?

$$\frac{v(5)-v(0)}{5-0} \approx \frac{140{,}000-300{,}000}{5} = -32{,}000$$

$$\frac{v(10)-v(6)}{10-6} \approx \frac{70{,}000-120{,}000}{4} = -12{,}500$$

Following this trend, we would expect the average rate of change of the value of the equipment from years 11 – 15 to be below –12,500 dollars per year.

TRY IT **Trends in Average Rates of Change**



Approximately how much did the value drop from year 11 to year 15?

5.10.4 Average Rate of Change

Cool Down

Here is the function f for Clare's moldy bread that you saw earlier.

d, time since mold spotting (days)	$f(d)$, area covered by mold (square millimeters)
0	1
1	2
2	4
3	8
4	16
5	32
6	64

1. What is the average rate of change for the mold over the 6 days?
2. How well does the average rate of change describe how the mold changes for these 6 days?
3. How could the average rate of change be more accurate?

5.10.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. A store receives 2,000 decks of popular trading cards. The number of decks of cards is a function, d, of the number of days, t, since the shipment arrived. Here is a table showing some values of d.

t	$d(t)$
0	2,000
5	1,283
10	823
15	528
20	338

Calculate the average rate of change for the following intervals:

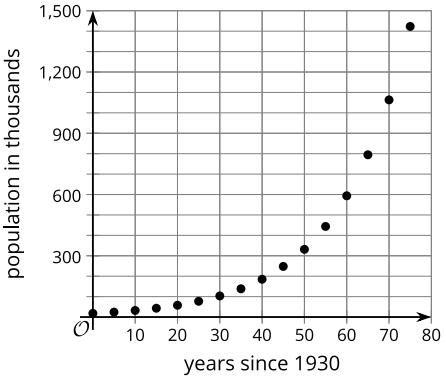
a. day 0 to day 5

b. day 15 to day 20

2. A study was conducted to analyze the effects on deer population in a particular area. Let f be an exponential function that gives the population of deer t years after the study began.
If $f(t) = a \cdot b^t$ and the population is increasing, **select three** statements that must be true.;

a. $b > 1$
b. $b < 1$
c. The average rate of change from year 0 to year 5 is less than the average rate of change from year 10 to year 15.
d. The average rate of change from year 0 to year 5 is greater than the average rate of change from year 10 to year 15.
e. $a > 0$

3. Function f models the population, in thousands, of a city t years after 1930.



a. The average rate of change of f from $t = 0$ to $t = 80$ is approximately 1,400 thousand people per year. Is this value a good way to describe the population change of the city over that time period?

a. Yes

b. No

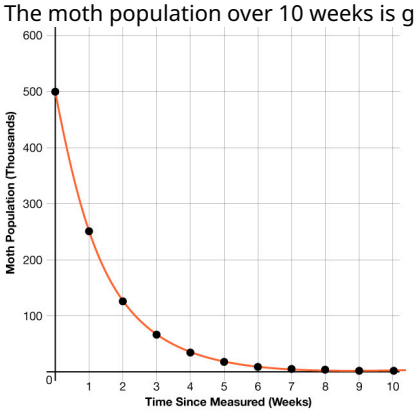
b. Is the rate of change between years 1940 and 1950 greater than or less than the rate of change between years 1960 and 1970?

a. greater than
b. less than

4. The function, f, gives the number of copies of a book that have sold w weeks after it was published. The equation $f(w) = 500 \cdot 2^w$ defines this function. **Select two** domains for which the average rate of change could be a good measure for the number of books sold.

a. $0 \leq w \leq 2$
b. $0 \leq w \leq 7$
c. $5 \leq w \leq 7$
d. $5 \leq w \leq 10$
e. $0 \leq w \leq 10$

5. The moth population over 10 weeks is graphed below.



a. After 4 weeks, the population was 40 thousand. What was the average rate of change from the start to the 4th week?

b. Is the rate of change between weeks 1 and 2 greater than or less than the rate of change between weeks 3 and 4?

a. greater than
b. less than

5.10.6 Lesson Summary

In this lesson, you learned how to:

- Calculate the average rate of change of a function over a specified period of time.
- Recognize how the average rate of change of an exponential function differs from that of a linear function.

Here are the **activities** that helped you reach those goals:

- 5.10.1: Calculate an Average Rate of Change from Two Points
  - In this activity, you recalled how to calculate an average rate of change from two points.
- 5.10.2: Explore Average Rates of Change in an Exponential Growth Function
  - In this activity, you began exploring average rates of change in an exponential growth function. In particular, the focus for this activity was on how exponential functions have different rates of change for different input intervals, which is not the case for linear functions.
  - 5.10.3: Self Check
  - 5.10.3: Additional Resources
- 5.10.3: Explore Average Rates of Change in an Exponential Decay Context
  - In this activity, you continued to explore average rates of change for exponential functions. This time, you focused on a previously encountered exponential decay context.
  - 5.10.4: Self Check
  - 5.10.4: Additional Resources
- 5.10.4: Average Rate of Change
  - In this activity, you found the average rate of change and explored how well the average rate of change describes how something changes over time.

After these activities, you completed the following **practice**:

- 5.10.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

5.11 MODELING EXPONENTIAL BEHAVIOR

5.11.0 Lesson Overview

In this lesson, you will study the successive bounce heights of balls, model the relationship between the number of bounces and the bounce heights, and use that model to answer questions about the balls' bounciness. As you compare and critique the different models, you construct viable arguments and critique the reasoning of others. Technology isn't required for this lesson, but there are opportunities for you to choose to use appropriate technology to solve problems.

When you finish this lesson, you will be able to:

- Use exponential functions to model situations that involve exponential growth or decay.
- Determine an appropriate model for a situation described by data.

Here are the **activities** that will help you reach those goals:

- 5.11.1: Changing the Graphing Window
- 5.11.2: Choosing an Appropriate Model
  - 5.11.2: Self Check
  - 5.11.2: Additional Resources
- 5.11.3: Modeling with Exponential Functions
- 5.11.4: Examine Exponential Decay in Context
  - 5.11.4: Self Check
  - 5.11.4: Additional Resources
- 5.11.5: Using Relationship Models to Answer Questions

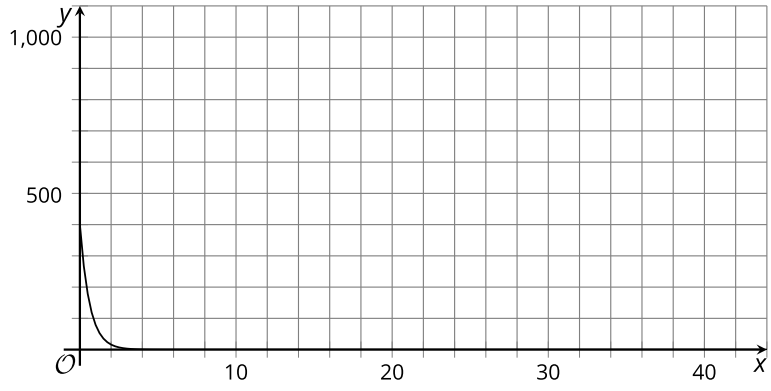
After that, you'll **practice and review**.

- 5.11.6: Practice
- 5.11.7: Lesson Summary

5.11.1 Changing the Graphing Window

Warm Up

Here is a graph of a function $f(x) = 400 \cdot (0.2)^x$.



1. Identify the approximate graphing window shown.

2. **2.** Suggest a new graphing window that would:
a. make the graph more informative or meaningful.
b. make the graph less informative or meaningful.

5.11.2 Choosing an Appropriate Model

Activity



Here are measurements for the maximum height of a tennis ball after bouncing several times on a concrete surface.

n **Bounce Number**	h **Height (Centimeters)**
0	150
1	80
2	43
3	20
4	11

1. Which is more appropriate for modeling the maximum height h, in centimeters, of the tennis ball after n bounces: A linear function or an exponential function? Use data from the table to support your answer.
2. Regulations say that a tennis ball, dropped on concrete, should rebound to a height between 53% and 58% of the height from which it is dropped. Does the tennis ball here meet this requirement? Be prepared to show your reasoning.
3. Write an equation that models the bounce height h after n bounces for this tennis ball.
4. About how many bounces will it take before the rebound height of the tennis ball is less than 1 centimeter? Be prepared to show your reasoning.
5. Graph the equation that models the bounce height h after n bounces for this tennis ball using the Desmos tool below.

Access multimedia content (http://openstax.org/books/algebra-1/pages/5-11-2-choosing-an-appropriate-model)

6. Use the graph of the equation that models the bounce height h after n bounces for this tennis ball to predict the difference in height after 6 bounces.

Video: Determining a Model and an Equation from a Real-World Scenario

Watch the following video to learn more about modeling exponential functions:

Determining a Model and an Equation from a Real-World Scenario (https://openstax.org/r/real-world-model)

Self Check

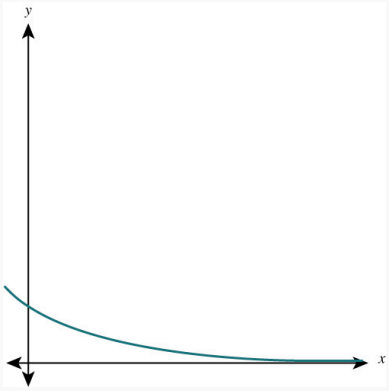
The table gives predictions of the average costs of a gasoline car after the year 2030.

Which graph best represents the relationship between time and the predicted cost?

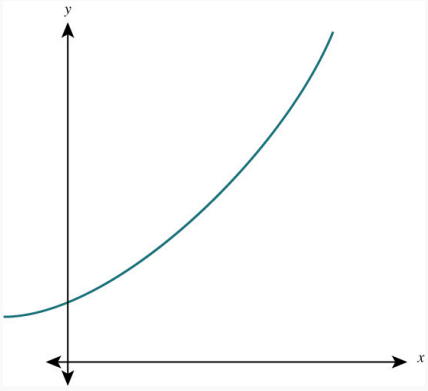
Time (years)	Cost (thousand of dollars)
0	35
5	40.2
10	46.3
15	53.2
20	61.2

Gasoline car

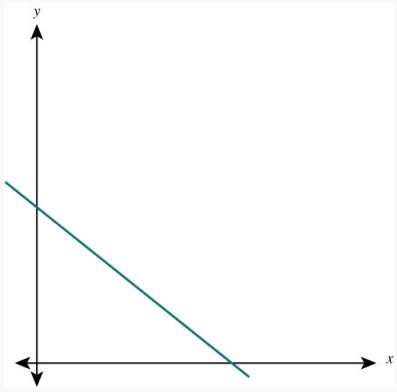
a.



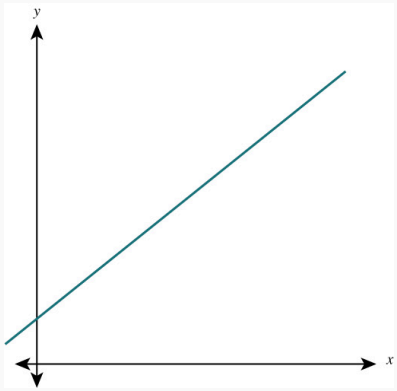
b.



c.



d.



Additional Resources

Using an Exponential Model

A lab researcher records the growth of the population of a yeast colony and finds that the population doubles every hour. The table below records the first four hours.

Hours into Study	0	1	2	3	4
Yeast Colony Population (Thousands)	5	10	20	40	80

The starting value is 5 (in thousands) and the population doubles, so the growth factor is 2.

EXAMPLE

You can use the data you already have to predict the population size of the yeast colony in the future.

1. What will the size of the population be in 5 hours?

The population will be 160,000. You will double the 80 which represents 80,000.

2. What will the size of the population be in 6 hours?

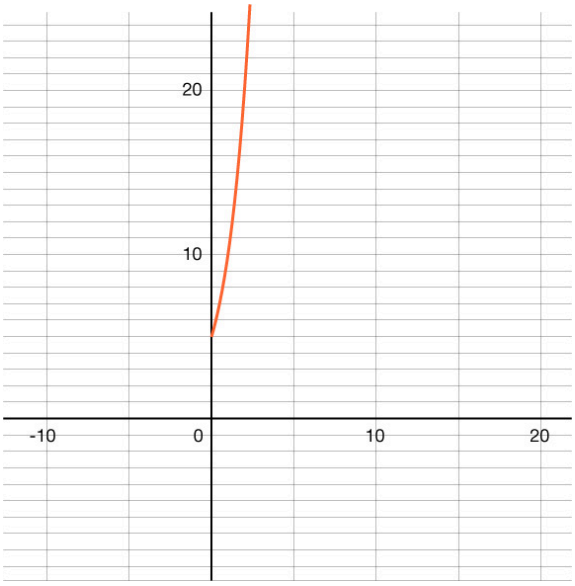
The population will be 320,000. You will double the 160 which represents 160,000.

3. What will the size of the population be in 10 hours?

You may have to use a calculator to figure this one out. You will double it four more times. The population will be 5,120,000.

The equation to model this growth is $P(t) = 5 \cdot 2^t$.

The graph is below:



TRY IT **Using an Exponential Model**

A ball bounces up and down from a starting height of 65 centimeters. As it bounces, its height decreases to $\frac{1}{3}$ the previous height. Complete a table to represent the first 4 bounces. Predict the height after 5 bounces. Then write a function, $h(x)$, to determine the height after x bounces.

5.11.3 Modeling with Exponential Functions

Activity

Your teacher will give your group three different kinds of balls.

Your goal is to measure the rebound heights, model the relationship between the number of bounces and the heights, and compare the bounciness of the balls.



1. Create a table similar to the one below and begin your investigation. Make sure to note which ball goes with which column.

n, Number of Bounces	a, Height for Ball 1 (cm)	b, Height for Ball 2 (cm)	c, Height for Ball 3 (cm)
0			
1			
2			
3			
4			

2. Which one appears to be the bounciest? Which one appears to be the least bouncy? Be prepared to show your reasoning.
3. For each one, write an equation expressing the bounce height in terms of the bounce number n.
4. Explain how the equations could tell us which one is the most bouncy.
5. If the bounciest one were dropped from a height of 300 cm, what equation would model its bounce height h?
6. Graph, on the same coordinate plane, the equation for each ball that models the bounce height h after n bounces using the Desmos tool below.

Access multimedia content (http://openstax.org/books/algebra-1/pages/5-11-3-modeling-with-exponential-functions)

7. Use the graph of the equation that models the bounce height h after n bounces for all 3 balls to predict the difference in the height of Ball 1 after 5 bounces.
8. Use the graph of the equation that models the bounce height h after n bounces for all 3 balls to predict the difference in the height of Ball 2 after 5 bounces.
9. Use the graph of the equation that models the bounce height h after n bounces for all 3 balls to predict the difference in the height of Ball 3 after 5 bounces.

Are you ready for more?

Extending Your Thinking

1. If Ball 1 were dropped from a point that is twice as high, would its bounciness be greater, less, or the same? Be prepared to show your reasoning.
2. Ball 4 is half as bouncy as the least bouncy ball. What equation would describe its height h in terms of the number of bounces n?
3. Ball 5 was dropped from a height of 150 centimeters. It bounced up very slightly once or twice and then began rolling. How would you describe its rebound factor? Be prepared to show your reasoning.

5.11.4 Examine Exponential Decay in Context

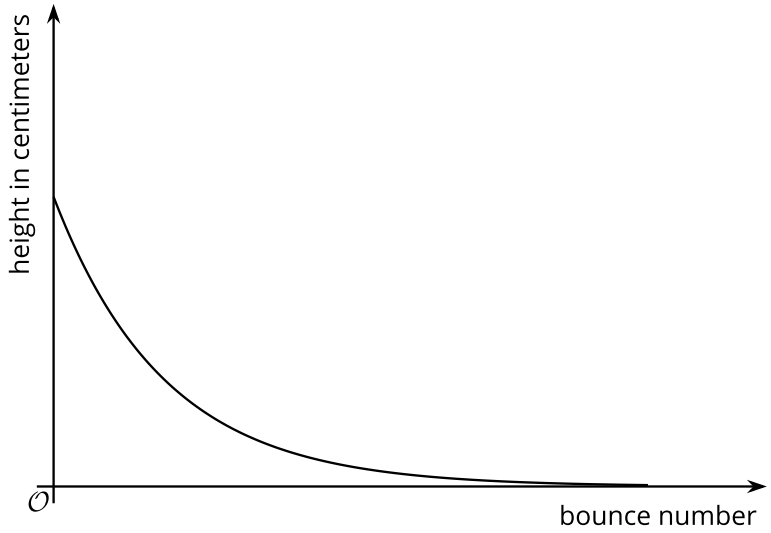
Activity

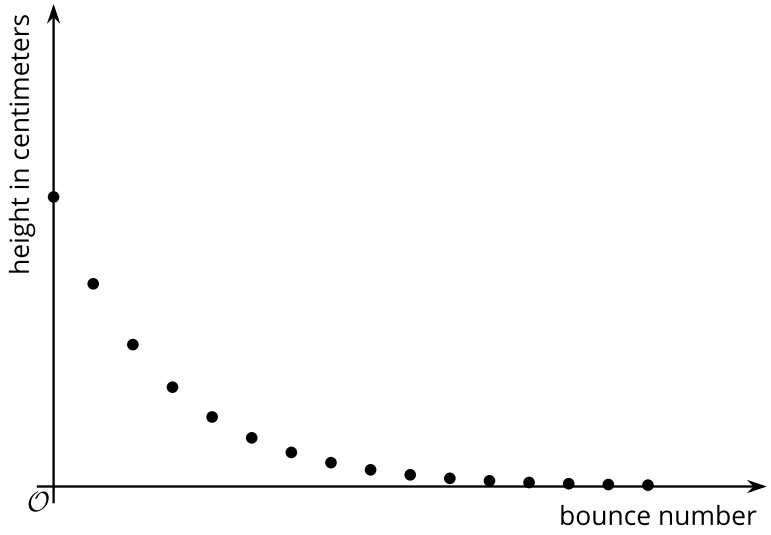
The table shows some heights of a ball after a certain number of bounces.

Bounce Number	Height in Centimeters
0	
1	
2	73.5

Bounce Number	Height in Centimeters
3	51.5
4	36

1. Is this ball more or less bouncy than the tennis ball in activity 5.11.2, which has the equation $h = 150 \cdot (0.53)^n$? Explain or show your reasoning.
2. From what height was the ball dropped? Explain or show your reasoning. Remember your units!
3. Write an equation that represents the bounce height of the ball, h, in centimeters after n bounces. Remember that an exponent may be entered using the "^" symbol.
4. Which graph would more appropriately represent the equation for h: Graph A or Graph B? Be prepared to show your reasoning.





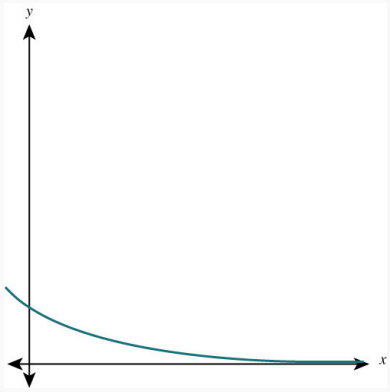
5. Will the n^{th} bounce of this ball be lower than the n^{th} bounce of the tennis ball with equation $h = 150 \cdot (0.53)^n$? Be prepared to show your reasoning.
6. The table shows an exponential decay that starts at 640. What exponential decay problem is it modeling? Use ^ to show your exponent.

x	0	1	2	3
y	640	160	40	10

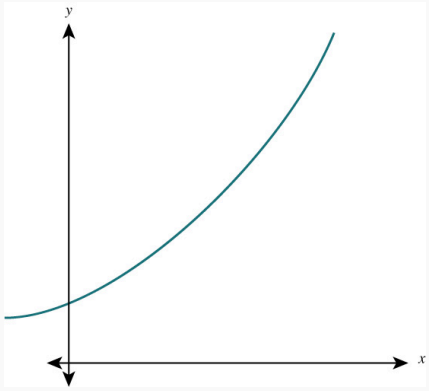
Self Check

The population of deer in a forest is divided in half every decade. Which of the following graphs could represent this situation?

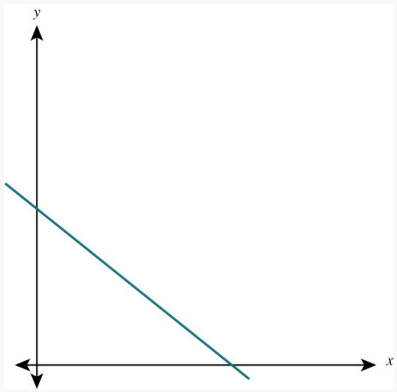
a.



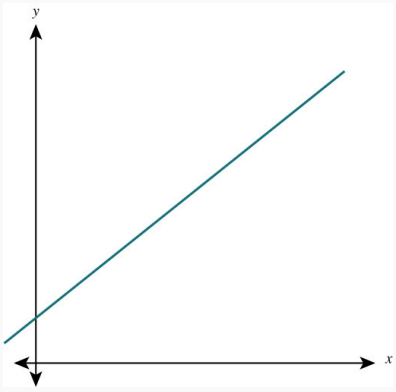
b.



c.



d.



Additional Resources

Writing Exponential Decay Functions

EXAMPLE 1

Data has been entered into a table that shows exponential decay. How would you write the equation that represents this model?

x	0	1	2	3
y	200	100	50	25

The formula for exponential decay is $y = a \cdot (1 - b)^x$ where a is the starting value and b is the amount that value decreases each unit of time x.

The pattern for the decay is starting at 200 and then decreasing by 12 with only 12 remaining. The equation is $y = 200 \cdot (1 - \frac{1}{2})^x$ or $y = 200 \cdot (\frac{1}{2})^x$.

EXAMPLE 2

A car sells for \$4,000 and loses $\frac{1}{8}$ of its value each year. Write a function that gives the car's value, $V(t)$, t years after it is sold.

Since the car loses $\frac{1}{8}$ of its value each year, it keeps $\frac{7}{8}$ of its value from year to year, so the growth factor is $\frac{7}{8}$.

$V(t) = 4000 \cdot (\frac{7}{8})^t$

TRY IT **Writing Exponential Decay Functions**

1. What exponential decay equation is modeled by the table of data given?

x	0	1	2	3
y	10,000	7,500	5,625	4,218.75

A computer is purchased for \$1,000 and loses $\frac{1}{5}$ of its value per year.

2. Write a function for the computer's value, $V(t)$, t years after it is purchased.

5.11.5 Using Relationship Models to Answer Questions

Cool Down

A ball is dropped from a certain height. The table shows the rebound heights of the ball after a series of bounces.

Bounce Number	Height in Centimeters
1	30
2	6
3	1
4	0

1. From what height, approximately, do you think the ball was dropped? Be prepared to show your reasoning.

5.11.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 4, use the following:

- Function h describes the height of a ball, in inches, after n by the equation $h(n) = 120 \cdot (\frac{4}{5})^n$.

1. What is $h(3)$? Round to the nearest integer. Think about what it represents in this situation.

2. Could $h(n)$ be 150?

a. Yes
b. No

3. Which ball loses its height more quickly, this ball, modeled by $h(n) = 120 \cdot (\frac{4}{5})^n$, or a tennis ball whose height in inches after n bounces is modeled by the function f where $f(n) = 50 \cdot (\frac{5}{9})^n$?

a. the ball modeled by $h(n)$
b. the ball modeled by $f(n)$

4. How many bounces would it take before the ball, modeled by $h(n)$, bounces less than 12 inches from the surface?

5. After a second bounce, a ball reached a height of 70 cm. The rebound factor for the ball was 0.8. From approximately what height, in cm, was the ball dropped?

a. 109
b. 21
c. 56
d. 560

6. Using the table below, find the height, in cm, at which the ball was dropped.

Number of Bounces	1	2	3	4
Height in cm	90	54	32	19

5.11.7 Lesson Summary

In this lesson, you learned how to:

- Use exponential functions to model situations that involve exponential growth or decay.
- Determine an appropriate model for a situation described by data.

Here are the **activities** that helped you reach those goals:

- 5.11.1: Changing the Graphing Window
  - In this activity, you thought about how changing the graphing window influences what you can see and understand about an exponential function
- 5.11.2: Choosing an Appropriate Model
  - In this activity, you examined the successive heights that a tennis ball reached after several bounces on a hard surface and considered how to model the relationship between the number of bounces and the height of the rebound. To do so, you determined the growth factor of successive bounce heights.
  - 5.11.2: Self Check
  - 5.11.2: Additional Resources
- 5.11.3: Modeling with Exponential Function
  - In this activity, you had an opportunity to gather and analyze data for bounce heights for multiple balls. You processed your data and decided on an appropriate factor to quantify the bounciness of each ball. You became more comfortable with the fact that the data are not exactly exponential and with choosing different ways for deciding on an appropriate exponential decay factor.
  - 5.11.3: Self Check
  - 5.11.3: Additional Resources
- 5.11.4: Examine Exponential Decay in Context
  - In this activity, you continued to examine exponential decay in the context of successive ball bounces. You used the given data to calculate a rebound factor and used it to write a function that models the relationship between number of bounces and bounce heights. You then used the function to answer questions about the ball and its bounces.
  - 5.11.4: Self Check
  - 5.11.4: Additional Resources
- 5.11.5: Using Relationship Models to Answer Questions
  - In this activity, you used data to determine from what height, approximately, you thought a ball was dropped and were able to explain your reasoning.

After these activities, you completed the following **practice**:

- 5.11.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

5.12 REASONING ABOUT EXPONENTIAL GRAPHS, PART 1

5.12.0 Lesson Overview

In this lesson, you will analyze the graph of an exponential function f given by $f(x) = a \cdot b^x$. In particular, you study the effect of b on the shape of the graph. To observe how b impacts the shape of the graph when $b > 1$ and when $0 < b < 1$, you simultaneously examine several functions of this form, all of which have the same value of a.

These lessons encourage greater attention to the growth factors of the different functions, rather than on interpreting the quantities in context. Since you are focusing mainly on understanding how varying b impacts the graph, you are looking at the structure of an exponential function and its graph.

When you finish this lesson, you will be able to:

- Describe the effect of changing a and b on a graph that represents $f(x) = a \cdot b^x$.
- Use equations and graphs to compare exponential functions.

Here are the **activities** that will help you reach those goals:

- 5.12.1: Modeling Exponential Decay
- 5.12.2: Equations and Their Graphs
  - 5.12.2: Self Check
  - 5.12.2: Additional Resources
- 5.12.3: Graphs Representing Exponential Decay
  - 5.12.3: Self Check
  - 5.12.2: Additional Resources
- 5.12.4: Possible Equation for a Function on a Graph

After these activities, you completed the following **practice** and **review**:

- 5.12.5: Practice
- 5.12.6: Lesson Summary

5.12.1 Modeling Exponential Decay

Warm Up

Jada received a gift of $180. In the first week, she spent a third of the gift money. She continues spending a third of what is left each week thereafter.

1. Which equation best represents the amount of gift money g, in dollars, she has after t weeks?
   a. $g = 180 - \frac{1}{3}t$
   b. $g = 180 \cdot (\frac{1}{3})^t$
   c. $g = \frac{1}{3} \cdot 180^t$
   d. $g = 180 \cdot (\frac{2}{3})^t$
2. Explain your reasoning.

5.12.2 Equations and Their Graphs

Activity

Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

1. Each of the following functions, f, g, h, and j, represents the amount of money in a bank account, in dollars, as a function of time x, in years. They are each written in the form $m(x) = a \cdot b^x$.

- $f(x) = 50 \cdot 2^x$
- $g(x) = 50 \cdot 3^x$
- $h(x) = 50 \cdot (\frac{3}{2})^x$
- $j(x) = 50 \cdot (0.5)^x$

a. Use graphing technology to graph each function on the same coordinate plane. It may be helpful to use a different color for each function.

Access multimedia content (http://openstax.org/books/algebra-1/pages/5-12-2-equations-and-their-graphs)

b. Explain how changing the value of b changes the graph.

2. Here are equations defining functions p, q, and r. They are also written in the form $m(x) = a \cdot b^x$.

- $p(x) = 10 \cdot 4^x$
- $q(x) = 40 \cdot 4^x$
- $r(x) = 100 \cdot 4^x$

a. Use graphing technology to graph each function on the same coordinate plane. It may be helpful to use a different color for each function.

Access multimedia content (http://openstax.org/books/algebra-1/pages/5-12-2-equations-and-their-graphs)

b. Explain how changing the value of a changes the graph.

3. Here are equations defining functions f, g, and h written in the form $m(x) = a \cdot b^x + d$.

- $f(x) = 4^x + 2$
- $g(x) = 4^x$
- $h(x) = 4^x - 2$

a. Use graphing technology to graph each function on the same coordinate plane. It may be helpful to use a different color for each function.

Access multimedia content (http://openstax.org/books/algebra-1/pages/5-12-2-equations-and-their-graphs)

b. Discuss with a partner how changing the value of d changes the asymptote of the graph.

Are you ready for more?

Extending Your Thinking

As before, consider bank accounts whose balances are given by the following functions:

$f(x) = 10 \cdot 3^x$

$g(x) = 3^{x+2}$

$h(x) = \frac{1}{2} \cdot 3^{x+3}$

Which function would you choose? Does your choice depend on x?

Video: Exponential Equations and How They Affect the Graphs

Watch the following video to learn more about exponential functions and how they affect the graphs:

Exponential Equations and How They Affect the Graphs (https://openstax.org/r/exponential-equations)

Self Check

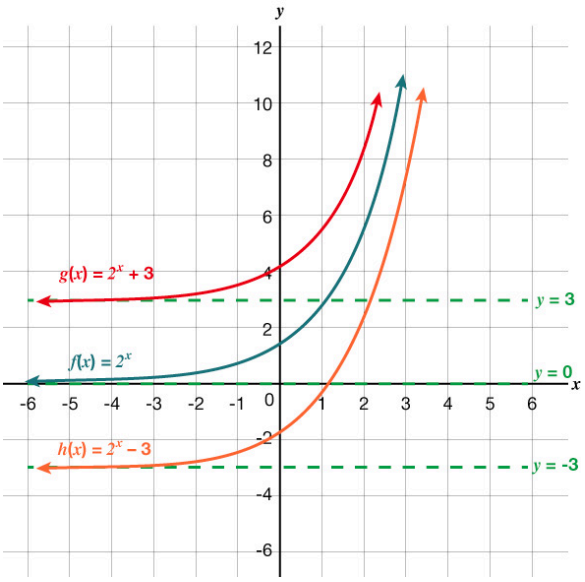
What is the horizontal asymptote for the function $f(x) = 2 \cdot 3^x - 5$?

a. $y = 3$
b. $y = -5$
c. $y = 2$
d. $y = 5$

Additional Resources

Vertical Shifts with Exponential Functions

The first transformation occurs when we add a constant d to the parent function $f(x) = b^x$, giving us a vertical shift d units in the same direction as the sign. For example, if we begin by graphing a parent function, $f(x) = 2^x$, we can then graph two vertical shifts alongside it, using $d = 3$: the upward shift, $g(x) = 2^x + 3$, and the downward shift, $h(x) = 2^x - 3$. Both vertical shifts are shown in the graph below.



Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$, remains unchanged.
- When the function is shifted up 3 units to $g(x) = 2^x + 3$:
  - The y-intercept shifts up 3 units to $(0, 4)$.
  - The asymptote shifts up 3 units to $y = 3$.
  - The range becomes $(3, \infty)$.
- When the function is shifted down 3 units to $h(x) = 2^x - 3$:
  - The y-intercept shifts down 3 units to $(0, -2)$.
  - The asymptote also shifts down 3 units to $y = -3$.
  - The range becomes $(-3, \infty)$.

TRY IT **Vertical Shifts with Exponential Functions**

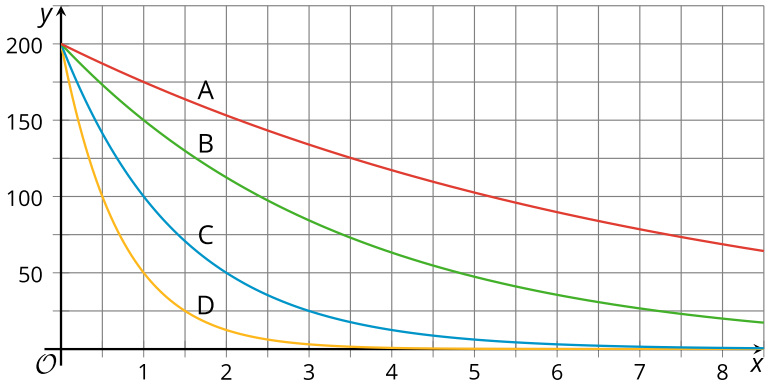
Where would the horizontal asymptote be located for the graph of $f(x) = 4 \cdot 2^x - 6$?

5.12.3 Graphs Representing Exponential Decay

Activity

CLASSROOM CONNECTION Log into student.desmos.com using the information provided by your teacher to complete the activity.

1. Examine the graph of the functions below.



Match each of the functions with a graph. Be prepared to explain your reasoning.

a. Which graph best represents $m(x) = 200 \cdot (\frac{1}{4})^x$?

b. Which graph best represents $n(x) = 200 \cdot (\frac{1}{2})^x$?

c. Which graph best represents $p(x) = 200 \cdot (\frac{3}{4})^x$?

d. Which graph best represents $q(x) = 200 \cdot (\frac{7}{8})^x$?

2. Functions f and g are defined by these equations:

- $f(x) = 1,000 \cdot (\frac{1}{10})^x$
- $g(x) = 1,000 \cdot (\frac{9}{10})^x$

a. Use the graphing tool or technology outside the course. On the same coordinate plane, sketch a graph that represents the functions, f and g. It may be helpful to use a different color for each function.

Access multimedia content (http://openstax.org/books/algebra-1/pages/5-12-3-graphs-representing-exponential-decay)

b. Which function is decaying more quickly?

c. Explain your reasoning regarding which function decays more quickly.

Building Character: Grit



Grit predicts accomplishing challenging goals of personal significance. For example, grittier students are more likely

to graduate from high school, and grittier cadets are more likely to complete their training at West Point.

Think about your current sense of grit. Are the following statements true for you?

- There is at least one subject or activity that I never get bored of thinking about.
- I finish whatever I begin.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

Self Check

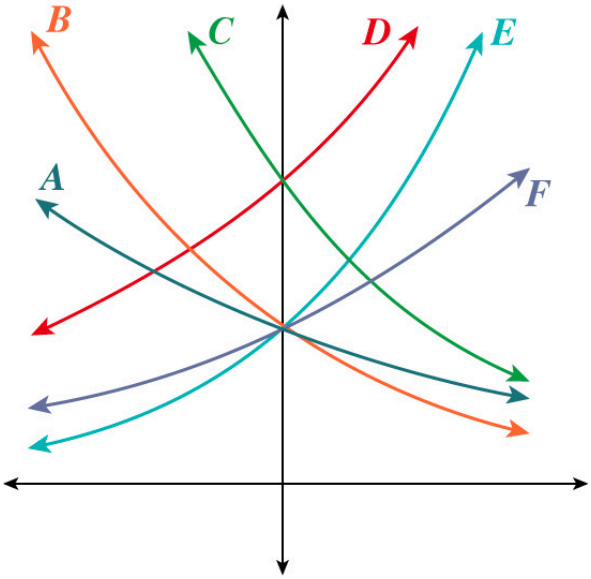
Which of the following functions will decay the fastest?

a. $j(x) = 0.5^x$
b. $h(x) = 1.34^x - 90$
c. $g(x) = 0.34^x + 2$
d. $f(x) = 0.86^x - 10$

Additional Resources

Matching Exponential Functions and Graphs

For each of the following, match each function with one of the graphs in the picture below.

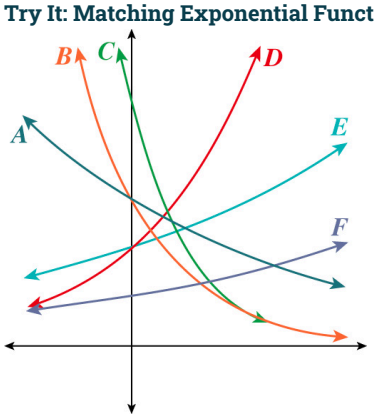


Remember, the equation for an exponential function is $y = a \cdot b^x$. The a will affect the y-intercept. When $b < 1$, there is exponential decay. The closer b is to 0, the faster it decays. The closer b is to 1, the slower it decays. When $b > 1$, the larger the value of b, the faster it grows.

Equation	y-intercept	Growth/Decay	Match
1. $f(x) = 2(0.68)^x$	$(0, 2)$	Decay	B
2. $f(x) = 2(1.28)^x$	$(0, 2)$	Growth	F
3. $f(x) = 2(0.81)^x$	$(0, 2)$	Decay	A
4. $f(x) = 4(1.28)^x$	$(0, 4)$	Growth	D

Equation	y-intercept	Growth/Decay	Match
5. $f(x) = 2(1.59)^x$	$(0, 2)$	Growth	E
6. $f(x) = 4(0.68)^x$	$(0, 4)$	Decay	C

TRY IT **Try It: Matching Exponential Functions and Graphs**

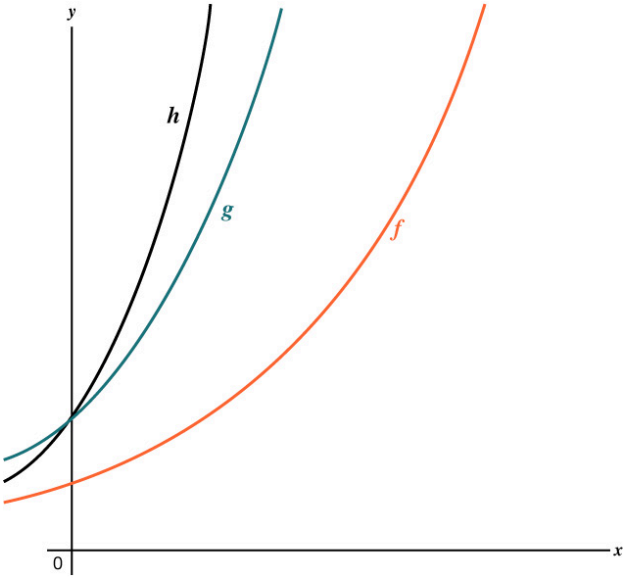


In the graph above, which function, $f(x) = a \cdot b^x$, has the largest value for a?

5.12.4 Possible Equation for a Function on a Graph

Cool Down

Examine the three graphs representing three exponential functions: f, g, and h.



The functions f and h are defined as:

- $f(x) = 10 \cdot 2^x$
- $h(x) = 20 \cdot 4^x$

1. Which of the following could define function g? Be prepared to show your reasoning.
   a. $g(x) = 20 \cdot (1.5)^x$
   b. $g(x) = 20 \cdot (2.5)^x$
   c. $g(x) = 10 \cdot (3.5)^x$
   d. $g(x) = 20 \cdot (4.5)^x$
2. Explain your reasoning.

5.12.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 - 5, use the the three exponential functions defined below:

- $f(x) = 100 \cdot 3^x$
- $g(x) = 100 \cdot (3.5)^x$
- $h(x) = 100 \cdot 4^x$

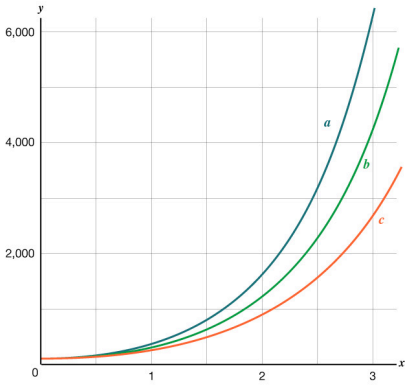
1. Which of these functions grows the least quickly?

a. $h(x)$
b. $g(x)$
c. $f(x)$

2. Which function grows the most quickly?

a. $h(x)$
b. $g(x)$
c. $f(x)$

The three given graphs represent the functions f, g, and h defined above.



3. Which graph corresponds to $f(x)$?

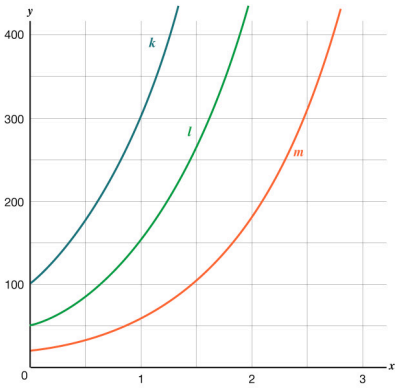
a. Graph A
b. Graph B
c. Graph C

4. Which graph corresponds to $g(x)$?

a. Graph A
b. Graph B
c. Graph C

5. Which graph corresponds to $h(x)$?

a. Graph A
b. Graph B
c. Graph C



For questions 6 - 8, use the given graphs of three exponential equations.

6. Which graph matches the equation $y = 20 \cdot 3^x$

a. Graph K
b. Graph L
c. Graph M

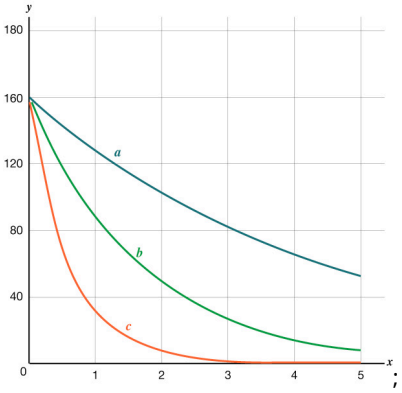
7. Which graph matches the equation $y = 50 \cdot 3^x$

a. Graph K
b. Graph L
c. Graph M

8. Which graph matches the equation $y = 100 \cdot 3^x$

a. Graph K
b. Graph L
c. Graph M

For questions 9 - 10, use the graph below.



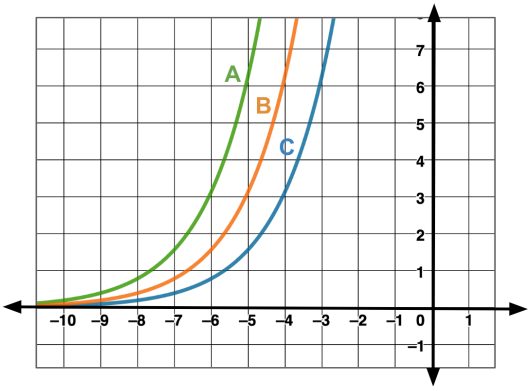
;

9. Written in the form $f(x) = a \cdot b^x$, what is the value of a for each function?

10. Which graph decreases the most quickly?

a. Graph A
b. Graph B
c. Graph C

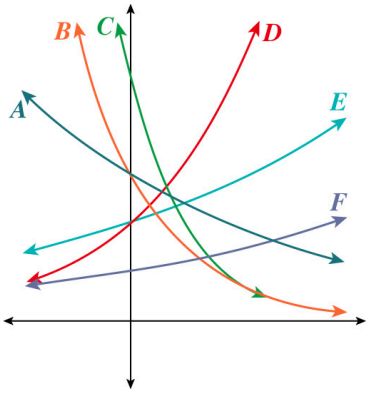
The graphs of $y = 50 \cdot 2^x$, $y = 200 \cdot 2^x$, and $y = 100 \cdot 2^x$ are shown.



11. Which is the graph of $y = 100 \cdot 2^x$?

a. Graph A
b. Graph B
c. Graph C

For questions 12 – 14, use the graph below:



12. Which of the functions graphed, $y = a \cdot b^x$, has the smallest value for a?

a. A
b. B
c. C
d. D
e. E
f. F

13. Which of the functions graphed, $y = a \cdot b^x$, has the lowest value for b?

a. A
b. B
c. C
d. D
e. E
f. F

14. What is the equation of the horizontal asymptote for $f(x) = 5 \cdot 2^x - 3$?

a. $y = 5$
b. $y = -3$
c. $y = 3$
d. $y = 2$

5.12.6 Lesson Summary

In this lesson, you learned how to:

- Describe the effect of changing a and b on a graph that represents $f(x) = a \cdot b^x$.
- Use equations and graphs to compare exponential functions.

Here are the **activities** that helped you reach those goals:

- 5.12.1: Modeling Exponential Decay
  - In this activity, you practiced modeling a situation characterized by exponential decay with an equation.
- 5.12.2: Equations and Their Graphs
  - In this activity, you examined how the numbers a and b influence the graph representing a function f defined by an equation of the form $f(x) = a \cdot b^x$.
  - 5.12.2: Self Check
  - 5.12.2: Additional Resources
- 5.12.3: Graphs Representing Exponential Decay
  - In this activity, you explored how the parameter b influences the graph of an exponential function, which prepared you to make sense of and model situations involving repeated percentage.
  - 5.12.3: Self Check
  - 5.12.3: Additional Resources
- 5.12.4: Possible Equation for a Function on a Graph
  - In this activity, you determined the graphed equation that defined a given function.

After these activities, you completed the following **practice** and **review**:

- 5.12.5: Practice
- 5.12.6: Lesson Summary

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

5.13 REASONING ABOUT EXPONENTIAL GRAPHS, PART 2

5.13.0 Lesson Overview

In this lesson, you will investigate the relationship between the parameters a and b in the expression $a \cdot b^x$ and a graph representing the function f given by $f(x) = a \cdot b^x$. You will start by identifying a function represented by a given graph and using the graph to make sense of a situation. You will also examine two abstract graphs, with unlabeled axes, and decide which one represents a given situation. In both cases, you will use your understanding of the connections between the parameters in an exponential expression and the features of an exponential graph to answer questions.

When you finish this lesson, you will be able to:

- Explain the meaning of the intersection of the graphs of two functions in terms of the situations they represent.
- Write an equation for the function given two points on a graph of an exponential function.

Here are the **activities** that will help you reach those goals:

- 5.13.1: Comparing Functions
- 5.13.2: Creating Exponential Functions from Graphs
  - 5.13.2: Self Check
  - 5.13.2: Additional Resources
- 5.13.3: Comparing Exponential Functions
  - 5.13.3: Self Check
  - 5.13.3: Additional Resources
- 5.13.4: Interpreting Features of an Exponential Graph

After that, you'll **practice and review**.

- 5.13.5: Practice
- 5.13.6: Lesson Summary

5.13.1 Comparing Functions

Warm Up

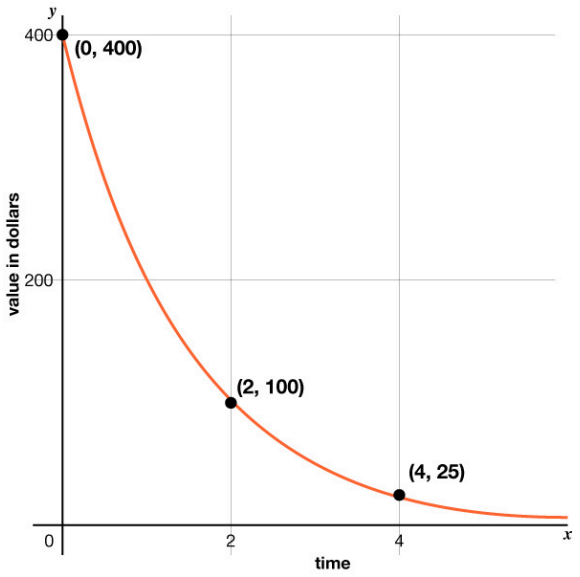
1. Which one doesn't belong?

$f(n) = 8 \cdot 2^n$
$g(n) = 2 \cdot 8^n$
$h(n) = 8 + 2n$
$j(n) = 8 \cdot (\frac{1}{2})^n$

5.13.2 Creating Exponential Functions from Graphs

Activity

Examine the graph representing an exponential function f. The function f gives the value of a computer, in dollars, as a function of time, x, measured in years since the time of purchase.



Based on the graph, what can you say about the following?

1. The purchase price of the computer

2. The value of f when x is 1

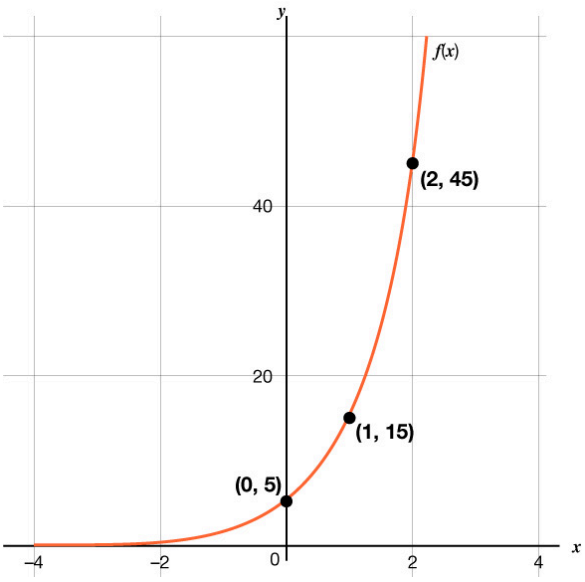
3. The meaning of $f(1)$

4. How the value of the computer is changing each year

5. An equation that defines f.

6. Whether the value of f will reach 0 after 10 years

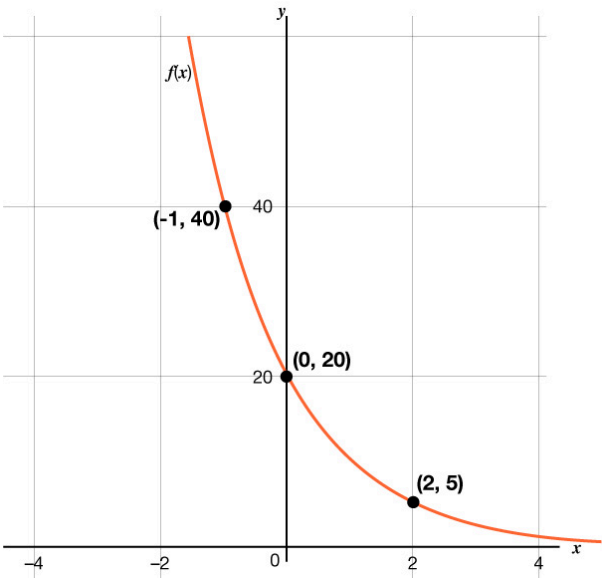
For questions 7 – 8, examine the graph of the exponential function below.



7. Write an equation that defines the function.

8. Find the value of the function when x is 5.

For questions 9 – 10, examine the graph of the exponential function below.



9. Write an equation that defines the function.

10. Find the value of the function when x is 5.

Are you ready for more?

Extending Your Thinking

Consider a function f defined by $f(x) = a \cdot b^x$.

If the graph of f goes through the points $(2, 10)$ and $(8, 30)$, would you expect $f(5)$ to be less than, equal to, or greater than 20?

If the graph of f goes through the points $(2, 30)$ and $(8, 10)$, would you expect $f(5)$ to be less than, equal to, or greater than 20?

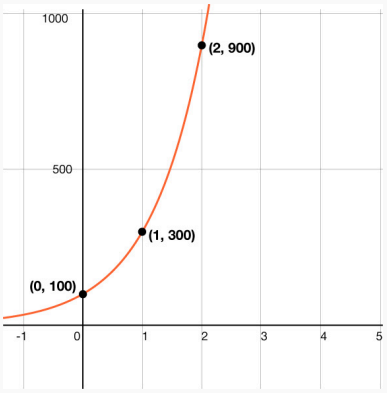
Video: Interpreting an Exponential Function from a Graph

Watch the following video to learn more about interpreting exponential functions:

[Interpreting an Exponential Function from a Graph (https://openstax.org/r/exponential-graph)](https://openstax.org/r/exponential-graph)

Self Check

Which of the following is the function for the curve graphed below?

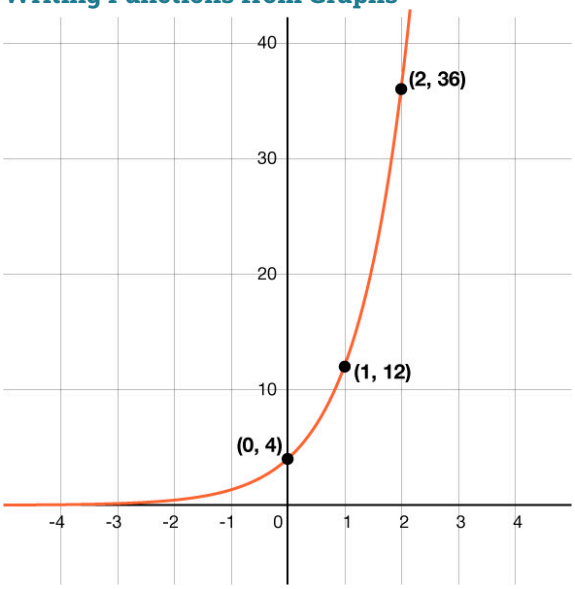


a. $f(x) = (100^3)^x$
b. $f(x) = 100 \cdot 3^x$
c. $f(x) = 3 \cdot 100^x$

d. $f(x) = 100 + 3^x$

Additional Resources

Writing Functions from Graphs

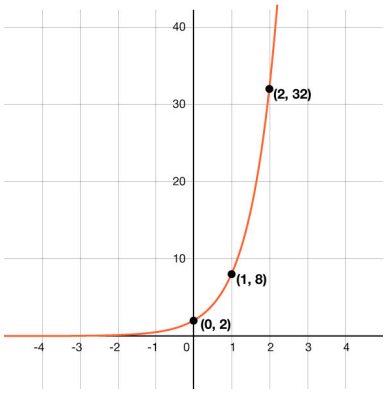


Find the equation of the function graphed above. Then find the value at $f(5)$.

1. Find the value of a, the starting point.	$a = 4$
2. Find the b, the growth factor.	$b = 3$
3. Write the equation of the function.	$f(x) = 4 \cdot 3^x$
4. Find $f(5)$.	$f(5) = 4 \cdot 3^5 = 972$

TRY IT **Try It: Writing Functions from Graphs**

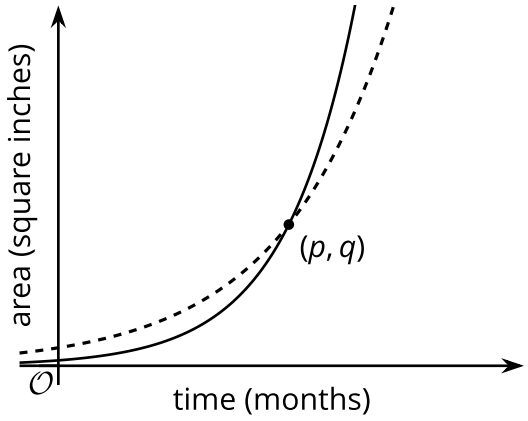
Use the graph below to write the equation of the function. Then find $f(5)$.



5.13.3 Comparing Exponential Functions

Activity

Here are graphs representing two functions and descriptions of two functions.



- Function f: The area of a wall that is covered by Mold A, in square inches, doubling every month.
- Function g: The area of a wall that is covered by Mold B, in square inches, tripling every month.

1. Which graph represents each function? Explain your answer.

2. When the mold was first spotted and measured, was there more of Mold A or Mold B? Explain how you know.

3. What does the point (p, q) tell us in this situation? Explain your answer.

Self Check

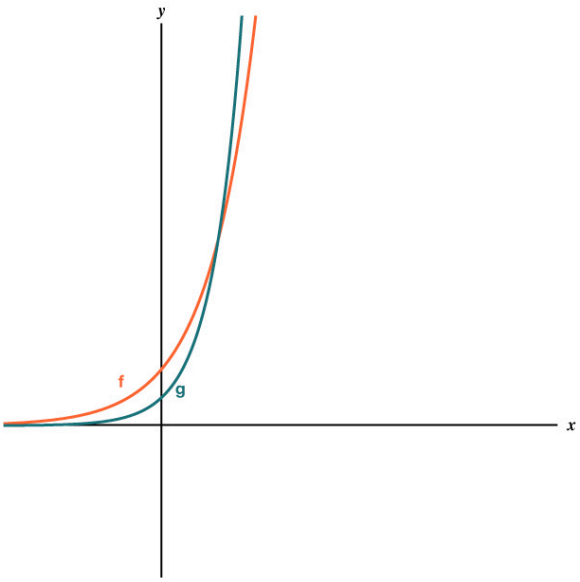
In the formula $f(x) = a \cdot b^x$, what changes on the graph when a changes?

a. The growth factor
b. The slope
c. The y-intercept
d. The x-intercept

Additional Resources

Reading Exponential Graphs

The graphs of f and g are located below and are functions of the form $y = a \cdot b^x$.



Which curve has the larger a value?

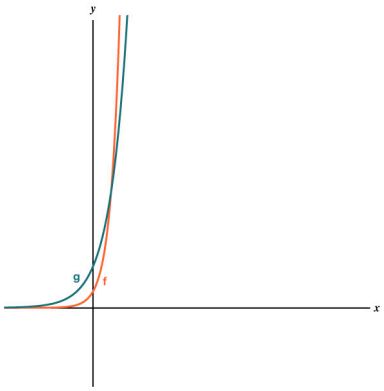
$f(x)$ because its y-intercept is higher.

Which curve has the larger b value?

$g(x)$ because it is growing more quickly.

TRY IT **Reading Exponential Graphs**

The graphs of f and g are located below and are functions of the form $y = ab^x$.

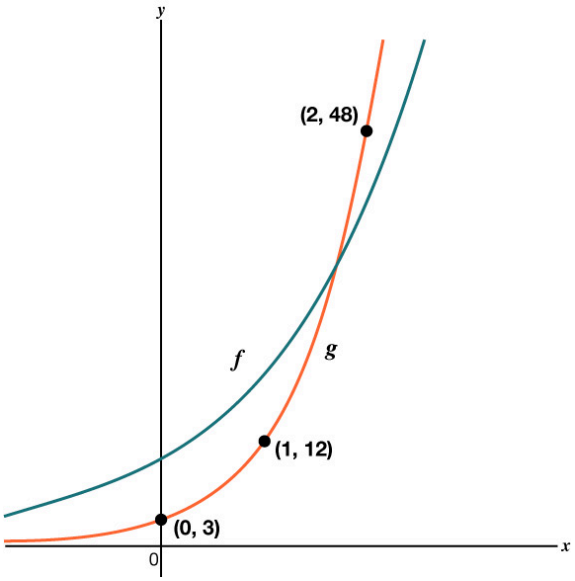


Which curve has the larger b value?

5.13.4 Interpreting Features of an Exponential Graph

Cool Down

Here are two graphs representing the function f given by $f(x) = 10 \cdot 2^x$ and the function g defined by $g(x) = a \cdot b^x$.



1. In function g, is b greater than or less than 2? Explain how you know.

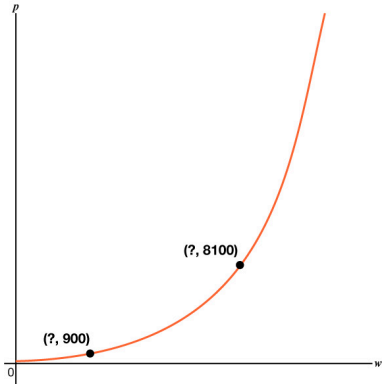
2. Write an equation that defines g. Explain your answer.

3. f and g represent the number, in thousands, of social media followers of two organizations as a function of years since 2010. What does the intersection of f and g mean in this context?

5.13.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Here is a graph of p, an insect population, w weeks after it was first measured. The population grows exponentially.



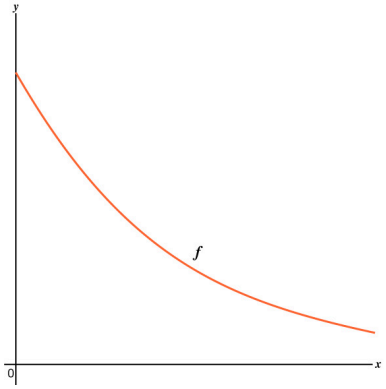
a. What is the weekly factor of growth for the insect population?

b. What was the population when it was first measured?

c. Write an equation relating p and w.

a. $p = -300 \cdot 3^w$
b. $p = \frac{300}{3^w}$
c. $p = 300 \cdot 3^w$
d. $p = 300 + 3^w$

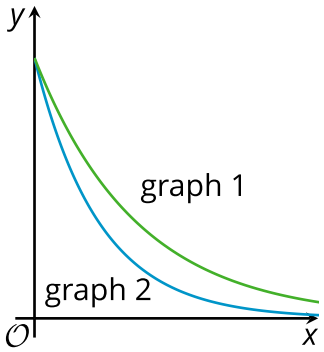
2. Here is a graph of the function f defined by $f(x) = a \cdot b^x$.



Select three possible values of b.

a. 0
b. $\frac{1}{10}$
c. $\frac{1}{2}$
d. $\frac{9}{10}$
e. 1
f. 1.3
g. $\frac{18}{5}$

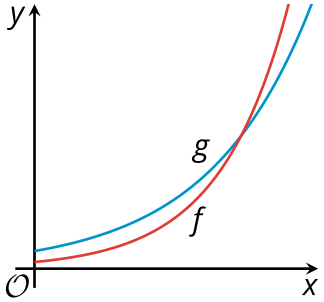
3. The function f is given by $f(x) = 50 \cdot (\frac{1}{2})^x$, and the function g is given by $g(x) = 50 \cdot (\frac{1}{3})^x$
Here are graphs of f and g.



Which graph is the graph of $f(x)$?

a. Graph 1
b. Graph 2

4. The function f is defined by $f(x) = 50 \cdot 3^x$. The function g is defined by $g(x) = a \cdot b^x$.
Here are graphs of f and g.



a. In $g(x)$, is a greater than or less than 50?

a. less than
b. greater than

b. In $g(x)$, is b greater than or less than 3?

a. less than
b. greater than

5.13.6 Lesson Summary

In this lesson, you learned how to:

- Explain the meaning of the intersection of the graphs of two functions in terms of the situations they represent.
- Write an equation for the function given two points on a graph of an exponential function.

Here are the **activities** that helped you reach those goals:

- 5.13.1: Comparing Functions
  - In this activity, you looked closely at functions and explained the ways in which they were similar or distinct.
- 5.13.2: Creating Exponential Functions from Graphs
  - In this activity, you analyzed graphs of exponential functions to identify the initial value and the growth factor. You then created corresponding equations for the functions.
  - 5.13.2: Self Check
  - 5.13.2: Additional Resources
- 5.13.3: Comparing Exponential Functions
  - In this activity, you were presented with two graphs without numerical values assigned to any points on the graphs. Using the graphs, you compared their starting values and growth factors in the context of the situation. You also analyzed the point of intersection.
  - 5.13.3: Self Check
  - 5.13.3: Additional Resources
- 5.13.4: Interpreting Features of an Exponential Graph
  - In this activity, you used the graph of a given exponential function to determine possible values in the equation of the other exponential function.

After these activities, you completed the following **practice**:

- 5.13.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

5.14 WHICH ONE CHANGES FASTER?

5.14.0 Lesson Overview

In this lesson, you will revisit the fact that exponential functions grow more quickly, eventually, than linear functions. You will examine tables and graphs and see that the exponential function quickly overtakes the linear function.

When you finish this lesson, you will be able to:

- Use tables, calculations, and graphs to compare growth rates of linear and exponential functions and predict how the quantities eventually change.

Here are the **activities** that will help you reach those goals:

- 5.14.1: Distinguish Linear and Exponential Growth
- 5.14.2: Using Tables to Compare Linear and Exponential Growth Functions
  - 5.14.2: Self Check
  - 5.14.2: Additional Resources
- 5.14.3: Compare Linear Functions with Exponential Functions
  - 5.14.3: Self Check
  - 5.14.3: Additional Resources
- 5.14.4: Using Function Notation to Compare Linear and Exponential Functions

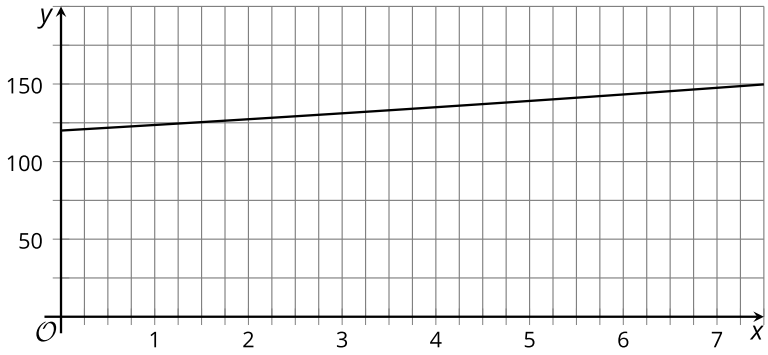
After these activities, you'll' **practice and review**:

- 5.14.5: Practice
- 5.14.6: Lesson Summary

5.14.1 Distinguish Linear and Exponential Growth

Warm Up

Here is a graph.



1. Which equation do you think the graph represents? Use the graph to support your reasoning.
- $y = 120 + (3.7) \cdot x$
- $y = 120 \cdot (1.03)^x$

2. What information might help you decide more easily whether the graph represents a linear or an exponential function?

5.14.2 Using Tables to Compare Linear and Exponential Growth Functions

Activity

A family has $1,000 to invest and is considering two options: investing in government bonds that offer 2% simple interest, or investing in a savings account at a bank, which charges a $20 fee to open an account and pays 2% compound interest. Both options pay interest annually.

Here are two tables showing what they would earn in the first couple of years if they do not invest additional amounts or withdraw any money.

years of investment	amount in dollars
0	$1,000
1	$1,020
2	$1,040

Bonds

years of investment	amount in dollars
0	$980
1	$999.60
2	$1,019.59

Savings Account

1. Bonds: How does the investment grow with simple interest?
2. Savings account: How are the amounts $999.60 and $1,019.59 calculated?
3. Write an equation to represent the relationship between the amount of money and the number of years of investment for each option.

a. Write an equation for the data in the Bonds table.

b. Write an equation for the data in the Savings Account table.

4. Which investment option should the family choose? Use your equations or calculations to support your answer.
5. Use the Desmos graphing tool to graph the two investment options and show how the money grows in each.

Access multimedia content (http://openstax.org/books/algebra-1/pages/5-14-2-using-tables-to-compare-linear-and-exponential-growth-functions)

Video: Using Tables to Compare Linear and Exponential Functions

Watch the following video to learn more about comparing linear and exponential functions:

Using Tables to Compare Linear and Exponential Functions (https://openstax.org/r/compare-functions-tables)

Self Check

Using a calculator, Joanna made the following table and then made the following conjecture: $3x$ is always greater than $(1.02)^x$. Is Joanna correct? Explain.

x	$(1.02)^x$	$3x$
1	1.02	3
2	1.0404	6
3	1.0612	9
4	1.0824	12
5	1.1041	15

a. Joanna is incorrect. If you continue the table to $x = 300$, $(1.02)^x$ is greater than $3x$.
b. Joanna is incorrect. Since $(1.02)^x$ is exponential growth and $3x$ is linear growth, eventually $(1.02)^x$ will be greater than $3x$.
c. Joanna is correct. If you graph the points in the table for $(1.02)^x$ and $3x$, the points for $3x$ are all above the points for $(1.02)^x$.
d. Joanna is correct. None of the values in the table for $(1.02)^x$ are greater than the values for $3x$.

Additional Resources

Comparing Linear and Exponential Growth Using Tables

Classify each of the following tables as linear, exponential growth, or exponential decay.

x	$f(x)$
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$
4	$\frac{1}{16}$
5	$\frac{1}{32}$

Table A

x	$f(x)$
1	1.4
2	2.5
3	3.6
4	4.7
5	5.8

Table B

x	$f(x)$
1	30
2	60
3	120
4	240
5	480

Table C

In each situation, the inputs or x-values are increasing by 1 each time. These can also be called consecutive terms.

If you have consecutive terms, then:

- If the difference between their corresponding outputs or y-values is always the same constant, then the input-output pairs in the table can be modeled by a linear function.
- If the quotient between their corresponding outputs or y-values is always the same constant, then the input-output pairs in the table can be modeled by an exponential function. A growth rate greater than one means that it is exponential growth, and a growth rate between zero and one means that it is exponential decay.

It is possible for a table to be a set of values that is not any of these relationships.

Using this information, you can see that in Table A, $\frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$. This holds true for the quotient of every other set of consecutive terms, too. Since the quotient is always the same, Table A represents an exponential function. Since the growth rate is $\frac{1}{2}$, or between zero and one, Table A represents exponential decay.

In Table B, the quotient of consecutive terms is not the same, but there is a common difference. The difference between the first two terms is $2.5 - 1.4 = 1.1$. This holds true for each of the other pairs of consecutive terms, so Table B represents a linear relationship.

In Table C, the quotient of consecutive terms is $\frac{60}{30} = 2$. This relationship holds true for all of the pairs of consecutive terms, which means this table represents an exponential relationship. Since the growth factor is greater than one, Table C represents exponential growth.

TRY IT **Try It: Comparing Linear and Exponential Growth Using Tables**

Which of the following tables represents exponential growth? Be prepared to show your reasoning.

x	$f(x)$
1	2
2	8
3	14
4	20

Table A

x	$f(x)$
1	432
2	144
3	48
4	16

Table B

x	$f(x)$
1	4
2	16
3	64
4	256

Table C

x	$f(x)$
1	6
2	4
3	9
4	2

Table D

5.14.3 Compare Linear Functions with Exponential Functions

Activity

Consider the functions $f(x) = 2x$ and $g(x) = (1.01)^x$.

1. Complete the table of values for the functions f and g.

x	$f(x)$	$g(x)$
1		
10		
50		

x	$f(x)$	$g(x)$
100		
500		

2. Based on the table of values, which function do you think grows faster? Be prepared to show your reasoning.
3. Which function do you think will reach a value of 2,000 first? Show your reasoning. If you get stuck, consider increasing x by 100 a few times and record the function values in the table.

Are you ready for more?

Extending Your Thinking

Consider the functions $g(x) = x^5$ and $f(x) = 5^x$. While it is true that $f(7) > g(7)$, for example, it is hard to check this using mental math. Find a value of x for which properties of exponents allow you to conclude that $f(x) > g(x)$ without a calculator.

Self Check

Terence looked down the second column of the table below and noticed that $\frac{3}{1} = \frac{9}{3} = \frac{27}{9} = \frac{81}{27}$. Because of his observation, he claimed that the input-output pairs in this table could be modeled with an exponential function. Explain why Terence is correct or incorrect.

x	$T(x)$
0	1
1	3
4	9
13	27
40	81

a. Terence is incorrect. This table represents a linear relationship since the ratio of change of each row is consistent.
b. Terence is incorrect. This table represents points that do not have an exponential or linear relationship because the x and y change by different amounts.
c. Terence is correct. This table can be modeled by $y = 1(3)^x$.
d. Terence is correct. Terms in a table that are connected by a factor are exponential.

Additional Resources

Determining and Comparing Linear and Exponential Functions

Mr. Smith has an apple orchard. He hires his daughter, Lucy, to pick apples and offers her two payment options:

Option A: \$1.50 per bushel of apples picked

Option B: 1 cent for coming to work, 3 cents for picking one bushel, 9 cents for picking two bushels, 27 cents for picking three bushels, and so on, with the amount of money tripling for each additional bushel picked

1. Which option is linear, and which option is exponential? How do you know?
2. Create a table to model each scenario.
3. If she picks 6 bushels, which option is better?
4. If she picks 12 bushels, which option is better?

5. How many bushels does she need to pick for Option B to be better than Option A?

Let's first think about linear and exponential functions in general. Let's also consider Options A and B when x represents the bushels of apples picked and $f(x)$ represents the total amount she earns based on x bushels of apples.

1. Since Option A is \$1.50 per bushel picked, it is linear. For Option B, the amount of money is tripling for each bushel picked, so it is exponential.

	Linear Model	Exponential Model
General Form	$f(x) = ax + b$	$f(x) = a(b)^x$
Meaning of Parameters a and b	a is the slope of the line or the constant rate of change; b is the y-intercept or the $f(x)$ value at $x = 0$.	a is the y-intercept or the $f(x)$ value when $x = 0$; b is the base or the constant quotient of change.
Example	$f(x) = 1.50x$	$f(x) = .01(3)^x$
Rule for Finding $f(x+1)$ from $f(x)$	Starting at $(0, 0)$, to find $f(x+1)$, add 1.5 to $f(x)$.	Starting at $(0, 0.1)$, to find $f(x+1)$, multiply $f(x)$ by 3.

x	$f(x)$
0	0
1	1.50
2	3.00
3	4.50

Linear Model Table

x	$f(x)$
0	0.01
1	0.03
2	0.09
3	0.27

Exponential Model Table

2. To create the table for Option A, add \$1.50 to each consecutive term. To create the table for Option B, multiply each term by 3 to get the next term.
Option A table

Number of bushels	Amount of money earned
1	1.50
2	3.00
3	4.50
4	6.00
5	7.50
6	9.00

Option B table

Number of bushels	Amount of money earned
1	0.03
2	0.09
3	0.27
4	0.81
5	2.43
6	7.29

3. To determine which scenario is better when she picks 6 bushels, you can look at the values in your table, or you can write the equation modeling each scenario and substitute in 6 for x. For 6 bushels, Option A is better at \$9.00.
4. To determine which scenario is better when she picks 12 bushels, you can extend the table or use the graph or equation. Option A is \$18.00 for 12 bushels, and Option B is \$5,314.41 for 12 bushels, so Option B is better.
5. To determine where Option B becomes the better option, you can extend the table, or you can graph the functions and find the x-value where Option B surpasses Option A. You can also use problems 3 and 4 to help you. The number of bushels is more than 6 but less than 12. Option B is better if you are going to pick 7 or more bushels of apples.

TRY IT **Determining and Comparing Linear and Exponential Functions**

Jayden has a dog-walking business. He has two plans. Plan 1 includes walking a dog once a day for a rate of \$5 per day. Plan 2 also includes one walk a day but charges 1 cent for 1 day, 2 cents for 2 days, 4 cents for 3 days, and 8 cents for 4 days, and it continues to double for each additional day. Mrs. Maroney needs Jayden to walk her dog every day for two weeks. Which plan should she choose? Show the work to justify your answer.

5.14.4 Using Function Notation to Compare Linear and Exponential Functions

Cool Down

The function f is given by $f(x) = 10x + 3$, and the function g is given by $g(x) = 2^x$. For each question, show your reasoning.

1. Which function reaches 50 first?

2. Which function reaches 100 first?

5.14.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Functions a, b, c, d, e, and f are given below. Which functions are linear? **Select the three** answers that apply.

a. $a(x) = 3x$
b. $b(x) = 3^x$
c. $c(x) = x^3$
d. $d(x) = 9 + 3x$
e. $e(x) = 9 \cdot 3^x$
f. $f(x) = 9 \cdot 3x$

2. Functions a, b, c, d, e, and f are given below. Which functions are exponential? **Select the two** answers that apply.

a. $a(x) = 3x$
b. $b(x) = 3^x$
c. $c(x) = x^3$
d. $d(x) = 9 + 3x$
e. $e(x) = 9 \cdot 3^x$
f. $f(x) = 9 \cdot 3x$

3. Here are 4 equations defining 4 different functions, a, b, c, and d. List them in order of increasing rate of change. That is, start with the one that grows the slowest and end with the one that grows the quickest.

- $a(x) = 5x + 3$
- $b(x) = 3x + 5$
- $c(x) = x + 4$
- $d(x) = 1 + 4x$

a. Option 1: $a(x)$, $b(x)$, $c(x)$, $d(x)$
b. Option 2: $d(x)$, $c(x)$, $b(x)$, $a(x)$
c. Option 3: $a(x)$,$d(x)$, $b(x)$, $a(x)$
d. Option 4: $c(x)$, $b(x)$, $d(x)$, $a(x)$

Use the given information for problems 4 – 6. Technology is required. Function f is defined by $f(x) = 3x + 5$, and function g is defined by $g(x) = (1.1)^x$.

	$f(x)$	$g(x)$
1		
5		
10		
20		

4. Which values will complete the table for $f(x)$?

a. 8, 15, 30, 60
b. 3, 15, 30, 60
c. 8, 20, 35, 65
d. 8, 11, 14, 17

5. Which of the following sets of values will complete the table for $g(x)$?

a. 1.1, 1.21, 1.33, 1.46
b. 1.1, 1.6, 2.59, 6.73

c. 1.1, 5.5, 10.10, 20.20
d. 8, 20, 35, 65

6. a. _____ is a linear function.

a. $f(x)$
b. $g(x)$
c. neither

b. _____ is an exponential function.

a. $f(x)$
b. $g(x)$
c. neither

7. Functions m and n are given by $m(x) = (1.05)^x$ and $n(x) = \frac{5}{8}x$. As x increases from 0, _____ reaches 30 first.

a. $m(x)$
b. $n(x)$
c. neither

8. Functions m and n are given by $m(x) = (1.05)^x$ and $n(x) = \frac{5}{8}x$. As x increases from 0, _____ reaches 100 first.

a. $m(x)$
b. $n(x)$
c. neither

9. The functions f and g are defined by $f(x) = 8x + 33$ and $g(x) = 2 \cdot (1.2)^x$. Which function eventually grows faster, f or g? Explain how you know.

a. $g(x)$ grows faster because it is an exponential function.
b. $g(x)$ grows faster because it is a linear function.
c. $f(x)$ grows faster because 33 is greater than 2.
d. $f(x)$ grows faster because 8 is greater than 1.2.

10. A line segment of length l is scaled by a factor of 1.5 to produce a segment with length m. The new segment is then scaled by a factor of 1.5 to give a segment of length n.

What scale factor takes the segment of length l to the segment of length n?

5.14.6 Lesson Summary

In this lesson, you learned how to:

- Use tables, calculations, and graphs to compare growth rates of linear and exponential functions and predict how the quantities eventually change.

Here are the **activities** that helped you reach those goals:

- 5.14.1: Distinguish Linear and Exponential Growth
  - In this activity, you analyzed a graph to determine if it was linear or exponential growth. You also had to justify your response and decide what information would make it easier to determine if the graph was linear or exponential.
- 5.14.2: Using Tables to Compare Linear and Exponential Growth Functions
  - In this activity, you compared linear and exponential growth in a context involving simple and compound interest. You were also required to analyze the necessary mathematics in the context of a real-world situation and justify your choices.
  - 5.14.2: Self Check
  - 5.14.2: Additional Resources
- 5.14.3: Compare Linear Functions with Exponential Functions
  - In this activity, you compared a linear function with an exponential one without a context, and you noticed that the exponential function grew much more slowly over a long period of time.
  - 5.14.3: Self Check
  - 5.14.3: Additional Resources
- 5.14.4: Using Function Notation to Compare Linear and Exponential Functions
  - In this activity, you explored approaches to estimate values to compare the two functions when only the equations are provided.

After these activities, you completed the following **practice and review**:

- 5.14.5: Practice
- 5.14.6: Lesson Summary

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

5.15 CHANGES OVER EQUAL INTERVALS

5.15.0 Lesson Overview

In this lesson, you will work to show that linear functions change by equal differences over equal intervals while exponential functions grow by equal factors over equal intervals.

When you finish this lesson, you will be able to:

- Calculate rates of change of functions given graphs, equations, or tables.
- Use rates of change to describe how a linear function and an exponential function change over equal intervals.

Here are the **activities** that will help you reach those goals:

- 5.15.1: Writing Equivalent Expressions
- 5.15.2: Linear Function Constant Rate of Change
  - 5.15.2: Self Check
  - 5.15.2: Additional Resources
- 5.15.3: Growth Rate of an Exponential Function
  - 5.15.3: Self Check
  - 5.15.3: Additional Resources
- 5.15.4: Comparing Linear and Exponential Growth

After that, you'll **practice and review**.

- 5.15.5: Practice
- 5.15.6: Lesson Summary

5.15.1 Writing Equivalent Expressions

Warm Up

For each given expression, write an equivalent expression with as few terms as possible.

1. $7p - 3 + 2(p + 1)$

2. $[4(n + 1) + 10] - 4(n + 1)$

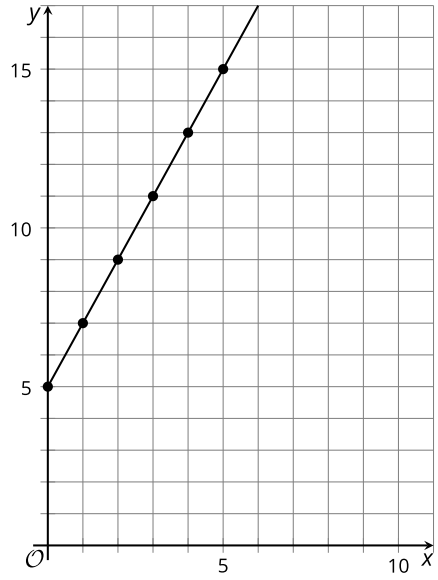
3. $9^5 \cdot 9^2 \cdot 9^x$

4. $\frac{2^{4n}}{2^n}$

5.15.2 Linear Function Constant Rate of Change

Activity

Here is a graph of $y = f(x)$ where $f(x) = 2x + 5$.



1. How do the values of f change whenever x increases by 1, for instance, when it increases from 1 to 2, or from 19 to 20? Be prepared to explain or show how you know.
2. Here is an expression we can use to find the difference in the values of f when the input changes from x to $x + 1$.

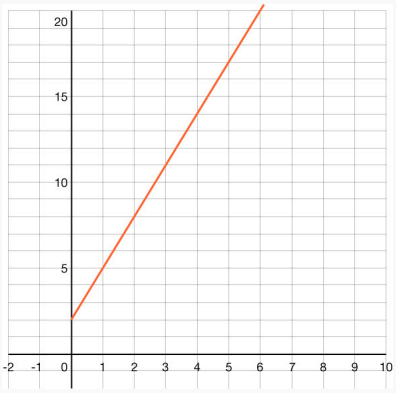
$[2(x + 1) + 5] - [2x + 5]$

Does this expression have the same value as what you found in the previous questions? Show your reasoning.

3. How do the values of f change whenever x increases by 4? Explain or show how you know.
4. Write an expression that shows the change in the values of f when the input value changes from x to $x + 4$.
5. Show or explain how that expression has a value of 8.

Self Check

Here is a graph of $y = f(x)$ where $f(x) = 3x + 2$.



How do the values of f change whenever x increases by 2?

a. The values of f increase by 1 whenever x increases by 2.
b. The values of f increase by 6 whenever x increases by 2.
c. The values of f increase by 3 whenever x increases by 2.
d. The values of f increase by 2 whenever x increases by 2.

Additional Resources

Determining Rate of Change of Linear Functions

For the table below, assume the function f is defined for all real numbers. Calculate $\triangle f = f(x+1) - f(x)$ in the last column. (The symbol $\triangle$ in this context means "change in.") What do you notice about $\triangle f$? Could the function be linear or exponential? Write a linear or an exponential function formula that generates the same input–output pairs as given in the table.

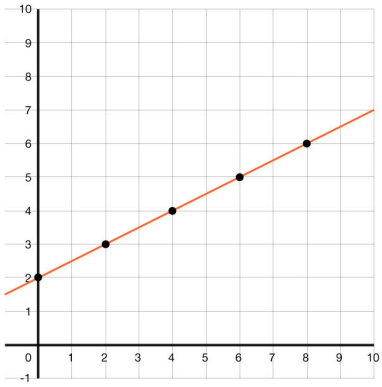
x	$f(x)$	$\triangle f = f(x+1) - f(x)$
1	-3	
2	1	
3	5	
4	9	
5	13	

For each row, you subtract $f(x)$ values. The first row would be $1-(-3)=4$, then $5-1=4$, then $9-5=4$, and then $13-9=4$. Think back to the lessons on the slope or rate of change of a linear function. The numerator was always the change in y or output, which is the same as the change in $f(x)$. Notice that all the changes in consecutive terms are the same, 4. This means that there is a constant rate of change and this particular function is linear.

Since this function is linear, an equation will have the form $f(x) = rate\ \ of\ \ change \cdot x + initial\ /value$. The rate of change is 4. The initial value is the same as the y-intercept, or where $x = 0$. $f(0)$ would be 4 less than the first term listed in this case, which means that $f(0) = -7$. The linear equation for this table would be $f(x) = 4x - 7$.

TRY IT **Determining Rate of Change of Linear Functions**

For the graph provided, assume that the function is defined for all real numbers. What is the rate of change? Write an equation that would define this function.



5.15.3 Growth Rate of an Exponential Function

Activity

Here is a table that shows some input and output values of an exponential function g. The equation $g(x) = 3^x$ defines the function.

x	$g(x)$
3	27
4	81
5	243
6	729
7	2,187
8	6,561
x	
$x+1$	

1. How does $g(x)$ change every time x increases by 1? Show or explain your reasoning.
2. Choose two new input values that are consecutive whole numbers and find their output values. Record them in the table. How do the output values change for those two input values?
3. Complete the table with the output when the input is x and when it is $x+1$.
4. Look at the change in output values as the x increases by 1. Does it still agree with your earlier findings? Show your reasoning.

Pause here for a class discussion. Then, work with your group on the next few questions.

5. Choose two x-values where one is 3 more than the other (for example, 1 and 4). How do the output values of g change as x increases by 3? (Each group member should choose a different pair of numbers and study the outputs.)
6. Complete this table with the output when the input is x and when it is $x+3$. Look at the change in output values as x increases by 3. Does it agree with your group's findings in the previous question? Show your reasoning.

x	$g(x)$
x	
$x+3$	

7. Summarize the process you used in questions 1 – 6 to analyze this data.
8. Explain if you found it helpful or not to analyze the data for simpler data (1 - 2) before analyzing it for more abstract data (questions 3 – 6).

Are you ready for more?

Extending Your Thinking

For integer inputs, we can think of multiplication as repeated addition and exponentiation as repeated multiplication:

Multiplication representing repeated addition	Exponentiation representing repeated multiplication
$3 \cdot 5 = 3 + 3 + 3 + 3 + 3$	$3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

We could continue this process with a new operation called tetration. It uses the symbol $\uparrow\uparrow$, and it is defined as repeated exponentiation:

Tetration representing repeated exponentiation
$3 \uparrow\uparrow 5 = 3^{3^{3^{3^3}}}$

1. Compute $2 \uparrow\uparrow 3$ and $3 \uparrow\uparrow 2$.
2. Compute $f(x) = 3 \uparrow\uparrow x$.
3. What is the relationship between $f(x)$ and $f(x+1)$?

Video: Analyzing Tables of Exponential Functions

Watch the following video to learn more about analyzing the tables of exponential functions:

Analyzing Tables of Exponential Functions (https://openstax.org/r/exponential-tables)

Self Check

Given $f(x) = 4^x$, what is $\frac{f(x+2)}{f(x+1)}$?

a. 16
b. There is not enough information to tell.
c. 4
d. 2

Additional Resources

Determining the Growth Rate of Exponential Functions

For the table below, assume the function f is defined for all real numbers. Calculate $\triangle f = f(x+1) - f(x)$ in the last column. (The symbol $\triangle$ in this context means "change in.") What do you notice about $\triangle f$? Could the function be linear or exponential? Write a linear or an exponential function formula that generates the same input–output pairs as given in the table.

x	$f(x)$	$\triangle f = f(x+1) - f(x)$
0	2	
1	6	
2	18	

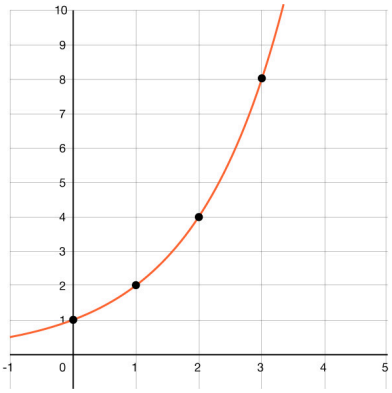
x	$f(x)$	$\triangle f = f(x+1) - f(x)$
3	54	
4	162	

For each row, you subtract $f(x)$ values. The first row would be $6 - 2 = 4$, then $18 - 6 = 12$, then $54 - 18 = 36$, and then $162 - 54 = 108$. Since the rate of change isn't consistent, it is not a linear function. Exponential functions have a growth factor or a repeated rate that terms are being multiplied by. If you divide consecutive terms, you will find the growth rate is 3. Notice that the change between consecutive terms also has a growth rate of 3.

Since this function is exponential, an equation will have the form $f(x) = initial\ \ value\ \ (growth\ rate)^x$. The growth rate is 3. The initial value is the same as the y-intercept, or where $x = 0$. $f(0)$ is given in the table, so the initial value is 2. The equation for this table would be $f(x) = 2(3)^x$.

TRY IT **Determining the Growth Rate of Exponential Functions**

For the graph provided, assume that the function is defined for all real numbers. What is the rate of change? Write an equation that would define this function.



5.15.4 Comparing Linear and Exponential Growth

Cool Down

1. The linear function g is defined by $g(x) = 4x$.
   a. Show that $g(3)$ and $g(2)$ have a difference of 4.
   b. Show that when the input value increases from x to $x + 1$, the output values $g(x + 1)$ and $g(x)$ have a difference of 4.

2. The exponential function h is defined by $h(x) = 4^x$.
   a. Show that $h(3)$ and $h(2)$ have a quotient of 4.
   b. Show that when the input value increases from x to $x + 1$, the output values $h(x + 1)$ and $h(x)$ have a quotient of 4.

Why Should I Care?



Exponential models can help you understand the power of investment. After two years, this investor made $210 off of an initial investment of $1000 because of the 10% interest rate.

Can you calculate how much money the investor will have made after twenty years?

5.15.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

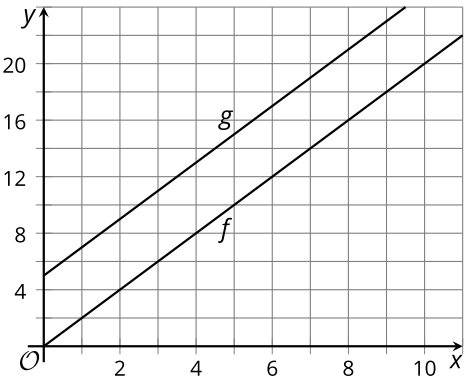
1. Whenever the input of a function f increases by 1, the output increases by 5. Which of these equations could define f?

a. $f(x) = x^5$
b. $f(x) = 5^x$
c. $f(x) = 5x + 3$
d. $f(x) = 3x + 5$

2. The function f is defined by $f(x) = 2^x$. Which of the following statements is true about the values of f? **Select the two statements** that apply.

a. When the input x increases by 1, the value of f increases by 2.
b. When the input x increases by 1, the value of f increases by a factor of 2.
c. When the input x increases by 3, the value of f increases by 8.
d. When the input x increases by 3, the value of f increases by a factor of 8.
e. When the input x increases by 4, the value of f increases by a factor of 4.

Use the graph below for questions 3 and 4.
The two lines on the coordinate plane are graphs of functions f and g.



3. Use the graph to explain why the value of f increases by 2 each time the input x increases by 1.

a. The y-axis counts by 2's so f must increase 2 each time the input increases 1.
b. There are two graphs so f increases by 2 each time the input x increases by 1.
c. $(0, 0)$ and $(1, 2)$ are two points on the graph of f. Between the two points, y increases by 2 and x increases by 1.
d. f does not increase by 2 each time x increases by 1.

4. Use the graph to explain why the value of g increases by 2 each time the input x increases by 1. **Select two** that apply.

a. f and g never intersect and stay an equal distance apart. f increases by 2 every time x increases by 1, so g must do so as well.
b. You can't use the graph because the points are not labeled.
c. There are two graphs so g increases by 2 each time the input x increases by 1.
d. $(0, 5)$and $(1, 7)$ are points on the graph of g. Between the two points, y increases by 2 and x increases by 1.
e. $(0, 0)$ and $(1, 2)$ are two points on the graph of g. Between the two points, y increases by 2 and x increases by 1.

5. The function h is given by $h(x) = 5^x$. Find the quotient $\frac{h(x+2)}{h(x)}$.

6. The function h is given by $h(x) = 5^x$. Find the quotient $\frac{h(x+3)}{h(x)}$.

a. 125

b. 25
c. 15
d. 5

7. For each of the functions f, g, h, p, and q, the domain is $0 \leq x \leq 100$. For which functions is the average rate of change a good measure of how the function changes for this domain? **Select three** that apply.

a. $f(x) = x + 2$
b. $g(x) = 2^x$
c. $h(x) = 111x - 23$
d. $p(x) = 50,000 \cdot 3^x$
e. $q(x) = 87.5$

8. What would be the rate of change of the following table?

x	$f(x)$
1	-4
2	-2
3	0
4	2
5	4

9. Given the following table, what would be $\frac{f(x+2)}{f(x+1)}$?

x	$f(x)$
1	2
2	4
3	8
4	16
5	32

10. Which of the following statements compare rates of $f(x) = 3x + 2$ and $g(x) = 3^x$?

a. In g you add 3 to get to the next term; in f you multiply by 3 to get to the next term.
b. In f you add 3 to get to the next term; in g you multiply by 3 to get to the next term.
c. In f and g you add 3 to get to the next term.
d. In f and g you multiply by 3 to get to the next term.

5.15.6 Lesson Summary

In this lesson, you learned how to:

- Calculate rates of change of functions given graphs, equations, or tables.
- Use rates of change to describe how a linear function and an exponential function change over equal intervals.

Here are the **activities** that helped you reach those goals:

- 5.15.1: Writing Equivalent Expressions
  - In this activity, you practiced writing equivalent expressions using the order of operations, combining like terms, and using properties of exponents.
- 5.15.2: Linear Function Constant Rate of Change
  - In this activity, you revisited and extended your understanding of the constant rate of change for a linear function.
  - 5.15.2: Self Check
  - 5.15.2: Additional Resources
- 5.15.3: Growth Rate of an Exponential Function
  - In this activity, you examined what happens to the values of an exponential function f when the input is increased by 1 and then when its value changes by a factor of b.
  - 5.15.3: Self Check
  - 5.15.3: Additional Resources
- 5.15.4: Comparing Linear and Exponential Growth
  - In this activity, you compared linear and exponential growth.

After these activities, you completed the following **practice**:

- 5.15.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

PROJECT 5: INTRODUCTION TO EXPONENTIAL FUNCTIONS

Project 5 Overview

In this project, you will use the main function types you have studied thus far in the course (linear and exponential) to model different populations. In the first activity, data for three city populations will be given and you will be asked to produce a linear or exponential model for each (if appropriate) and then make predictions for populations at other dates.

In the second activity, you will examine the world population. The task is more open-ended and only limited data is provided, likely requiring you to gather more data.

Throughout the project, you will engage in different parts of the modeling cycle. Be careful - you will have to think carefully about how to choose an appropriate linear or exponential model.

Project Activities

1. Analyze Data
2. Writing Equations to Model Populations
3. Open-ended Modeling with Data Investigation

Learning Targets

When you finish this project, you will have demonstrated how to:

- Determine how well a chosen model fits the given information.
- Determine whether to use a linear function or an exponential function to model real-world data.

Analyze Data

The table below shows the populations of three cities during different years. Take a moment to think of at least one thing you notice and one thing you wonder.

City	1950	1960	1970	1980	1990	2000
Paris	6,300,000	7,400,000	8,200,000	8,700,000	9,300,000	9,700,000
Austin	132,000	187,000	254,000	346,000	466,000	657,000
Chicago	3,600,000	3,550,000	3,400,000	3,000,000	2,800,000	2,900,000

Write Equations to Model Populations

Use the following prompt and table to answer questions 1 – 7.

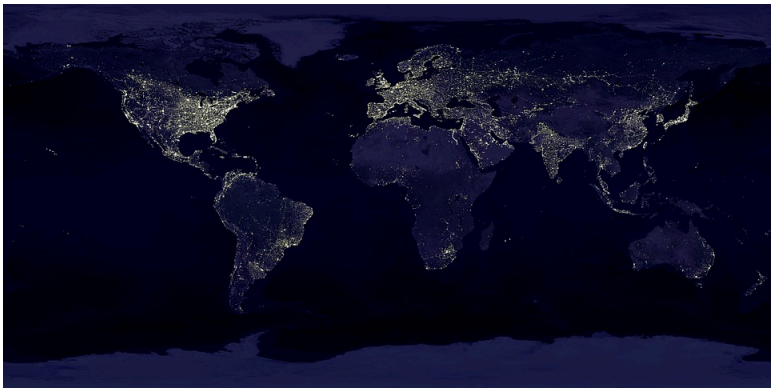
The table below shows the population data for three cities at different times between 1950 and 2000.

City	1950	1960	1970	1980	1990	2000
Paris	6,300,000	7,400,000	8,200,000	8,700,000	9,300,000	9,700,000
Austin	132,000	187,000	254,000	346,000	466,000	657,000
Chicago	3,600,000	3,550,000	3,400,000	3,000,000	2,800,000	2,900,000

1. What does the data tell us, if anything, about the current population in the cities or what the population will be in 2050?
2. How would you describe the population change in each city during this time period? Write one to two sentences for

each city. Then discuss with your group.
3. What kind of model (linear, exponential, both, or neither) do you think is appropriate for each city's population?
4. For each population that you think can be modeled by a linear and/or exponential function:
   a. Write an equation for the function(s).
   b. Graph the function(s).
5. Compare the graphs of your functions with the actual population data. How well do the models fit the data?
6. Use your models to predict the population in each city in 2010, the current year, and 2050.
7. Do you think that these predictions are (or will be) accurate? Be prepared to show your reasoning.

Open-ended Modeling with Data Investigation



year	1804	1927	1960	1974	1987	1999	2011
world population in billions	1	2	3	4	5	6	7

1. Would a linear function be appropriate for modeling the world population growth over the last 200 years? Explain. If you think it is appropriate, find a linear model.
2. Would an exponential function be appropriate for modeling the world population growth over the last 200 years? Explain. If you think it is appropriate, find an exponential model.
3. From 1950 to the present day, by about what percentage has the world population grown each year?
4. From 1950 to the present day, by about how many people has the world population grown each year?
5. If the growth trend continues, what will the world population be in 2050? How long do you think the growth will continue? Be prepared to show your reasoning using graphic displays that support your arguments..

Unit 5 Wrap Up

Throughout this unit, you learned that Operation Hope founder John Hope Bryant helped underserved populations become part of the economy and work toward financial freedom. Teaching people about finance charges and the growth of investments involves looking at exponential models and comparing rates of change.

Similarly, in this unit, you learned about exponential relationships. You learned that in an exponential relationship, the y values are being multiplied by the same number called the growth factor. You learned that exponential growth functions increase as x increases and exponential decay functions decrease as x increases.

You also learned to identify key features of exponential graphs and to interpret graphs related to real world contexts.

Finally, you compared exponential functions to linear functions. While linear functions have a constant rate of change, exponential functions use the average rate of change over different intervals to describe how the data is changing. You compared linear growth to exponential growth using tables and graphs to verify that exponential functions eventually overtake the linear function.



Access our Unit 5 Student Self-Assessment (https://openstax.org/r/unit-5-self-assesment) to reflect on the skills you learned during this unit. Congratulations on completing Unit 5!

Building Character: Grit

Throughout this unit, you learned that **grit** is important to your success. It is your ability to use your passion and persevere to accomplish long term goals. Take time to find a purpose that will require you to find value in yourself and serve others. Grit is held up with determination and toughness which allows you to follow through on things you started. Look at your struggles as opportunities to learn.

UNIT 6 OVERVIEW AND READINESS

Unit 6 Overview

Why Should I Care?

Do you think you would want to open your own business someday? Watch the video to learn a bit about how Ozzy Gamez and Juan Quezada opened the Plant Shop Chicago and how they can use polynomials to make sure the business is successful.

Unit 6 Overview (https://openstax.org/r/plant-shop-polynomials)

In this unit, you will learn about polynomials. Understanding how to perform operations with polynomials can help you plan a budget for the school fundraiser so you can decide the best price to charge for the products and how to account for the shipping costs.

Building Character: Judgment

Ozzy Gamez grew up in Belize, and Juan Quezada grew up in Mexico. However, they had a mutual love of plants. Ozzy and Juan turned their love of plants into a thriving business. Plant Shop Chicago opened its doors in 2021 with the goal of growing the plant community. It takes hard work, planning, and a keen sense of judgment to grow a successful business. Plant Shop Chicago uses digital tools to make the business run efficiently. They could use polynomials to plan their revenue and costs so they can make a profit.

Throughout this unit, you will work on developing your **judgment**. We expect good judgment from doctors when they are treating us, teachers when they are grading assignments, and employers when they are reviewing resumes. A key factor in making sound judgements is not jumping to conclusions. Your judgment will never be perfect, but you can strive to be aware of bias and make corrections when necessary.

Think about yourself. When you face a problem that requires judgment, how many of these things are true?

- I try not to jump to conclusions.
- I am aware that this is a matter of judgment: I ask myself what someone I trust would think of this problem.
- I independently ask several people for their judgment, and I consider their reasons.
- I try to separate my hopes and fears from the facts and beliefs supporting the judgment.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

Am I Ready to Learn This?

Getting Ready for Unit 6

To be ready for this unit, you need to be able to:

- Simplify expressions with exponents.
- Simplify expressions using the Distributive Property.
- Find the greatest common factor (GCF) of two or more numbers.

Next, you will answer some questions that will help you see how ready you are for this unit. If you need help, some mini-lessons will help you brush up on anything you need to work on.

Simplify Expressions Using Properties of Exponents: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #1: Simplify Expressions Using Properties of Exponents

Which expression is correct?

a. $\frac{x^8}{x^4} = x^2$

b. $\frac{y^6}{y^2} = y^3$

c. $x^2 \cdot x^3 = x^6$

d. $y \cdot y^3 = y^4$

Simplify Expressions Using Properties of Exponents

In the expression $y^4 \cdot y^2$:

$y^4 = y \cdot y \cdot y \cdot y$

The exponent 4 means there are 4 factors of y.

$y^2 = y \cdot y$

The exponent 2 means there are 2 factors of y.

$y^4 \cdot y^2 = y \cdot y \cdot y \cdot y \cdot y \cdot y$

There are 6 factors of y, which can be written as y^6.

$y^4 \cdot y^2 = y^6$

Notice the exponent is the sum of the exponents of the factors.

When you multiply exponential expressions with the same base, you keep the base and add the exponents.

Product Property for Exponents

If a is a real number and m and n are integers, then

$a^m \cdot a^n = a^{m+n}$

To multiply with like bases, add the exponents.

In the expression $\frac{y^5}{y^3}$:

$y^5 = y \cdot y \cdot y \cdot y \cdot y$

The exponent 5 means there are 5 factors of y.

$y^3 = y \cdot y \cdot y$

The exponent 3 means there are 3 factors of y.

$\frac{y^5}{y^3} = \frac{y \cdot y \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y}}{\cancel{y} \cdot \cancel{y} \cdot \cancel{y}} = y \cdot y$

There are 2 factors of y, which can be written as y^2.

$\frac{y^5}{y^3} = y^2$

Notice the exponent is the difference of the exponents of the factors.

When you divide exponential expressions with the same base, you keep the base and subtract the exponents.

Quotient Property for Exponents

If a is a real number, $a \neq 0$, and m and n are integers, then

$\frac{a^m}{a^n} = a^{m-n}$, $m > n$ and $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$, $n > m$

Here's how to simplify expressions using properties of exponents.

1. $x^2 \cdot x^3$

The bases are the same. To multiply, use the Product Property for Exponents.

$x^2 \cdot x^3 = x^{2+3} = x^5$. To multiply with like bases, add the exponents.

2. $y \cdot y^4$

The bases are the same. To multiply, use the Product Property for Exponents.

Remember: A variable with no exponent shown has an exponent of 1, $y = y^1$.

$y \cdot y^4 = y^1 \cdot y^4 = y^{1+4} = y^5$. To multiply with like bases, add the exponents.

3. $\frac{x^5}{x^2}$

The bases are the same. To divide, use the Quotient Property of Exponents.

$\frac{x^5}{x^2} = x^{5-2} = x^3$. To divide with like bases, subtract the exponents.

4. $\frac{x^4yz}{x^2y^2z}$

The expression contains like bases. Use the Quotient Property of Exponents.

$\frac{x^4yz}{x^2y^2z} = \frac{x^4y^1z^1}{x^2y^2z^1} = \frac{x^{4-2}z^{1-1}}{y^{2-1}} = \frac{x^2z^0}{y^1} = \frac{x^2}{y}$. To divide with like bases, subtract the exponents.

Remember: Any nonzero number to the zero power is 1.

TRY IT **Simplify Expressions Using Properties of Exponents**

Simplify the expressions.

1. $x^3 \cdot x^4$
2. $\frac{x^9}{x^3}$
3. $\frac{x^4y^2z^5}{x^6yz^3}$

Check Your Understanding

Which expression is equivalent to y^6 ?

a. $y^2 \cdot y^3$

b. $y \cdot y^6$

c. $\frac{y^{12}}{y^2}$

d. $\frac{y^{12}}{y^6}$

Videos: Simplifying Expressions with Exponents

Khan Academy: Exponent properties involving products

Watch this video to see how to multiply expressions with exponents.

Exponent properties involving products | Numbers and operations | 8th grade | Khan Academy (https://www.youtube.com/embed/zM_p7tfWvLU)

Khan Academy: Exponents properties with quotients

Watch this video to see how to divide expressions with exponents.

Exponent properties involving quotients (examples) | 8th grade | Khan Academy (https://www.youtube.com/embed/AR1uqNbjM5s)

Simplify Expressions Using the Distributive Property: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #2: Simplify Expressions Using the Distributive Property

Which expression is equivalent to $-2(x-4)$?

a. $2x-8$
b. $-2x+8$
c. $-2x-8$
d. $-2x-4$

Simplify Expressions Using the Distributive Property

The Distributive Property can be used to remove parentheses as you simplify expressions.

DISTRIBUTIVE PROPERTY

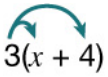
If a, b, c are real numbers, then $a(b+c) = ab+ac$

Also,
$$(b+c)a = ba+ca$$
$$a(b-c) = ab-ac$$
$$(b-c)a = ba-ca$$

Simplify:

1. $3(x+4)$

It may help to use arrows to show the Distributive Property.



Use the Distributive Property.

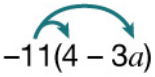
$3 \cdot x + 3 \cdot 4$

Multiply.

$3x+12$

2. $-11(4-3a)$

Use arrows to show the Distributive Property.



Use the Distributive Property

$-11 \cdot 4 - (-11) \cdot 3a$

Multiply.

$-44-(-33a)$

Simplify.

$-44+33a$

Watch signs when using the Distributive Property with negative signs.

TRY IT **Simplify Expressions Using the Distributive Property**

Simplify: $-2(4y+1)$.

Check Your Understanding

Which expression is equivalent to $4(5-2a)$?

a. $20-2a$

b. $20+2a$

c. $20-8a$

d. $20+8a$

Videos: Using the Distributive Property to Simplify Expressions

Watch the first 2 minutes and 40 seconds of the video to see how to use the Distributive Property to simplify expressions.

Khan Academy: Distributive property with variables

How to use the distributive property with variables | 6th grade | Khan Academy (https://www.youtube.com/embed/Jp25LHI9wII)

Find the Greatest Common Factor: Mini-Lesson Review

MINI LESSON QUESTION **Question #3: Find the Greatest Common Factor**

Which is the greatest common factor of 48 and 80?

a. 40

b. 24

c. 8

d. 16

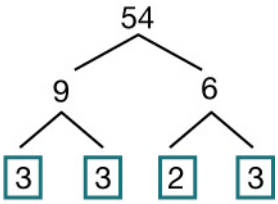
Find the Greatest Common Factor

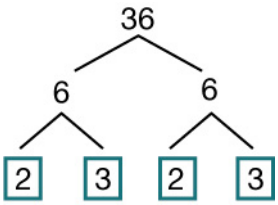
The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

Find the greatest common factor of 54 and 36.

Step 1 - Factor each number into primes.

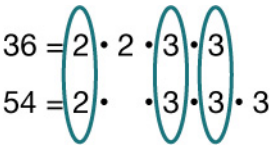
Factor 54 and 36.





Step 2 - List the factors. In each column, circle the common factors.

Circle the 2, 3, and 3 that are shared by both numbers.



Step 3 - Bring down the common factors that all the expressions share.

Bring down the 2, 3, and 3.

GCF $= 2 \cdot 3 \cdot 3$

Step 4 - Multiply the factors.

Multiply the common factors 2, 3, and 3.

GCF $= 18$

The GCF of 54 and 36 is 18.

TRY IT **Find the Greatest Common Factor**

Find the greatest common factor (GCF) of 84 and 56.

Check Your Understanding

Which is the greatest common factor of 96 and 72?

a. 24

b. 12

c. 36

d. 48

Videos: Greatest Common Factor of Two Integers

Watch the following video to learn more about finding the greatest common factor of two integers.

Khan Academy: Greatest common factor explained

Greatest common factor explained | Factors and multiples | Pre-Algebra | Khan Academy (https://www.youtube.com/embed/jFd-6EPfnec)

Unit 6 Overview: Wrap Up

How did you do? Take a moment to reflect on how many answers you got right and wrong on the previous pages. Did you complete the mini-lesson reviews? If not, completing these reviews can help you get a better understanding of these concepts:

- Simplify expressions with exponents.
- Simplify expressions using the Distributive Property.
- Find the greatest common factor (GCF) of two or more numbers.

If you are still struggling, be sure to reach out to your teacher for additional help. Nobody is good at problems like this on the first try. Keep at it, and it will get easier.

UNIT 6 INQUIRY PROJECT: AREA MODEL MULTIPLICATION

Inquiry Project Overview

In this project, you will revisit using an area model as a tool for finding products. You will start with a numerical area model, focusing on two-digit numbers and reviewing that you can partition a rectangle into smaller areas whose sum is the total area of the rectangle. Then, you will then apply this area model to algebraic expressions where an area model can be used to multiply polynomials. First, you will be asked to play with a simulation for a few minutes. This open play time allows for you to learn how to use a simulation for learning and discover how to interact with it. You will have an opportunity to share how you used the simulation during the open play time. Be sure you are aware of the controls and how to use the simulation.

When you engage with the activity sheet, you should be pausing to discuss some of your responses with a partner. Be prepared to stop and share your individual responses and partner discussions with the whole class. By the end of the lesson, you should have completed the activity sheets using the simulation, and should be able to use an area model to multiply two two-digit numbers or two polynomials, and explain why an area model can be used to find the product of two factors.

Project Activities

1. Discovering an Area Model
2. Understanding an Area Model
3. Using an Area Model
4. Applying an Area Model to Polynomials

Learning Targets

When you finish this project, you will have demonstrated how to:

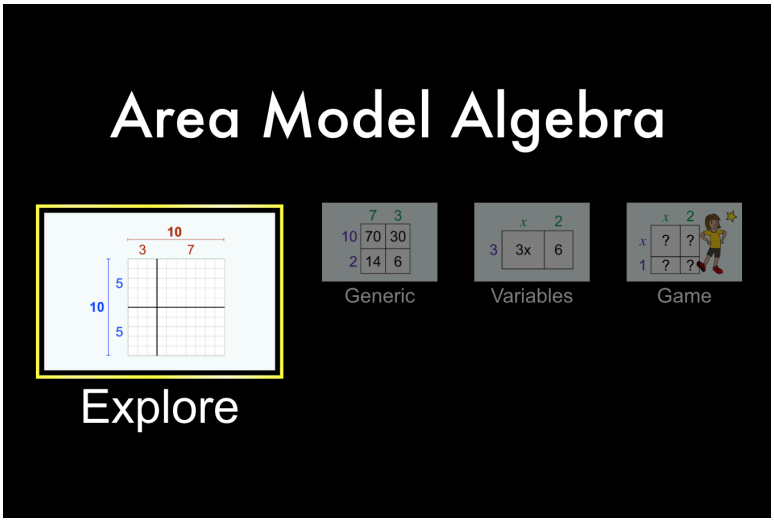
- Multiply two two-digit numbers using an area model.
- Multiply two polynomials using an area model.

Discovering an Area Model

Access the interactive simulator (https://phet.colorado.edu/sims/html/area-model-algebra/latest/area-model-algebra_en.html?screens=1,2,3)to begin this project.

In this activity, whenever you see ⚐, stop and share your responses with your partner. If you have different responses, try to come to a consensus.

Today you are going to investigate something called an interactive simulation. It is a tool that will help you visualize how area model multiplication works.



Step 1 - Open the interactive simulator (https://phet.colorado.edu/sims/html/area-model-algebra/latest/area-model-algebra_en.html?screens=1,2,3), Area Model Algebra in your web browser. You will use it to complete this project. It will look like the image above.

Step 2 - Play with any of the screens on Area Model Algebra for 5 minutes.

Step 3 - Complete the following questions.

1. What does the area of a rectangle have to do with multiplication? ⚠
2. Write down two things you notice or have questions about, and one thing a neighbor noticed that is interesting to you.

Understanding an Area Model

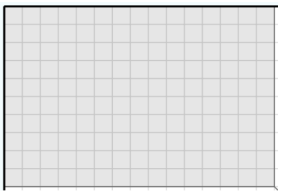
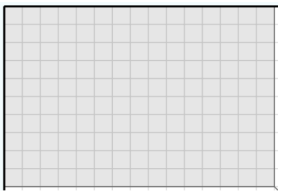
Access the interactive simulator (https://phet.colorado.edu/sims/html/area-model-algebra/latest/area-model-algebra_en.html?screens=1,2,3)to begin this project.

In this activity, whenever you see ⚠, stop and share your responses with your partner. If you have different responses, try to come to a consensus.

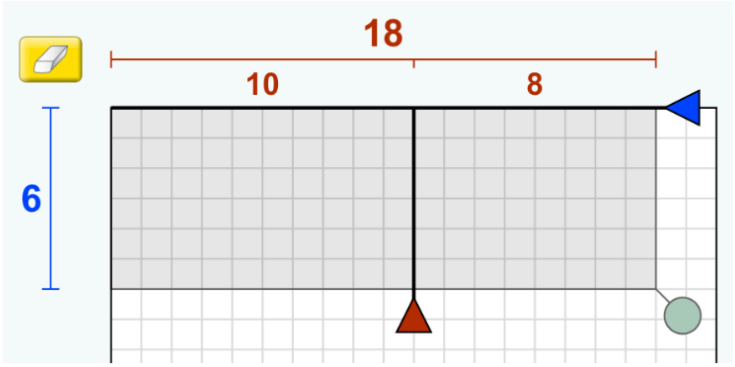
For questions 1 – 8, use the following instructions.

Use the simulation screen labeled “Explore”. Note you can switch between simulation screens using the toolbar located below the area model.

1. Explain what the red and blue sliders do to the outside of the rectangle.
2. Explain what the red and blue sliders do to the inside of the rectangle.
3. Describe what changes and what doesn’t change when the red and blue sliders are moved. ⚠
4. Multiply 10×15 using an area model. Find two different ways to partition the 10x15 rectangle. Use the sim to support you in filling out this table. ⚠

Problem	Labeled Area Model with Partial Products	List partial products and write as a sum	Total area of rectangle
10×15			
10×15			

5. Use the sim to model 6×18. What is the total area of this rectangle? Represent the total area in multiple ways.



6. In an area model, how do the partial products (interior numbers) get calculated?
7. In an area model, what are two different ways the total area could get calculated?
8. How do the partial products relate to the total area of the area model?

Using an Area Model

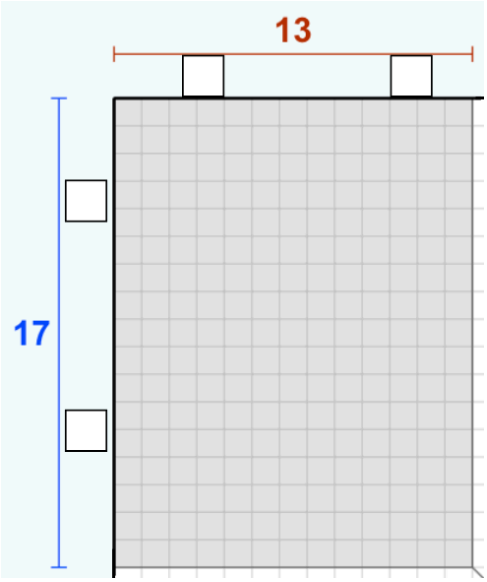
Access the interactive simulator (https://phet.colorado.edu/sims/html/area-model-algebra/latest/area-model-algebra_en.html?screens=1,2,3)to begin this project.

In this activity, whenever you see ◈, stop and share your responses with your partner. If you have different responses, try to come to a consensus.

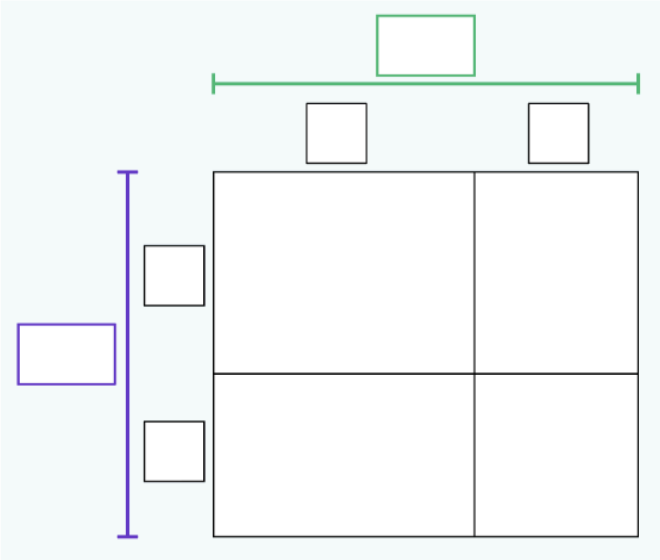
For questions 1 – 4, use the prompt and instructions below.

Suppose we want to find the product of 17 and 13 using an area model. Use the simulation screens labeled "Explore" and "Generic". Note you can switch between using the toolbar located below the area model.

1. Draw and label your partitions on the scaled area model below.



2. Draw and label your partitions on the generic model below.



3. Discuss with your group: How does your area model compare to those in your group? What is the same? What is different? ◈
4. Justify how you know your model represents 17x13. Does your area model represent 17×13? How do you know? ◈
5. Challenge yourself to work through levels 1-2 of the Area Model Numbers Game (https://phet.colorado.edu/sims/html/area-model-algebra/latest/area-model-algebra_en.html?screens=4ß)!
6. What are three different ways you could partition 17?
7. Write your own 2-digit times 2-digit multiplication problem that uses an area model, and find the total area.
8. What is a convenient way to break up a multiplication problem into an area model, and why is it convenient for you?

Applying an Area Model to Polynomials

Access the interactive simulator (https://phet.colorado.edu/sims/html/area-model-algebra/latest/area-model-algebra_en.html?screens=1,2,3)to begin this project.

In this activity, whenever you see ◈, stop and share your responses with your partner. If you have different responses,

try to come to a consensus.

For questions 1 – 3, use the instructions below.

> Use the simulation screen labeled “Variables”. Note you can switch between simulation screens using the toolbar located below the area model.

Use the *Variables Screen* to answer the following questions.

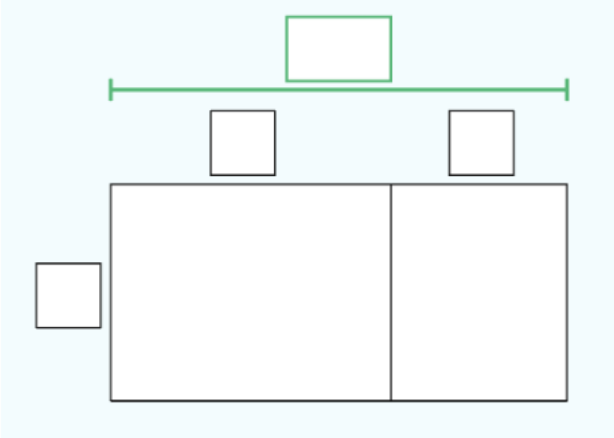
1. Play around with entering your own variable expressions, then answer the following.

a. In an area model, how do the partial products (interior numbers) get calculated?

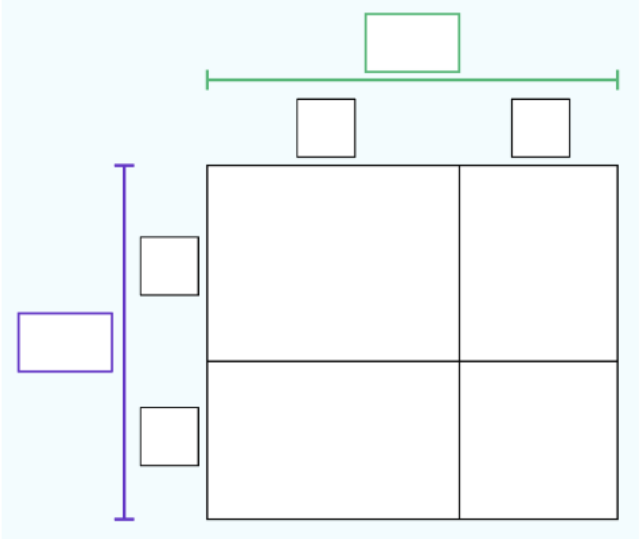
b. In an area model, how does the total area get calculated?

c. How do the partial products relate to the total area of the area model?

2. Use the sim to model 6(x+8). What is the total area of this rectangle? Represent the total area in multiple ways.



3. Use the sim to model $(x + 3)(x - 5)$. What is the total area of this rectangle? Represent the total area in multiple ways.

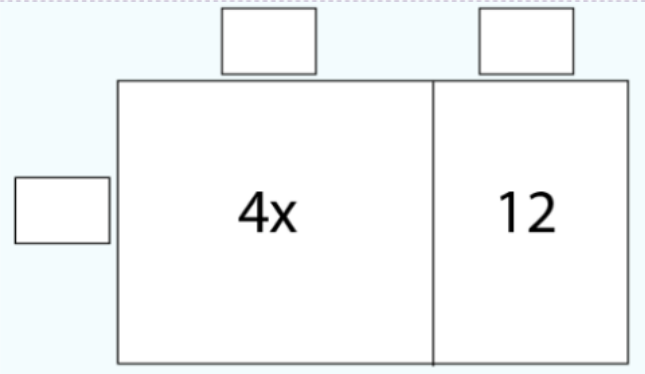


4. Challenge yourself to work through levels 1 - 4 of the Area Model Algebra Game (https://phet.colorado.edu/sims/html/area-model-algebra/latest/area-model-algebra_en.html?screens=4)!

5. How is multiplying variable expressions using an area model similar to multiplying numbers using an area model?

Extension

Challenge yourself to work backwards and factor an algebraic expression!



The Area Model sim is playing tricks on you! It gives you the partial products, but not the side lengths. What numbers and/or variables must be on the outside of this rectangle?

6.1 ADD AND SUBTRACT POLYNOMIALS

6.1.0 Lesson Overview

In this lesson, you will review what you have learned about polynomials to classify them into like terms to solve addition and subtraction problems. You will solve addition and subtraction problems using polynomial functions. You will be able to compare your work by evaluating these functions and solutions for given values.

Polynomials are frequently used in calculating physics problems and other real-world situations involving projectiles and the motion of objects. Mastering the polynomial operations in this unit will help you in future studies.

When you finish this lesson, you will be able to:

- Determine the degree of polynomials.
- Add and subtract polynomials.
- Evaluate a polynomial function for a given value.
- Add and subtract polynomial functions.

Here are the **activities** that will help you reach those goals:

- 6.1.1: Understanding Polynomials
- 6.1.2: Adding and Subtracting Polynomials
  - 6.1.2: Self Check
  - 6.1.2: Additional Resources
- 6.1.3: Evaluating a Polynomial Function for a Given Value
  - 6.1.3: Self Check
  - 6.1.3: Additional Resources
- 6.1.4: Adding and Subtracting Polynomial Functions
  - 6.1.4: Self Check
  - 6.1.4: Additional Resources
- 6.1.5: Reviewing Polynomial Addition and Subtraction

After that, you'll **practice and review**.

- 6.1.6: Practice
- 6.1.7: Lesson Summary

6.1.1 Understanding Polynomials

Warm Up

A **monomial** is an algebraic expression with exactly one term: $5x^2$

A **binomial** is an algebraic expression with exactly two terms: $5x^2 + 6y^3$

A **trinomial** is an algebraic expression with exactly three terms: $5x^2 + 6y^3 - 7c$

Monomials, binomials, and trinomials are all different types of polynomials. A **polynomial** is a monomial or two or more monomials combined by addition or subtraction.

The **degree of a polynomial** is the highest degree of all its terms.

CLASSROOM CONNECTION Log into student.desmos.com using the information provided by your teacher to complete the activity.

Examine the following expressions and use them to answer questions 1 – 6.

x^2
$c+4$
mn
$8d^2+3e-g$
$b-h$
15
$n^2-7f+ab$
z

1. Which of the polynomials listed above are monomials?
2. Which of the polynomials listed above are binomials?
3. Which of the polynomials listed above are trinomials?
4. Which of the polynomials listed above have a degree of 0?
5. Which of the polynomials listed above have a degree of 1?
6. Which of the polynomials listed above have a degree of 2?

6.1.2 Adding and Subtracting Polynomials

Activity

We will solve these two problems as a class.

1. Do these terms have the same variables with the same exponents? Explain your answer.

$6x^4y^2 \qquad 8x^4y^2$

2. Add the terms together: $6x^4y^2+8x^4y^2$. What is the sum?

In problems 3 – 7, simplify the monomial addition and subtraction problems on your own.

3. $12q+9q$

4. $-14c^3+5c^3$

5. $8g^2h^2+(-6g^2h^2)$

6. $8mn^3p-(-5mn^3p)$

7. $-16a^2+3b^2+8a^2$

8. Can the following monomials be simplified using like terms? Explain. $17a^2bc^3+6a^2b^2c^3$

In problems 9 - 14, simplify the polynomial addition and subtraction problems on your own.

9. $(5k+7m-2)+(7k+9m-4)$

10. $(12w-11)-(5w+6)$

11. $(5k^2+7k-2)+(7k^2+9k-4)$

You can use the following video if you need some help.

Problem Animation: Adding, Subtracting, and Evaluating Polynomial Functions (https://openstax.org/r/polynomial-operations-animation)

12. $(9h^2 - 4h + 17) + (2h^2 - 9)$

13. $(8f^2 + 6fv - 3v^2) - (6f^2 - 3v^2)$

14. $(8z^3 - 6zx + 3x^2) + (12z^3 - 4zx)$

Are you ready for more?

Extending Your Thinking

Simplify.

$(6a^3 - 2a^2b) - 2(ab^2 + 3b^3) + (2a^2b + 2ab^2)$

Self Check

Find the sum.

$(14y^2 + 6y - 4) + (3y^2 + 8y + 5)$

a. $11y^2 + 14y + 1$
b. $17y^2 + 14y - 9$
c. $11y^2 - 2y - 9$
d. $17y^2 + 14y + 1$

Additional Resources

Adding and Subtracting Monomials

We have learned how to simplify expressions by combining like terms. Remember, like terms must have the same variables with the same exponent. Since monomials are terms, adding and subtracting monomials is the same as combining like terms. If the monomials are like terms, we just combine them by adding or subtracting the coefficients. Remember that like terms must have the same variables with the same exponents.

These terms have the same variables and the same exponents:

- $5m^3n^2 \qquad 2m^3n^2$

So we can add the monomials:

- $5m^3n^2 + 2m^3n^2 = 7m^3n^2$

The variables and exponents of the monomials do not change when we add or subtract them.

EXAMPLE 1

$18a + 2a$

Step 1 - Combine like terms.

$18a + 2a = 20a$

EXAMPLE 2

$37t - 15t$

Step 1 - Combine like terms.

$37t - 15t = 22t$

EXAMPLE 3

$25y^2 + 15y^2$

Step 1 - Combine like terms.

$25y^2 + 15y^2 = 40y^2$

TRY IT **Adding and Subtracting Monomials**

$16pq^3 - (-7pq^3)$

Adding and Subtracting Polynomials

We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials. Look for the like terms—those with the same variables and the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together.

Let's look at some examples.

EXAMPLE 1

Find the sum: $(7y^2 - 2y + 9) + (4y^2 - 8y - 7)$.

Step 1 - Identify like terms by finding those with the same variable and same exponent.

$(7y^2 - 2y + 9) + (4y^2 - 8y - 7)$

Step 2 - Use the Commutative Property to rearrange like terms together. Be sure to take the appropriate sign with each term (+/−).

$7y^2 + 4y^2 - 2y - 8y + 9 - 7$

Step 3 - Combine like terms.

$11y^2 - 10y + 2$

EXAMPLE 2

Find the sum: $(22g - 14h + 3) + (10g + 8h - 9)$

Step 1 - Identify like terms.

$(22g - 14h + 3) + (10g + 8h - 9)$

Step 2 - Use the Commutative Property to rearrange like terms together. Be sure to take the appropriate sign with each term (+/−).

$(22g + 10g - 14h + 8h + 3 + 9)$

Step 3 - Combine like terms.

$32g - 6h + 12$

EXAMPLE 3

Find the sum: $(8j + 7k) + (2s^2 - 3k + 26)$

Step 1 - Identify like terms.

$(8j + 7k - 3) + (2s^2 - 3k + 26)$

Step 2 - Rearrange like terms together.

$2s^2 + 7k - 3k + 8j - 3 + 26$

Step 3 - Combine like terms.

EXAMPLE 4

Find the difference: $(15g - 4h + 8) - (10g + 13h - 1)$

Step 1 - Identify like terms.

$(15g - 4h + 8) - (10g + 13h - 1)$

Step 2 - Rearrange like terms together.

Remember to distribute the subtraction to all terms in the second polymnomial.

$15g - 10g - 4h - 13h + 8 + 1$

Step 3 - Combine like terms.

$5g - 17h + 9$

EXAMPLE 5

Find the difference: $(16y^2 + 12y + 9) - (3y^2 + 2y - 12)$

Step 1 - Identify like terms.

$(16y^2 + 12y + 9) - (3y^2 + 2y - 12)$

Step 2 - Rearrange like terms together.

$(16y^2 - 3y^2 + 12y - 2y + 9 + 12)$

Step 3 - Combine like terms.

$(13y^2 + 10y + 21)$

TRY IT **Adding and Subtracting Polynomials**

Find the sum and difference. If necessary, be careful with the signs as you distribute while subtracting the polynomials.

1. $(7x^2 - 4x + 5) + (x^2 - 7x + 3)$
2. $(9w^2 - 7w + 5) - (2w^2 - 4)$
3. $(8a + 12b) - (10a + 3b)$

Write down your answer, then select the **solution** button to compare your work.

6.1.3 Evaluating a Polynomial Function for a Given Value

Activity

Evaluate each function for $h(-4)$, $h(0)$, and $h(6)$.

1. $h(x) = 3x^2 - 7x - 9$
2. $h(x) = -4x^2 - 3x + 2$
3. $h(x) = -2x^2 - 5x + 15$

Evaluate each function for $d(-2)$, $d(0)$, and $d(3)$.

4. $d(x) = 8x^2 - 2x + 6$
5. $d(x) = -5x^2 - 4x + 10$
6. $d(x) = -3x^2 + 6x + 7$
7. Work with a partner. One partner should write their own polynomial function similar to the ones in this activity. The other partner should select 3 different values between –10 and 10. Exchange your function and values with your partner. Each member should evaluate the polynomial function for the given values. When finished, compare your answers. Did you find the same solutions? Share any differences you found.

Self Check

For the function $g(x) = 5x^2 - 8x + 4$, find $g(-3)$.

a. 73
b. 25
c. 13
d. -17

Additional Resources

Evaluating a Polynomial Function for a Given Value

A **polynomial function** is a function whose range values are defined by a polynomial. For example, $f(x) = x^2 + 5x + 6$ and $g(x) = 3x - 4$ are polynomial functions because $x^2 + 5x + 6$ and $3x - 4$ are polynomials.

To evaluate a polynomial function, we will substitute the given value for the variable and then simplify using the order of operations.

Let's look at some examples.

For the function $f(x) = 5x^2 - 8x + 4$, find:

1. $f(4)$
2. $f(-2)$
3. $f(0)$

The following is a step-by-step breakdown.

1. $f(x) = 5x^2 - 8x + 4$

Step 1 - To find $f(4)$, substitute 4 for x.

$f(4) = 5(4)^2 - 8(4) + 4$

Step 2 - Simplify the exponents.

$f(4) = 5 \cdot 16 - 8(4) + 4$

Step 3 - Multiply.

$f(4) = 80 - 32 + 4$

Step 4 - Simplify.

$f(4) = 52$

2. $f(x) = 5x^2 - 8x + 4$

Step 1 - To find $f(-2)$, substitute -2 for x.

$f(-2) = 5(-2)^2 - 8(-2) + 4$

Step 2 - Simplify the exponents.

$f(-2) = 5 \cdot 4 - 8(-2) + 4$

Step 3 - Multiply.

$f(-2) = 20 + 15 + 4$

Step 4 - Simplify.

$f(-2) = 40$

3. $f(x) = 5x^2 - 8x + 4$

Step 1 - To find $f(0)$, substitute 0 for x.

$f(0) = 5(0)^2 - 8(0) + 4$

Step 2 - Simplify the exponents.

$f(0) = 5 \cdot 0 - 8(0) + 4$

Step 3 - Multiply.

$f(0) = 0 + 0 + 4$

Step 4 - Simplify.

$f(0) = 4$

TRY IT **Try It: Evaluating a Polynomial Function for a Given Value**

For the function $f(x) = 3x^2 + 2x - 15$, find:

1. $f(3)$
2. $f(-5)$
3. $f(0)$

6.1.4 Adding and Subtracting Polynomial Functions

Activity

Add the polynomial functions. Evaluate the sum for each given value.

$z(x) = 8x^2 + 4x - 6$

$w(x) = -5x^2 + 2x + 3$

1. $(z + w)(x) =$ _____

2. $(z + w)(6) =$ _____

3. $(z + w)(-9) =$ _____

4. $(z + w)(0)$ _____

Subtract the polynomial functions. Evaluate the sum for each given value.

$m(x) = -6x^2 + 2x - 8$

$n(x) = -x^2 - 7x - 6$

5. $(m - n)(x) =$ _____

6. $(m - n)(-2) =$ _____

7. $(m - n)(0) =$ _____

8. $(m - n)(4) =$ _____

9. What is a good way to check that you found the correct sum or difference when adding or subtracting polynomial functions?

Video: Combining and Evaluating Polynomial Functions

Watch the following video to learn more about how to add and subtract polynomial functions and evaluate them at a given value:

Combining and Evaluating Polynomial Functions (https://openstax.org/r/polynomial-operations)

Self Check

For functions $f(x) = 5x^2 - 4x - 1$ and $g(x) = x^2 + 3x + 8$, find:

$(f + g)(x) =$ _____

$(f+g)(5) =$ _____

a. $(f+g)(x) = 6x^2 - x + 7$

$(f+g)(5) = 32$

b. $(f+g)(x) = 4x^2 - 7x - 9$

$(f+g)(5) = 56$

c. $(f+g)(x) = 6x^2 - x + 7$

$(f+g)(5) = 152$

d. $(f+g)(x) = 4x^2 - x + 7$

$(f+g)(5) = 102$

Additional Resources

Adding and Subtracting Polynomial Functions

Just as polynomials can be added and subtracted, polynomial functions can also be added and subtracted.

For functions $f(x)$ and $g(x)$,

- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$

Let's look at some examples.

For the functions $f(x) = 3x^2 - 5x + 7$ and $g(x) = x^2 - 4x - 3$, find:

a. $(f+g)(x)$
b. $(f+g)(3)$
c. $(f-g)(x)$
d. $(f-g)(-2)$

The following is a step-by-step breakdown.

a. $(f+g)(x) = f(x) + g(x)$

Step 1 - Substitute

$f(x) = 3x^2 - 5x + 7$ and $g(x) = x^2 - 4x - 3$.

$(f+g)(x) = (3x^2 - 5x + 7) + (x^2 - 4x - 3)$

Step 2 - Rewrite without the parentheses.

$(f+g)(x) = 3x^2 - 5x + 7 - x^2 - 4x - 3$

Step 3 - Put like terms together.

$(f+g)(x) = 3x^2 + x^2 - 5x - 4x + 7 - 3$

Step 4 - Combine like terms.

$(f+g)(x) = 4x^2 - 9x + 4$

In part (a), we found $(f+g)(x)$, and now are asked to find $(f+g)(3)$.

b. $(f+g)(x) = 4x^2 - 9x + 4$

Step 1 - To find $(f+g)(3)$, substitute $x = 3$.

$(f+g)(3) = 4(3)^2 - 9 \cdot 3 + 4$

Step 2 - Simplify.

$(f+g)(3) = 4 \cdot 9 - 9 \cdot 3 + 4$

$(f+g)(3) = 36 - 27 + 4$

$(f+g)(3) = 13$

Notice that we could have found $(f+g)(3)$ by first finding the values of $f(3)$ and $g(3)$ separately and then adding the results.

Step 1 - Find $f(3)$.

$f(x) = 3x^2 - 5x + 7$

$f(3) = 3(3)^2 - 5(3) + 7$

$f(3) = 19$

Step 2 - Find $g(3)$.

$g(x) = x^2 - 4x - 3$

$g(3) = -6$

Step 3 - Find $(f+g)(3)$.

$(f+g)(x) = f(x) + g(x)$

$(f+g)(3) = f(3) + g(3)$

Step 4 - Substitute $f(3) = 19$ and $g(3) = -6$.

$(f+g)(3) = 19 + (-6)$

$(f+g)(3) = 13$

c. $(f-g)(x) = f(x) - g(x)$

Step 1 - Substitute

$f(x) = 3x^2 - 5x + 7$ and $g(x) = x^2 - 4x - 3$.

$(f-g)(x) = (3x^2 - 5x + 7) - (x^2 - 4x - 3)$

Step 2 - Rewrite without the parentheses.

$(f-g)(x) = 3x^2 - 5x + 7 - x^2 + 4x + 3$

Step 3 - Put like terms together.

$(f-g)(x) = 3x^2 - x^2 - 5x + 4x + 7 + 3$

Step 4 - Combine like terms.

$(f-g)(x) = 2x^2 - x + 10$

d. $(f-g)(x) = 2x^2 - x + 10$

Step 1 - To find $(f-g)(-2)$, substitute $x = -2$.

$(f-g)(-2) = 2(-2)^2 - (-2) + 10$

Step 2 - Simplify.

$(f-g)(-2) = 2 \cdot 5 - (-2) + 10$

$(f-g)(-2) = 20$

Adding and Subtracting Polynomial Functions

For functions $f(x) = 2x^2 - 4x + 3$ and $g(x) = x^2 - 2x - 6$, find:

a. $(f+g)(x)$
b. $(f+g)(3)$
c. $(f-g)(x)$
d. $(f-g)(-2)$

6.1.5 Reviewing Polynomial Addition and Subtraction

Cool Down

Draw a tic-tac-toe board on an extra sheet of paper. Use the polynomial functions shown here for the activity.

$f(x) = 8x^2 - 6x - 2$

$h(x) = 6x^2 - 4x - 1$

For each polynomial problem below, place an X on your tic-tac-toe board corresponding to the location of its matching solution.

$16x^2 - 9x^2 =$ _____

$(9x^2 - 5x - 3) + (2x^2 + 8x - 3) =$ _____

$f(5) - h(-2) =$ _____

$(7x^2 - 3x + 7) - (3x^2 + 6x - 4) =$ _____

$f(4) =$ _____

For each polynomial problem below, place an O on your tic-tac-toe board corresponding to the location of its matching solution.

$11x^3 + 12x^3 =$ _____

$f(x) - h(x) =$ _____

$f(x) + h(x) =$ _____

$h(-3) =$ _____

$7x^2$	$14x^2 - 10x - 3$	$11x^2 + 3x - 6$
$2x^2 - 2x - 1$	$23x^3$	137
65	102	$4x^2 - 9x + 11$

Did the Xs or Os win the tic-tac-toe game?

6.1.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 - 5, simplify each expression completely.

1. $14ab^2 - 8ab^2$

a. $8ab^2$
b. $14ab^2$
c. $22ab^2$
d. $6ab^2$

2. $(8k^2 - 3k + 14) + (8k + 17)$

3. $(7b^2 - 2b - 12) - (4b^2 + 9b - 7)$

a. $11b^2 - 11b - 5$
b. $3b^2 - 11b - 5$
c. $3b^2 + 7b - 19$
d. $11b^2 + 7b - 19$

4. $(7mn^3 - 2m^2n - 12mn) - (3mn^3 - 4mn)$

a. $4mn^3 - 10m^2n$
b. $4mn^3 - 2m^2n - 16mn$
c. $4mn^3 - 2m^2n - 8mn$
d. $10mn^3 - 2m^2n - 16mn$

5. $(r^3 - r^2s) - 2(rs^2 + s^3) + (r^2s + 2rs^2)$

a. $r^3 + rs^2 - 2s^3$
b. $r^3 - 2s^3$
c. $r^3 - rs^2 - 2s^3$
d. $r^3 + 2s^3$

Use the following polynomial functions to answer questions 6 – 10.

- $p(x) = 12x^2 + 3x - 4$
- $d(x) = 3x^2 - 11x - 6$

6. $p(4)$

a. 48
b. 192
c. 200
d. –2

7. $d(-3)$

a. –12
b. 54
c. 113
d. 95

8. $p(x) + d(x)$

9. $p(x) - d(x)$

10. $p(5) - d(-6)$

a. 479
b. 168
c. 311
d. 143

6.1.7 Lesson Summary

In this lesson, you learned how to:

- Determine the degree of polynomials.
- Add and subtract polynomials.
- Evaluate a polynomial function for a given value.
- Add and subtract polynomial functions.

Here are the **activities** that helped you reach those goals:

- 6.1.1: Understanding Polynomials
  - In this activity, you classified different types of polynomials. You also found the degree of different polynomials.
- 6.1.2: Adding and Subtracting Polynomials
  - In this activity, you added and subtracted different types of polynomials by grouping like terms.
  - 6.1.2: Self Check
  - 6.1.2: Additional Resources
- 6.1.3: Evaluating a Polynomial Function for a Given Value
  - In this activity, you substituted given values into various polynomial functions and simplified to find solutions.
  - 6.1.3: Self Check
  - 6.1.3: Additional Resources
- 6.1.4: Adding and Subtracting Polynomial Functions
  - In this activity, you used the skills learned in previous activities to add and subtract polynomial functions. You were able to compare your work by evaluating the original and new polynomial functions using the same value.
  - 6.1.4: Self Check
  - 6.1.4: Additional Resources
- 6.1.5: Reviewing Polynomial Addition and Subtraction
  - In this activity, you reviewed all of the skills you learned adding and subtracting polynomials to solve a tic-tac-toe puzzle.

After these activities, you completed the following **practice**:

- 6.1.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

6.2 MULTIPLYING POLYNOMIALS

6.2.0 Lesson Overview

In this lesson, you will multiply different types of polynomials. You will learn methods to multiply binomials by other binomials. You will learn different methods to multiply polynomials by other polynomials, including using polynomial functions.

Polynomials are frequently used in calculating physics problems and other real-world situations involving projectiles and the motion of objects. Mastering the polynomial operations in this unit will help you in future studies.

When you finish this lesson, you will be able to:

- Multiply monomials.
- Multiply a polynomial by a monomial.
- Multiply binomials.
- Multiply a polynomial by a polynomial.
- Multiply special products.
- Multiply polynomial functions.

Here are the **activities** that will help you reach those goals:

- 6.2.1: Multiplying Monomials
- 6.2.2: Multiplying Binomials
  - 6.2.2: Self Check
  - 6.2.2: Additional Resources
- 6.2.3: Multiplying a Polynomial by a Polynomial
  - 6.2.3: Self Check
  - 6.2.3: Additional Resources
- 6.2.4: Multiplying Special Products
  - 6.2.4: Self Check
  - 6.2.4: Additional Resources
- 6.2.5: Multiplying Polynomial Functions

After that, you'll **practice and review**.

- 6.2.6: Practice
- 6.2.7: Lesson Summary

6.2.1 Multiplying Monomials

Warm Up

Since monomials are algebraic expressions, we can use the properties of exponents to multiply monomials.

1. $(3x^2)(-4x^3)$

2. $(\frac{5}{6}x^3y)(12xy^2)$

Multiplying a polynomial by a monomial is done by applying the Distributive Property of Multiplication over Addition.

Find the product of each expression.

3. $7x(3x+6)$

4. $7b^2(9b-4)$

5. $2k(5k^2+3k-1)$

6. $4r^2s(2r^2+8rs-12s^2)$

Building Character: Judgement



A key factor in making sound judgements is not jumping to conclusions. Your judgment will never be perfect, but you can strive to be aware of bias and make corrections when necessary.

Think about yourself. When you face a problem that requires judgment, how many of these things are true?

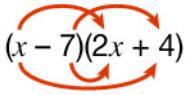
- I independently ask several people for their judgment, and I consider their reasons.
- I try to separate my hopes and fears from the facts and beliefs supporting the judgment.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

6.2.2 Multiplying Binomials

Activity

The binomials in the expression $(x-7)(2x+4)$ are solved three different ways in the table. Describe how the two binomials are being multiplied.

Solution	Description of Solution.
$(x-7)(2x+4)$ $x(2x+4)-7(2x+4)$ $2x^2+4x-14x-28$ $2x^2-10x-28$	**First,** _____ **Second,** _____ **Next,** _____ So, the solution is $2x^2-10x-28$.
$x-7$ $\times \quad 2x+4$ $4x-28$ $2x^2-14x$ $2x^2-10x-28$	**First,** _____ **Second,** _____ **Third,** _____ **Next,** _____ So, the solution is $2x^2-10x-28$.
 $2x^2+4x-14x-28$ $2x^2-10x-28$	**First,** _____ **Second,** _____ **Third,** _____ **Next,** _____ Combine like terms, and the solution is $2x^2-10x-28$.

The video below can also help break down solutions if you find yourself stuck.

Problem Animation: Multiplying Binomials Using Algebra Tiles (https://openstax.org/r/multiply-binomials)

Multiply each binomial.

1. $(a+6)(a-4)$

2. $(5b+9)(4b+3)$

3. $(5x^2-1)(2x-8)$

4. $(7mn+1)(11mn-3)$

Are you ready for more?

Extending Your Thinking

Multiply each binomial. Add like terms to simplify the expression.

$(5x^2y-xy)(3x+y)+(6x+4y)(-3x^2y+2xy)$

Self Check

Multiply the binomials.

$(5y+2)(6y-3)$

a. $30y^2+3y+6$
b. $30y^2-27y-6$
c. $27y-6$
d. $30y^2-3y-6$

Additional Resources

Multiplying Binomials Using the Distributive Property

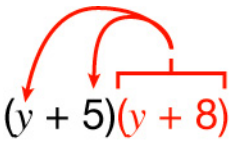
Just like there are different ways to represent multiplication of numbers, there are several different ways to represent how to multiply a binomial times a binomial using the Distributive Property of Multiplication over Addition.

Distribute to multiply the binomials.

EXAMPLE 1

$(y+5)(y+8)$

Step 1 - Distribute the binomial $(y+8)$.



$y(y+8)+5(y+8)$

Step 2 - Now, distribute the individual terms.

$y^2+8y+5y+40$

Step 3 - Combine like terms.

$y^2+13y+40$

EXAMPLE 2

$(4y^2+3)(2y^2-5)$

Step 1 - Distribute the binomial $(2y^2-5)$.

$4y^2(2y^2\text{-}5)+3(2y^2\text{-}5)$

Step 2 - Distribute the individual terms.

$8y^2-20y^2+6y^2-15$

Step 3 - Combine like terms.

$8y^4-14y^2-15$

Multiplying Binomials Using the FOIL Method

If you multiply binomials often enough, you may notice a pattern. Notice that the first term in the result is the product of the **first** terms in each binomial. The second and third terms are the product of multiplying the two **outer** terms and then the two **inner** terms. And the last term results from multiplying the two **last** terms.

We abbreviate "First, Outer, Inner, Last" as FOIL. The letters stand for "First, Outer, Inner, Last." We use this as another way to multiply binomials. The word FOIL is easy to remember and ensures we find all four products.

Let's multiply $(x+3)(x+7)$ using both methods.

Distributive Property	FOIL
$(x + 3)(x + 7)$	$(x + 3)(x + 7)$
$x(x + 7) + 3(x + 7)$	
$x^2 + 7x + 3x + 21$ $F \quad O \quad I \quad L$	$x^2 + 7x + 3x + 21$ $F \quad O \quad I \quad L$
$x^2 + 10x + 21$	$x^2 + 10x + 21$

We summarize the steps of the FOIL method below. The FOIL method only applies to multiplying binomials, not other polynomials!

How to use the FOIL method to multiply two binomials:

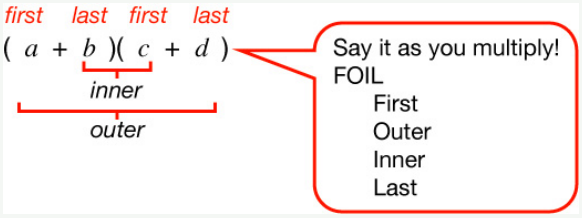
Step 1 - Multiply the *First* terms.

Step 2 - Multiply the *Outer* terms.

Step 3 - Multiply the *Inner* terms.

Step 4 - Multiply the *Last* terms.

Step 5 - Combine like terms, when possible.



When you multiply by the FOIL method, drawing the arrows will help your brain focus on the pattern and make it easier to apply.

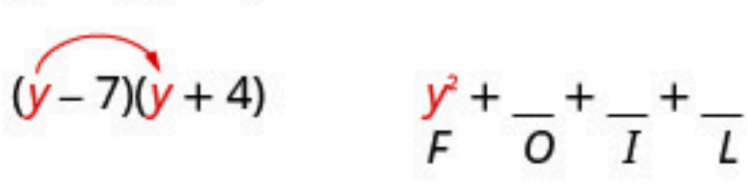
Now we will review examples where we use the FOIL pattern to multiply two binomials.

Multiply the binomials using the FOIL method.

EXAMPLE 3

$(y - 7)(y + 4)$

Step 1 - Multiply the *First* terms.



Step 2 - Multiply the *Outer* terms.



$y^2 + 4y + _ + _$

F O I L

Step 3 - Multiply the *Inner* terms.



$y^2 + 4y - 7y + _$

F O I L

Step 4 - Multiply the *Last* terms.



$y^2 + 4y - 7y - 28$

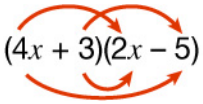
F O I L

Step 5 - Combine like terms.

$y^2 - 3y - 28$

EXAMPLE 4

$(4x + 3)(2x - 5)$



Step 1 - Multiply the *First* terms, $4x \cdot 2x$.

$8x^2 + _ + _ + _$

F O I L

Step 2 - Multiply the *Outer* terms, $4x \cdot (-5)$.

$8x^2 - 20x + _ + _$

F O I L

Step 3 - Multiply the *Inner* terms, $3 \cdot 2x$.

$8x^2 - 20x + 6x + _$

F O I L

Step 4 - Multiply the *Last* terms, $3 \cdot (-5)$.

$8x^2 - 20x + 6x - 15$

F O I L

Step 5 - Combine like terms.

$8x^2 - 14x - 15$

The FOIL method is usually the quickest method for multiplying two binomials, but it **only** works for binomials. You can use the Distributive Property to find the product of any two polynomials.

Multiplying Binomials Using Vertical Alignment

Another representation that works for all polynomials is Vertical Alignment. It is much like the method you use to multiply whole numbers. Look carefully at this example of multiplying two-digit numbers.

$$\begin{array}{rl} 23 & \\ \times\, 46 & \\ \hline 138 & \text{partial product} \\ 920 & \text{partial product} \\ \hline 1058 & \text{product} \end{array}$$

Start by multiplying 23 by 6 to get 138.

Next, multiply 23 by 4, lining up the partial product in the correct columns.

Use a 0 as a place holder in the ones column because you will be multiplying by 4 tens.

Finally, add the partial products.

EXAMPLE 5

Multiply using Vertical Alignment: $(3y - 1)(2y - 6)$. It does not matter which binomial goes on the top.

Step 1 - Multiply $3y - 1$ by -6.

Step 2 - Multiply $3y - 1$ by $2y$.

Step 3 - Add like terms.

Notice the partial products are the same as the terms in the FOIL method.

$$\begin{array}{r} (3y - 1) \\ \times (2y - 6) \\ \hline -18y + 6 \\ 6y^2 - 2y \\ \hline 6y^2 - 20y + 6 \end{array}$$

We have now used three ways to multiply binomials. Be sure to practice each process, and try to decide which one you prefer.

TRY IT **Multiplying Binomials**

In questions 1 – 2, multiply the binomials using any of the methods described.

1. $(x + 8)(x + 9)$

2. $(3c^2 + 4)(5c - 2)$

6.2.3 Multiplying a Polynomial by a Polynomial

Activity

For questions 1 – 5, match each polynomial expression to its simplified solution. Be prepared to explain or show your work.

1. $(c + 1)(c^2 + 3c + 6)$

2. $(c - 6)(c^2 + c + 1)$

3. $(c+5)(c^2+2c-4)$

4. $(c^2-3)(c^2-4c-1)$

5. $(c^2+10)(c^2-5c+2)$

6. Now, fill in the missing blanks to complete the polynomial multiplication.

$$\begin{aligned}(2x-4)\left(x^2+\square x-5\right) &= 2x\left(x^2+6x-5\right)-4\left(x^2+6x-\square\right)\\ &= 2x^3+\square x^2-10x-4x^2-24x+\square\\ &= \square x^3+8x^2-\square x+20\end{aligned}$$

Video: Multiplying Polynomials Using Different Methods

Watch the following video to learn more about how to multiply polynomials using both the Distributive Property and Vertical Alignment.

Multiplying Polynomials Using Different Methods (https://openstax.org/r/multiply-polynomials)

Self Check

Multiply.

$(d+4)(2d^2-3d+5)$

a. $8d^2-12d+20$
b. $2d^3-3d^2+5d$
c. $2d^3+5d^2-7d+20$
d. $2d^3-11d^2-17d+20$

Additional Resources

Multiplying a Polynomial by a Polynomial

We have multiplied monomials by monomials, monomials by polynomials, and binomials by binomials. Now we're ready to multiply a polynomial by a polynomial. Remember, FOIL will not work in this case, but we can use other representations of the Distributive Property such as Vertical Alignment or distributing individual terms or polynomials.

EXAMPLE 1

Multiply $(b+3)(2b^2-5b+8)$ using the Distributive Property by distributing either a binomial or an individual term.

Step 1 - Distribute.

$(2b^2-5b+8)$.

$b(2b^2-5b+8)+3(2b^2-5b+8)$

Step 2 - Multiply.

$2b^3-5b^2+8b+6b^2-15b+24$

Step 3 - Combine like terms.

$2b^3+b^2-7b+24$

EXAMPLE 2

Multiply $(b+3)(2b^2-5b+8)$ using vertical alignment.

It is easier to put the polynomial with fewer terms on the bottom because we get fewer partial products this way.

Step 1 - Multiply.

$(2b^2-5b+8)$ by 3.

Step 2 - Multiply.

$(2b^2 - 5b + 8)$ by b.

Step 3 - Add like terms.

$$\begin{array}{r} 2b^2 - 5b + 8 \\ \times \quad b + 3 \\ \hline 6b^2 - 15b + 24 \\ 2b^3 - 5b^2 + 8b \\ \hline 2b^3 + b^2 - 7b + 24 \end{array}$$

TRY IT **Multiplying a Polynomial by a Polynomial**

For questions 1 – 2, multiply $(z - 3)(z^2 - 5z + 2)$ using the listed method.

1. The Distributive Property

2. Vertical Alignment

Write down your answer, then select the **solution** button to compare your work.

6.2.4 Multiplying Special Products

Activity

Multiply each expression. Identify each as a binomial square or a conjugate pair (difference of squares).

1. $(3x - 7)^2$

2. $(6z + 4)(6z - 4)$

3. $(4a + 2b)(4a + 2b)$

4. $(3w - 9p)(3w + 9p)$

5. $(2y - 8n)(2y + 8n)$

6. $(5k + 2)(5k + 2)$

Self Check

Multiply.

$(3f - 6g)^2$

a. $9f^2 + 36g^2$
b. $9f^2 - 36fg - 36g^2$
c. $9f^2 - 36fg + 36g^2$
d. $9f^2 - 18fg + 36g^2$

Additional Resources

Multiplying Binomial Squares

Mathematicians like to look for patterns that will make their work easier. A good example of this is squaring binomials. While you can always get the product by writing the binomial twice and multiplying the two binomials, there is less work to do if you learn to use a pattern. Let's start by looking at three examples and look for a pattern.

Look at these results. Do you see any patterns?

$(x+9)^2$	$(y-7)^2$	$(2x+3)^2$
$(x+9)(x+9)$	$(y-7)(y-7)$	$(2x+3)(2x+3)$
$x^2+9x+9x+81$	$y^2-7y-7y+49$	$4x^2+6x+6x+9$
$x^2+18x+81$	$y^2-14y+49$	$4x^2+12x+9$

What about the number of terms? In each example, we squared a binomial, and the result was a trinomial.

$(a+b)^2 =$ ____ + ____ + ____

Now look at the *first term* in each result. Where did it come from?

The first term is the product of the first terms of each binomial. Since the binomials are identical, it is just the square of the first term!

$(a+b)^2 = a^2 +$ ____ + ____

To get the first term of the product, square the first term.

Where did the *last term* come from? Look at the examples and find the pattern.

The last term is the product of the last terms, which is the square of the last term.

$(a+b)^2 =$ ____ + ____ $+b^2$

To get the last term of the product, square the last term.

Finally, look at the *middle term*. Notice it came from adding the "outer" and the "inner" terms—which are the same! So the middle term is double the product of the two terms of the binomial.

$(a+b)^2 =$ ____ $+ 2ab +$ ____

$(a+b)^2 =$ ____ $+ 2ab +$ ____

To get the middle term of the product, multiply the terms and double their product.

Putting it all together:

Binomial Squares Pattern

If a and b are real numbers,

$(a+b)^2 = a^2 + 2ab + b^2$ $\quad \underbrace{(a+b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} + \underbrace{2ab}_{2\text{(product of terms)}} + \underbrace{b^2}_{\text{(last term)}^2}$

$(a-b)^2 = a^2 - 2ab + b^2$ $\quad \underbrace{(a-b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} - \underbrace{2ab}_{2\text{(product of terms)}} + \underbrace{b^2}_{\text{(last term)}^2}$

To square a binomial, square the first term, square the last term, double the product of the two terms.

TRY IT **Multiplying Binomial Squares**

1. Multiply $(x+5)^2$.

2. Multiply $(2x-3y)^2$.

Write down your answer, then select the **solution** button to compare your work.

Conjugate Pairs

We just saw a pattern for squaring binomials that we can use to make multiplying some binomials easier. Similarly, there is a pattern for another product of binomials.

A pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference, is called a **conjugate pair** and is of the form $(a - b), (a + b)$.

CONJUGATE PAIR

A conjugate pair is two binomials of the form

$(a - b), (a + b)$

The pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

There is a nice pattern for finding the product of conjugates. You could, of course, simply FOIL to get the product, but using the pattern makes your work easier. Let's look for the pattern by using FOIL to multiply some conjugate pairs.

$(x + 9)(x - 9)$ $(y - 8)(y + 8)$ $(2x - 5)(2x + 5)$

$x^2 - 9x + 9x - 81$ $y^2 + 8y - 8y - 64$ $4x^2 + 10x - 10x - 25$

$x^2 - 81$ $y^2 - 64$ $4x^2 - 25$

What do you observe about the products?

The product of the two binomials is also a binomial! Most of the products resulting from FOIL have been trinomials.

Each *first term* is the product of the first terms of the binomials, and since they are identical, it is the square of the first term.

$(a + b)(a - b) = a^2 -$ ____

To get the first term, square the first term.

The *last term* came from multiplying the last terms, and it is the square of the last term.

$(a + b)(a - b) = a^2 - b^2$

To get the last term, square the last term.

Why is there no middle term? Notice the two middle terms you get from FOIL combine to 0 in every case, the result of one addition and one subtraction.

The product of conjugates is always of the form $a^2 - b^2$. This is called a **difference of squares**.

This leads to the pattern:

PRODUCT OF CONJUGATES PATTERN

If *a* and *b* are real numbers,

Product of Conjugates Pattern

If a and b are real numbers,

$(a - b)(a + b) = a^2 - b^2$ $(a - b)(a + b) =$ a^2 − b^2

difference

conjugates squares

The product is called a difference of squares.

To multiply conjugates, square the first term, square the last term, write it as a difference of squares.

We just developed special product patterns for Binomial Squares and for the Product of Conjugates. The products look similar, so it is important to recognize when it is appropriate to use each of these patterns and to notice how they differ. Look at the two patterns together and note their similarities and differences.

COMPARING THE SPECIAL PRODUCT PATTERNS

Binomial Squares	Product of Conjugates
$(a+b)^2 = a^2 + 2ab + b^2$	$(a-b)(a+b) = a^2 - b^2$
$(a-b)^2 = a^2 - 2ab + b^2$	
Squaring a binomial.	Multiplying conjugates.
Product is a **trinomial**.	Product is a **binomial**.
Inner and outer terms with FOIL are **the same** .	Inner and outer terms with FOIL are **opposites** .
Middle term is **double the product** of the two terms.	There is **no** middle term.

TRY IT **Conjugate Pairs**

1. Multiply $(2x+5)(2x-5)$.

2. Multiply $(5m-9n)(5m+9n)$.

6.2.5 Multiplying Polynomial Functions

Cool Down

For questions 1 – 4, use the following polynomial function to complete the multiplication.

$f(x) = 3x + 2b$
$g(x) = 3x - 2b$
$h(x) = 7x + 2$
$j(x) = 9x^2 - 12x + 4$

1. $f(x) \cdot g(x) =$

2. $f(x) \cdot f(x) =$

3. $h(x) \cdot j(x) =$

4. $h(x) \cdot h(x) =$

6.2.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 8, simplify each expression completely.

1. $(5h^2)(-3h^2)$

a. $-15h^8$
b. $-15h$
c. $-15h^4$
d. $-15h^2$

2. $(-\frac{4}{5}d^2s^2)(6ds^2)$

a. $-\frac{24}{30}d^2s^4$
b. $-\frac{24}{5}d^3s^4$
c. $(-\frac{24}{30}d^3s^4)$
d. $(-\frac{24}{5}d^2s^4)$

3. $(n-4)(3n+7)$

a. $3n^2+5n-28$
b. $3n^2-5n-28$
c. $3n^2-12n+28$
d. $3n^2-19n+28$

4. $(3x^2+3)(7x-5)$

5. $(m-6)(2m^2-7m+8)$

a. $2m^3-19m^2+50m-48$
b. $2m^3+19m^2+34m-48$
c. $-6m^2-42m-48$
d. $2m^3-7m^2+8m$

6. $(c-4)(c^2+5c-9)$

a. $-4c^3+c^2-29c+36$
b. $-4c^2-20c+36$
c. $c^3+c^2-29c+36$
d. c^3+5c^2-9c

7. $(5x-3)^2$

8. $(11z+7)(11z-7)$

a. $121z^2+49$
b. $121z^2+144z-49$
c. $121z^2-49$
d. $121z^2-144z-49$

For questions 9–10, use the following polynomial functions.

- $f(x)=5x+4v$
- $g(x)=5x-4v$

9. $f(x)\cdot g(x)$

a. $25x^2-16v^2$
b. $25x^2+40xv+16v^2$
c. $25x^2-40xv+16v^2$
d. $25x^2+16v^2$

10. $f(x) \cdot f(x)$

a. $25x^2 - 16v^2$
b. $25x^2 + 40xv + 16v^2$
c. $25x^2 - 40xv + 16v^2$
d. $25x^2 + 16v^2$

6.2.7 Lesson Summary

In this lesson, you learned how to:

- Multiply monomials.
- Multiply a polynomial by a monomial.
- Multiply binomials.
- Multiply a polynomial by a polynomial.
- Multiply special products.
- Multiply polynomial functions.

Here are the **activities** that helped you reach those goals:

- 6.2.1: Multiplying Monomials
  - In this activity, you learned how to multiply monomials by other monomials. You used properties of exponents when multiplying variables with different exponents.
- 6.2.2: Multiplying Binomials
  - In this activity, you multiplied binomials using the Distributive Property, the FOIL method, and Vertical Alignment.
  - 6.2.2: Self Check
  - 6.2.2: Additional Resources
- 6.2.3: Multiplying a Polynomial by a Polynomial
  - In this activity, you multiplied polynomials using the Distributive Property and Vertical Alignment.
  - 6.2.3: Self Check
  - 6.2.3: Additional Resources
- 6.2.4: Multiplying Special Products
  - In this activity, you learned how to use patterns to quickly multiply binomial squares. You also learned how to multiply conjugate pairs using the Difference of Squares Pattern.
  - 6.2.4: Self Check
  - 6.2.4: Additional Resources
- 6.2.5: Multiplying Polynomial Functions
  - In this activity, you continued to practice multiplying polynomials in polynomial function form.

After these activities, you completed the following **practice**:

- 6.2.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

6.3 DIVIDING POLYNOMIALS

6.3.0 Lesson Overview

In this lesson, you will review exponent properties when dividing by monomials. You will also learn multiple methods to divide a polynomial by a binomial. You will master a method to determine the remainder of a division problem involving polynomials, as well as a method to determine if a binomial is a factor of a larger polynomial. You will also gain fluency in dividing polynomials as functions.

Polynomials are frequently used in calculating physics problems and other real-world situations involving projectiles and the motion of objects. Mastering the polynomial operations in this unit will help you in future studies.

When you finish this lesson, you will be able to:

- Divide monomials.
- Divide a polynomial by a monomial.
- Divide polynomials using long division.
- Divide polynomials using synthetic division.
- Divide polynomial functions.
- Use the Remainder and Factor Theorems.

Here are the **activities** that will help you reach those goals:

- 6.3.1: Dividing by Monomials
- 6.3.2: Dividing Polynomials Using Long Division
  - 6.3.2: Self Check
  - 6.3.2: Additional Resources
- 6.3.3: Dividing Polynomials Using Synthetic Division
  - 6.3.3: Self Check
  - 6.3.3: Additional Resources
- 6.3.4: Dividing Polynomial Functions and the Remainder Theorem
  - 6.3.4: Self Check
  - 6.3.4: Additional Resources
- 6.3.5: Using the Factor Theorem

After that, you'll **practice and review**.

- 6.3.6: Practice
- 6.3.7: Lesson Summary

6.3.1 Dividing by Monomials

Warm Up

Look at the division expression $-54a^6b^3 \div 9a^4b^2$.

1. Rewrite the division expression as a fraction.
2. Since we have a monomial divided by a monomial, we can rewrite the fraction as a product of several fractions where the factors in the numerator and denominator "match."
3. Now we can simplify the fraction multiplication and use the Quotient Property to divide.

For questions 4 – 6, divide a polynomial by a monomial. Divide each term of the polynomial by the monomial.

4. Find the quotient: $(18x^3y - 36xy^2) \div (-3xy)$.
5. Find the qoutient: $(48a^2b^2 + 30ab) \div (6ab)$.
6. Find the quotient: $(-72g^4h^6 - 45g^7h^3) \div (9g^3h^3)$.

6.3.2 Dividing Polynomials Using Long Division

Activity

Recall the process of dividing $960 \div 20$ as a class using long division. Use this same method to divide the polynomials below. After each problem, check in with a partner. With your partner, discuss your work and use the words dividend, divisor, and quotient in your explanations.

1. Find the quotient using long division.

$(m^2 + 9m + 20) \div (m + 4)$

Look through the video for help.

Problem Animation: Long Division of Polynomials (https://openstax.org/r/polynomial-long-division-animation)

2. Find the quotient using long division.

$(p^2 + 2p - 8) \div (p^2 - 7p + 10)$

3. Find the quotient using long division.

$(5x + 5) \div (x + 5)$

Notice in the next problem, there is no x^3 term in the dividend. Add $0x^3$ as a placeholder when using long division.

4. Find the quotient using long division.

$(x^4 - 11x^2 - 7x - 6) \div (x + 3)$

In this next problem, you will need to add two placeholders to divide.

5. Find the quotient using long division.

$(125x^3 - 8) \div (5x - 2)$

Are you ready for more?

Extending Your Thinking

Find the quotient using long division.

$(125y^5 - 80y^4 - 31y^3 + 37y^2 + 10y - 24) \div (5y - 4)$

Video: Dividing Polynomials Using Long Division

Watch the following video to learn more about how to divide polynomials using long division.

Dividing Polynomials Using Long Division (https://openstax.org/r/polynomial-long-division)

Self Check

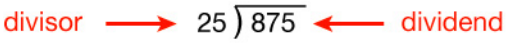
Find the quotient: $(3x^3 - 19x^2 - 9x - 35) \div (x - 7)$.

a. $x^2 + 2x + 5$
b. $3x^2 + 2x - 23$
c. $3x^2 + 2x + 5$
d. $3x^2 + 40x + 5$

Additional Resources

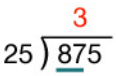
Dividing Polynomials Using Long Division

To divide a polynomial by a binomial, we follow a procedure very similar to long division of numbers. So let's look carefully at the steps we take when we divide a 3-digit number, 875, by a 2-digit number, 25.



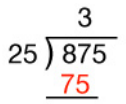
Step 1 - Starting from the leftmost digit in the dividend and moving right one digit at a time, find the smallest value possible that is larger than the divisor.

Divide this value, 87, by the divisor, 25. Write the whole-number result with no remainder in the space for the quotient.

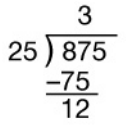


Step 2 -**Multiply** this digit in the quotient by the divisor, $3 \cdot 25$.

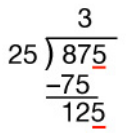
Write the result below the corresponding digits in the dividend.



Step 3 -**Subtract** the two values.

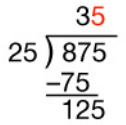


Step 4 - Bring down the next digit in the dividend.



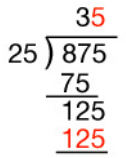
Step 5 - Repeat the division by dividing the new value, 125, by the divisor.

Write the result, 5, over the corresponding digit in the quotient.



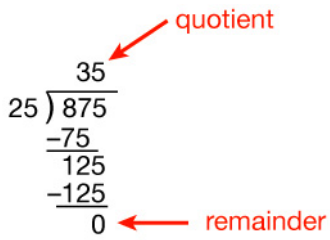
Step 6 - Multiply this digit in the quotient by the divisor, $5 \cdot 25$.

Write the result at the bottom below the previous number.



Step 7 -**Subtract** the two values. Since there are no more digits to bring down, this value is the remainder.

The quotient is on top.



We check division by multiplying the quotient by the divisor.

If we did the division correctly, the product should equal the dividend.

- $35 \cdot 25$
- 875 ✓

Now we will divide a trinomial by a binomial. As you read through the Example, notice how similar the steps are to the numerical Example above.

EXAMPLE 1

Find the quotient: $(x^2 + 9x + 20) \div (x + 5)$.

Step 1 - Write it as a long division problem. Be sure the dividend is in standard form.

$$x + 5 \overline{)\, x^2 + 9x + 20}$$

Step 2 - Divide x^2 by x. It may help to ask yourself, "What do I need to multiply x by to get x^2?"

$$\begin{array}{r} x \\ x + 5 \overline{)\, x^2 + 9x + 20} \end{array}$$

Step 3 - Put the answer, x, in the quotient over the x term. Multiply x times $x + 5$. Line up the like terms under the dividend.

$$\begin{array}{r} x \\ x + 5 \overline{)\, x^2 + 9x + 20} \\ x^2 + 5x \end{array}$$

Step 4 - Subtract $x^2 + 5x$ from $x^2 + 9x$. You may find it easier to change the signs and then add. Then bring down the last term, 20.

$$\begin{array}{r} x \\ x + 5 \overline{)\, x^2 + 9x + 20} \\ -x^2 + (-5x) \\ \hline 4x + 20 \end{array}$$

Step 5 - Divide $4x$ by x. It may help to ask yourself, "What do I need to multiply x by to get $4x$?" Put the answer, 4, in the quotient over the constant term.

$$\begin{array}{r} x + 4 \\ x + 5 \overline{)\, x^2 + 9x + 20} \\ -x^2 + (-5x) \\ \hline 4x + 20 \end{array}$$

Step 6 - Multiply 4 times $x + 5$.

$$\begin{array}{r} x + 4 \\ x + 5 \overline{)\ x^2 + 9x + 20} \\ \underline{-x^2 + (-5x)} \\ 4x + 20 \\ \underline{4x + 20} \end{array}$$

Step 7 - Subtract $4x + 20$ from $4x + 20$.

$$\begin{array}{r} x + 4 \\ x + 5 \overline{)\ x^2 + 9x + 20} \\ \underline{-x^2 + (-5x)} \\ 4x + 20 \\ \underline{-4x + (-20)} \\ 0 \end{array}$$

Step 8 - Check: Multiply the quotient by the divisor. You should get the dividend.

$(x + 4)(x + 5)$

$x^2 + 9x + 20$✓

When we divided 875 by 25, we had no remainder. But sometimes division of numbers does leave a remainder. The same is true when we divide polynomials. In the next Example, we'll have division that leaves a remainder. We write the remainder as a fraction with the divisor as the denominator.

Look back at the dividend in the previous Example. The terms were written in descending order of degrees, and there were no missing degrees. The dividend in the next Example will be $x^4 - x^2 + 5^x - 6$. It is missing an x^3 term. We will add in $0x^3$ as a placeholder.

EXAMPLE 2

Find the quotient: $(x^4 - x^2 + 5x - 6) \div (x + 2)$.

Notice that there is no x^3 term in the dividend. We will add $0x^3$ as a placeholder.

Step 1 - Write it as a long division problem.

Be sure the dividend is in standard form with placeholders for missing terms.

$$x + 2 \overline{)\ x^4 + 0x^3 - x^2 + 5x - 6}$$

Step 2 - Divide x^4 by x.

Put the answer, x^3, in the quotient over the x^3 term. Multiply x^3 times $x + 2$.

Line up the like terms. Subtract and then bring down the next term.

$$\begin{array}{r} x^3 \\ x + 2 \overline{)\ x^4 + 0x^3 - x^2 + 5x - 6} \\ \underline{-(x^4 + 2x^3)} \\ -2x^3 - x^2 \end{array}$$

Step 3 - Divide $-2x^3$ by x.

Put the answer, $-2x^2$, in the quotient over the x^2 term.

Multiply $-2x^2$ times $x+1$. Line up the like terms.

Subtract and bring down the next term.

$$
\begin{array}{r}
x^3-2x^2 \\
x+2\overline{)\,x^4+0x^3-x^2+5x-6} \\
\underline{-(x^4+2x^3)} \\
-2x^3-x^2 \\
\underline{-(-2x^3-4x^2)} \\
3x^2+5x
\end{array}
$$

Step 4 - Divide $3x^2$ by x.

Put the answer, $3x$, in the quotient over the x term.

Multiply $3x$ times $x+1$.

Line up the like terms. Subtract and bring down the next term.

$$
\begin{array}{r}
x^3-2x^2+3x \\
x+2\overline{)\,x^4+0x^3-x^2+5x-6} \\
\underline{-(x^4+2x^3)} \\
-2x^3-x^2 \\
\underline{-(-2x^3-4x^2)} \\
3x^2+5x \\
\underline{-(3x^2-6x)} \\
-x-6
\end{array}
$$

Step 5 - Divide $-x$ by x. Put the answer, -1, in the quotient over the constant term.

Multiply -1 times $x+1$. Line up the like terms.

Change the signs and then add.

Write the remainder as a fraction with the divisor as the denominator.

$$
\begin{array}{r}
x^3-2x^2+3x-1-\dfrac{4}{x+2} \\
x+2\overline{)\,x^4+0x^3-x^2+5x-6} \\
\underline{-(x^4+2x^3)} \\
-2x^3-x^2 \\
\underline{-(-2x^3-4x^2)} \\
3x^2+5x \\
\underline{-(3x^2-6x)} \\
-x-6 \\
\underline{-(-x-2)} \\
-4
\end{array}
$$

Step 6 - To check, multiply $(x+2)(x^3-2x^2+3x-1-\frac{4}{x+2})$.

The result should be x^4-x^2+5x-6.

In the next Example, we will divide by $2a+3$.

As we divide, we will have to consider the coefficients as well as the variables.

EXAMPLE 3

Find the quotient: $(8a^3 + 27) \div (2a + 3)$.

This time, we will show the division all in one step. We need to add two placeholders to divide.

$$
\begin{array}{r}
4a^2 - 6a^2 + 9 \\
2a + 3\overline{)\ 8a^3 + \ 0a^2 + \ 0a + 27} \\
\underline{-(8a^3 + 12a^2)} \qquad \longleftarrow\ 4a^2(2a+3) \\
-12a^2 + \ 0a \\
\underline{-(-12a^2 - \ 18a)} \qquad \longleftarrow\ 6a(2a+3) \\
18a + 27 \\
\underline{-(18a + 27)} \qquad \longleftarrow\ 9(2a+3) \\
0
\end{array}
$$

To check, multiply $(2a + 3)(4a^2 - 6a + 9)$.

The result should be $8a^3 + 27$.

EXAMPLE 4

Find the quotient: $(2x + 14) \div (x + 7)$. Can you see a quick way to solve this?

\begin{array}{r} 2 \\ x+7{\overline{\smash{\big)}\,2x+14}}\\ \underline{-~(2x+14)}\\ 0\\ \end{array}

The answer is 2.

TRY IT **Dividing Polynomials Using Long Division**

Find the quotients.

1. $(y^2 + 10y + 21) \div (y + 3)$
2. $(x^4 - 7x^2 + 7x + 6) \div (x + 3)$
3. $(x^3 - 64) \div (x - 4)$

6.3.3 Dividing Polynomials Using Synthetic Division

Activity

In this activity we're going to use synthetic division to divide a polynomial by a binomial.

Part 1: Use Long Division

You are going to solve a division problem using long division.

1. $(x^2 - 13x + 42) \div (x - 6)$

Part 2: Use Synthetic Division

You are now going to solve the same division problem using synthetic division.

2. $(x^2 - 13x + 42) \div (x - 6)$

Work with a partner to find each quotient. Discuss any differences you have in determining the correct solution.

3. $(4x^3 + 5x^2 - 5x + 3) \div (x + 2)$

4. $(2x^3 - 11x^2 + 16x - 12) \div (x - 4)$

5. $(x^4 - 9x^2 + 2x + 6) \div (x + 3)$

6. $(3x^4 - 11x^3 + 2x^2 + 10x + 6) \div (x - 3)$

Self Check

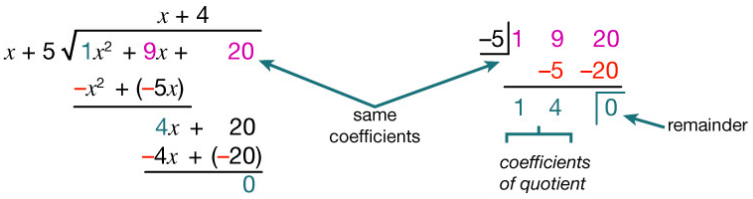
Use synthetic division to find the quotient and remainder when $x^4 + x^2 + 6x - 10$ is divided by $x + 2$.

a. $x^3 - 2x^2 + 5x - 4$; remainder of -2
b. $x^3 - 2x^2 + 5x - 4$; remainder of 2
c. $x^3 - 2x^2 + 5x - 4$; remainder of 0
d. $x^3 - x^2 + 8x$; remainder of 26

Additional Resources

Dividing Polynomials Using Synthetic Division

Mathematicians like to find patterns to make their work easier. Since long division can be tedious, let's look back at the long division we did in a previous example and look for some patterns. We will use this as a basis for what is called synthetic division. The same problem in the synthetic division format is shown next.



Synthetic division basically just removes unnecessary repeated variables and numbers. Here, all the x and x^2 are removed as well as the $-x^2$ and $-4x$ because they are opposite the term above.

The first row of the synthetic division is the coefficients of the dividend. The -5 is the opposite of the 5 in the divisor.

The second row of the synthetic division is the numbers shown in red in the division problem.

The third row of the synthetic division is the numbers shown in blue in the division problem.

Notice the quotient and remainder are shown in the third row.

Synthetic division only works when the divisor is of the form $x - c$.

The following example will explain the process.

EXAMPLE 1

Use synthetic division to find the quotient and remainder when $2x^3 + 3x^2 + x + 8$ is divided by $x + 2$.

Step 1 - Write the dividend with decreasing powers of x.

$2x^3 + 3x^2 + x + 8$

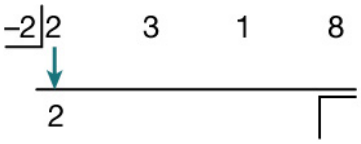
Step 2 - Write the coefficients of the terms as the first row of the synthetic division.

| 2 3 1 8

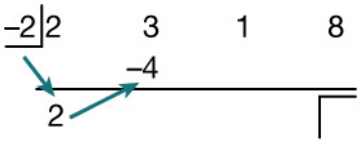
Step 3 - Write the divisor as $x - c$ and place c in the synthetic division in the divisor box.



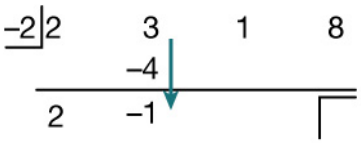
Step 4 - Bring down the first coefficient to the third row.



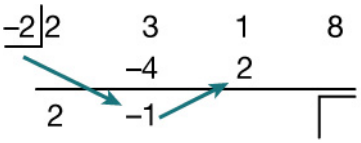
Step 5 - Multiply that coefficient by the divisor and place the result in the second row under the second coefficient.



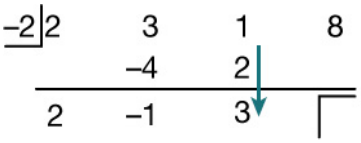
Step 6 - Add the second column, putting the result in the third row.



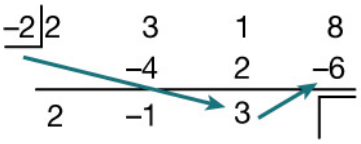
Step 7 - Multiply that result by the divisor and place the result in the second row under the third coefficient.



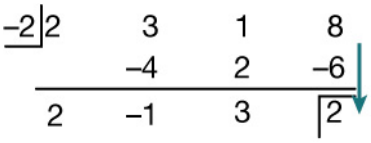
Step 8 - Add the third column, putting the result in the third row.



Step 9 - Multiply that result by the divisor and place the result in the third row under the third coefficient.



Step 10 - Add the final column, putting the result in the third row.



Step 11 - The quotient is $2x^2 - 1x + 3$, and the remainder is 2.

The division is complete. The numbers in the third row give us the result. The $2\ {-1}\ 3$ are the coefficients of the quotient. The quotient is $2x^2 - 1x + 3$. The 2 in the box in the third row is the remainder.

Compare your answer:

$$\begin{aligned}
(\text{quotient})(\text{divisor}) + \text{remainder} &= \text{dividend} \\
\left(2x^2 - 1x + 3\right)(x+2) + 2 &\stackrel{?}{=} 2x^3 + 3x^2 + x + 8 \\
2x^3 - x^2 + 3x + 4x^2 - 2x + 6 + 2 &\stackrel{?}{=} 2x^3 + 3x^2 + x + 8 \\
2x^3 + 3x^2 + x + 8 &= 2x^3 + 3x^2 + x + 8 \checkmark
\end{aligned}$$

EXAMPLE 2

Use synthetic division to find the quotient and remainder when $x^4 - 16x^2 + 3x + 12$ is divided by $x + 4$.

The polynomial $x^4 - 16x^2 + 3x + 12$ has its terms in order with descending degree, but we notice there is no x^3 term. We will add a 0 as a placeholder for the x^3 term. In $x - c$ form, the divisor is $x - (-4)$.

-4	1	0	-16	3	12
		-4	16	0	-12
	1	-4	0	3	0

We divided a 4^{th} degree polynomial by a 1st degree polynomial, so the quotient will be a 3^{rd} degree polynomial. Reading from the third row, the quotient has the coefficients $1\ {-4}\ 0\ 3$, which is $x^3 - 4x^2 + 3$. The remainder is 0.

TRY IT **Dividing Polynomials Using Synthetic Division**

1. Use synthetic division to find the quotient and remainder when $3x^3 + 10x^2 + 6x - 2$ is divided by $x + 2$.

2. Use synthetic division to find the quotient and remainder when $x^4 - 16x^2 + 5x + 20$ is divided by $x + 4$.

6.3.4 Dividing Polynomial Functions and the Remainder Theorem

Activity

Divide the polynomial functions. Find the quotient value.

1. $f(x) = x^2 - 5x - 24$

$g(x) = x + 3$

$(\frac{f}{g})(x) =$ _____

$f(-3) =$ _____

2. $f(x) = x^2 - 15x + 54$

$g(x) = x - 9$

$(\frac{f}{g})(x) =$

_____ $f(9) =$ _____

3. $f(x) = x^2 + 2x - 3$

$g(x) = x + 3$

$(\frac{f}{g})(x) =$ _____

$f(-3) =$ _____

The Remainder Theorem states that if a polynomial function $f(x)$ is divided by $x - c$, then the remainder is $f(c)$. This

means we can always compare the remainder by finding $f(c)$ when the divisor is written in the form $x - c$.

4. Use the Remainder Theorem to find the remainder when $f(x) = x^3 - 7x + 12$ is divided by $x + 3$.

5. Use the Remainder Theorem to find the remainder when $f(x) = 2x^3 - 6x - 24$ is divided by $x - 3$.

Self Check

For functions $f(x) = x^3 + 2x^2 - 19x + 12$ and $g(x) = x - 3$:

Find $(\frac{f}{g})(x)$ and $(\frac{f}{g})(0)$.

a. $(\frac{f}{g})(x) = x^2 - 5x + 4$

$(\frac{f}{g})(0) = -4$

b. $(\frac{f}{g})(x) = x^2 + 5x + 4$

$(\frac{f}{g})(0) = 4$

c. $(\frac{f}{g})(x) = x^2 + 5x - 4$

$(\frac{f}{g})(0) = 12$

d. $(\frac{f}{g})(x) = x^2 + 5x - 4$

$(\frac{f}{g})(0) = -4$

Additional Resources

Dividing Polynomial Functions

Just as polynomials can be divided, polynomial functions can also be divided.

DIVISION OF POLYNOMIAL FUNCTIONS

For functions $f(x)$ and $g(x)$, where $g(x) \neq 0$,

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$

EXAMPLE 1

For functions $f(x) = x^2 - 5x - 14$ and $g(x) = x + 2$:

a. Find $(\frac{f}{g})(x)$.

Step 1 - $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ Substitute for $f(x)$ and $g(x)$.

$(\frac{f}{g})(x) = \frac{f(x)=x^2-5x-14}{x+2}$

Step 2 - Divide the polynomials using long division.

$$
\begin{array}{r}
x-7 \\
x+2\overline{)\,x^2-5x-14} \\
\underline{-(x^2+2x)} \\
-7x-14 \\
\underline{-(-7x-14)} \\
0
\end{array}
$$

$(\frac{f}{g})(x) = x - 7$

b. Find $(\frac{f}{g})(-4)$.

Step 1 - In part (1), we found $(\frac{f}{g})(x)$.

$(\frac{f}{g})(x) = x - 7$

Step 2 - To find $(\frac{f}{g})(-4)$, substitute $x = -4$.

$(\frac{f}{g})(-4) = -4 - 7$

$(\frac{f}{g})(-4) = -11$

TRY IT **Dividing Polynomial Functions**

For functions $f(x) = x^2 - 5x - 36$ and $g(x) = x + 4$:

1. Find $(\frac{f}{g})(x)$.
2. Find $(\frac{f}{g})(-5)$.

Using the Remainder Theorem

Let's look at some division problems and their remainders. They are summarized in the chart below. If we take the dividend from each division problem and use it to define a function, we get the functions shown in the chart. When the divisor is written as $x - c$, the value of the function at c, $f(c)$, is the same as the remainder from the division problem.

Dividend	Divisor	Remainder	Function	$f(c)$
$x^4 - 7x^2 + 7x + 6$	$x - (-3)$	3	$f(x) = x^4 - 7x^2 + 7x + 6$	$f(-3) = 3$
$f(x) = 3x^3 - 2x^2 - 10x + 8$	$x - (-3)$	-61	$f(x) = 3x^3 - 2x^2 - 10x + 8$	$f(-3) = -61$
$x^4 - 16x^2 + 3x + 15$	$x - (-4)$	3	$f(x) = x^4 - 16x^2 + 3x + 15$	$f(-4) = 3$

To see this more generally, we realize we can check a division problem by multiplying the quotient times the divisor and adding the remainder. In function notation, we could say: To get the dividend $f(x)$, we multiply the quotient, $q(x)$, times the divisor, $x - c$, and add the remainder, r.

$f(x) = q(x)(x - c) + r$

If we evaluate this at c, we get:

$f(c) = q(c)(c - c) + r$

$f(c) = q(c)(0) + r$

$f(c) = r$

This leads us to the Remainder Theorem.

REMAINDER THEOREM

If the polynomial function $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

EXAMPLE 2

Use the Remainder Theorem to find the remainder when $f(x) = x^3 + 3x + 19$ is divided by $x + 2$.

To use the Remainder Theorem, we must use the divisor in the $x - c$ form. We can write the divisor $x + 2$ as $x - (-2)$. So, our c is -2.

To find the remainder, we evaluate $f(c)$, which is $f(-2)$.

$f(x) = x^3 + 3x + 19$

Step 1 - To evaluate $f(-2)$, substitute $x = -2$.

$f(-2) = (-2)^3 + 3(-2) + 19$

Step 2 - Simplify.

$f(-2) = (-8) - 6 + 19$

$f(-2) = 5$

The remainder is 5 when $f(x) = x^3 + 3x + 19$ is divided by $x + 2$.

Step 3 - Check using synthetic division.

$$\begin{array}{r|cccc} -2 & 1 & 0 & 3 & 19 \\ & & -2 & 4 & -14 \\ \hline & 1 & -2 & 7 & \boxed{5} \end{array}$$

The remainder is 5.

TRY IT **Using the Remainder Theorem**

Use the Remainder Theorem to find the remainder when $f(x) = x^3 + 4x + 15$ is divided by $x + 2$.

6.3.5 Using the Factor Theorem

Cool Down

In all the division problems we completed that had a remainder of zero, we say that the divisor was a factor of the dividend.

Whenever a divisor, $x - c$, divides a polynomial function, $f(x)$, and results in a remainder of zero, we say $x - c$ is a factor of $f(x)$.

This leads us to the Factor Theorem.

FACTOR THEOREM

For any polynomial function $f(x)$,

- if $x - c$ is a factor of $f(x)$, then $f(c) = 0$.

- if $f(c) = 0$, then $x - c$ is a factor of $f(x)$.

Use the Factor Theorem to determine whether the binomial is a factor of the given polynomial function. Be prepared to show your reasoning.

1. Is $(x - 5)$ a factor of $f(x) = x^3 - 125$?

2. Is $(x - 6)$ a factor of $f(x) = x^3 - 216$?

3. Is $(x + 3)$ a factor of $f(x) = x^3 + 8x^2 + 21x + 18$?

4. Is $(x + 4)$ a factor of $f(x) = x^3 + x^2 - 14x + 8$?

6.3.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 - 2, find the quotients using division.

1. $\frac{45x^5y^9}{-60x^8y^6}$

a. $-\frac{3}{4}x^3y^3$
b. $-\frac{3y^3}{4x^3}$
c. $-\frac{4y^3}{3x^3}$
d. $-\frac{3x^3}{4y^3}$

2. $(48y - 24x) \div (2y - x)$

a. -24
b. $24y$
c. $-24x$
d. 24

For questions 3 and 4, divide using long division.

3. $(m^3 + 1000) \div (m + 10)$

a. $m^2 - 10m + 100$
b. $m^2 + 10m - 100$
c. $m^2 - 100$
d. $m^2 + 10m$

4. $(125y^3 - 64) \div (5y + 4)$

a. $25y^2 - 20y + 16$
b. $25y^2 + 20y + 16$
c. $25y^2 - 20y + 16 - \frac{64}{5y+4}$
d. $25y^2 - 20y + 16 - \frac{128}{5y+4}$

For questions 5 - 6, divide using synthetic division.

5. $(b^3 - 3b^2 - 4b + 12) \div (b + 2)$

a. $b^2 + 5b - 6$; remainder 0
b. $b^2 - 5b$; remainder 6
c. $b^2 - 5b + 6$; remainder 0
d. $b^2 + 5b$; remainder 6

6. $(3x^4 - 11x^3 + 2x^2 + 10x + 6) \div (x - 3)$

a. $3x^3 - 2x^2 - 4x$; remainder -2
b. $3x^3 - 2x^2 - 4x - 2$; remainder 0
c. $3x^3 - 20x^2 - 62x - 180$; remainder 0
d. $3x^3 + 2x^2 + 4x + 2$; remainder 0

For questions 7 and 8, use the following polynomial functions.

- $f(x) = x^2 - 13x + 36$
- $g(x) = x - 4$

7. Find $(\frac{f}{g})(x)$.

a. $x - 9$
b. $x - 3$
c. $x - 12$
d. $x + 9$

8. Find $(\frac{f}{g})(-1)$.

a. 8
b. -4
c. -10
d. -8

9. Use the Remainder Theorem to find the remainder.
$f(x) = x^2 - 4x - 9$ is divided by $x + 2$

a. –9
b. –13
c. 21
d. 3

10. Use the Factor Theorem to determine if $x - 3$ is a factor of $x^3 - 7x^2 + 11x + 3$.

a. Yes
b. No

6.3.7 Lesson Summary

In this lesson, you learned how to:

- Divide monomials.
- Divide a polynomial by a monomial.
- Divide polynomials using long division.
- Divide polynomials using synthetic division.
- Divide polynomial functions.
- Use the Remainder and Factor Theorems.

Here are the **activities** that helped you reach those goals:

- 6.3.1: Dividing by Monomials
  - In this activity, you used the properties of exponents to divide monomials by monomials. You also used distributed division to divide polynomials by monomials.
- 6.3.2: Dividing Polynomials Using Long Division
  - IIn this activity, you used long division, in the same way as with constants, to divide polynomials by binomials. You also learned how to interpret a remainder in these division problems.
  - 6.3.2: Self Check
  - 6.3.2: Additional Resources
- 6.3.3: Dividing Polynomials Using Synthetic Division
  - In this activity, you learned a quicker, more efficient method to divide polynomials by binomials.
  - 6.3.3: Self Check
  - 6.3.3: Additional Resources
- 6.3.4: Dividing Polynomial Functions and the Remainder Theorem
  - In this activity, you divided polynomials in function form. You also learned an efficient way to determine the remainder in polynomial division. This can be used as an additional check for polynomial division problems.
  - 6.3.4: Self Check
  - 6.3.4: Additional Resources
- 6.3.5: Using the Factor Theorem
  - In this activity, you found a method to determine if a binomial of the form $x - c$ is a factor in a larger polynomial.

After these activities, you completed the following **practice**:

- 6.3.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

6.4 GREATEST COMMON FACTOR AND FACTOR BY GROUPING

6.4.0 Lesson Overview

In this lesson, you will use your skills finding common factors in different terms to find the greatest common factor (GCF) of two or more expressions. You will then use the GCF to factor these expressions. In some cases, you will need to group terms together so that the expressions can be factored. These skills will allow you to rewrite expressions into a simpler, fully factored form.

When you finish this lesson, you will be able to:

- Find the greatest common factor of two or more expressions.
- Factor the greatest common factor from a polynomial.
- Factor by grouping.

Here are the **activities** that will help you reach those goals:

- 6.4.1: Finding Factors
- 6.4.2: Finding the GCF of Two or More Expressions
  - 6.4.2: Self Check
  - 6.4.2: Additional Resources
- 6.4.3: Factoring the GCF from Polynomials
  - 6.4.3: Self Check
  - 6.4.3: Additional Resources
- 6.4.4: Factoring Polynomials by Grouping
  - 6.4.4: Self Check
  - 6.4.4: Additional Resources
- 6.4.5: Can You Factor the Polynomial?

After that, you'll **practice and review**.

- 6.4.6: Practice
- 6.4.7: Lesson Summary

6.4.1 Finding Factors

Warm Up

Finding the factors of a number, monomial, or polynomial is an important step used in the activities of this lesson.

1. Factor 56 into prime numbers to find all of its factors.

2. Find all of the factors of $42m^3y^2$.

3. Find all of the common factors between $6n^2p^2$ and $9ns^2$.

4. Find the least common multiple (LCM) of $3x^2$ and $8x^4$.

6.4.2 Finding the GCF of Two or More Expressions

Activity

For the first part of the activity, you will work with a partner. Your teacher will hand out two pieces of paper, one to each of you. It is important for the activity that you do not show your partner your paper.

1. When you receive your paper, look at your expression. Write down all of the factors in your expression.

For example, if your expression was $100x^3y^2$, you would write down $2{\cdot}2{\cdot}5{\cdot}5{\cdot}x{\cdot}x{\cdot}x{\cdot}y{\cdot}y$. Do not share these factors with your partner yet.

2. When you and your partner have finished finding the factors of your expression, you will try to determine the GCF of

these two terms without actually knowing the other's expression! Take turns with your partner sharing one factor and the quantity of that factor. For example, you could say, "My expression has two factors that are 5" or "My expression has three factors that are x." Write down the factors your partner has shared with you.

3. Once you and your partner have finished taking turns sharing factors, you should know all of the factors in your partner's expression. Using this information, determine your partner's expression and find the GCF of these two expressions. Once you have found the GCF, compare your answer with your partner's. Did you have the same answer?

Write the GCF that you found.

Now, let's look at an example of finding the GCF of two expressions as a class:

4. $18g^4h^2, 63g^3h^3$

Work individually on the next part of the activity.

Find the greatest common factor of each of the following sets of expressions.

5. $10p^3q, 12pq^2$

6. $8a^2b^3, 10ab^2$

7. $10a^3, 12a^2, 14a$

8. $35x^3y^2, 10x^4y, 5x^5y^3$

Self Check

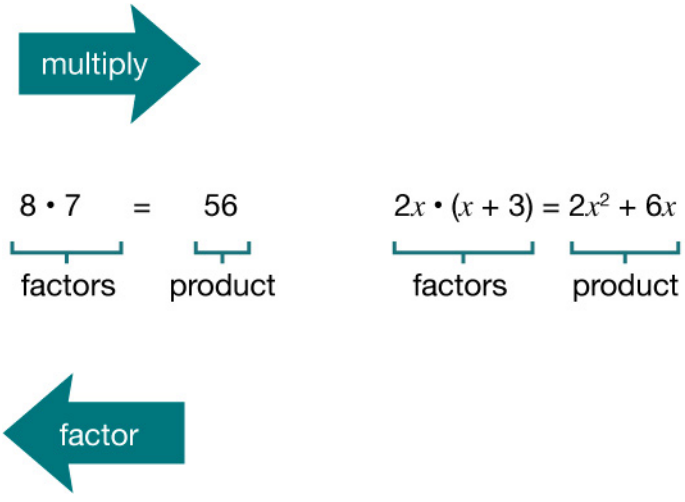
Find the greatest common factor of $25m^4, 35m^3, 20m^2$.

a. $10m$
b. $5m^2$
c. $5m$
d. $10m^2$

Additional Resources

Finding the GCF of Two or More Expressions

Earlier, we multiplied factors together to get a product. Now, we will reverse this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called **factoring**.



Now we will factor expressions and find the **greatest common factor** of two or more expressions.

GREATEST COMMON FACTOR

The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

We will summarize the steps we use to find the greatest common factor.

How to find the greatest common factor (GCF) of two expressions:

Step 1- Factor each coefficient into primes. Write all variables with exponents in expanded form.

Step 2- List all factors, matching common factors in a column. In each column, circle the common factors.

Step 3- Bring down the common factors that all expressions share.

Step 4- Multiply the factors.

The following example will show the steps to find the greatest common factor of three expressions.

EXAMPLE

Find the greatest common factor of $21x^3, 9x^2, 15x$.

Step 1 - Factor each coefficient into primes and write the variables with exponents in expanded form.

Step 2 - Circle the common factors in each column.

Step 3 - Bring down the common factors.

$$\begin{aligned} 21x^3 &= 3 \cdot 7 \cdot x \cdot x \cdot x \\ 9x^2 &= 3 \cdot 3 \cdot x \cdot x \\ 15x &= 3 \cdot 5 \cdot x \\ \hline \text{GCF} &= 3 \cdot x \end{aligned}$$

Step 4 - Multiply the factors.

GFC = $3x$

The GCF of $21x^3$, $9x^2$, and $15x$ is $3x$.

TRY IT **Finding the GCF of Two or More Expressions**

Find the greatest common factor of $14x^3, 105x, 70x^2$.

6.4.3 Factoring the GCF from Polynomials

Activity

To factor a GCF from a polynomial and represent it in factored form, we must first find the GCF.

Let's look at an example: $7z^5 - 21sz^3$.

Work individually on the following problems.

Factor the GCF from each polynomial. Remember that if the leading coefficient is negative, then the GCF will be negative.

1. $5x^3 - 25x^2$
2. $8x^3y - 10x^2y^2 + 12xy^3$
3. $-5y^3 + 35y^2 - 15y$

4. $15x^3y - 3x^2y^2 + 6xy^3$

5. $-4p^3q - 12p^2q^2 + 16pq^2$

6. $5x(x+1) + 3(x+1)$

7. $3b(b-2) - 13(b-2)$

8. $6m(m-5) - 7(m-5)$

Self Check

Factor: $8a^3b + 2a^2b^2 - 6ab^3$.

a. $2ab(4a^2 + ab - 3b^2)$
b. $2b(4a^3 + a^2b - 3ab^2)$
c. $ab(8a^2 + 2ab - 6b^2)$
d. $-2ab(4a^2 + ab - 3b^2)$

Additional Resources

Factoring the GCF from Polynomials

It is sometimes useful to represent a number as a product of factors, for example, 12 as $2 \cdot 6$ or $3 \cdot 4$. In algebra, it can also be useful to represent a polynomial in factored form. We will start with a product, such as $3x^2 + 15x$, and end with its factors, $3x(x+5)$. To do this, we apply the Distributive Property "in reverse." We state the Distributive Property here just as you saw it in earlier chapters and "in reverse."

DISTRIBUTIVE PROPERTY

If a, b, and c are real numbers, then

$$a(b+c) = ab + ac \text{ and } ab + ac = a(b+c)$$

The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to factor a polynomial? You just find the GCF of all the terms and write the polynomial as a product!

EXAMPLE 1

Factor: $8m^3 - 12m^2n + 20mn^2$.

How to factor the greatest common factor from a polynomial:

Step 1 - Find the GCF of all the terms of the polynomial. For this problem, find the GCF of $8m^3$, $12m^2n$, and $20mn^2$.

$$\begin{aligned} 8m^3 &= 2 \cdot 2 \cdot 2 \cdot m \cdot m \cdot m \\ 12m^2n &= 2 \cdot 2 \cdot 3 \cdot m \cdot m \cdot n \\ 20mn^2 &= 2 \cdot 2 \cdot 5 \cdot m \cdot n \cdot n \\ \text{GCF} &= 2 \cdot 2 \cdot m \\ \text{GCF} &= 4m \end{aligned}$$

Step 2 - Rewrite each term as a product using the GCF. In this case, rewrite $8m^3$, $12m^2n$, and $20mn^2$ as products of their GCF, $4m$.

- $8m^3 = 4m \cdot 2m^2$
- $12m^2n = 4m \cdot 3mn$

- $20mn^2 = 4m \cdot 5n^2$

$8m^3 - 12m^2n + 20mn^2$

$4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$

Step 3 - Use the "reverse" Distributive Property to factor the expression.

$4m(2m^2 - 3mn + 5n^2)$

Step 4 - Check by multiplying the factors.

$4m(2m^2 - 3mn + 5n^2)$

$4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$

$8m^3 - 12m^2n + 20mn^2$

When the leading coefficient is negative, we factor the negative out as part of the GCF.

EXAMPLE 2

Factor: $-4a^3 + 36a^2 - 8a$.

The leading coefficient is negative, so the GCF will be negative.

Step 1 - Rewrite each term using the GCF, $-4a$.

$-4a \cdot a^2 - (-4a) \cdot 9a + (-4a) \cdot 2$

Step 2 - Factor the GCF.

$-4a(a^2 - 9a + 2)$

Step 3 - Check.

$-4a(a^2 - 9a + 2)$

$-4a \cdot a^2 - (-4a) \cdot 9a + (-4a) \cdot 2$

$-4a^3 + 36a^2 - 8a$

So far, our greatest common factors have been monomials. In the next example, the greatest common factor is a binomial.

EXAMPLE 3

Factor: $3y(y + 7) - 4(y + 7)$.

The GCF is the binomial $y + 7$.

$3y(y + 7) - 4(y + 7)$

Step 1 - Factor the GCF, $(y + 7)$.

$(y + 7)(3y - 4)$

Step 2 - Check on your own by multiplying.

TRY IT **Factoring the GCF from Polynomials**

Factor.

1. $9xy^2 + 6x^2y^2 + 21y^3$
2. $-4b^3 + 16b^2 - 8b$
3. $4m(m + 3) - 7(m + 3)$

6.4.4 Factoring Polynomials by Grouping

Activity

If there is no common term in all of the terms of a polynomial, it may be possible to factor by grouping. To do this, separate the polynomial into two parts with two terms in each part. Then look for the GCF in each part. If the polynomial can be factored, you will find that a common factor emerges from both parts.

Let's look at an example. Factor the following polynomial.

$ms - 4m + 7s - 28$

Video: Factoring by Grouping

Watch the following video to learn more about how to factor by grouping.

Factoring by Grouping (https://openstax.org/r/factor-grouping)

Work on the following exercises individually.

Factor by grouping:

1. $xy + 7y + 8x + 56$

2. $x^2 + 2x - 5x - 10$

You can use the following video if you need some help

Problem Animation: Factoring a Trinomial Using the Grouping Method (https://openstax.org/r/factor-trinomial-grouping)

3. $20x^2 - 16x - 15x + 12$

4. $x^2 + 4x - 7x - 28$

5. $2x^2 - 14x - 5x + 35$

6. $xy - 6x - 4y + 24$

7. $9x^2 + 12x - 15x - 20$

8. $xy - 9x + 2y - 18$

Self Check

Factor by grouping: $42m^2 - 18m - 35m + 15$.

a. $(7m - 5)(6m - 3)$
b. $(7m + 3)(6m + 5)$
c. $(7m - 3)(6m - 5)$
d. $(14m - 3)(3m - 5)$

Additional Resources

Factoring Polynomials by Grouping

Sometimes there is no common factor of all the terms of a polynomial. When there are four terms, we separate the polynomial into two parts with two terms in each part. Then we look for the GCF in each part. If the polynomial can be factored, you will find that a common factor emerges from both parts. Not all polynomials can be factored. Just like some numbers are prime, some polynomials are prime.

EXAMPLE 1

Factor by grouping: $xy + 3y + 2x + 6$.

Step 1 - Group terms with common factors.

Is there a greatest common factor of all four terms?

$xy + 3y + 2x + 6$

No, so let's separate the first two terms from the second two.

$$\underbrace{xy + 3y}\ \underbrace{+\ 2x + 6}$$

Step 2 - Factor out the common factor in each group.

Factor the GCF from the first two terms.

$$\underbrace{y(x + 3) + 2x + 6}$$

Factor the GCF from the second two terms.

$y(x + 3) + 2(x + 3)$

Step 3 - Factor the common factor from the expression.

Notice that each term has a common factor of $(x + 3)$.

$y(x\ +\ 3) + 2(x\ +\ 3)$

Factor out the common factor.

$(x + 3)(y + 2)$

Step 4 - Check.

Multiply $(x + 3)(y + 2)$. Is the product the original expression?

$(x + 3)(y + 2)$

$xy + 2x + 3y + 6$

$xy + 3y + 2x + 6$✓

How to factor by grouping:

Step 1 - Group terms with common factors.

Step 2 - Factor out the common factor in each group.

Step 3 - Factor the common factor from the expression.

Step 4 - Check by multiplying the factors.

EXAMPLE 2

Factor by grouping: $x^2 + 3x - 2x - 6$.

Step 1 - There is no GCF in all four terms.

$x^2 + 3x - 2x - 6$

Step 2 - Separate into two parts.

$x^2 + 3x \qquad - 2x - 6$

Step 3 - Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.

$x(x + 3) - 2(x + 3)$

Step 4 - Factor out the common factor.

$(x + 3)(x - 2)$

Step 5 - Check on your own by multiplying.

TRY IT **Factoring Polynomials by Grouping**

Factor by grouping:

1. $xy + 8y + 3x + 24$
2. $6x^2 - 3x - 4x + 2$

6.4.5 Can You Factor the Polynomial?

Cool Down

1. Which of these polynomials has a greatest common factor of 1?

a. $10q^2 + 14q + 20$

b. $12x^3 - 2x$

c. $12x^3y - 7g^8h^3 + 11s^2$

d. $-2x - 4$

Determine if the expression can be factored by grouping. If it can, factor the expression. If not, answer "prime."

2. $cd + 6c + 4d + 24$

3. $x^2 - y + 4w - 3$

6.4.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 3, find the GCF of each set of expressions.

1. $12m^2n^3, 30m^5n^3$

a. $6m^2n^3$
b. $2m^5n^3$
c. $3m^2n^3$
d. $6m^3n^3$

2. $28x^2y^4, 42x^4y^4$

3. $27p^2q^3, 45p^3q^4, 9p^4q^3$

a. $9p^3q^3$
b. $9p^2q^3$
c. $27p^2q^3$
d. $9pq^3$

For questions 4 – 6, factor the expression.

4. $15x^3y - 3x^2y^2 + 6xy^3$

5. $-7a^3 + 21a^2 - 14a$

a. $-7a(a^2 - 3a + 2)$
b. $-7a^2(a - 1)$
c. $7a(a^2 - 3a + 2)$
d. $-7(a^2 - 3a + 2)$

6. $8n(n - 4) + 5(n - 4)$

a. $(8n - 4)(n + 5)$
b. $(n - 4)(8n + 5)$
c. $(n - 4)(8n - 5)$
d. $(n + 4)(8n + 5)$

For questions 7–10, factor by grouping

7. $u^2 - u + 6u - 6$

a. $(u - 1)(u + 6)$
b. $(u + 1)(u - 6)$
c. $(u - 1)(u - 6)$
d. $(u + 1)(u + 6)$

8. $16q^2 + 20q - 28q - 35$

a. $(4q - 5)(4q - 7)$
b. $(4q + 5)(4q - 7)$
c. $(4q - 5)(4q + 7)$
d. $(4q + 5)(4q + 7)$

9. $r^2 - 3r - r + 3$

a. $(r + 3)(r - 1)$
b. $(r - 3)(r + 1)$
c. $(r + 3)(r + 1)$
d. $(r - 3)(r - 1)$

10. $6y^2 + 7y + 24y + 28$

6.4.7 Lesson Summary

In this lesson, you learned how to:

- Find the greatest common factor of two or more expressions.
- Factor the greatest common factor from a polynomial.
- Factor by grouping.

Here are the **activities** that helped you reach those goals:

- 6.4.1: Finding Factors
  - In this activity, you reviewed how to find the factors of a number, monomial, or group of terms. This is an important skill used in factoring polynomials later in the lesson.
- 6.4.2: Finding the GCF of Two or More Expressions
  - In this activity, you found all the factors of two or more expressions and used that list to find their greatest common factor.
  - 6.4.2: Self Check
  - 6.4.2: Additional Resources
- 6.4.3: Factoring the GCF from Polynomials
  - In this activity, you found the GCF of two or more expressions and then factored it out of those terms using the “reverse” of the Distributive Property.
  - 6.4.3: Self Check
  - 6.4.3: Additional Resources
- 6.4.4: Factoring Polynomials by Grouping
  - In this activity, you recognized terms in an expression that shared factors and grouped them together to factor the polynomial.
  - 6.4.4: Self Check
  - 6.4.4: Additional Resources
- 6.4.5: Can you Factor the Polynomial?
  - In this activity, you used everything you learned in the lesson to determine if polynomials can be factored.

After these activities, you completed the following **practice**:

- 6.4.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

6.5 FACTOR TRINOMIALS

6.5.0 Lesson Overview

In this lesson, you will learn many different methods to factor a trinomial. You will learn to use these techniques at different times to factor many different types of trinomials. After the lesson, you will be able to select the best method for you to be able to factor a given trinomial. The factoring skills that you gain in these lessons will allow you to move on to master more complicated concepts in the future!

When you finish this lesson, you will be able to:

- Factor trinomials of the form $x^2 + bx + c$.
- Factor trinomials of the form $ax^2 + bx + c$ using trial and error.
- Factor trinomials of the form $ax^2 + bx + c$ using the "ac" method.
- Factor trinomials using substitution.

Here are the **activities** that will help you reach those goals:

- 6.5.1: Undoing FOIL
- 6.5.2: Factoring Trinomials with Leading Coefficients of 1
  - 6.5.2: Self Check
  - 6.5.2: Additional Resources
- 6.5.3: Factoring Trinomials Using Trial and Error
  - 6.5.3: Self Check
  - 6.5.3: Additional Resources
- 6.5.4: Factoring Trinomials Using the "ac" Method and Substitution
  - 6.5.4: Self Check
  - 6.5.4: Additional Resources
- 6.5.5: Choosing Your Own Method to Factor Trinomials

After that, you'll **practice and review**.

- 6.5.6: Practice
- 6.5.7: Lesson Summary

6.5.1 Undoing FOIL

Warm Up

We are going to factor the trinomial $q^2 + 10q + 24$ by reversing the process of FOIL.

1. Find all the different combinations of factors of 24.
One example is provided in the table.

Factors of 24	Sum of factors
$1, 24$	$1 + 24 = 25$

We're going to look at these factors to see which of them would fit in the boxes and multiply to make the trinomial

$q^2 + 10q + 24$ that we started with in the beginning.

$(q + \square)(q + \square)$

2. Which pair of factors fits the situation? What is the factored expression?

6.5.2 Factoring Trinomials with Leading Coefficients of 1

Activity

Let's factor this trinomial.

$t^2 + 14t + 24$

Think:

What are the factors of 24?

Which of these factors add to 14?

Here is the table we made in the previous activity:

Factors of 24	Sum of Factors
1, 24	$1 + 24 = 25$
2, 12	$2 + 12 = 14$
3, 8	$3 + 8 = 11$
4, 6	$4 + 6 = 10$

You can use the following video if you need some help

Problem Animation: Factoring a Trinomial with a Leading Coeffienct of 1 (https://openstax.org/r/factor-trinomial-coefficient-1)

Factor each trinomial. If the trinomial cannot be factored, write "prime."

1. $9m + m^2 + 18$
2. $a^2 - 11ab + 10b^2$
3. $u^2 - 9uv - 12v^2$
4. $5p - 6 + p^2$
5. $m^2 - 64mn - 65n^2$

Video: Factoring Trinomials of the Form $x^2 + bx + c$

Watch the following video to learn more about how factor trinomials of the form $x^2 + bx + c$.

Factoring Trinomials (https://openstax.org/r/factor-trinomials)

Self Check

Factor: $y^2 - 16y + 63$.

a. $(y - 7)(y - 9)$

b. $(y+7)(y+9)$
c. $(y+7)(y-9)$
d. $(y-7)(y+9)$

Additional Resources

How to Factor a Trinomial of the Form $x^2 + bx + c$

To figure out how we would factor a trinomial of the form $x^2 + bx + c$, such as $x^2 + 5x + 6$, and factor it to $(x+2)(x+3)$, let's start with two general binomials of the form $(x+m)$ and $(x+n)$.

$(x+m)(x+n)$

Step 1 - FOIL to find the product.

$x^2 + mx + nx + mn$

Step 2 - Factor the GCF from the middle terms.

$x^2 + (m+n)x + mn$

Our trinomial is of the form $x^2 + bx + c$.

$$\underbrace{x^2 + bx + c}_{x^2 + (m+n)x + mn}$$

This tells us that to factor a trinomial of the form $x^2 + bx + c$, we need two factors $(x+m)$ and $(x+n)$ where the two numbers m and n multiply to c and add to b.

EXAMPLE 1

Factor: $x^2 + 11x + 24$.

Step 1 - Write the factors as two binomials with the first terms x.

Write two sets of parentheses and put x as the first term.

$x^2 + 11x + 24$

$(x \quad)(x \quad)$

Step 2 - Find two numbers, m and n, that

- multiply to c: $m \cdot n = c$
- add to b: $m + n = b$

Find two numbers that multiply to 24 and add to 11.

Factors of 24	Sum of factors
$1, 24$	$1 + 24 = 25$
$2, 12$	$2 + 12 = 14$
$3, 8$	$3 + 8 = 11$✓
$4, 6$	$4 + 6 = 10$

Step 3 - Use m and n as the last terms of the factors.

Use 3 and 8 as the last terms of the binomials.

$(x+3)(x+8)$

Step 4 - Check by multiplying the factors.

$(x+3)(x+8)$

$x^2+8x+3x+24$

$x^2+11x+24$✓

Let's summarize the steps we used to find the factors.

How to factor trinomials of the form x^2+bx+c:

Step 1 - Write the factors as two binomials with first terms x.

$x^2 + bx + c$
$(x \quad)(x \quad)$

Step 2 -Find two numbers, m and n, that

- multiply to c: $m \cdot n = c$
- add to b: $m+n=b$

Step 3 - Use m and n as the last terms of the factors.

$(x+m)(x+n)$

Step 4 - Check by multiplying the factors.

In the first example, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let's look first at trinomials with only the middle term negative.

How do you get a **positive product** and a **negative sum**? We use two negative numbers.

EXAMPLE 2

Factor: $y^2-11y+28$.

Again, with the positive last term, 28, and the negative middle term, $-11y$, we need two negative factors. Find two numbers that multiply to 28 and add to -11.

Step 1 - Write the factors as two binomials with first terms y.

$(y \quad)(y \quad)$

Step 2 - Find two numbers that multiply to 28 and add to –11.

Factors of 28	Sum of factors
$-1,-28$	$-1+(-28)=-29$
$-2,-14$	$-2+(-14)=-16$
$-4,-7$	$-4+(-7)=-11$✓

Step 3 - Use –4 and –7 as the last two terms of the binomials.

$(y-4)(y-7)$

Step 4 - Check.

$(y-4)(y-7)$

$y^2-7y-4y+28$

$y^2-11y+28$✓

Now, what if the last term in the trinomial is negative? Think about FOIL. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful when you choose factors to make sure you get the correct sign for the middle term, too. How do you get a negative product and a positive sum? We use one positive and one negative number. When we factor trinomials, we must have the terms written in descending order—in order from highest degree to lowest degree.

EXAMPLE 3

Factor: $2x+x^2-48$.

Step 1 - Put the terms in decreasing degree order.

$x^2+2x-48$

Step 2 - Write the factors as two binomials with first terms x.

$(x\quad)(x\quad)$

Step 3 - Find two numbers that multiply to −48 and add to 2.

Factors of –48	Sum of factors
$-1, 48$	$-1+48=47$
$-2, 24$	$-2+24=22$
$-3, 16$	$-3+16=13$
$-4, 12$	$-4+12=8$
$-6, 8$	$-6+8=2^*$

Since the middle term of the trinomial is positive, the sum of the factors must also be positive. This means that we are only looking at the factors in which the larger number is positive. This is why factors such as 1 and −48 are not included in the table.

Step 4 - Use –6 and 8 as the last terms of the binomials.

$(x-6)(x+8)$

Step 5 - Check.

$(x-6)(x+8)$

$x^2-6x+8x-48$

$x^2+2x-48$ ✓

Sometimes you'll need to factor trinomials of the form $x^2+bxy+cy^2$ with two variables, such as $x^2+12xy+36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, $x\cdot x$. The y^2 in the last term means that the second terms of the binomial factors must each contain y. To get the coefficients b and c, you use the same process summarized above.

EXAMPLE 4

Factor: $r^2-8rs-9s^2$.

We need r in the first term of each binomial and s in the second term. The last term of the trinomial is negative, so the

factors must have opposite signs.

Step 1 - Note that the first terms are r and the last terms contain s.

$(r \quad s)(r \quad s)$

Step 2 - Find the numbers that multiply to −9 and add to −8.

Factors of −9	Sum of factors
$-1, 9$	$-1 + 9 = 8$
$1, -9$	$1 + (-9) = -8$✓
$3, -3$	$3 + (-3) = 0$

Step 3 - Use 1 and –9 as coefficients of the last terms.

$(r + s)(r - 9s)$

Step 4 - Check.

$(r - 9s)(r + s)$
$r^2 + rs - 9rs - 9s^2$
$r^2 - 8rs - 9s^2$

It is important to keep in mind that some trinomials are prime. The only way to be certain a trinomial is prime is to list all the possibilities and show that none of them work. If there are no factor pairs that fit all the requirements, then the trinomial is prime.

Let's summarize the method we just developed to factor trinomials of the form $x^2 + bx + c$.

STRATEGY FOR FACTORING TRINOMIALS OF THE FORM $x^2 + bx + c$

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

$$x^2 + bx + c$$
$$(x + m)(x + n)$$

When c is positive, m and n have the same sign.

b positive	b negative
m, n positive	m, n negative
$x^2 + 5x + 6$	$x^2 - 6x + 8$
$(x + 2)(x + 3)$	$(x - 4)(x - 2)$
same signs	same signs

When c is negative, m and n have opposite signs.

$x^2 + x - 12$	$x^2 - 2x - 15$
$(x + 4)(x - 3)$	$(x - 5)(x + 3)$
opposite signs	opposite signs

Notice that, in the case when m and n have opposite signs, the sign of the one with the larger absolute value matches the sign of b.

TRY IT **How to Factor a Trinomial of the Form $x^2 + bx + c$**

Factor each trinomial.

1. $q^2 + 10q + 24$

2. $-7n + 12 + n^2$

3. $m^2 - 13mn + 12n^2$

6.5.3 Factoring Trinomials Using Trial and Error

Activity

How to factor trinomials of the form $ax^2 + bx + c$ using trial and error:

Step 1 - Write the trinomial in descending order of degrees as needed.

Step 2 - Factor any GCF.

Step 3 - Find all the factor pairs of the first term.

Step 4 - Find all the factor pairs of the third term.

Step 5 - Test all the possible combinations of the factors until the correct product is found.

Step 6 - Check by multiplying.

Remember these helpful rules when factoring trinomials:

- Factor out the GCF first.
  - This is an important rule since it makes the remaining trinomial much simpler to factor. It may change the trinomial from the form $ax^2 + bx + c$ to the form $x^2 + bx + c$. This would allow you to use the "undoing FOIL" process instead of trial and error.
- When the middle term is negative and the last term is positive, the signs in the binomials must both be negative.
  - Only two negative numbers will result in a negative sum and a positive product.
- If the leading coefficient is negative, so is the GCF.
  - Factoring out the negative sign from the leading coefficient will make the trinomial simpler to factor.
- When the leading term is not 1 and there is no GCF, then the first two terms of the binomial factors will multiply to equal the leading term.
  - This means you can use trial and error to find the correct binomial factors.

Work with a partner. Factor each trinomial. Ask your partner if you get stuck. Compare your answers and discuss any issues you had.

1. $2a^2 + 5a + 3$

2. $8x^2 - 14x + 3$

3. $18x^2 - 3xy - 10y^2$

4. $15n^3 - 85n^2 + 100n$

5. $-10y^4 - 55y^3 - 60y^2$

6. $56q^3 + 320q^2 - 96q$

Self Check

Factor the trinomial completely: $6y^3 + 18y^2 - 60y$.

a. $6y(y + 2)(y + 5)$
b. $6y(y - 2)(y + 5)$
c. $(6y^2 - 12y)(y + 5)$
d. $6y(y + 2)(y - 5)$

Additional Resources

Factoring Trinomials Using Trial and Error

Our next step is to factor trinomials whose leading coefficient is not 1, trinomials of the form $ax^2 + bx + c$. Remember to always check for a GCF first! Sometimes, after you factor the GCF, the leading coefficient of the trinomial becomes 1 and you can factor it by the methods we've used so far. Let's do an example to see how this works.

EXAMPLE 1

Factor completely: $4x^3 + 16x^2 - 20x$.

Step 1 - The trinomial is already written in descending order of degrees.

Step 2 - Is there a greatest common factor?

$4x^3 + 16x^2 - 20x$

Yes, $GCF = 4x$. Factor it.

$4x(x^2 + 4x - 5)$

Is it a binomial, trinomial, or more than three terms?

$4x(x^2 + 4x - 5)$

It is now a trinomial with a leading coefficient of 1. So "undo FOIL."

$4x(x)(x)$

Step 3 - Use a table like the one shown to find two numbers that multiply to -5 and add to 4.

Factors of -5	Sum of factors
$-1, 5$	$-1 + 5 = 4^*$
$1, -5$	$1 + (-5) = -4$

Step 4 - Use −1 and 5 as coefficients of the last terms.

$4x(x - 1)(x + 5)$

Step 5 - Check.

$$\begin{aligned} &4x\,(x-1)\,(x+5)\\ &4x(x^2+5x-x-5)\\ &4x(x^2+4x-5)\\ &4x^3+16x^2-20x\checkmark \end{aligned}$$

What happens when the leading coefficient is not 1 and there is no GCF? There are several methods that can be used to factor these trinomials. First we will use the trial and error method.

EXAMPLE 2

Factor the trinomial $3x^2 + 5x + 2$.

The trinomial is already written in descending order of degrees. There is no GCF to factor.

From our earlier work, we expect this will factor into two binomials.

$$\begin{aligned} &3x^2 \;+\; 5x \;+\; 2\\ &\quad(\quad)(\quad) \end{aligned}$$

We know the first terms of the binomial factors will multiply to give us $3xa^2$. The only factors of $3x^2$ are $1x$, $3x$. We can place them in the binomials.

$3x^2 + 5x + 2$
$1x, 3x$

$(x \quad)(3x \quad)$

Check: Does $1x \cdot 3x = 3x^2$?

We know the last terms of the binomials will multiply to 2. Since this trinomial has all positive terms, we only need to consider positive factors. The only factors of 2 are $1, 2$. But we now have two cases to consider because it will make a difference if we write $1, 2$ or $2, 1$.

$3x^2 + 5x + 2$
$1x, 3x \qquad 2, 1$

$(x + 1)(3x + 2)$ or $(x + 2)(3x + 1)$

Which factors are correct? To decide that, we multiply the inner and outer terms.

$3x^2 + 5x + 2$
$1x, 3x \qquad 2, 1$

$(x + 1)(3x + 2)$ $3x$ $2x$ $5x$ or $(x + 2)(3x + 1)$ $6x$ $1x$ $7x$

Since the middle term of the trinomial is $5x$, the factors in the first case will work. Let's use FOIL to check.

$$(x + 1)(3x + 2)$$
$$3x^2 + 2x + 3x + 2$$
$$3x^2 + 5x + 2 \checkmark$$

Our result of the factoring is:

$$3x^2 + 5x + 2$$
$$(x + 1)(3x + 2)$$

How to factor trinomials of the form $ax^2 + bx + c$ using trial and error.

Step 1 - Write the trinomial in descending order of degrees as needed.

Step 2 - Factor any GCF.

Step 3 - Find all the factor pairs of the first term.

Step 4 - Find all the factor pairs of the third term.

Step 5 - Test all the possible combinations of the factors until the correct product is found.

Step 6 - Check by multiplying.

Remember, when the middle term is negative and the last term is positive, the signs in the binomials must both be negative.

When we factor an expression, we always look for a greatest common factor first. If the expression does not have a greatest common factor, there cannot be one in its factors either. This may help us eliminate some of the possible factor combinations.

EXAMPLE 3

Factor completely using trial and error: $18x^2 - 37xy + 15y^2$.

Step 1 - The trinomial is already in descending order.

$18x^2 - 37xy + 15y^2$

Step 2 - Find the factors of the first term.

$18x^2 - 37xy + 15y^2$
$1x \cdot 18x$
$2x \cdot 9x$
$3x \cdot 6x$

Step 3 - Find the factors of the last term. Consider the signs. Since 15 is positive and the coefficient of the middle term is negative, we use the negative factors.

$18x^2 - 37xy + 15y^2$

$1x \cdot 18x$ $\quad$ $-1y \cdot -15y$
$2x \cdot 9x$ $\quad$ $-3y \cdot -5y$
$3x \cdot 6x$

Step 4 - Consider all the combinations of factors.

$18x^2 - 37xy + 15y^2$	
Possible factors	**Product**
$(x - 1y)(18x - 15y)$	Not an option
$(x - 15y)(18x - 1y)$	$18x^2 - 271xy + 15y^2$
$(x - 3y)(18x - 5y)$	$18x^2 - 59xy + 15y^2$
$(x - 5y)(18x - 3y)$	Not an option
$(2x - 1y)(9x - 15y)$	Not an option
$(2x - 15y)(9x - 1y)$	$18x^2 - 137xy + 15y^2$
$(2x - 3y)(9x - 5y)$	$18x^2 - 37xy + 15y^{2*}$
$(2x - 5y)(9x - 3y)$	Not an option
$(3x - 1y)(6x - 15y)$	Not an option
$(3x - 15y)(6x - 1y)$	Not an option
$(3x - 3y)(6x - 5y)$	Not an option

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

Step 5 - The correct factors are those whose product is the original trinomial.

$(2x - 3y)(9x - 5y)$

Step 6 - Check by multiplying.

$$(2x - 3y)(9x - 5y)$$
$$18x^2 - 10xy - 27xy + 15y^2$$
$$18x^2 - 37xy + 15y^2 \checkmark$$

In other trinomials, don't forget to look for a GCF first, and remember if the leading coefficient is negative, so is the GCF.

TRY IT **Factoring Trinomials Using Trial and Error**

Factor each trinomial completely using trial and error.

1. $5x^3 + 15x^2 - 20x$

2. $10y^2 - 37y + 7$
3. $6n^3 - 34n^2 + 40n$

6.5.4 Factoring Trinomials Using the “ac” Method and Substitution

Activity

Let’s learn a new method that always works on any trinomial of the form $ax^2 + bx + c$.

How to factor trinomials of the form $ax^2 + bx + c$ using the “ac” method:

Step 1 - Factor any GCF. Don’t forget!

Step 2 - Find the product ac.

Step 3 - Find two numbers, m and n, that

- multiply to ac: $m \cdot n = a \cdot c$.
- add to b: $m + n = b$.

Step 4 - Split the middle term using m and n.

$ax^2 + mx + nx + c$

Step 5 - Factor by grouping.

Step 6 - Check by multiplying the factors.

Factor each trinomial using the “ac” method.

1. $16x^2 - 32x + 12$

2. $6y^2 + y - 15$

3. $48z^3 - 102z^2 - 45z$

There will be times when a trinomial does not seem to be in the form $ax^2 + bx + c$.

In some of these cases, we can make a substitution that allows the trinomial to be written in the $ax^2 + bx + c$ form. This is called factoring by substitution.

It is standard to use u for the substitution. The substitution for u might be a variable with an exponent, such as x^2, or it might be a binomial, such as $(x - 3)$.

Factor each trinomial using the substitution method.

4. $y^4 - y^2 - 20$

5. $(x - 5)^2 + 6(x - 5) + 8$

6. $x^4 - 3x^2 - 28$

Self Check

Factor using the “ac” method: $6x^2 + 13x + 2$.

a. $(x - 2)(6x + 1)$
b. $(x - 2)(6x - 1)$
c. $(6x + 2)(x + 1)$
d. $(x + 2)(6x + 1)$

Additional Resources

Factoring Trinomials Using the "ac" Method

Another way to factor trinomials of the form $ax^2 + bx + c$ is the "ac" method. (The "ac" method is sometimes called the grouping method.) The "ac" method is actually an extension of the methods you used in the last section to factor trinomials with a leading coefficient of 1. This method is very structured (that is, step by step), and it always works!

EXAMPLE 1

Factor using the "ac" method: $6x^2 + 7x + 2$.

Step 1 - Factor any GCF.

Is there a greatest common factor?

$6x^2 + 7x + 2$

No!

Step 2 - Find the product ac.

$ax^2 + bx + c$
$6x^2 + 7x + 2$

$6 \cdot 2$

12

Step 3 - Find two numbers, m and n, that

- multiply to ac: $m \cdot n = a \cdot c$.
- add to b: $m + n = b$.

Find two numbers that multiply to 12 and add to 7. Both factors must be positive.

$3 \cdot 4 = 12$

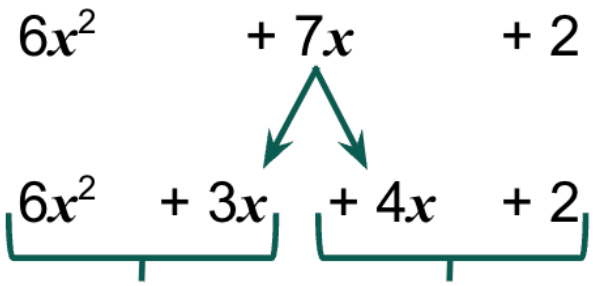
$3 + 4 = 7$

Step 4 - Split the middle term using m and n.

$$ax^2 + bx + c$$

$$ax^2 + \overbrace{mx + nx}^{bx} + c$$

Rewrite $7x$ as $3x + 4x$. It would give the same result if we used $4x + 3x$.



Notice that $6x^2 + 3x + 4x + 2$ is equal to $6x^2 + 7x + 2$. We just split the middle term to get a more useful form.

Step 5 - Factor by grouping.

$3x(2x + 1) + 2(2x + 1)$

$(2x+1)(3x+2)$

Step 6 - Check by multiplying the factors.

$(2x+1)(3x+2)$

$6x^2+4x+3x+2$

$6x^2+7x+2$✓

The "ac" method is summarized here:

How to factor trinomials of the form ax^2+bx+c using the "ac" method:

Step 1 - Factor any GCF. Don't forget!

Step 2 - Find the product ac.

Step 3 - Find two numbers, m and n, that

- multiply to ac: $m \cdot n = a \cdot c$.
- add to b: $m+n=b$.

Step 4 - Split the middle term using m and n.

$ax^2+mx+nx+c$

Step 5 - Factor by grouping.

Step 6 - Check by multiplying the factors.

TRY IT **Factoring Trinomials Using the "ac" Method**

Factor using the "ac" method: $10y^2-55y+70$.

Factoring Trinomials Using Substitution

Sometimes a trinomial does not appear to be in the ax^2+bx+c form. However, we can often make a thoughtful substitution that will allow us to make it fit the ax^2+bx+c form. This is called factoring by substitution. It is standard to use u for the substitution.

In the form ax^2+bx+c, the middle term has a variable, x, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

EXAMPLE 2

Factor by substitution: x^4-4x^2-5.

The variable part of the middle term is x^2, and its square, x^4, is the variable part of the first term. (We know $(x^2)^2=x^4$). If we let $u=x^2$, we can put our trinomial in the ax^2+bx+c form we need to factor it.

Step 1 - Rewrite the trinomial to prepare for the substitution.

$(x^2)^2-4(x^2)-5$

Step 2 - Let $u=x^2$ and substitute.

u^2-4u-5

Step 3 - Factor the trinomial.

$(u+1)(u-5)$

Step 4 - Replace u with x^2.

$(x^2+1)(x^2-5)$

Step 5 - Check:

$$\begin{aligned}&\left(x^2+1\right)\left(x^2-5\right)\\&x^4-5x^2+x^2-5\\&x^4-4x^2-5\checkmark\end{aligned}$$

TRY IT **Try It: Factoring Trinomials Using Substitution**

Factor by substitution: $(x-2)^2+7(x-2)+12$.

In this case, the expression to be substituted is not a monomial. Let $u=x-2$.

6.5.5 Choosing Your Own Method to Factor Trinomials

Cool Down

Work with a partner. For each trinomial, explain which method you would use to factor. When you have discussed your reasoning, factor the trinomial and compare solutions.

1. $x^2-14x-32$

2. $6r^2+30r+36$

3. $(x+3)^2-9(x+3)-36$

6.5.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 3, factor.

1. $u^2 - 9u + 18$

a. $(u + 3)(u - 6)$
b. $(u + 3)(u + 6)$
c. $(u - 3)(u - 6)$
d. $(u + 9)(u + 2)$

2. $8 - 6x + x^2$

a. $(x + 4)(x - 2)$
b. $(x - 4)(x - 2)$
c. $(x - 8)(x + 1)$
d. $(x + 4)(x + 2)$

3. $a^2 + 5ab - 24b^2$

a. $(a + 8b)(a - 3b)$
b. $(a + 8b)(a + 3b)$
c. $(a - 8b)(a - 3b)$
d. $(a - 8b)(a + 3b)$

For questions 4 – 5, factor using the trial and error method.

4. $4b^2 + 5b + 1$

a. $(b + 1)(4b + 1)$
b. $(5b + 1)(b + 1)$
c. $(5b + 1)(b - 1)$
d. $(b - 1)(4b - 1)$

5. $30x^2 - 53xy - 21y^2$

a. $(3x - y)(10x - 21y)$
b. $(5x + y)(6x - 21y)$
c. $(15x + y)(2x - 21y)$
d. $(3x + y)(10x - 21y)$

6. Factor completely using the "ac" method: $4y^2 + 8y + 3$.

a. $(2y + 1)(2y - 3)$
b. $(2y - 1)(2y + 3)$
c. $(2y + 1)(2y + 3)$
d. $(2y - 1)(2y - 3)$

7. Factor completely using the "ac" method: $18w^2 - 39w + 18$.

a. $3(3w + 2)(2w - 3)$
b. $3(3w + 2)(2w + 3)$
c. $3(3w - 2)(2w - 3)$
d. $(9w - 2)(2w - 3)$

8. Factor by substitution: $h^4 + 4h^2 - 12$.

a. $(h^2 - 2)(h^2 + 6)$
b. $(h - 2)(h + 6)$
c. $(h + 2)(h - 6)$
d. $(h^2 + 2)(h^2 - 6)$

9. Factor by substitution: $(y-4)^2+8(y-4)+15$.

a. $(y-5)(y-3)$
b. $(y-1)(y+1)$
c. $(y+5)(y+3)$
d. $(y+1)^2$

10. Factor using any method: $13z^2+39z-26$.

a. $13(z^2+3z-2)$
b. $13(2z+1)(z-2)$
c. $13(2z-1)(z+2)$
d. $13(2z+1)(z+2)$

6.5.7 Lesson Summary

In this lesson, you learned how to:

- Factor trinomials of the form $x^2 + bx + c$.
- Factor trinomials of the form $ax^2 + bx + c$ using trial and error.
- Factor trinomials of the form $ax^2 + bx + c$ using the "ac" method.
- Factor using substitution.

Here are the **activities** that helped you reach those goals:

- 6.5.1: Undoing FOIL
  - In this activity, you learned the steps used to factor a trinomial by reversing the FOIL method.
- 6.5.2: Factoring Trinomials with Leading Coefficients of 1
  - In this activity, you learned how to factor any trinomial of the form $x^2 + bx + c$. By finding the factors of the c term and checking to see if they add up to the b term, you are able to find the factors of a trinomial of this form.
  - 6.5.2: Self Check
  - 6.5.2: Additional Resources
- 6.5.3: Factoring Trinomials Using Trial and Error
  - In this activity, you learned that you sometimes need to factor out the GCF from a trinomial as the first step to factoring. You also learned some tricks to help determine the factors of a trinomial, such as when the middle term is negative and the last term is positive, the signs in the binomial factors must both be negative. You practiced finding the factors of a trinomial of the form $ax^2 + bx + c$ using a trial and error method.
  - 6.5.3: Self Check
  - 6.5.3: Additional Resources
- 6.5.4: Factoring Trinomials Using the "ac" Method and Substitution
  - In this activity, you learned a method that can be used to factor any trinomial, the "ac" method. By finding two numbers that multiply to ac and add to b, you can factor the trinomial by grouping. You also learned a substitution method that allows you to factor trinomials that do not appear to fit into the $ax^2 + bx + c$ form at first.
  - 6.5.4: Self Check
  - 6.5.4: Additional Resources
- 6.5.5: Choosing Your Own Method to Factor Trinomials
  - In this activity, you combined everything you learned in the lesson to choose your own preferred method to factor a trinomial.

After these activities, you completed the following **practice**:

- 6.5.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

6.6 FACTOR SPECIAL PRODUCTS

6.6.0 Lesson Overview

In this lesson, you will develop two different methods to factor special polynomials. While these polynomials can be factored using the methods from the previous lesson, these special cases allow for a more streamlined approach to factoring. One important skill that you will develop is the ability to recognize when you have a polynomial that fits into these special cases.

When you finish this lesson, you will be able to:

- Factor perfect square trinomials.
- Factor differences of squares.

Here are the **activities** that will help you reach those goals:

- 6.6.1: Identifying Perfect Square Trinomials
- 6.6.2: Factoring Perfect Square Trinomials
  - 6.6.2: Self Check
  - 6.6.2: Additional Resources
- 6.6.3: Factoring the Difference of Squares
  - 6.6.3: Self Check
  - 6.6.3: Additional Resources
- 6.6.4: Sorting Polynomials by Factoring Methods

After that, you'll **practice and review**.

- 6.6.5: Practice
- 6.6.6: Lesson Summary

6.6.1 Identifying Perfect Square Trinomials

Warm Up

Identify which of the polynomials is a perfect square trinomial.

Helpful hints for recognizing perfect square trinomials:

- Are the first and last terms squares?
- Does the middle term fit the pattern of $2ab$?

1. $25v^2 + 20v + 4$

2. $49s^2 + 154s + 121$

3. $25r^2 + 65rs + 36s^2$

4. $64z^2 - 16z + 1$
You can use the following video if you need some help.
Problem Animation: Factoring a Perfect Square Trinomial (https://openstax.org/r/perfect-square-trinomial)

5. $49x^2 + 28xy + 9y^2$

6.6.2 Factoring Perfect Square Trinomials

Activity

Remember the patterns for perfect square trinomials:

$a^2 + 2ab + b^2 = (a + b)^2$

$a^2 - 2ab + b^2 = (a - b)^2$

Factor each perfect square trinomial. If necessary, factor out the GCF first.

1. $64y^2 - 80y + 25$

2. $16y^2 + 24y + 9$

3. $25n^2 - 120n + 144$

4. $100y^2 - 20y + 1$

5. $10jk^2 + 80jk + 160j$

6. $75u^4 - 30u^3v + 3u^2v^2$

Are you ready for more?

Extending Your Thinking

Factor this perfect square trinomial.

$90p^4r^2 + 300p^3qr^2 + 250p^2q^2r^2$

Self Check

Factor: $64m^2 + 112mn + 49n^2$.

a. $(8m + 7n)^2$
b. $(8m - 7n)^2$
c. $(8n + 7m)^2$
d. $m(8 + 7n)^2$

Additional Resources

Factoring Perfect Square Trinomials

Some trinomials are perfect squares. They result from multiplying a binomial by itself. We squared a binomial using the Binomial Squares pattern in a previous lesson.

$(a + b)^2$

$(3x + 4)^2$

$a^2 + 2 \cdot a \cdot b + b^2$

$(3x)^2 + 2(3x \cdot 4) + 4^2$

$9x^2 + 24x + 16$

The trinomial $9x^2 + 24x + 16$ is called a perfect square trinomial. It is the square of the binomial $3x + 4$.

In this chapter, you will start with a perfect square trinomial and factor it into its prime factors.

You could factor this trinomial using the methods described in the last section since it is of the form $ax^2 + bx + c$. But if you recognize that the first and last terms are squares and the trinomial fits the perfect square trinomials pattern, you will save yourself a lot of work.

Here is the pattern—the reverse of the binomial squares pattern.

PERFECT SQUARE TRINOMIALS PATTERN

If a and b are real numbers

$a^2 + 2ab + b^2 = (a + b)^2$

$a^2 - 2ab + b^2 = (a - b)^2$

To use this pattern, you have to recognize that a given trinomial fits it. Check first to see if the leading coefficient is a perfect square, a^2. Next, check that the last term is a perfect square, b^2. Then check the middle term: Is it the product, $2ab$? If everything checks, you can easily write the factors.

EXAMPLE 1

Factor: $9x^2 + 12x + 4$.

Step 1 - Does the trinomial fit the perfect square trinomials pattern, $a^2 + 2ab + b^2$?

- Is the first term a perfect square? Write it as a square, a^2 .
  - Is $9x^2$ a perfect square?
  - Yes. Write it as $(3x)^2$.
- Is the last term a perfect square? Write it as a square, b^2 .
  - Is 4 a perfect square?
  - Yes. Write it as $(2)^2$.
- Check the middle term. Is it $2ab$?
  - Is $12x$ twice the product of $3x$ and 2^2? Does it match?
  - Yes, so we have a perfect square trinomial!

$9x^2 + 12x + 4$

$(3x)^2$

$(3x)^2$ $(2)^2$

$(3x)^2$ $(2)^2$

$2(3x)(2)$

$12x$

Step 2 - Write the square of the binomial.

$9x^2 + 12x + 4$

$a^2 + 2 \cdot a \cdot b + b^2$

$(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$

$(a + b)^2$

$(3x + 2)^2$

Step 3 - Check by multiplying.

$(3x + 2)^2$

$(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$

$9x^2 + 12x + 4$ ✓

The sign of the middle term determines which pattern we will use. When the middle term is negative, we use the pattern $a^2 - 2ab + b^2$, which factors to $(a - b)^2$.

The steps are summarized here.

How to factor perfect square trinomials:

Step 1 - Does the trinomial fit the perfect square trinomials pattern?

- Is the first term a perfect square? Write it as a square.

- Is the last term a perfect square? Write it as a square.
- Check the middle term. Is it $2ab$?

Step 2 - Write the square of the binomial.

Step 3 - Check by multiplying.

$a^2 + ab + b^2$ $\qquad$ $a^2 - ab + b^2$

$(a)^2$ $\qquad$ $(a)^2$

$(a)^2$ $\qquad$ $(b)^2$ $\qquad$ $(a)^2$ $\qquad$ $(b)^2$

$(a)^2$ $\qquad$ $(b)^2$ $\qquad$ $(a)^2$ $\qquad$ $(b)^2$

$2 \cdot a \cdot b$ $\qquad$ $2 \cdot a \cdot b$

$(a + b)^2$ $\qquad$ $(a - b)^2$

We'll work one now where the middle term is negative.

EXAMPLE 2

Factor: $81y^2 - 72y + 16$.

The first and last terms are squares. See if the middle term fits the pattern of a perfect square trinomial. The middle term is negative, so the binomial square would be $(a - b)^2$.

Step 1 - Are the first and last terms perfect squares?

$81y^2 - 72y + 16$

$(9y)^2$ $\qquad$ $(4)^2$

Step 2 - Check the middle term.

$(9y^2)$ $\qquad$ $(4)^2$

$2(9y)(4)$

$72y$

Does it match $(a - b)^2$?

$a^2 \quad - \quad 2 \quad a \quad b \quad + \quad b^2$

$(9y)^2 - 2 \cdot 9y \cdot 4 + 4^2$

Yes.

Step 3 - Write as the square of a binomial.

$(9y - 1)^2$

Step 4 - Check by multiplying.

$(9y - 1)^2$

$(9y)^2 - 2 \cdot 9y \cdot 4 + 4^2$

$81y^2 - 72y + 16$ ✓

The next example will be a perfect square trinomial with two variables.

EXAMPLE 3

Factor: $36x^2 + 84xy + 49y^2$.

Step 1 - Test each term to verify the pattern.

$a^2 \quad + \quad 2 \quad (a) \quad (b) \quad + \quad b^2$

$6x^2 + 2(6x)(7y) + (7y)^2$

Step 2 - Factor.

$(6x + 7y)^2$

Step 3 - Check by multiplying.

$(6x + 7y)^2$

$(6x)^2 + 2 \cdot 6x \cdot 7y + (7y)^2$

$36x^2 + 84xy + 49y^2$ ✓

Remember, the first step in factoring is to look for a greatest common factor. Perfect square trinomials may have a GCF in all three terms, and it should be factored out first. And, sometimes, once the GCF has been factored, you will recognize a perfect square trinomial.

EXAMPLE 4

Factor: $100x^2y - 80xy + 16y$.

Step 1 - Is there a GCF? Yes, $4y$, so factor it out.

$4y(25x^2 - 20x + 4)$

Step 2 - Is this a perfect square trinomial? Verify the pattern.

$a^2 \quad - \quad 2 \quad a \quad b \quad + \quad b^2$

$4y\left[(5x)^2 - 2 \cdot 5x \cdot 2 + 2^2\right]$

Step 3 - Write as the square of a binomial.

$4y(5x - 2)^2$

Remember: Keep the factor $4y$ in the final product.

Step 4 - Check by multiplying.

$4y(5x - 2)^2$

$4y[(5x)^2 - 2 \cdot 5x \cdot 2 + 2^2]$

$4y(25x^2 - 20x + 4)$

$100x^2y - 80xy + 16y$ ✓

TRY IT **Factoring Perfect Square Trinomials**

Factor.

1. $4x^2 + 12x + 9$
2. $49x^2 + 84xy + 36y^2$
3. $8x^2y - 24xy + 18y$

6.6.3 Factoring the Difference of Squares

Activity

Remember that $(a-b)$ and $(a+b)$ are a conjugate pair. When you multiply them together, you end up with the difference of squares, a^2-b^2.

Let's look at the pattern used to factor a difference of squares:

$a^2-b^2=(a-b)(a+b)$

Work with a partner to factor using the difference of squares.

As you factor each binomial, take turns with your partner, alternating the completion of each step one at a time. If you do not know the next step, ask your partner for help.

1. $81y^2-1$

2. $196m^2-25n^2$

3. $6p^2q^2-54p^2$

4. $7a^4c^2-7b^4c^2$

5. $a^2+6a+9-9b^2$

6. $x^2+6x+9-4y^2$

7. Can a binomial with an addition operation be factored using the difference of squares? Explain.

Video: Factoring the Difference of Squares

Watch the following video to learn more about how to factor the difference of squares when there is also a greatest common factor.

Factoring the Difference of Squares (https://openstax.org/r/difference-squares)

Self Check

Factor using the difference of squares: $121p^2-9q^2$.

a. $(11p+3q)^2$
b. $(11p-3q)^2$
c. $(11q-3p)(11q+3p)$
d. $(11p-3q)(11p+3q)$

Additional Resources

Factoring the Difference of Squares

The other special product you saw in the previous chapter was the Product of Conjugates pattern. You used this to multiply two binomials that were conjugates. Here's an example:

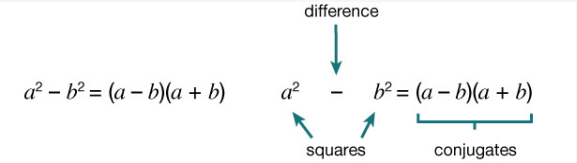
$$\begin{array}{l}(a\ -\ b)(a\ +\ b)\\(3x-4)(3x+4)\end{array}$$

$$\begin{array}{l}(a)^2\ -\ (b)^2\\(3x)^2-(4)^2\\9x^2-16\end{array}$$

A difference of squares pattern factors to a product of conjugates.

DIFFERENCE OF SQUARES PATTERN

If a and b are real numbers,



Remember, "difference" refers to subtraction. So, to use this pattern, you must make sure you have a binomial in which two squares are being subtracted.

How to factor the difference of squares:

Step 1 - Does the binomial fit the pattern?

Is this a difference?

Are the first and last terms perfect squares?

$a^2 - b^2$

__ – __

Step 2 - Write them as squares.

$(a)^2 - (b)^2$

Step 3 - Write the product of conjugates.

$(a - b)(a + b)$

Step 4 - Check by multiplying.

EXAMPLE 1

Factor: $64y^2 - 1$.

Step 1 - Does the binomial fit the pattern?

- Is this a difference?
  Yes.
- Are the first and last terms perfect squares?
  Yes.

Step 2 - Write them as the difference of squares.

$a^2 - b^2$

$(8y)^2 - 1^2$

Step 3 - Write the product of conjugates.

$(a - b)(a + b)$

$(8y - 1)(8y + 1)$

Step 4 - Check by multiplying.

$(8y - 1)(8y + 1)$

$64y^2 - 1$ ✓

It is important to remember that sums of squares do not factor into a product of binomials. There are no binomial factors that multiply together to get a sum of squares. After removing any GCF, the expression $a^2 + b^2$ is prime!

The next example shows variables in both terms.

EXAMPLE 2

Factor: $144x^2 - 49y^2$.

Step 1 - Is this a difference of squares?

$144x^2 - 49y^2$

Yes.

$(12x)^2 - (7y)^2$

Step 2 - Factor as the product of conjugates.

$(12x - 7y)(12x + 7y)$

Step 3 - Check by multiplying.

$(12x - 7y)(12x + 7y)$

$44x^2 - 49y^2$ ✓

As always, you should look for a common factor first whenever you have an expression to factor. Sometimes a common factor may "disguise" the difference of squares, and you won't recognize the perfect squares until you factor out the GCF.

Also, to completely factor the binomial in the next example, we'll factor a difference of squares twice!

EXAMPLE 3

Factor: $48x^4y^2 - 243y^2$.

Step 1 - Is there a GCF?

$48x^4y^2 - 243y^2$

Yes, $3y^2$—factor it out!

$3y^2(16x^4 - 81)$

Step 2 - Is the binomial a difference of squares? Yes.

$3y^2[(4x^2)^2 - (9)^2]$

Step 3 - Factor as a product of conjugates.

$3y^2(4x^2 - 9)(4x^2 + 9)$

Step 4 - Notice the first binomial is also a difference of squares!

$3y^2[(2x)^2 - (3)^2](4x^2 + 9)$

Step 5 - Factor it as the product of conjugates.

$3y^2(2x - 3)(2x + 3)(4x^2 + 9)$

The last factor, the sum of squares, cannot be factored.

Step 6 - Check by multiplying.

$3y^2(2x - 3)(2x + 3)(4x^2 + 9)$

$3y^2(4x^2 - 9)(4x^2 + 9)$

$3y^2(16x^4 - 81)$

$48x^4y^2 - 243y^2$ ✓

EXAMPLE 4

Factor: $x^2 - 6x + 9 - y^2$.

Step 1 - Factor by grouping the first three terms.

$$\underbrace{x^2 - 6x + 9} - y^2$$

Step 2 - Use the perfect square trinomial pattern.

$(x-3)^2 - y^2$

Step 3 - Is there a difference of squares?

Yes—write them as squares.

$$\begin{matrix} a^2 & - & b^2 \end{matrix}$$
$(x-3)^2 - y^2$

Step 4 - Factor as the product of conjugates.

$$\begin{matrix} (a & - & b) & (a & + & b) \end{matrix}$$
$((x-3)-y)((x-3)+y)$

$(x-3-y)(x-3+y)$

You may want to rewrite the solution as $(x-y-3)(x+y-3)$.

TRY IT **Factoring the Difference of Squares**

Factor using the difference of squares.

1. $121m^2 - 1$
2. $2x^4y^2 - 32y^2$
3. $x^2 - 10x + 25 - y^2$

6.6.4 Sorting Polynomials by Factoring Methods

Cool Down

Examine the following polynomials:

$27q^2 - 3$
$x^2 - 10x + 25 - y^2$
$16x^2 - 72x + 81$
$36y^2 + 12y + 1$
$4p^2 - 100$
$x^2 + 12x + 36 - y^2$

Sort the polynomials by whether you would factor them using the difference of squares or the perfect square trinomial method.

Draw a table similar to this on a sheet of paper. Record your results. If a polynomial requires both methods, place it in both columns.

Difference of Squares Method	Perfect Square Trinomial Method

Why Should I Care?



Ozzie Gomez and Juan Casava own a store called Plant Shop Chicago. They use polynomials everyday to make sure the operations of their small business go smoothly.

One of the most important parts of running a business is understanding where your money is going. You need to know how much you are spending and how much you are making. For example, Ozzie Gomez and Juan Cassava can use polynomials to calculate how many different plants they can buy and how much money they can make off of each sale.

6.6.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 5, factor using perfect square trinomials.

1. $9y^2 + 24y + 16$

a. $(4y + 3)^2$
b. $(3y + 4)^2$
c. $(3y - 4)^2$
d. $(4y - 3)^2$

2. $16z^2 - 72z + 81$

a. $(4z + 9)^2$
b. $(9z + 4)^2$
c. $(9z - 4)^2$
d. $(4z - 9)^2$

3. $27p^2q + 90pq + 75q$

a. $3q(3p - 5)^2$
b. $3q(3p + 5)^2$
c. $9q(p + 5)^2$
d. $(27p + 5)^2$

4. $64z^2 - 16z + 1$

a. $(8z - 1)^2$
b. $(z + 8)^2$
c. $(8z + 1)^2$
d. $(z - 8)^2$

5. $64x^2y - 96xy + 36y$

a. $4y(4x + 3)^2$
b. $4y(3x - 4)^2$
c. $2y(8x - 6)^2$
d. $4y(4x - 3)^2$

For questions 6 – 10, factor using the difference of squares.

6. $25v^2 - 1$

a. $(v - 5)(v + 5)$
b. $(v - 5)(5v + 1)$
c. $(5v - 1)(5v + 1)$
d. $(5v - 1)(v + 5)$

7. $4 - 49x^2$

a. $(7 - 2x)(7 + 2x)$
b. $(2 - 7x)(2 - 7x)$
c. $(2 - 7x)(2 + 7x)$
d. $(2 - 7x)(7 + 2x)$

8. $169c^2 - 36d^2$

a. $(13c - 6d)(13c - 6d)$
b. $(13d - 6c)(13d + 6c)$
c. $(13c + 6d)^2$
d. $(13c - 6d)(13c + 6d)$

9. $121x^2 - 144y^2$

a. $(11x - 12y)(11x - 12y)$
b. $(11x - 12y)(11x + 12y)$
c. $(12x + 11y)(12x + 11y)$
d. $(12x - 11y)(12x + 11y)$

10. $x^2 - 16x + 64 - y^2$

a. $(x - 8 - y)(x + 8 + y)$
b. $(x - 8 - y)(x - 8 + y)$
c. $(x + 8 - y)(x + 8 + y)$
d. $(x - 8 - y)^2$

6.6.6 Lesson Summary

In this lesson, you learned how to:

- Factor perfect square trinomials.
- Factor differences of squares.

Here are the **activities** that helped you reach those goals:

- 6.6.1: Identifying Perfect Square Trinomials
  - In this activity, you learned how to recognize perfect square trinomials. They always follow a pattern of $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$.
- 6.6.2: Factoring Perfect Square Trinomials
  - In this activity, you learned how to factor perfect square trinomials. You can recognize that they will always factor to a form of $(a + b)^2$ or $(a - b)^2$. In some cases, you first need to factor the GCF out of the existing terms.
  - 6.6.2: Self Check
  - 6.6.2: Additional Resources
- 6.6.3: Factoring the Difference of Squares
  - In this activity, you learned how to factor using the difference of squares. In these problems, the middle term will always cancel to the form of $(a^2 - b^2)$. There are no binomial factors that multiply together to get a sum of squares, so $a^2 + b^2$ is prime.
  - 6.6.3: Self Check
  - 6.6.3: Additional Resources
- 6.6.4: Sorting Polynomials by Factoring Methods
  - In this activity, you practiced recognizing which of these two methods you would use to factor based on the polynomial you were given. You learned that it is possible to have a polynomial that requires both of these methods to be factored completely.

After these activities, you completed the following **practice**:

- 6.6.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

6.7 GENERAL STRATEGY FOR FACTORING POLYNOMIALS

6.7.0 Lesson Overview

In this lesson, you will review each strategy learned throughout this unit to assess the best strategy to factor polynomials. You will be presented with different polynomials, and, without additional context, you will find the fully factored form of each.

When you finish this lesson, you will be able to:

- Recognize and use the appropriate method to factor a polynomial completely.

Here are the **activities** that will help you reach those goals:

- 6.7.1: Factoring Strategies for Polynomials
- 6.7.2: Reviewing General Strategies to Factor a Polynomial
  - 6.7.2: Self Check
  - 6.7.2: Additional Resources
- 6.7.3: Implementing General Strategies for Factoring
  - 6.7.3: Self Check
  - 6.7.3: Additional Resources
- 6.7.4: Is the Polynomial Completely Factored?

After that, you'll **practice and review**.

- 6.7.5: Practice
- 6.7.6: Lesson Summary

6.7.1 Factoring Strategies for Polynomials

Warm Up

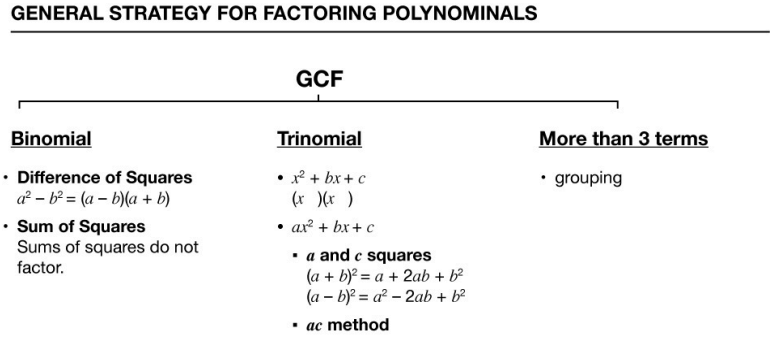
1. Working with a partner, list as many strategies as you can for factoring polynomials.
2. Work with your partner to write a sample polynomial that would be fitting for each strategy listed.

6.7.2 Reviewing General Strategies to Factor a Polynomial

Activity

Take a moment to look at all parts of the diagram. Are there any parts that you do not understand?

Use this time to ask your teacher if there is something you would like clarified here.



Use the diagram above as needed for the activity.

Determine the best strategy to factor the polynomial. Be prepared to show your reasoning. Remember to factor out the GCF first, if necessary, before choosing a strategy. If a polynomial is prime, write “prime.” An example is provided.

Polynomial	Reasoning	Strategy
$4x^2 + 20xy + 25y^2$	After GCF, trinomial of form $a^2 + 2ab + b^2$	trinomial square pattern

1. $8y^3 + 16y^2 - 24y$
2. $16x^3 - 36x$
3. $6pq^2 - 9pq - 6p$
4. $4x^2 - 12xy + 9y^2 - 25$
5. $a^2 + 9$
6. $9x^2 - 24xy + 16y^2$
7. $27y^2 - 48$

Are you ready for more?

Extending Your Thinking

Factor the polynomial below. Write down every factoring strategy that you use.

$9x^2 - 12xy + 4y^2 - 49$

Self Check

Which strategy would you use to factor $40x^2y + 44xy - 24y$? What is the factored result?

a. Factor out the GCF. Use the perfect square trinomial strategy.

$4y(5x + 2)(2x - 3)$

b. Factor out the GCF. Use the "ac" method.

$4y(5x - 2)(2x + 3)$

c. Factor out the GCF. Use the difference of squares.

$4y(5x - 2)(5x + 2)$

d. Factor out the GCF. Use the undo FOIL method.

$4y(5x + 2)(2x - 3)$

Additional Resources

Reviewing General Strategies to Factor a Polynomial

You have now become acquainted with all the methods of factoring that you will need in this course. The following chart summarizes all the factoring methods we have covered and outlines a strategy you should use when factoring polynomials.

GENERAL STRATEGY FOR FACTORING POLYNOMINALS

GCF

Binomial

- **Difference of Squares**
  $a^2 - b^2 = (a - b)(a + b)$
- **Sum of Squares**
  Sums of squares do not factor.

Trinomial

- $x^2 + bx + c$
  $(x \quad)(x \quad)$
- $ax^2 + bx + c$
  - ***a* and *c* squares**
    $(a + b)^2 = a + 2ab + b^2$
    $(a - b)^2 = a^2 - 2ab + b^2$
  - ***ac* method**

More than 3 terms

- grouping

These are the steps to use when deciding how to factor a polynomial. We will review an example after looking at these steps.

Use a general strategy for factoring polynomials:

Step 1 - Is there a greatest common factor? Factor it out.

Step 2 - Is the polynomial a binomial, a trinomial, or are there more than three terms?

If it is a binomial:

- Is it a sum?
  - Of squares? Sums of squares do not factor.
  - Of cubes? Use the sum of cubes pattern.
- Is it a difference?
  - Of squares? Factor as the product of conjugates.

If it is a trinomial:

- Is it of the form $x^2 + bx + c$? Undo FOIL.
- Is it of the form $ax^2 + bx + c$?
  - If a and c are squares, check if it fits the trinomial square pattern.
  - Use trial and error or the "ac" method.

If it has more than three terms:

- Use the grouping method.

Step 3 - Check.

- Is it factored completely?
- Do the factors multiply back to the original polynomial?

EXAMPLE 1

Factor completely: $24y^2 - 150$.

Be careful when you are asked to factor a binomial because there are several options!

Step 1 - Is there a GCF?

$24y^2 - 150$

Yes, 6.

Step 2 - Factor out the GCF.

$6(4y^2 - 25)$

Step 3 - In the parentheses, is it a binomial, a trinomial, or are there more than three terms? Binomial.

Step 4 - Is it a sum?

No.

Step 5 - Is it a difference? Of squares or cubes?

Yes, squares.

$6((2y)^2 - (5)^2)$

Step 6 - Write as a product of conjugates.

$6(2y - 5)(2y + 5)$

Step 7 - Is the expression factored completely?

Neither binomial can be factored.

Step 8 - Check your answer by multiplying.

$6(2y - 5)(2y + 5)$

$6(4y^2 - 25)$

$24y^2 - 150$✓

EXAMPLE 2

Use the diagram and questions as references. Which strategy would you use to factor the following polynomials in the left column of the table? Be prepared to show your reasoning.

Look at each polynomial. Review the steps to decide which strategy to use.

Polynomial	Strategy	Reasoning
$x^2 - 3x - 10$		
$4a^2 - 12ab + 9b^2$		
$4y^2 - 25$		

TRY IT **Reviewing General Strategies to Factor a Polynomial**

Identify the strategy you would use to factor each polynomial.

1. $4x^2y - 16xy + 12y$
2. $4x^2 + 8bx - 4ax - 8ab$

6.7.3 Implementing General Strategies for Factoring

Activity

Work with a partner on these problems. After each solution, discuss if there are multiple strategies that would work to factor the polynomial.

Find a strategy that fits and factor completely:

1. $5y^3 - 15y^2 - 270y$

You can use the following video if you need some help.

Problem Animation: Using Multiple Processes to Factor a Polynomial (https://openstax.org/r/factor-polynomial)

2. $75m^3 + 12m$

3. $16x^2 + 24xy - 4x - 6y$

4. $2n^2 + 13n - 7$

5. $121r^2 - s^2$

6. $25w^2 - 60w + 36$

7. $(3x + 1)^2 - 6(3x + 1) + 9$

8. $16x^2 - 24xy + 9y^2 - 64$

Self Check

Which strategy would you use to factor $9x^2 - 24xy + 16y^2$? What is the factored result?

a. trinomial square pattern; $(3x - 4y)^2$
b. difference of squares; $(4x - 3y)^2$
c. grouping; $(3x - 4y)(3x + 4y)$
d. trinomial square pattern; $(3x + 4y)^2$

Additional Resources

Implementing General Strategies for Factoring

Let's practice recognizing and implementing these strategies as we factor polynomials.

Remember, a polynomial is completely factored if, other than monomials, its factors are prime!

EXAMPLE 1

Find a strategy that fits and factor completely: $7x^3 - 21x^2 - 70x$.

Step 1 - Is there a GCF?

$7x^3 - 21x^2 - 70x$

Yes, $7x$.

Step 2 - Factor out the GCF.

$7x(x^2 - 3x - 10)$

Step 3 - In the parentheses, is it a binomial, a trinomial, or are there more than three terms?

Trinomial with leading coefficient 1.

Step 4 - "Undo" FOIL.

$7x(x \quad)(x \quad)$

$7x(x + 2)(x - 5)$

Step 5 - Is the expression factored completely?

Yes. Neither binomial can be factored.

Step 6 - Check your answer by multiplying.

$7x(x + 2)(x - 5)$

$7x(x^2 - 5x + 2x - 10)$

$7x(x^2 - 3x - 10)$

$7x^3 - 21x^2 - 70x$✓

EXAMPLE 2

Find a strategy that fits and factor completely: $4x^2 + 8bx - 4ax - 8ab$.

Step 1 - Is there a GCF?

$4x^2 + 8bx - 4ax - 8ab$

Factor out the GCF, 4.

$4(x^2 + 2bx - ax - 2ab)$

Step 2 - There are four terms. Use grouping.

$4[x(x + 2b) - a(x + 2b)]$

$4(x + 2b)(x - a)$

Step 3 - Is the expression factored completely?

Yes.

Step 4 - Check your answer by multiplying.

$4(x + 2b)(x - a)$

$4(x^2 - ax + 2bx - 2ab)$

$4x^2 + 8bx - 4ax - 8ab$✓

EXAMPLE 3

Find a strategy that fits and factor completely: $40x^2y + 44xy - 24y$.

Step 1 - Is there a GCF?

$40x^2y + 44xy - 24y$

Factor out the GCF, $4y$.

$4y(10x^2 + 11x - 6)$

Step 2 - Factor the trinomial with $a \neq 1$ using the "ac" method or trial and error.

$4y(5x - 2)(2x + 3)$

Step 3 - Is the expression factored completely?

Yes.

Step 4 - Check your answer by multiplying.

$4y(5x - 2)(2x + 3)$

$4y(10x^2 + 11x - 6)$

$40x^2y + 44xy - 24y$✓

TRY IT **Implementing General Strategies for Factoring**

Find a strategy that fits and factor completely:

1. $27y^2 - 48$
2. $6x^2 - 12xc + 6bx - 12bc$

6.7.4 Is the Polynomial Completely Factored?

Cool Down

1. Explain what it means to factor a polynomial completely.

For questions 2 – 7, determine if each polynomial is completely factored. If the answer is no, write the completely factored form.

2. $3x(8x^2 - 9x - 3)$
3. $60x^2y - 75xy + 30y$
4. $2c(4c^2 + 20cd + 81d^2)$
5. $5n^2 + 5n + 12$
6. $5q^2 - 15q - 90$
7. $64x^2 + 16xy + y^2$

6.7.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 - 10, factor each of the expressions completely.

1. $36x^2y + 15xy - 6y$

a. $3y(3x - 2)(4x + 1)$
b. $3y(3x - 2)(4x - 1)$
c. $(3x + 2)(4x - 1)$
d. $3y(3x + 2)(4x - 1)$

2. $49b^2 - 36a^2$

a. $(6b - 7a)(6b + 7a)$
b. $(7b - 6a)^2$
c. $(7b - 6a)(7b + 6a)$
d. $(7b + 6a)^2$

3. $49b^2 - 112b + 64$

a. $(7b + 8)(7b - 8)$
b. $(7b + 8)^2$
c. $(8b + 7)(8b - 7)$
d. $(7b - 8)2$

4. $7b^2 + 7b - 42$

a. $7(b + 3)(b - 2)$
b. $7(b - 3)(b - 2)$
c. $7(b - 3)(b + 2)$
d. $7(b + 3)(b + 2)$

5. $15pq - 15p + 12q - 12$

a. $3(5p + 4)(q - 1)$
b. $3(5p + 4)(q + 1)$
c. $3(4p + 5)(q - 1)$
d. $3(5p - 4)(q - 1)$

6. $4x^2 + 40x + 84$

a. $4(x + 3)(x - 7)$
b. $(4x + 3)(x + 7)$
c. $4(x - 3)(x - 7)$
d. $4(x + 3)(x + 7)$

7. $m^2 + 14mn + 49n^2$

a. $(7m + n)^2$
b. $(m + 7n)^2$
c. $(m + 7n)(m - 7n)$
d. $(7m + n)(7m - n)$

8. $(4x - 5)^2 - 7(4x - 5) + 12$

a. $(4x - 4)(4x - 3)$
b. $(x - 4)(x - 3)$
c. $(4x - 9)(4x - 8)$
d. $(4x - 4)(4x + 3)$

9. $8m^2 - 32$

a. $4(2m - 2)(2m + 2)$
b. $8(m + 2)^2$

c. $8(m-2)^2$
d. $8(m-2)(m+2)$

10. $9x^2 - 6xy + y^2 - 49$

a. $(3x - y + 7)^2$
b. $(3x - y + 7)(3x - y - 7)$
c. $(3x - y - 7)^2$
d. $(x - 3y + 7)(x - 3y - 7)$

6.7.6 Lesson Summary

In this lesson, you learned how to:

- Recognize and use the appropriate method to factor a polynomial completely.

Here are the **activities** that helped you reach those goals:

- 6.7.1: Factoring Strategies for Polynomials
  - In this activity, you identified all the polynomial factoring strategies you have learned in this unit. You created sample polynomials that would fit into each strategy.
- 6.7.2: Reviewing General Strategies to Factor a Polynomial
  - In this activity, you reviewed the strategies used to factor polynomials. When given a sample polynomial, you assessed it and determined an appropriate strategy for factoring.
  - 6.7.2: Self Check
  - 6.7.2: Additional Resources
- 6.7.3: Implementing General Strategies for Factoring
  - In this activity, you used every strategy you have learned to factor different polynomials. You combined strategies to completely factor polynomials of many different forms.
  - 6.7.3: Self Check
  - 6.7.3: Additonal Resources
- 6.7.4: Is the Polynomial Completely Factored?
  - In this activity, you analyzed different polynomials and assessed if they could be factored further or if each factor in the polynomial was prime.

After these activities, you completed the following **practice**:

- 6.7.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

PROJECT 6: POLYNOMIALS AND RECTANGLES

Project 6 Overview

In this culminating project, you will use polynomials to model dimensions and areas of rectangles. Both activities in the lesson use the dimensions and areas of rectangles to apply the skills of working with polynomials learned throughout the unit.

In the first activity, you will be given diagrams of rectangles with side lengths labeled and asked to complete a table to find the missing measures of length, width, or area.

In the second activity, you will be given a table of measures of rectangles, similar to the table in the previous activity. However, you will not have the visual of the rectangles. Again, you will be given the area of rectangles and asked to find the dimensions.

In the third activity, you will create your own table with given measures. When you are done, you will trade tables with a partner and work to find the missing measures in their created problem. After the activity, be prepared to discuss the methods you used to create the table.

Throughout the project, you will have an opportunity to apply the skills of working with polynomials as you add, multiply, divide, and factor polynomials to find the missing measures of rectangles.

Project Activities

1. Areas of Rectangles
2. Dimensions and Areas of Rectangles
3. Create Your Own Rectangle

Learning Targets

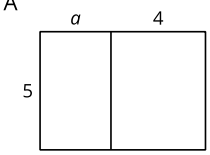
When you finish this project, you will have demonstrated how to:

- Use polynomials to model rectangles.

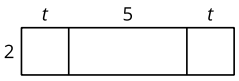
Areas of Rectangles

Complete the table with the length, width, and area of each rectangle. Two dimensions are entered in the table.

A



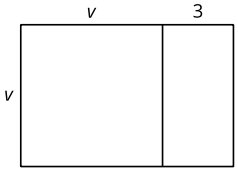
B



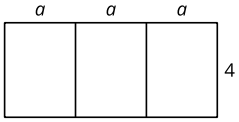
C



D



E



Rectangle	Length (units)	Width (units)	Area (square units)
A	$a+4$		
B		2	

Rectangle	Length (units)	Width (units)	Area (square units)
C			
D			
E			

Dimensions and Areas of Rectangles

Use the given information in the table to find the missing length, width, or area of each rectangle. Be prepared to discuss the strategies you used as you found the missing measures.

Rectangle	Length (units)	Width (units)	Area (square units)
A	$x+3$	$x-4$	
B		$3x-5$	$9x^2-25$
C			$x^2+8x+16$
D	$x-2$		$2x^2-3x-2$
E		$x+6$	$x^2+4x-12$
F			x^2-16
G	$3x-5$	$3x-5$	
H	$3x-1$	$x+2$	
I	$2x-3$		$4x^2-12x+9$
J			$x^2+6x-16$
K	$2x+3$	$2x-3$	
L			$4x^2-12xy+9y^2-25$

Create Your Own Rectangles

You will use the table to create an activity for your partner to find missing dimensions, similar to the previous activity you completed.

Step 1 - On a separate piece of paper, create 3 different rectangles with length and width formed by degree one binomials. Find the area of each rectangle.

Step 2 -

For rectangle A:

- Fill in the table with the binomials that form the length and width of one of the rectangles you created. Leave the area column blank for rectangle A.

For rectangle B:

- Fill in the table with the binomial that represents the length of the second rectangle you created and with the binomial or trinomial that represents the area of the rectangle. Leave the width column blank for rectangle B.

For rectangle C:

- Fill in the table with the binomial or trinomial that represents the area of the third rectangle you created. Leave the length and width columns blank for rectangle C.

Rectangle	Length (units)	Width (units)	Area (square units)
A			
B			
C			

Step 3 - Trade tables with your partner.

Step 4 - Each partner works to find the missing information for each of the rectangles their partner created.

Step 5 - Partners compare answers to see if they match. Discuss discrepancies.

Unit 6 Wrap Up

Throughout this unit, you learned that Ozzy Gamez and Juan Quezada can use polynomials to make decisions that will help their business, Plant Shop Chicago, run smoothly. Understanding polynomials can help them plan revenue and costs so they can make a profit.

In this unit, you learned how to classify polynomials and determine their degree. You added and subtracted polynomials by combining like terms. You multiplied polynomials using the Distributive Property, vertical alignment, and the FOIL method. You learned to recognize special case products, such as binomial squares and conjugate pairs, to facilitate their multiplication. You divided by monomials using the Quotient of Powers Property, and you divided polynomials by polynomials using long division and synthetic division.

You factored the GCF, greatest common factor, from a polynomial by using the "reverse" of the Distributive Property. You learned to factor a polynomial by grouping. You learned methods for factoring trinomials, such as undoing FOIL and trial and error. You learned to recognize patterns of perfect square trinomials and difference of squares. Finally, you learned to analyze a polynomial by its form and to select an appropriate strategy for factoring.



Access our Unit 6 Student Self-Assessment (https://openstax.org/r/unit-6-self-assesment) to reflect on the skills you learned during this unit. Congratulations on completing Unit 6!

Building Character: Judgment

Throughout this unit, you learned that **judgment** is important for making sound decisions without bias. You can improve judgment through experience and by trying to understand the qualities that contribute to good judgment.

UNIT 7 OVERVIEW AND READINESS

Unit 7 Overview

Why Should I Care?

Have you ever heard of, or participated in, a “Punkin Chunkin” contest? Watch the following video to learn more, and how quadratics may play a role.

Unit 7 Overview (https://openstax.org/r/punkin-chunkin)

In this unit, you will learn about quadratic functions systematically and use quadratics to model real-world scenarios like ballistic motion. You will look at patterns that grow quadratically and contrast them with linear and exponential growth. Then you will examine other quadratic relationships via tables, graphs, and equations, gaining appreciation for some of the special features of quadratic functions and the situations they represent.

You will also analyze equivalent quadratic expressions to understand how these expressions help to reveal important behavior of the associated quadratic function and its graph. You will gain an appreciation for the factored, standard, and vertex forms of a quadratic function and use these forms to solve problems.

Building Character: Proactivity

Punkin Chunkin competitions encourage teams to design and build machines to throw pumpkins the greatest distance. To win, teams must work together and use creative thinking. In teams like these, one person often steps up and leads the group to accomplish the tasks they need to finish. They don’t wait to take direction from others; they set the tone themselves. This is an example of proactivity. Working together as a group to brainstorm ideas and come up with solutions to problems before they happen is another way to be proactive in a team setting.

Throughout this unit, you will work on developing your **proactivity**. Proactivity is the ability to take initiative. Proactive people don’t accept the world as it is or wait for direction from others to initiate change. Instead, they take matters into their own hands, acting to make the world better for themselves and others. They speak truth to power, take charge to solve problems, and champion new ideas. There’s evidence that proactivity is one of the qualities that distinguished the greatest American presidents from their peers. And there’s also evidence that it’s a trait you can develop.

Think about your current sense of proactivity. How many of the following statements are true for you?

- I am always looking for better ways of doing things.
- I love to challenge the status quo.
- I enjoy being a champion for ideas, even against opposition.
- I like to be the one to take initiative.

Don’t worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

Am I Ready to Learn This?

Getting Ready for Unit 7

To be ready for this unit, you need to be able to:

- Multiply binomials.
- Factor trinomials.
- Compare linear and exponential functions.

Next, you will answer some questions that will help you see how ready you are for this unit. If you need help, some mini-lessons will help you brush up on anything you need to work on.

Multiply Binomials: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #1: Multiply Binomials

Multiply the binomials.

$(3x + 1)(2x - 4)$

a. $6x^2 - 12x - 4$
b. $6x^2 + 2x - 4$
c. $6x^2 + 10x - 4$
d. $6x^2 - 10x - 4$

Multiplying Binomials

You can multiply polynomials using the distributive property.

EXAMPLE 1

Multiply $(x + 3)(x + 7)$ using the distributive property.

Step 1 - Multiply each term in the first binomial by the second binomial.

$(x + 3)(x + 7)$

$x(x + 7) + 3(x + 7)$

Step 2 - Use the distributive property to eliminate the parentheses.

$x^2 + 7x + 3x + 21$

Step 3 - Combine like terms.

$x^2 + 10x + 21$

There is a pattern when multiplying two polynomials that are binomials.

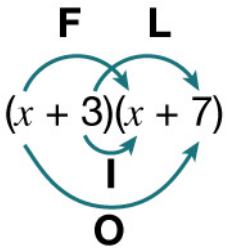
- The first term in the result is the product of the first terms in each binomial.
- The second and third terms are the product of multiplying the two outer terms and then the two inner terms.
- The last term results from multiplying the two last terms.

The method "First, Outer, Inner, Last" is abreviated as FOIL. It represents multiplying the First terms, Last terms, Inner terms, and Last terms of the binomials. The word "FOIL" is easy to remember and ensures you find all four products.

EXAMPLE 2

Multiply $(x + 3)(x + 7)$ using FOIL.

Step 1 - Multiply the First terms, Outer terms, Inner terms, and Last terms.



$x^2 + 7x + 3x + 21$

Step 2 - Combine like terms.

$x^2 + 10x + 21$

EXAMPLE 3

Multiply $(x - 3)(2x + 7)$ using FOIL.

Step 1 - Multiply the First terms, Outer terms, Inner terms, and Last terms.

$2x^2 + 7x - 6x - 21$

Step 2 - Combine like terms.

$2x^2 + x - 21$

TRY IT **Multiplying Binomials**

Multiply $(7x + 1)(2x - 3)$ using FOIL.

Check Your Understanding

Multiply the binomials.

$(2x + 5)(3x - 2)$

a. $6x^2 + 11x - 10$

b. $6x^2 - 11x - 10$

c. $6x^2 + 15x - 10$

d. $6x^2 + 19x - 10$

Video: Multiplying Binomials

Watch the following video to learn more about multiplying binomials.

Khan Academy: Multiplying Binomials

Example 1: Multiplying a binomial by a binomial | Algebra I | Khan Academy (https://www.youtube.com/embed/ZMLFfTX615w)

Factor Trinomials: Mini-Lesson Review

MINI LESSON QUESTION **Question #2: Factor Trinomials**

Factor: $y^2 - y - 42$.

a. $(y - 7)(y - 6)$

b. $(y + 7)(y + 6)$

c. $(y + 7)(y - 6)$

d. $(y - 7)(y + 6)$

Factor Trinomials

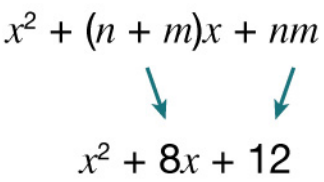
EXAMPLE 1

To factor the trinomial $x^2 + 8x + 12$, you can reverse the process of FOIL.

The trinomial factors to two binomials of the form $(x + m)(x + n)$.

- Using FOIL: $(x + m)(x + n) = x^2 + nx + mx + mn$.
- Using the distributive property: $x^2 + (n + m)x + mn$.

Take a look at the trinomial $x^2 + 8x + 12$:



To factor the trinomial, we need factors of 12, m and n, so that the sum, $n + m$, is 8.

Use a table to find all the different combinations of factors of 12. Then, find the sum of the factors.

Factors of 12	Sum of factors
$1, 12$	$1 + 12 = 13$
$2, 6$	$2 + 6 = 8$✓
$3, 4$	$3 + 4 = 7$

The numbers 2 and 6 have a product of 12 and a sum of 8. They are the factors we need:

$x^2 + 8x + 12 = (x + 6)(x + 2)$

EXAMPLE 2

Factor the trinomial $x^2 - 5x - 24$.

To factor the trinomial, we need factors of -24, m and n, so that the sum, $n + m$ is -5.

- The product of the factors is negative, so the factors must have opposite signs.
- The sum is negative, so the negative sign will go with the factor with the greater absolute value.

Factors of 12	Sum of factors
$1, -24$	$1 + (-24) = -23$
$2, -12$	$2 + (-12) = -10$
$3, -8$	$3 + (-8) = -5$✓
$4, -6$	$4 + (-6) = -2$

The numbers 3 and -8 have a product of -24 and a sum of -5.

$x^2 - 5x - 24 = (x + 3)(x - 8)$

EXAMPLE 3

Factor the trinomial $x^2 - 9x + 18$.

To factor the trinomial, we need factors of 18, m and n, so that the sum, $n + m$, is -9.

- The product of the factors is positive, so the factors must have the same signs.

- The sum is negative, so both factors are negative.

Factors of 18	Sum of factors
$-1, -18$	$-1 + (-18) = -19$
$-2, -9$	$-2 + (-9) = -11$
$-3, -6$	$-3 + (-6) = -9$✓

The numbers -3 and -6 have a product of 18 and a sum of -9.

$x^2 - 9x + 18 = (x - 3)(x - 6)$

› **TRY IT** **Factoring Trinomials**

Factor the trinomial $x^2 + 2x - 24$.

Check Your Understanding

Factor: $y^2 - 13y + 36$.

a. $(y - 4)(y - 9)$

b. $(y - 2)(y + 18)$

c. $(y + 4)(y + 9)$

d. $(y - 6)(y - 6)$

Video: Factoring Trinomials

Watch the following video to learn more about factoring trinomials.

Khan Academy: Factoring Quadratics

Factoring quadratics as (x+a)(x+b) (example 2) | Mathematics II | High School Math | Khan Academy (https://www.youtube.com/embed/1kfq0aR3ASs)

Compare Linear and Exponential Functions: Mini-Lesson Review

? **MINI LESSON QUESTION** **Question #3: Compare Linear and Exponential Functions**

Which statement best describes the following tables?

x	y
1	3
2	6

Table A

x	y
3	9
4	12
5	15

Table A

x	y
1	3
2	6
3	12
4	24
5	48

Table B

a. Table A represents an exponential function with a constant difference of 3; Table B represents a linear function with a constant growth factor of 2.
b. Table A represents a linear function with a constant growth factor of 2; Table B represents an exponential function with a constant difference of 3.
c. Table A represents an exponential function with a constant growth factor of 2; Table B represents a linear function with a constant difference of 3.
d. Table A represents a linear function with a constant difference of 3; Table B represents an exponential function with a constant growth factor of 2.

Compare Linear and Exponential Functions

Look at the patterns in the two tables.

x	y
1	4
2	8
3	12
4	16
5	20

x	y
1	4
2	8
3	16
4	32
5	64

In each situation, the x-values are increasing by one. These can also be called consecutive terms.

- If the difference in the y-values of consecutive terms is always the same constant, then the values in the table can be modeled by a linear function. The constant is the slope, or rate of change.
- If the quotient for the y-values of consecutive terms is always the same constant, then the values in the table can be modeled by an exponential function. The constant is the growth factor.
- It is possible for a table to be a set of values that is neither of these relationships.

Using this information you can see that in the first table, the difference in consecutive y-values is the same constant: 8-4 $= 4$, $12 - 8 = 4$, $16 - 12 = 4$, $20 - 16 = 4$. The differences are the same, so the table represents a linear function.

In the second table, the quotient of consecutive terms is the same: $\frac{8}{4} = 2$, $\frac{16}{8} = 2$, $\frac{32}{16} = 2$, $\frac{64}{32} = 2$. The quotient is the same, so the table represents an exponential function.

Linear growth happens at a constant rate. The rate of change, or slope, in the first table is 4. The linear function is of the form $y = mx + b$, where m is the slope.

Exponential growth does not happen at a constant rate. The growth factor is constant, but the rate of growth changes as the value of the independent variable, x, changes. The exponential function is of the form $y = ab^x$, where b is the growth factor.

TRY IT **Compare Linear and Exponential Functions.**

Look at the patterns in the two tables.

Table A	
x	y
1	5
2	15
3	45
4	135
5	405

Table B	
x	y
1	5
2	15
3	25
4	35
5	45

Which of the following tables represents linear growth and which represents exponential growth? Be prepared to show your reasoning.

Check Your Understanding

Which statement best describes the following tables?

x	y
1	2
2	6
3	18
4	54
5	162

Table A

x	y
1	2
2	6
3	10
4	14
5	18

Table B

a. Table A represents an exponential function with a constant growth factor of 3; Table B represents a linear function with a constant difference of 4.

b. Table A represents a linear function with a constant difference of 4; Table B represents an exponential function with a constant growth factor of 3.

c. Table A represents a linear function with a constant growth factor of 3; Table B represents an exponential function with a constant difference of 4.

d. Table A represents an exponential function with a constant difference of 4; Table B represents a linear function with a constant growth factor of 3.

Video: Linear and Exponential Growth

Watch the following video to learn more about linear and exponential growth.

Khan Academy: Exponential VS Linear Growth

Examples of linear and exponential relationships (https://www.youtube.com/embed/_vlXdx-CqM0)

Unit 7 Overview: Wrap Up

- Multiply binomials.
- Factor trinomials.

- Compare linear and exponential functions.

If you are still struggling, be sure to reach out to your teacher for additional help. Nobody is good at problems like this the first try. Keep at it and it will get easier.

7.1 PATTERNS OF CHANGE

7.1.0 Lesson Overview

In this lesson, you will use your knowledge of geometric patterns to create tables and graphs that will represent the area of a garden.

When you finish this lesson, you will be able to:

- Create drawings, tables, and graphs that represent the area of a garden.
- Recognize a situation represented by a graph that increases and then decreases.

Here are the **activities** that will help you reach those goals:

- 7.1.1: Noticing New Patterns of Change
- 7.1.2: The Relationship Between Length and Area
  - 7.1.2: Self Check
  - 7.1.2: Additional Resources
- 7.1.3: Plotting Measurements
  - 7.1.3: Self Check
  - 7.1.3: Additional Resources
- 7.1.4: The Input and Output of a Function

After that, you'll **practice and review**.

- 7.1.5: Practice
- 7.1.6: Lesson Summary

7.1.1 Noticing New Patterns of Change

Warm Up

1. Look at the patterns in the three tables. What do you notice? What do you wonder?

x	y
1	0
2	5
3	10
4	15
5	20

x	y
1	3
2	6
3	12
4	24
5	48

x	y
1	8
2	11
3	10
4	5
5	-4

7.1.2 The Relationship between Length and Area

Activity

Noah has 50 meters of fencing to completely enclose a rectangular garden in the backyard.



1. Draw some possible diagrams of Noah's garden. Label the length and width of each rectangle.
2. Find the length and width of a rectangle that would produce the largest possible area. Explain why you think that pair of length and width gives the largest possible area.

Self Check

Sonya has 70 meters of fencing to surround a rectangular garden. What is the largest possible area that can be enclosed, in square meters?

a. 1225
b. 306.25
c. 35
d. 17.5

Additional Resources

Largest Possible Area

Tyrel is enclosing a grassy area for his goats. He has 80 feet of fencing and wants to make a rectangular enclosed area. What are the dimensions of the enclosure with the largest possible area?

First, make a table with possible length and width dimensions and their areas. Remember, each $2l + 2w = 80$ and $A = lw$.

Length (meters)	Width (meters)	Area (square meters)
5	35	175
10	30	300
15	25	375

Length (meters)	Width (meters)	Area (square meters)
20	20	400
25	15	375
30	10	300
35	5	175

Notice that the closer the rectangle is to a square, the larger the area.

The largest area occurs when the length and width are both 20 feet for an area of 400 square feet.

TRY IT **Largest Possible Area**

Elena has 100 feet of fencing to make a rectangular pen for her cow. What is the largest possible area she can make with this amount of fencing?

7.1.3 Plotting Measurements

Activity

Access multimedia content (http://openstax.org/books/algebra-1/pages/7-1-3-plotting-measurements)

1. Use the graphing tool or technology outside the course. Plot some values for the length and area of the garden with perimeter 50 meters on the coordinate plane.
   Use your table, or here is a sample table of values if you need one:

Length (meters)	Area (square meters)
5	100
10	150
12	156
12.5	156.25
18	126
20	100
24	24

Values for a Rectangular Garden with a Perimeter of 50 Meters

2. What do you notice about how the plotted points increase and decrease?

3. The points $(3, 66)$ and $(22, 66)$ each represent the length and area of the garden. Plot these two points on a coordinate plane, if you haven't already done so. What do these points mean in this situation?
4. Could the point $(1, 25)$ represent the length and area of the garden?

Are you ready for more?

Extending Your Thinking

1. What happens to the area when you interchange the length and width? For example, compare the areas of a rectangle of length 11 meters and width 14 meters with a rectangle of length 14 meters and width 11 meters.
2. What patterns would you notice if you were to plot more length and area pairs on the graph?

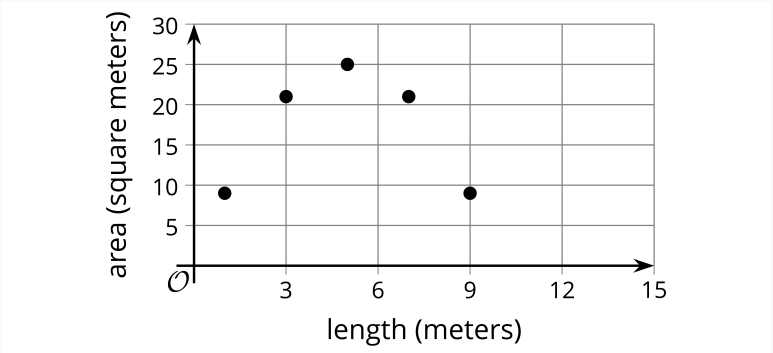
Video: The Meaning of Points on a Graph

Watch the following video to learn more about the meaning of points on a graph.

The Meaning of Points on a Graph (https://openstax.org/r/graph-points)

Self Check

The graph below shows a rectangle's side length and its area. What is true about the point $(3, 21)$?

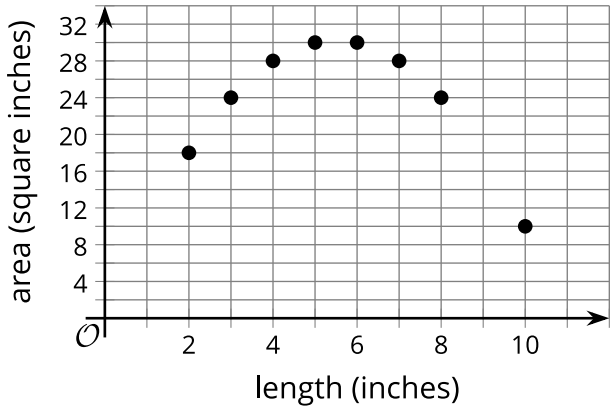


a. The area of the rectangle is 3 square meters.
b. It is not possible to find the width of the rectangle when the length is 3 meters.
c. The length of the rectangle is 3 meters, and the area is 21 square meters.
d. The width of the rectangle is 3 meters, and the area is 21 square meters.

Additional Resources

Determining the Meaning of a Point on a Graph

The graph below shows the relationship between a rectangle's side length and area.



What does the point $(7, 28)$ mean in the situation?

The x-axis represents the length of the rectangle in inches. So, when $x = 7$, the rectangle has a side length of 7 inches.

The y-axis represents the area of the rectangle. So, when $y = 28$, the rectangle has an area of 28 square inches.

This also means the rectangle has a width of 4 inches since $7 \times 4 = 28$.

TRY IT **Determining the Meaning of a Point on a Graph**

Using the same graph above, determine the meaning of the point $(3, 24)$ and the width of the rectangle.

7.1.4 The Input and Output of a Function

Cool Down

A **rectangular**yard is enclosed by 100 meters of fencing. The table shows some possible values for the length and width of the yard.

Length (meters)	Width (meters)	Area (square meters)
10	40	400
20	30	
25	25	625
35	15	525
40		

1. Complete the table with the missing values.
2. If the values for length and area are plotted, what would the graph look like?
3. How is the relationship between the length and the area of the rectangle different from other kinds of relationships we've seen before?

7.1.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Here are a few pairs of positive numbers whose sum is 50.

Row	First number	Second number	Product
a	1	49	
b	2	48	
c	10	40	

a. Find the product in Row a.

b. Find the product in Row b.

c. Find the product in Row c.

Use the following table to answer questions 2 – 5.

Here are some lengths and widths of a rectangle whose perimeter is 20 meters.

Row	Length (meters)	Width (meters)	Area (square meters)
a	1	9	
b	3	7	
c	5		
d	7		
e	9		

2. Complete the **Width** column of the table.

a. Width in Row c

b. Width in Row d

c. Width in Row e

3. Complete the **Area** column of the table. What do you notice about the areas?

a. Area in Row a

b. Area in Row b

c. Area in Row c

d. Area in Row d

e. Area in Row e

4. Without calculating, predict whether the area of the rectangle will be greater than or less than 25 square meters if the length is 5.25 meters.

a. greater than
b. less than

5. Plot the points for length and area from your table. Do the values change in a linear way? Do they change in an exponential way?

a. linear
b. exponential
c. linear and exponential
d. not linear and not exponential

Use the following to answer questions 6 – 8.

Copies of a book are arranged in a stack. Each copy of a book is 2.1 cm thick.

Row	Copies of book	Stack height in cm
a	0	
b	1	
c	2	
d	3	
e	4	

6. Complete the table.

a. Stack height in Row a

b. Stack height in Row b

c. Stack height in Row c

d. Stack height in Row d

e. Stack height in Row e

7. What is the difference in the height of the stack of books when a new copy of the book is added?

8. How high is a stack of b books, in centimeters?

a. 2.1^b
b. $2.1(b-1)$
c. $2.1+b$
d. $(2.1)b$

7.1.6 Lesson Summary

In this lesson, you learned how to:

- Create drawings, tables, and graphs that represent the area of a garden.
- Recognize a situation represented by a graph that increases and then decreases.

Here are the **activities** that helped you reach those goals:

- 7.1.1: Noticing New Patterns of Change
  - In this activity, you determined patterns in tables based on how the y-values changed as the x-values changed.
- 7.1.2: The Relationship Between Length and Area
  - In this activity, you examined different combinations of lengths and widths of a rectangle and their respective areas given an amount of fencing. You realized that the closer the rectangle was to a square, the larger the area.
  - 7.1.2: Self Check
  - 7.1.2: Additional Resources
- 7.1.3: Plotting Measurements
  - In this activity, you looked at how a function that wasn't linear or exponential changed by increasing and then decreasing.
  - 7.1.3: Self Check
  - 7.1.3: Additional Resources
- 7.1.4: The Input and Output of a Function
  - In this activity, you completed a table where values again increased and then decreased, rather than always increasing or always decreasing like in linear and exponential functions.

After these activities, you completed the following **practice**:

- 7.1.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

7.2 INTRODUCTION TO QUADRATIC RELATIONSHIPS

7.2.0 Lesson Overview

In this lesson, you will work with problems that show patterns of change and be able to describe the patterns in mathematical terms. A new term, quadratic, will be used to describe patterns with squared terms.

When you finish this lesson, you will be able to:

- Describe how a pattern is growing.
- Tell whether a pattern is growing linearly, exponentially, or quadratically.
- Explain that an expression with a squared term is called quadratic.

Here are the **activities** that will help you reach those goals:

- 7.2.1: Squares in a Geometric Pattern
- 7.2.2: Constant and Exponential Change
  - 7.2.2: Self Check
  - 7.2.2: Additional Resources
- 7.2.3: Quadratic Expressions
  - 7.2.3: Self Check
  - 7.2.3: Additional Resources
- 7.2.4: Quadratic Relationships

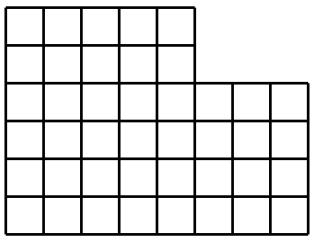
After that, you'll **practice and review**.

- 7.2.5: Practice
- 7.2.6: Lesson Summary

7.2.1 Squares in a Geometric Pattern

Warm Up

How does each expression represent the number of small squares in the figure?



1. Expression A: $6 \cdot 8 - 2 \cdot 3$
2. Expression B: $4 \cdot 8 + 2 \cdot 5$
3. Expression C: $8 + 8 + 8 + 8 + 5 + 5$
4. Expression D: $5 \cdot 6 + 3 \cdot 4$

Building Character: Proactivity



Proactivity is the ability to take initiative. There's evidence that proactivity is one of the qualities that distinguished the greatest American presidents from their peers.

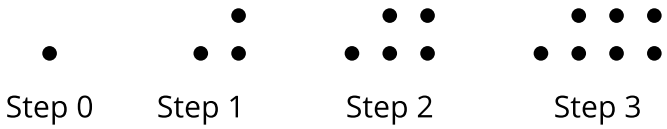
Think about your current sense of proactivity. Are the following statements true for you??

- I am always looking for better ways of doing things.
- I love to challenge the status quo.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

7.2.2 Constant and Exponential Change

Pattern 1



Pattern 2



1. Look at Pattern 1.

a. How is the number of dots in each step changing?

b. How many dots will be in Step 5 in the pattern, and how will they be arranged?

2. Look at Pattern 2.

a. How is the number of dots in each step changing?

b. How many dots will be in Step 5 in the pattern, and how will they be arranged?

Consider the following table.

Step	Number of dots in Pattern 1	Number of dots in Pattern 2
0	1	0
1	3	1
2	5	4
3	7	9
4		
5		
10		
12		
n		

3. Complete the table with the number of dots for Pattern 1.

a. In Step 4, what is the number of dots in Pattern 1?

b. In Step 5, what is the number of dots in Pattern 1?

c. In Step 10, what is the number of dots in Pattern 1?

d. In Step 12, what is the number of dots in Pattern 1?

e. In Step n, what is the number of dots in Pattern 1?

4. Complete the table with the number of dots for Pattern 2.

a. In Step 4, what is the number of dots in Pattern 2?

b. In Step 5, what is the number of dots in Pattern 2?

c. In Step 10, what is the number of dots in Pattern 2?

d. In Step 12, what is the number of dots in Pattern 2?

e. In Step n, what is the number of dots in Pattern 2?

5. Use the graphing tool or technology outside the course. Plot the number of dots at each step number for Pattern 1 and Pattern 2.

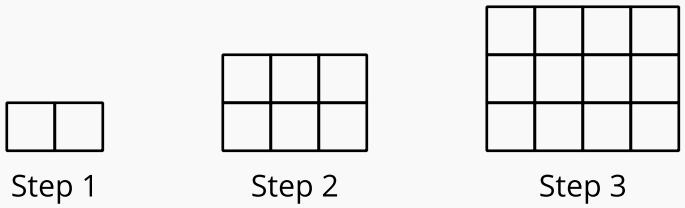
Access multimedia content (http://openstax.org/books/algebra-1/pages/7-2-2-constant-and-exponential-change)

a. Pattern 1

b. Pattern 2

6. Explain why the graphs of the two patterns look the way they do.

Self Check



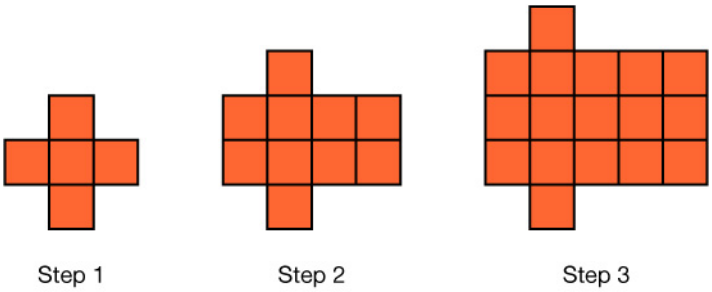
How many squares will be in Step 5?

a. 35
b. 30
c. 25
d. 20

Additional Resources

Writing Expressions for Patterns

EXAMPLE 1



Find the number of squares in Step 5 -

First, look to see the pattern. Every step, another row and another column are added, plus the two extra squares.

- Step 1 - 1 row of $3 + 2$
- Step 2 - 2 rows of $4 + 2$
- Step 3 - 3 rows of $5 + 2$

So the width is always 1 by n, or n. The length is $n + 2$, and then another 2 is added.

Then, draw pictures or create a table:

Step	# of squares
1	5
2	10
3	17

Step	# of squares
4	26
5	37

At Step 5, there are 37 squares.

EXAMPLE 2

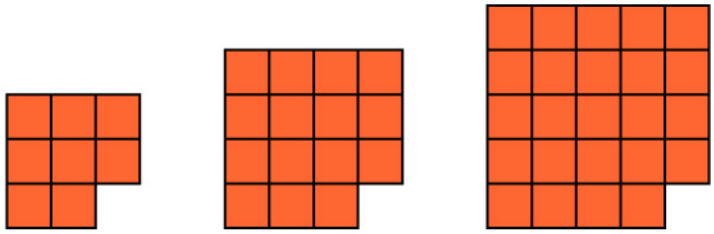
Find the expression for Step n.

Multiply the length and width, and then add the 2 extra squares:

$n(n+2)+2 = n^2 + 2n + n$

TRY IT **Writing Expressions for Patterns**

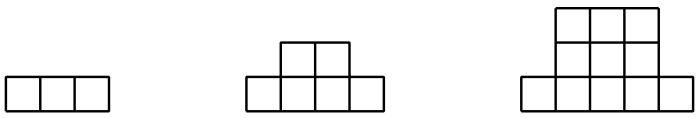
Use the figure below to find the number of squares in Step 5 and then Step n.



Step 1 Step 2 Step 3

7.2.3 Quadratic Expressions

Activity



Step 1 Step 2 Step 3

Here is a pattern of squares.

1. Is the number of small squares growing linearly? Explain how you know.

2. Answer questions a–e to complete the table.

Step	Number of small squares
1	3
2	6

Step	Number of small squares
3	11
4	a. _____
5	b. _____
10	c. _____
12	d. _____
n	e. _____

a. In Step 4, what is the number of small squares?

b. In Step 5, what is the number of small squares?

c. In Step 10, what is the number of small squares?

d. In Step 12, what is the number of small squares?

e. In Step n, what is the number of small squares?

3. Is the number of small squares growing exponentially? Explain how you know.

The n^{th} expression, $n^2 + 2$, is an example of a **quadratic expression**. A quadratic expression can be written using a squared term. It can be written in other ways as well, but the expression should be able to be written with the highest exponent on the variable as 2.

- A quadratic relationship is not like a linear relationship because as one quantity increases by a certain amount, the second quantity doesn't increase by the same amount.
- A quadratic relationship is not like an exponential relationship because as one quantity increases by a certain amount, the second quantity doesn't change by the same factor. (In this case, if the number of squares grew exponentially, this would mean that, from each step to the next, the number of squares is multiplied by the same factor.)

Are you ready for more?

Extending Your Thinking

1. Han wrote $n(n+2) - 2(n-1)$ for the number of small squares in the design at Step n in the pattern at the beginning of the activity.
Explain why Han is correct.

2. On your own, draw and label what Step 3 would look like when Han was writing his expression.

Video: Quadratic Expressions

Watch the following video to learn more about quadratic expressions.

Quadratic Expressions (https://openstax.org/r/quadratic-expressions)

Self Check

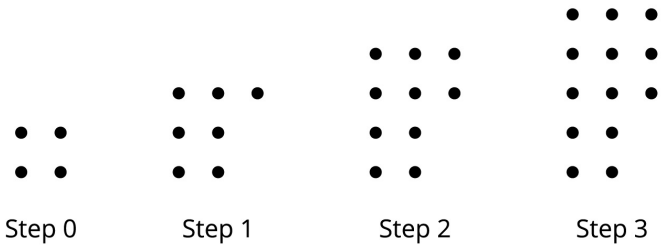
Which of the following is an example of a quadratic expression?

a. $3(n+2)$
b. $2^n + 3$
c. $n^2 + 3$
d. $2n + 4$

Additional Resources

Writing Expression from Patterns

Find the number of dots in Step 5, and write an expression to find n number of dots.



Step 1 - Make a table to help you find the pattern.

Step number	Number of dots
0	4
1	7
2	10
3	13

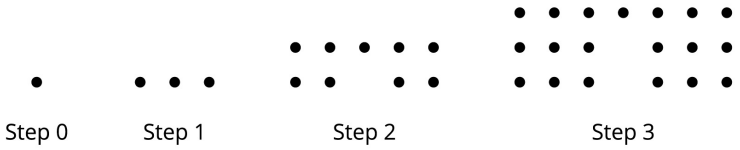
Step 2 - Notice that the start has 2 rows and 2 columns, or 4 dots. Each step adds a row of 2 dots, plus 1 more dot. So, each step adds 3 dots.

Step 3 - Continuing this linear pattern, Step 5 would have 19 dots.

Step 4 - An expression could be $3n + 4$. This is a linear expression.

TRY IT **Writing Expression from Patterns**

Find the number of dots in Step 5, and write an expression to find n number of dots.



7.2.4 Quadratic Relationships

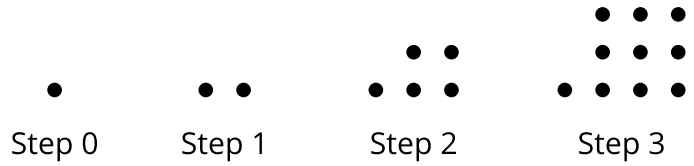
Cool Down

Here are three patterns of dots.

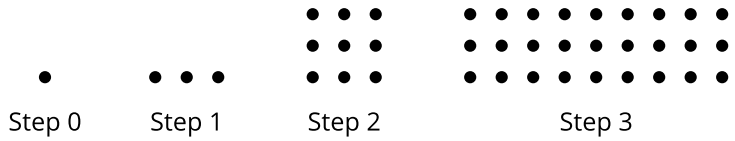
Pattern A



Pattern B



Pattern C



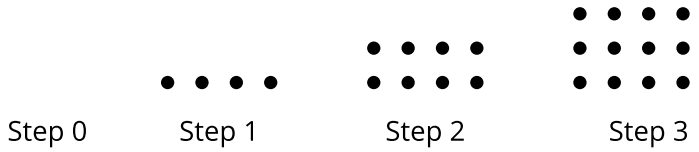
1. Which pattern shows a quadratic relationship between the step number and the number of dots? Explain how you know algebraically and in complete sentences.

7.2.5 Practice

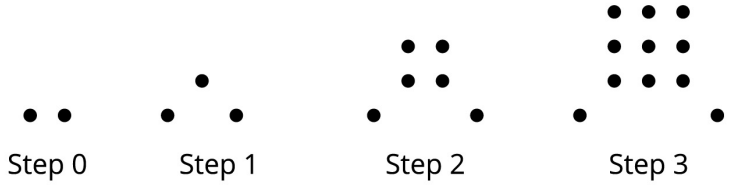
Complete the following questions to practice the skills you have learned in this lesson.

1. Here are two patterns of dots.

Pattern A



Pattern B



a. How many dots will there be in Step 4 of Pattern A?

b. How many dots will there be in Step 4 of Pattern B?

c. Which pattern shows a quadratic relationship between the step number and the number of dots?

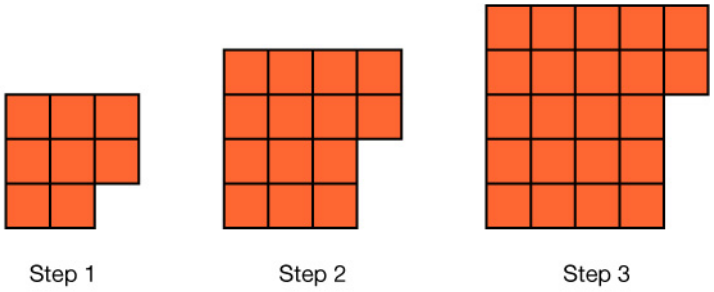
   a. Pattern A
   b. Pattern B

2. Each expression represents the total number of dots in a pattern where n represents the step number.

Select two expressions that represent a quadratic relationship between the step number and the total number of dots. (If you get stuck, consider sketching the first few steps of each pattern as described by the expression.)

a. n^2
b. $2n$
c. $n \cdot n$
d. $n + n$
e. $n + 2$
f. $n \div 2$

3. Use the pattern below to answer the questions.



a. How many blocks are in Step 4?

b. How many blocks are in Step 5?

c. How many blocks are in Step n?

a. $(n+2)^2$
b. $(n+2)^2 - n$
c. $n+6$
d. $n^2 - 2$

7.2.6 Lesson Summary

In this lesson, you learned how to:

- Describe how a pattern is growing.
- Tell whether a pattern is growing linearly, exponentially, or quadratically.
- Explain that an expression with a squared term is called quadratic.

Here are the **activities** that helped you reach those goals:

- 7.2.1: Squares in a Geometric Pattern
  - In this activity, you explored how different expressions can represent the same figure and still be equivalent.
- 7.2.2: Constant and Exponential Change
  - In this activity, you wrote an expression to describe a pattern, and then you graphed the values of the step and the number of dots in each figure. One was linear, while the other was not.
  - 7.2.2: Self Check
  - 7.2.2: Additional Resources
- 7.2.3: Quadratic Expressions
  - In this activity, you explored a quadratic relationship, including what makes it quadratic and not linear or exponential.
  - 7.2.3: Self Check
  - 7.2.3: Additional Resources
- 7.2.4: Quadratic Relationships
  - In this activity, you examined three patterns and chose which one represents a quadratic relationship.

After these activities, you completed the following **practice**:

- 7.2.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

7.3 DETERMINING IF A FUNCTION IS QUADRATIC

7.3.0 Lesson Overview

In this lesson, you will continue to work with patterns but begin to see these relationships as quadratic functions and write equations to define them.

When you finish this lesson, you will be able to:

- Recognize quadratic functions written in different ways.
- Use information from a pattern of shapes to write a quadratic function.
- Know that, in a pattern of shapes, the step number is the input and the number of squares is the output.

Here are the **activities** that will help you reach those goals:

- 7.3.1: Quadratic Expressions and Area
- 7.3.2: Writing Equations for Patterns with Squares
  - 7.3.2: Self Check
  - 7.3.2: Additional Resources
- 7.3.3: Quadratic Sequences
  - 7.3.3: Self Check
  - 7.3.3: Additional Resources
- 7.3.4: Quadratic Function?

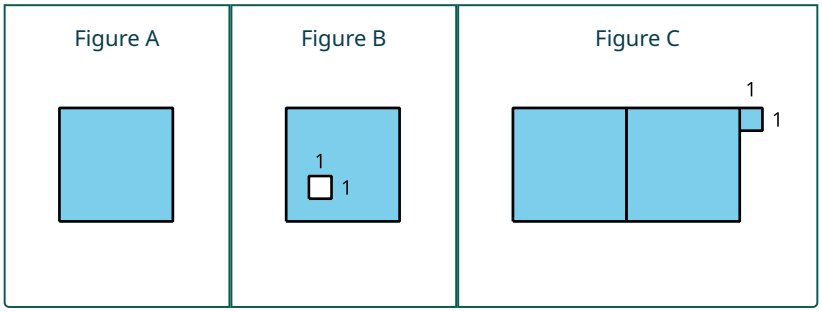
After that, you'll **practice and review**.

- 7.3.5: Practice
- 7.3.6: Lesson Summary

7.3.1 Quadratic Expressions and Area

Warm Up

Figure A is a large square. Figure B is a large square with a smaller square removed. Figure C is composed of two large squares with one smaller square added.



Write an expression to represent the area of each shaded figure when the side length of the large square is as shown in the first column.

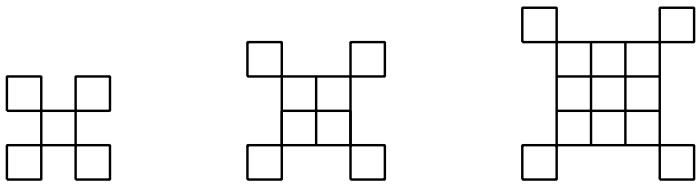
Copy and complete the table. Compare your work by answering 1 - 12.

Side length of large square	Area of A	Area of B	Area of C
4			
x			
$4x$			
$x+3$			

1. When the side length of the large square is 4, what is the area of figure A?
2. When the side length of the large square is 4, what is the area of figure B?
3. When the side length of the large square is 4, what is the area of figure C?
4. When the side length of the large square is x, what is the area of figure A?
5. When the side length of the large square is x, what is the area of figure B?
6. When the side length of the large square is x, what is the area of figure C?
7. When the side length of the large square is $4x$, what is the area of figure A?
8. When the side length of the large square is $4x$, what is the area of figure B?
9. When the side length of the large square is $4x$, what is the area of figure C?
10. When the side length of the large square is $x+3$, what is the area of figure A?
11. When the side length of the large square is $x+3$, what is the area of figure B?
12. When the side length of the large square is $x+3$, what is the area of figure C

7.3.2 Writing Equations for Patterns with Squares

Activity



Step 1 Step 2 Step 3

1. If the pattern continues, what will we see in Step 5 and Step 18?

a. Sketch or describe the figure in each of these steps.

b. How many small squares are in each of these steps? Explain how you know.

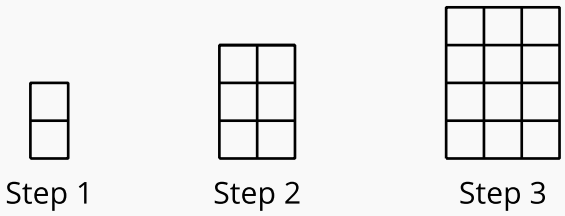
2. Write an equation to represent the relationship between the step number n and the number of squares y. Be prepared to explain how each part of your equation relates to the pattern. (If you get stuck, try making a table.)
3. Sketch the first three steps of a pattern that can be represented by the equation $y = n^2 - 1$.

Are you ready for more?

Extending Your Thinking

1. For the original step pattern in the statement, $y = n^2 + 4$, write an equation to represent the relationship between the step number, n, and the perimeter, P.
2. For the step pattern you created in Part 3 of the activity, $y = n^2 - 1$, write an equation to represent the relationship between the step number, n, and the perimeter, P.
3. Are these linear functions?

Self Check



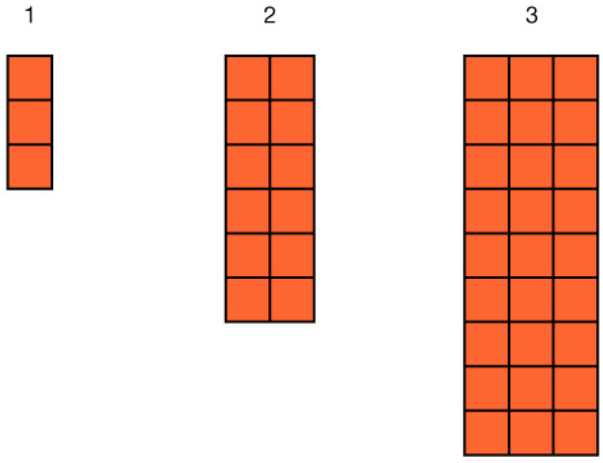
Which equation represents the relationship between the step number n and the number of squares y?

a. $y = n(n + 1)$
b. $y = n^2 + 1$
c. $y = 2n + 1$
d. $y = n + 1$

Additional Resources

Patterns Represented by Quadratics

See Steps 1 – 3 of a pattern of squares below. Write an equation representing the relationship between the step number n and the number of squares y.



Make a table for the number of columns and rows of each step and the number of squares:

Step	# of columns	# of rows	# of squares
1	1	3	3
2	2	6	12
3	3	9	27
n	n	$3n$	$3n^2$

First, count how many columns there are in each step (the length):

For each, there are n columns.

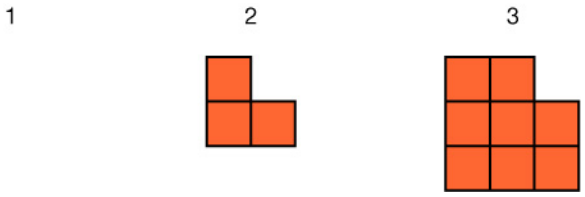
Next, count how many rows there are in each step (the height):

For each, there are $3n$ rows.

The area of this figure is then $l \times h$ or $n \times 3n$, which becomes $y = 3n^2$.

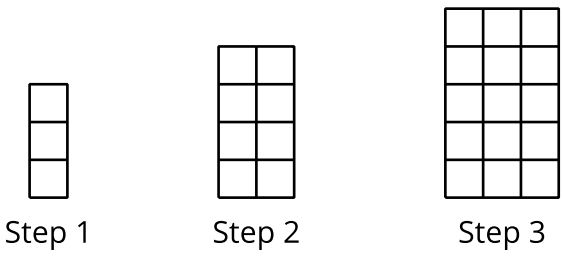
TRY IT **Patterns Represented by Quadratics**

See Steps 1 – 3 of a pattern of squares below. Write an equation representing the relationship between the step number n and the number of squares y.



7.3.3 Quadratic Sequences

Activity



1. Sketch the next step in the pattern. Explain what you sketched.
2. Kiran says that the pattern is growing linearly because as the step number goes up by 1, the number of rows and the number of columns also increase by 1. Do you agree? Be prepared to show your reasoning.

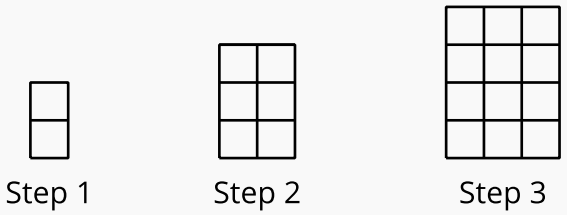
3. To represent the number of squares after n steps, Diego and Jada wrote different equations. Diego wrote the equation $f(n) = n(n+2)$. Jada wrote the equation $f(n) = n^2 + 2n$. Is either Diego or Jada correct? Be prepared to show your reasoning.

Video: Quadratic Sequences

Watch the following video to learn more about quadratic sequences.

Quadratic Sequences (https://openstax.org/r/quadratic-sequences)

Self Check



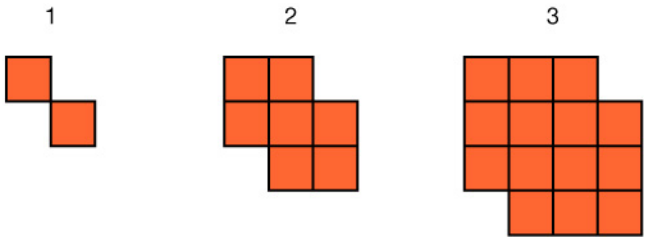
Which equation represents the relationship between the step number n and the number of squares y?

a. $y = 2n$
b. $y = n^2 + 1$
c. $y = n^2 + n$
d. $y = n + 1$

Additional Resources

More Complex Quadratic Patterns

See Steps 1 – 3 of a pattern of squares below. Write an equation representing the relationship between the step number n and the number of squares y.



First, try to find the pattern if the missing squares were actually there.

Make a table:

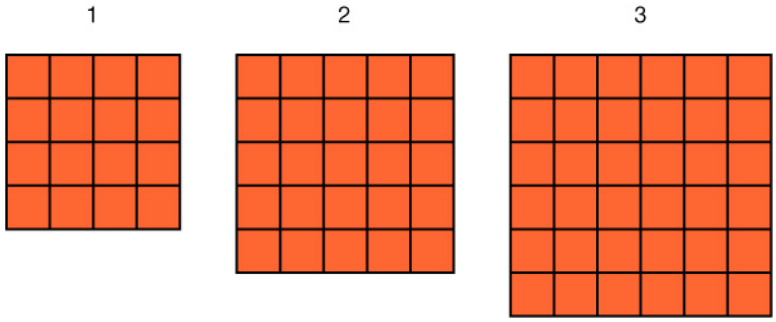
Step	# of columns	# of rows	# of squares
1	2	2	2
2	2	2	7
3	4	4	14
n	$n+1$	$n+1$	$(n+1)^2 - 2$

There would be $(n+1)(n+1)$ squares, but there are 2 missing from each step, so the equation would be:

$y = (n+1)^2 - 2$

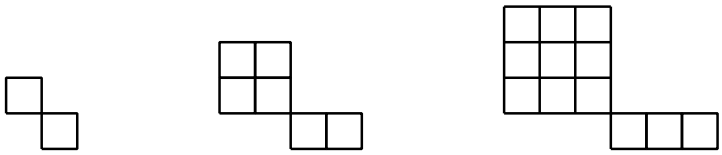
TRY IT **More Complex Quadratic Patterns**

See Steps 1 – 3 of a pattern of squares below. Write an equation representing the relationship between the step number n and the number of squares y.



7.3.4 Is the Pattern a Quadratic Function?

Cool Down

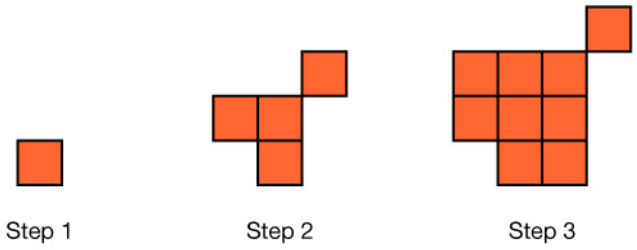


1. Write an equation to represent the relationship between the step number and the number of squares in the pattern. Briefly describe how each part of the equation relates to the pattern.
2. Is the relationship between the step number and the number of squares a quadratic function? Explain how you know.

7.3.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Sketch or describe the figure in Step 4 and Step 15.



a. How many small squares are in Step 4?

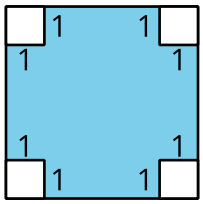
b. How many small squares are in Step 15?

c. Which equation represents the relationship between the step number, n, and the number of small squares, y, in each step?

a. $y = n^2 - n$
b. $y = n^2 - 1$
c. $y = n^2 + 1$
d. $y = n^2$

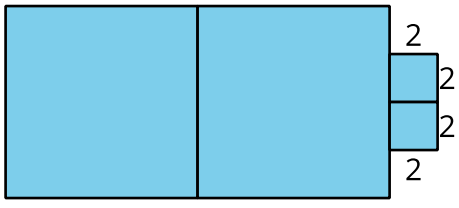
2. Each figure is composed of large squares and small squares. The side length of the large square is x. Choose the correct expression that represents the area, y, of the shaded part of each figure.

Figure A



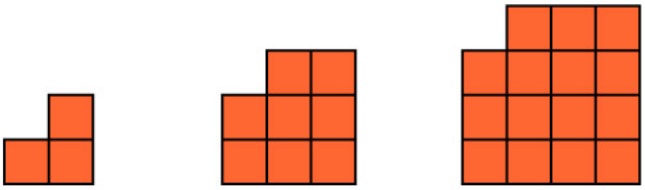
a. $y = x^2 - 8$
b. $y = x^2 - 4$
c. $y = x^2 - 1$
d. $y = x^2$

Figure B



a. $y = 2x^2$
b. $y = x^2 + 8$
c. $y = 2x^2 + 8$
d. $y = x^2$

3. Use the sequence below to answer the following two questions.



Step 1 Step 2 Step 3

a. How many squares are in Step 5?

b. Choose the correct expression that represents the area, y, of the shaded part of each figure for n steps.

a. $(n+1)^2-1$
b. $(n+1)^2$
c. n^2-1
d. $n-1$

7.3.6 Lesson Summary

In this lesson, you learned how to:

- Recognize quadratic functions written in different ways.
- Use information from a pattern of shapes to write a quadratic function.
- Know that, in a pattern of shapes, the step number is the input and the number of squares is the output.

Here are the **activities** that helped you reach those goals:

- 7.3.1: Quadratic Expressions and Area
  - In this activity, you represented the areas of shaded figures formed from squares with quadratic expressions.
- 7.3.2: Writing Equations for Patterns with Squares
  - In this activity, you looked at visual patterns to write quadratic functions that represent sequences.
  - 7.3.2: Self Check
  - 7.3.2: Additional Resources
- 7.3.3: Quadratic Sequences
  - In this activity, you explored more quadratic functions from visual sequences that involved quadratic expressions written in different forms.
  - 7.3.3: Self Check
  - 7.3.3: Additional Resources
- 7.3.4: Quadratic Function?
  - In this activity, you wrote a quadratic function from a pattern that did not have a constant.

After these activities, you completed the following **practice**:

- 7.3.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

7.4 COMPARING QUADRATIC AND EXPONENTIAL FUNCTIONS

7.4.0 Lesson Overview

In this lesson, you will investigate how quantities that grow quadratically compare to those that grow exponentially. You will use different mathematical models to show the relationship between the functions to determine which one grows faster.

When you finish this lesson, you will be able to:

- Explain using graphs, tables, or calculations how exponential functions eventually grow faster than quadratic functions.

Here are the **activities** that will help you reach those goals:

- 7.4.1: Comparing Expressions
- 7.4.2: Exploring Linear, Exponential, and Quadratic Growth
  - 7.4.2: Self Check
  - 7.4.2: Additional Resources
- 7.4.3: Comparing Exponential and Quadratic Functions
  - 7.4.3: Self Check
  - 7.4.3: Additional Resources
- 7.4.4: Comparing Exponential and Quadratic Expressions

After that, you'll **practice and review**.

- 7.4.5: Practice
- 7.4.6: Lesson Summary

7.4.1 Comparing Expressions

Warm Up

List these quantities in order, from least to greatest, without evaluating each expression. Be prepared to explain your reasoning.

2^{10}

10^2

2^9

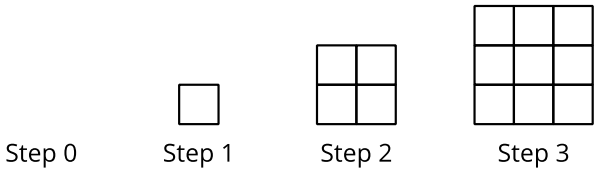
9^2

7.4.2 Exploring Linear, Exponential, and Quadratic Growth

In this activity, you will compare two different patterns. In each pattern, the number of small squares is a function of the step number, n.

1. In Pattern A, the length and width of the rectangle grow by 1 small square from each step to the next.

Pattern A



a. Write an equation to represent the number of small squares at Step n in Pattern A.

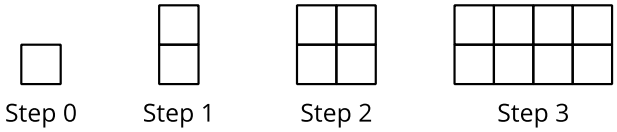
b. Is the function in Pattern A linear, quadratic, or exponential?

c. Complete the table for Pattern A.

Step Number n	Number of Small Squares $f(n)$
0	
1	
2	
3	
4	
5	
6	
7	
8	

2. In Pattern B, the number of small squares doubles from each step to the next.

Pattern B



a. Write an equation to represent the number of small squares at Step n in Pattern B.

b. Is the function in Pattern B linear, quadratic, or exponential?

c. Complete the table for Pattern B.

Step Number n	Number of Small Squares $g(n)$
0	
1	
2	
3	
4	
5	
6	
7	
8	

3. How would the two patterns compare if they continue to grow? Make one to two observations.

Self Check

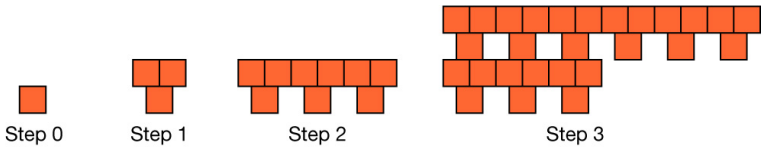
Which of the following functions has a constant growth factor?

a. $f(x) = x + 4$
b. $f(x) = 4^x$
c. $f(x) = 4x^2$
d. $f(x) = 4x$

Additional Resources

Determining Growth Factors

Is the pattern below represented by a function that is linear, exponential, or quadratic?



Step 1 - Draw a table. In the right column, determine the growth factor.

Step Number x	Number of Squares $f(x)$	Growth Factor
0	1	3
1	3	3
2	9	3
3	27	3

Step 2 - Write the function to represent the pattern.

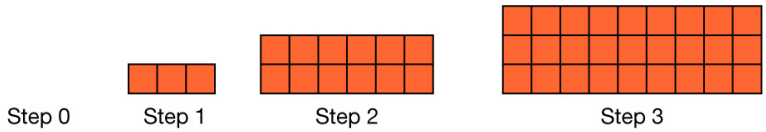
$f(x) = 3^x$

Step 3 - Determine what type of function this is.

The function is exponential.

TRY IT **Determining Growth Factors**

Is the pattern below represented by a function that is linear, exponential, or quadratic?



7.4.3 Comparing Exponential and Quadratic Functions

Activity

1. Here are two functions: $p(x) = 6x^2$ and $q(x) = 3^x$. Investigate the output of p and q for different values of x. For large enough values of x, one function will have a greater value than the other. Which function will have a greater value as x increases?
   Support your answer with tables, graphs, or other representations.

Are you ready for more?

Extending Your Thinking

1. Janna says that some exponential functions grow more slowly than the quadratic function as x increases. Do you agree with Janna? Be prepared to show your reasoning.
2. Could you have an exponential function $g(x) = b^x$ so that $g(x) < f(x)$ for all values of x?

Video: Comparing Exponential and Quadratic Functions

Watch the following video to learn more about functions.

Comparing Exponential and Quadratic Functions (https://openstax.org/r/exponential-vs-quadratic)

Self Check

Which of the following has the greatest value as x increases?

a. $y = (\frac{1}{4})^x$
b. $y = 4x$
c. $y = 4^x$
d. $y = 4 \cdot x^2$

Additional Resources

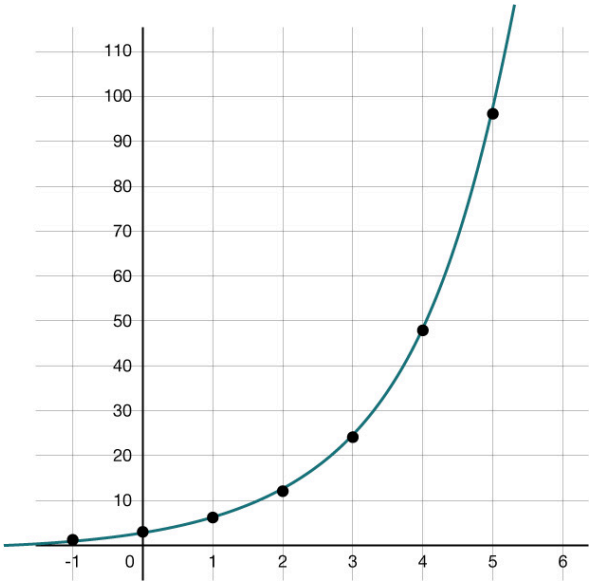
Exponential versus Quadratic Functions

For each of the two functions, do the following:

a. Complete the table of values.
b. Sketch a graph.
c. Decide whether each function is linear, quadratic, or exponential, and be prepared to explain how you know.

1. $f(x) = 3 \cdot 2^x$

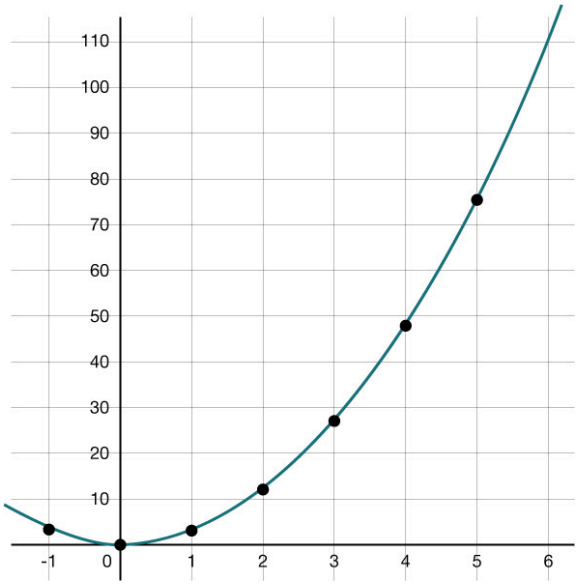
x	-1	0	1	2	3	5
$f(x)$	$\frac{3}{2}$ or equivalent	3	6	12	24	96



This function is exponential. It has a growth factor of 2.

2. $g(x) = 3 \cdot x^2$

x	-1	0	1	2	3	5
$g(x)$	3	0	3	12	27	75



This function is not linear because it does not have a constant rate of change, and it is not exponential because it does not have a constant growth factor. The function is quadratic because it can be represented by $g(x) = 3x^2$, which has x to

the second power, and that is the greatest power.

Note: Between $f(x)$ and $g(x)$, $f(x)$ will have the greater value as x increases.

TRY IT **Exponential versus Quadratic Functions**

Which of the following has the greater value as x increases, $f(x)$ or $g(x)$?

$f(x) = 3x^2$

$g(x) = 3^x$

7.4.4 Comparing Exponential and Quadratic Expressions

Cool Down

1. Tyler completes the table comparing values of the expressions $5x^2$ and 2^x.

x	$5x^2$	2^x
1	5	2
2	20	4
3	45	8
4	80	16

Tyler concludes that $5x^2$ will always result in greater values than 2^x for the same value of x. Do you agree? Explain or show your reasoning.

7.4.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

Use the following to answer questions 1 – 7.

The table shows values of the expressions $10x^2$ and 2^x.

x	$10x^2$	2^x
1	10	2
2	40	4
3	90	8
4	160	16
8		
10		
12		

1. Complete the statement.

a. When the value of x increases by 1, the expression $10x^2$:

a. Changes logarithmically. It has decreased by one-half.
b. Grows by decreasing amounts but by increasing growth factors.
c. Changes exponentially. It doubles.
d. Grows by increasing amounts but by decreasing growth factors.

b. When the value of x increases by 1, the expression 2^x:

a. Changes logarithmically. It has decreased by one-half.
b. Grows by decreasing amounts but by increasing growth factors.
c. Changes exponentially. It doubles.
d. Grows by increasing amounts but by decreasing growth factors.

2. Will the value of $10x^2$ be greater than or less than 2^x when x is 8?

a. greater than
b. less than

3. Will the value of 2^x be greater than or less than $10x^2$ when x is 10?

a. greater than
b. less than

4. Will the value of $10x^2$ be greater than or less than 2^x when x is 12?

a. greater than
b. less than

5. Determine the following values of $10x^2$.

a. What is the value of $10x^2$ when x is 8?

b. What is the value of $10x^2$ when x is 10?

c. What is the value of $10x^2$ when x is 12?

6. Determine the following values of 2^x.

a. What is the value of 2^x when x is 8?

b. What is the value of 2^x when x is 10?

c. What is the value of 2^x when x is 12?

7. Choose the best statement that describes how the values of the exponential expression 2^x and the quadratic expression $10x^2$ change as x becomes greater and greater.

a. There is no change in the values of the expressions when x increases.
b. They both grow at the same rate.
c. The value of the quadratic expression grows much more quickly as x increases.
d. The value of the exponential expression grows much more quickly as x increases.

8. Answer the following questions about functions $f(x)$ and $g(x)$.

a. What type of expression is $f(x) = 1.5^x$?

a. linear
b. quadratic
c. exponential

b. What type of expression is $g(x) = 500x^2 + 345x$?

a. linear
b. quadratic
c. exponential

c. The values of which function will eventually be greater for larger and larger values of x?

a. Function $f(x)$
b. Function $g(x)$

9. The table shows the values of 4^x and $100x^2$ for some values of x.

x	4^x	$100x^2$
1	4	100
2	16	400
3	64	900
4	256	1,600
5	1,024	2,500

Looking at the patterns in the table, choose the best answer to explain why, as the value of x increases, the values of the exponential expression 4^x will eventually overtake the values of the quadratic expression $100x^2$.

a. The value of $100x^2$ grows by a factor of x every time and the value of 4^x grows by different factors.
b. The value of $100x^2$ grows by different factors and the value of 4^x grows by a factor of x every time.
c. The value of $100x^2$ grows by a factor of 4 every time and the value of 4^x grows by different factors.
d. The value of $100x^2$ grows by different factors and the value of 4^x grows by a factor of 4 every time.

7.4.6 Lesson Summary

In this lesson, you learned how to:

- Explain using graphs, tables, or calculations that exponential functions eventually grow faster than quadratic functions.

Here are the **activities** that helped you reach those goals:

- 7.4.1: Comparing Expressions
  - In this activity, you compared exponential expressions and ordered them from least to greatest by reasoning about the base numbers and exponents.
- 7.4.2: Exploring Linear, Exponential, and Quadratic Growth
  - In this activity, you examined patterns and then determined expressions that represented the patterns. You then determined if each expression was linear, exponential, or quadratic.
  - 7.4.2: Self Check
  - 7.4.2: Additional Resources
- 7.4.3: Comparing Exponential and Quadratic Functions
  - In this activity, you compared an exponential function and a quadratic function through their tables of values and graphs. You then determined which function had a larger value as the x-values increased.
  - 7.4.3: Self Check
  - 7.4.3: Additional Resources
- 7.4.4: Comparing Exponential and Quadratic Expressions
  - In this activity, you determined the point at which a particular exponential function is always greater than a particular quadratic function.

After these activities, you completed the following **practice**:

- 7.4.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

7.5 BUILDING QUADRATIC FUNCTIONS TO DESCRIBE SITUATIONS, PART 1

7.5.0 Lesson Overview

In this lesson, you will construct quadratic functions to represent various situations, investigate the movement of free-falling objects, analyze the vertical distances that falling objects travel over time, and see that they can be described by quadratic functions.

When you finish this lesson, you will be able to:

- Explain the meaning of the terms in a quadratic expression that represents the height of a falling object.
- Use tables, graphs, and equations to represent the height of a falling object.

Here are the **activities** that will help you reach those goals:

- 7.5.1: Numerical Patterns
- 7.5.2: A Time-Distance Quadratic Model
  - 7.5.2: Self Check
  - 7.5.2: Additional Resources
- 7.5.3: Distance as a Quadratic Function of Elapsed Time
  - 7.5.3: Self Check
  - 7.5.3: Additional Resources
- 7.5.4: A Quadratic Expression that Represents the Relationship Between Distance and Time

After that, you'll **practice and review**.

- 7.5.5: Practice
- 7.5.6: Lesson Summary

7.5.1 Numerical Patterns

Warm Up

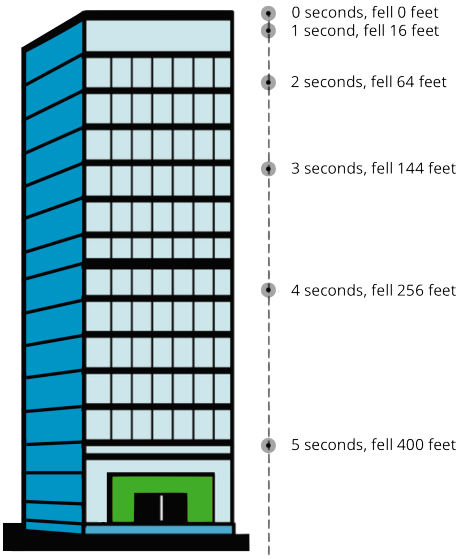
1. Study the table. What do you notice? What do you wonder?

x	0	1	2	3	4	5
y	0	16	64	144	256	400

7.5.2 A Time-Distance Quadratic Model

Activity

A rock is dropped from the top floor of a 500-foot-tall building. A camera captures the distance the rock traveled, in feet, after each second.



1. How many feet will the rock have fallen after 6 seconds? Be prepared to show your reasoning.
2. Jada noticed that the distances fallen are all multiples of 16. She wrote down: $16 = 16 \cdot 1$ $64 = 16 \cdot 4$ $144 = 16 \cdot 9$ $256 = 16 \cdot 16$ $400 = 16 \cdot 25$

Then, she noticed that 1, 4, 9, 16, and 25 are 1^2, 2^2, 3^2, 4^2, 5^2.

a. Use Jada's observations to predict the distance fallen in feet after 7 seconds. (Assume the building is tall enough that an object dropped from the top of it will continue falling for at least 7 seconds.) Be prepared to explain your reasoning.

b. Write an equation for the function, with d representing the distance dropped after t seconds.

Self Check

An object is dropped from a building and falls so that the distance in feet it falls, $d(t)$, is found by $d(t) = 16t^2$, after t seconds. How many feet has it fallen after 10 seconds?

a. 2,560
b. 1,600
c. 320
d. 160

Additional Resources

Finding Distance Fallen

The Empire State Building is 1250 ft high (excluding the lightning rod). A visitor drops a penny from the top, and it falls straight down. How far has it fallen after 5 seconds?

Recall the previous activity's formula $h(t) = 16t^2$.

Substitute $t = 5$.

$h(5) = 16(5)^2 = 16 \cdot 25 = 400$ feet.

TRY IT **Finding Distance Fallen**

Will the penny dropped from the Empire State Building reach the ground after 9 seconds? Be prepared to show your reasoning.

7.5.3 Distance as a Quadratic Function of Elapsed Time

Activity

Galileo Galilei, an Italian scientist, and other medieval scholars studied the motion of free-falling objects. The law they discovered can be expressed by the equation $d = 16t^2$, which gives the distance fallen in feet as a function of time, t, in seconds.

An object is dropped from a height of 576 feet.

1. How many feet does it fall in 0.5 seconds?

2. To find out where the object is the first few seconds after it was dropped, Elena and Diego created different tables.

Elena's table:

Time (Seconds)	Distance Fallen (Feet)
0	0
1	16
2	64
3	
4	
t	

Diego's table:

Time (Seconds)	Distance from the Ground (Feet)
0	576
1	560
2	512
3	
4	
t	

a. How are the two tables alike?

b. How are the two tables different?

3. Complete Elena's table. Be prepared to explain your reasoning.

a. What is the distance the object has fallen 3 seconds after it was dropped?

b. What is the distance the object has fallen 4 seconds after it was dropped?

c. What is the distance the object has fallen t seconds after it was dropped?

4. Complete Diego's table. Be prepared to explain your reasoning.

a. What is the object's distance from the ground 3 seconds after it was dropped?

b. What is the object's distance from the ground 4 seconds after it was dropped?

c. What is the object's distance from the ground t seconds after it was dropped?

Video: Distance as a Quadratic Function of Elapsed Time

Watch the following video to learn more about quadratic functions.

Distance as a Quadratic Function of Elapsed Time (https://openstax.org/r/quadratic-distance)

Self Check

A small ball is dropped from a 350-foot-tall building. Which equation could represent the ball's height, $h(t)$, in feet, relative to the ground, as a function of time, t, in seconds?

a. $h(t) = \frac{350}{16t^2}$
b. $h(t) = 350 + 16t^2$
c. $h(t) = 350 - 16t^2$
d. $h(t) = 350 - 16t$

Additional Resources

Height as a Quadratic Function

The general formula for a free falling object is:

$$h(t) = -16t^2 + v_0 t + h_0$$

Let's look at the meaning of the coefficients in the formula:

In the formula, -16 is the constant that has been determined by a combination of Newton's Laws and Earth's weight and is used with the units feet/second^2.

v_0 is the vertical speed of the object. If the object is simply dropped, the vertical speed is 0. If the object were thrown upward, this value would be positive, and if the object were thrown down, this value would be negative.

h_0 is the initial height of the object.

A man drops his keys from a restaurant at the top of a 196-foot-tall building. If the keys are free falling, will they reach the ground within 3 seconds?

Step 1 - Write an equation for the height at a given time, t.

$$h(t) = 196 - 16t^2$$

Step 2 - Substitute $t = 3$ into the function.

$$h(3) = 196 - 16(3)^2$$

Step 3 - Evaluate.

$$h(3) = 196 - 144 = 52$$

At $t = 3$ seconds, the keys will still have 52 feet to travel to hit the ground.

TRY IT **Height as a Quadratic Function**

Will the keys in the situation above reach the ground after 5 seconds?

7.5.4 A Quadratic Expression that Represents the Relationship between Distance and Time

Cool Down

The expression $16t^2$ represents the distance in feet an object falls after t seconds. Assume an object is dropped from a height of 906 feet.

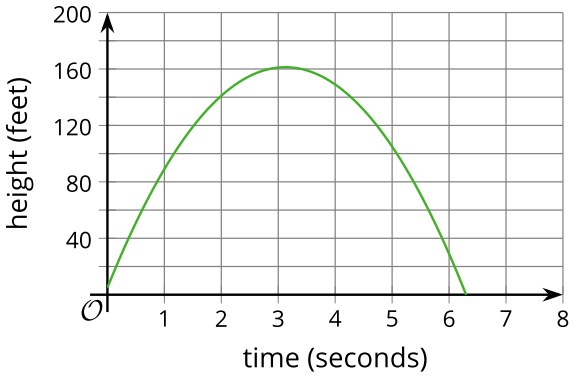
1. What is the height of the object in feet 2 seconds after it is dropped?
2. Write an expression representing the height of the object in feet t seconds after it is dropped.

7.5.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. A rocket is launched in the air and its height, in feet, is modeled by the function h.

Here is a graph representing h.



Select three true statements about the situation.

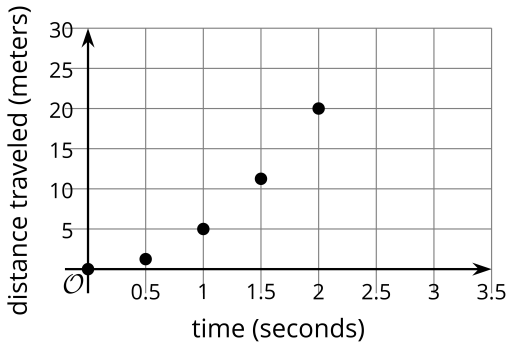
a. The rocket is launched from a height less than 20 feet above the ground.
b. The rocket is launched from about 20 feet above the ground.
c. The rocket reaches its maximum height after about 3 seconds.
d. The rocket reaches its maximum height after about 160 seconds.
e. The maximum height of the rocket is about 160 feet.

2. A baseball travels d meters t seconds after being dropped from the top of a building. The distance traveled by the baseball can be modeled by the equation $d = 5t2$.

Use this table to answer questions a – g.

t (seconds)	d (meters)
0	
0.5	
1	
1.5	
2	

a. How many meters (d) did the baseball travel in 0 seconds (t)?
b. How many meters (d) did the baseball travel in 0.5 seconds (t)?
c. How many meters (d) did the baseball travel in 1 second (t)?
d. How many meters (d) did the baseball travel in 1.5 seconds (t)?
e. How many meters (d) did the baseball travel in 2 seconds (t)?
f. Is this graph an accurate representation of the data in the table?



a. Yes
b. No

g. Is the baseball traveling at a constant speed?

a. Yes
b. No

3. A small ball is dropped from a 100-foot-tall building. Which equation could represent the ball's height, $h(t)$, in feet, relative to the ground, as a function of time, t, in seconds?

a. $h(t) = 100 - \frac{16}{t}$
b. $h(t) = 100 + 16t^2$
c. $h(t) = 100 - 16t^2$
d. $h(t) = 100 - 16t$

7.5.6 Lesson Summary

In this lesson, you learned how to:

- Explain the meaning of the terms in a quadratic expression that represents the height of a falling object.
- Use tables, graphs, and equations to represent the height of a falling object.

Here are the **activities** that helped you reach those goals:

- 7.5.1: Numerical Patterns
  - In this activity, you spent time examining a given table and discovered predictable patterns of the values. The big thing you learned is that the discovered patterns help determine the function rule.
- 7.5.2: A Time-Distance Quadratic Model
  - In this activity, you built a very simple quadratic model using given time-distance data of a free-falling rock. The big thing you learned is that the motion of a falling object is commonly modeled with a quadratic function.
  - 7.5.2: Self Check
  - 7.5.2: Additional Resources
- 7.5.3: Distance as a Quadratic Function of Elapsed Time
  - In this activity, you continued to explore how quadratic functions can model the movement of a falling object by evaluating the function seen earlier in the lesson. Then you built a new function to represent the distance from the ground of a falling object t seconds after it is dropped.
  - 7.5.3: Self Check
  - 7.5.3: Additional Resources
- 7.5.4: A Quadratic Expression that Represents the Relationship Between Distance and Time
  - In this activity, you wrote expressions that represent the distance in feet of an object when it falls after t seconds.

After these activities, you completed the following **practice**:

- 7.5.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

7.6 BUILDING QUADRATIC FUNCTIONS TO DESCRIBE SITUATIONS, PART 2

7.6.0 Lesson Overview

In this lesson, you will build on your knowledge of using simple quadratic functions to describe how an object falls over time, given the effect of constantly changing speed. You will construct quadratic functions to represent projectile motions, objects that are launched directly upward. Working through the activities, you will discover that gravity affects the speed of a projectile, and the linear model will need to be represented differently to adjust for the change.

When you finish this lesson, you will be able to:

- Create quadratic functions and graphs that represent a situation.
- Relate the vertex of a graph and the zeros of a function to a situation.
- Know that the domain of a function can depend on the situation it represents.

Here are the **activities** that will help you reach those goals:

- 7.6.1: Using Linear Functions to Describe Constant Speed
- 7.6.2: The Force of Gravity Change in Quadratic Functions
  - 7.6.2: Self Check
  - 7.6.2: Additional Resources
- 7.6.3: Using Quadratic Functions to Describe Height
  - 7.6.3: Self Check
  - 7.6.3: Additional Resources
- 7.6.4: Interpreting Graphs of Quadratic Functions

After that, you'll **practice and review.**

- 7.6.5: Practice
- 7.6.6: Lesson Summary

7.6.1 Using Linear Functions to Describe Constant Speed

Warm Up

A cannon is 10 feet off the ground. It launches a cannonball straight up with a velocity of 406 feet per second.

Imagine that there is no gravity and that the cannonball continues to travel upward with the same velocity.

1. Answer questions a–f using the table with the heights of the cannonball at different times.

Seconds	Distance above ground (feet)
0	10
1	a. _____
2	b. _____
3	c. _____
4	d. _____
5	e. _____
t	f. _____

a. How far above the ground is the cannonball at 1 second?

b. How far above the ground is the cannonball at 2 seconds?

c. How far above the ground is the cannonball at 3 seconds?

d. How far above the ground is the cannonball at 4 seconds?

e. How far above the ground is the cannonball at 5 seconds?

f. How far above the ground is the cannonball at t seconds?

2. Write an equation to model the distance in feet, d, of the ball t seconds after it was fired from the cannon, if there was no gravity.

7.6.2 The Force of Gravity Change in Quadratic Functions

Activity

Earlier, in activity 7.6.1, you completed a table that represents the height of a cannonball, in feet, as a function of time, in seconds, if there was no gravity.

This table shows the actual heights of the ball at different times.

Seconds	0	1	2	3	4	5
Distance above ground (feet)	10	400	758	1084	1378	1640

1. Compare the values in this table with those in the table you completed earlier. What are two similarities or differences that you notice?

2. Use the graphing tool or technology outside the course.

Access multimedia content (http://openstax.org/books/algebra-1/pages/7-6-2-the-force-of-gravity-change-in-quadratic-functions)

a. Plot the two sets of data you have on the same coordinate plane.

b. How are the two graphs alike? How are they different?

3. What is the starting position for both graphs?

4. What was the constant rate of change for the cannonball without gravity?

The constant rate of change is also the **initial velocity**. Recall that the effect of gravity is represented by $16t^2$. Gravity causes the bend in the curve for the second table and graph because it pulls the cannonball down toward the ground.

5. With a partner, write an equation to model the actual distance d, in feet, of the ball t seconds after it was fired from the cannon. Consider gravity and how the points changed on the graph.

Self Check

An arrow is shot up into the air from 5 feet, at an initial velocity of 220 feet per second. Which equation can represent the height, $h(t)$, for any time, t?

a. $h(t) = 220 + 5t - 16t^2$
b. $h(t) = 5 + 220t - 16t^2$
c. $h(t) = 5 + 220t + 16t^2$
d. $h(t) = 5 + 220t$

Additional Resources

The Effect of Gravity on Motion

A firework is sent up in the air, at an initial velocity of 130 feet per second, from a height of 6 feet. Write an equation that

represents the height, $h(t)$, of the firework for any second, t. Remember to consider the effect of gravity.

Step 1 - Identify the initial velocity, v_0.

130 ft per second

Step 2 - Identify the starting height, h_0.

6

Step 3 - Write an equation of this form: $h(t) = h_0 + v_0 t - 16t^2$.

$h(t) = 6 + 130t - 16t^2$

TRY IT **The Effect of Gravity on Motion**

Write an equation if a firework is shot from 2 feet off the ground at an initial velocity of 110 feet per second. Write an equation that represents the height, $h(t)$, of the firework for any second, t. Remember to consider the effect of gravity.

7.6.3 Using Quadratic Functions to Describe Height

Activity

1. The function defined by $d = 50 + 312t - 16t^2$ gives the height in feet of a cannonball t seconds after the ball leaves the cannon.

a. What does the term 50 tell us about the cannonball?

b. What does the term $312t$ tell us about the cannonball?

c. What does $-16t^2$ tell us about the cannonball?

2. Use the graphing tool or technology outside the course. Graph the function. Adjust the graphing window to the following boundaries: $0 < x < 25$ and $0 < y < 2000$.

Access multimedia content (http://openstax.org/books/algebra-1/pages/7-6-3-using-quadratic-functions-to-describe-height)

3. Observe the graph and:

a. Describe the shape of the graph. What does it tell us about the movement of the cannonball?

The shape of the quadratic function is called a **parabola**. It has a "U" shape.

b. Estimate the maximum height the ball reaches. When does this happen?

The peak of the parabola, where the maximum occurs, is also called the **vertex** of the graph.

c. Estimate when the ball hits the ground.

The **zero** of a function is where the function equals 0, or where it crosses the x-axis. In this case, there is only one zero where the cannonball hits the ground. The parabola does cross the x-axis in the negative x-values, but that does not make sense in this problem.

4. What domain is appropriate for this function? Be prepared to show your reasoning.

Are you ready for more?

Extending Your Thinking

1. If the cannonball were fired at 800 feet per second, would it reach a mile in height? Remember there are 5280 feet in a mile. Be prepared to show your reasoning.

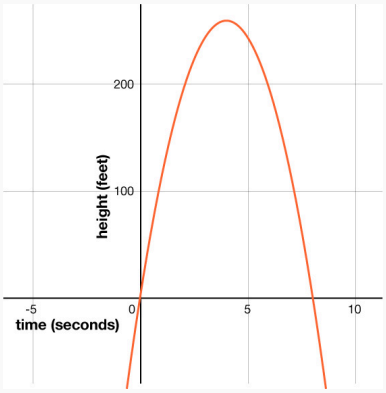
Video: Using Quadratic Functions to Describe Height

Watch the following video to learn more about quadratic functions.

7.6.3 (https://openstax.org/r/more-quadratic-functions)

Self Check

A toy rocket is launched from a platform 4 feet above the ground at a velocity of 128 feet per second. Its height is graphed below. Approximately what is the rocket's maximum height?

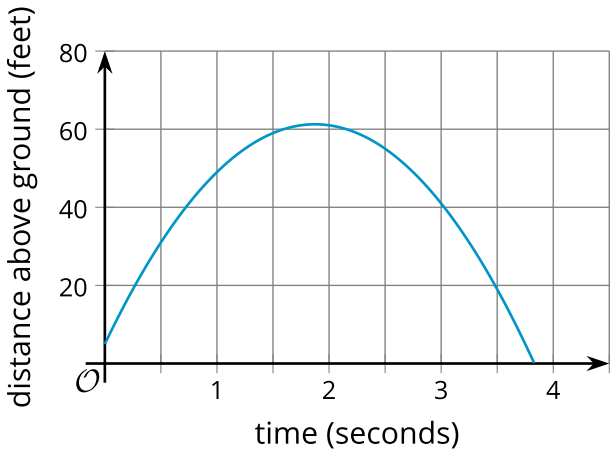


a. 128
b. 110
c. 260
d. 8

Additional Resources

Reading Graphs of Quadratics

An object is thrown upward from a height of 5 feet with a velocity of 60 feet per second. Its height, $h(t)$, in feet after t seconds is modeled by the function $h(t) = 5 + 60t - 16t^2$.



Step 1 -Identify the starting height.

5 feet

Step 2 -Identify the maximum height by looking at the vertex or peak of the parabola.

approximately 62 feet

Step 3 -Identify when the object reaches its maximum by matching the maximum with its matching time from the x-axis.

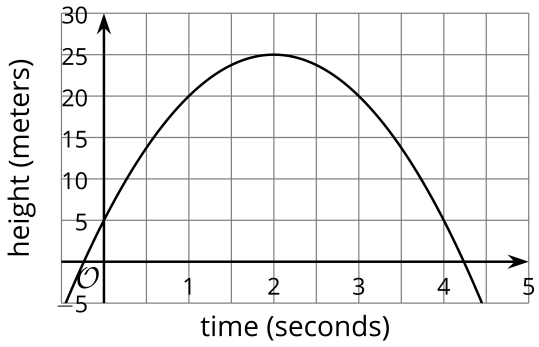
approximately 1.75 seconds

Step 4 -When does the object hit the ground?

Look at where the object has a height of 0 feet, or the zero of the function. This happens at approximately 3.75 seconds. Notice there would be another zero on the graph of the function to the left of $x = 0$, but that does not make sense in the problem because the object was thrown from 5 feet.

TRY IT **Reading Graphs of Quadratics**

A rock is launched into the air, and its height is represented by the graph below.

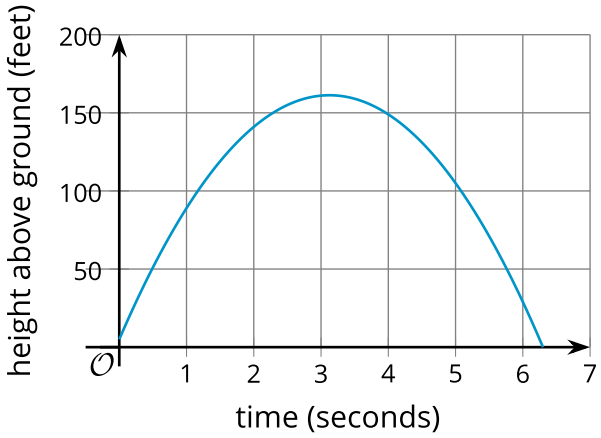


What is the maximum height of the rock?

7.6.4 Interpreting Graphs of Quadratic Functions

Cool Down

The height in feet, h, of a stomp rocket (propelled by a short blast of air) above the ground after t seconds is given by the equation $h(t) = 5 + 100t - 16t^2$. Here is a graph that represents h.



1. How does the 5 in the equation relate to the graph?
2. What does $100t$ in the equation mean in terms of the rocket?
3. What does the $-16t^2$ mean in terms of the rocket?
4. About when does the rocket hit the ground?

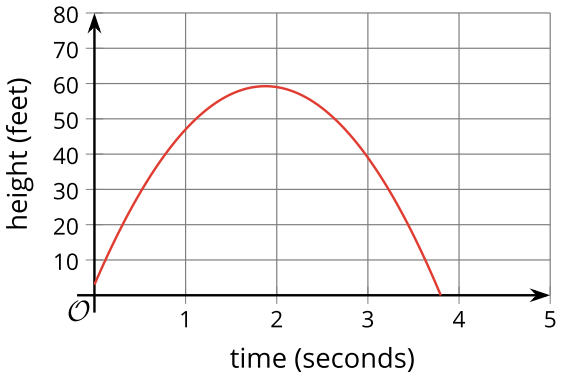
7.6.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. The height of a diver above the water is given by $h(t) = -5t^2 + 10t + 3$, where t is time measured in seconds and $h(t)$ is measured in meters. **Select two** statements that are true about the situation.

a. The diver begins 5 meters above the water.
b. The diver begins 3 meters above the water.
c. The function has 1 zero that makes sense in this situation.
d. The function has 2 zeros that make sense in this situation.
e. The graph that represents h starts at the origin and curves upward.
f. The diver begins at the same height as the water level.

2. The height of a baseball, in feet, is modeled by the function h given by the equation $h(t) = 3 + 60t - 16t^2$. The graph of the function is shown.



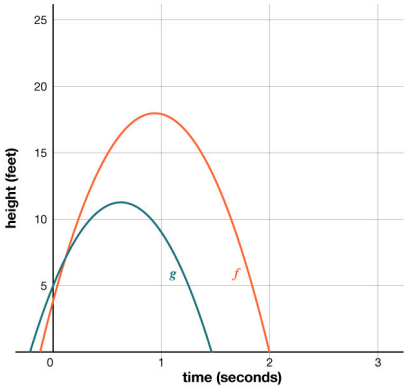
a. About when does the baseball reach its maximum height? Round to the nearest second.

b. About how high is the maximum height of the baseball? Round to the nearest ten.

c. When does the baseball hit the ground? Round to the nearest second.

3. Two rocks are launched straight up in the air. The height of Rock A is given by the function f, where $f(t) = 4 + 30t - 16t^2$. The height of Rock B is given by g, where $g(t) = 5 + 20t - 16t^2$. In both functions, t is time measured in seconds and height is measured in feet.

The following is a graph of the functions. Use the graph to answer the questions a – c.



a. Which graph is the height of Rock A, $f(t) = 4 + 30t - 16t^2$?

a. Green Graph
b. Red Graph

b. Which graph is the height of Rock B, $g(t) = 5 + 20t - 16t^2$?

a. Green Graph
b. Red Graph

c. Determine which rock hits the ground first.

a. Rock A
b. Rock B

4. Each expression represents an object's distance from the ground, in meters, as a function of time, t, in seconds.

- Object A: $-5t^2 + 25t + 50$
- Object B: $-5t^2 + 50t + 25$

a. Which object was launched with the greater vertical speed?

a. Object A
b. Object B

b. Which object was launched from the greater height?

a. Object A
b. Object B

7.6.6 Lesson Summary

In this lesson, you learned how to:

- Create quadratic functions and graphs that represent a situation.
- Relate the vertex of a graph and the zeros of a function to a situation.
- Know that the domain of a function can depend on the situation it represents.

Here are the **activities** that helped you reach those goals:

- 7.6.1: Using Linear Functions to Describe Constant Speed
  - In this activity, you considered what happens if an object is launched up in the air unaffected by gravity. It reinforced your knowledge that an object that travels at a constant speed can be described with a linear function.
- 7.6.2: The Force of Gravity Change in Quadratic Functions
  - In this activity, you learned a model that accounts for the fact that an object that is launched straight up at a constant speed does not keep going at the same rate when the influence of gravity is taken into account. The big lesson was that adding a quadratic term to a linear function has an effect of "bending" the graph, as the output values are no longer changing at a constant rate.
  - 7.6.2: Self Check
  - 7.6.2: Additional Resources
- 7.6.3: Using Quadratic Functions to Describe Height
  - In this activity, you explored another model of a projectile motion. You graphed and interpreted a quadratic function in context and began considering a reasonable domain for the function.
  - 7.6.3: Self Check
  - 7.6.3: Additional Resources
- 7.6.4: Interpreting Graphs of Quadratic Functions
  - In this activity, you examined a quadratic function and its graph and identified what each term of the equation meant in the context of the situation graphed.

After these activities, you completed the following **practice:**

- 7.6.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

7.7 DOMAIN, RANGE, VERTEX, AND ZEROS OF QUADRATIC FUNCTIONS

7.7.0 Lesson Overview

In this lesson, you will encounter a quadratic relationship in an economic context. You will use a table of values to study the relationship between price and revenue, write an equation to define the function, and consider what the graph of the function tells us about the situation being modeled.

When you finish this lesson, you will be able to:

- Choose a domain that makes sense in a revenue situation.
- Model revenue with quadratic functions and graphs.
- Relate the vertex of a graph and the zeros of a function to a revenue situation.

Here are the **activities** that will help you reach those goals:

- 7.7.1: Comparing Graphs of Functions
- 7.7.2: Modeling Real-World Data with Quadratic Functions
  - 7.7.2: Self Check
  - 7.7.2: Additional Resources
- 7.7.3: The Domain, Vertex, and Zero of Quadratic Functions
  - 7.7.3: Self Check
  - 7.7.3: Additional Resources
- 7.7.4: Relate the Domain of a Function to Its Graph

After that, you'll **practice and review**.

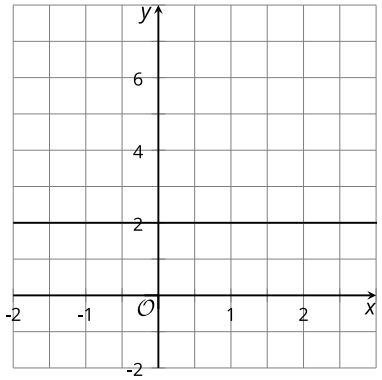
- 7.7.5: Practice
- 7.7.6: Lesson Summary

7.7.1 Comparing Graphs of Functions

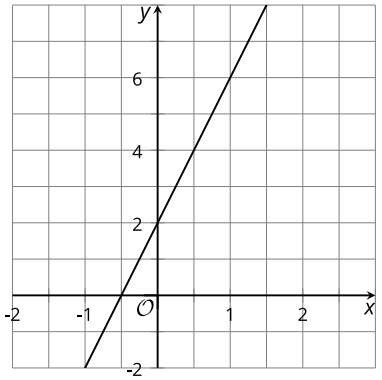
Warm Up

1. Which one doesn't belong?

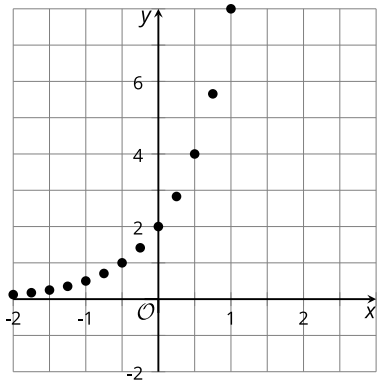
Graph A



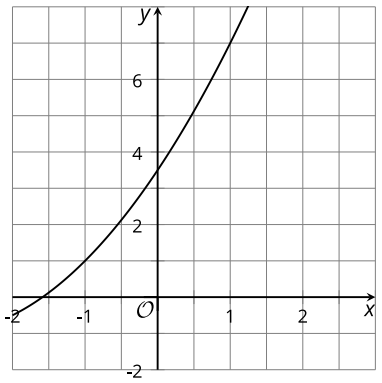
Graph B



Graph C



Graph D



7.7.2 Modeling Real-World Data with Quadratic Functions

Activity

For questions 1 – 6, use the following information to complete the table. Show the predicted number of downloads at each listed price. The first row has been completed for you.

A company that sells movies online is deciding how much to charge customers to download a new movie. Based on data from previous sales, the company predicts that if they charge x dollars for each download, then the number of downloads, in thousands, is $18 - x$.

Price (dollars per download)	Number of downloads (thousands)	Revenue (thousands of dollars)
3	15	45
5	1. ____	

Price (dollars per download)	Number of downloads (thousands)	Revenue (thousands of dollars)
10	1. ____	
12	1. ____	
15	1. ____	
18	1. ____	
x	1. ____	

1. When the price is $\$5$ per download, what is the number of downloads in thousands?

2. When the price is $\$10$ per download, what is the number of downloads in thousands?

3. When the price is $\$12$ per download, what is the number of downloads in thousands?

4. When the price is $\$15$ per download, what is the number of downloads in thousands?

5. When the price is $\$18$ per download, what is the number of downloads in thousands?

6. When the price is $\$x$ per download, what is the number of downloads in thousands?

For questions 7 – 12 complete the table by finding the **revenue** at each price. The first row has been completed for you.

Price (dollars per download)	Number of downloads (thousands)	Revenue (thousands of dollars)
3	15	45
5		1. ____
10		1. ____
12		1. ____
15		10. ____
18		11. ____
x		12. ____

7. When the price is $\$5$ per download, what is the revenue in thousands of dollars?

8. When the price is $\$10$ per download, what is the revenue in thousands of dollars?

9. When the price is $\$12$ per download, what is the revenue in thousands of dollars?

10. When the price is $\$15$ per download, what is the revenue in thousands of dollars?

11. When the price is $\$18$ per download, what is the revenue in thousands of dollars?

12. When the price is $\$x$ per download, what is the revenue in thousands of dollars?

13. Is the relationship between the price of the movie and the revenue (in thousands of dollars) linear, quadratic, or exponential? Explain how you know.
14. Use the graphing tool or technology outside the course. Plot the points that represent the revenue, r, as a function of the price of one download in dollars, x, using the Desmos tool below.

Access multimedia content (http://openstax.org/books/algebra-1/pages/7-7-2-modeling-real-world-data-with-quadratic-functions)

15. What price would you recommend the company charge for a new movie? Be prepared to show your reasoning.

Are you ready for more?

Extending Your Thinking

1. The function that uses the price (in dollars per download) x to determine the number of downloads (in thousands) $18 - x$ is an example of a demand function, and its graph is known. Economists are interested in factors that can affect the demand function and therefore the price suppliers wish to set.
   What are some things that could increase the number of downloads predicted for the same given prices?
2. If the demand shifted so that we predicted $20 - x$ thousand downloads at a price of x dollars per download, what do you think would happen to the price that gives the maximum revenue? Check what actually happens.

Self Check

A company that sells music downloads is deciding how much to charge customers to download an album. Based on previous sales, the company predicts that if they charge x dollars for each download, then the number of downloads, in thousands, is $13 - x$.

Which of the following represents the revenue for downloads?

a. $13 - x^2$
b. $13x - x^2$
c. $\frac{13-x}{x}$
d. $13 - x$

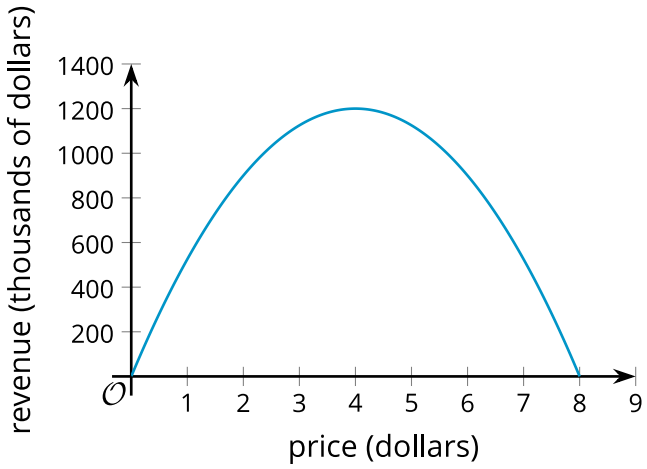
Additional Resources

Revenue Represented with Quadratics

Quadratic functions often come up when studying revenue. (Revenue means the money collected when someone sells something.)

Suppose we are selling raffle tickets and deciding how much to charge for each ticket. When the price of the tickets is higher, typically fewer tickets will be sold.

Each ticket has a price of d dollars, and it is possible to sell $600 - 75d$ tickets. We can find the revenue by multiplying the price by the number of tickets expected to be sold. A function that models the revenue, r, collected is $r(d) = d(600 - 75d)$. Here is a graph that represents the function.



It makes sense that the revenue goes down after a certain point, since if the price is too high nobody will buy a ticket. From the graph, we can tell that the greatest revenue, $\$1,200$, comes from selling the tickets for $\$4$ each. So we know, the range of this parabola reaches a maximum value of $1,200.

We can also see that the domain of the function r is between 0 and 8. This makes sense because the cost of the tickets can't be negative, and if the price were more than $\$8$, the model would not work, as the revenue collected cannot be negative. (A negative revenue would mean the number of tickets sold is negative, which is not possible.)

TRY IT **Revenue Represented with Quadratics**

Based on past concerts, a venue expects to sell $(500 - 10p)$ tickets when each ticket is p dollars.

Write the function, $r(p)$, to represent the revenue for the concert.

7.7.3 The Domain, Vertex, and Zero of Quadratic Functions

Activity

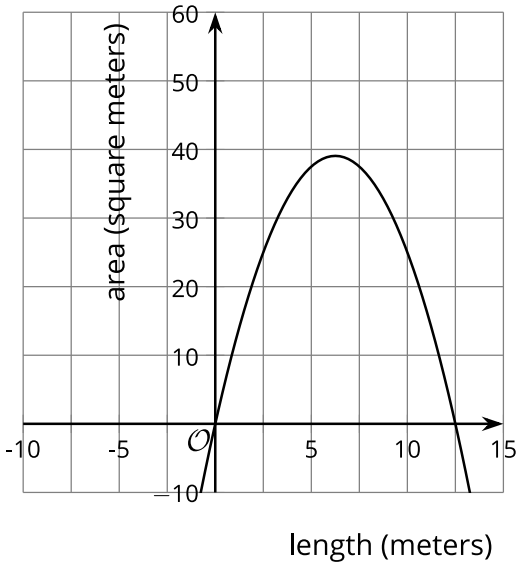
Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

Here are three sets of descriptions and equations that represent some familiar quadratic functions. The graphs show what a graphing technology may produce when the equations are graphed. For a more accurate answer, use the graphing tool or technology outside the course. Graph the equation that represents this scenario using the Desmos tool below. (Students had access to Desmos.)

- Describe a **domain** that is appropriate for the situation. Think about any upper or lower limits for the input, as well as whether all numbers make sense as the input. Write the domain as an inequality. Then, describe how the graph should be modified to show the domain that makes sense.
- Describe a **range** that is appropriate for the situation. Think about any upper or lower limits for the input, as well as whether all numbers make sense as the input. Write the range as an inequality. What does the range mean in this situation? Then, describe how the graph should be modified to show the range that makes sense.
- Identify or estimate the The **vertex (of a graph)**. Describe what it means in the situation.
- Identify or estimate the **zero of the function**. Describe what they mean in the situation.

Use the following information to answer questions 1 – 4.

The area of a rectangle with a perimeter of 25 meters and a side length x: $A(x) = x \cdot \frac{(25-2x)}{2}$



1. What is an inequality that represents the domain?

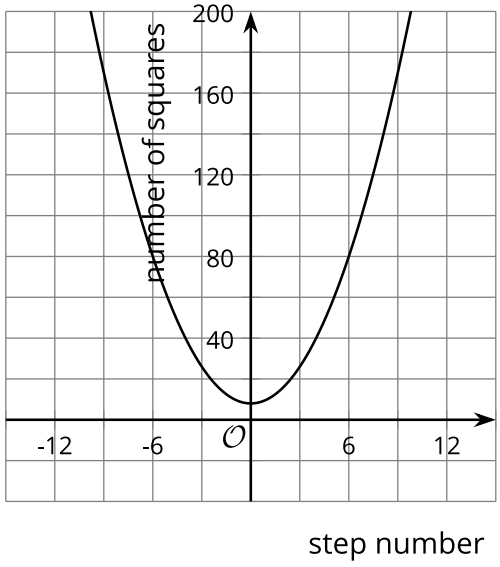
2. What is an inequality that represents the range?

3. What is the vertex of the graph?

4. What are the zeros of the graph?

Use the following information to answer questions 5 – 8.

The number of squares as a function of step number n: $f(n) = n^2 + 4$



5. What is an inequality that represents the domain?

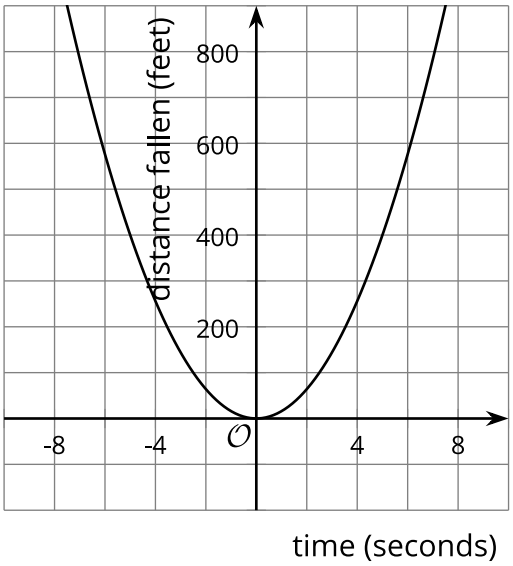
6. What is an inequality that represents the range?

7. What is the vertex of the graph?

8. What are the zeros of the graph?

Use the following information to answer questions 9 - 12.

The distance in feet that an object has fallen t seconds after being dropped: $g(t) = 16t^2$



9. What is an inequality that represents the domain?

10. What is an inequality that represents the range?

11. What is the vertex of the graph?

12. What are the zeros of the graph?

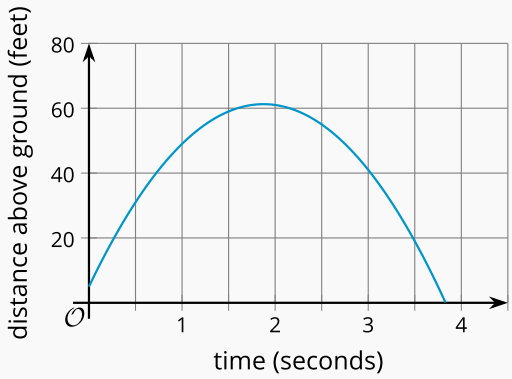
Video: Identifying Domain, Vertex, and Zero of Quadratic Functions

Watch the following video to learn more about the domain, vertex, and zero of quadratic functions.

Identifying Domain, Vertex, and Zero of Quadratic Functions (https://openstax.org/r/quadratic-properties)

Self Check

Find the approximate domain of the graph below:

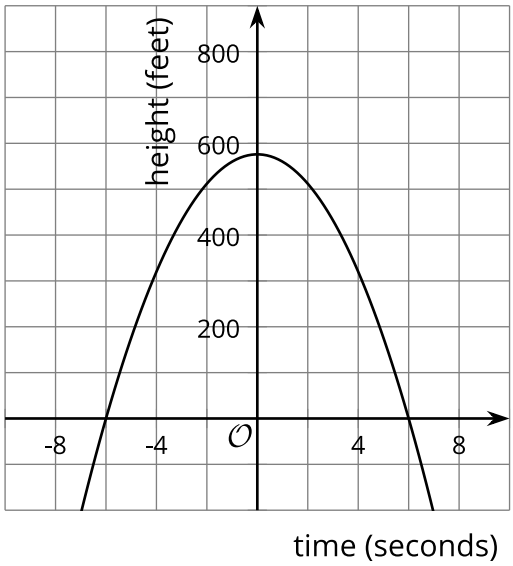


a. $x \geq 0$
b. $0 \leq x \leq 3.8$
c. $0 \leq x \leq 60$
d. $0 \leq x \leq 1.8$

Additional Resources

The Meaning of Quadratic Characteristics

The height in feet of an object t seconds after being dropped is graphed below: $h(t) = 576 - 16t^2$.



Find the domain of the function.

- The **domain** is the t values, $0 \le t \le 6$ or only numbers from 0 up through 6. Negative values of time don't make sense here, so the part of the graph to the left side of the vertical axis has no meaning. The object hits the ground 6 seconds after being dropped, so values greater than 6 are not meaningful.

Find the range of the function.

- The **range** of the height of the function is from the initial height down to the ground. In this case, since the object is being dropped from a height of 576 feet and falls to the ground, the range of the function is [0, 576]. This means that the object's height will be between 0 feet (when it hits the ground) and 576 feet (when it is initially dropped).

Find the vertex and describe its meaning.

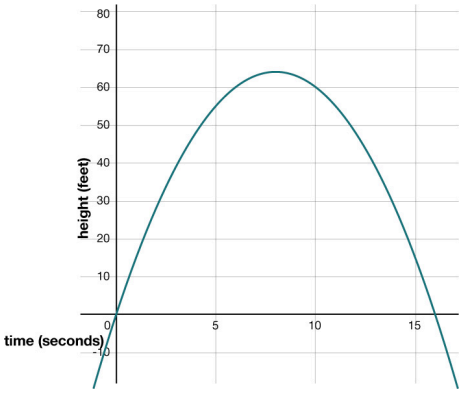
- The **vertex (of a graph)** is the peak of the parabola, or the maximum, at the point $(0, 576)$. It tells us the time when the height is the greatest.

Find the zeros and describe their meaning.

- The **zero of the function** is 6. It tells us that the height of the object is 0 feet at 6 seconds after being dropped, or the time when the object hits the ground.

TRY IT **The Meaning of Quadratic Characteristics**

For questions 1 – 2, use the following graph.



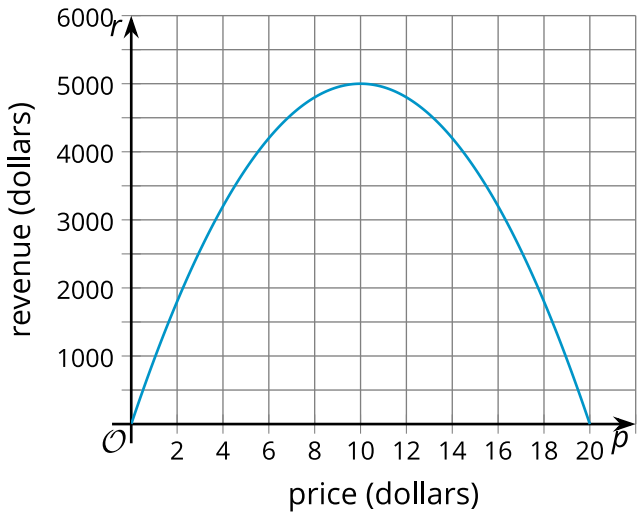
Describe the zeros of graph.

Describe what the zeros mean for the graph.

7.7.4 Relate the Domain of a Function to Its Graph

Cool Down

This graph represents the revenue in dollars that a company expects if they sell their product for p dollars.



1. Based on this model, which price would generate more revenue for the company, $\$5$ or $\$17$?
2. At what price (in dollars) should the company sell their product if they wish to make as much revenue as possible?
3. How much revenue (in dollars) would they make?
4. What is an appropriate domain for the function? Explain how you know.

7.7.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

Use the following scenario to answer questions 1 – 5.

Based on past musical productions, a theater predicts selling $400 - 8p$ tickets when each ticket is sold at p dollars.

1. Answer questions a–g to complete the middle column in the table to find out how many tickets the theater expects to sell and what revenues it expects to receive at the given ticket prices.

Ticket price (dollars)	Number of tickets sold	Revenue (dollars)
5	360	1800
10	a. ____	
15	b. ____	
20	c. ____	
30	d. ____	
45	e. ____	
50	f. ____	
p	g. ____	

a. When the price is $\$10$ per ticket, what is the number of tickets sold?

b. When the price is $\$15$ per ticket, what is the number of tickets sold?

c. When the price is $\$20$ per ticket, what is the number of tickets sold?

d. When the price is $\$30$ per ticket, what is the number of tickets sold?

e. When the price is $\$45$ per ticket, what is the number of tickets sold?

f. When the price is $\$50$ per ticket, what is the number of tickets sold?

g. When the price is p per ticket, what is the number of tickets sold?

a. $p(8p - 400)$
b. $p(400 - 8p)$
c. $400 - 8p$
d. $392p$

2. Answer questions a-g to complete the right column in the table to find out what revenues the theater expects to receive at the given ticket prices.

Ticket price (dollars)	Number of tickets sold	Revenue (dollars)
5	360	1800
10		a. ____
15		b. ____
20		c. ____
30		d. ____
45		e. ____
50		f. ____
p		g. ____

a. When the price is $\$10$ per ticket, what is the revenue in dollars?

b. When the price is $\$15$ per ticket, what is the revenue in dollars?

c. When the price is $\$20$ per ticket, what is the revenue in dollars?

d. When the price is $\$30$ per ticket, what is the revenue in dollars?

e. When the price is $\$45$ per ticket, what is the revenue in dollars?

f. When the price is $\$50$ per ticket, what is the revenue in dollars?

g. When the price is p per ticket, what is the revenue in dollars?

a. $392p$
b. $p(400 - 8p)$
c. $400 - 8p$
d. $p(8p - 400)$

3. For which ticket price, listed in the table, will the theater earn no revenue?

4. Calculate the lowest price of a ticket that the theater should sell the tickets for if it must earn at least $\$3,200$ in revenue to break even (to not lose money) on the musical production. (The calculated value may not be listed in the table.)

5. Calculate the highest price of a ticket that the theater should sell the tickets for if it must earn at least $\$3,200$ in revenue to break even (to not lose money) on the musical production. (The calculated value may not be listed in the table.)

6. A company sells running shoes. If the price of a pair of shoes in dollars is p, the company estimates that it will sell $50,000 - 400p$ pairs of shoes.

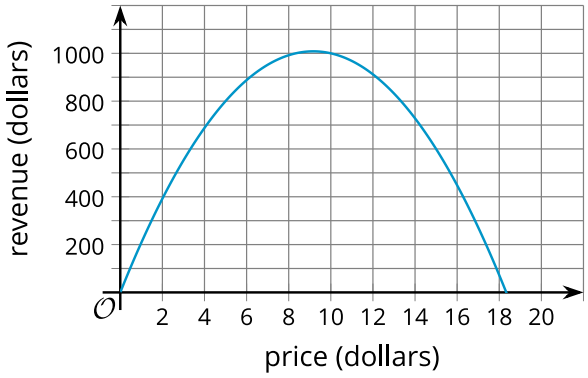
Which expression represents the revenue in dollars from selling running shoes if a pair of shoes is priced at p dollars?

a. $(49,600p) \cdot p$
b. $(50,000p - 400p)$
c. $(50,000p - 400) \cdot p$
d. $(50,000 - 400p) \cdot p$

Use the following scenario to answer questions 7 – 10.

The function f represents the revenue in dollars the school can expect to receive if it sells $220 - 12x$ coffee mugs for x dollars each.

Here is the graph of f.



7. Select **three** statements that describe this situation.

a. At $2 per coffee mug, the revenue will be $196.
b. The school expects to sell 160 mugs if the price is $5 each.
c. The school will lose money if it sells the mugs for more than $10 each.
d. The school will earn about $1,000 if it sells the mugs for $10 each.
e. The revenue will be more than $700 if the price is between $5 and $14.
f. The expected revenue will increase if the price per mug is greater than $10.

8. To the nearest hundred dollars, what is the maximum revenue for the mugs?

9. Which is the approximate domain?

a. $x \geq 0$
b. $0 \leq x \leq 1000$
c. $0 \leq x \leq 18$
d. $0 \leq x \leq 9$

10. Which is the approximate range?

a. $-\infty < y \leq 1000$
b. $y \leq 1000$
c. $0 \leq y \leq 1000$
d. $0 \leq y < \infty$

7.7.6 Lesson Summary

In this lesson, you learned how to:

- Choose a domain that makes sense in a revenue situation.
- Model revenue with quadratic functions and graphs.
- Relate the vertex of a graph and the zeros of a function to a revenue situation.

Here are the **activities** that helped you reach those goals:

- 7.7.1: Comparing Graphs of Functions
  - In this activity, you analyzed and compared the features of the graphs of four different functions.
- 7.7.2: Modeling Real-World Data with Quadratic Functions
  - In this activity, you analyzed the relationship between the price of downloading a movie and the number of downloads. You modeled and graphed the relationship as a quadratic function that showed the revenue as a function of the price of the download. You used this function, referred to as the demand function, to predict profit price points.
  - 7.7.2: Self Check
  - 7.7.2: Additional Resources
- 7.7.3: The Domain, Vertex, and Zero of Quadratic Functions
  - In this activity, you looked at appropriate domains for quadratic functions given the situations they represented. You identified and estimated the vertex of the graph and the zeros of the functions. You used modeling and graphing to interpret upper or lower limits for the input, as well as whether all numbers made sense.
  - 7.7.3: Self Check
  - 7.7.3: Additional Resources
- 7.7.4: Relate the Domain of a Function to Its Graph
  - In this activity, you analyzed a graph that represented the revenue that a company would expect to see if they sell their product at a certain price. You used this information to determine an appropriate domain for the function of the graph.

After these activities, you completed the following **practice**:

- 7.7.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

7.8 EQUIVALENT QUADRATIC EXPRESSIONS

7.8.0 Lesson Overview

In this lesson, you will use familiar tools like tables and graphs to represent quadratic functions. These tools will help you discover that finding the sum of the partial areas in a rectangle is the same as applying the distributive property to multiply out the terms in each factor.

When you finish this lesson, you will be able to:

- Rewrite quadratic expressions in different forms by using an area diagram or the distributive property.

Here are the **activities** that will help you reach those goals:

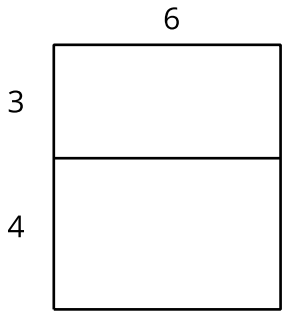
- 7.8.1: Area Diagrams
- 7.8.2: Using the Distributive Property to Write Equivalent Expressions
  - 7.8.2: Self Check
  - 7.8.2: Additional Resources
- 7.8.3: Using Diagrams to Find Equivalent Quadratic Expressions
  - 7.8.3: Self Check
  - 7.8.2: Additional Resources
- 7.8.4: Writing Equivalent Expressions

After that, you'll **practice and review.**

- 7.8.5: Practice
- 7.8.6: Lesson Summary

7.8.1 Area Diagrams

Warm Up



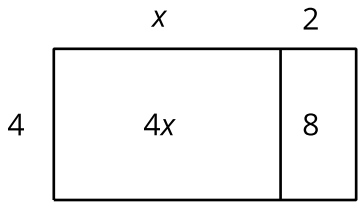
1. Explain why the diagram shows that $6(3+4) = 6 \cdot 3 + 6 \cdot 4$.

2. Draw a diagram to show that $5(x+2) = 5x + 10$.

7.8.2 Using the Distributive Property to Write Equivalent Expressions

Activity

Applying the distributive property to multiply out the factors of, or expand, $4(x+2)$ gives us $4x+8$, so we know the two expressions are equivalent. We can use a rectangle with side lengths $(x+2)$ and 4 to illustrate the multiplication.



1. Draw a diagram to show that $n(2n+5)$ and $2n^2+5n$ are equivalent expressions.

2. For each expression, use the distributive property to write an equivalent expression. If you get stuck, consider drawing a diagram.

a. $6(\frac{1}{3}n+2)$

b. $p(4p+9)$

c. $5r(r+\frac{3}{5})$

d. $(0.5w+7)w$

Self Check

Which expression is equivalent to $4t(t-\frac{3}{4})$?

a. $4t^2-\frac{3}{4}$
b. $4t^2-3t$
c. $4t^2-\frac{3}{4}t$
d. $4t^2-3$

Additional Resources

Distributive Property

Distributive Property

If a, b, c are real numbers, then:

$a(b+c)=ab+ac$

Other forms:

$a(b-c)=ab-ac$

$(b+c)a=ba+ca$

EXAMPLE 1

Simplify $3(x+4)$

Step 1 - Distribute the 3 to both the x and the 4.



Step 2 - Multiply the terms.

$3(x)+3(4)$

Step 3 - Simplify.

$3x+12$

EXAMPLE 2

Simplify $m(7-2m)$

Step 1 - Distribute the m to both the 7 and the $2m$.



Step 2 - Multiply the terms. Recall $m \cdot m = m^2$.

$m(7) - m(2m)$

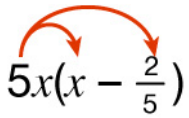
Step 3 - Simplify.

$7m - 2m^2$

EXAMPLE 3

Simplify $5x(x - \frac{2}{5})$

Step 1 - Distribute the $5x$ to both the x and the $\frac{2}{5}$.



Step 2 - Multiply the terms. Recall $x \cdot x = x^2$.

$5x(x) - 5x(\frac{2}{5})$

Step 3 - Simplify.

$5x^2 - 2x$

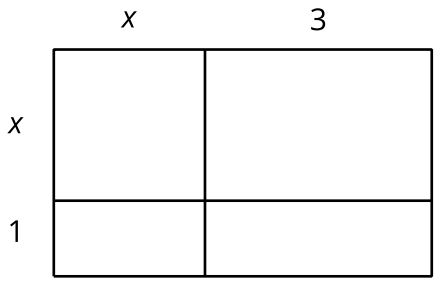
TRY IT **Distributive Property**

Simplify $7y(4y - \frac{3}{7})$

7.8.3 Using Diagrams to Find Equivalent Quadratic Expressions

Activity

1. Here is a diagram of a rectangle with side lengths $x + 1$ and $x + 3$. Use this diagram to show that $(x + 1)(x + 3)$ and $x^2 + 4x + 3$ are equivalent expressions.



2. Draw diagrams to help you write an equivalent expression for each of the following:

a. $(x+5)^2$

b. $2x(x+4)$

c. $(2x+1)(x+3)$

d. $(x+m)(x+n)$

3. Write an equivalent expression for each expression without drawing a diagram. Use the distributive property to help. Remember FOIL for multiplying binomials: Multiply the First terms, the Outer terms, the Inner terms, and the Last terms.

a. $(x+2)(x+6)$

b. $(x+5)(2x+10)$

Are you ready for more?

Extending Your Thinking

1. Image 1

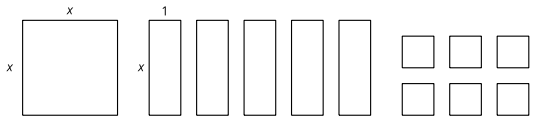
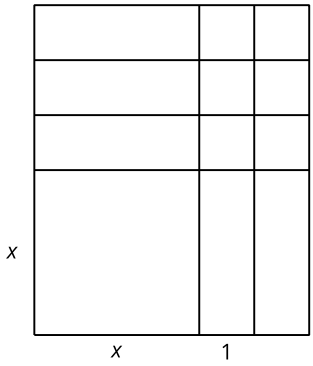


Image 2



Are Image 1 and Image 2 equivalent? Explain or show your reasoning.

2. What does this tell you about an equivalent expression for x^2+5x+6?

3. Is there a different non-zero number of 1-by-1 squares that we could use instead that would allow us to arrange the combined figures into a single large rectangle?

Video: Using Diagrams to Find Equivalent Quadratic Expressions

Watch the following video to learn more about using diagrams to find equivalent quadratic expressions.

Using Diagrams to Find Equivalent Quadratic Expressions (https://openstax.org/r/quadratic-diagrams)

Self Check

Which expression is equivalent to $(x+2)(x+4)$?

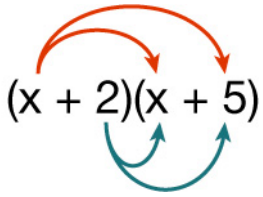
a. x^2+2x+8
b. $7x^2+8$
c. x^2+6x+8
d. x^2+4x

Additional Resources

Multiplying Binomials

Write an expression equivalent to $(x + 2)(x + 5)$.

Step 1 - Distribute. Use FOIL. Multiply the First terms, Outer terms, Inner terms, and Last terms.



Step 2 - Write out the four terms.

$x^2 + 5x + 2x + 10$

Step 3 - Combine like terms.

$x^2 + 7x + 10$

TRY IT **Multiplying Binomials**

Write an expression equivalent to $(x + 7)(x + 3)$.

7.8.4 Writing Equivalent Expressions

Cool Down

1. Draw a diagram to show that $(3x + 1)(x + 2)$ is equivalent to $3x^2 + 7x + 2$.
2. Is $(x + 4)^2$ equivalent to $2x^2 + 8x + 8$? Explain or show your reasoning.

Why Should I Care?



Anna's class entered a contest where they had to build a catapault and use it to launch a pumpkin at a distant target. But when Anna's class shot their pumpkin for the first time, it went wildly off course.

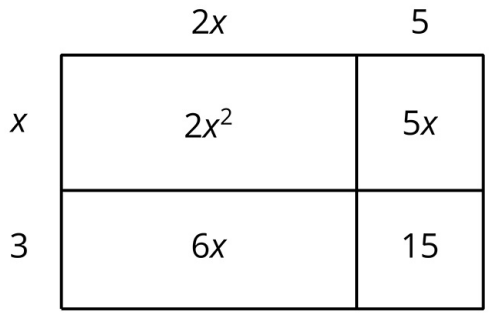
Anna thought about their problem and came up with a solution. If Anna's class used a quadratic equation, they could mathematically find the best place and angle for their pumpkin catapault. It worked! Their accuracy went from 2% to 99%.

You can use quadratic equations for any object that travels in a curved path, like footballs, frisbees, and golf balls.

7.8.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Choose the expression that best describes the diagram.



a. $(2x + 5)(x + 3)$ is equivalent to $30x^2$.
b. $(2x + 5)(x + 3)$ is equivalent to $2x^2 + 11x + 15$.
c. $(2x + 5)(x + 3)$ is equivalent to $2x^2 + 14x + 23$.
d. $(2x + 5)(x + 3)$ is equivalent to $2x^2 + 5x + 8$.

In questions 2a–2d, match each expanded quadratic expression with an equivalent expression written as a product.

2a. $x^2 + 12x + 32$

a. $(x + 2)(x + 3)$
b. $(x + 8)(x + 4)$
c. $(x + 4)(x + 2)$
d. $(x + 2)(x + 6)$

2b. $x^2 + 5x + 6$

a. $(x + 2)(x + 3)$
b. $(x + 8)(x + 4)$
c. $(x + 4)(x + 2)$
d. $(x + 2)(x + 6)$

2c. $x^2 + 6x + 8$

a. $(x + 2)(x + 3)$
b. $(x + 8)(x + 4)$
c. $(x + 4)(x + 2)$
d. $(x + 2)(x + 6)$

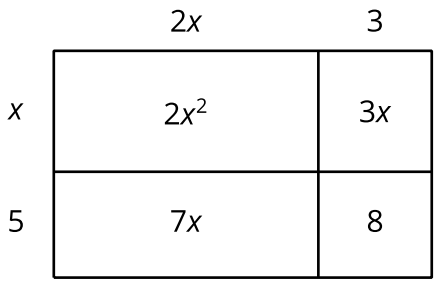
2d. $x^2 + 8x + 12$

a. $(x + 2)(x + 3)$
b. $(x + 8)(x + 4)$
c. $(x + 4)(x + 2)$
d. $(x + 2)(x + 6)$

3. **Select three** expressions that are equivalent to $x^2 + 4x$.

a. $x(x + 4)$
b. $(x + 2)^2$
c. $(x + x)(x + 4)$
d. $(x + 2)^2 - 4$
e. $(x + 4)x$

Tyler drew a diagram to expand $(x + 5)(2x + 3)$.



4a. Tyler's diagram is incorrect. Choose the sentence that describes Tyler's mistake.

a. Tyler should have labeled the top x and 3 and the side $2x$ and 5.
b. Tyler should have labeled the top $2x$ and 5 and the side x and 3.
c. Tyler multiplied 5 and 2 and 5 and 3 instead of adding them.
d. Tyler added 5 and 2 and 5 and 3 instead of multiplying them.

4b. What is the correct expanded form of $(x + 5)(2x + 3)$?

a. $2x^2 + 13x + 15$
b. $2x^2 + 10x + 8$
c. $30x^2$
d. $3x + 8$

7.8.6 Lesson Summary

In this lesson, you learned how to:

- Rewrite quadratic expressions in different forms by using an area diagram or the distributive property.

Here are the **activities** that helped you reach those goals:

- 7.8.1: Area Diagrams
  - In this activity, you drew area diagrams to represent multiplication.
- 7.8.2: Using the Distributive Property to Write Equivalent Expression
  - In this activity, you used diagrams to write equivalent expressions, and then used the distributive property to find the product of a monomial times a binomial.
  - 7.8.2: Self Check
  - 7.8.2: Additional Resources
- 7.8.3: Using Diagrams to Find Equivalent Quadratic Expressions
  - In this activity, you continued to use diagrams to find equivalent expressions, and then used the distributive property to multiply binomials by binomials.
  - 7.8.3: Self Check
  - 7.8.3: Additional Resources
- 7.8.4: Writing Equivalent Expressions
  - In this activity, you multiplied binomial expressions that had coefficients in front of x.

After these activities, you completed the following **practice:**

- 7.8.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

7.9 STANDARD FORM AND FACTORED FORM

7.9.0 Lesson Overview

In this lesson, you will draw and interpret area diagrams and use the distributive property to write quadratic equations in factored and standard forms.

When you finish this lesson, you will be able to:

- Rewrite quadratic expressions given in factored form in standard form using either the distributive property or a diagram.
- Know the difference between "factored form" and "standard form."

Here are the **activities** that will help you reach those goals:

- 7.9.1: Solving Equations Using Opposite Operations
- 7.9.2: Finding Products of Difference
  - 7.9.2: Self Check
  - 7.9.2: Additional Resources
- 7.9.3: Standard and Factored Forms of Quadratic Expressions
  - 7.9.3: Self Check
  - 7.9.3: Additional Resources
- 7.9.4: Expressions in Standard Form

After that, you'll **practice and review**.

- 7.9.5: Practice
- 7.9.6: Lesson Summary

7.9.1 Solving Equations Using Opposite Operations

Warm Up

Solve each equation mentally. Then, enter your answer to check it.

1. $40 - 8 = 40 + n$

2. $25 + -100 = 25 - n$

3. $-3 - \frac{1}{2} = 3 + \frac{1}{2}n$

4. $72 - n = 72 + 6$

7.9.2 Finding Products of Differences

Activity

1. Show that $(x - 1)(x - 1)$ and $x^2 - 2x + 1$ are equivalent expressions by drawing a diagram or applying the **distributive property**. Show your reasoning.

2. For each expression, write an equivalent expression. Show your reasoning.

a. $(x + 1)(x - 1)$

b. $(x - 2)(x + 3)$

c. $(x - 2)^2$

Video: Finding Products of Differences

Watch the following video to learn more about finding products of differences.

Finding Products of Differences (https://openstax.org/r/products-differences)

Self Check

Which of the following expressions is equivalent to $(x-2)^2$?

a. x^2+4x+4
b. x^2-4x+4
c. x^2+4
d. x^2-4

Additional Resources

Multiplying Binomials with Negative Numbers

In a previous lesson, you learned how to square a binomial.

$(a+b)^2=a^2+2ab+b^2$ and $(a-b)^2=a^2-2ab+b^2$

You can use this to solve the following example.

EXAMPLE 1

Find the expression equivalent to $(x-3)^2$.

This means to multiply $(x-3)$ by $(x-3)$. So, $(x-3)^2=(x-3)(x-3)$.

Now, distribute. Remember, you can use FOIL.



Distribute the x to the second binomial, then distribute the -3 to the second binomial.

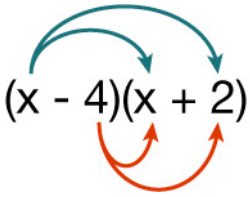
Then, combine like terms.

The expression becomes $x^2-3x-3x+9=x^2-6x+9$.

EXAMPLE 2

Find the expression equivalent to $(x-4)(x+2)$.

First, distribute. Remember, you can use FOIL.



Multiply the x by $(x+2)$, then multiply the -4 by $(x+2)$ and combine like terms:

$x^2+2x-4x-8=x^2-2x+8$

TRY IT **Multiplying Binomials with Negative Numbers**

Find the expression equivalent to $(x+5)(x-3)$.

7.9.3 Standard and Factored Forms of Quadratic Expressions

Activity

The quadratic expression $x^2 + 4x + 3$ is written in **standard form**.

Here are some other quadratic expressions. The expressions on the left are written in standard form and the expressions on the right are not.

Written In Standard Form	Not Written In Standard Form
$x^2 - 1$	$(2x + 3)x$
$x^2 + 9x$	$(x + 1)(x - 1)$
$\frac{1}{2}x^2$	$3(x - 2)^2 + 1$
$4x^2 - 2x + 5$	$-4(x^2 + x) + 7$
$3x^2 - x + 6$	$(x + 8)(-x + 5)$

1. What are some characteristics of expressions in standard form?
2. $(x + 1)(x - 1)$ and $(2x + 3)x$ in the right column are quadratic expressions written in **factored form**. Why do you think that form is called factored form?

Are you ready for more?

Extending Your Thinking

1. Which quadratic expression can be described as being both standard form and factored form? Explain how you know.

Self Check

Which of the following expressions is a quadratic expression written in standard form?

a. $7(x - 6)^2 + 1$
b. $x(x - 5)$
c. $x^2 - 9x + 20$
d. $x - 4$

Additional Resources

Standard Form of Quadratics

Examples of Standard Form Quadratics

x^2

$x^2 - 9$

$x^2 - 3x$

$x^2 + 4x - 45$

These examples have an x^2 and lead with it. Some have an x term or a constant term but it is not needed to be in standard form. Standard form of a quadratic expression is of the form $ax^2 + bx + c$, where $a \neq 0$.

Non-Examples of Standard Form Quadratics

$7 - x$

$9(x - 2)$

$x(x + 10)$

These examples do not have an x^2 term.

TRY IT **Standard Form of Quadratics**

Determine if $4x(x - 2)$ is a quadratic in standard form.

7.9.4 Expressions in Standard Form

Cool Down

For each expression, write an equivalent expression in **standard form**. Show your reasoning.

1. $(2x + 5)(x + 1)$

2. $(x - 2)(x + 2)$

7.9.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Choose the standard form of the quadratic expression.

a. $(x+4)(x-1)$

a. $-x^2-3x+4$
b. x^2+3x-4
c. $-x^2+3x-4$
d. x^2-3x+4

b. $(2x-1)(3x-1)$

a. $-6x^2+5x-1$
b. $-6x^2-5x+1$
c. $6x^2-5x+1$
d. $6x^2+5x-1$

2. Consider the expression $8-6x+x^2$.

a. Is the expression in standard form?

a. Yes
b. No

b. Is the expression equivalent to $(x-4)(x-2)$?

a. Yes
b. No

3. Find the standard form of the quadratic expression $(x-2)(x+8)$. Use the standard form to answer questions a, b, and c.

a. What is the value of a in this expression ax^2+bx+c?

b. What is the value of b in this expression ax^2+bx+c?

c. What is the value of c in this expression ax^2+bx+c?

7.9.6 Lesson Summary

In this lesson, you learned how to:

- Rewrite quadratic expressions given in factored form in standard form using either the distributive property or a diagram.
- Know the difference between “factored form” and “standard form.”

Here are the **activities** that helped you reach those goals:

- 7.9.1: Solving Equations Using Opposite Operations
  - In this activity, you solved equations that involve using inverse operations.
- 7.9.2: Finding Products of Differences
  - In this activity, you multiplied binomial expressions that involve subtraction.
  - 7.9.2: Self Check
  - 7.9.2: Additional Resources
- 7.9.3: Standard and Factored Forms of Quadratic Expressions
  - In this activity, you identified if an expression is a quadratic in standard form.
  - 7.9.3: Self Check
  - 7.9.3: Additional Resources
- 7.9.4: Expressions in Standard Form
  - In this activity, you continued to identify standard quadratic expressions.

After these activities, you completed the following **practice**:

- 7.9.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

7.10 GRAPHS OF FUNCTIONS IN STANDARD AND FACTORED FORMS

7.10.0 Lesson Overview

In this lesson, you will continue to work with quadratic expressions and functions and learn how to show (algebraically) that two different expressions can define the same function. You will discover the connections between different forms of quadratic expressions and features of the graphs that represent the expressions. You will learn that that some numbers in the expressions are related to the x- and y-intercepts of graphs representing expressions in standard and factored form.

When you finish this lesson, you will be able to:

- Explain the meaning of the intercepts on a graph of a quadratic function in terms of the situation it represents.
- Explain how the numbers in the factored form of a quadratic expression relate to the intercepts of its graph.

Here are the **activities** that will help you reach those goals:

- 7.10.1: Graphs of Linear Equations
- 7.10.2: Quadratic Forms and Their Graphs
  - 7.10.2: Self Check
  - 7.10.2: Additional Resources
- 7.10.3: The x- and y-intercepts of Quadratic Expressions
  - 7.10.3: Self Check
  - 7.10.3: Additional Resources
- 7.10.4: Using the Standard Form of an Equation to Find the x- and y-intercepts

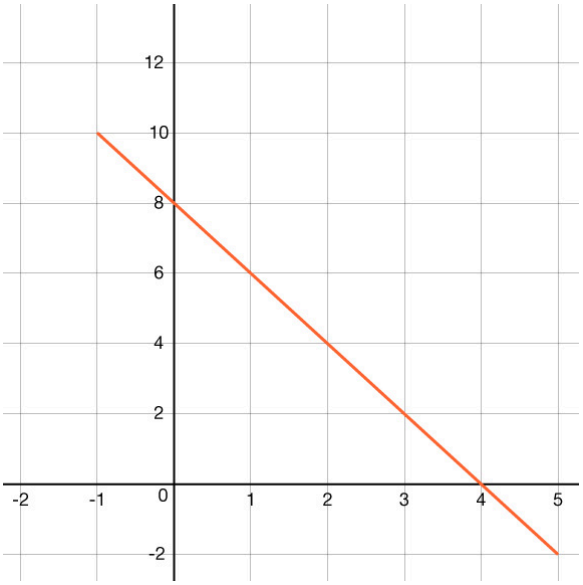
After that, you'll **practice and review.**

- 7.10.5: Practice
- 7.10.6: Lesson Summary

7.10.1 Graphs of Linear Equations

Warm Up

Here is a graph of the equation $y = 8 - 2x$.



1. Where do you see the 8 from the equation in the graph?
2. Where do you see the –2 from the equation in the graph?
3. What is the value of x for the x-intercept of the graph? How does this relate to the equation?

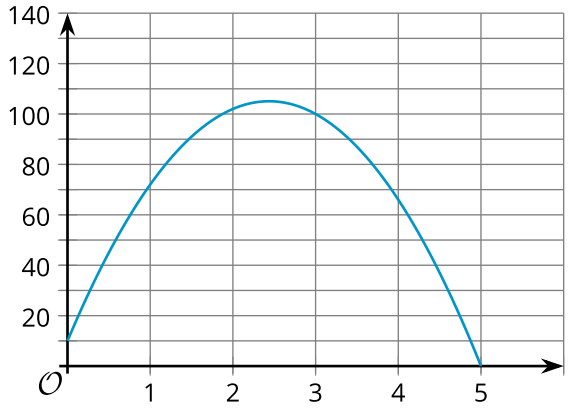
7.10.2 Quadratic Forms and Their Graphs

Activity

In an earlier lesson, we saw that an equation such as $h(t) = 10 + 78t - 16t^2$ can model the height of an object thrown upward from a height of 10 feet with a vertical velocity of 78 feet per second.



1. Is the expression $-16t^2 + 78t + 10$ written in standard form? Be prepared to show your reasoning.
2. Paola said that the equation $g(t) = -2(8t + 1)(t - 5)$ also defines the same function, written in factored form. Is Paola correct? Be prepared to show your reasoning.
3. Here is a graph representing both $g(t) = -2(8t + 1)(t - 5)$ and $h(t) = 10 + 78t - 16t^2$.



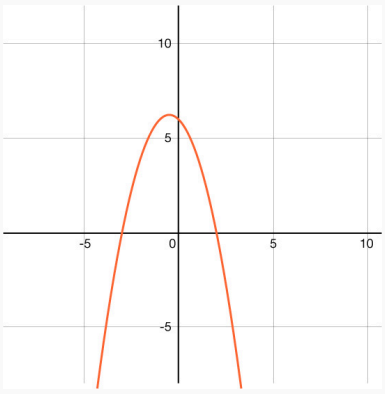
a. What is the y-value of the y-intercept?

b. What is the x-value of the x-intercept?

c. What do each of these points mean in this situation?

Self Check

The quadratic function $y = -(x - 2)(x + 3)$ is graphed below.



What are all of the x-intercepts?

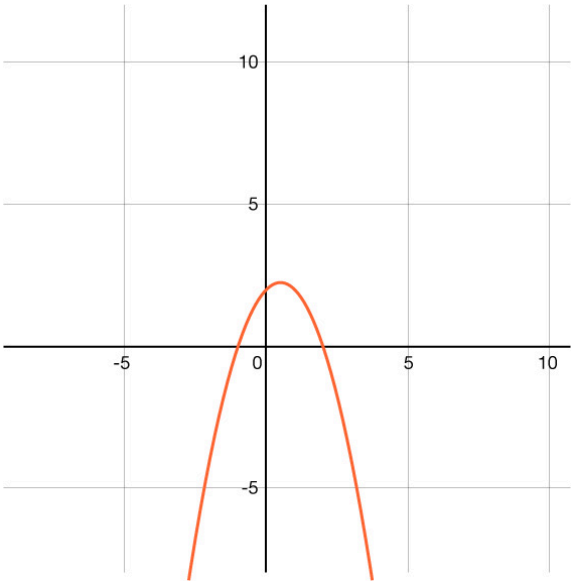
a. $x = 6$
b. $x = -3, x = 2$
c. $x = 2, x = 6$
d. $x = -3$

Additional Resources

Characteristics of Quadratic Graphs

The graph of $y = (x + 1)(x - 2)$ is shown below.

What are the x- and y-intercepts?



The x-intercepts are where the graph crosses the x-axis.

This happens at $(-1, 0)$ and $(2, 0)$.

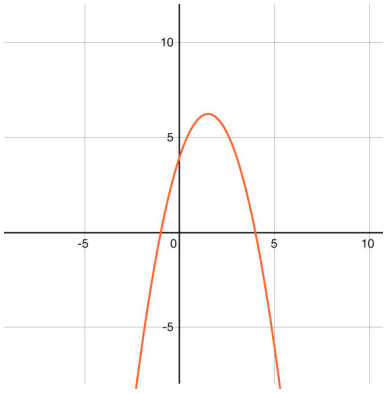
The y-intercept is where the graph crosses the y-axis.

This happens at $(0, 2)$.

TRY IT **Characteristics of Quadratic Graphs**

The graph of $y = -(x - 4)(x + 1)$ is below.

What are the x- and y-intercepts of the graph?



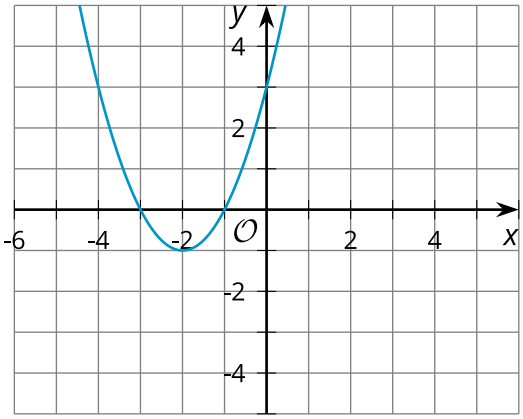
7.10.3 The *x*- and *y*-intercepts of Quadratic Expressions

Activity

Here are pairs of expressions in standard and factored forms. Each pair of expressions define the same quadratic function, which can be represented with the given graph.

Identify the x-intercepts and the y-intercept of each graph.

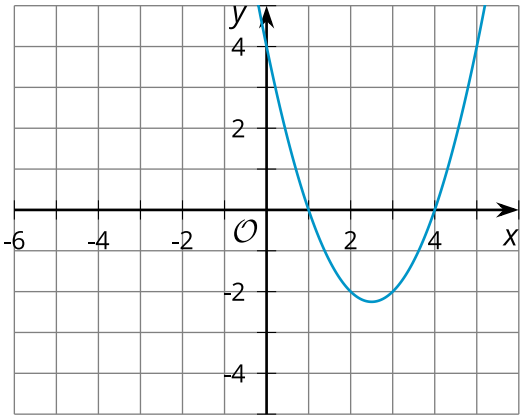
1. Function f
$x^2 + 4x + 3$
$(x + 3)(x + 1)$



a. Identify the x-intercepts of the graph.

b. Identify the y-value of the y-intercept of the graph.

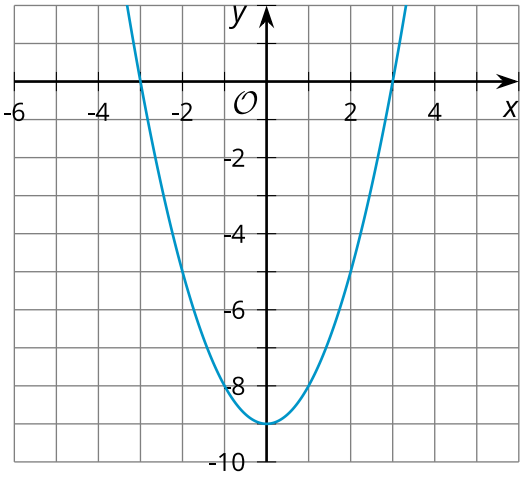
2. Function g
$x^2 - 5x + 4$
$(x - 4)(x - 1)$



a. Identify the x-intercepts of the graph.

b. Identify the y-value of the y-intercept of the graph.

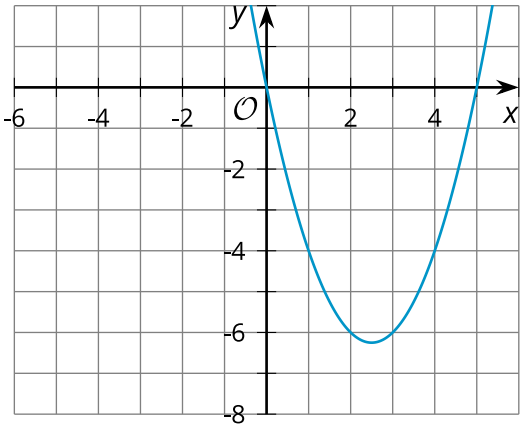
3. Function h
$x^2 - 9$
$(x - 3)(x + 3)$



a. Identify the x-intercepts of the graph.

b. Identify the y-value of the y-intercept of the graph.

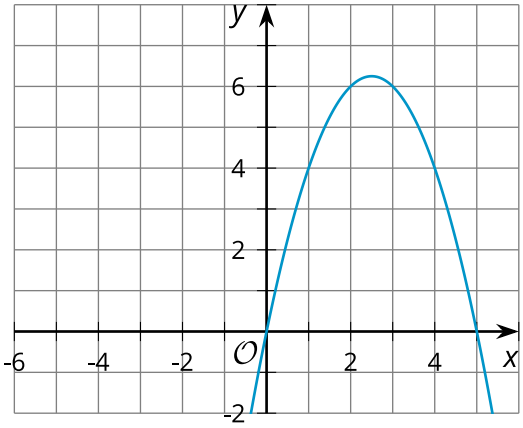
4. Function j
$x^2 - 5x$
$x(x - 5)$



a. Identify the x-intercepts of the graph.

b. Identify the y-value of the y-intercept of the graph.

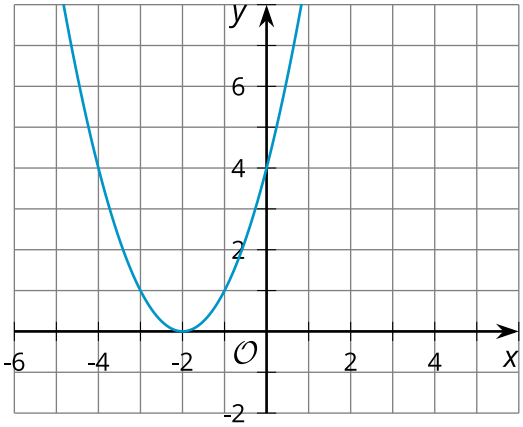
5. Function k
$5x - x^2$
$x(5 - x)$



a. Identify the x-intercepts of the graph.

b. Identify the y-value of the y-intercept of the graph.

6. Function l
$x^2 + 4x + 4$
$(x + 2)(x + 2)$



a. Identify the x-intercept of the graph. **Answer:**

b. Identify the y-value of the y-intercept of the graph.

7. What do you notice about the x-intercepts, the y-intercept, and the numbers in the expressions defining each function? Make a couple of observations.

8. Here is an expression that models function p, another quadratic function: $(x-9)(x-1)$.

a. Predict the x-intercepts of the graph that represents this function.

b. Identify the y-value of the y-intercept of the graph that represents this function.

Video: Determining the x- and y-intercepts of Quadratic Expressions

Watch the following video to learn more about the x- and y-intercepts of quadratic expressions.

Determining the x- and ð ¦-intercepts of Quadratic Expressions (https://openstax.org/r/quadratic-intercepts)

Self Check

What are the x-values of the x-intercept(s) of

$y=(x-9)(x+2)$?

a. $x=9, x=2$
b. $x=-2, x=-9$
c. $x=-2, x=9$
d. $x=2, x=-9$

Additional Resources

Finding x-intercepts Without Graphs

When a quadratic is in factored form, the x-intercepts of a quadratic equation will be:

- the opposite sign of the constant in the factor because if $x+a=0$, then $x=-a$.
- $x=0$ is an option.
- If a factor is repeated, there is only one x-intercept.

EXAMPLE

Find the x-intercepts of $y=(x-3)(x+2)$.

The x-intercepts occur when $y=0$.

Set each factor to 0 and solve.

$x-3=0 \qquad x+2=0$
$x=3 \qquad x=-2$

Notice that for the factor $(x-3)$ or $(x-3)(x+(-3))$, the constant is – 3, so the x-intercept is $x=+3$ or the point $(3,0)$.

For the factor $(x+2)$, the constant is $+2$, so the x-intercept is $x=-2$ or the point $(-2,0)$.

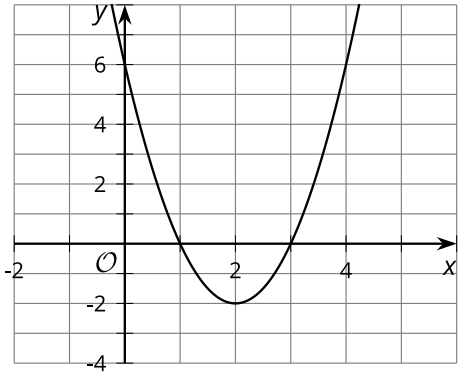
TRY IT **Finding x-intercepts Without Graphs**

Find the x-intercepts of $y=(x-1)(x+5)$.

Are you ready for more?

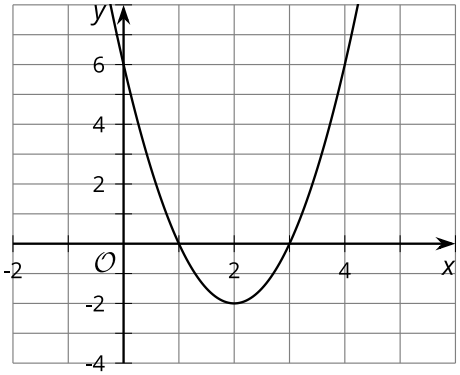
Extending Your Thinking

1. Find the values of a, p, and q that will make $y = a(x - p)(x - q)$ be the equation represented by the graph.



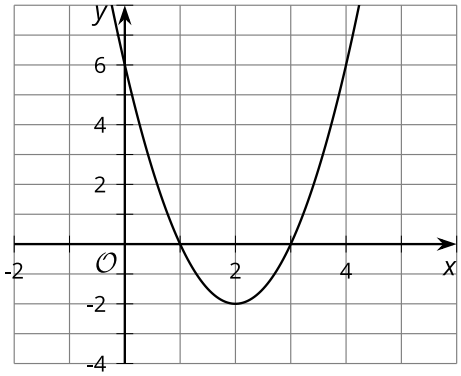
What is the value of a?
What is the value of a?

2. Find the values of a, p, and q that will make $y = a(x - p)(x - q)$ be the equation represented by the graph.



What is the value of p?

3. Find the values of a, p, and $qy = a(x - p)(x - q)$ be the equation represented by the graph.



What is the value of q ?

7.10.4 Using the Standard Form of an Equation to Find the x- and y-intercepts

Cool Down

The equations $y = x^2 + 6x + 8$ and $y = (x + 2)(x + 4)$ both define the same quadratic function.

1. Without graphing, identify the x-intercepts of the graph. Explain how you know.

2. Without graphing, identify the y-value of the y-intercept of the graph. Be prepared to show your reasoning.

7.10.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. A quadratic function f is defined by $f(x) = (x - 7)(x + 3)$.

a. Without graphing, identify the x-intercepts of the graph of f.

a. $(0, -7), (0, 3)$
b. $(0, 7), (0, -3)$
c. $(-7, 0), (3, 0)$
d. $(7, 0), (-3, 0)$

b. Use the standard form of $(x - 7)(x + 3)$ to identify the y-value of the y-intercept of the graph of f.

2. Answer these questions for the equation $y = x^2 - 5x + 4$.

a. What is the y-value of the y-intercept of this equation's graph?

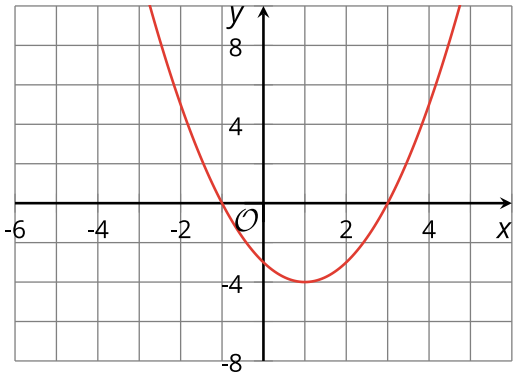
b. An equivalent way to write this equation is $y = (x - 4)(x - 1)$. What are the x-intercepts of this equation's graph?

a. $(0, -1), (0, -4)$
b. $(0, 1), (0, 4)$
c. $(-1, 0), (-4, 0)$
d. $(1, 0), (4, 0)$

3. Noah said that if we graph $y = (x - 1)(x + 6)$, the x-intercepts will be at $(1, 0)$ and $(-6, 0)$. Is Noah correct?

a. Yes
b. No

4. **Select two** x-intercepts of this graph.



a. 1
b. -1
c. 3
d. -3

5. Use the standard form of $y = (x - 2)(x - 4)$ to identify the y-value of the y-intercept of the graph.

7.10.6 Lesson Summary

In this lesson, you learned how to:

- Explain the meaning of the intercepts on a graph of a quadratic function in terms of the situation it represents.
- Explain how the numbers in the factored form of a quadratic expression relate to the intercepts of its graph.

Here are the **activities** that helped you reach those goals:

- 7.10.1: Graphs of Linear Equations
  - In this activity, you located the x- and y-intercepts of linear graphs.
- 7.10.2: Quadratic Forms and Their Graphs
  - In this activity, you looked at different forms of a quadratic and their graphs and picked out characteristics. These characteristics included x- and y-intercepts.
  - 7.10.2: Self-Check
  - 7.10.2: Additional Resources
- 7.10.3: The X- and Y-Intercepts of Quadratic Expressions
  - In this activity, you continued looking at different forms and graphs of quadratics and identifying their intercepts. You then made observations about how the graphs and forms are related in helping identify intercepts.
  - 7.10.3: Self Check
  - 7.10.3: Additional Resources
- 7.10.4: Using the Standard Form of an Equation to Find the X- and Y-Intercepts
  - In this activity, you used the standard form of a quadratic equation to find the y-intercept and the factored form of the same equation to find the x-intercepts.

After these activities, you completed the following **practice:**

- 7.10.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

7.11 GRAPHING FROM THE FACTORED FORM

7.11.0 Lesson Overview

In this lesson, you will evaluate expressions at certain x-values to see if they produce an output of 0. You will learn what the factored form can tell us about the vertex and the y-intercept of a graph representing a quadratic function.

When you finish this lesson, you will be able to:

- Graph a quadratic function given in factored form.
- Find the vertex and y-intercept of the graph of a quadratic function in factored form without graphing it first.

Here are the **activities** that will help you reach those goals:

- 7.11.1: Finding Coordinates
- 7.11.2: The Factored Form of a Quadratic Expression and the X-Intercepts of Its Graph
  - 7.11.2: Self Check
  - 7.11.2: Additional Resources
- 7.11.3: Sketching a Graph of a Quadratic Function Using at Least Three Identifiable Points
  - 7.11.3: Self Check
  - 7.11.3: Additional Resources
- 7.11.4: Using the Vertex and Axis of Symmetry of Quadratics
  - 7.11.4: Self Check
  - 7.11.4: Additional Resources
- 7.11.5: Graphs of Quadratic Functions

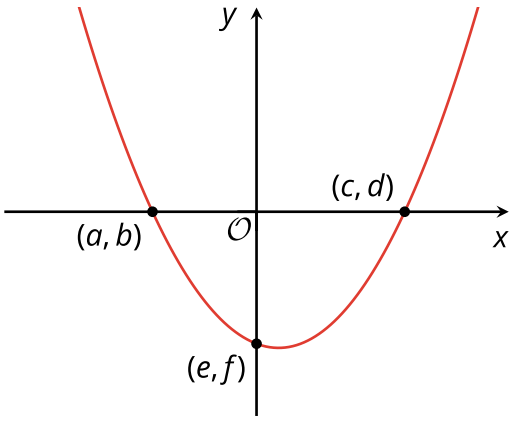
After that, you'll **practice and review.**

- 7.11.6: Practice
- 7.11.7: Lesson Summary

7.11.1 Finding Coordinates

Warm Up

Here is a graph of a function w defined by $w(x) = (x + 1.6)(x - 2)$. Three points on the graph are labeled.



Use the graph to answer the following questions.

1. Find the value of a. Be prepared to explain your reasoning.
2. Find the value of b. Be prepared to explain your reasoning.
3. Find the value of c. Be prepared to explain your reasoning.
4. Find the value of d. Be prepared to explain your reasoning.

5. Find the value of e. Be prepared to explain your reasoning.

6. Find the value of f. Be prepared to explain your reasoning.

7.11.2 The Factored Form of a Quadratic Expression and the x-intercepts of Its Graph

Activity

Consider two functions defined by $f(x) = x(x + 4)$ and $g(x) = x(x - 4)$

1. Consider the function defined by $f(x) = x(x + 4)$.
Then, answer questions a–g, and create a table of values in which to enter the given x values and their outputs using the function.

a. When the value of $x = -4$, what is the value of $f(x)$?

b. When the value of $x = -3$, what is the value of $f(x)$?

c. When the value of $x = 0$, what is the value of $f(x)$?

d. When the value of $x = 1$, what is the value of $f(x)$?

e. When the value of $x = 2$, what is the value of $f(x)$?

f. When the value of $x = 3$, what is the value of $f(x)$?

g. When the value of $x = 5$, what is the value of $f(x)$?

2. Now, answer questions a–d to analyze the function defined by $f(x) = x(x + 4)$.

a. What are the x-intercepts of the graph? Be prepared to show your reasoning.

b. What is the value of the x-coordinate in the vertex of the graph?

c. What is the value of the y-coordinate in the vertex of the graph?

d. What is the vertex of the graph of function f? Be prepared to show your reasoning.

3. Consider the function defined by $g(x) = x(x - 4)$. Then, answer questions a–g, and create a table of values in which to enter the given x values and their outputs using the function.

a. When the value of $x = -4$, what is the value of $g(x)$?

b. When the value of $x = -3$, what is the value of $g(x)$?

c. When the value of $x = 0$, what is the value of $g(x)$?

d. When the value of $x = 1$, what is the value of $g(x)$?

e. When the value of $x = 2$, what is the value of $g(x)$?

f. When the value of $x = 4$, what is the value of $g(x)$?

g. When the value of $x = 5$, what is the value of $g(x)$?

4. Now, answer questions a–d to analyze the function defined by $g(x) = x(x - 4)$.

a. What are the x-intercepts of the graph? Be prepared to show your reasoning.

b. What is the value of the x-coordinate in the vertex of the graph?

c. What is the value of the y-coordinate in the vertex of the graph?

d. What is the vertex of the graph of function g? Be prepared to show your reasoning.

5. Use the graphing tool or technology outside the course.

Access multimedia content (http://openstax.org/books/algebra-1/pages/7-11-2-the-factored-form-of-a-quadratic-expression-and-the-x-intercepts-of-its-graph)
a. Plot the points from the tables on the same coordinate plane. (Consider using different colors or markings for each set of points so you can tell them apart.)

b. Make a couple of observations about how the two graphs compare.

Self Check

The table for function f is below. What is the vertex of the graph of f?

x	-3	-2	-1	0	1	2
$f(x)$	0	-1	-3	-1	0	2

a. $(1, 0)$
b. $(-1, -3)$
c. $(0, -1)$
d. $(-3, 0)$

Additional Resources

Determining Parts of a Graph from a Table

Identify the x-intercepts and vertex of the quadratic from the table:

x	-5	-3	-1	0	1
$f(x)$	16	0	-2	0	16

The x-intercepts are where the y-values are 0.

This occurs at $(-3, 0)$ and $(0, 0)$.

The vertex occurs where the function will create a mirror image, or be symmetrical for all points. This often occurs halfway between the two zeros (when there are two zeros).

For this function, the vertex is at $(-1, -2)$.

TRY IT **Try It: Determining Parts of a Graph from a Table**

Determine the vertex from the table:

x	5	6	7	8	9	10
$f(x)$	0	4	8	4	0	-2

7.11.3 Sketching a Graph of a Quadratic Function Using at Least Three Identifiable Points

Activity

1. The functions f, g, and h are given. Refer to these functions to complete questions a–f.

Equation	x -intercept	x-coordinate of the vertex
$f(x) = (x + 3)(x - 5)$	a. _____	b. _____
$g(x) = 2x(x - 3)$	c. _____	d. _____
$h(x) = (x + 4)(4 - x)$	e. _____	f. _____

a. Predict the x-intercepts for function $f(x) = (x + 3)(x - 5)$.

b. Predict the x-coordinate of the vertex for function $f(x) = (x + 3)(x - 5)$.

c. Predict the x-intercepts for function $g(x) = 2x(x - 3)$.

d. Predict the x-coordinate of the vertex for function $g(x) = 2x(x - 3)$.

e. Predict the x-intercepts for function $h(x) = (x + 4)(4 - x)$.

f. Predict the x-coordinate of the vertex for function $h(x) = (x + 4)(4 - x)$.

2. Use the graphing tool or technology outside the course. Graph the functions f, g, and h:
$f(x) = (x + 3)(x - 5)$
$g(x) = 2x(x - 3)$
$h(x) = (x + 4)(4 - x)$

Use the graphs to check your predictions.

Access multimedia content (http://openstax.org/books/algebra-1/pages/7-11-3-sketching-a-graph-of-a-quadratic-function-using-at-least-three-identifiable-points)

3. Without using technology, sketch a graph that represents the equation $y = (x - 7)(x + 11)$ and that shows the x-intercepts and the vertex. Think about how to find the y-coordinate of the vertex. Be prepared to explain your reasoning.

Are you ready for more?

Extending Your Thinking

1. The quadratic function f is given by $f(x) = x^2 + 2x + 6$.
Find $f(-2)$.

2. Find $f(0)$.

3. What is the x-coordinate of the vertex of the graph of this quadratic function?

4. Does the graph have any x-intercepts? Explain or show how you know.

Video: Sketching a Graph of a Quadratic Function

Watch the following video to learn more about sketching a graph of a quadratic function using at least three identifiable points.

Sketching a Graph of a Quadratic Function (https://openstax.org/r/graph-quadratic)

Self Check

Find the coordinate of the vertex of $f(x) = (x - 2)(x - 4)$.

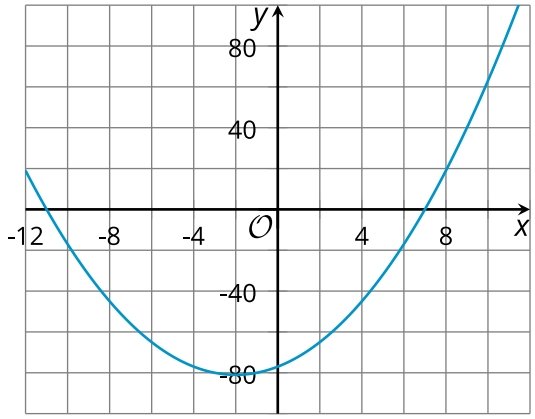
a. $(0, 8)$
b. $(4, 0)$
c. $(3, -1)$
d. $(2, 0)$

Additional Resources

Graphing Quadratics With Points

The function f given by $f(x) = (x + 1)(x - 1)$ is written in factored form. Recall that this form is helpful for finding the zeros of the function (where the function has the value 0) and telling us the x-intercepts on the graph representing the function.

Here is a graph representing f. It shows two x-intercepts at $x = -1$ and $x = 3$.



If we use –1 and 3 as inputs to f, what are the outputs?

- $f(-1) = (-1 + 1)(-1 - 3) = (0)(-4) = 0$
- $f(3) = (3 + 1)(3 - 3) = (4)(0) = 0$

Because the inputs –1 and 3 produce an output of 0, they are the zeros of the function f. And because both x values have 0 for their y value, they also give us the x-intercepts of the graph (the points where the graph crosses the x-axis, which always have a y-coordinate of 0). So, the zeros of a function have the same values as the x-coordinates of the x-intercepts of the graph of the function.

The factored form can also help us identify the vertex of the graph, which is the point where the function reaches its minimum value. Notice that the x-coordinate of the vertex is 1, and that 1 is halfway between –1 and 3. Once we know the x-coordinate of the vertex, we can find the y-coordinate by evaluating the function $f(1) = (1 + 1)(1 - 3) = (2)(-2) = -4$. So the vertex is at $(1, -4)$.

When a quadratic function is in standard form, the y-intercept is clear: its y-coordinate is the constant term c in $ax^2 + bx + c$. To find the y-intercept from factored form, we can evaluate the function at $x = 0$, because the y-intercept is the point where the graph has an input value of 0:

$f(0) = (0 + 1)(0 - 3) = (1)(-3) = -3$

TRY IT **Graphing Quadratics with Points**

Find the zeros and vertex of the function f given by $f(x) = (x - 4)(x + 2)$. Use the points to graph f.

7.11.4 Using the Vertex and Axis of Symmetry of Quadratics

Activity

While answering the questions below, plot each point on a coordinate plane with grid paper by hand. Do not use technology.

For questions 1 – 4, use the function $f(x) = (x - 2)(x + 4)$.

1. Find the x-coordinates of the x-intercepts of $f(x)$.

2. Find the coordinate of the vertex.

3. On your graph, sketch a quadratic through the key features on your graph and draw a vertical line through the vertex using a different color.
4. What is the equation of the vertical line that goes through the vertex?
5. What does the vertical line that goes through the vertex do to the parabola?
6. On your graph, label your x-intercepts, vertex, and axis of symmetry. Identify your vertex as a maximum or minimum.
7. Given a vertex of $(2, -4)$ and a point $(1, 3)$, what is another point on the quadratic?
(Hint: It is helpful to draw a picture and use the axis of symmetry).

Self Check

What is the equation of the axis of symmetry for $f(x) = (x + 1)(x - 3)$?

a. $x = -4$
b. $x = 3$
c. $x = -1$
d. $x = 1$

Additional Resources

Axis of Symmetry of a Quadratic

In the graph of a quadratic function, the axis of symmetry is an imaginary line that cuts the parabola in half to create two symmetrical sides. It goes through the vertex of the parabola so the equation of the axis of symmetry is the same as the x-coordinate of the vertex.

Figure 1

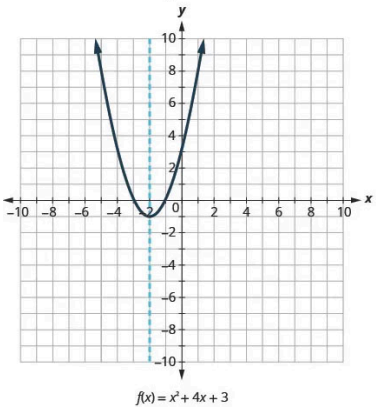
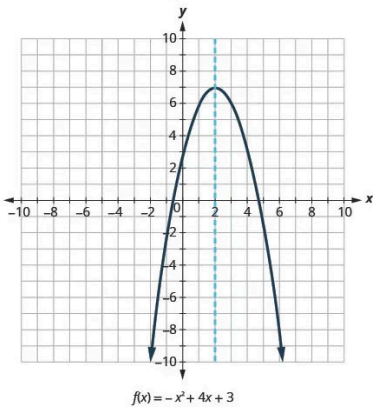


Figure 2



Notice that in Figure 1, the vertex is at $(-2, -2)$ and the axis of symmetry is at $x = -2$.

In Figure 2, the vertex is at $(2, 7)$ and the axis of symmetry is at $x = 2$.

EXAMPLE

Find the vertex and axis of symmetry for $f(x) = (x + 2)(x - 6)$.

Step 1 - Find the x-intercepts.

Set each factor equal to 0.

$x + 2 = 0$

$x = -2$

$x - 6 = 0$

$x = 6$

Step 2 - Find the x-coordinate of the vertex.

The vertex is halfway between the two x-intercepts.

$x = 6 + (-2)^2 = 2$

$x = \frac{6+(-2)}{2} = 2$

Step 3 - Find the y-coordinate of the vertex.

Substitute the x-coordinate into $f(x)$.

$f(2) = (2 + 2)(2 - 6) = (4)(-4) = -6$

The vertex is at $(2, -16)$.

Step 4 - Find the axis of symmetry.

The axis of symmetry is the equation where x equals the x-coordinate of the function.

$x = 2$

TRY IT **Axis of Symmetry of a Quadratic**

For questions 1 – 2, use the equation of the quadratic, $f(x) = (x - 4)(x - 10)$.

1. What is the vertex of the quadratic?
2. What is the axis of symmetry of the quadratic?

7.11.5 Graphs of Quadratic Functions

Cool Down

The function f is given by $f(x) = (x - 2)(x + 4)$.

Without using graphing technology, answer questions 1 – 4.

1. What are the x-intercepts of the graph representing f?
2. What is the x-coordinate of the vertex of the graph?
3. What is the y-coordinate of the vertex of the graph?
4. What is the y-intercept?
5. Use the graphing tool or technology outside the course. Sketch a graph that represents f.
Access multimedia content (http://openstax.org/books/algebra-1/pages/7-11-5-graphs-of-quadratic-functions)

7.11.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. **Select two** true statements about the graph that represents $y = 2x(x - 11)$.

   a. Its x-intercepts are at $(-2, 0)$ and $(11, 0)$.
   b. Its x-intercepts are at $(0, 0)$ and $(11, 0)$.
   c. Its x-intercepts are at $(2, 0)$ and $(-11, 0)$.
   d. It has only one x-intercept.
   e. The x-coordinate of its vertex is -4.5.
   f. The x-coordinate of its vertex is 11.
   g. The x-coordinate of its vertex is 4.5.
   h. The x-coordinate of its vertex is 5.5.

2. **Select two** equations whose graphs have a vertex with x-coordinate 2.

   a. $y = (x - 2)(x - 4)$
   b. $y = (x - 2)(x + 2)$
   c. $y = (x - 1)(x - 3)$
   d. $y = x(x + 4)$
   e. $y = x(x - 4)$

3. Three different equations are given. Refer to these equations to complete questions a–f.

Equation	x-intercept	x-coordinate of the vertex
$y = x(x - 2)$	a. _____	b. _____
$y = (x - 4)(x + 5)$	c. _____	d. _____
$y = -5x(3 - x)$	e. _____	f. _____

   a. What are the x-intercepts of the graph that represents $y = x(x - 2)$?

      a. $(0, 0)$ and $(-2, 0)$
      b. $(0, 0)$ and $(0, -2)$
      c. $(0, 0)$ and $(2, 0)$
      d. $(0, 0)$ and $(0, 2)$

   b. What is the x-coordinate of the vertex of the graph that represents $y = x(x - 2)$?

   c. What are the x-intercepts of the graph that represents $y = (x - 4)(x + 5)$?

      a. $(0, 5)$ and $(0, -4)$
      b. $(5, 0)$ and $(-4, 0)$
      c. $(0, -5)$ and $(0, 4)$
      d. $(-5, 0)$ and $(4, 0)$

   d. What is the x-coordinate of the vertex of the graph that represents $y = (x - 4)(x + 5)$?

   e. What are the x-intercepts of the graph that represents $y = -5x(3 - x)$?

      a. $(0, 0)$ and $(0, 3)$
      b. $(0, 0)$ and $(3, 0)$
      c. $(0, 0)$ and $(-3, 0)$
      d. $(0, 0)$ and $(0, -3)$

   f. What is the x-coordinate of the vertex of the graph that represents $y = -5x(3 - x)$?

4. What is the equation of the axis of symmetry of $y = (x - 4)(x + 5)$?

5. Answer questions a–c about the function $y = (x - 2)(x - 4)$.

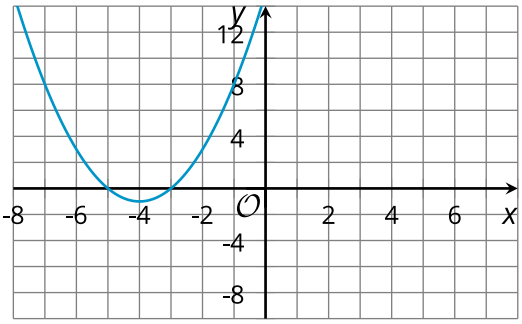
a. What are the x-intercepts of the vertex of the graph $y = (x - 2)(x - 4)$.

a. $(0, -2)$ and $(0, -4)$
b. $(-2, 0)$ and $(-4, 0)$
c. $(0, 2)$ and $(0, 4)$
d. $(2, 0)$ and $(4, 0)$

b. Is $(-1, 3)$ a point on the graph?

a. Yes
b. No

c. Does the graph represent the equation $y = (x - 2)(x - 4)$?



a. Yes
b. No

7.11.7 Lesson Summary

In this lesson, you learned how to:

- Graph a quadratic function given in factored form.
- Find the vertex and y-intercept of the graph of a quadratic function in factored form without graphing it first.

Here are the **activities** that helped you reach those goals:

- 7.11.1: Finding Coordinates
  - In this activity, you determined the missing values of coordinates given a function and a graph of a quadratic.
- 7.11.2: The Factored Form of a Quadratic Expression and the x-intercepts of Its Graph
  - In this activity, you were given a quadratic function in factored form. You used this function to complete a table and determine the x-intercepts and vertex of the graph of the function.
  - 7.11.2: Self Check
  - 7.11.2: Additional Resources
- 7.11.3: Sketching a Graph of a Quadratic Function Using at Least Three Identifiable Points
  - In this activity, you continued to determine the intercepts and vertex of a quadratic and used these points to graph the quadratic function.
  - 7.11.3: Self Check
  - 7.11.3: Additional Resources
- 7.11.4: Using the Vertex and Axis of Symmetry of Quadratics
  - In this activity, you determined the vertex and axis of symmetry of a quadratic and used these to graph the quadratic function.
  - 7.11.4: Self Check
  - 7.11.4: Additional Resources
- 7.11.5: Graphs of Quadratic Functions
  - In this activity, you once again identified specific points from a quadratic function and used them to graph the function.

After these activities, you completed the following **practice**:

- 7.11.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

7.12 GRAPHING THE STANDARD FORM, PART 1

7.12.0 Lesson Overview

In this lesson, you will experiment with the graphs of quadratic functions expressed in standard form and reason about how the parameters of the expressions—specifically the coefficient of the squared term and the constant term—relate to features of the graphs.

When you finish this lesson, you will be able to:

- Explain how the a and c in $ax^2 + bx + c$ affect the graph of the equation.
- Explain how graphs, tables, and equations that represent the same quadratic function are related.

Here are the **activities** that will help you reach those goals:

- 7.12.1: Using Coefficients and Constant Terms to Identify Graphs of Linear Equations
- 7.12.2: Transformations with Quadratic Functions
  - 7.12.2: Self Check
  - 7.12.2: Additional Resources
- 7.12.3: Understanding the Behaviors of a Graph in Relation to Its Quadratic Expression
  - 7.12.3: Self Check
  - 7.12.3: Additional Resources
- 7.12.4: Representations of Quadratic Functions
  - 7.12.4: Self Check
  - 7.12.4: Additional Resources
- 7.12.5: Identifying Graphs of Quadratic Equations

After that, you'll **practice and review**.

- 7.12.6: Practice
- 7.12.7: Lesson Summary

7.12.1 Using Coefficients and Constant Terms to Identify Graphs of Linear Equations

Warm Up

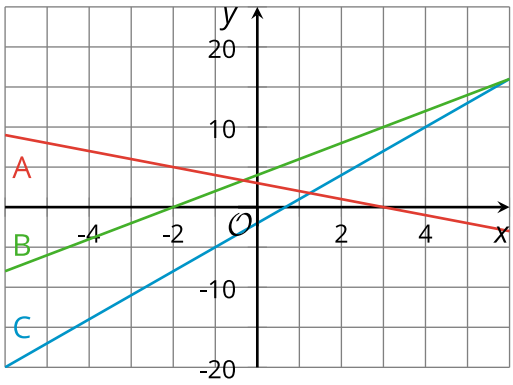
Graphs A, B, and C represent three linear equations:

$y = 2x + 4$

$y = 3 - x$

$y = 3x - 2$

Use the following graph to answer questions 1 – 3.



1. Which graph corresponds to equation $y = 2x + 4$?
2. Which graph corresponds to equation $y = 3 - x$?
3. Which graph corresponds to equation $y = 3x - 2$?

Building Character: Proactivity



Proactive people don't accept the world as it is or wait for direction from others to initiate change. Instead, they take matters into their own hands, acting to make the world better for themselves and others. They speak truth to power, take charge to solve problems, and champion new ideas.

Think about your current sense of proactivity. Are the following statements true for you??

- I enjoy being a champion for ideas, even against opposition.
- I like to be the one to take initiative.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

7.12.2 Transformations with Quadratic Functions

Activity

Use the graphing tool or technology outside the course. Graph $f(x) = x^2$, and then experiment with each of the changes to the function in questions 1 – 7. Record your observations of what you noticed or wondered about each set of changes to the function (include sketches, if helpful).

Access multimedia content (http://openstax.org/books/algebra-1/pages/7-12-2-transformations-with-quadratic-functions)

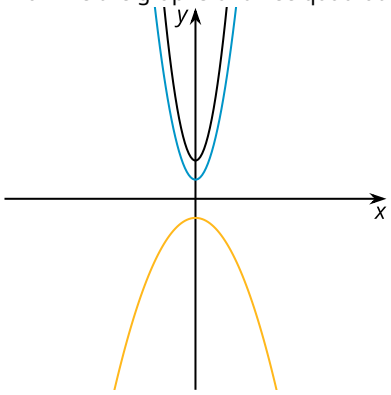
1. Add different constant terms to x^2 (for example: $x^2 + 5$, $x^2 + 10$, $x^2 - 3$).
2. Multiply x^2 by different positive coefficients greater than 1 (for example: $3x^2$, $7.5x^2$).
3. Multiply x^2 by different negative coefficients less than or equal to -1 (for example: $-x^2$,$-4x^2$).
4. Multiply x^2 by different coefficients between -1 and 1 (for example: $12x^2$, $-0.25x^2$).
5. Add different constants to x before squaring the term (for example $(x - 1)^2$, $(x + 2)^2$, $(x - 3)^2$).
6. Multiply x by different positive coefficients greater than 1 (for example: $(3x)^2$, $(7.5x)^2$).
7. Multiply x by different positive coefficients between 0 and 1 (for example: $(0.5x)^2$, $(0.1x)^2$).

For quadratics, the **parent function** is $f(x) = x^2$. From there, any moves created by making changes to the function like in questions 1 – 7 above are called **transformations**.

Are you ready for more?

Extending Your Thinking

1. Examine the graphs of three quadratic functions.



What can you say about the coefficient of x^2 in the expression that defines f (in black at the top)?

2. What can you say about the coefficient of x^2 in the expression that defines g (in blue in the middle)?

3. What can you say about the coefficient of x^2 in the expression that defines h (in yellow at the bottom)?

4. How do each of the functions compare to each other?

Self Check

How is the graph of $f(x) = (x - 4)^2$ transformed from the graph of $f(x) = x^2$?

a. Left 4
b. Right 4
c. Down 4
d. Up 4

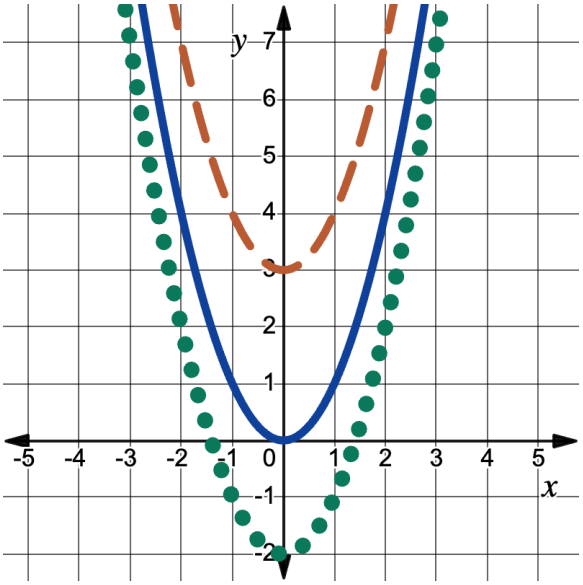
Additional Resources

Graphing Quadratic Functions Using Transformations

Graph a Quadratic Function of the Form $f(x) = x^2 + k$ Using a Vertical Shift

The graph of $f(x) = x^2 + k$ shifts the graph of $f(x) = x^2$ vertically k units.

- If $k > 0$, shift the parabola vertically up k units.
- If $k < 0$, shift the parabola vertically down $|k|$ units.



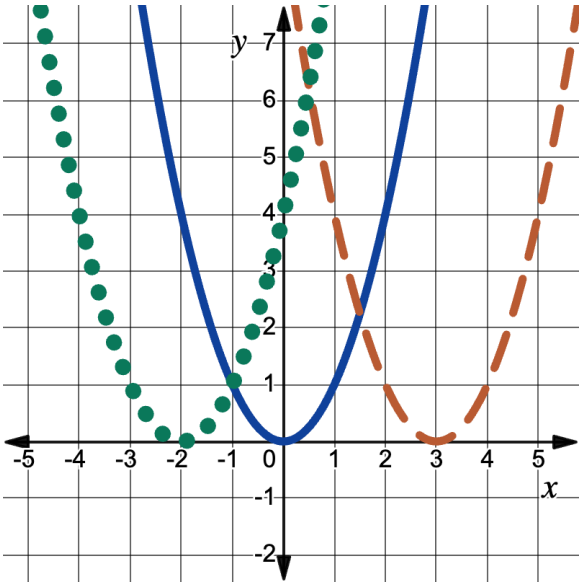
For the functions graphed above, notice that the parent function, $f(x) = x^2$, is graphed using a blue, solid line.

1. The parabola that has been shifted **vertically up** 3 units represents the function $f(x) = x^2 + 3$ since $k = 3$ and $3 > 0$ (orange, dashed graph).
2. The parabola that has been shifted **vertically down** 2 units represents the function $f(x) = x^2 - 2$ since $k = -2$ and $-2 < 0$ (green, dotted graph).

GRAPH A QUADRATIC FUNCTION OF THE FORM $f(x) = (x - h)^2$ USING A HORIZONTAL SHIFT

The graph of $f(x) = (x - h)^2$ shifts the graph of $f(x) = x^2$ horizontally h units.

- If $h > 0$, shift the parabola horizontally right h units.
- If $h < 0$, shift the parabola horizontally left $|h|$ units.



Again, for the functions graphed above, the parent function, $f(x) = x^2$, is graphed using a blue, solid line.

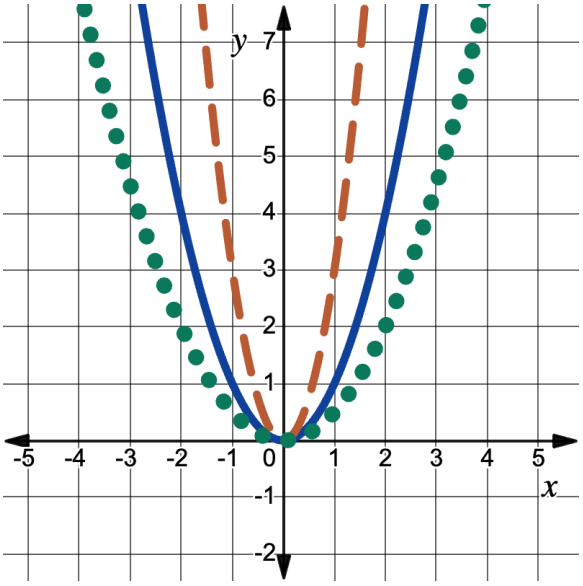
1. The parabola that has been shifted **horizontally right** 3 units represents the function $f(x) = (x - 3)^2$ since $h = 3$ and $3 > 0$ (orange, dashed graph).

2. The parabola that has been shifted **horizontally left** 2 units represents the function $f(x) = (x + 2)^2$ since $h = -2$ and $-2 < 0$ (green, dotted graph).

GRAPH OF A QUADRATIC FUNCTION OF THE FORM $f(x) = ax^2$

The coefficient a in the function $f(x) = ax^2$ affects the graph of $f(x) = x^2$ by stretching or compressing it.

- If $0 < |a| < 1$, the graph of $f(x) = ax^2$ will be "wider" than the graph of $f(x) = x^2$.
- If $|a| > 1$, the graph of $f(x) = ax^2$ will be "skinnier" than the graph of $f(x) = x^2$.



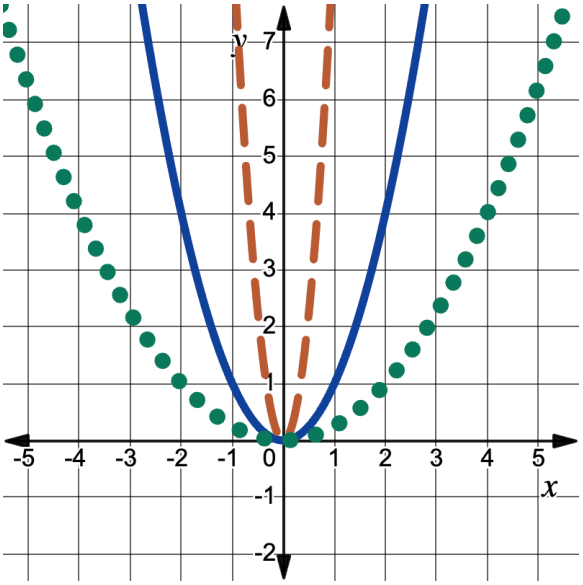
Again, the parent function, $f(x) = x^2$, is graphed using a blue, solid line.

1. The parabola that has been **vertically compressed** by a scale factor of 2 represents the function $f(x) = \frac{1}{2}x^2$ since $a = \frac{1}{2}$ and $0 < \frac{1}{2} < 1$ (green, dotted graph).
2. The parabola that has been **vertically stretched** by a scale factor of 3 represents the function $f(x) = 3x^2$ since $a = 3$ and $3 > 1$ (orange, dashed graph).

GRAPH OF A QUADRATIC FUNCTION OF THE FORM $f(x) = (bx)^2$

The coefficient b in the function $f(x) = (bx)^2$ affects the graph of $f(x) = x^2$ by stretching or compressing it.

1. If $0 < |b| < 1$, the graph of $f(x) = (bx)^2$ will appear "wider" than the graph of $f(x) = x^2$.
2. If $|b| > 1$, the graph of $f(x) = (bx)^2$ will appear "skinnier" than the graph of $f(x) = x^2$.



Again, the parent function, $f(x) = x^2$, is graphed using a blue, solid line.

1. The parabola that has been **horizontally compressed** by a scale factor of 3 represents the function $f(x) = (3x)^2$ since $b = 3$ and $3 > 1$ (orange, dashed graph).
2. The parabola that has been **horizontally stretched** by a scale factor of 2 represents the function $f(x) = \left(\frac{1}{2}x\right)^2$ since $b = \frac{1}{2}$ and $0 < \frac{1}{2} < 1$ (green, dotted graph).

TRY IT **Graphing Quadratic Functions Using Transformations**

How does the graph of $f(x) = x^2 + 3$ change from $f(x) = x^2$?

7.12.3 Understanding the Behaviors of a Graph in Relation to Its Quadratic Expression

Activity

1. Use the table with different values of x to answer questions a–g for the values of $x^2 + 10$. (You may also use a spreadsheet tool, if available.)

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$x^2 + 10$	a. ___	b. ___	c. ___	d. ___	e. ___	f. ___	g. ___

a. When $x = -3$ and $x^2 = 9$, what is the value of $x^2 + 10$?

b. When $x = -2$ and $x^2 = 4$, what is the value of $x^2 + 10$?

c. When $x = -1$ and $x^2 = 1$, what is the value of $x^2 + 10$?

d. When $x = 0$ and $x^2 = 0$, what is the value of $x^2 + 10$?

e. When $x = 1$ and $x^2 = 1$, what is the value of $x^2 + 10$?

f. When $x = 2$ and $x^2 = 4$, what is the value of $x^2 + 10$?

g. When $x = 3$ and $x^2 = 9$, what is the value of $x^2 + 10$?

2. Use the table with different values of x to answer questions a–g for the values of $x^2 - 3$. (You may also use a spreadsheet tool, if available.)

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$x^2 - 3$	a. ___	b. ___	c. ___	d. ___	e. ___	f. ___	g. ___

a. When $x = -3$ and $x^2 = 9$, what is the value of $x^2 - 3$?

b. When $x = -2$ and $x^2 = 4$, what is the value of $x^2 - 3$?

c. When $x = -1$ and $x^2 = 1$, what is the value of $x^2 - 3$?

d. When $x = 0$ and $x^2 = 0$, what is the value of $x^2 - 3$?

e. When $x = 1$ and $x^2 = 1$, what is the value of $x^2 - 3$?

f. When $x = 2$ and $x^2 = 4$, what is the value of $x^2 - 3$?

g. When $x = 3$ and $x^2 = 9$, what is the value of $x^2 + 10$?

Select the **solution** button to see the solutions to questions 1 – 2 organized into a table.

3. In the earlier activity "Transformations with Quadratic Functions," you observed the effects on the graph of adding or subtracting a constant term from x^2. Study the values in the table created from questions 1 – 2 in this activity. Use these values to explain why the graphs in the earlier activity changed the way they did when a constant term is added or subtracted.

4. Use the table with different values of x to answer questions a–g for the values of $2x^2$. (You may also use a spreadsheet tool, if available.)

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$2x^2$	a. ____	b. ____	c. ____	d. ____	e. ____	f. ____	g. ____

a. When $x = -3$ and $x^2 = 9$, what is the value of $2x^2$?

b. When $x = -2$ and $x^2 = 4$, what is the value of $2x^2$?

c. When $x = -1$ and $x^2 = 1$, what is the value of $2x^2$?

d. When $x = 0$ and $x^2 = 0$, what is the value of $2x^2$?

e. When $x = 1$ and $x^2 = 1$, what is the value of $2x^2$?

f. When $x = 2$ and $x^2 = 4$, what is the value of $2x^2$?

g. When $x = 3$ and $x^2 = 9$, what is the value of $2x^2$?

5. Use the table with different values of x to answer questions a–g for the values of $\frac{1}{2}x^2$. (You may also use a spreadsheet tool, if available.)

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$\frac{1}{2}x^2$	a. ____	b. ____	c. ____	d. ____	e. ____	f. ____	g. ____

a. When $x = -3$ and $x^2 = 9$, what is the value of $\frac{1}{2}x^2$?

b. When $x = -2$ and $x^2 = 4$, what is the value of $\frac{1}{2}x^2$?

c. When $x = -1$ and $x^2 = 1$, what is the value of $\frac{1}{2}x^2$?

d. When $x = 0$ and $x^2 = 0$, what is the value of $\frac{1}{2}x^2$?

e. When $x = 1$ and $x^2 = 1$, what is the value of $\frac{1}{2}x^2$?

f. When $x = 2$ and $x^2 = 4$, what is the value of $\frac{1}{2}x^2$?

g. When $x = 3$ and $x^2 = 9$, what is the value of $\frac{1}{2}x^2$?

6. Use the table with different values of x to answer questions a–g for the values of $-2x^2$. (You may also use a spreadsheet tool, if available.)

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$-2x^2$	a. _____	b. _____	c. _____	d. _____	e. _____	f. _____	g. _____

a. When $x = -3$ and $x^2 = 9$, what is the value of $-2x^2$?

b. When $x = -2$ and $x^2 = 4$, what is the value of $-2x^2$?

c. When $x = -1$ and $x^2 = 1$, what is the value of $-2x^2$?

d. When $x = 0$ and $x^2 = 0$, what is the value of $-2x^2$?

e. When $x = 1$ and $x^2 = 1$, what is the value of $-2x^2$?

f. When $x = 2$ and $x^2 = 4$, what is the value of $-2x^2$?

g. When $x = 3$ and $x^2 = 9$, what is the value of $-2x^2$?

Select the **solution** button to see the solutions to questions 4 – 6 organized into a table.

7. In the earlier activity "Transformations with Quadratic Functions," you observed the effects on the graph of multiplying x^2 by different coefficients. Study the values in the table created from questions 1 and 2 in this activity. Use these values to explain why the graphs in the earlier activity changed the way they did when x^2 is multiplied by a number greater than 1, by a negative number less than or equal to -1, and by numbers between -1 and 1.

Video: Understanding the Behaviors of a Graph in Relation to Its Quadratic Expression

Watch the following video to learn more about the behaviors of a graph in relation to its quadratic expression.

Understanding the Behaviors of a Graph in Relation to Its Quadratic Expression (https://openstax.org/r/quadratic-behavior)

Self Check

Which curve is narrower than $f(x) = x^2$?

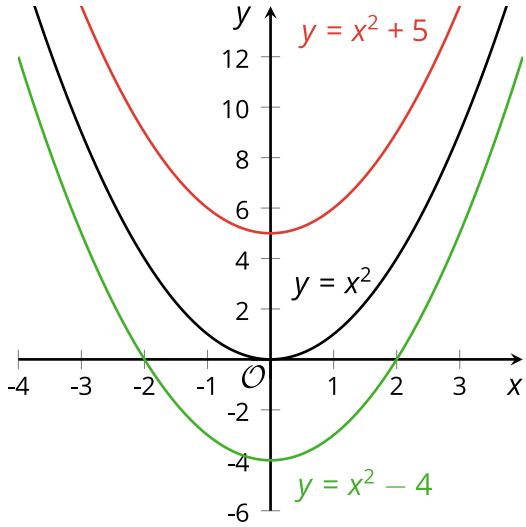
a. $f(x) = (x + 3)^2$
b. $f(x) = \frac{1}{3}x^2$
c. $f(x) = 3x^2$
d. $f(x) = x^2 + 3$

Additional Resources

Comparing Transformations to Quadratic Graphs

Remember that the graph representing any quadratic function is a shape called a parabola. People often say that a parabola "opens upward" when the lowest point on the graph is the vertex (where the graph changes direction), and "opens downward" when the highest point on the graph is the vertex. Each coefficient in a quadratic expression written in standard form $ax^2 + bx + c$ tells us something important about the graph that represents it.

The graph of $f(x) = x^2$ is a parabola opening upward with vertex at $(0, 0)$. Adding a constant term 5 gives $f(x) = x^2 + 5$ and raises the graph by 5 units. Subtracting 4 from x^2 gives $f(x) = x^2 - 4$ and moves the graph 4 units down.



x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$x^2 + 5$	14	9	6	5	6	9	14
$x^2 - 4$	5	0	-3	-4	-3	0	5

A table of values can help us see that adding 5 to x^2 increases all the output values of $f(x) = x^2$ by 5, which explains why the graph moves up 5 units. Subtracting 4 from x^2 decreases all the output values of $f(x) = x^2$ by 4, which explains why the graph shifts down by 4 units.

In general, the constant term of a quadratic expression in standard form influences the vertical position of the graph. An expression with no constant term (such as x^2 or $x^2 + 9x$) means that the constant term is 0, so the y-intercept of the graph is on the x-axis. It's not shifted up or down relative to the x-axis.

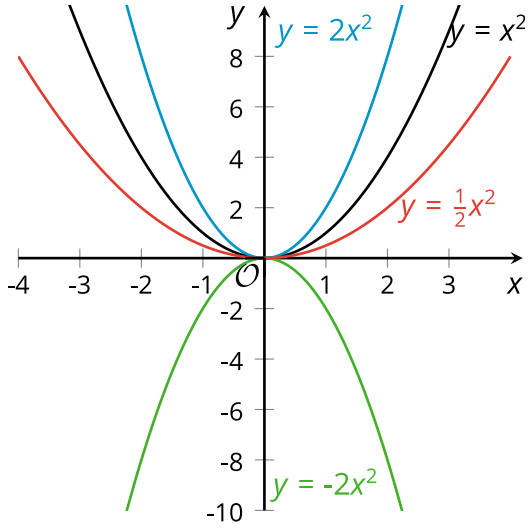
Coefficient of the Squared Term

The coefficient of the squared term in a quadratic function also tells us something about its graph. The coefficient of the squared term in $f(x) = x^2$ is 1. Its graph is a parabola that opens upward.

Multiplying x^2 by a number greater than 1 makes the graph steeper, so the parabola is narrower than that representing x^2.

Multiplying x^2 by a number less than 1 but greater than 0 makes the graph less steep, so the parabola is wider than that representing x^2.

Multiplying x^2 by a number less than 0 makes the parabola open downward.



x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$2x^2$	18	8	2	0	2	8	18
$-2x^2$	-18	-8	-2	0	-2	-8	-18

If we compare the output values of $2x^2$ and $-2x^2$, we see that they are opposites, which suggests that one graph would be a reflection of the other across the x-axis.

In the previous examples, parameters were being applied to the output values of the function. In other words, after an input value was squared, something was then added, subtracted, or multiplied with the squared value. The resulting impact on the graph was to vertically alter it.

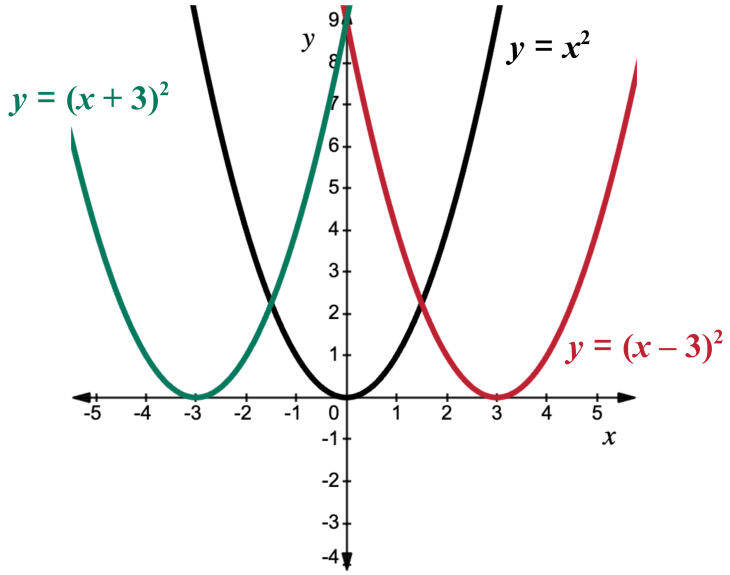
However, what happens when the input value is altered before applying the exponent? In these situations, the impact on the graph is to produce a horizontal transformation.

Horizontal Shifts

Return to the quadratic parent function, $f(x) = x^2$. The function is equivalent to $f(x) = (x - 0)^2$. While a bit obvious, this implies there is no change to the input value before being squared.

If the number used to change the input is greater than 0, then the graph of the function is shifted to the right.

Conversely, if the number used to change the input is less than 0, the graph of the function is shifted to the left. Note the impact to the subtraction operation when negative values are subtracted from the input.



x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$(x-3)^2$	36	25	16	9	4	1	0
$(x--3)^2=(x+3)^2$	0	1	4	9	16	25	36

Notice in this table of values that the change to the input values takes place before squaring the term. The constant is directly affecting the x-value before applying the function rule.

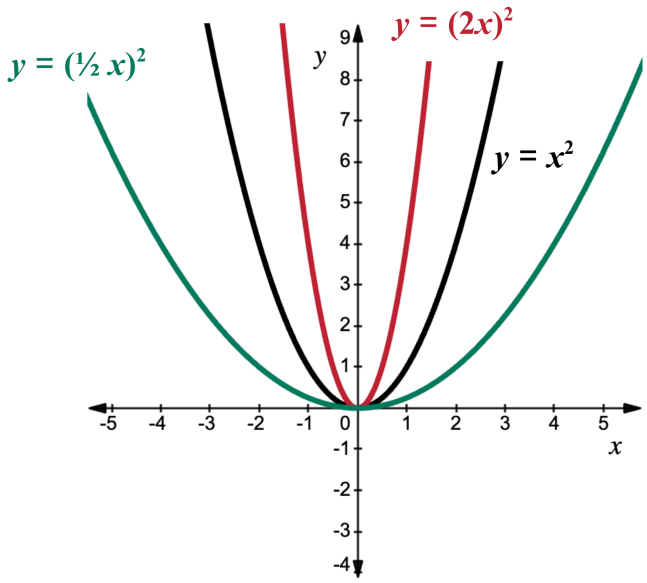
Examine the vertex of each graph. When we compare the graphs of $(x-3)^2$ and $(x+3)^2$ to the graph of the parent function, we see the parabola has been shifted to the right 3 units and left 3 units, respectively. Thus, a horizontal shift in the graph results when a constant has been subtracted from the input value.

Horizontal Dilations and Reflections

Now we will investigate the impact of a coefficient on the input value.

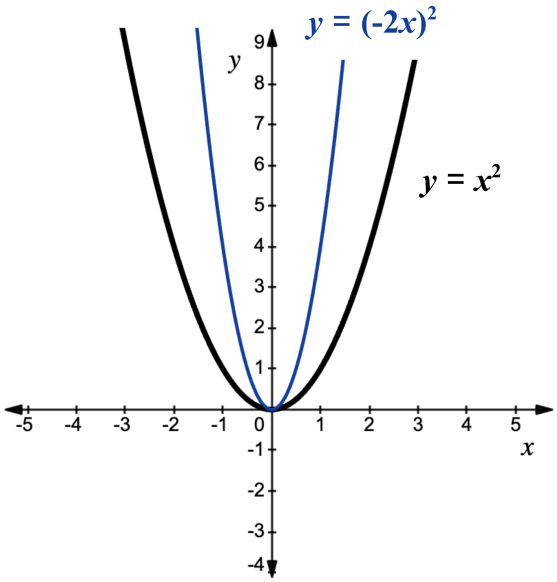
Multiplying the input value by a value greater than 1 and then applying the exponent, results in a parabola that has been horizontally compressed.

Multiplying the input value by a value that is between 0 and 1, yields a parabola that has been horizontally stretched.



x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$(2x)^2$	36	16	4	0	4	16	36
$(\frac{1}{2}x)^2$	2.25	1	0.25	0	0.25	1	2.25
$(-2x)^2$	36	16	4	0	4	16	36

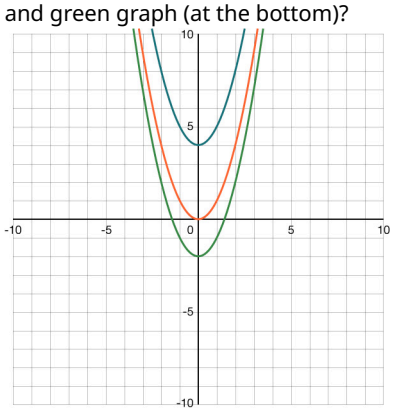
If the input value is multiplied by a negative number, then the parabola is reflected over the y-axis. However, due to the symmetric nature of parabolas, this is difficult to discern visually.



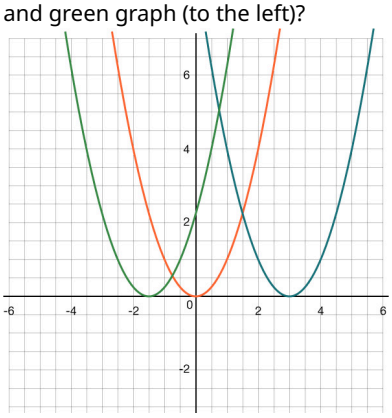
Multiplying the input value by a number changes the quantity that is squared. As a result, the graph of the equation is horizontally stretched or compressed.

TRY IT **Comparing Transformations to Quadratic Graphs**

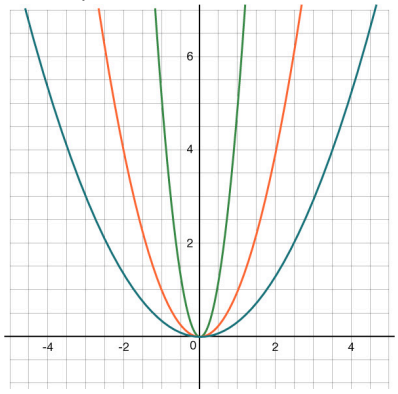
1. If the red graph (in the middle) is the parent function, $f(x) = x^2$, what are the equations of the blue graph (at the top) and green graph (at the bottom)?



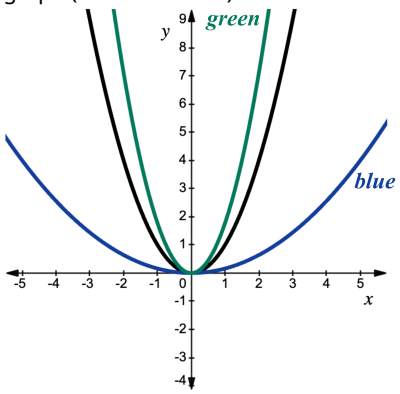
2. If the red graph (in the middle) is the parent function, $f(x) = x^2$, what are the equations of the blue graph (to the right) and green graph (to the left)?



3. The red graph (in the middle) is $f(x) = x^2$. The equations of the other graphs are $f(x) = \frac{1}{3}x^2$ and $f(x) = 5x^2$. Which function matches the green graph (on the inside)? The blue graph (on the outside)?



4. The black graph (in the middle) is the parent function $f(x) = x^2$. The equations of the other graphs are $f(x) = (\frac{2}{5}x)^2$ and $f(x) = (\frac{4}{3}x)^2$. Which function matches the green graph (on the inside)? The blue graph (on the outside)?



7.12.4 Representations of Quadratic Functions

Activity

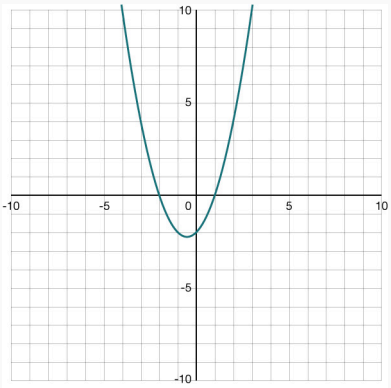
Your teacher will give your group a set of cards (https://openstax.org/r/set-cards-7-12-4). Each card contains a graph or an equation.

- Take turns with your partner to sort the cards into sets so that each set contains two equations and a graph that all represent the same quadratic function.
- For each set of cards that you put together, explain to your partner how you know they belong together.
- For each set that your partner puts together, listen carefully to their explanation. If you disagree, discuss your thinking and work to reach an agreement.

Once all the cards are sorted and discussed, record the equivalent equations, sketch the corresponding graph, and write a brief note or explanation about why the representations were grouped together.

Self Check

Examine the graph below.



Which function matches the graph?

a. $f(x) = (x - 2)(x - 1)$
b. $f(x) = (x + 2)(x + 1)$
c. $f(x) = (x + 2)(x - 1)$
d. $f(x) = (x - 2)(x + 1)$

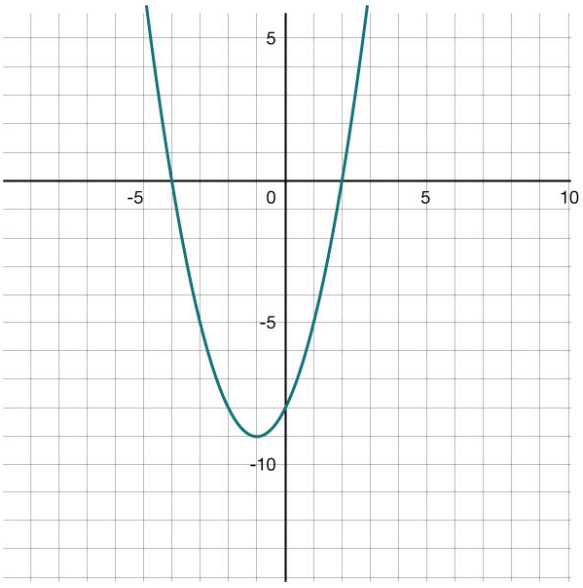
Additional Resources

Matching Quadratic Functions and Their Graphs

Below is a quadratic function represented in standard form, factored form, and with a graph.

Standard Form: $f(x) = x^2 + 2x - 8$

Factored Form: $f(x) = (x + 4)(x - 2)$



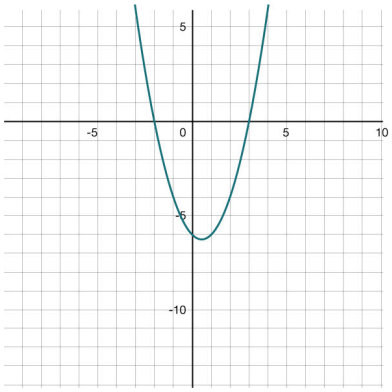
Notice the factored form helps find the zeros: $x = -4$ and $x = 2$.

The factored form gives the zeros since the zeros (x-intercepts) are located where each factor equals 0.

The standard form, $f(x) = x^2 + 2x - 8$, helps give the y-intercept. The y-intercept occurs when $x = 0$. Here, $f(0) = -8$. On the graph, the y-intercept is $(0, -8)$.

TRY IT **Matching Quadratic Functions and Their Graphs**

Given the graph below and its function in standard form and factored form, identify all intercepts.



Standard form: $f(x) = x^2 - x - 6$ Factored form: $f(x) = (x - 3)(x + 2)$

7.12.5 Identifying Graphs of Quadratic Equations

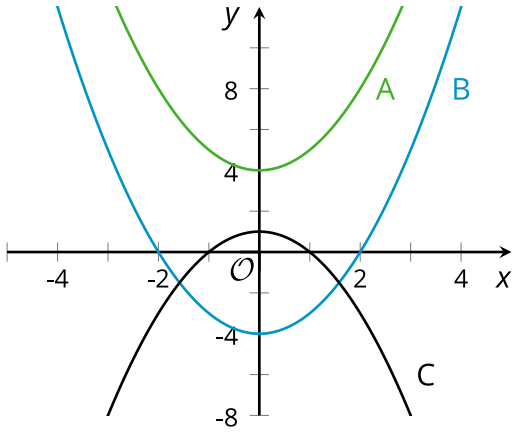
Cool Down

Here are graphs that represent three quadratic functions, defined by:

$f(x) = x^2 - 4$

$g(x) = 1 - x^2$

$h(x) = x^2 + 4$

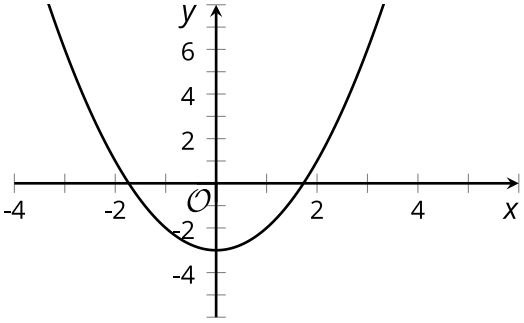


1. Which graph represents the equation $f(x) = x^2 - 4$?
2. Which graph represents the equation $g(x) = 1 - x^2$?
3. Which graph represents the equation $h(x) = x^2 + 4$?
4. Which part of the equation tells us that Graph C opens downward?

7.12.6 Practice

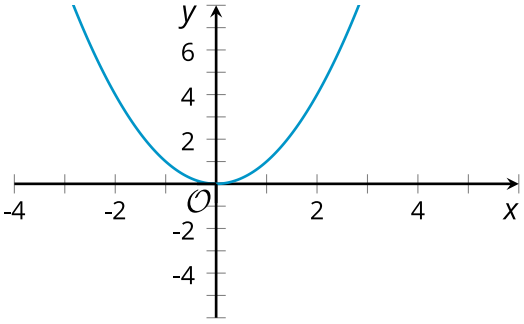
Complete the following questions to practice the skills you have learned in this lesson.

1. Choose the quadratic equation that is represented by the graph.



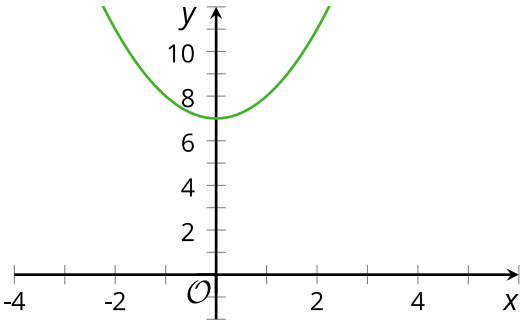
a. $f(x) = x^2 + 7$
b. $f(x) = x^2$
c. $f(x) = x^2 - 3$
d. $f(x) = x^2 + 5$

2. Choose the quadratic equation that is represented by the graph.



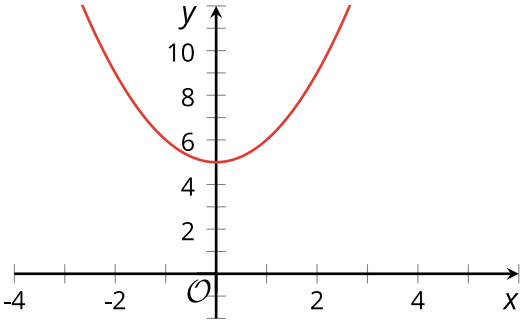
a. $f(x) = x^2 + 7$
b. $f(x) = x^2$
c. $f(x) = x^2 - 3$
d. $f(x) = x^2 + 5$

3. Choose the quadratic equation that is represented by the graph.



a. $f(x) = x^2 + 7$
b. $f(x) = x^2$
c. $f(x) = x^2 - 3$
d. $f(x) = x^2 + 5$

4. Choose the quadratic equation that is represented by the graph.



a. $f(x) = x^2 + 7$
b. $f(x) = x^2$
c. $f(x) = x^2 - 3$
d. $f(x) = x^2 + 5$

5. The following two equations are equivalent.

- factored form $f(x) = (x + 2)(x + 3)$
- standard form $f(x) = x^2 + 5x + 6$

a. Which equation helps find the x-intercepts most efficiently?

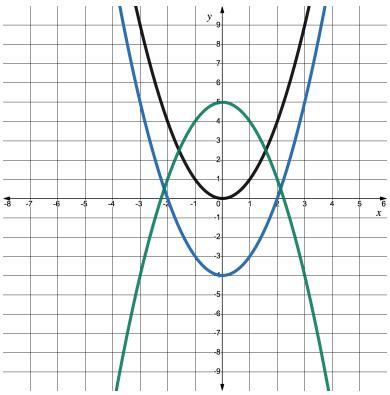
a. standard form
b. factored form

b. Which equation helps find the y-intercept most efficiently?

a. standard form
b. factored form

6. There are three equations represented on this graph:
$f(x) = x^2$ (black graph, at the top, opening upward)
$f(x) = x^2 - 4$
$f(x) = -x^2 + 5$;

Use the graph to answer questions a and b.



a. Which graph represents the equation $f(x) = x^2 - 4$?

a. Blue graph (in the middle, opening upward)
b. Green graph (at the bottom, opening downward)

b. Which graph represents the equation $f(x) = -x^2 + 5$?

a. Blue graph (in the middle, opening upward)
b. Green graph (at the bottom, opening downward)

7. **Select two** equations whose graphs have a positive y-intercept.

a. $f(x) = x^2 + 3x - 2$
b. $f(x) = x^2 - 10x$
c. $f(x) = (x - 1)^2$
d. $f(x) = 5x^2 - 3x - 5$
e. $f(x) = (x + 1)(x + 2)$

8. Which function represents the graph of $f(x) = x^2$ shifted 5 units to the right?

a. $f(x) = (x - 5)^2$
b. $f(x) = (x + 5)^2$
c. $f(x) = x^2 - 5$
d. $f(x) = x^2 + 5$

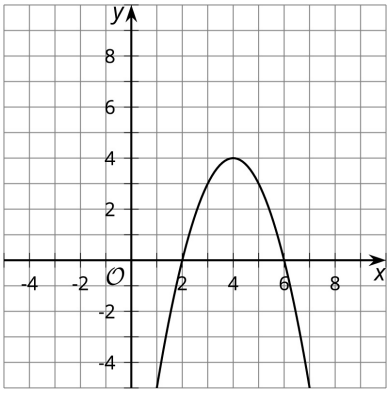
9. **Select three** equations whose graphs open downward.

a. $f(x) = (1 - x)^2$
b. $f(x) = 1 - x^2$
c. $f(x) = -(x - 1)^2$
d. $f(x) = -1 + x^2$
e. $f(x) = (1 - x)(x + 2)$

10. Which curve is the narrowest?

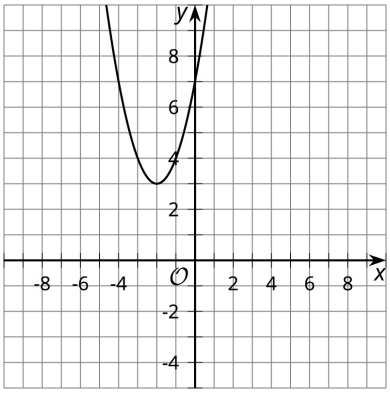
a. $f(x) = (\frac{1}{3}x)^2$
b. $f(x) = \frac{1}{3}x^2$
c. $f(x) = (3x)^2$
d. $f(x) = 3x^2$

11. Which equation is represented by the graph?



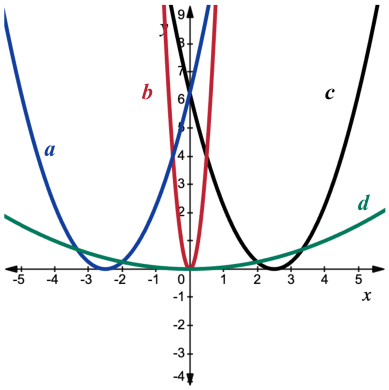
a. $f(x) = -(x + 4)^2 + 4$
b. $f(x) = -(x - 4)^2 + 4$
c. $f(x) = (x + 4)^2 + 4$
d. $f(x) = (x - 4)^2 + 4$

12. Which equation is represented by the graph?



a. $f(x) = (x + 3)^2 + 2$
b. $f(x) = (x + 2)^2 + 3$
c. $f(x) = (x - 3)^2 + 2$
d. $f(x) = (x - 2)^2 + 3$

For questions 13 - 14, use the following graph.



13. Which of the graphs has been horizontally stretched?

a. Graph a
b. Graph b
c. Graph c
d. Graph d

14. Which function most closely represents the equation for graph c?

a. $f(x) = (\frac{1}{4}x)^2$
b. $f(x) = (x + 2.5)^2$
c. $f(x) = (4x)^2$
d. $f(x) = (x - 2.5)^2$

7.12.7 Lesson Summary

In this lesson, you learned how to:

- Explain how the a and c in $ax^2 + bx + c$ affect the graph of the equation.
- Explain how graphs, tables, and equations that represent the same quadratic function are related.

Here are the **activities** that helped you reach those goals:

- 7.12.1: Using Coefficients and Constant Terms to Identify Graphs of Linear Equations
  - In this activity, you matched three linear equations with their graphs by comparing slopes and y-intercepts.
- 7.12.2: Transformations with Quadratic Functions
  - In this activity, you explored how transformations affect the graphs of quadratic functions.
  - 7.12.2: Self Check
  - 7.12.2: Additional Resources
- 7.12.3: Understanding the Behaviors of a Graph in Relation to Its Quadratic Expression
  - In this activity, you examined the values in tables and how the values from a quadratic expression affect its graph.
  - 7.12.3: Self Check
  - 7.12.3: Additional Resources
- 7.12.4: Representations of Quadratic Functions
  - In this activity, you used the standard form and factored form of quadratic functions as well as their graphs to identify intercepts to match the equivalent forms and their graphs.
  - 7.12.4: Self Check
  - 7.12.4: Additional Resources
- 7.12.5: Identifying Graphs of Quadratic Equations
  - In this activity, you matched the graphs of quadratic functions with their equations.

After these activities, you completed the following **practice**:

- 7.12.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

7.13 GRAPHING THE STANDARD FORM, PART 2

7.13.0 Lesson Overview

In this lesson, you will work on activities that focus on the coefficient of the linear term, the b in $ax^2 + bx + c$, and how changes to it affect the graph. You will discover that adding a linear term to the squared term translates the graph in both horizontal and vertical directions. This understanding will show that writing an expression such as $x^2 + b$ in factored form can help us reason how to sketch the graph.

When you finish this lesson, you will be able to:

- Explain how the b in $ax^2 + bx + c$ affects the graph of the equation.
- Match equations given in standard and factored form with their graph.

Here are the **activities** that will help you reach those goals:

- 7.13.1: Equivalent Expressions
- 7.13.2: The Linear Term in a Quadratic Expression
  - 7.13.2: Self Check
  - 7.13.2: Additional Resources
- 7.13.3: Writing Equations that Represent a Graph
  - 7.13.3: Self Check
  - 7.13.3: Additional Resources
- 7.14.4: Writing Quadratic Equations from Real Solutions
  - 7.14.4: Self Check
  - 7.14.4: Additional Resources
- 7.13.5: Graphing Quadratic Equations

After that, you'll **practice and review**.

- 7.13.6: Practice
- 7.13.7: Lesson Summary

7.13.1 Equivalent Expressions

Warm Up

1. Complete each row with an equivalent expression in standard form or factored form.

Standard Form	Factored Form
x^2	a. ____
d. ____	$x(x + 9)$
$x^2 - 18x$	b. ____
e. ____	$x(6 - x)$
$-x^2 + 10x$	c. ____
f. ____	$-x(x + 2.75)$

a. What is the factored form of x^2?

b. What is the factored form of $x^2 - 18x$?

c. What is the factored form of $-x^2 + 10x$?

d. What is the standard form of $x(x + 9)$?

e. What is the standard form of $x(6 - x)$?

f. What is the standard form of $-x(x + 2.75)$?

2. What do the quadratic expressions in each column have in common (besides the fact that everything in the left column is in standard form and everything in the other column is in factored form)? Be prepared to share your observations.

7.13.2 The Linear Term in a Quadratic Expression

Activity

1. Use the graphing tool or technology outside the course. Graph $y = x^2$, and then experiment with adding different linear terms (for example, $x^2 + 4x$, $x^2 + 20x$, $x^2 - 50x$). Record your observations.

Access multimedia content (http://openstax.org/books/algebra-1/pages/7-13-2-the-linear-term-in-a-quadratic-expression)

Use your observations from question 1 to answer questions 2 – 5 and complete the table.

equation	x -intercepts	x-coordinate of vertex
$y = x^2 + 6x$	2a. ____	2b. ____
$y = x^2 - 10x$	3a. ____	3b. ____
$y = -x^2 + 50x$	4a. ____	4b. ____
$y = -x^2 - 36x$	5a. ____	5b. ____

2. Find the x-intercepts and the x-coordinate of the vertex for the equation $y = x^2 + 6x$.

a. What are the x-intercepts for the equation $y = x^2 + 6x$?

b. What is the x-coordinate of the vertex for the equation $y = x^2 + 6x$?

3. Find the x-intercepts and the x-coordinate of the vertex for the equation $y = x^2 - 10x$.

a. What are the x-intercepts for the equation $y = x^2 - 10x$?

b. What is the x-coordinate of the vertex for the equation $y = x^2 - 10x$?

4. Find the x-intercepts and the x-coordinate of the vertex for the equation $y = -x^2 + 50x$.

a. What are the x-intercepts for the equation $y = -x^2 + 50x$?

b. What is the x-coordinate of the vertex for the equation $y = -x^2 + 50x$?

5. Find the x-intercepts and the x-coordinate of the vertex for the equation $y = -x^2 - 36x$.

a. Find the x-intercepts for the equation $y = -x^2 - 36x$.

b. What is the x-coordinate of the vertex for the equation $y = -x^2 - 36x$?

Some quadratic expressions have no linear terms. For questions 6 – 7, find the x-intercepts and the x-coordinate of the vertex of the graph representing each equation:

$y = x^2 - 25$

$y = x^2 + 16$

(Note it is possible for the graph to not intersect the x-axis.) If you get stuck, try graphing the equations.

6. Find the x-intercepts and the x-coordinate of the vertex for the equation $y = x^2 - 25$.

a. What are the x-intercepts for the equation $y = x^2 - 25$?

b. What is the x-coordinate of the vertex for the equation $y = x^2 - 25$?

7. Find the x-intercepts and the x-coordinate of the vertex for the equation $y = x^2 + 16$.

a. What are the x-intercepts for the equation $y = x^2 + 16$?

b. What is the x-coordinate of the vertex for the equation $y = x^2 + 16$?

Self Check

Find the x-coordinates of all of the x-intercepts of the function $f(x) = x^2 - 5x$.

a. $x = 0, x = -5$
b. $x = 0$
c. $x = 0, x = 5$
d. $x = 5$

Additional Resources

Quadratics Without Linear Terms

EXAMPLE

Find the x-intercepts and the vertex for $f(x) = x^2 - 4$.

Step 1 - Write the function in factored form. Use difference of squares:

$f(x) = x^2 - 4 = (x - 2)(x + 2)$

Step 2 - Set each factor equal to 0.

$x - 2 = 0$

$x = 2$

$x + 2 = 0$

$x = -2$

The x-intercepts are $(2, 0)$ and $(-2, 0)$.

Step 3 - Find the x-value in the middle of the two x-intercepts. This occurs at $x = 0$. This is the x-coordinate of the vertex.

To find the point at the vertex, substitute into the function:

$f(0) = 0^2 - 4 = -4$

Vertex: $(0, -4)$

TRY IT **Quadratics Without Linear Terms**

Find the x-intercepts and the vertex for $f(x) = x^2 - 64$.

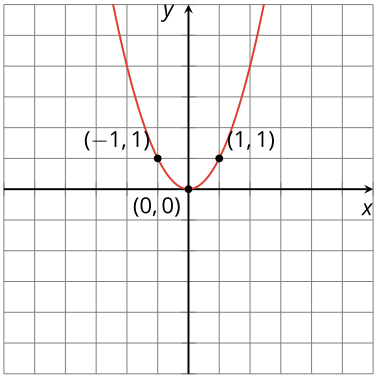
7.13.3 Writing Equations that Represent a Graph

Activity

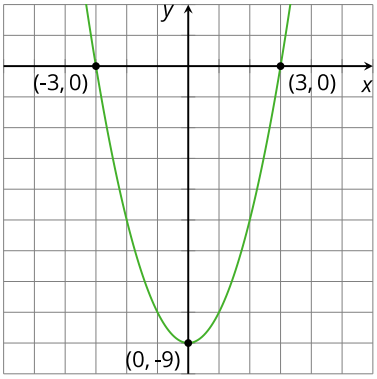
Use the graphing tool or technology outside the course. Identify the function that matches each graph. Make sure your graph goes through all three points shown!

Access multimedia content (http://openstax.org/books/algebra-1/pages/7-13-3-writing-equations-that-represent-a-graph)

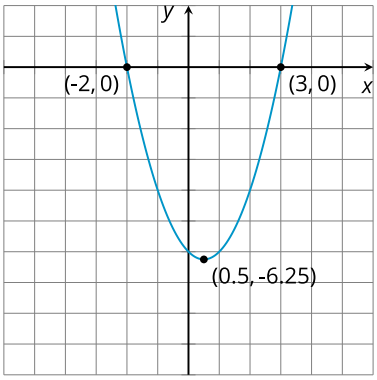
1. Identify the function of Graph A.



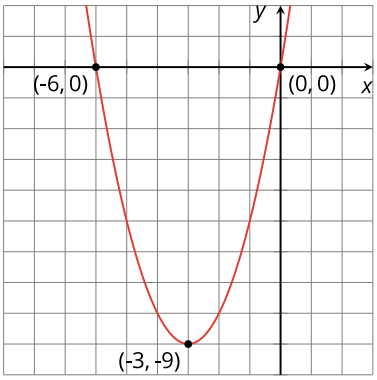
2. Identify the function of Graph B.



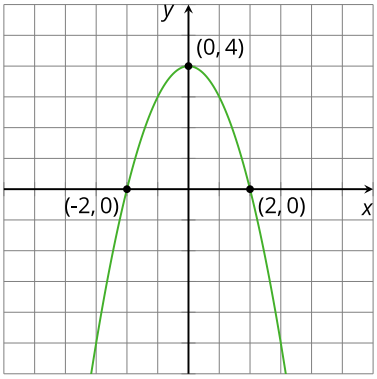
3. Identify the function of Graph C.



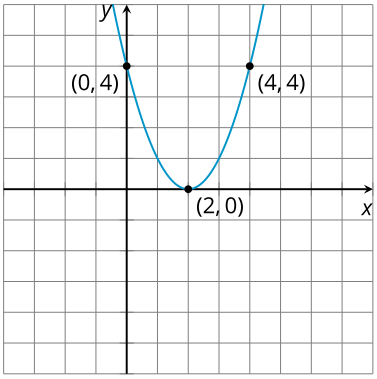
4. Identify the function of Graph D.



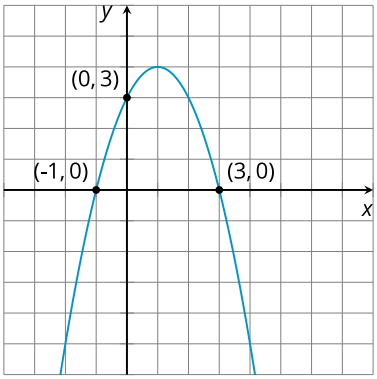
5. Identify the function of Graph E.



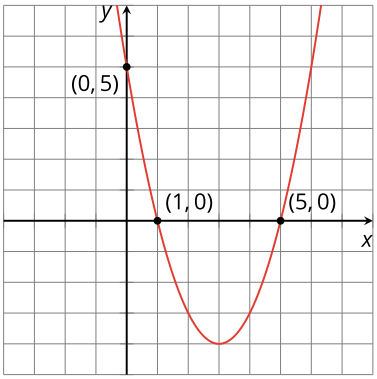
6. Identify the function of Graph F.



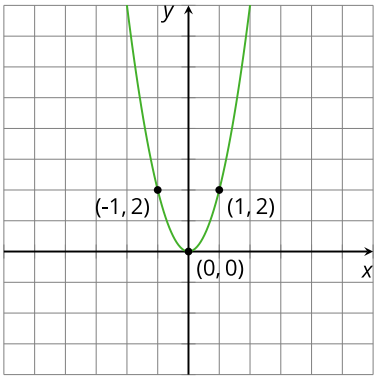
7. Identify the function of Graph G.



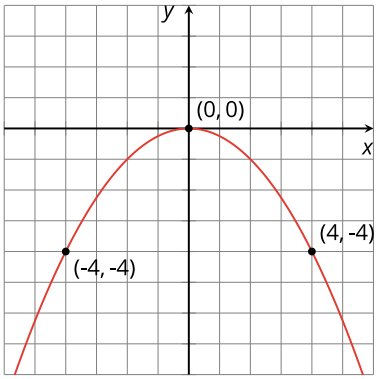
8. Identify the function of Graph H.



9. Identify the function of Graph I.



10. Identify the function of Graph J.



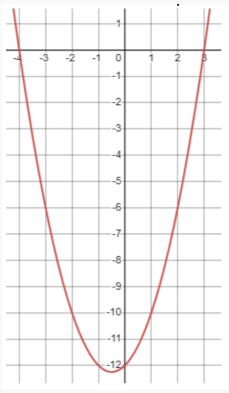
Video: Writing Equations that Represent a Quadratic Graph

Watch the following video to learn more about writing equations that represent a quadratic graph.

Writing Equations that Represent a Quadratic Graph (https://openstax.org/r/quadratic-graph-equations)

Self Check

Which equation describes the graph below?

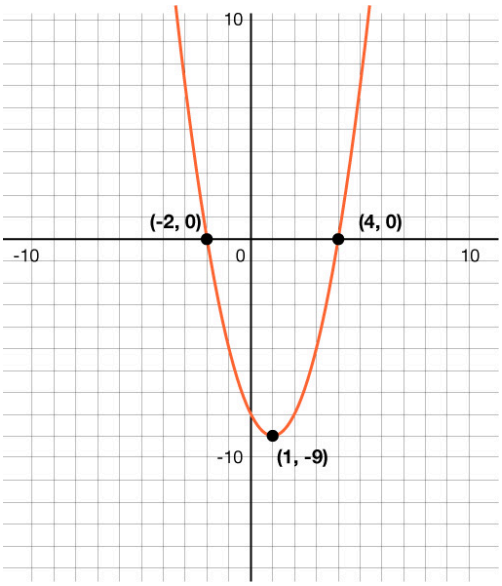


a. $f(x) = x(x + 3)$
b. $f(x) = x + 12$
c. $f(x) = (x - 3)(x + 4)$
d. $f(x) = (x + 3)(x - 4)$

Additional Resources

Writing Quadratic Equations from Graphs

Write an equation for the graph below:



First, locate the x-intercepts at $(-2, 0)$ and $(4, 0)$.

Next, write the factor for each x-intercept.

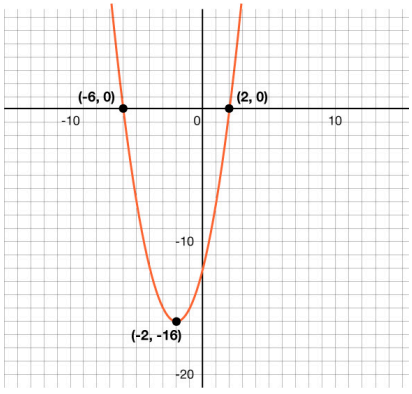
The factors that result in the given zeros are $(x + 2)$ and $(x - 4)$.

The equation for this quadratic function is $f(x) = (x + 2)(x - 4)$.

Use graphing technology to check your answer.

TRY IT **Writing Quadratic Equations from Graphs**

Write an equation for the graph below:



7.13.4 Writing Quadratic Equations from Real Solutions

For questions 1 – 4, use the following information:

A parabola crosses the x-axis at (-1,0) and (2,0). We want to write a quadratic equation with those x-intercepts. We can do this without graphing.

1. When $y = 0$, what must x be?
2. Rewrite the equations from number 1 so they both equal zero.

3. What equation do you get when you multiply the previous equations together?
4. Multiply the binomials.

Self Check

The solution to a quadratic equation is $x = 4$ and $x = -8$. What is the equation?

$x^2 + 32x - 12 = 0$
$x^2 + 4x - 32 = 0$
$x^2 - 32x + 4 = 0$
$x^2 - 4x + 32 = 0$

Additional Resources

When you write a quadratic in factored form, that helps you find the x-intercepts.

$y = x^2 + 2x - 3$

$y = (x + 3)(x - 1)$

When we set $y = 0$, we find the places where the parabola crosses the x-axis.

$0 = (x + 3)(x - 1)$

So, $0 = x + 3$ and $0 = x - 1$

$x = -3$ and 1.

-3 and 1 are called the real solutions to the quadratic equation.

We can also start with the real solutions and use them to write quadratic equations.

When we know that $x = -3$ and 1, then that means $0 = x + 3$ and $0 = x - 1$.

So, $0 = (x + 3)(x - 1)$.

And, using the FOIL method to multiply this out, we come to

$0 = x^2 + 2x - 3$.

When y is not equal to zero, we have the following quadratic equation

$y = x^2 + 2x - 3$, which is exactly where we started!

TRY IT **Writing Quadratic Equations from Real Solutions**

1. Write a quadratic from the real solutions $x = 1$ and $x = 7$ without graphing.

7.13.5 Graphing Quadratic Equations

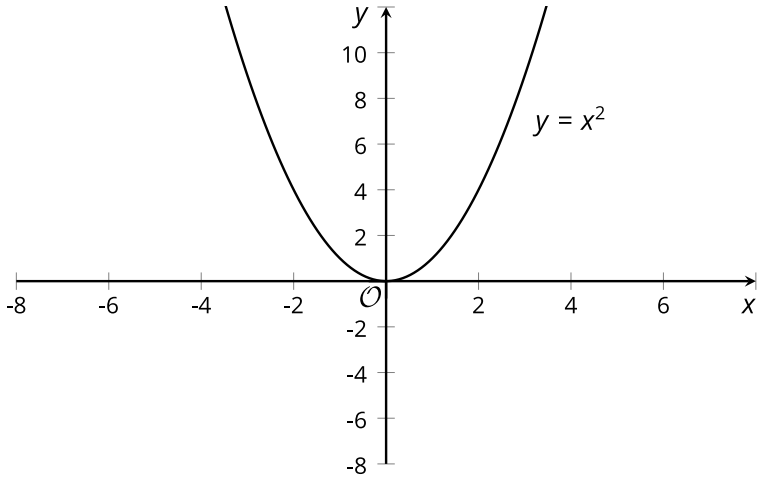
Cool Down

1. Consider the quadratic equation $y = x^2 - 4x$. Consider the graph of the equation when answering questions a and b.

a. Where are the x-intercepts located?

b. What is the x-coordinate of the vertex?

2. Here is a graph of $y = x^2$.

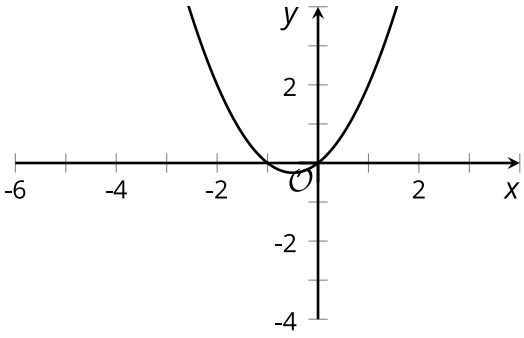


Sketch this graph on your own paper, then sketch a graph of $y = x^2 + 5x$ on the same graph. Briefly explain how you know where to sketch the second graph.

7.13.6 Practice

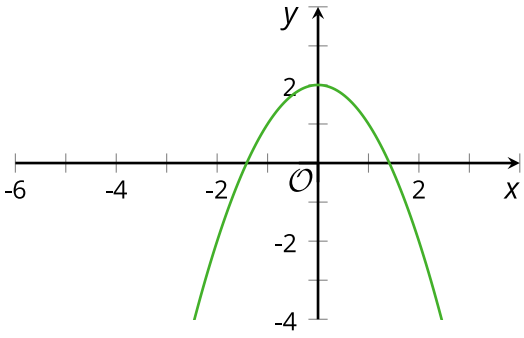
Complete the following questions to practice the skills you have learned in this lesson.

1. Choose the quadratic equation that represents each graph.



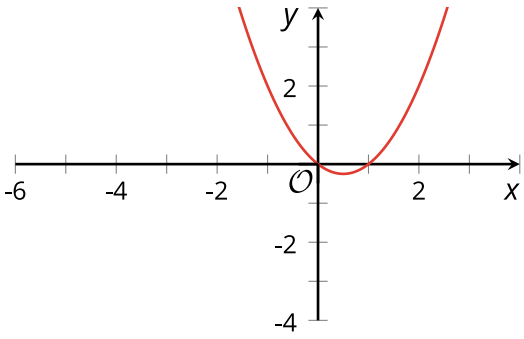
a.

a. $y = x^2 + 3x$
b. $y = x^2 - x$
c. $y = -x^2 + 2$
d. $y = x^2 + x$



b.

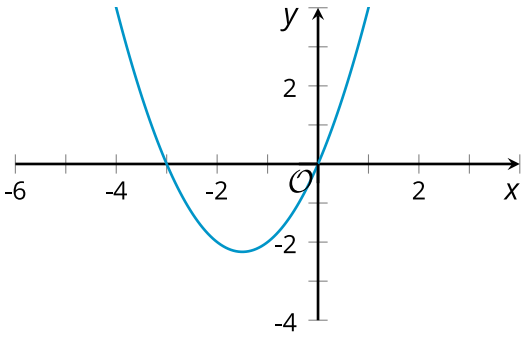
a. $y = x^2 + 3x$
b. $y = x^2 - x$
c. $y = -x^2 + 2$
d. $y = x^2 + x$



c.

a. $y = x^2 + 3x$
b. $y = x^2 - x$
c. $y = -x^2 + 2$
d. $y = x^2 + x$

d.



a. $y = x^2 + 3x$
b. $y = x^2 - x$
c. $y = -x^2 + 2$
d. $y = x^2 + x$

Without graphing the equations, answer questions 2 – 5 to complete the table.

Equation	x-intercepts	x-coordinate of vertex
$y = x^2 + 12x$	2a. _____	2b. _____
$y = x^2 - 3x$	3a. _____	3b. _____
$y = -x^2 + 16x$	4a. _____	4b. _____
$y = -x^2 - 24x$	5a. _____	5b. _____

2. Find the x-intercepts and the x-coordinate of the vertex for the equation $y = x^2 + 12x$.

a. What are the x-intercepts for the equation $y = x^2 + 12x$?

a. $(0, 0), (-24, 0)$
b. $(0, 0), (16, 0)$
c. $(0, 0), (3, 0)$
d. $(0, 0), (-12, 0)$

b. What is the x-coordinate of the vertex for the equation $y = x^2 + 12x$?

3. Find the x-intercepts and the x-coordinate of the vertex for the equation $y = x^2 - 3x$.

a. What are the x-intercepts for the equation $y = x^2 - 3x$?

a. $(0, 0), (-24, 0)$
b. $(0, 0), (16, 0)$
c. $(0, 0), (3, 0)$
d. $(0, 0), (-12, 0)$

b. What is the x-coordinate of the vertex for the equation $y = x^2 - 3x$?

4. Find the x-intercepts and the x-coordinate of the vertex for the equation $y = -x^2 + 16x$.

a. What are the x-intercepts for the equation $y = -x^2 + 16x$?

a. $(0, 0), (-24, 0)$
b. $(0, 0), (16, 0)$
c. $(0, 0), (3, 0)$
d. $(0, 0), (-12, 0)$

b. What is the x-coordinate of the vertex for the equation $y = -x^2 + 16x$?

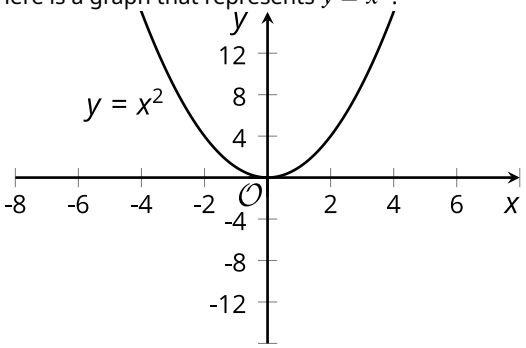
5. Find the x-intercepts and the x-coordinate of the vertex for the equation $y = -x^2 - 24x$.

a. What are the x-intercepts for the equation $y = -x^2 - 24x$?

a. $(0, 0), (-24, 0)$
b. $(0, 0), (16, 0)$
c. $(0, 0), (3, 0)$
d. $(0, 0), (-12, 0)$

b. What is the x-coordinate of the vertex for the equation $y = -x^2 - 24x$?

6. Here is a graph that represents $y = x^2$.



a. Describe what would happen to the graph if the original equation were changed to $y = x^2 - 6x$.

a. The graph would move to the left and up.
b. The graph would move to the right and up.
c. The graph would move to the left and down.
d. The graph would move to the right and down.

b. Predict the x-intercepts of the graph if the original equation were changed to $y = x^2 - 6x$.

a. $(0, 0), (0, -6)$
b. $(0, 0), (-6, 0)$
c. $(0, 0), (0, 6)$
d. $(0, 0), (6, 0)$

c. In which quadrant is the vertex located if the original equation were changed to $y = x^2 - 6x$?

a. First
b. Second
c. Third
d. Fourth

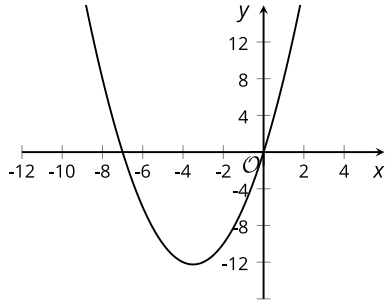
7. **Select two** equations whose graph opens upward.

a. $y = -x^2 + 9x$
b. $y = 10x - 5x^2$
c. $y = (2x - 1)^2$
d. $y = (1 - x)(2 + x)$
e. $y = x^2 - 8x - 7$

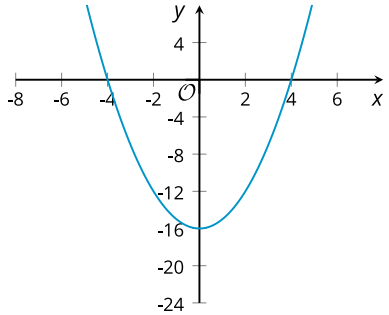
8. Technology required. Use the three graphs to answer questions a–c. Then, use graphing technology to check each

equation.

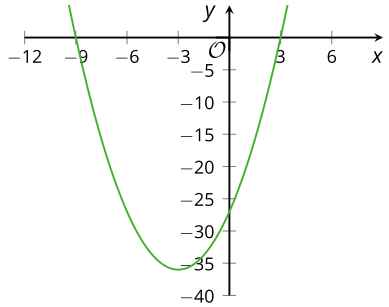
Graph A



Graph B



Graph C



a. Which graph represents the equation $f(x) = x(x + 7)$ or $f(x) = x^2 + 7x$? Then, use graphing technology to check your answer.

   a. Graph A
   b. Graph B
   c. Graph C

b. Which graph represents the equation $g(x) = (x + 4)(x - 4)$ or $g(x) = x^2 - 16$? Then, use graphing technology to check your answer.

   a. Graph A
   b. Graph B
   c. Graph C

c. Which graph represents the equation $h(x) = (x - 3)(x + 9)$ or $h(x) = x^2 + 6x - 27$? Then, use graphing technology to check your answer.

   a. Graph A
   b. Graph B
   c. Graph C

9. Choose the quadratic equation written from the real solutions $x = -4$ and $x = -3$.

   a. $y = x^2 - 7x - 12$
   b. $y = x^2 + 7x - 12$
   c. $y = x^2 + 7x + 12$
   d. $y = x^2 - 7x + 12$

7.13.7 Lesson Summary

In this lesson, you learned how to:

- Explain how the b in $ax^2 + bx + c$ affects the graph of the equation.
- Match equations given in standard and factored form with their graph.

Here are the activities that helped you reach those goals:

- 7.13.1: Equivalent Expressions
  - In this activity, you compared standard and factored forms of quadratic equations and made observations about the two types of forms.
- 7.13.2: The Linear Term in a Quadratic Expression
  - In this activity, you looked at different quadratic equations of the form $f(x) = x^2 + bx$ and $f(x) = x^2 + c$ and identified the x-intercepts and vertex.
  - 7.13.2: Self Check
  - 7.13.2: Additional Resources
- 7.13.3: Writing Equations that Represent a Graph
  - In this activity, you used graphing technology to find the equation that would graph a quadratic function through three given points.
  - 7.13.3: Self Check
  - 7:13.3: Additional Resources
- 7.13.4: Writing Quadratic Equations from Real Solutions
  - In this activity, you used the real solutions of a quadratic to write the quadratic equation that has those solutions.
  - 7.13.4: Self Check
  - 7:13.4: Additional Resources
- 7.13.5: Graphing Quadratic Equations
  - In this activity, you graphed quadratic functions in different forms by identifying the x-intercepts and vertex.

After these activities, you completed the following **practice**:

- 7.13.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

7.14 GRAPHS THAT REPRESENT SITUATIONS

7.14.0 Lesson Overview

In this lesson, you will interpret equations and graphs of quadratic functions in terms of the situations they represent. Additionally, you will use your analyses to solve problems and to compare quadratic functions given in different representations.

When you finish this lesson, you will be able to:

- Explain how a quadratic equation and its graph relate to a situation.

Here are the **activities** that will help you reach those goals:

- 7.14.1: Evaluating Quadratic Functions in the Real World
- 7.14.2: Interpreting a Functional Relationship Between Two Quantities
  - 7.14.2: Self Check
  - 7.14.2: Additional Resources
- 7.14.3: Analyzing Functions Using Different Representations
  - 7.14.3: Self Check
  - 7.14.3: Additional Resources
- 7.14.4: Using Quadratic Functions to Solve Problems
- 7.14.5: Identifying Key Features of Graphed Functions

After that, you'll **practice and review**.

- 7.14.6: Practice
- 7.14.7: Lesson Summary

7.14.1 Evaluating Quadratic Functions in the Real World

Warm Up

The height, in inches, of a frog's jump is modeled by the equation $h(t) = 60t - 75t^2$, where the time t, after the frog jumps, is measured in seconds.



1. For $h(t) = 60t - 75t^2$:

a. Find $h(0)$.

b. Find $h(0.8)$.

2. What do these values of $h(0)$ and $h(0.8)$ mean in terms of the frog's jump?
3. After it jumped, how much time passed, in seconds, before the frog reached its maximum height?
4. Explain how you know this is the time the frog reached the maximum height.

7.14.2 Interpreting a Functional Relationship between Two Quantities

Activity

The equation $h(t) = 2 + 23.7t - 4.9t^2$ represents the height of a pumpkin that is catapulted up in the air as a function of time, t, in seconds. The height is measured in meters above ground. The pumpkin is shot up at a vertical velocity of 23.7 meters per second.

1. What do you think the 2 in the equation $h(t) = 2 + 23.7t - 4.9t^2$ tells us in this situation?
2. What do you think the $-4.9t^2$ in the equation $h(t) = 2 + 23.7t - 4.9t^2$ tells us in this situation?
3. If we graph the equation, will the graph open upward or downward? Why?
4. Think about where the vertical intercept would be. What is the y-value of that vertical intercept?
5. What about the horizontal intercepts?
6. Use the graphing tool or technology outside the course. Graph the equation $h(t) = 2 + 23.7t - 4.9t^2$.

Access multimedia content (http://openstax.org/books/algebra-1/pages/7-14-2-interpreting-a-functional-relationship-between-two-quantities)

7. Now that you have graphed the equation, answer questions a–c using the graph.

a. What is the vertical intercept and what does this point mean in our catapulted pumpkin situation?

b. What are the horizontal intercepts and what do these points mean in our catapulted pumpkin situation?

c. What is the vertex of the parabola and what does this point mean in our catapulted pumpkin situation?

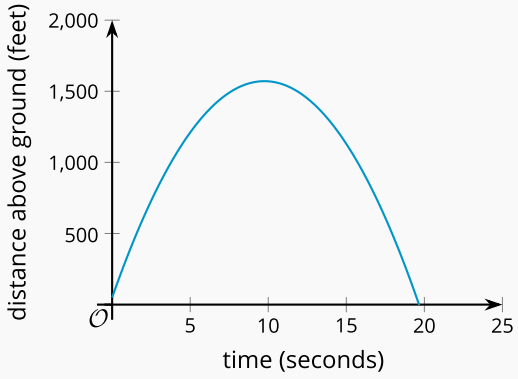
Are you ready for more?

Extending Your Thinking

1. What approximate vertical velocity would this pumpkin need for it to stay in the air for about 10 seconds? (Assume that it is still shot from 2 meters above the ground and that the effect of gravity pulling it down is the same.)

Self Check

The function that gives the height, h, in feet of a cannonball t seconds after the ball leaves the cannon is graphed below.



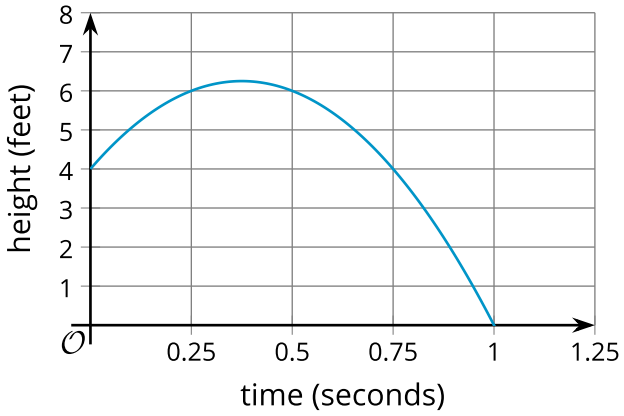
What is the point $(10, 1, 600)$ and what does it mean in this situation?

a. Y-intercept, the height the cannonball was launched
b. Maximum height, the highest height the cannonball reaches
c. Maximum height, the height the cannonball was launched
d. Y-intercept, the highest height the cannonball reaches

Additional Resources

The Meaning of Values on a Graph

Let's say a tennis ball is hit straight up in the air, and its height in feet above the ground is modeled by the equation $f(t) = 4 + 12t - 16t^2$ where t represents the time in seconds after the ball is hit. Here is a graph that represents the function, from the time the tennis ball was hit until the time it reached the ground.



In the graph, we can see some information we already know, and some new information:

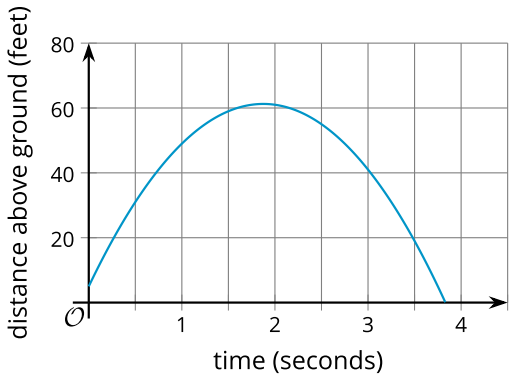
- The 4 in the equation means the graph of the function intersects the vertical axis at 4. It shows that the tennis ball was 4 feet off the ground at $t = 0$, when it was hit.
- The horizontal intercept is $(1, 0)$. It tells us that the tennis ball hit the ground 1 second after it was hit.
- The vertex of the graph is at approximately $(0.4, 6.3)$. This means that about 0.4 second after the ball was hit, it reached the maximum height of about 6.3 feet.

The equation can be written in factored form as $f(t) = (-16t - 4)(t - 1)$. From this form, we can see that the zeros of the function are $t = 1$ and $t = \frac{1}{4}$. The negative zero, -14, is not meaningful in this situation because the time before the ball was hit is irrelevant.

TRY IT **The Meaning of Values on a Graph**

An object is thrown upward from a height of 5 feet with a velocity of 60 feet per second. Its height h(t) in feet after t seconds is modeled by the function $h(t) = 5 + 60t - 16t^2$ and graphed below.

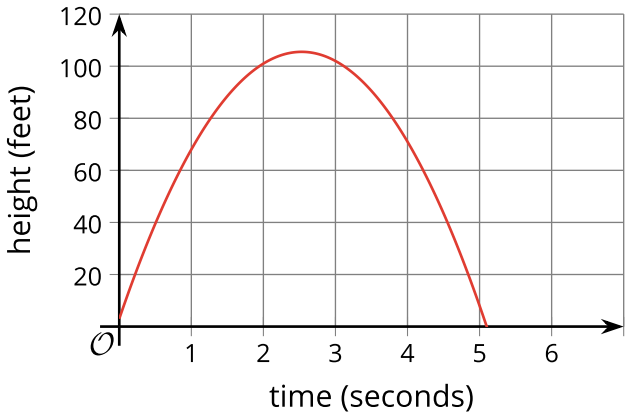
What is the value of the vertical intercept (y-intercept) and what does it mean in the function?



7.14.3 Analyzing Functions Using Different Representations

Activity

Here is a graph that represents the height of a baseball, h, in feet as a function of time, t, in seconds after it was hit by Player A.



The function g defined by $g(t) = (-16t - 1)(t - 4)$ also represents the height in feet of a baseball t seconds after it was hit by Player B. Without graphing function g, answer the following questions and explain or show how you know.

1. Which player's baseball stayed in flight longer?
2. Which player's baseball reached a greater maximum height?
3. Based on the graph, what is the approximate height, in feet, of Player A's baseball when hit? How do you know?
4. What is the height, in feet, of Player B's baseball when hit?

Video: Comparing Quadratic Functions

Watch the following video to learn more about comparing quadratic functions in different forms.

Comparing Quadratic Functions (https://openstax.org/r/compare-quadratics)

Self Check

The functions describing the time that baseballs stayed in the air, in seconds, that four baseball players hit are below.

- $f(t) = (-16t - 1)(t - 5)$
- $g(t) = (-16t - 1)(t - 10)$
- $h(t) = (t - 7)(-16t - 1)$

- $p(t) = (-8t - 1)(2t - 5)$

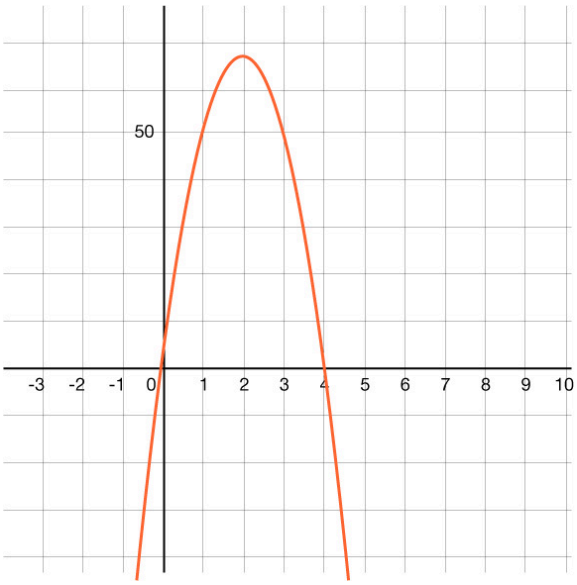
Which function represents the ball that stayed in the air the longest?

a. $p(t) = (-8t - 1)(2t - 5)$
b. $h(t) = (t - 7)(-16t - 1)$
c. $g(t) = (-16t - 1)(t - 10)$
d. $f(t) = (-16t - 1)(t - 5)$

Additional Resources

Comparing the Graphs and Equations of Quadratic Functions

Jai kicks a football. The graph that represents the height of the football, h, as a function of time, in seconds, is below.



Kellan kicks a football and the height in feet, h, is represented by the function $h(t) = (-16t - 1)(t - 6)$.

Which player's football stayed in the air longer?

Jai's football landed after 4 seconds since the zero is at $x = 4$. The other zero, where x is negative, does not make sense in this problem.

Looking at the function of Kellan's ball, find the zeros by setting each factor equal to zero.

- $-16t - 1 = 0$ becomes $t = \frac{-1}{16}$. A negative time does not make sense in this problem.
- $t - 6 = 0$ becomes $t = 6$.

Kellan's ball was in the air for 6 seconds, so his ball stayed in the air 2 seconds longer than Jai's.

TRY IT **Comparing the Graphs and Equations of Quadratic Functions**

Using the same information above, which player's football reached a higher height?

7.14.4 Using Quadratic Functions to Solve Problems

Activity

This activity is an extension of the expectations in the TEKS.

Your teacher will give you either a problem card or a data card. Do not show or read your card to your partner.

If your teacher gives you the data card:

1. Silently read the information on your card.
2. Ask your partner, "What specific information do you need?" and wait for your partner to ask for information. Only give information that is on your card. (Do not figure out anything for your partner!)
3. Before telling your partner the information, ask, "Why do you need to know (that piece of information)?"
4. Read the problem card and solve the problem independently.
5. Share the data card and discuss your reasoning.
6. Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity (steps 1 – 5) for the next problem, trading roles with your partner.
7. When you and your partner have solved both problems, enter your answers to the questions below.

If your teacher gives you the problem card:

1. Silently read your card and think about what information you need to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain to your partner how you are using the information to solve the problem.
4. When you have enough information, share the problem card with your partner, and solve the problem independently.
5. Read the data card and discuss your reasoning.
6. Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity (steps 1 – 5) for the next problem, trading roles with your partner.
7. When you and your partner have solved both problems, enter your answers to the questions below.

Use your experience with trying to solve the problem cards by answering the reflection questions below.

1. When answering the questions on Problem Card 1, what information did you ask your partner to provide? What information did you need to know?

2. When answering the questions on Problem Card 2, what information did you ask your partner to provide? What information did you need to know?

7.14.5 Identifying Key Features of Graphed Functions

Cool Down

The equation $y = (-16t - 2)(t - 1)$ represents the height in feet of a beach ball thrown by a child as a function of time, t, in seconds.

1. Since the equation is in factored form, we can tell that the graph intersects the horizontal axis at two points. Use this information to find the zeros of the function. Explain or show your reasoning.
a. Find one of the zeros of the functions using $(-16t - 2)$.
b. Find one of the zeros of the functions using $(t - 1)$.

2. What do the zeros tell us in this situation?

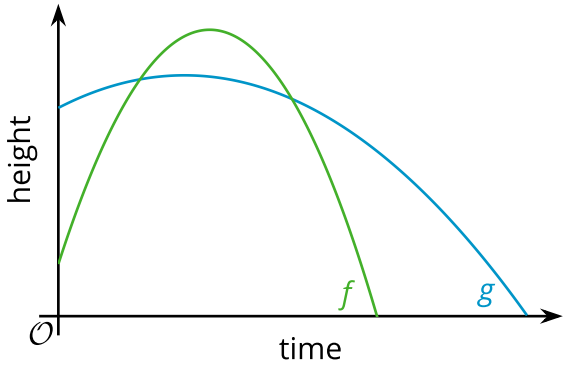
3. Are both zeros meaningful?

4. From what height, in feet, is the beach ball thrown? Explain or show your reasoning.

7.14.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Here are graphs of functions f and g.
Each represents the height of an object being launched into the air as a function of time.



a. Which object was launched from a higher point?

   a. The object represented by function g
   b. The object represented by function f

b. Which object reached a higher point?

   a. The object represented by function g
   b. The object represented by function f

c. Which object was launched with the greater upward velocity?

   a. The object represented by function g
   b. The object represented by function f

d. Which object landed last?

   a. The object represented by function g
   b. The object represented by function f

2. The function h given by $h(t) = (1 - t)(8 + 16t)$ models the height of a ball in feet, t seconds after it was thrown.

a. Find one of the zeros of the functions using $(8 + 16t)$.

b. Find one of the zeros of the functions using $(1 - t)$.

c. What do the zeros tell us in this situation?

   a. The negative zero represents the time, in seconds, when the basketball is on the ground or hits the ground.
   b. The positive zero represents the time, in seconds, when the basketball reaches the maximum height.
   c. The negative zero represents the height, in feet, when the time is zero.
   d. The positive zero means the time, in seconds, when the basketball is on the ground or hits the ground.

d. Are both zeros meaningful?

   a. Yes
   b. No

e. From what height, in feet, is the ball thrown?

f. When, in seconds, does the ball reach its highest point?

g. How high, in feet, does the ball go?

3. The height in feet of a thrown football is modeled by the equation $f(t) = 6 + 30t - 16t^2$, where time t is measured in seconds.

a. Which term in the equation describes how many feet above the ground the football was thrown?

   a. The squared term $16t^2$
   b. The squared term $-16t^2$
   c. The linear term $30t$
   d. The constant term 6

b. Which term in the equation describes the initial upward velocity of the football in feet per second?

   a. The squared term $16t^2$
   b. The squared term $-16t^2$
   c. The linear term $30t$
   d. The constant term 6

c. How does the squared term $-16t^2$ affect the value of the function f?

   a. The value of the function is zero when $-16t^2$ is zero.
   b. The squared term has no effect on the value of the function.
   c. The squared term increases the value of the function.
   d. The squared term decreases the value of the function.

d. What does this term $-16t^2$ reveal about the situation?

   a. How long the ball is in the air before it hits the ground.
   b. How far the football was thrown above the ground.
   c. The initial upward velocity of the football.
   d. The influence of gravity pulling the ball down to the ground.

4. The height in feet of an arrow is modeled by the equation $h(t) = (1 + 2t)(18 - 8t)$, where t is seconds after the arrow is shot.

a. At what time, in seconds, does the arrow hit the ground?

b. At what height, in feet, is the arrow shot?

5. Two objects are launched into the air.

- The height, in feet, of Object A is given by the equation $f(t) = 4 + 32t - 16t^2$.
- The height, in feet, of Object B is given by the equation $g(t) = 2.5 + 40t - 16^2$.
- In both functions, t is seconds after launch.

a. Which object was launched from a greater height?

   a. Object A
   b. Object B

b. Which object was launched with a greater upward velocity?

   a. Object A
   b. Object B

7.14.7 Lesson Summary

In this lesson, you learned how to:

- Explain how a quadratic equation and its graph relate to a situation.

Here are the activities that helped you reach those goals:

- 7.14.1: Evaluating Quadratic Functions in the Real World
  - In this activity, you found the meaning of values in a frog's jump such as maximum height.
- 7.14.2: Interpreting a Functional Relationship Between Two Quantities
  - In this activity, you found values in a graph and interpreted their meanings based on the situation graphed.
  - 7.14.2: Self Check
  - 7.14.2: Additional Resources
- 7.14.3: Analyzing Functions Using Different Representations
  - In this activity, you compared the heights of two baseballs hit by using a graph of one baseball player and the function of another.
  - 7.14.3: Self Check
  - 7.14.3: Additional Resources
- 7.14.4: Using Quadratic Functions to Solve Problems
  - In this activity, you determined the information you would need to answer questions about the launch of a toy rocket.
- 7.14.5: Identifying Key Features of Graphed Functions
  - In this activity, you used an equation to determine the zeros of an equation and the height from which a beach ball was thrown.

After these activities, you completed the following **practice**:

- 7.14.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

7.15 VERTEX FORM

7.15.0 Lesson Overview

In this lesson, you will be introduced to quadratic expressions in vertex form and learn that this form allows us to easily see where the vertex of a graph is located.

When you finish this lesson, you will be able to:

- Recognize the "vertex form" of a quadratic equation.
- Relate the numbers in the vertex form of a quadratic equation to its graph.

Here are the **activities** that will help you reach those goals:

- 7.15.1: Analyzing Two Sets of Equations
- 7.15.2: The Vertex Form
  - 7.15.2: Self Check
  - 7.15.2: Additional Resources
- 7.15.3: The Coordinates of the Vertex
  - 7.15.3: Self Check
  - 7.15.3: Additional Resources
- 7.15.4: The Graph of a Function in Vertex Form

After that, you'll **practice and review.**

- 7.15.5: Practice
- 7.15.6: Lesson Summary

7.15.1 Analyzing Two Sets of Equations

Warm Up

Set 1:

$f(x) = x^2 + 4x$

$g(x) = x(x + 4)$

$h(x) = (x + 2)^2 - 4$

Set 2:

$p(x) = -x^2 + 6x - 5$

$q(x) = (5 - x)(x - 1)$

$r(x) = -1(x - 3)^2 + 4$

1. After observing the two sets, what do you notice?

2. After observing the two sets, what do you wonder?

7.15.2 The Vertex Form

Activity

Here are two sets of equations for quadratic functions you saw earlier. In each set, the expressions that define the output are equivalent. In other words, all the equations in Set 1 define the same function. The same is true for Set 2: all the equations define the same function but they are just written in different forms.

Set 1:

$f(x) = x^2 + 4x$

$g(x) = x(x + 4)$

$h(x) = (x + 2)^2 - 4$

Set 2:

$p(x) = -x^2 + 6x - 5$

$q(x) = (5 - x)(x - 1)$

$r(x) = -1(x - 3)^2 + 4$

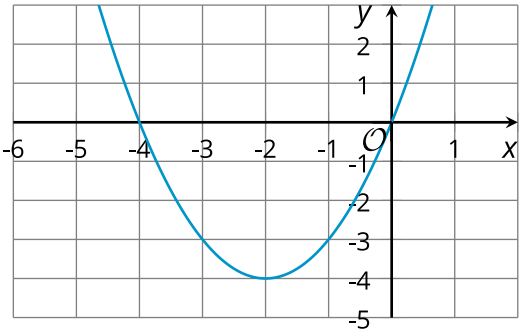
The expression that defines h is written in **vertex form**. We can show that it is equivalent to the expression defining f by expanding and simplifying the expression:

$$\begin{aligned}(x + 2)^2 - 4 &= (x + 2)(x + 2) - 4 \\ &= x^2 + 2x + 2x + 4 - 4 \\ &= x^2 + 4x, \text{ which is } f(x)\end{aligned}$$

If this simplified expression is factored, then we arrive at the expression that defines function g: $x^2 + 4x = x(x + 4)$. The expression that defines g is written in **factored form**. The simplified expression that defines function f is written in **standard form**.

1. Show that the expressions defining r and p are equivalent.

For questions 2 – 3, examine the graphs representing the quadratic functions.

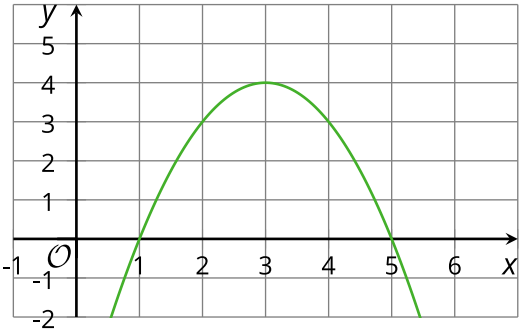


2. Examine the graph of the function h.

a. What is the x-coordinate of the vertex of the graph of h?

b. What is the y-coordinate of the vertex of the graph of h?

3. Examine the graph of the function r.

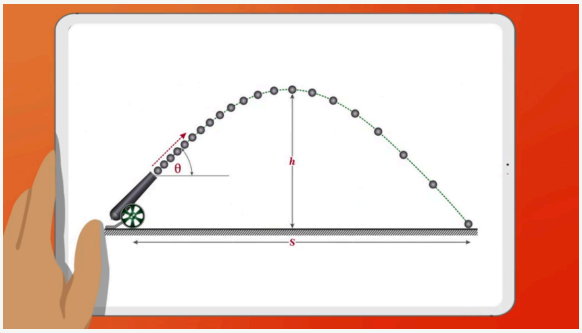


a. What is the x-coordinate of the vertex of the graph of r?

b. What is the y-coordinate of the vertex of the graph of r?

4. Why do you think expressions such as those defining h and r are said to be written in vertex form?

Why Should I Care?



When you launch an object, it does not travel in a straight line. Instead, it follows a curved path called a parabola. This means you have to account for distance, height, and angle if you want to hit your target. You can use quadratic equations to find these measurements.

The next time you throw a ball, think about the parabolic path that it follows.

Self Check

Which equation is equivalent to $y = (x - 3)^2 - 9$?

a. $y = x^2 + 6x - 9$
b. $y = x^2 - 6x$
c. $y = x^2 - 3x - 9$
d. $y = x^2 - 18$

Additional Resources

Equivalent Forms of Quadratics

Quadratics can be written in different equivalent forms:

- Standard form: $y = ax^2 + bx + c$
- Factored form: $y = (x + a)(x + b)$
- Vertex form: $y = a(x - h)^2 + k$

Example

Write $y = (x + 2)^2 - 9$ in standard and factored forms.

Here is how to find the equivalent standard form:

Step 1 - Expand the vertex form to multiply the binomials.

$y = (x + 2)(x + 2) - 9$

Step 2 - Multiply the binomials.

$y = x^2 + 2x + 2x + 4 - 9$

Step 3 - Combine like terms.

$y = x^2 + 4x - 5$

Here is how to find the factored form:

Step 1 - From the standard form, find the factors of the trinomial expression $y = x^2 + 4x - 5$.

Step 2 - Examine the trinomial expression.

$x^2 + 4x - 5$

Step 3 - Factor the trinomial.

$(x+5)(x-1)$

The factored form is $y=(x+5)(x-1)$.

TRY IT **Equivalent Forms of Quadratics**

Find the equation in standard form that is equivalent to $y=(x-2)^2+4$.

7.15.3 The Coordinates of the Vertex

Activity

For questions 1 – 2, use the Desmos graphing tool or technology outside the course.

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1. Graph $y=x^2$. Then, add different numbers to x before it is squared (for example, $y=(x+4)^2$, $y=(x-3)^2$) and observe how the graph changes.

2. Graph $y=(x-1)^2$. Then, experiment with each of the following changes to the function and see how they affect the graph and the vertex.

a. Add different constant terms to $(x-1)^2$ (for example: $y=(x-1)^2+5$, $y=(x-1)^2-9$).

b. Multiplying $(x-1)^2$ by different coefficients (for example: $y=3(x-1)^2$, $y=-2(x-1)^2$, $y=0.5(x-1)^2$.

3. Without graphing, find the coordinates of the vertex of $y=(x+10)^2$ and determine if it opens up or down.

a. What is the x-coordinate of the vertex?

b. What is the y-coordinate of the vertex?

c. Does the graph open up or down?

Select the **solution** button to see the solutions to question 3 organized into a table.

4. Without graphing, find the coordinates of the vertex of $y=(x-4)^2+8$ and determine if it opens up or down.

a. What is the x-coordinate of the vertex?

b. What is the y-coordinate of the vertex?

c. Does the graph open up or down?

Select the **solution** button to see the solutions to questions 3 – 4 organized into a table.

What patterns do you notice? Can you use those ideas to find the vertices for the next equation? What do you know about whether it will open up or down?

5. Without graphing, find the coordinates of the vertex of $y=-(x-4)^2+8$ and determine if it opens up or down.

a. What is the x-coordinate of the vertex?

b. What is the y-coordinate of the vertex?

c. Does the graph open up or down?

Select the **solution** button to see the solutions to questions 3 – 5 organized into a table

6. The next equation, $y=x^2-7$, looks a little different from the previous ones. Can you use what you've learned to find the coordinates of its vertex and determine if it opens up or down.

a. What is the x-coordinate of the vertex?

b. What is the y-coordinate of the vertex?

c. Does the graph open up or down?

Select the **solution** button to see the solutions to questions 3 – 6 organized into a table

7. Are the patterns you noticed still happening? See if you can apply those patterns to the following equations.

a. For the equation $y = \frac{1}{2}(x+3)^2 - 5$, what is the x-coordinate of the vertex?

b. For the equation $y = \frac{1}{2}(x+3)^2 - 5$, what is the y-coordinate of the vertex?

c. For the equation $y = \frac{1}{2}(x+3)^2 - 5$, does the graph open up or down?

d. For the equation $y = -(x+100)^2 + 50$, what is the x-coordinate of the vertex?

e. For the equation $y = -(x+100)^2 + 50$, what is the y-coordinate of the vertex?

f. For the equation $y = -(x+100)^2 + 50$, does the graph open up or down?

Select the **solution** button to see the solutions to questions 3 – 7 organized into a table.

8. Use the completed rows of the table to hypothesize about the coordinates of the vertex of the equation in the last row, and whether the parabola opens up or down.

a. For the equation $y = a(x+m)^2 + n$, what is the x-coordinate of the vertex?

b. For the equation $y = a(x+m)^2 + n$, what is the y-coordinate of the vertex?

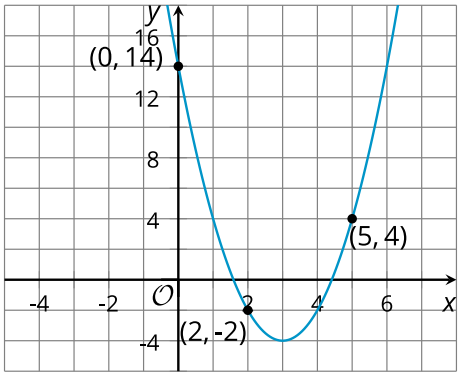
c. For the equation $y = a(x+m)^2 + n$, can you determine if the graph opens up or down with the information you are given?

Are you ready for more?

Extending Your Thinking

Examine the graph below, then use it to answer questions 1 – 3.

1.



What is the x-coordinate of the vertex of this graph?

2. What is the y-coordinate of the vertex of this graph?

3. Which quadratic equation best fits the function in the graph?

Video: Learning About the Vertex Form of a Quadratic

Watch the following video to learn more about the vertex form of a quadratic.

Learning About the Vertex Form of a Quadratic (https://openstax.org/r/vertex-form)

Self Check

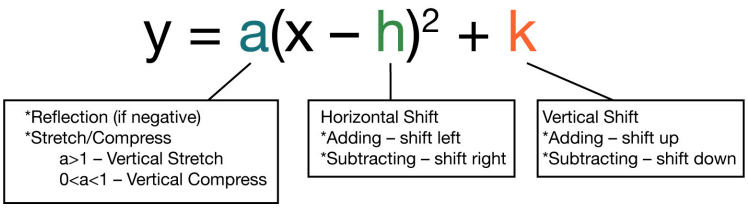
Find the vertex for $y = 3(x-2)^2 - 5$.

a. $(2, 5)$
b. $(-2, -5)$
c. $(2, -5)$
d. $(3, -5)$

Additional Resources

Using Vertex Form

The vertex form of a quadratic is below:



Notice that the value of h is the opposite of what it is in the parenthesis. This seems odd: when we add k, the vertex of the parabola moves up k units. But when we subtract h within the squared term, the vertex of the parabola moves right h units, not left.

Let's explore why. We know the y-coordinate of the vertex is $y = k$. To find that using this equation, we need to know where $a(x - h)^2$ is zero, because that will make $y = k$. Set $a(x - h)^2 = 0$. This happens when $x - h = 0$, which is when $x = h$. That means h is the x-coordinate of the vertex.

EXAMPLE

Find the vertex of $y = 3(x - 4)^2 - 2$.

The parabola of the quadratic will shift right 4 (remember this horizontal shift is counterintuitive and translates in the opposite direction of what the expression "looks like") and down 2.

So, the vertex is at $(4, -2)$.

TRY IT **Using Vertex Form**

Find the vertex of $y = (x - 2)^2 + 5$.

7.15.4 The Graph of a Function in Vertex Form

Cool Down

Answer questions 1 – 3 about the function f, given by $f(x) = (x + 3)^2 - 1$.

1. What is the x-coordinate of the vertex of the graph of f?
2. What is the y-coordinate of the vertex of the graph of f?
3. Does the graph open up or down?

7.15.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. **Select three** quadratic expressions written in vertex form.

   a. $y = (x - 2)^2 + 1$
   b. $y = x^2 - 4$
   c. $y = x(x + 1)$
   d. $y = (x + 3)(x + 3)$
   e. $y = (x - 4)^2 + 6$

Examine the following two equations and use them to answer questions 2 – 5. One defines function m and the other defines function p.

- $m(x) = x(x + 6)$
- $p(x) = (x + 3)^2 - 9$

2. How do you know that the expressions defining m and p are equivalent?

   a. Both expressions are equal to $x^2 - 3$.
   b. Both expressions are equal to $2x + 6$.
   c. Both expressions are equal to $x^2 - 6x$.
   d. Both expressions are equal to $x^2 + 6x$.

3. Recall $m(x) = x(x + 6)$ and $p(x) = (x + 3)^2 - 9$. What is the x-coordinate of the vertex of the graph of m?

4. Recall $m(x) = x(x + 6)$ and $p(x) = (x + 3)^2 - 9$. What is the y-coordinate of the vertex of the graph of m?

5. Recall $m(x) = x(x + 6)$ and $p(x) = (x + 3)^2 - 9$. What are the x-intercepts of the graph of p?

   a. $(0, 0)$ and $(0, 6)$
   b. $(0, 0)$ and $(0, -6)$
   c. $(0, 0)$ and $(6, 0)$
   d. $(0, 0)$ and $(-6, 0)$

For questions 6 – 11, enter the coordinates of the vertex of the graph that represents the equation.

6. Find the coordinates of the vertex for the equation $y = (x - 3)^2 + 5$.

   a. What is the x-coordinate of the vertex?

   b. What is the y-coordinate of the vertex?

7. Find the coordinates of the vertex for the equation $y = (x + 7)^2 + 3$.

   a. What is the x-coordinate of the vertex?

   b. What is the y-coordinate of the vertex?

8. Find the coordinates of the vertex for the equation $y = (x - 4)^2$.

   a. What is the x-coordinate of the vertex?

   b. What is the y-coordinate of the vertex?

9. Find the coordinates of the vertex for the equation $y = x^2 - 1$.

   a. What is the x-coordinate of the vertex?

   b. What is the y-coordinate of the vertex?

10. Find the coordinates of the vertex for the equation $y = 2(x + 1)^2 - 5$.

    a. What is the x-coordinate of the vertex?

    b. What is the y-coordinate of the vertex?

11. Find the coordinates of the vertex for the equation $y = -2(x + 1)^2 - 5$.

a. What is the x-coordinate of the vertex?

b. What is the y-coordinate of the vertex?

For questions 12 – 14, answer the questions about the x- and y-coordinates of the vertex, and whether the function opens up or down.

12. For function $f(x) = (x - 4)^2 - 5$:

a. What is the x-coordinate of the vertex?

b. What is the y-coordinate of the vertex?

c. For the function $f(x) = (x - 4)^2 - 5$, does the graph open up or down?

   a. Up
   b. Down

13. For function $g(x) = -x^2 + 5$:

a. What is the x-coordinate of the vertex?

b. What is the y-coordinate of the vertex?

c. For the function $g(x) = -x^2 + 5$, does the graph open up or down?

   a. Up
   b. Down

14. For function $h(x) = 2(x + 1)^2 - 4$:

a. What is the x-coordinate of the vertex?

b. What is the y-coordinate of the vertex?

c. For the function $h(x) = 2(x + 1)^2 - 4$, does the graph open up or down?

   a. Up
   b. Down

7.15.6 Lesson Summary

In this lesson, you learned how to:

- Recognize the "vertex form" of a quadratic equation.
- Relate the numbers in the vertex form of a quadratic equation to its graph.

Here are the **activities** that helped you reach those goals:

- 7.15.1: Analyzing Two Sets of Equations
  - In this activity, you looked at equations in different forms and compared and contrasted them.
- 7.15.2: The Vertex Form
  - In this activity, you explored the different forms from 7.15.1 and proved that the different forms, factored, standard, and vertex, are equivalent.
  - 7.15.2: Self Check
  - 7.15.2: Additional Resources
- 7.15.3: The Coordinates of the Vertex
  - In this activity, you used graphing technology to explore how changing the numbers, or parameters, in the vertex form affected the graph of a quadratic.
  - 7.15.3: Self Check
  - 7.15.3: Additional Resources
- 7.15.4: The Graph of a Function in Vertex Form
  - In this activity, you examined a quadratic function in vertex form and identified the vertex and if the graph of the function opens up or down.

After these activities, you completed the following **practice:**

- 7.15.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

7.16 GRAPHING FROM THE VERTEX FORM

7.16.0 Lesson Overview

In this lesson, you will continue working with the vertex form of quadratic equations. You will discover that you can anticipate what happens on the graph on either side of the vertex by reasoning about the parts of an expression in vertex form.

When you finish this lesson, you will be able to:

- Graph a quadratic function given in vertex form, showing the maximum or minimum and the y-intercept.
- Find the maximum or the minimum of a quadratic function given in vertex form without first graphing it.

Here are the **activities** that will help you reach those goals:

- 7.16.1: Three Forms of Quadratic Expressions
- 7.16.2: Graph Functions in Vertex Form
  - 7.16.2: Self Check
  - 7.16.2: Additional Resources
- 7.16.3: Quadratic Equations and Graphs
  - 7.16.3: Self Check
  - 7.16.3: Additional Resources
- 7.16.4: Sketching a Graph of an Equation

After that, you'll **practice and review**.

- 7.16.5: Practice
- 7.16.6: Lesson Summary

7.16.1 Three Forms of Quadratic Expressions

Warm Up

Expressions in different forms can be used to define the same function. Here are three ways to define a function f.

Standard form	Factored form	Vertex form
$f(x) = x^2 - 4x + 3$	$f(x) = (x-3)(x-1)$	$f(x) = (x-2)^2 - 1$
$f(x) = ax^2 + bx + c$	$f(x) = a(x-r_1)(x-r_2)$	$f(x) = a(x-h)^2 + k$

Which form would you use if you want to find the following features of the graph of f? Be prepared to explain your reasoning.

1. The x-intercepts

2. The vertex

3. The y-intercept

7.16.2 Graph Functions in Vertex Form

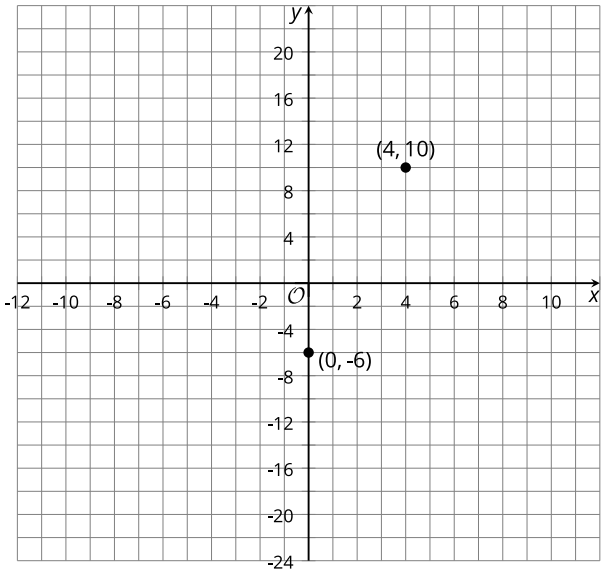
Activity

Examine these two equations that define quadratic functions.

$p(x) = -(x-4)^2 + 10$

$q(x) = \frac{1}{2}(x-4)^2 + 10$

1. The graph of p passes through $(0, -6)$ and $(4, 10)$, as shown on the coordinate plane.



a. Find the coordinates of another point on the graph of p. Explain or show your reasoning.

b. Use the graphing tool or technology outside the course. Plot $p(x) = -(x - 4)^2 + 10$ and the points $(0, -6)$, $(4, 10)$, and $(8, -6)$ to label the graph.

Access multimedia content (http://openstax.org/books/algebra-1/pages/7-16-2-graph-functions-in-vertex-form)

A line containing the vertex is called the parabola's **axis of symmetry**.

2. Find the x- and y-coordinates of the vertex and sketch the graph.

a. What is the x-coordinate of the vertex of $q(x) = \frac{1}{2}(x - 4)^2 + 10$?

b. What is the the y-coordinate of the vertex of $q(x) = \frac{1}{2}(x - 4)^2 + 10$?

c. Use the graphing tool or technology outside the course. On the same coordinate plane containing the points and equation from question 1, plot the coordinates of the vertex of the function q.

d. Find two other points that are on the graph of q. Try to identify points that are symmetrical over the axis of symmetry.

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e. Use the graphing tool or technology outside the course. Sketch and label the graph of q. Explain or show your reasoning.

3. Recall that $p(x) = -(x - 4)^2 + 10$. Pilar says, "Once I know the vertex is $(4, 10)$, I can find out, without graphing, whether the vertex is the maximum or the minimum of function p. I would just compare the coordinates of the vertex with the coordinates of a point on either side of it."

Complete the table by answering questions a and b.

x	3	4	5
$p(x)$		10	

a. Evaluate $p(3)$.

b. Evaluate $p(5)$.

c. Explain how Pilar might have reasoned whether the vertex is the minimum or maximum.

Are you ready for more?

Extending Your Thinking

1. Choose the equation for a quadratic function whose graph has the vertex at $(2, 3)$ and contains the point $(0, -5)$.
   a. $y = -2(x-2)^2 + 3$
   b. $y = -3(x-2)^2 + 2$
   c. $y = -5(x-2)^2$
   d. $y = 2(x-2)^2 + 3$
2. Use the graphing tool or technology outside the course. Sketch a graph of your function.

Self Check

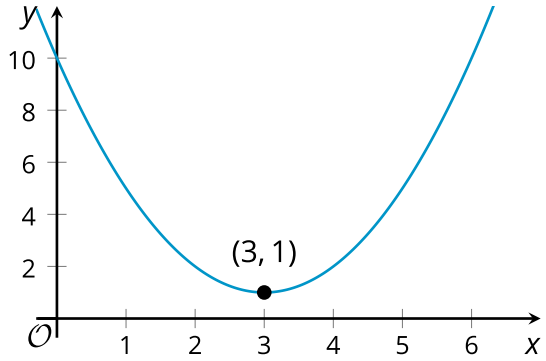
The equation of a quadratic function is $y = (x-1)^2 + 5$ and the point $(3, 9)$ also lies on the parabola. What is another point that lies on the parabola?

a. $(-1, 5)$
b. $(-1, 9)$
c. $(3, 1)$
d. $(-5, 9)$

Additional Resources

Using Key Points to Graph Quadratics

We saw that vertex form is especially helpful for finding the vertex of a graph of a quadratic function. For example, we can tell that the function p given by $p(x) = (x-3)^2 + 1$ has a vertex at $(3, 1)$.



We also noticed that, when the squared expression $(x-3)^2$ has a positive coefficient, the graph opens upward. You can use the vertex $(3, 1)$ to determine that minimum function value is 1.

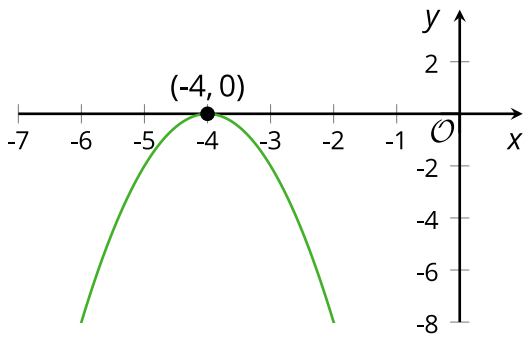
But why does the function p take on its minimum value when x is 3?

Here is one way to explain it: When $x = 3$, the squared term $(x-3)^2$ equals 0, as $(3-3)^2 = 0^2 = 0$. When x is any other value besides 3, the squared term $(x-3)^2$ is a positive number greater than 0. (Squaring any number results in a positive number.) This means that the output when $x \neq 3$ will always be greater than the output when $x = 3$, so the function p has a minimum value at $x = 3$.

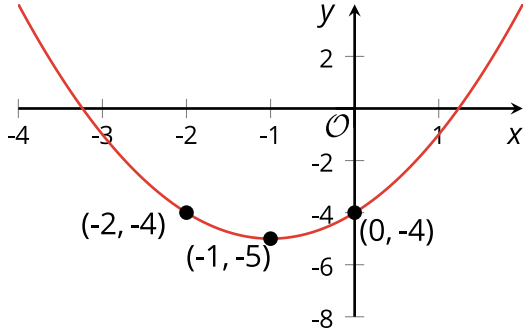
This table shows some values of the function for some values of x. Notice that the output is the least when $x = 3$ and it increases both as x increases and as it decreases.

x	0	1	2	3	4	5	6
$(x-3)^2 + 1$	10	5	2	1	2	5	10

The squared term sometimes has a negative coefficient, for instance: $h(x) = -2(x+4)^2$. The x value that makes $(x+4)^2$ equal 0 is -4, because $(-4+4)^2 = 0^2 = 0$. Any other x value makes $(x+4)^2$ greater than 0. But when $(x+4)^2$ is multiplied by a negative number (-2), the resulting expression, $-2(x+4)^2$, ends up being negative. This means that the output when $x - 4$ will always be less than the output when $x = -4$, so the function h has its maximum value when $x = -4$.



Remember that we can find the y-intercept of the graph representing any function we have seen. The y-coordinate of the y-intercept is the value of the function when $x = 0$. If g is defined by $g(x) = (x+1)^2 - 5$, then the y-intercept is $(0, -4)$ because $g(0) = (0+1)^2 - 5 = -4$. Its vertex is at $(-1, -5)$. Another point on the graph with the same y-coordinate is located the same horizontal distance from the vertex but on the other side.

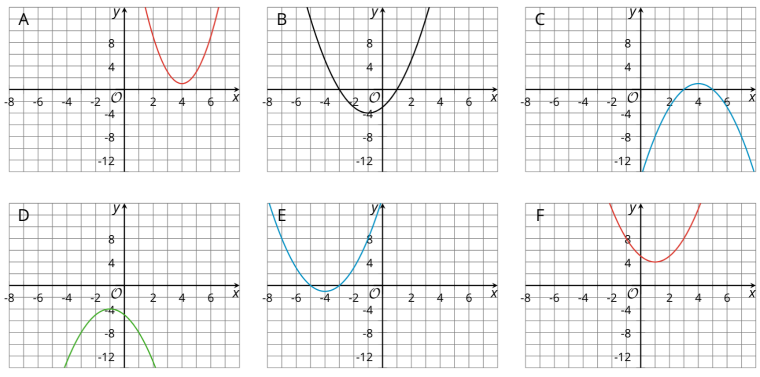


TRY IT **Using Key Points to Graph Quadratics**

A quadratic function is represented by the equation $y = (x+3)^2 - 5$. If $(5, 61)$ is another point on the graph, find a third point on the curve.

7.16.3 Quadratic Equations and Graphs

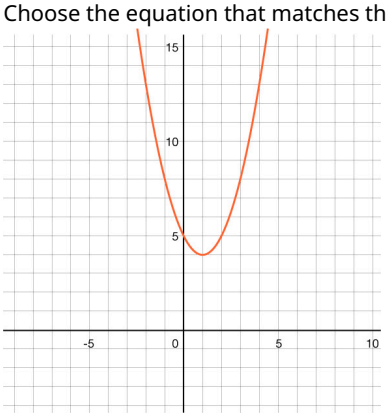
Activity



Your teacher will give you a set of cards (https://openstax.org/r/set-cards-7-16-3). Each card contains an equation or a graph that represents a quadratic function. Take turns matching each equation to a graph that represents the same function.

- For each pair of cards that you match, explain to your partner how you know they belong together.
- For each pair of cards that your partner matches, listen carefully to their explanation. If you disagree, discuss your thinking and work to reach an agreement.
- Once all the cards are matched, record the equation, the letter of the equation, and a sketch of the corresponding graph, and write a brief note or explanation about how you knew they were a match.

1. Choose the equation that matches this graph.



a. $f(x) = (x-1)^2 + 4$

b. $g(x) = -(x-4)^2 + 1$

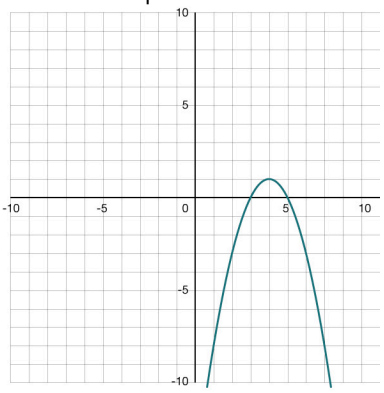
c. $h(x) = (x+1)^2 - 4$

d. $p(x) = -(x+1)^2 - 4$

e. $q(x) = 2(x-4)^2 + 1$

f. $r(x) = (x+4)^2 - 1$

2. Choose the equation that matches this graph.



a. $f(x) = (x-1)^2 + 4$

b. $g(x) = -(x-4)^2 + 1$

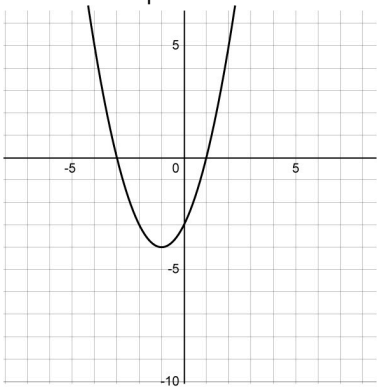
c. $h(x) = (x+1)^2 - 4$

d. $p(x) = -(x+1)^2 - 4$

e. $q(x) = 2(x-4)^2 + 1$

f. $r(x) = (x+4)^2 - 1$

3. Choose the equation that matches this graph.



a. $f(x) = (x-1)^2 + 4$

b. $g(x) = -(x-4)^2 + 1$

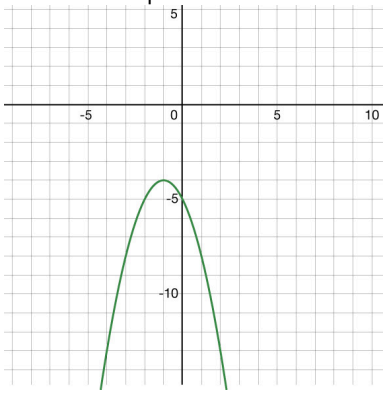
c. $h(x) = (x+1)^2 - 4$

d. $p(x) = -(x+1)^2 - 4$

e. $q(x) = 2(x-4)^2 + 1$

f. $r(x) = (x+4)^2 - 1$

4. Choose the equation that matches this graph.



a. $f(x) = (x-1)^2 + 4$

b. $g(x) = -(x-4)^2 + 1$

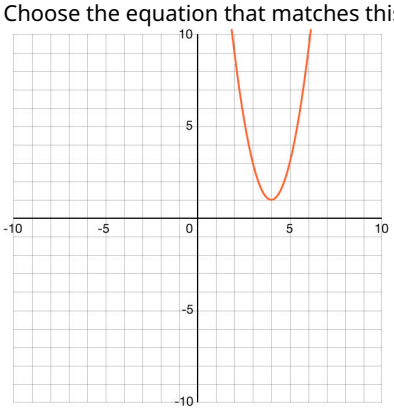
c. $h(x) = (x+1)^2 - 4$

d. $p(x) = -(x+1)^2 - 4$

e. $q(x) = 2(x-4)^2 + 1$

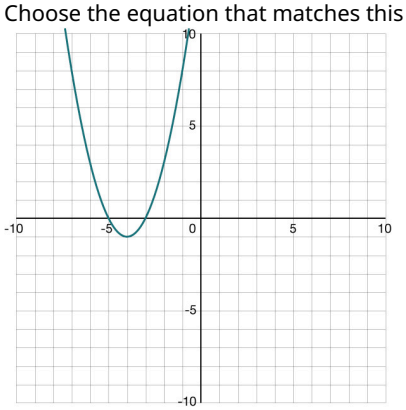
f. $r(x) = (x+4)^2 - 1$

5. Choose the equation that matches this graph.



a. $f(x) = (x-1)^2 + 4$
b. $g(x) = -(x-4)^2 + 1$
c. $h(x) = (x+1)^2 - 4$
d. $p(x) = -(x+1)^2 - 4$
e. $q(x) = 2(x-4)^2 + 1$
f. $r(x) = (x+4)^2 - 1$

6. Choose the equation that matches this graph.



a. $f(x) = (x-1)^2 + 4$
b. $g(x) = -(x-4)^2 + 1$
c. $h(x) = (x+1)^2 - 4$
d. $p(x) = -(x+1)^2 - 4$
e. $q(x) = 2(x-4)^2 + 1$
f. $r(x) = (x+4)^2 - 1$

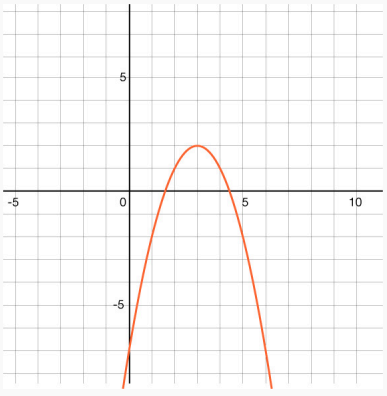
Video: Learning About Quadratic Equations and Graphs

Watch the following video to learn more about quadratic equations and graphs.

Learning About Quadratic Equations and Graphs (https://openstax.org/r/quadratic-equations)

Self Check

Which of the following equations matches the graph below?



a. $y = (x+3)^2 + 2$
b. $y = -(x+3)^2 + 2$
c. $y = -(x-3)^2 + 2$
d. $y = (x-3)^2 + 2$

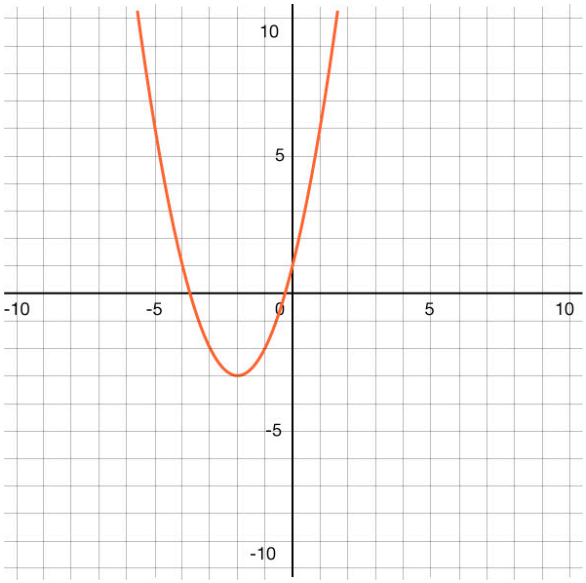
Additional Resources

Matching Graphs and Equations

When matching graphs of quadratic functions and their equations, look for the matching vertex. Then look if the parabola is opening up or down. Finally, use another point to determine if the graph has been compressed or stretched.

Recall this type of transformation will affect the a-value of the equation.

Looking at the graph below, choose the equation that matches the graph.



a. $y = (x-2)^2 - 3$
b. $y = (x+2)^2 - 3$
c. $y = -(x+2)^2 - 3$
d. $y = -(x-2)^2 - 3$

Step 1 - Find the vertex.

The vertex is at $(-2, -3)$.

Step 2 - Does the parabola open up or down?

The parabola opens up. So the leading coefficient is positive.

The correct answer is b. $y = (x+2)^2 - 3$.

Remember the vertex form is $y = a(x-h)^2 + k$. If the x-variable is being added in the parenthesis, the parabola has been shifted left.

The $(x+2)$ moves the parabola left 2.

The -3 shifts it down 3 from the origin.

Step 3 - Use the vertex form of the equation and write the parts you know: the coordinates of the vertex.
$y = a(x+2)^2 - 3$

Step 4 - To solve for a, find another point on the graph of the parabola and plug in those values for x and y in the equation you wrote.

Substitute the point $(-1, -2)$ into the equation to check the value of a. We have hypothesized $a = 1$, and with the coordinates from the second point, $x = -1$ and $y = -2$.

Step 5 - Solve for a.

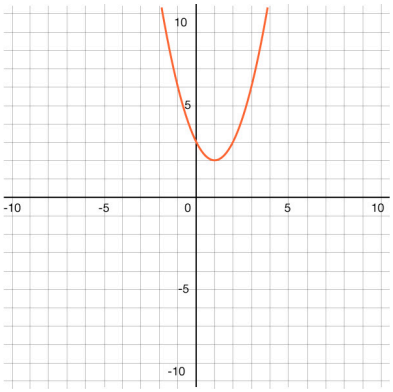
$-2 = a(-1+2)^2 - 3$. Now solve for a to find that $a = 1$.

Step 6 - Plug the value for a back into the equation.

So, $y = (x+2)^2 - 3$.

TRY IT **Matching Graphs and Equations**

Match the graph with its equation in vertex form.



a. $y = (x + 1)^2 + 2$

b. $y = -(x + 1)^2 + 2$

c. $y = -(x - 1)^2 + 2$

d. $y = (x - 1)^2 + 2$

7.16.4 Sketching A Graph of an Equation

Cool Down

1. What is the x-coordinate of the vertex of the graph defined by $y = (x - 3)^2 + 2$?
2. What is the y-coordinate of the vertex of the graph defined by $y = (x - 3)^2 + 2$?
3. Find the coordinates of two other points on the graph. The points should be symmetric across the vertex. Show your reasoning.
4. Use the graphing tool or technology outside the course. Draw a graph that represents the equation $y = (x - 3)^2 + 2$.
   Access multimedia content (http://openstax.org/books/algebra-1/pages/7-16-4-sketching-a-graph-of-an-equation)

7.16.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. What is the x-coordinate of the vertex of the graph that represents $y = (x - 2)^2 - 8$?

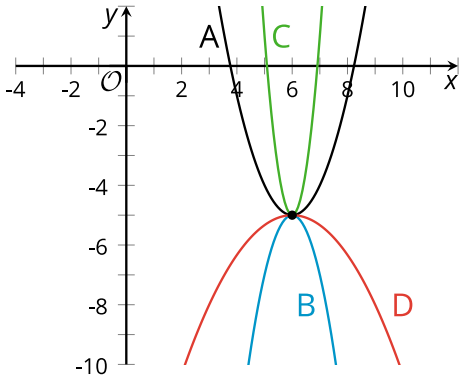
2. What is the y-coordinate of the vertex of the graph that represents $y = (x - 2)^2 - 8$?

3. Where is the y-intercept for the equation $y = (x - 2)^2 - 8$ located?

4. Which of the following is another point on the graph of the equation $y = (x - 2)^2 - 8$?

a. $(-4, -4)$
b. $(4, 4)$
c. $(-4, 4)$
d. $(4, -4)$

5. Which graph is a representation of $y = (x - 6)^2 - 5$?

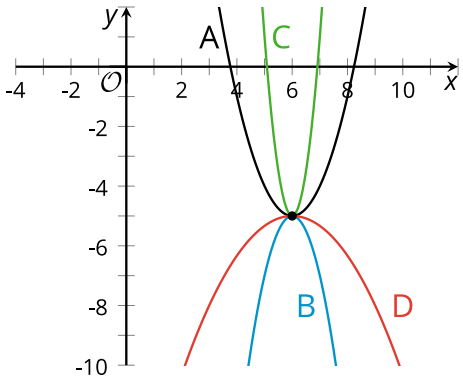


a. Graph A
b. Graph B
c. Graph C
d. Graph D

6. The function v is defined by $v(x) = 12(x + 5)^2 - 7$.
Without graphing, determine if the vertex of the graph representing v shows the minimum or maximum value of the function.

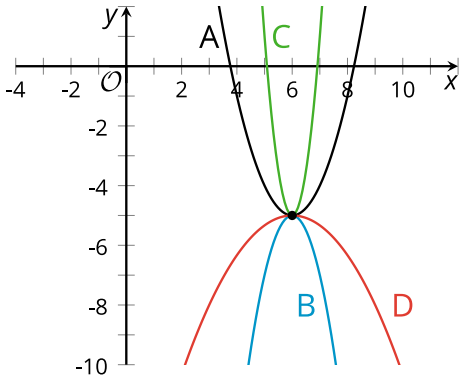
a. The vertex shows the minimum value.
b. The vertex shows the maximum value.

7. Choose the equation that matches the graph of parabola D .



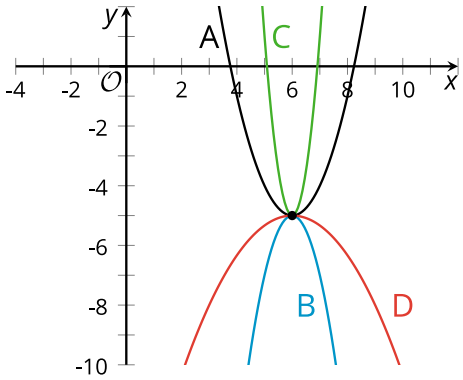
a. $y = -\frac{1}{3}(x-6)^2 - 5$
b. $y = 6(x-6)^2 - 5$
c. $y = (x-6)^2 - 5$
d. $y = -2(x-6)^2 - 5$

8. Choose the equation that matches the graph B.



a. $y = -\frac{1}{3}(x-6)^2 - 5$
b. $y = 6(x-6)^2 - 5$
c. $y = (x-6)^2 - 5$
d. $y = -2(x-6)^2 - 5$

9. Choose the equation that matches the graph C.



a. $y = -\frac{1}{3}(x-6)^2 - 5$
b. $y = 6(x-6)^2 - 5$
c. $y = (x-6)^2 - 5$
d. $y = -2(x-6)^2 - 5$

7.16.6 Lesson Summary

In this lesson, you learned how to:

- Graph a quadratic function given in vertex form, showing the maximum or minimum and the y-intercepts.
- Find the maximum or the minimum of a quadratic function given in vertex form without first graphing it.

Here are the activities that helped you reach those goals:

- 7.16.1: Three Forms of Quadratic Expressions
  - In this activity, you identified which forms of quadratics would be best to use to find different characteristics of the graphs of the functions.
- 7.16.2: Graph Functions in Vertex Form
  - In this activity, you completed the graphs of quadratic functions using knowledge of finding the vertex from vertex form and the symmetry of parabolas.
  - 7.16.2: Self Check
  - 7.16.2: Additional Resources
- 7.16.3: Quadratic Equations and Graphs
  - In this activity, you matched graphs of quadratic functions with the appropriate equations in vertex form.
  - 7.16.3: Self Check
  - 7.16.3: Additional Resources
- 7.16.4: Sketching A Graph of an Equation
  - In this activity, you sketched the graph of a quadratic function using the vertex form. You graphed the vertex and two other points to complete the graph.

After these activities, you completed the following **practice**:

- 7.16.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

7.17 CHANGING THE VERTEX

7.17.0 Lesson Overview

In this lesson, you will determine what expressions would produce graphs with particular vertices. We will look more closely at how changing the h and the k in a quadratic expression of the form $a(x-h)^2+k$ translates the graph, and recall that the coefficient a affects the direction of the opening of a parabola.

When you finish this lesson, you will be able to:

- Describe how changing a number in the vertex form of a quadratic function affects its graph.

Here are the **activities** that will help you reach those goals:

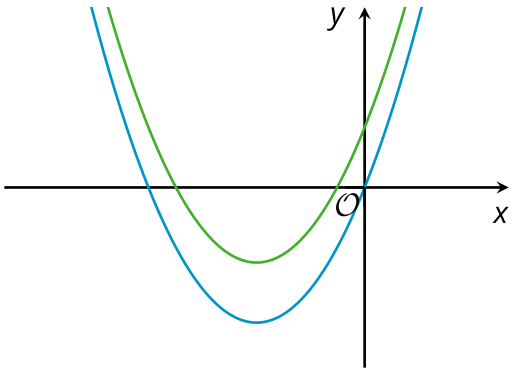
- 7.17.1: Quadratic Expressions in Different Forms
- 7.17.2: Modify Expressions to Translate Graphs
  - 7.17.2: Self Check
  - 7.17.2: Additional Resources
- 7.17.3: Changing the Parameters of a Quadratic Expression
  - 7.17.3: Self Check
  - 7.17.3: Additional Resources
- 7.17.4: Transforming the Parent Function
- 7.17.5: Equations of Graph Translations

After that, you'll **practice and review.**

- 7.17.6: Practice
- 7.17.7: Lesson Summary

7.17.1 Quadratic Expressions in Different Forms

Warm Up



Here are graphs representing the functions f and g given by $f(x)=x(x+6)$ and $g(x)=x(x+6)+4$.

1. Which graph represents $f(x)=x(x+6)$? How do you know?
2. Which graph represents $g(x)=x(x+6)+4$? How do you know?
3. Where does the graph of f meet the x-axis? Explain how you know.

7.17.2 Modify Expressions to Translate Graphs

Activity

Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

1. How would you change the equation $f(x) = x^2$ so that the vertex of the graph of the new equation is located at the following coordinates and the graph opens as described?

a. $(0, 11)$, opens up

b. $(7, 11)$, opens up

c. $(7, -3)$, opens down

2. Use the graphing tool or technology outside the course. Verify your predictions.

Access multimedia content (http://openstax.org/books/algebra-1/pages/7-17-2-modify-expressions-to-translate-graphs)

3. Kiran graphed the equation $f(x) = x^2 + 1$ and noticed that the vertex is at $(0, 1)$. He changed the equation to $f(x) = (x - 3)^2 + 1$ and saw that the graph shifted 3 units to the right and the vertex is now at $(3, 1)$. Next, he graphed the equation $f(x) = x^2 + 2x + 1$ and observed that the vertex is at $(-1, 0)$. Kiran thought, "If I change the squared term x^2 to $(x - 5)^2$, the graph will move 5 units to the right and the vertex will be at $(4, 0)$."

a. Do you agree with Kiran?

b. Explain or show your reasoning why you agree or disagree with Kiran.

4. $f(x) = (x + 4)^2 - 1$ is a parabola with a vertex at $(-4, 1)$ that opens upward.

a. If we multiply $f(x)$ by 2, we get $2f(x) = 2[(x + 4)^2 - 1] = 2(x + 4)^2 - 2$. How does this change the graph?

b. If we multiply $f(x)$ by $\frac{1}{2}$, we get $\frac{1}{2}f(x) = \frac{1}{2}[(x + 4)^2 - 1] = \frac{1}{2}(x + 4)^2 - \frac{1}{2}$. How does this change the graph?

5. Use the graphing tool or technology outside the course. Verify your prediction using Desmos tool below.

Access multimedia content (http://openstax.org/books/algebra-1/pages/7-17-2-modify-expressions-to-translate-graphs)

6. Sophia graphed the equation $f(x) = x^2 - 6x + 9$. She wondered what would happen to the graph if everywhere she had an x, she replaced it with $3x$.

a. Sophia predicted that the parabola would be narrower, and the x-intercepts and y-intercepts would change. Write why you agree or disagree with Sophia's prediction.

b. Sophia also wondered what would happen if she replaced the x in $f(x) = x^2 - 6x + 9$ with $\frac{1}{3}x$, how that would change the graph. She predicted that the parabola would be wider, and the x-intercepts and y-intercepts would change. Write why you agree or disagree with Sophia's prediction.

7. Use the graphing tool or technology outside the course. Verify your prediction using Desmos tool below.

Access multimedia content (http://openstax.org/books/algebra-1/pages/7-17-2-modify-expressions-to-translate-graphs)

Self Check

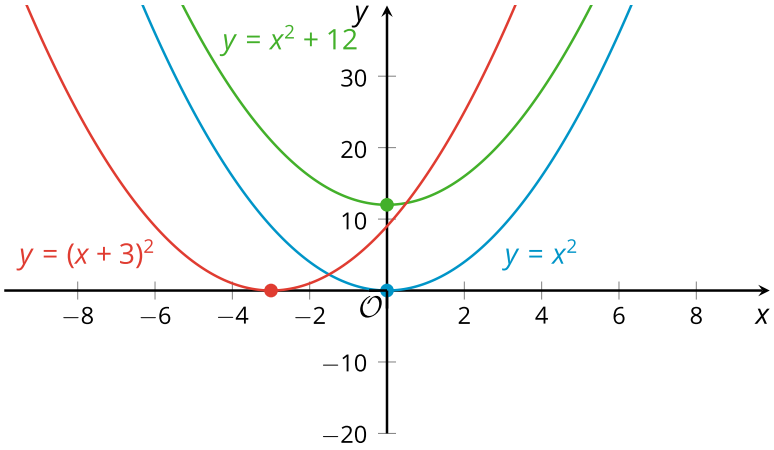
What transformations and shifts need to be made to change $y = x^2$ to $y = -(x + 5)^2 - 10$?

a. Reflect over the x-axis, left 5, and up 10
b. Reflect over the x-axis, left 5, and down 10
c. Left 5 and down 10
d. Right 5 and down 10

Additional Resources

Transforming Graphs of Quadratics

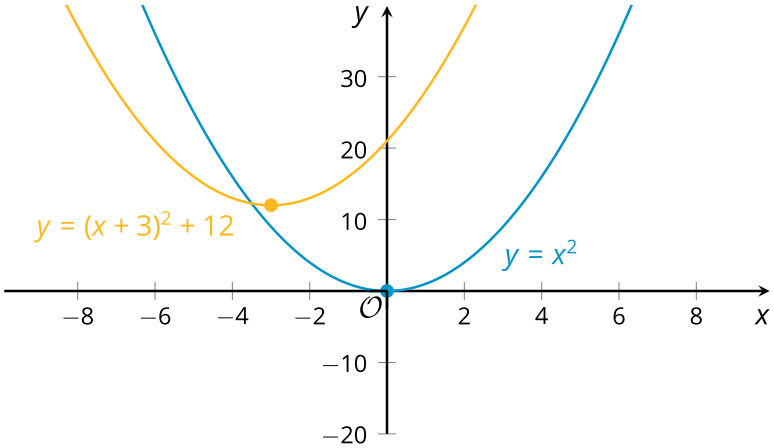
The graphs of $f(x) = x^2$, $f(x) = x^2 + 12$, and $f(x) = (x + 3)^2$ all have the same shape but their locations are different. The graph that represents $f(x) = x^2$ has its vertex at $(0, 0)$.



Notice that adding 12 to x^2 raises the graph by 12 units, so the vertex of that graph is at $(0, 12)$. Replacing x^2 with $(x + 3)^2$ shifts the graph 3 units to the left, so the vertex is now at $(-3, 0)$.

We can also shift a graph both horizontally and vertically.

The graph that represents $f(x) = (x + 3)^2 + 12$ will look like that for $f(x) = x^2$ but it will be shifted 12 units up and 3 units to the left. Its vertex is at $(-3, 12)$.

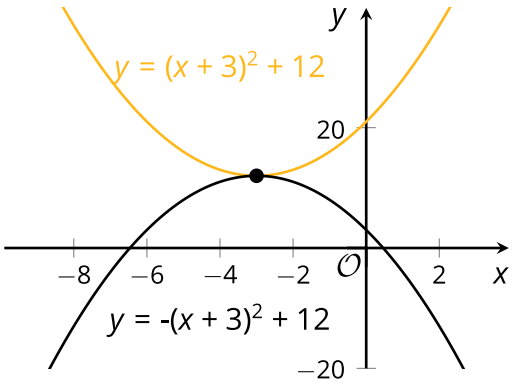


Translating Functions

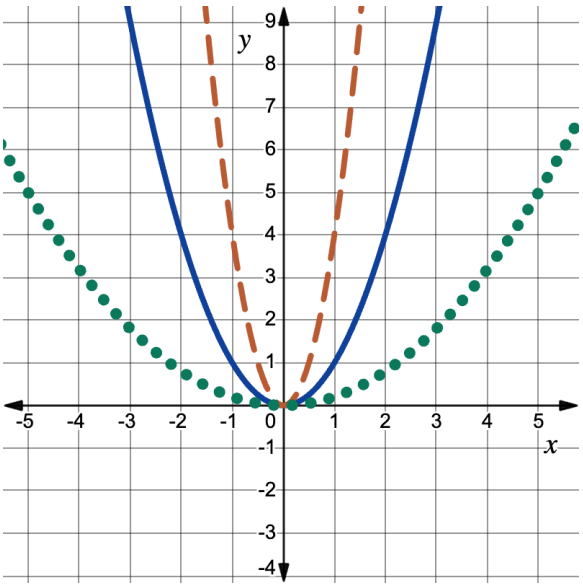
A vertical shift transforms the parent function into another function by moving the graph up or down "d" units.

A horizontal shift transforms the parent function into another function by moving the graph left or right "c" units.

The graph representing the equation $f(x) = -(x + 3)^2 + 12$ has the same vertex at $(-3, 12)$, but because the squared term $(x + 3)^2$ is multiplied by a negative number, the graph is flipped over horizontally, so that it opens downward.



Quadratic equations can also be transformed through dilations that stretch or compress the parabola. The graphs for the parent quadratic function, $f(x) = x^2$, and $f(x) = 4x^2$, and $f(x) = \frac{1}{5}x^2$ are all parabolas. The second and third equation, however, have been vertically dilated from the parent function.

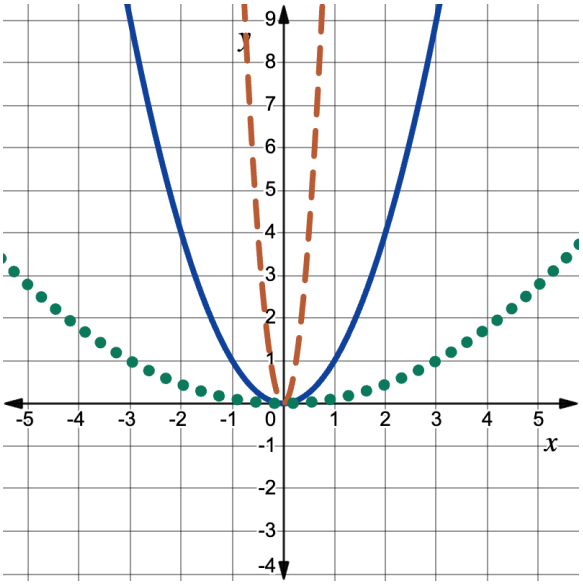


The orange dashed parabola represents $f(x) = 4x^2$ and depicts the result of multiplying each output of the function by 4. This dilation represents a vertical stretch.

The green dotted parabola represents $f(x) = \frac{1}{5}x^2$ and depicts a graph where each of the output values from the function are multiplied by $\frac{1}{5}$. This dilation represents a vertical compression.

The scale factor of the coefficient we multiply by determines the impact on the graph. If the scale factor is greater than 1, then the dilation is a vertical stretch. If the scale factor is between 0 and 1, then the dilation is a vertical compression.

When quadratic equations are dilated horizontally, the result looks similar but the impact to the function is very different because horizontal dilations affect the input values before the function is applied. The graphs for the parent quadratic function, $f(x) = x^2$, and $f(x) = (4x)^2$ and $f(x) = (\frac{1}{3}x)^2$ are displayed below.



The orange dashed parabola represents $f(x) = (4x)^2$ and depicts the result of multiplying each input of the function by 4 and then squaring. This dilation represents a horizontal compression.

The green dotted parabola represents $f(x) = (\frac{1}{3}x)^2$ and depicts a graph where each of the input values from the function are multiplied by $\frac{1}{3}$. This dilation represents a horizontal stretch.

Notice that the scale factor determines the impact on the graph here, too. If the scale factor is greater than 1, then the dilation is a horizontal compression. If the scale factor is between 0 and 1, then the dilation is a horizontal stretch.

Dilating Functions

$af(bx)$

A vertical dilation transforms the parent function into another function by stretching or compressing the output values by a scale factor of "a."

A horizontal dilation transforms the parent function into another function by stretching or compressing the input values by a scale factor of "b."

TRY IT **Transforming Graphs of Quadratics**

1. What transformations and shifts need to be made to change $f(x) = x^2$ to $f(x) = -(x-2)^2 + 7$?
2. What transformations need to be made to $f(x) = x^2$ so it becomes $f(x) = (3x)^2$?

7.17.3 Changing the Parameters of a Quadratic Expression

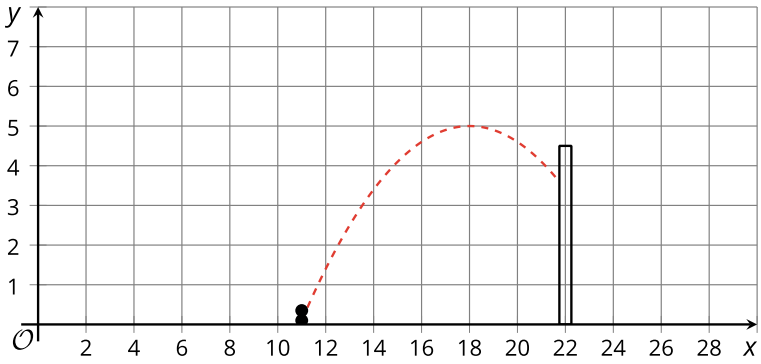
Activity

Mai is learning to create computer animation by programming. In one part of her animation, she uses a quadratic function to model the path of the main character, an animated peanut, jumping over a wall.



Mai uses the equation $f(x) = -0.1(x - h)^2 + k$ to represent the path of the jump. f represents the height of the peanut as a function of the horizontal distance it travels, x.

On the screen, the base of the wall is located at $(22, 0)$, with the top of the wall at $(22, 4.5)$. The dashed curve in the picture shows the graph of one equation Mai tried, where the peanut fails to make it over the wall.



1. What is the value of h in this equation?
2. What is the value of k in this equation?
3. Starting with Mai's equation, choose values for h and k that will guarantee the peanut stays on the screen but also makes it over the wall. Be prepared to explain your reasoning.

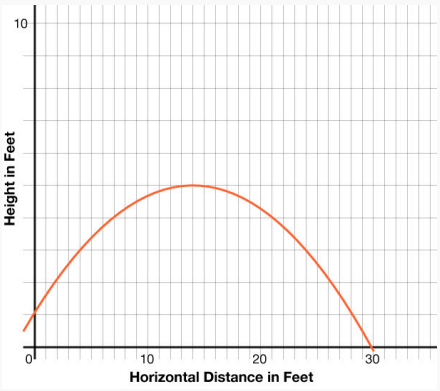
Video: Changing the Parameters of a Quadratic Expression

Watch the following video to learn more about changing the parameters of a quadratic expression.

Changing the Parameters of a Quadratic Expression (https://openstax.org/r/quadratic-parameters)

Self Check

Cash kicked a football off a tee. The path of the football is shown below and represented by the function $f(x) = -0.02(x - h)^2 + k$, where x represents the horizontal distance and $f(x)$ represents the height.



Find the vertex and its meaning.

a. $(30, 0)$, the football travels 30 feet before hitting the ground.
b. $(14, 5)$ the football hit its highest height of 14 feet after traveling 5 feet.
c. $(14, 5)$, the football hit its highest height of 5 feet after traveling 14 feet.
d. $(0, 1)$, the ball was kicked from a height of 1 foot.

Additional Resources

Determining the Vertex From a Graph

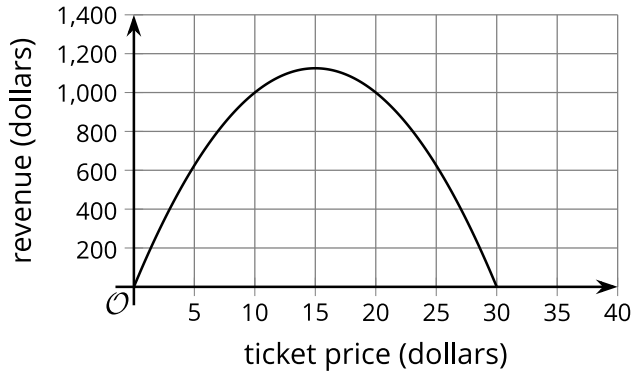
Zaren made a paper football for a game at indoor recess and flicked it in the air from a table. The path of the paper football is shown below and represented by the function $f(x) = -0.02(x - h)^2 + k$.



Find the value of the vertex and its meaning:

The peak of the graph is at $(17, 3)$. This means the maximum horizontal distance traveled was 17 inches when the height of the object was 3 inches.

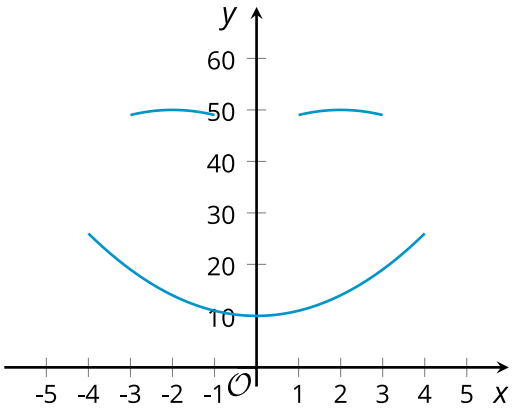
TRY IT **Determining the Vertex From a Graph**



The revenue made from a high school play is shown in the graph above. Find the vertex and tell what it means in this situation.

7.17.4 Transforming the Parent Function

Activity



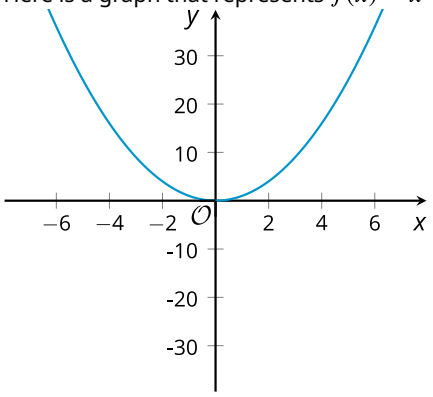
Do you see two "eyes" and a smiling "mouth" on the graph? The three arcs on the graph all represent quadratic functions that were initially defined by $y = x^2$, but whose equations were later modified.

1. Write an equation to represent the smile in the smiley face.
2. Write an equation to represent the left eye in the smiley face.
3. Write an equation to represent the right eye in the smiley face.
4. What is the domain for the mouth? Write the domain as an inequality.
5. What is the domain for the left eye? Write the domain as an inequality.
6. What is the domain for the right eye? Write the domain as an inequality.

7.17.5 Equations of Graph Translations

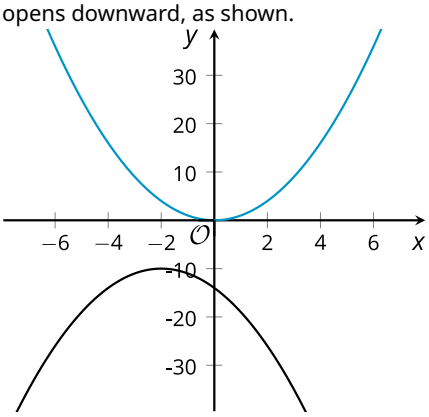
Cool Down

1. Here is a graph that represents $f(x) = x^2$.



Use the graphing tool or technology outside the course. On the same coordinate plane, sketch a graph that represents $f(x) = (x - 6)^2 + 10$.

2. The graph representing $f(x) = x^2$ is shifted 2 units to the left, 10 units down, and flipped so that it opens downward, as shown.



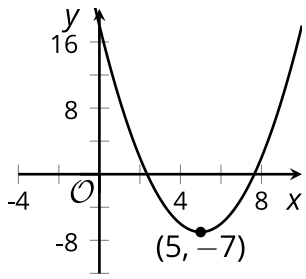
Which equation defines this curve?

a. $f(x) = -(x + 2)^2 - 10$
b. $f(x) = (x + 2)^2 - 10$
c. $f(x) = -(x + 2)^2 + 10$
d. $f(x) = (x + 2)^2 + 10$

7.17.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Here is the graph of quadratic function f.

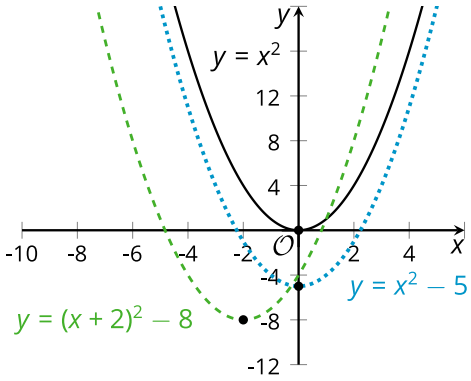


- Andre uses the expression $(x-5)^2+7$ to define f.
- Noah uses the expression $(x+5)^2-7$ to define f.

Do you agree with either of them?

a. Yes
b. No

2. Here are the graphs of $f(x)=x^2$, $f(x)=x^2-5$, and $f(x)=(x+2)^2-8$.



a. How do the three graphs compare?

a. All three graphs contain the point $(0,0)$.
b. They all have a negative a coefficient in the vertex form of the equation.
c. The shape is the same but they are in different locations.
d. They all have the same vertex.

b. Compare the graphs of $f(x)=x^2$ and $f(x)=x^2-5$. What role does the -5 play in the comparison?

a. Subtracting 5 from the squared term shifts the graph down by 5 units.
b. Subtracting 5 from the squared term shifts the graph up by 5 units.
c. Subtracting 5 from the squared term shifts the graph right by 5 units.
d. Subtracting 5 from the squared term shifts the graph left by 5 units.

c. Compare the graphs of $f(x)=x^2$ and $f(x)=(x+2)^2-8$. What roles do the $+2$ and -8 play in the comparison?

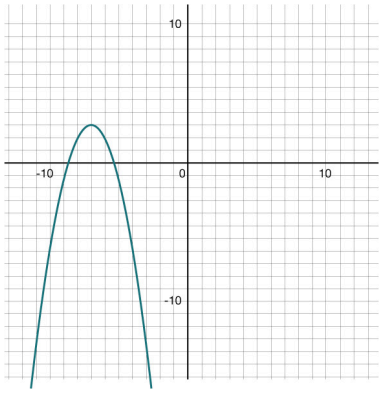
a. Adding 2 to x before squaring shifts the graph of $f(x)=x^2$ to the down by 2 units and subtracting 8 from the squared term shifts the graph right by 8 units.
b. Adding 2 to x before squaring shifts the graph of $f(x)=x^2$ to the right by 2 units and subtracting 8 from the squared term shifts the graph up by 8 units.
c. Adding 2 to x before squaring shifts the graph of $f(x)=x^2$ to the up by 2 units and subtracting 8 from the squared term shifts the graph left by 8 units.

d. Adding 2 to x before squaring shifts the graph of $f(x) = x^2$ to the left by 2 units and subtracting 8 from the squared term shifts the graph down by 8 units.

3. **Select the three** equations with a graph whose vertex has both a positive x and a positive y.

a. $f(x) = x^2$
b. $f(x) = (x-1)^2$
c. $f(x) = (x-3)^2 + 2$
d. $f(x) = 2(x-4)^2 - 5$
e. $f(x) = 0.5(x+2)^2 + 6$
f. $f(x) = -(x-4)^2 + 3$
g. $f(x) = -2(x-3)^2 + 1$

4. Which equation represents the graph shown?



a. $f(x) = (x-7)^2 + 3$
b. $f(x) = (x+7)^2 + 3$
c. $f(x) = -(x-7)^2 + 3$
d. $f(x) = -(x+7)^2 + 3$

5. The graph representing $f(x) = x^2$ is shifted 4 units to the left, 16 units down, and flipped so it opens downward (reflects over the x-axis). Which equation defines the curve?

a. $f(x) = (x+4)^2 - 16$
b. $f(x) = -(x-4)^2 - 16$
c. $f(x) = -(x+4)^2 - 16$
d. $f(x) = (x-4)^2 - 16$

6. The graph of $f(x) = x^2$ is transformed by a horizontal dilation of $\frac{1}{2}$. What is the equation of the function that results from this translation?

a. $f(x) = (\frac{1}{2}x)^2$
b. $f(x) = (x - \frac{1}{2})^2$
c. $f(x) = x^2 + \frac{1}{2}$
d. $f(x) = \frac{1}{2}x^2$

7. A function $g(x) = (2x)^2 - 5$ was transformed from the parent quadratic function. Which of the following answer choices describes the transformations that were applied?

Select the **two** descriptions that apply.

a. A horizontal shift 2 units to the right
b. A horizontal dilation by a factor of 2
c. A horizontal shift left 5 units
d. A vertical shift down 5 units

e. A vertical dilation by a factor of 2

8. Which of the following represents a function that has experienced a horizontal compression?

a. $f(x) = x^2$
b. $f(x) = (\frac{5}{2}x)^2$
c. $f(x) = (\frac{1}{5}x)^2$
d. $f(x) = (-1x)^2$

7.17.7 Lesson Summary

In this lesson, you learned how to:

- Describe how changing a number in the vertex form of a quadratic function affects its graph.

Here are the **activities** that helped you reach those goals:

- 7.17.1: Quadratic Expressions in Different Forms
  - In this activity, you matched graphs with their equations by comparing values.
- 7.17.2: Modify Expressions to Translate Graphs
  - In this activity, you shifted graphs by changing the vertex and writing the correct vertex form given the vertex and the direction of the parabola opening.
  - 7.17.2: Self Check
  - 7.17.2: Additional Resources
- 7.17.3: Changing the Parameters of a Quadratic Expression
  - In this activity, you changed parameters that found the equation needed to represent the context given. You found the equation of the parabola that modeled the path a peanut needed to take to jump over the wall in a simulation.
  - 7.17.3: Self Check
  - 7.17.3: Additional Resources
- 7.17.4: Transforming the Parent Function
  - In this activity, you were given pieces of parabolas and worked to find the equation of each piece and justify your reasoning.
- 7.17.5: Equations of Graph Translations
  - In this activity, you looked at both graphs and equations and declared the translations that were made from the parent function $f(x) = x^2$ to the new equation.

After these activities, you completed the following **practice:**

- 7.17.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

PROJECT 7: DESIGN A FOUNTAIN

Project 7 Overview

In this culminating project, you will integrate ideas from the unit by designing a fountain with given criteria. You will use the fact that the shape made by water coming out of a jet or fountain can be modeled with a quadratic equation.

In the first activity, you will work through a simplified example with one fountain and one jet. Then, you will think about what information is needed to write the equation and create a sketch to draw the path. Next, you will determine reasonable measurements to write the equation. Understanding the use of the vertex form of a quadratic equation will be helpful.

The second activity gives you an opportunity to design a fountain with three to five jets. You will be given certain measurements and criteria to meet to write the equation for paths of the jets. When you are done, you will present your equations and an explanation of what information you used and how you wrote the equation of the path of the jets.

Project Activities

1. Introduction to Fountain Design
2. Designing a Fountain

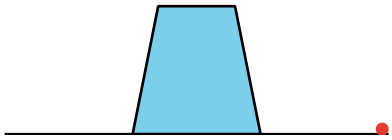
Learning Targets

When you finish this project, you will have demonstrated how to:

- Use quadratic functions to solve a real-world problem.

Introduction to Fountain Design

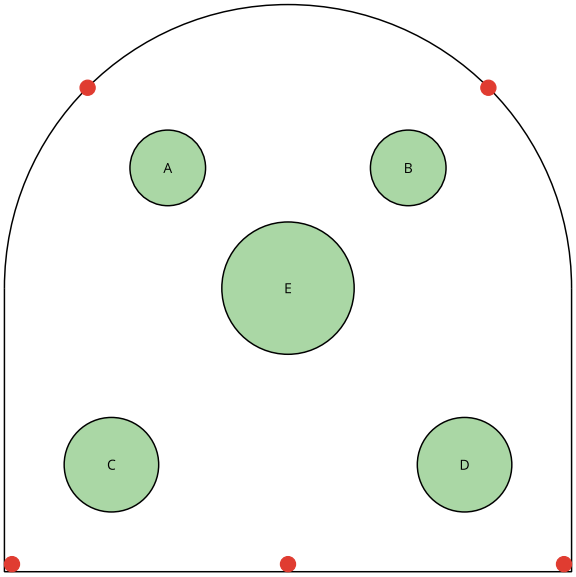
The trapezoid represents the statue and the red dot is the location of the water jet.



1. What would you need to know in order to write an equation for a path for the water that would make it go over the statue?
2. What is a reasonable height of the statue and a reasonable distance from the jet to the statue?
3. What strategy would you use to find the equation for the path of the water?

Design a Fountain

A nearby park has a decorative pool, and the board of commissioners of the park is considering installing water jets in the pool to make it into a fountain. They have asked you to design the water jets. Here is a diagram of the pool from above:



- A and B are statues which are 4 feet tall.
- C and D are wider statues which are 5.5 feet tall.
- E is a statue with a vase on top, which is 8 feet tall.
- The pool is 15 feet wide.

The diagram is drawn to scale. The dots along the edge show where jets can be placed. You don't have to put jets in all five places. The jets will be at water level.

You will need to figure out how many jets there should be, where they should go, and what path the water should make. Your design needs to meet these criteria:

- The jets can only go in the places indicated by the dots.
- There must be at least two jets.
- The water can't hit any of the statues.
- Some water must go into the vase.
- The water can't go higher than 10 feet.

When you have your final design, you should create a presentation to explain it to the commissioners. The presentation should show the paths of the water and include your mathematical descriptions of the paths.

How did your work creating one fountain in part 1 help with your process in part 2 where you created multiple fountains?

Unit 7 Wrap Up

Throughout this unit, you learned about machines that students of Howard Bernard created for a Punkin Chunkin competition. While working within a budget as a constraint, students created a machine that would launch a pumpkin the farthest. While creating the machines, students relied on quadratic functions to represent the paths of the pumpkins and tweak their designs.

In the beginning of this unit, you explored different patterns that weren't linear and exponential and discovered how to recognize quadratic functions from geometric patterns. You then used quadratic functions to describe a variety of quadratic situations.

You also learned to write equivalent quadratic expressions in standard, factored, and vertex forms.

Finally, you learned how to graph quadratics using the different forms and that each form helps to find key characteristics about quadratic graphs.



Access our Unit 7 Student Self-Assessment (https://openstax.org/r/unit-7-self-assesment) to reflect on the skills you learned during this unit. Congratulations on completing Unit 7!

Building Character: Proactivity

Throughout this unit, you learned that **proactivity** is important for taking initiative and brainstorming ideas. You discovered that proactivity is a trait that can be learned and comes from a willingness to initiate change. Proactive people are always looking to improve ways of doing things and are often leaders in different settings. They encourage brainstorming of ideas within teams to create solutions to problems.

UNIT 8 OVERVIEW AND READINESS

Unit 8 Overview

Why Should I Care?

Imagine you are at a park. This park has a large water fountain. In this fountain, streams of water shoot up and arc from the middle to the edge of the fountain's pool. The streams of water always land inside the pool. How does this work?

Also at this park, you see a group of children playing catch with a ball. What do the fountain and the kids playing with the ball have in common?

The answer? Quadratic equations!

Watch the video to learn a bit about how we can see quadratic equations in everyday things, from water fountains to playing with a ball.

Unit 8 Overview (https://openstax.org/r/everyday-quadratics)

In this unit, you will learn about quadratic equations. Understanding quadratic equations can help you understand the world around you. For example, they can be used to determine the maximum height a thrown ball will reach and the time it takes for the ball to hit the ground.

Building Character: Curiosity

In the video, you learned that Juan David Crespo used quadratic equations to plan how high and far each stream of water will go in a fountain. But how exactly does he do it? Are you interested in learning more?

Throughout this unit, you will work on developing your **curiosity**. Curiosity is the desire to learn and understand new things. When you are curious about something, you process it on a deeper level than just basic understanding. People typically spend more time learning about things they are curious about and more readily remember the information. Curious people tend to be happier individuals.

Think about your current sense of curiosity. How many of the following statements are true for you?

- I got so absorbed in learning that I lost track of time.
- I took the initiative to learn more about one of my interests.
- When I didn't know the answer to a question, I couldn't rest until I figured it out.
- I explored a completely new idea or topic—just for the fun of it.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

Am I Ready to Learn This?

Getting Ready for Unit 8

To be ready for this unit, you need to be able to:

- Write equivalent expressions using the distributive property.
- Find the x-intercepts of the graph of a quadratic equation given in factored form.
- Match quadratic graphs and their functions.

Next, you will answer some questions that will help you see how ready you are for this unit. If you need help, some mini-lessons will help you brush up on anything you need to work on.

Write Equivalent Expressions: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #1: Write Equivalent Expressions

Which expression is equivalent to $(x-3)(x+4)$?

a. $x^2-3x-12$
b. x^2-x-12
c. x^2+x-12

d. $x^2 + 4x$

Write Equivalent Expressions Using the Distributive Property

The standard form of a quadratic expression is $ax^2 + bx + c$, where a, b, and c are constants, and a is not 0.

EXAMPLE

Write $(x - 3)(x + 5)$ as an equivalent expression in standard form using the distributive property.

Step 1 - Multiply each term in the first binomial by the second binomial.

$(x - 3)(x + 5)$

$x(x + 5) - 3(x + 5)$

Step 2 - Use the distributive property to eliminate the parentheses.

$x^2 + 5x - 3x - 15$

Step 3 - Combine like terms.

$x^2 + 2x - 15$

TRY IT **Write Equivalent Expressions Using the Distributive Property**

Write $(x + 2)(x - 7)$ as an equivalent expression in standard form using the distributive property.

Check Your Understanding

Which expression is equivalent to $(x + 5)(x - 1)$?

a. $x^2 + 4x - 5$

b. $x^2 - 4x + 5$

c. $x^2 + 5x - 1$

d. $x^2 + 5x - 5$

Video: Multiplying Binomials

Watch the following video to learn more about multiplying binomials using the distributive property to write equivalent expressions.

Khan Academy: Multiplying Binomials

Example 1: Multiplying a binomial by a binomial | Algebra I | Khan Academy (https://www.youtube.com/embed/ZMLFfTX615w)

Find Intercepts: Mini-Lesson Review

MINI LESSON QUESTION **Question #2: Find Intercepts**

What are the x-intercept(s) of the graph of $y = (2x - 9)(x + 2)$?

a. $x = 2, x = -9$

b. $x = -2, x = 9$

c. $x = -2, x = 4.5$

d. $x = 2, x = -4.5$

Find the x-Intercepts

The x-intercepts of the graph of an equation occur when $y = 0$.

EXAMPLE 1

For the equation $y = (x - 9)(x + 2)$, look at the factors $(x - 9)$ and $(x + 2)$.

Notice in the first factor, $(x - 9)$, $x = 9$ makes the factor equal to 0.

Substitute 9 for x in the equation.

$y = (9 - 9)(9 + 2)$

$y = (0)(11)$

$y = 0$

Notice in the second factor, $(x + 2)$, $x = -2$ makes the factor equal to 0.

Substitute -2 for x in the equation.

$y = (-2 - 9)(-2 + 2)$

$y = (-11)(0)$

$y = 0$

In the equation $y = (x - 9)(x + 2)$, $y = 0$ when $x = 9$ and $x = -2$.

The x-intercepts occur when $x = 9$ and $x = -2$.

To find the x-intercepts of a quadratic equation in factored form, set each factor equal to zero and solve for x.

EXAMPLE 2

Find the x-intercepts for $y = (2x + 7)(x - 4)$.

The x-intercepts occur when $y = 0$.

Set each factor to 0 and solve.

$$\begin{array}{rclrcl} 2x + 7 & = & 0 & x - 4 & = & 0 \\ x & = & -3.5 & x & = & 4 \end{array}$$

TRY IT **Find the x-Intercepts**

Find the x-intercepts for $y = (2x - 5)(x + 3)$.

Check Your Understanding

What are the x-intercept(s) of the graph of $y = (2x + 1)(x - 3)$?

a. $x = -3, x = 0.5$

b. $x = 3, x = -0.5$

c. $x = -3, x = 1$

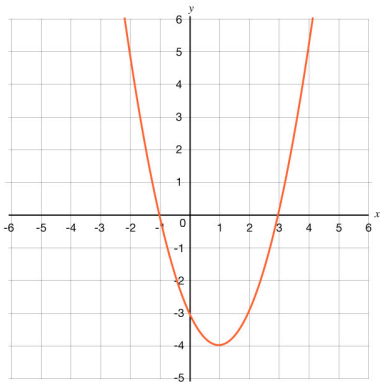
d. $x = 3, x = -1$

Match Quadratic Graphs and Their Equations: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #3: Match Quadratic Graphs and Their Equations

Examine the graph below.



Which function matches the graph?

a. $f(x) = (x + 1)(x + 3)$

b. $f(x) = (x - 1)(x - 3)$

c. $f(x) = (x + 1)(x - 3)$

d. $f(x) = (x - 1)(x + 3)$

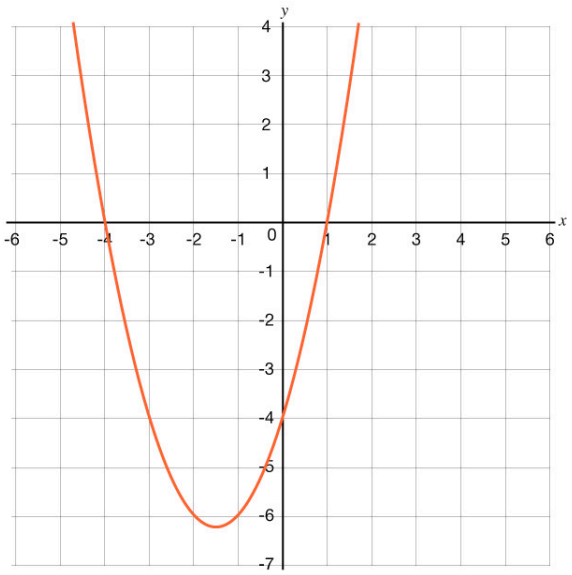
Match Quadratic Graphs and Their Equations

The factored form helps find the zeros of a quadratic function.

The factored form gives the zeros, and the x-intercepts are located where each factor equals 0.

EXAMPLE

Examine the graph. What are the x-intercepts of the graph?



The graph crosses the x-axis at $x = -4$ and $x = 1$. These are the x-intercepts.

Look at each function to see which matches the graph. Look for the factors that have zeros as $x = -4$ and $x = 1$.

a. $f(x) = (x + 4)(x + 1)$

- This function has zeros at $x = -4$ and $x = -1$. The zero $x = -1$ is not a zero on the graph.

b. $f(x) = (x - 4)(x - 1)$

- This function has zeros at $x = 4$ and $x = 1$. The zero $x = 4$ is not a zero on the graph.

c. $f(x) = (x - 4)(x + 1)$

- This function has zeros at $x = 4$ and $x = -1$. The signs of the zeros of this function are the opposite of the signs of the zeros on the graph.

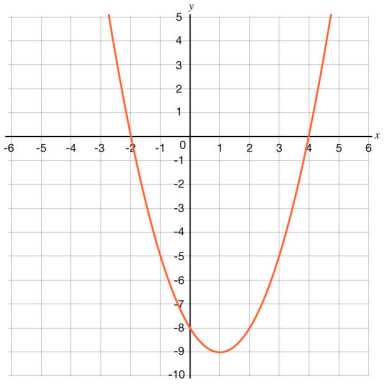
d. $f(x) = (x + 4)(x - 1)$

- This function has zeros at $x = -4$ and $x = -1$. This function has zeros that match the zeros on the graph.

The function $f(x) = (x + 4)(x - 1)$ matches the graph.

TRY IT **Match Quadratic Graphs and Their Equations**

Examine the graph.



Which function matches the graph?

a. $f(x) = (x + 2)(x + 4)$

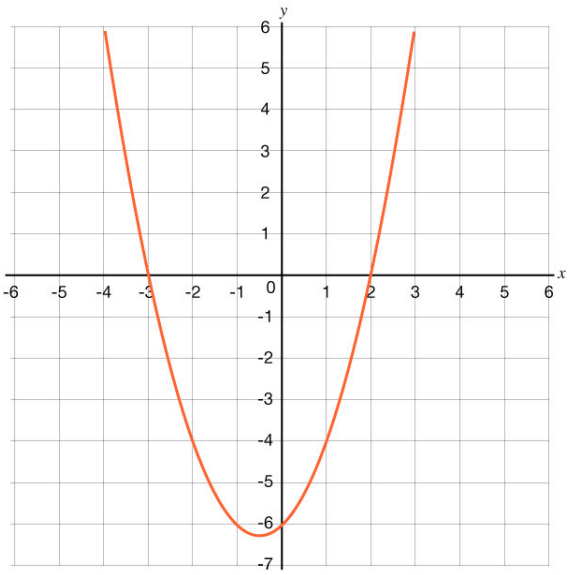
b. $f(x) = (x - 2)(x - 4)$

c. $f(x) = (x + 2)(x - 4)$

d. $f(x) = (x - 2)(x + 4)$

Check Your Understanding

Examine the graph.



Which function matches the graph?

a. $f(x) = (x - 3)(x - 2)$

b. $f(x) = (x - 3)(x + 2)$

c. $f(x) = (x + 3)(x - 2)$

d. $f(x) = (x + 3)(x + 2)$

Unit 8 Overview: Wrap Up

- Write equivalent expressions using the distributive property.
- Find the x-intercepts of the graph of a quadratic equation given in factored form.
- Match quadratic graphs and their functions.

If you are still struggling, be sure to reach out to your teacher for additional help. Nobody is good at problems like this the first try. Keep at it, and it will get easier.

8.1 FINDING UNKNOWN INPUTS

8.1.0 Lesson Overview

In this lesson, you will encounter a problem that cannot be easily solved by familiar strategies. After exploring different ways to solve the problem, you will write a quadratic equation to model the situation. You will also interpret what the solution to that equation means in the given context.

When you finish this lesson, you will be able to:

- Explain the meaning of a solution to an equation in terms of a situation.
- Write a quadratic equation that represents a situation.

Here are the **activities** that will help you reach those goals:

- 8.1.1: Understanding a Situation with Quadratic Equations
- 8.1.2: Modeling a Quadratic Problem
  - 8.1.2: Self Check
  - 8.1.2: Additional Resources
- 8.1.3: Formulating a Quadratic Equation to Represent the Model
  - 8.1.3: Self Check
  - 8.1.3: Additional Resources
- 8.1.4: Interpreting a Solution

After that, you'll **practice and review**.

- 8.1.5: Practice
- 8.1.6: Lesson Summary

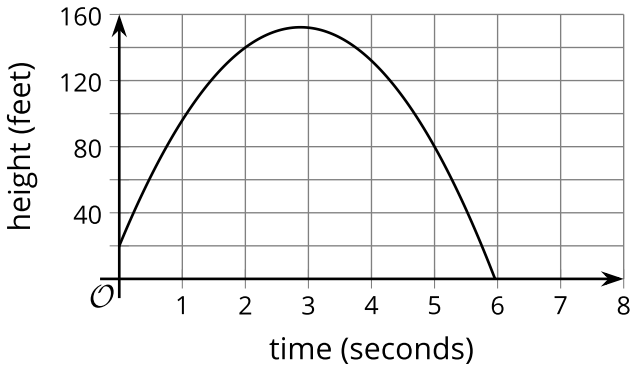
8.1.1 Understanding a Situation with Quadratic Equations

Warm Up

A mechanical device is used to launch a potato vertically into the air. The potato is launched from a platform 20 feet above the ground, with an initial vertical velocity of 92 feet per second.

The function $h(t) = -16t^2 + 92t + 20$ models the height of the potato over the ground, in feet, t seconds after launch.

Here is the graph representing the function.



For each question, explain your reasoning.

1. What is the height of the potato 1 second after launch?
2. Will the potato still be in the air 8 seconds after launch?

3. Will the potato reach 120 feet? If so, when will it happen?
4. When will the potato hit the ground, or have a height of zero feet?

8.1.2 Modeling a Quadratic Problem

Activity

Your teacher will give you a picture that is 7 inches by 4 inches, a piece of colored paper that measures 4 inches by 2.5 inches representing framing material, a ruler, and a pair of scissors.

Cut the framing material to create a rectangular frame for the picture. The frame should have the same width all the way around and have no overlaps. All of the framing material should be used with no leftover pieces. The frame should not cover any part of the picture.

Let's review the information we know.

- The picture measures 7 inches by 4 inches.
- There are 10 square inches of framing material.
- The frame must be uniform in width.
- All of the framing material must be used.
- The frame should not cover any part of the picture.

Describe the process you used to create the frame.

Now that you have completed the activity, can you think of any mathematical method to solve the problem?

Are you ready for more?

Extending Your Thinking

Juan says, "The perimeter of the picture is 22 inches. If I cut the framing material into 9 pieces, each one being 2.5 inches by $\frac{4}{9}$ inch, I'll have more than enough material to surround the picture because those pieces would yield $9 \cdot 2.5$, or 22.5 inches for the frame."

Do you agree with Juan? Be prepared to show your reasoning.

Self Check

Suppose you want to frame a picture that measures 8 inches by 3 inches. The frame is created from 12 square inches of framing material. The frame must be uniform in width, and all of the framing material must be used. Which statement is true?

a. The total area of the framed picture will be 36 square inches.
b. The perimeter of the framed picture will be 22 inches.
c. The total area of the framed picture will be 24 square inches.
d. The frame will measure exactly 8 inches by 3 inches.

Additional Resources

Modeling a Quadratic Problem

This activity challenges you to complete a very difficult task!

Let's review the information we know from the picture in the Self Check.

- The picture measures 8 inches by 3 inches.
- There are 12 square inches of framing material.
- The frame must be uniform in width.
- All of the framing material must be used.
- The frame should not cover any part of the picture.

What other information can you learn from the situation?

- The perimeter of the picture is $8 + 8 + 3 + 3$, or 22 inches. The framed picture will have a perimeter greater than this.
- If the picture is 8 inches by 3 inches, then the area of the picture is $8 \cdot 3$, or 24 square inches.
- Since there are 12 square inches of framing material and all of it must be used, the framed picture will have an area of $24 + 12$, or 36 square inches.

The more you know about the situation, the closer you are to being able to solve this problem mathematically instead of guessing and checking to see if you are correct.

We will use this information in later activities to set up a quadratic equation that can be used to solve the problem.

TRY IT **Modeling a Quadratic Problem**

Suppose you want to frame a different picture that measures 10 inches by 6 inches. The frame is created from 36 square inches of framing material. The frame must be uniform in width, and all of the framing material must be used.

1. What is the perimeter of the picture?

2. What is the area of the picture?

3. What will be the total area of the framed picture?

Suppose you want to frame a different picture that measures 10 inches by 6 inches. The frame is created from 36 square inches of framing material. The frame must be uniform in width, and all of the framing material must be used.

4. What will be the total area of the framed picture?

5. What is the area of the picture?

8.1.3 Formulating a Quadratic Equation to Represent the Model

Activity

Here is a diagram that shows the picture with a frame that is the same width all the way around.

The picture is 7 inches by 4 inches. The frame is created from 10 square inches of framing material (in the form of a rectangle measuring 4 inches by 2.5 inches).



1. Write an expression to represent the length of the framed picture on its shorter side. Use x to represent the width of the frame.

2. Write an expression to represent the length of the framed picture on its longer side. Use x to represent the width of the frame.

3. Use the expressions from the previous exercises to write an equation to represent the relationship between the measurements of the framed picture and its total area. Be prepared to explain what each part of your equation represents.
4. What would a solution to this equation mean in this situation?

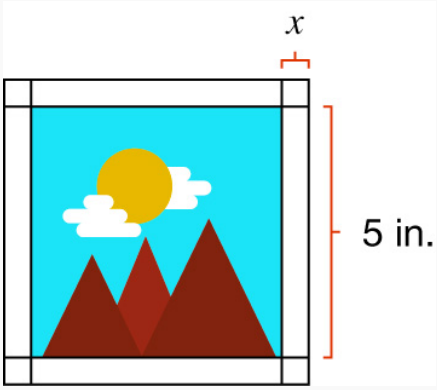
Video: Formulating a Quadratic Equation to Represent the Model

Watch the following video to learn more about formulating a quadratic equation to represent the model.

Formulating a Quadratic Equation to Represent the Model (https://openstax.org/r/quadratic-model)

Self Check

Suppose a different picture is 5 inches by 5 inches. The frame is created from 11 square inches of framing material. The frame will be uniform in width, and all of the framing material is used. Which equation would you need to solve to determine the value of x, the width of the frame?



a. $4x^2 + 20x = 36$
b. $x^2 + 25 = 11$
c. $x^2 + 10x + 25 = 36$
d. $4x^2 + 20x + 25 = 36$

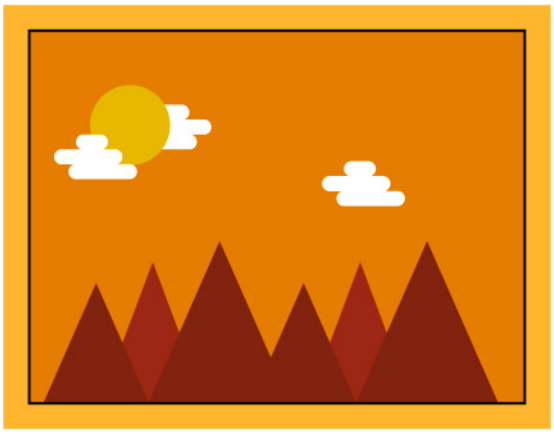
Additional Resources

Formulating a Quadratic Equation to Represent the Model

Let's look at a similar situation to the previous activities but using a picture with different measurements.

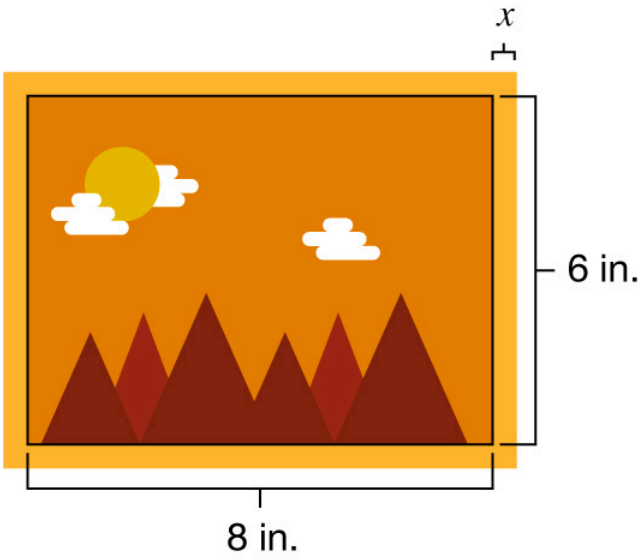
Imagine a picture 8 inches by 6 inches. The frame is created from 32 square inches of framing material. The frame will be uniform in width, and all of the framing material is used.

How do we write an equation representing the total area of the framed picture?



Let's break down the measurements that we know.

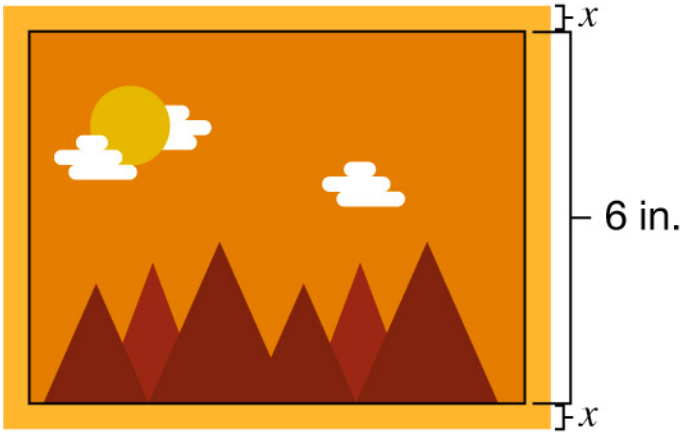
Let x represent the thickness of the frame in inches.



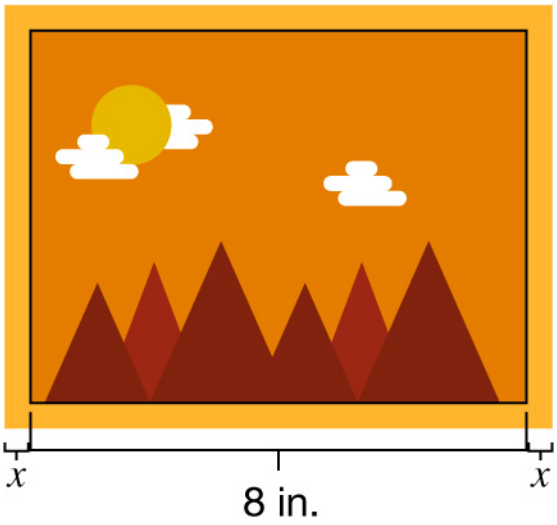
Since the picture is 8 inches by 6 inches, it has an area of 48 square inches.

The frame is created from 32 square inches of framing material, and we must use all of it. So, the total area of the framed picture is $48 + 32$, or 80 square inches.

The length of one side of the framed picture is $x + 6 + x = 2x + 6$.



The length of the other side of the framed picture is $x + 8 + x = 2x + 8$.



So, the area of the framed picture is $(2x + 8)(2x + 6)$. We can set this equal to 80 square inches.

$(2x + 8)(2x + 6) = 80$

Let's multiply.

$4x^2 + 12x + 16x + 48 = 80$

$4x^2 + 28x + 48 = 80$

So, the equation representing the total area of the framed picture is $4x^2 + 28x + 48 = 80$.

This equation is a quadratic equation. A **quadratic equation** is defined as one that can be written in the form $ax^2 + bx + c = 0$, where a is not 0.

TRY IT **Formulating a Quadratic Equation to Represent the Model**

A picture measures 9 inches by 7 inches. Its frame is created from 36 square inches of framing material. The frame will be uniform in width, and all of the framing material is used.

Write an equation representing the total area of the framed picture.

8.1.4 Interpreting a Solution

Cool Down

A framed picture has a total area y, in square inches. The thickness of the frame is represented by x, in inches. The equation $y = (2x + 8)(2x + 10)$ relates these two variables.

1. What is the width, or shorter side, of the picture without the frame?
2. What is the length, or longer side, of the picture without the frame?
3. What would a solution to the equation $100 = (2x + 8)(2x + 10)$ mean in this situation?

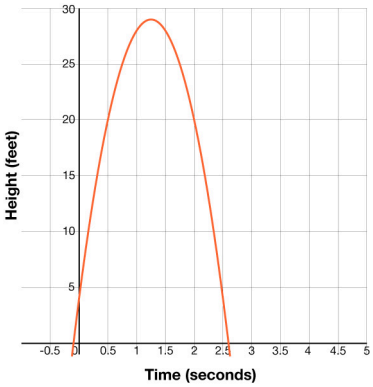
8.1.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

Use the following scenario to answer questions 1 – 3.

A water balloon is launched by a catapult into the air. The water balloon is launched from 4 feet above the ground, with an initial vertical velocity of 40 feet per second.

The function $h(t) = -16t^2 + 40t + 4$ models the height of the water balloon over the ground, in feet, t seconds after launch.



1. What is the height of the water balloon 1 second after launch?

a. 34 feet
b. 28 feet
c. 22 feet
d. 18 feet

2. Will the water balloon still be in the air 2 seconds after the launch?

a. No
b. Yes

3. Using the graph, estimate when the water balloon hit the ground.

a. 2.6 seconds
b. 2.1 seconds
c. 1.8 seconds
d. 1.2 seconds

Use the following scenario to answer questions 4 – 5.

Suppose you want to frame a different picture that measures 12 inches by 9 inches. The frame is created from 46 square inches of framing material. The frame must be uniform in width, and all of the framing material must be used.

4. Which of the following is true about the situation described?

a. The perimeter of the framed picture is greater than 42 inches.
b. The area of the framed picture is 148 square inches.
c. The perimeter of the framed picture is 42 inches.
d. The perimeter of the picture is 40 inches.

5. What is the total area of the framed picture?

a. 176 square inches
b. 154 square inches
c. 108 square inches
d. 46 square inches

Use the following scenario to answer questions 6 – 8.

Suppose you want to frame a different picture that measures 9 inches by 8 inches. The frame is created from 38 square inches of framing material. The frame must be uniform in width, and all of the framing material must be used. Let x represent the thickness of the frame.

6. What is the total area of the framed picture?

a. 134 square inches
b. 110 square inches
c. 72 square inches
d. 34 square inches

7. Write an equation to represent the relationship between the measurements of the picture and of the frame and the area of the framed picture.

a. $4x^2 + 34x + 72 = 110$
b. $x^2 + 34x + 72 = 110$
c. $4x^2 + 34x = 72$
d. $x^2 + 34x = 38$

8. What does the solution to the equation in the previous question represent in this situation?

a. The solution represents the total area of the framed picture.
b. The solution represents the perimeter of the frame when all of the framing material is used.
c. The solution represents the thickness of the frame when all of the framing material is used.
d. The solution represents the area of the framing material that should be used.

Use the following scenario to answer questions 9 – 10.

A framed picture has a total area y, in square inches. The thickness of the frame is represented by x, in inches. The equation $y = (2x + 5)(2x + 7)$ relates these two variables.

9. What are the width and length of the picture without the frame?

a. 5 inches and 7 inches
b. 2 inches and 7 inches
c. 2 inches and 5 inches
d. 2 inches and 2 inches

10. Which of the following represents the area of the frame?

a. 35
b. $x + 35$
c. $x + y$
d. $y - 35$

8.1.6 Lesson Summary

In this lesson, you learned how to:

- Explain the meaning of a solution to an equation in terms of a situation.
- Write a quadratic equation that represents a situation.

Here are the **activities** that helped you reach those goals:

- 8.1.1: Understanding a Situation with Quadratic Equations
  - In this activity, you analyzed a graph representing a situation represented by a quadratic equation. You then used the graph to describe the situation.
- 8.1.2: Modeling a Quadratic Problem
  - In this activity, you attempted to solve a problem that was represented by a quadratic equation using any method possible, including guess and check.
  - 8.1.2: Self Check
  - 8.1.2: Additional Resources
- 8.1.3: Formulating a Quadratic Equation to Represent the Model
  - In this activity, you learned how to write a quadratic equation to represent the given situation. You also learned how to interpret the solution to this equation.
  - 8.1.3: Self Check
  - 8.1.3: Additional Resources
- 8.1.4: Interpreting a Solution
  - In this activity, you used a quadratic equation to draw conclusions about a given situation.

After these activities, you completed the following **practice**:

- 8.1.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

8.2 WHEN AND WHY DO WE WRITE QUADRATIC EQUATIONS?

8.2.0 Lesson Overview

In this lesson, you will analyze and interpret given equations, write equations to represent relationships and constraints, and work to solve these equations. Ultimately, you will see that sometimes solutions to quadratic equations cannot be easily or precisely found by graphing or reasoning.

When you finish this lesson, you will be able to:

- Recognize the factored form of a quadratic expression and know when it can be useful for solving problems.
- Use a graph to find the solutions to a quadratic equation but also know its limitations.

Here are the **activities** that will help you reach those goals:

- 8.2.1: Interpreting Equations in Context
- 8.2.2: Exploring Non-Graphing Strategies to Solve Quadratic Equations
  - 8.2.2: Self Check
  - 8.2.2: Additional Resources
- 8.2.3: Solving a Quadratic Equation Set Equal to Zero
  - 8.2.3: Self Check
  - 8.2.3: Additional Resources
- 8.2.4: Solving a Real-World Problem Using Quadratic Equations

After that, you'll **practice and review**.

- 8.2.5: Practice
- 8.2.6: Lesson Summary

8.2.1 Interpreting Equations in Context

Warm Up



Tickets to a play cost \$12, and for each purchase there is a \$2.50 fee. The expression $12t + 2.50$ represents the cost to purchase tickets for the play, where t is the number of tickets. Be prepared to explain your response to each question.

1. A family paid \$62.50 for tickets. How many tickets were bought?

2. A teacher paid \$278.50 for tickets for her students. How many tickets were bought?

8.2.2 Exploring Non-Graphing Strategies to Solve Quadratic Equations

Activity

In a previous lesson, you saw an equation that defines the height of a potato as a function of time after it was launched from a mechanical device. Here is a different function modeling the height of a potato, in feet, t seconds after being fired from a different device:

$f(t) = -16t^2 + 80t + 64$

1. What equation would we solve to find the time at which the potato hits the ground?
2. Use any method except graphing to find a solution to the equation from the previous question.

Building Character: Curiosity



Curiosity is the desire to learn and understand new things. When you are curious about something, you process it on a deeper level than just basic understanding.

Think about your current sense of curiosity. Are the following statements true for you??

- I got so absorbed in learning that I lost track of time.
- I took the initiative to learn more about one of my interests.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

Self Check

After a flashlight is turned on, it gets brighter as it warms up, and then it grows dimmer as the battery dies. The equation modeling the brightness it emits, in lumens, h hours after being turned on is shown here.

$f(h) = -8h^2 + 30h + 50$

Which equation would you solve to determine when the flashlight dies and stops emitting lumens?

a. $-8h^2 + 30h + 50 = 0$
b. $-8h^2 + 30h = 50$
c. $8h^2 + 30h + 50 = 0$

d. $-8h^2 + 30h - 50 = 0$

Additional Resources

Using Various Methods to Solve Quadratic Equations

A tennis ball is thrown out of the window of a building. The equation modeling the height of the tennis ball, in feet, t seconds after being thrown is shown here.

$f(t) = -24t^2 + 33t + 30$

Let's find the equation we would solve to find the time at which the ball hits the ground.

When the ball hits the ground, its height will be 0. So, we set the equation to 0.

$-24t^2 + 33t + 30 = 0$

The equation above represents the ball when it hits the ground. The solution to this equation represents the time it takes from when the ball is thrown for it to hit the ground.

Let's try to solve the equation.

Step 1 - Write the equation to be solved.

$-24t^2 + 33t + 30 = 0$

Step 2 - Subtract 30 from each side of the equation.

$-24t^2 + 33t = -30$

Step 3 - Divide each side of the equation by 3.

$-8t^2 + 11t = -10$

Let's try to solve the remaining equation using guess and check.

When $t = 3$, the expression $-8t^2 + 11t$ has a value of $-8(3)^2 + 11(3)$, or -39. This value is not equal to the -10 we needed to find.

When $t = 2$, the expression $-8t^2 + 11t$ has a value of $-8(2)^2 + 11(2)$, or -10. This value is equal to the -10 we needed to find.

The equation is true when $t = 2$, so the ball took 2 seconds to hit the ground. This method works but can be more difficult when the solution is a decimal. Graphing can also work, but it is time consuming and results in an approximate solution.

TRY IT **Using Various Methods to Solve Quadratic Equations**

An arc of water flows from a water fountain down to a pool below it. The height of the arc above the pool, in inches, can be modeled using the equation $f(x) = -2x^2 + 5x + 3$, where x is the horizontal distance the water has traveled, in inches, from the water fountain.

1. Which equation would you solve to determine how far the arc of water has traveled horizontally when it reaches the pool below?

2. Solve the equation using a non-graphing method.

8.2.3 Solving a Quadratic Equation Set Equal to Zero

Activity

The expressions $p(200 - 5p)$ and $-5p^2 + 200p$ define the same function. The function models the revenue a school would earn from selling raffle tickets at p dollars each.

1. At what price or prices would the school collect $0 revenue from raffle sales? Explain or show your reasoning.
2. The school staff noticed that there are two ticket prices that would result in a revenue of $500. What equation could you use to determine what those two prices are?

Are you ready for more?

Extending Your Thinking

1. Can you find the following prices without graphing? If the school charges $10, it will collect $1500 in revenue. Find another price that would generate $1500 in revenue.
2. If the school charges $28, it will collect $1680 in revenue. Find another price that would generate $1680 in revenue.
3. Find the price that would produce the maximum possible revenue. Be prepared to show your reasoning.

Video: Solving a Quadratic Equation Set Equal to Zero

Watch the following video to learn more about solving a quadratic equation set equal to zero.

8.2.3 (https://openstax.org/r/quadratics-set-to-zero)

Self Check

The expressions $s(60-12s)$ and $-12s^2+60s$ define the same function. A soap bubble machine can launch many soap bubbles into the air before they begin to pop. The function given models the number of soap bubbles floating in the air in s seconds.

According to the function, how many seconds have passed before all of the bubbles have been made and also popped?

a. 7 seconds
b. 6 seconds
c. 5 seconds
d. 0 seconds

Additional Resources

Solving a Quadratic Equation Set Equal to Zero

The expressions $h(40-10h)$ and $-10h^2+40h$ define the same function. The function models the height of a weather balloon in miles h hours after it is released.

1. What equation can be used to determine the times the weather balloon is on the ground?

Solution

When the weather balloon is on the ground, the expression will equal 0. So, the equation $h(40-10h)=0$ represents the weather balloon when it is on the ground.

2. How can we solve the equation $h(40-10h)=0$?

Solution

We know that any value multiplied by 0 equals 0. So, if h or $(40-10h)$ equals 0, then the left side of the equation will also equal 0.

The weather balloon is on the ground at 0 hours. Now, let's find the solution to $40-10h=0$.

$40-10h=0$

$40=10h$

$4=h$

The weather balloon is also on the ground after 4 hours.

3. A scientist wants to determine the two different times when the weather balloon is 30 miles above Earth. Write an equation to represent these times.

Solution

The times when the weather balloon is 30 miles above Earth are represented by the equation $h(40 - 10h) = 30$ or the equation $-10h^2 + 40h = 30$.

4. If the weather balloon is 30 miles above Earth after 1 hour, find another time that the weather balloon is 30 miles above Earth.

Solution

We know the total trip takes the weather balloon 4 hours. Since the graph representing the flight of the weather balloon is symmetric, if it takes 1 hour to reach 30 miles up, then 1 hour from touching the ground, it will be at 30 miles up again. So, the weather balloon will be 30 miles above Earth after 3 hours.

TRY IT **Solving a Quadratic Equation Set Equal to Zero**

The expressions $p(420 - 60p)$ and $-60p^2 + 420p$ define the same function. The function models revenue earned at a baseball game from the sale of baseball magazines sold at p dollars each.

1. At what price or prices would the game organizers collect $0 revenue from magazine sales? Explain or show your reasoning.

2. The organizers determined there are two magazine prices that would result in a revenue of $600. What equation would you use to determine what those two prices are?

8.2.4 Solving a Real-World Problem Using Quadratic Equations

Cool Down

A movie theater models the revenue from ticket sales in one day as a function of the ticket price, p. Here are two expressions defining the same revenue function.

$p(120 - 4p)$

$120p - 4p^2$

1. According to this model, how high would the ticket price have to be for the theater to make $0 in revenue? Be prepared to show your reasoning.

2. What equation can you write to find out what ticket price(s) would allow the theater to make $600 in revenue?

8.2.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. The expression $9t + 7.25$ represents the cost to purchase tickets for a concert, where t is the number of tickets. A group of friends paid $79.25 for tickets. How many tickets were bought?

a. 12 tickets
b. 10 tickets
c. 8 tickets
d. 6 tickets

Use the following scenario to answer questions 2 – 4.

A model rocket is launched from a launchpad. The function modeling the height of the rocket, in feet, t seconds after being launched is shown.

$f(t) = -15t^2 + 58t + 8$

2. Which equation would you solve to find the time at which the rocket hits the ground?

a. $15t^2 + 58t + 8 = 0$
b. $-15t^2 + 58t = 0$
c. $-15t^2 + 58t = 8$
d. $-15t^2 + 58t + 8 = 0$

3. Use a non-graphing method to find the solution to the previous equation to determine the time that has passed when the rocket hits the ground.

a. 5 seconds
b. 4 seconds
c. 3 seconds
d. 2 seconds

4. How high off the ground is the launchpad from which the rocket lifts off?

a. 15 feet
b. 8 feet
c. 2 feet
d. 0 feet

Use the following scenario to answer questions 5 – 8.

The expressions $t(90 - 3t)$ and $-3t^2 + 90t$ define the same function. The function models the number of lumens emitted by a spotlight in an art exhibit over the course of t seconds.

5. Which equation represents the time when the spotlight is not emitting any light?

a. $-3t^2 + 90t = 30$
b. $-3t^2 + 90t = 0$
c. $-3t^2 + 90t = 675$
d. $t(90 - 3t) = -3t^2 + 90t$

6. At what time(s) does the spotlight not emit any light?

a. 15 seconds
b. Only 0 seconds
c. 10 and 25 seconds
d. 0 and 30 seconds

7. The artist planned for the spotlight to emit 600 lumens at 10 seconds from the start. At what other time will the spotlight emit 600 lumens of light?

a. 30 seconds
b. 25 seconds

c. 20 seconds
d. 0 seconds

8. At 5 seconds, the spotlight will emit 375 lumens. Find another time that the spotlight emits 375 lumens.

a. 25 seconds
b. 15 seconds
c. 10 seconds
d. 0 seconds

Use the following scenario to answer questions 9 – 10.

A bowling alley models the revenue from sales in one day as a function of the cost to rent a lane per hour, c. Here are two expressions defining the same revenue function.

$c(160 - 10c)$

$160c - 10c^2$

9. According to this model, how high would the cost to rent a lane have to be for the bowling alley to make $0 in revenue?

a. 16.0
b. 14.0
c. 10.0
d. 8.0

10. What equation can you write to determine the lane cost that would allow the bowling alley to make $640 in revenue?

a. $160c - 10c^2 = 0$
b. $160c - 10c^2 + 640 = 0$
c. $c(160 - 10c) = 640$
d. $c(160 - 10c) = 0$

8.2.6 Lesson Summary

In this lesson, you learned how to:

- Recognize the factored form of a quadratic expression and know when it can be useful for solving problems.
- Use a graph to find the solutions to a quadratic equation but also know its limitations.

Here are the **activities** that helped you reach those goals:

- 8.2.1: Interpreting Equations in Context
  - In this activity, you interpreted an equation representing a real-world situation to answer different questions about that situation.
- 8.2.2: Exploring Non-Graphing Strategies to Solve Quadratic Equations
  - In this activity, you tried to solve a quadratic equation using trial and error. You learned that a strategy such as this might be difficult to use when the solutions are harder to find.
  - 8.2.2: Self Check
  - 8.2.2: Additional Resources
- 8.2.3: Solving a Quadratic Equation Set Equal to Zero
  - In this activity, you continued exploring how to solve quadratic equations. You learned that a quadratic equation that has been set equal to zero provides more opportunities for how to solve it. You set quadratic equations equal to zero when possible to find solutions in context.
  - 8.2.3: Self Check
  - 8.2.3: Additional Resources
- 8.2.4: Solving a Real-World Problem Using Quadratic Equations
  - In this activity, you solved a real-world problem using the skills learned in the lesson. By developing a better understanding of quadratic equations, you also better understand the context of a real-world situation.

After these activities, you completed the following **practice**:

- 8.2.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

8.3 SOLVING QUADRATIC EQUATIONS BY REASONING

8.3.0 Lesson Overview

In this lesson, you will begin to solve quadratic equations by reasoning about what values would make the equations true and by using structure in the equations. You will explore in depth the idea that some quadratic equations have two solutions.

When you finish this lesson, you will be able to:

- Find solutions to quadratic equations by reasoning about the values that make the equation true.
- Explain that quadratic equations may have one or two solutions.
- Implement the square root property.

Here are the **activities** that will help you reach those goals:

- 8.3.1: Determining the Number of Solutions
- 8.3.2: Recognizing Pairs of Solutions
  - 8.3.2: Self Check
  - 8.3.2: Additional Resources
- 8.3.3: Solving More Complex Quadratic Equations
  - 8.3.3: Self Check
  - 8.3.3: Additional Resources
- 8.3.4: Finding the Solutions

After that, you'll **practice and review**.

- 8.3.5: Practice
- 8.3.6: Lesson Summary

8.3.1 Determining the Number of Solutions

Warm Up

How many solutions does each equation have? Be prepared to show your reasoning.

1. $x^2 = 9$

2. $x^2 = 0$

3. $x^2 - 1 = 3$

4. $2x^2 = 50$

5. $(x + 1)(x + 1) = 0$

6. $x(x - 6) = 0$

7. $(x - 1)(x - 1) = 4$

8. How are the solutions to questions 2 and 5 different from the other solutions?

A **double root** is a root that appears twice in the solution of an algebraic equation.

8.3.2 Recognizing Pairs of Solutions

Activity

A whole number that has two factors that are equal to each other is called a perfect square. Those equal factors that multiply to be the number are called the **square root of a number**. For instance, 3 is a square root of 9 because $3 \cdot 3 = 9$..

SQUARE ROOT PROPERTY

If $x^2 = k$, then

$x = \sqrt{k}$ and $x = -\sqrt{k} \rightarrow x = \pm\sqrt{k}$

Each of these equations has two solutions. Use your knowledge of square roots and the square root property. What are the solutions? Explain or show your reasoning.

1. $n^2 + 4 = 404$
2. $432 = 3n^2$
3. $\frac{n^2}{5} - 20 = 0$
4. $2n^2 - 7 = 65$

Are you ready for more?

Extending Your Thinking

1. How many solutions does the equation $(x-3)(x+1)(x+5) = 0$ have? What are the solutions?
2. How many solutions does the equation $(x-2)(x-7)(x-2) = 0$ have? What are the solutions?
3. Write a new equation that has 10 solutions.

Self Check

Solve for p. Find all solutions.

$3p^2 + 8 = 251$

a. $p = 9$ and $p = -9$
b. $p = 11$
c. $p = 11$ and $p = -11$
d. $p = 9$

Additional Resources

Using the Square Root Property to Solve Quadratic Equations

Let's look again at the square root property.

SQUARE ROOT PROPERTY

If $x^2 = k$, then

$x = \sqrt{k}$ and $x = -\sqrt{k} \rightarrow x = \pm\sqrt{k}$

Notice that the square root property gives two solutions to an equation of the form $x^2 = k$, the principal square root of k and its opposite. We could also write the solution as $x = \pm\sqrt{k}$. We read this as x equals positive and negative the square root of k.

EXAMPLE 1

Let's solve $x^2 = 9$ using the square root property.

$x^2 = 9$

$x = \pm\sqrt{9}$

$x = \pm 3$

So, $x = 3$ and $x = -3$.

If the number in the radical is not a perfect square, leave that part inside the radical. Let's solve another example.

EXAMPLE 2

Solve $x^2 - 50 = 0$.

Step 1 - Isolate the quadratic term and make its coefficient 1.

To do this, add 50 to both sides of the equation to get x^2 by itself.

$x^2 - 50 = 0$

$x^2 = 50$

Step 2 - Use the square root property.

Remember to write the symbol.

$x = \pm\sqrt{50}$

Step 3 - Simplify the radical by pulling out the factors that are perfect squares.

$x = \pm\sqrt{25}\cdot\sqrt{2}$

$x = \pm 5\sqrt{2}$

$x = 5\sqrt{2}\,,\ x = -5\sqrt{2}$

Step 4 - Check the solutions.

$$\begin{aligned} x^2 - 50 &= 0 \\ (5\sqrt{2})^2 - 50 &\stackrel{?}{=} 0 \\ 25\cdot 2 - 50 &\stackrel{?}{=} 0 \\ 0 &= 0\checkmark \end{aligned}$$

$$\begin{aligned} x^2 - 50 &= 0 \\ (-5\sqrt{2})^2 - 50 &\stackrel{?}{=} 0 \\ 25\cdot 2 - 50 &\stackrel{?}{=} 0 \\ 0 &= 0\checkmark \end{aligned}$$

The steps for solving a quadratic equation using the square root property are listed here.

How to solve a quadratic equation using the square root property.

Step 1 - Isolate the quadratic term and make its coefficient 1.

Step 2 - Use the square root property.

Step 3 - Simplify the radical.

Step 4 - Check the solutions.

Before using the square root property in Step 2, the coefficient of the variable term must equal 1. In the next example, we must divide both sides of the equation by the coefficient 3 before using the square root property.

EXAMPLE 3

Solve $3z^2 = 108$.

Step 1 - Isolate the quadratic term and make its coefficient 1.

When the quadratic term is isolated, divide by 3 to make its coefficient 1. Simplify.

$3z^2 = 108$

$\frac{3z^2}{3} = \frac{108}{3}$

$z^2 = 36$

Step 2 - Use the square root property.

$z = \pm\sqrt{36}$

Step 3 - Simplify the radical.

Rewrite to show two solutions.

$z = \pm 6$

$z = 6, z = -6$

Step 4 - Check the solutions.

$3z^2 = 108$ $\quad$ $3z^2 = 108$

$3(6)^2 \stackrel{?}{=} 108$ $\quad$ $3(-6)^2 \stackrel{?}{=} 108$

$3(36) \stackrel{?}{=} 108$ $\quad$ $3(36) \stackrel{?}{=} 108$

$108 = 108$✓ $\quad$ $108 = 108$✓

TRY IT **Using the Square Root Property to Solve Quadratic Equations**

Solve the following quadratic equations using the square root property.

1. $2n^2 = 98$
2. $\frac{n^2}{2} - 30 = 68$

8.3.3 Solving More Complex Quadratic Equations

Activity

Solving an equation of the form $a(x-h)^2 = k$ is a similar process to the equations we have previously solved.

Notice that the quadratic term, x, in the original form $ax^2 = k$ is replaced with $(x-h)$.



Once the binomial is isolated, the square root property can be used. You can reason the solution by taking the square root and then appropriately isolating the variable using the two different square roots.

Solve the following quadratic equations. Explain or show your reasoning.

1. $144 = (n+1)^2$

2. $(u-6)^2 = 64$

3. $(n-5)^2 - 30 = 70$

4. $4(x+6)^2 - 5 = 31$

Are you ready for more?

Extending Your Thinking

1. Solve the following quadratic equation. Be prepared to show your reasoning.
$(4x-3)^2 + 11 = 47$

Video: Solving More Complex Quadratic Equations

Watch the following video to learn more about how to solve a quadratic equation of the form $a(x-h)^2=k$.

8.3.3 (https://openstax.org/r/more-solving-quadratics)

Self Check

Solve the following quadratic equation.

$(x+3)^2-16=48$

a. $x=5$
b. $x=-8$ and $x=8$
c. $x=-11$ and $x=5$
d. $x=11$ and $x=-5$

Additional Resources

Solving Quadratic Equations of the Form $a(x-h)^2=k$

We can use the square root property to solve an equation of the form $a(x-h)^2=k$ as well.

Notice that the quadratic term, x, in the original form $ax^2=k$ is replaced with $(x-h)$.

$ax^2=k \quad a(x-h)^2=k$

Like before, the first step is to isolate the term that has the variable squared. In this case, a binomial is being squared.

Once the binomial is isolated, by dividing each side by the coefficient of a, the square root property can be used on $(x-h)^2$.

Solve $4(y-7)^2=48$.

Step 1 - Isolate the quadratic term and make its coefficient 1.

So, divide both sides of the equation by 4.

$4(y-7)^2=48$

$(y-7)^2=12$

Step 2 - Use the square root property on the binomial.

$y-7=\pm\sqrt{12}$

Step 3 - Simplify the radical by pulling out the factors that are perfect squares.

Solve for y. Rewrite the two solutions.

$y-7=\pm\sqrt{12}$

$y-7=\pm\sqrt{(4\cdot 3)}$

$y-7=\pm 2\sqrt{3}$

$y=7\pm 2\sqrt{3}$

$y=7+2\sqrt{3}\,,\ y=7-2\sqrt{3}$

Step 4 Check the solutions.

$$\begin{aligned} 4(y-7)^2 &= 48 \\ 4(7+2\sqrt{3}-7)^2 &\stackrel{?}{=} 48 \\ 4(2\sqrt{3})^2 &\stackrel{?}{=} 48 \\ 4(12)^2 &\stackrel{?}{=} 48 \\ 48 &= 48 \checkmark \end{aligned} \qquad \begin{aligned} 4(y-7)^2 &= 48 \\ 4(7-2\sqrt{3}-7)^2 &\stackrel{?}{=} 48 \\ 4(-2\sqrt{3})^2 &\stackrel{?}{=} 48 \\ 4(12)^2 &\stackrel{?}{=} 48 \\ 48 &= 48 \checkmark \end{aligned}$$

TRY IT **Solving Quadratic Equations of the Form $a(x-h)^2 = k$**

Solve the following quadratic equations.

1. $3(a-3)^2 = 363$
2. $\frac{1}{2}(b+2)^2 = 200$

8.3.4 Finding the Solutions

Cool Down

Use what you have learned in the lesson to solve the following quadratic equations.

1. $5m^2 = 80$

2. $(m-6)^2 = 100$

3. $\frac{2}{3}y^2 - 8 = -2$

4. $\frac{1}{6}(x+3)^2 - 8 = 16$

8.3.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 - 2, determine the number of solutions. Then solve the equation.

1. $3x^2 = 48$

a. 2 solutions; $x = 12$ and $x = -12$
b. 2 solutions; $x = 4$ and $x = -4$
c. 1 solution; $x = 4$
d. 0 solutions

2. $(x + 2)(x + 2) = 0$

a. 2 solutions; $x = 2$ and $x = -2$
b. 1 solution; $x = 2$
c. 1 solution; $x = -2$
d. 0 solutions

For questions 3 – 10, solve the quadratic equation.

3. $4m^2 = 36$

a. $m = 2$ and $m = -2$
b. $m = 9$ and $m = -9$
c. $m = 3$
d. $m = 3$ and $m = -3$

4. $d^2 - 25 = 0$

a. $d = 25$
b. $d = -5$
c. $d = 5$
d. $d = 5$ and $d = -5$

5. $\frac{2}{3}z^2 = 54$

a. $z = 7$ and $z = -7$
b. $z = 9$ and $z = -9$
c. $z = 7$ and $z = -9$
d. $z = 6$ and $z = -6$

6. $4p^2 + 10 = 266$

a. $p = 8$ and $p = -8$
b. $p = 16$ and $p = -16$
c. $p = 12$ and $p = -12$
d. $p = 18$ and $p = -18$

7. $(a - 7)^2 + 5 = 105$

a. $a = 3$ and $a = -3$
b. $a = -10$ and $a = 10$
c. $a = -17$ and $a = 3$
d. $a = 17$ and $a = -3$

8. $(b - 1)^2 - 9 = 55$

a. $b = -9$ and $b = 7$
b. $b = 9$ and $b = -7$
c. $b = 8$ and $b = -8$
d. $b = 9$ and $b = -9$

9. $\frac{1}{7}(x + 3)^2 - 5 = 2$

a. $x = -2$ and $x = 12$
b. $x = -4$ and $x = 4$
c. $x = 4$ and $x = -10$
d. $x = -4$ and $x = 10$

10. $5(x - 8)^2 - 7 = 118$

a. $x = 5$ and $x = -11$
b. $x = 5$ and $x = -5$
c. $x = -3$ and $x = -13$
d. $x = 13$ and $x = 3$

8.3.6 Lesson Summary

In this lesson, you learned how to:

- Find solutions to quadratic equations by reasoning about the values that make the equation true.
- Explain that quadratic equations may have one or two solutions.
- Implement the square root property.

Here are the **activities** that helped you reach those goals:

- 8.3.1: Determining the Number of Solutions
  - In this activity, you analyzed various quadratic equations to determine the number of unique solutions that made the equation true.
- 8.3.2: Recognizing Pairs of Solutions
  - In this activity, you learned how to use the square root property to reason about the values that made a quadratic equation true.
  - 8.3.2: Self Check
  - 8.3.2: Additional Resources
- 8.3.3: Solving More Complex Quadratic Equations
  - In this activity, you continued to explore how to solve quadratic equations as they took on a more complicated form, such as $a(x-h)^2 = k$.
  - 8.3.3: Self Check
  - 8.3.3: Additional Resources
- 8.3.4: Finding the Solutions
  - In this activity, you solved quadratic equations of any format from this lesson without any additional context for how to begin finding the solution.

After these activities, you completed the following **practice**:

- 8.3.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

8.4 SOLVING QUADRATIC EQUATIONS WITH THE ZERO PRODUCT PROPERTY

8.4.0 Lesson Overview

In this lesson, you will learn about the zero product property. You will use it to reason about the solutions to quadratic equations that each have a quadratic expression in factored form on one side and 0 on the other side.

When you finish this lesson, you will be able to:

- Explain the meaning of the "zero product property."
- Find solutions to quadratic equations when one side is a product of factors and the other side is zero.

Here are the **activities** that will help you reach those goals:

- 8.4.1: Introducing the Zero Product Property
- 8.4.2: Solving Equations of Increasing Complexity Using Reasoning
  - 8.4.2: Self Check
  - 8.4.2: Additional Resources
- 8.4.3: Applying the Zero Product Property to Solve a Real-World Projectile Problem
  - 8.4.3: Self Check
  - 8.4.3: Additional Resources
- 8.4.4: Solving Equations Using the Zero Product Property and Reasoning

After that, you'll **practice and review**.

- 8.4.5: Practice
- 8.4.6: Lesson Summary

8.4.1 Introducing the Zero Product Property

Warm Up

What values of the variables make each equation true?

1. Find the value of a.
   $6 + 2a = 0$
2. Find the value of b.
   $7b = 0$
3. Find the value of c.
   $7(c - 5) = 0$
4. Determine which of the statements are true if $g \cdot h = 0$. HINT: There may be more than one statement that is true.
5. What did you notice about one of the factors in each of the previous problems?

We will use the **zero product property** to help solve **quadratic equations** throughout this lesson.

ZERO PRODUCT PROPERTY

If the product of two expressions is 0, then one or both of the expressions equal 0.

If $ab = 0$, then $a = 0$ or $b = 0$, or $a = b = 0$.

8.4.2 Solving Equations of Increasing Complexity Using Reasoning

Activity

For each equation, find its solution or solutions. Be prepared to explain your reasoning.

For questions 1 – 3, review the process of solving equations.

1. $x - 3 = 0$

2. $x + 11 = 0$

3. $2x + 11 = 0$

For questions 4 – 7, recall the **zero product property** means that each (or all) of the factors could be equal to zero when their product is zero. Solve each equation by setting each factor equal to zero and solving.

4. $x(2x + 11) = 0$

5. $(x - 3)(x + 11) = 0$

6. $(x - 3)(2x + 11) = 0$

7. $x(x + 3)(3x - 4) = 0$

Are you ready for more?

Extending Your Thinking

1. Use factors of 48 to find as many solutions as you can to the equation $(x - 3)(x + 5) = 48$.

2. Once you think you have all the solutions, explain why these must be the only solutions.

Self Check

Solve the equation.

$(x + 4)(8x - 16) = 0$

a. $x = 2$ and $x = -4$
b. $x = 2$ and $x = -2$
c. $x = 4$ and $x = -2$
d. $x = 4$ and $x = -4$

Additional Resources

Solving Equations of Increasing Complexity Using Reasoning

Let's look again at the **zero product property**.

ZERO PRODUCT PROPERTY

If the product of two expressions is 0, then one or both of the expressions equal 0.

If $ab = 0$, then $a = 0$ or $b = 0$, or $a = b = 0$.

We can use the zero product property to help reason through finding the solutions to complex quadratic equations.

We know that if factors are multiplied together and the result is 0, then one of those factors must equal 0.

EXAMPLE 1

$(x - 7)(2x + 12) = 0$

This equation has two different factors that, when multiplied, result in a product of 0.

To find possible solutions to this equation, we can set each factor equal to 0 and solve.

$(x - 7) = 0 \qquad (2x + 12) = 0$

The value $x = 7$ makes the first factor equal to 0. The value $x = -6$ makes the second factor equal to 0.

So, the solutions to the original equation are $x = 7$ and $x = -6$.

EXAMPLE 2

$x(x-9)(3x-8)=0$

This equation has three different factors, but the reasoning is the same. We can still set each factor equal to 0 and solve.

The first factor is just x. So, $x=0$ is a solution.

The second factor gives a solution of $x=9$ when we solve $(x-9)=0$.

The third factor gives a solution of $x=\frac{8}{3}$ when we solve $(3x-8)=0$.

So, the solutions to $x(x-9)(3x-8)=0$ are $x=0$, $x=\frac{8}{3}$, and $x=9$.

TRY IT **Solving Equations of Increasing Complexity Using Reasoning**

Use the zero product property to reason through finding the solutions to the following quadratic equations.

1. $(x-6)(4x+20)=0$

Use the zero product property to reason through finding the solutions to the following quadratic equations.

2. $(x+1)(5x+14)=0$

8.4.3 Applying the Zero Product Property to Solve a Real-World Projectile Problem

Activity

We have seen quadratic functions modeling the height of a projectile as a function of time.

Here are two ways to define the same function that approximates the height of a projectile in meters, t seconds after launch:

$h(t)=-5t^2+27t+18$ $h(t)=(-5t-3)(t-6)$

1. Which way of defining the function allows us to use the **zero product property** to find out when the height of the object is 0 meters (when the object is on the ground)?
2. Without graphing, determine at what time(s) the height of the object is 0 meters (when the object is on the ground). Show your reasoning.
3. Using the function in standard form, determine the height of the object before it started its motion, at $t=0$ seconds. Be prepared to show your reasoning.

Video: Applying the Zero Product Property

Watch the following video to learn more about applying the zero product property to solve a real-world projectile problem.

Applying the Zero Product Property (https://openstax.org/r/zero-product)

Self Check

The equivalent functions below model the height of a projectile in meters, t seconds after launch.

$h(t)=-8t^2+27t+20$ $h(t)=(-8t-5)(t-4)$

Without graphing, determine at what time the height of the object is 0 meters (when the object is on the ground).

a. 8 seconds
b. 4 seconds
c. $\frac{5}{8}$ seconds
d. 0 seconds

Additional Resources

Applying the Zero Product Property to Solve a Real-World Projectile Problem

We have previously looked at functions representing the motion of projectiles. We analyzed two equivalent expressions that define the same quadratic function, one in quadratic form and the other in factored form.

Now we're going to look at a similar problem, but with the additional knowledge of how the function relates to the **zero product property**.

Let's look at an example.

The equivalent functions below model the height, in feet, of a baseball t seconds after being hit.

$h(t) = -15t^2 + 44t + 3 \qquad h(t) = (-15t - 1)(t - 3)$

Using the zero product property, the quadratic expression in factored form allows you to more easily find out when the height of the baseball is 0 feet.

Without graphing, we can determine at what time the baseball reaches the ground.

Let's apply the zero product property to solve $(-15t - 1)(t - 3) = 0$.

Set each factor equal to 0 and solve.

$$\begin{aligned} -15t - 1 &= 0 \\ -15t - 1 + 1 &= 0 + 1 \\ -15t &= 1 \\ \frac{-15t}{-15} &= \frac{1}{-15} \\ t &= -\frac{1}{15} \end{aligned} \qquad \begin{aligned} t - 3 &= 0 \\ t - 3 + 3 &= 0 + 3 \\ t &= 3 \end{aligned}$$

So, $t = -\frac{1}{15}$ and $t = 3$. These values of t represent the **roots** of the function. Because the object was launched at $t = 0$, the negative solution doesn't make sense in this context of time. So the baseball reaches the ground in 3 seconds.

There is one benefit of knowing the function in quadratic form. By substituting 0 in for t, we can easily determine at what height the baseball was hit, when $t = 0$ seconds.

$h(t) = -15t^2 + 44t + 3$

$h(0) = -15(0)^2 + 44(0) + 3$

$h(0) = 3$

So, the baseball was hit at a height of 3 feet. Note that this point, $(0, 3)$, is the y-intercept of the function when graphed.

TRY IT **Applying the Zero Product Property to Solve a Real-World Projectile Problem**

The functions below model the height of a projectile in feet, t seconds after launch.

- $h(t) = -7t^2 + 34t + 48$
- $h(t) = (-7t - 8)(t - 6)$

1. Without graphing, determine at what time the height of the object is 0 feet.

2. Determine the height of the object before it is launched.

8.4.4 Solving Equations Using the Zero Product Property and Reasoning

Cool Down

1. Find all solutions to $(x + 5)(2x - 3) = 0$. Explain or show your reasoning.

8.4.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 4, solve each equation. Identify every correct answer.

1. $x(3x + 12) = 0$

a. $x = 0$ and $x = -4$
b. $x = 4$ and $x = -4$
c. $x = 0$ and $x = -3$
d. $x = 0$ and $x = -12$

2. $(x - 7)(x + 9) = 0$

a. $x = -7$ and $x = 9$
b. $x = 7$ and $x = -9$
c. $x = 0$, $x = 7$, and $x = -9$
d. $x = 7$ and $x = -7$

3. $(x - 8)(4x + 15) = 0$

a. $x = 8$ and $x = \frac{15}{4}$
b. $x = 8$ and $x = \frac{4}{15}$
c. $x = 8$ and $x = -\frac{4}{15}$
d. $x = 8$ and $x = -\frac{15}{4}$

4. $x(x - 6)(5x - 20) = 0$

a. $x = 0$ and $x = 6$
b. $x = 4$ and $x = 6$
c. $x = 0$, $x = 4$, and $x = 6$
d. $x = 0$, $x = -4$, and $x = 6$

For questions 5 - 6, use the following.

The functions below model the height of a projectile in feet, t seconds after launch.

- $h(t) = -7t^2 + 34t + 48$
- $h(t) = (-7t - 8)(t - 6)$

5. Without graphing, determine at what time the height of the object is 0 feet.

a. 0 seconds
b. $-\frac{8}{7}$ seconds
c. 6 seconds
d. $\frac{8}{7}$ seconds

6. Determine the height of the object before it is launched.

a. 48 feet
b. 6 feet
c. $\frac{8}{7}$ feet
d. 0 feet

For questions 7 – 10, solve each equation. Identify every correct answer.

7. $x(6x - 18) = 0$

a. $x = -3$
b. $x = 0$ and $x = 3$
c. $x = 3$
d. $x = 0$ and $x = -3$

8. $(x + 13)(x - 9) = 0$

a. $x = -13$ and $x = 9$
b. $x = -9$ and $x = 13$
c. $x = -13$, $x = 0$, and $x = -9$
d. $x = -9$ and $x = -13$

9. $(x - 7)(x - 7) = 0$

a. $x = -7$
b. $x = 0$ and $x = 7$
c. $x = -7$ and $x = 7$
d. $x = 7$

10. $(x - \frac{5}{6})(6x + 30) = 0$

a. $x = -6$ and $x = \frac{5}{6}$
b. $x = -5$ and $x = \frac{5}{6}$
c. $x = 5$ and $x = \frac{5}{6}$
d. $x = 5$ and $x = -\frac{5}{6}$

8.4.6 Lesson Summary

In this lesson, you learned how to:

- Explain the meaning of the "zero product property."
- Find solutions to quadratic equations when one side is a product of factors and the other side is zero.

Here are the **activities** that helped you reach those goals:

- 8.4.1: Introducing the Zero Product Property
  - In this activity, you learned how the zero product property affects factors being multiplied together in an equation set equal to 0.
- 8.4.2: Solving Equations of Increasing Complexity Using Reasoning
  - In this activity, you implemented the zero product property to solve increasingly complex quadratic equations.
  - 8.4.2: Self Check
  - 8.4.2: Additional Resources
- 8.4.3: Applying the Zero Product Property to Solve a Real-World Projectile Problem
  - In this activity, you applied the zero product property to solve a real-world problem involving projectiles. You increased your knowledge of these types of problems that you have seen in previous lessons.
  - 8.4.3: Self Check
  - 8.4.3: Additional Resources
- 8.4.4: Solving Equations Using the Zero Product Property and Reasoning
  - In this activity, you demonstrated all of the skills you learned in this lesson to continue solving challenging quadratic equations.

After these activities, you completed the following **practice**:

- 8.4.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

8.5 HOW MANY SOLUTIONS?

8.5.0 Lesson Overview

In this lesson, you will build on the idea that both graphing and rewriting quadratic equations in the form of **expression = 0** are useful strategies for solving equations. You will also review the ties between the zeros of a function and the horizontal intercepts of its graph, which you began exploring in an earlier unit.

When you finish this lesson, you will be able to:

- Explain why dividing by a variable to solve a quadratic equation is not a good strategy.
- Explain why quadratic equations can have no solutions and explain why there are none.

Here are the **activities** that will help you reach those goals:

- 8.5.1: Reviewing Zero Product Property
- 8.5.2: Interpreting Graphs to Solve Quadratic Equations
  - 8.5.2: Self Check
  - 8.5.2: Additional Resources
- 8.5.3: Choosing an Effective Strategy to Solve Quadratic Equations
  - 8.5.3: Self Check
  - 8.5.3: Additional Resources
- 8.5.4: Analyzing Errors When Solving Quadratic Equations

After that, you'll **practice and review**.

- 8.5.5: Practice
- 8.5.6: Lesson Summary

8.5.1 Reviewing Zero Product Property

Warm Up

Decide whether each statement is true or false.

1. 3 is the only solution to $x^2 - 9 = 0$.

2. A solution to $x^2 + 25 = 0$ is –5.

3. $x(x - 7) = 0$ has two solutions.

4. 5 and –7 are the solutions to $(x - 5)(x + 7) = 12$.

8.5.2 Interpreting Graphs to Solve Quadratic Equations

Activity

Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

Julio is solving three equations by graphing.

$(x - 5)(x - 3) = 0$

$(x - 5)(x - 3) = -1$

$(x - 5)(x - 3) = -4$

To solve the first equation, $(x - 5)(x - 3) = 0$, he graphed $y = (x - 5)(x - 3)$ and then looked for the x-intercepts of the graph.

1. Explain why the x-intercepts can be used to solve $(x-5)(x-3)=0$.
2. Graph $y=(x-5)(x-3)$. What are the solutions?

Access multimedia content (http://openstax.org/books/algebra-1/pages/8-5-2-interpreting-graphs-to-solve-quadratic-equations)

To solve the second equation, Julio rewrote it as $(x-5)(x-3)+1=0$. He then graphed $y=(x-5)(x-3)+1$.

3. Graph $y=(x-5)(x-3)+1$. Then, use the graph to solve the equation. Be prepared to explain how you used the graph for solving.

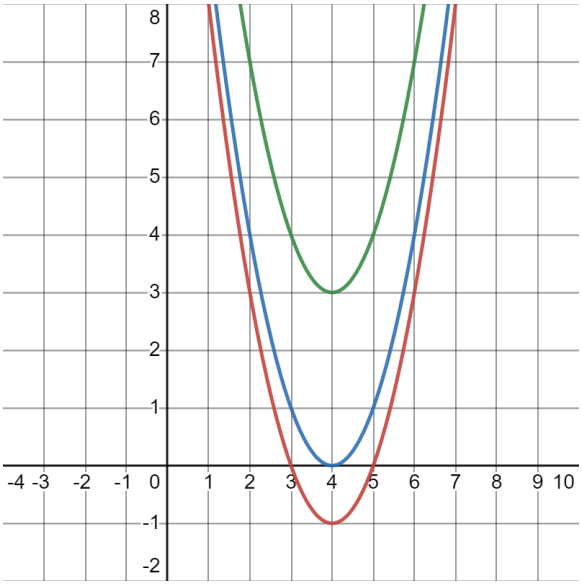
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Solve the third equation using Julio's strategy of graphing the equations.

4. What are the solutions to $y=(x-5)(x-3)+4$?

Access multimedia content (http://openstax.org/books/algebra-1/pages/8-5-2-interpreting-graphs-to-solve-quadratic-equations)

Think about the graphing strategy used to solve the equations and the solutions you found. View the Desmos graphs of all three equations.



5. Why might it be helpful to rearrange each equation to equal 0 on one side and then graph the expression on the non-zero side?
6. How many solutions does each of the three equations have?

Are you ready for more?

Extending Your Thinking

1. $y=(x-3)(x-5)$ and $y=-1$ on the same coordinate plane.
   Access multimedia content (http://openstax.org/books/algebra-1/pages/8-5-2-interpreting-graphs-to-solve-quadratic-equations)
   How do the equations $y=(x-3)(x-5)$ and $y=-1$ help solve the equation $(x-3)(x-5)=-1$?
2. Graph $y=(x-3)(x-5)$ and $y=0$ on the same coordinate plane.
   Access multimedia content (http://openstax.org/books/algebra-1/pages/8-5-2-interpreting-graphs-to-solve-quadratic-equations)
   How do the equations $y=(x-3)(x-5)$ and $y=0$ help solve the equation $(x-3)(x-5)=0$?

3. Graph $y = (x - 3)(x - 5)$ and $y = 3$ on the same coordinate plane.
   Access multimedia content (http://openstax.org/books/algebra-1/pages/8-5-2-interpreting-graphs-to-solve-quadratic-equations)
   How do the equations $y = (x - 3)(x - 5)$ and $y = 3$ help solve the equation $(x - 3)(x - 5) = 3$?
4. Use the graphs you created in questions 1 – 3 to help you find a few other equations of the form $(x - 3)(x - 5) = z$ that have whole-number solutions.
5. Some of the values of z that you might have found are –1, 0, 3, 8, 15, 24, etc. Find a pattern in the values of z that give whole-number solutions.
6. Without solving, determine if $(x - 5)(x - 3) = 120$ and $(x - 5)(x - 3) = 399$ have whole-number solutions. Be prepared to show your reasoning.

Self Check

Use graphing technology to determine the number of solutions for the equation $(x + 3)(x - 6) = -22$. Then find any solutions.

a. The equation has no solutions.
b. There are two solutions. $x = -3$ or $x = 6$
c. There are two solutions. $x = -6$ or $x = 3$
d. There is one solution. $x = 5.5$

Additional Resources

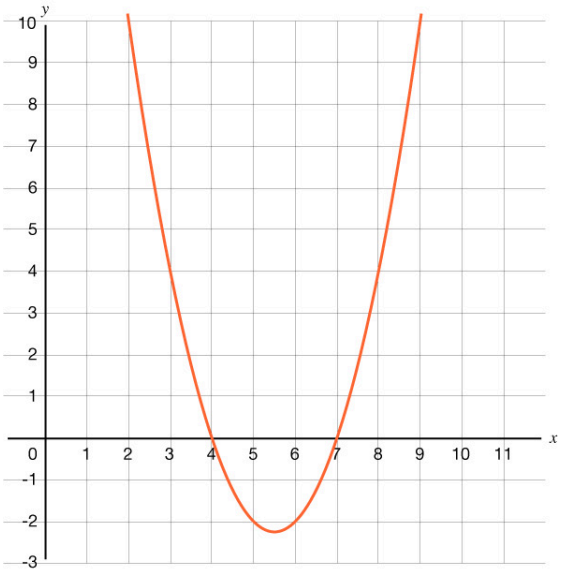
Interpreting Graphs to Solve Quadratic Equations

Let's analyze and interpret the graphs of different quadratic equations.

EXAMPLE 1

First, let's look at the equation $(x - 4)(x - 7) = 0$.

We will graph the equation $y = (x - 4)(x - 7)$.



This graph has two x-intercepts.

The solutions to $(x - 4)(x - 7) = 0$ are x-values that make the expression have a value of 0.

At the x-intercepts, the y-value of the expression $(x - 4)(x - 7)$ is 0, so the x-intercepts give the solutions.

The x-intercepts and solutions to the equation are 4 and 7.

Note that the values of 4 and 7 could be referred to as **solutions** of the equation, x**-intercepts** of the graph, **roots** of the equation, or **zeros** of the equation.

EXAMPLE 2

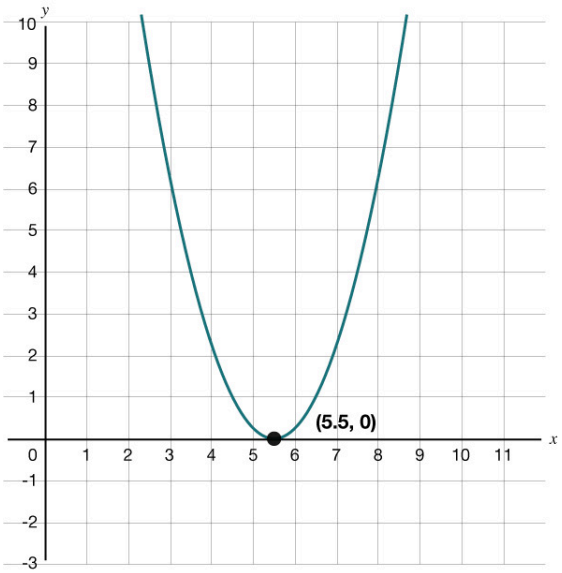
Let's look at a different example.

$(x-4)(x-7)=-2.25$

First, we rewrite the equation to set it equal to 0.

$(x-4)(x-7)+2.25=0$

We will graph the equation $y=(x-4)(x-7)+2.25$.



This graph has one x-intercept. It is an example of a double root at 5.5. A **double root** is a root that appears twice in the solution of an algebraic equation.

The solution is the x-coordinate of the x-intercept.

So, the solution to $(x-4)(x-7)=-2.25$ is 5.5.

EXAMPLE 3

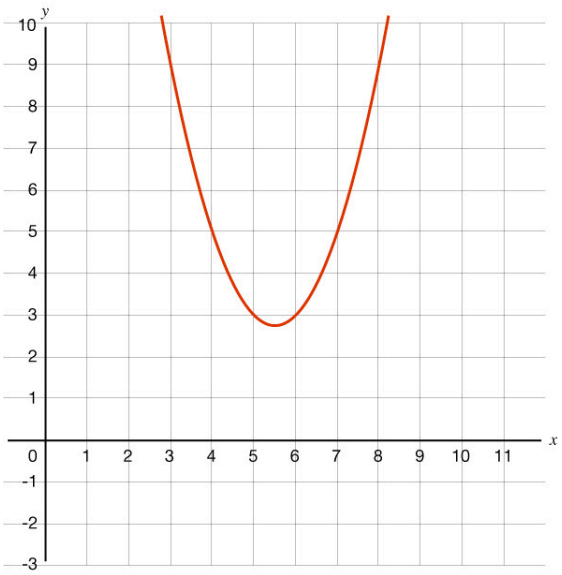
Let's look at one last example.

$(x-4)(x-7)=-5$

First, we rewrite the equation to set it equal to 0.

$(x-4)(x-7)+5=0$

We will graph the equation $y=(x-4)(x-7)+5$.



The graph has no x-intercepts.

So, there are no real solutions for $(x-4)(x-7)+5=0$.

Remember, when a **quadratic expression** is equal to 0 and we want to find the unknown values, we can think of it as finding the zeros of a **quadratic function**, or finding the x-intercepts of the graph of that **function**.

TRY IT **Interpreting Graphs to Solve Quadratic Equations**

Determine the number of solutions for the equation $(x-4)(x-9)=-6.25$. Then find any solutions.

Use the Desmos graphing tool or technology outside the course.

Access multimedia content (http://openstax.org/books/algebra-1/pages/8-5-2-interpreting-graphs-to-solve-quadratic-equations)

8.5.3 Choosing an Effective Strategy to Solve Quadratic Equations

Activity

Solve each equation. Be prepared to explain or show your reasoning.

1. $x^2=121$

2. $x^2-31=5$

3. $(x-4)(x-4)=0$

4. $(x+3)(x-1)=5$

5. $(x+1)^2=-4$

6. $(x-4)(x-1)=990$

Video: Choosing an Effective Strategy to Solve Quadratic Equations

Watch the following video to learn more about choosing an effective strategy to solve quadratic equations.

Choosing an Effective Strategy to Solve Quadratic Equations (https://openstax.org/r/quadratic-strategy)

Self Check

Solve the equation using any method covered in the lesson.

$(x-6)(x+5)=-30.25$

a. $x=1.5$

b. $x = 0.5$ and $x = 1.5$
c. $x = 0.5$
d. No solutions

Additional Resources

Choosing an Effective Strategy to Solve Quadratic Equations

You have learned various strategies to solve quadratic equations in different forms. If possible, try to use reasoning before graphing an equation.

Let's look at some examples.

Using Reasoning and the Square Root Property

Equations such as these can be solved by reasoning and using the **square root property**.

EXAMPLE 1

$x^2 = 81$

$x = 9$ and $x = -9$

EXAMPLE 2

$x^2 + 7 = 56$

$x^2 = 49$

$x = 7$ and $x = -7$

EXAMPLE 3

$(x + 1)^2 = 9$

$x = 2$ and $x = -4$

Each equation above is different in some way, but they all can be solved using the same process.

Using Reasoning and the Zero Product Property

Equations such as these can be solved by reasoning and using the **zero product property**.

EXAMPLE 1

$(x - 4)(x + 5) = 0$

$x = 4$ and $x = -5$

EXAMPLE 2

$(x + 3)(2x + 7) = 0$

$x = -3$ and $x = -\frac{7}{2}$

Using Graphing to Solve

Some equations might require graphing to be solved. One method is to rearrange the equation such that it is in the form of *expression* $= 0$. Then graph $y =$*expression*, and find the x-intercepts to identify the solutions.

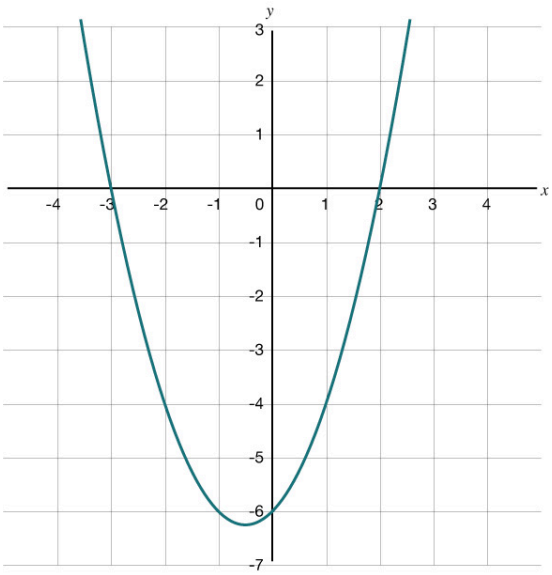
Given: $(x + 2)(x - 1) = 4$

Step 1 - Rewrite the equation to set it equal to 0.

$(x+2)(x-1)-4=0$

Step 2 - Graph the equation with y = the expression that equals zero.

Graph $y=(x+2)(x-1)-4$.



Step 3 - Identify the x-intercepts from the graph.

This graph has two x-intercepts at $(-3,0)$ and $(2,0)$.

The original equation $(x+2)(x-1)=4$ has two solutions at $x=-3$ and $x=2$.

TRY IT **Choosing an Effective Strategy to Solve Quadratic Equations**

Solve each equation using any method.

1. $(x+1)^2+6=22$
2. $(x+3)(x-8)=0$

8.5.4 Analyzing Errors When Solving Quadratic Equations

Cool Down

1. Consider $(x-5)(x+1)=7$. Priya reasons that if this is true, then either $x-5=7$ or $x+1=7$. So, the solutions to the original equation are 12 and 6. Do you agree? If not, where was the mistake in Priya's reasoning?

2. Consider $x^2-10x=0$. Diego says that to solve we can just divide each side by x to get $x-10=0$, so the solution is 10. Mai says, "I wrote the expression on the left in factored form, which gives $x(x-10)=0$, and ended up with two solutions: 0 and 10."
Do you agree with either strategy? Be prepared to show your reasoning.

3. Consider the equation $x^2=-16$. Mona says that there are two solutions, $x=4$ or $x=-4$.
Do you agree? If not, where was the mistake in Mona's reasoning?

4. Consider the equation $(x-3)(x-3)=0$. Dominic says that there are two solutions, $x=3$ or $x=-3$.
Do you agree? If not, where was the mistake in Dominic's reasoning?

8.5.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 - 2, decide whether each statement is true or false.

1. $x(x-5)=0$ has only one solution.

a. false
b. true

2. A solution to $x^2-49=0$ is $x=-7$.

a. false
b. true

3. Use graphing technology to determine the number of solutions for the equation $(x-4)(x+5)=-20$. Then find any solutions.

a. Two solutions; $x=-1$ and $x=0$
b. Two solutions; $x=1$ and $x=0$
c. One solution; $x=1$
d. No solution

4. Use graphing technology to determine the number of solutions for the equation $(x+2)(x+5)=-2$. Then find any solutions.

a. Two solutions; $x=3$ and $x=4$
b. Two solutions; $x=-3$ and $x=-4$
c. Two solutions; $x=-3$ and $x=4$
d. No solution

For questions 5 – 8, choose the best method to solve the equation and then use the method to find the solution.

5. $x(x-6)=0$

a. $x=6$ and $x=-6$
b. $x=0$ and $x=6$
c. $x=6$
d. $x=0$ and $x=-6$

6. $\frac{1}{2}(x-4)^2=32$

a. $x=12$ and $x=-4$
b. $x=-8$ and $x=8$
c. $x=-12$ and $x=4$
d. $x=-4$ and $x=4$

7. $(x+7)(x-2)=-8$

a. $x=-6$ and $x=1$
b. $x=-8$ and $x=-1$
c. $x=-7$ and $x=2$
d. No solutions

8. $(x-4)(x+9)=-42.25$

a. $x=-5$ and $x=-1$
b. $x=-3$ and $x=-2$
c. $x=-2.5$
d. No solutions

9. Which statement is true of the equation $x^2+36=0$?

a. The solutions are $x=6$ and $x=-6$.

b. The solution is $x = 6$.
c. The solutions are $x = 0$ and $x = 6$.
d. The equation has no solutions.

10. Which statement is true of the equation $x^2 + 9x = 0$?

a. The solutions are $x = 9$ and $x = -9$.
b. The solutions are $x = 0$ and $x = -9$.
c. The solution is $x = -9$.
d. The equation has no solutions.

8.5.6 Lesson Summary

In this lesson, you learned how to:

- Explain why dividing by a variable to solve a quadratic equation is not a good strategy.
- Explain why quadratic equations can have no solutions and explain why there are none.

Here are the **activities** that helped you reach those goals:

- 8.5.1: Reviewing Zero Product Property
  - In this activity, you reviewed how the zero product property allows you to find solutions to quadratic equations in which an expression of factors is set equal to 0.
- 8.5.2: Interpreting Graphs to Solve Quadratic Equations
  - In this activity, you learned how to use technology to graph quadratic equations and interpret those graphs to find solutions.
  - 8.5.2: Self Check
  - 8.5.2: Additional Resources
- 8.5.3: Choosing an Effective Strategy to Solve Quadratic Equations
  - In this activity, you practiced solving quadratic equations without any outside context about the best strategy to use.
  - 8.5.3: Self Check
  - 8.5.3: Additional Resources
- 8.5.4: Analyzing Errors When Solving Quadratic Equations
  - In this activity, you learned to watch out for common errors made when solving quadratic equations. This includes not taking the square root of a negative number and accurately determining how many solutions an equation might have.

After these activities, you completed the following **practice**:

- 8.5.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

8.6 REWRITING QUADRATIC EXPRESSIONS IN FACTORED FORM, PART 1

8.6.0 Lesson Overview

In this lesson, you will begin to rewrite quadratic expressions from standard to factored form. You will look at expressions of the form $(x + m)(x + n)$ and $(x - m)(x - n)$ where m and n are positive.

When you finish this lesson, you will be able to:

- Explain how the numbers in a quadratic expression in factored form relate to the numbers in an equivalent expression in standard form.
- Rewrite a quadratic expression in standard form when given the quadratic expression in factored form.
- Rewrite a quadratic expression in factored form when given the quadratic expression in the form of $x^2 + bx + c$.

Here are the **activities** that will help you reach those goals:

- 8.6.1: Finding and Reasoning Unknown Factors
- 8.6.2: Using Diagrams to Understand Equivalent Expressions
  - 8.6.2: Self Check
  - 8.6.2: Additional Resources
- 8.6.3: Rewriting Quadratic Expressions in Standard Form
  - 8.6.3: Self Check
  - 8.6.3: Additional Resources
- 8.6.4: Finding the Missing Numbers in Pairs of Equivalent Expressions

After that, you'll **practice and review**.

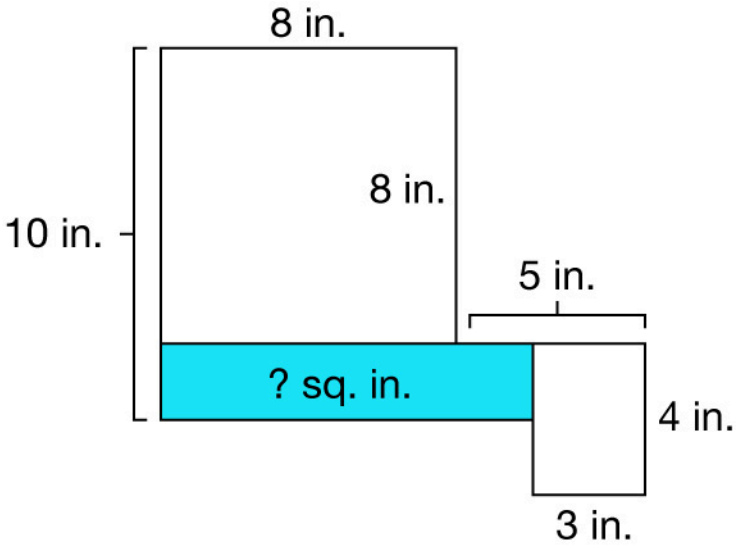
- 8.6.5: Practice
- 8.6.6: Lesson Summary

8.6.1 Finding and Reasoning Unknown Factors

Warm Up

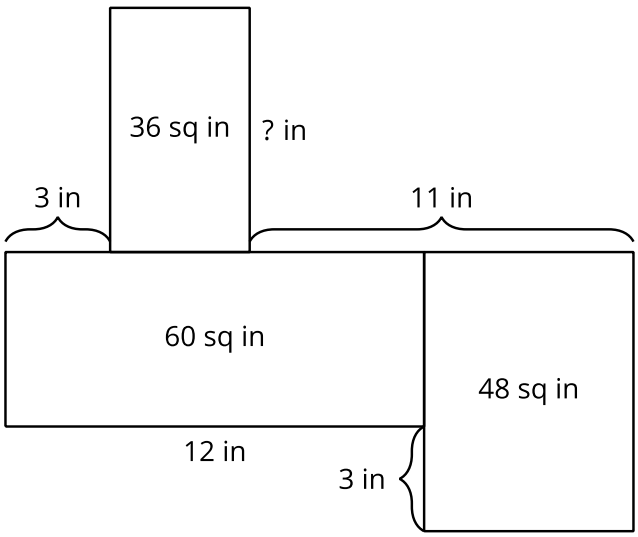
Examine the two puzzles that involve side lengths and areas of rectangles. Can you find the missing area in Figure A and the missing length in Figure B? Be prepared to explain your reasoning.

Figure A



1. What is the missing area in Figure A above?

Figure B



2. What is the missing length in Figure B above?

8.6.2 Using Diagrams to Understand Equivalent Expressions

Activity

Look at these two expressions.

$(x + 7)(x + 9)$

$x^2 + 16x + 63$

We can show they are equivalent by using a diagram to break apart the different pieces.

	x	**7**
x	x^2	$7x$
9	$9x$	63

The binomial factors $(x + 7)$ and $(x + 9)$ have been placed in the cells along the horizontal and vertical edge of the diagram.

To find the internal cells, the terms in each factor are multiplied together.

Then, the like terms in the product cells are combined. Notice that $7x + 9x$ results in the linear term of our second expression, $16x$ (from $x^2 + 16x + 63$).

Use a diagram to show that each pair of expressions is equivalent. It is helpful to rewrite any subtraction expressions as adding the opposite before starting, as has been modeled in questions 2 and 5.

1. $x(x + 3)$ and $x^2 + 3x$
2. $x(x + (-6))$ and $x^2 - 6x$
3. $(x + 2)(x + 4)$ and $x^2 + 6x + 8$
4. $(x + 4)(x + 10)$ and $x^2 + 14x + 40$
5. $(x + (-5))(x + (-1))$ and $x^2 - 6x + 5$
6. $(x - 1)(x - 7)$ and $x^2 - 8x + 7$

7. Review the pairs of expressions that involve the product of two sums or two differences that you just diagramed. How is each expression in factored form (the first expression placed into the cells along the horizontal and vertical edge of the diagram) related to the equivalent expression in standard form (the second expression that results from the product cells)?

Self Check

An expression in standard form is shown. The diagram below is equivalent to which expression in factored form?

$x^2 - 13x + 42$

	x	**-6**
x	x^2	$-6x$
-7	$-7x$	42

a. $(x - 6)(x - 7)$
b. $(x - 6)(x + 7)$
c. $(x + 6)(x + 7)$
d. $(x - 13)(x + 42)$

Additional Resources

Using Diagrams to Understand Equivalent Expressions

EXAMPLE 1

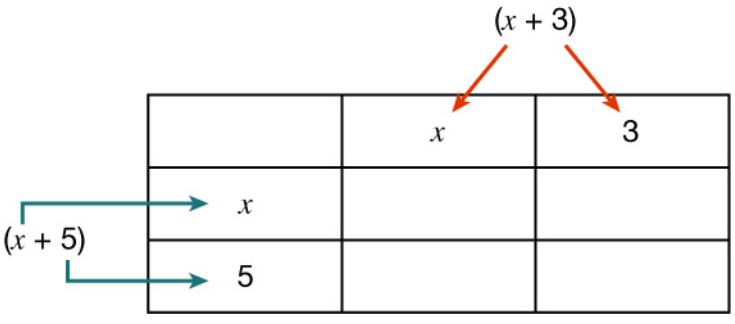
Look at these two expressions.

$(x + 3)(x + 5)$

$x^2 + 8x + 15$

We can show they are equivalent by using a diagram to break apart the different pieces.

Step 1 - First, let's populate the diagram.



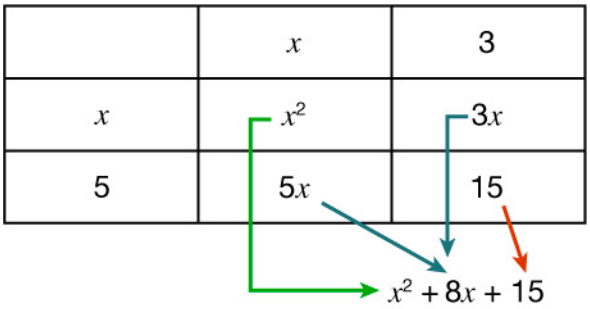
Step 2 - Then we multiply the corresponding rows and columns.

	x	**3**
x	$x \cdot x = x^2$	$3 \cdot x = 3x$
5	$5 \cdot x = 5x$	$3 \cdot 5 = 15$

This leaves the diagram shown here.

	x	**3**
x	x^2	$3x$
5	$5x$	15

Step 3 - We can use this diagram to find our expression in standard form.



So, the expressions are equivalent.

EXAMPLE 2

The same can be done with two factors using subtraction as well.

$(x - 6)(x - 9)$

$x^2 - 15x + 54$

It is helpful to rewrite the factored form as the addition of negative numbers when subtracting occurs in the factors.

So $(x - 6)(x - 9)$ becomes $(x + (-6))(x + (-9))$. Now the process is the same.

Step 1 - Let's populate the table.

	x	**–6**
x		
–9		

Step 2 - Multiply the columns and rows.

	x	**–6**
x	x^2	$-6x$
–9	$-9x$	54

Step 3 - Use the diagram to write the expression in standard form.

$x^2 - 15x + 54$

The expressions are equivalent.

When the coefficients of the x-terms are both 1 in the factored form:

Take notice that the linear term (x-term) in the expression in standard form is the sum of the two numbers in the expression in factored form.

- In the example above, the term $-15x$ is the sum of the two numbers –9 and –6.

Also, the constant term is the product of the two numbers in the expression in factored form.

- In the example above, the term 54 is the product of the two numbers –9 and –6.

Recognizing this will be helpful for switching between standard and factored forms. Remember, this pattern only occurs when the coefficients of the x-terms in the factors are 1 (because they result in a quadratic term that has a coefficient of 1 as well, x^2).

TRY IT **Using Diagrams to Understand Equivalent Expressions**

1. Use a diagram to write $(x + 2)(x + 7)$ in standard form.

	x	**–3**
x	x^2	$-3x$
–8	$-8x$	24

Based on the diagram, what is an equivalent factored form of this expression?

2. The diagram shown breaks down $x^2 - 11x + 24$.

	x	**–3**
x	x^2	$-3x$
–8	$-8x$	24

Based on the diagram, what is an equivalent factored form of this expression?

8.6.3 Rewriting Quadratic Expressions in Standard Form

Activity

1. Each row in the table contains a pair of equivalent expressions.
   Complete the table with the missing expressions. If you get stuck, consider drawing a diagram.

Factored form	Standard form
$x(x+7)$	
	x^2+9x
	x^2-8x
$(x+6)(x+2)$	
	$x^2+13x+12$
$(x-6)(x-2)$	
	$x^2-7x+12$
	x^2+6x+9
	$x^2+10x+9$
	$x^2-10x+9$
	x^2-6x+9
	$x^2+(m+n)x+mn$

Are you ready for more?

Extending Your Thinking

1. A mathematician threw a party. She told her guests, "I have a riddle for you. I have three daughters. The product of their ages is 72. The sum of their ages is the same as my house number. How old are my daughters?"
   The guests went outside to look at the house number. They thought for a few minutes, and then they said, "This riddle can't be solved!"
   The mathematician said, "Oh yes, I forgot to tell you the last clue. My youngest daughter prefers strawberry ice cream."
   With this last clue, the guests could solve the riddle. How old are the mathematician's daughters? Be prepared to show your reasoning.

Video: Rewriting Quadratic Expressions

Watch the following video to learn more about rewriting quadratic expressions into factored form.

Rewriting Quadratic Expressions (https://openstax.org/r/rewrite-quadratics)

Self Check

Find the factored form of $x^2 - 18x + 80$.

a. $(x + 8)(x + 10)$
b. $(x - 4)(x - 20)$
c. $(x - 12)(x - 6)$
d. $(x - 8)(x - 10)$

Additional Resources

Rewriting Quadratic Expressions in Standard Form

We have had a lot of practice rewriting factored form into standard form. To reverse this process, many of the same skills are needed.

Let's look at an example.

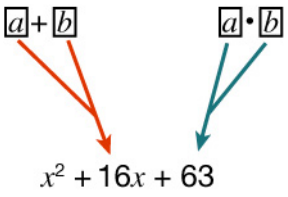
EXAMPLE 1

Write $x^2 + 16x + 63$ in factored form.

To do this, we must think about how you converted from factored form into standard form. First, check if the leading coefficient (for the quadratic term) is 1. If so, then:

- To find the coefficient of the middle term, 16, we added two numbers together from the diagram.
- To find the last term or constant, 63, we multiplied two numbers together from the diagram.

So, which numbers, a and b, have a sum of 16 and a product of 63?



The positive factors of 63 include:

1, 63

3, 21

7, 9

With some reasoning, we find that 7 and 9 are the numbers we want.

So, the factored form of $x^2 + 16x + 63$ is $(x + 7)(x + 9)$.

Let's look at another example.

EXAMPLE 2

Write $x^2 - 10x + 16$ in factored form.

The sign of the middle term is negative, but we follow the same process since the coefficient of x^2 is 1.

Let's rewrite the expression as $x^2 + -10x + 16$.

Which numbers have a sum of –10 and a product of 16?

The factors of 16 include:

1, 16

–1, –16

2, 8

–2, –8

4, 4

–4, –4

The numbers we need are –2 and –8 because they have a product of 16 and a sum of –10.

So, the factored form is $(x + (-2))(x + (-8))$. This can be rewritten as $(x - 2)(x - 8)$.

TRY IT **Rewriting Quadratic Expressions in Standard Form**

Find the factored form of each expression.

1. $x^2 + 9x + 18$
2. $x^2 - 12x + 35$

8.6.4 Finding the Missing Numbers in Pairs of Equivalent Expressions

Cool Down

The problems below consist of pairs of equivalent expressions: one in standard form and the other in factored form. Note that the leading coefficients on the quadratic terms are all 1.

Find the missing numbers.

1. $x^2 + _x + _$ and $(x + 1)(x + 17)$

2. $x^2 + 9x + 14$ and $(x + 2)(x + _)$

3. $x^2 - 7x + 10$ and $(x - 2)(x - _)$

4. $x^2 - 9x + 20$ and $(x - _)(x - _)$

8.6.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Write $(x + 6)(x + 5)$ in standard form.

a. $x^2 - 11x + 30$
b. $x^2 + 30$
c. $x^2 + 11x + 30$
d. $x^2 + 10x - 30$

2. Write $(x - 3)(x - 7)$ in standard form.

a. $x^2 - 10x + 21$
b. $x^2 + 10x + 21$
c. $x^2 + 10x - 21$
d. $x^2 + 21$

3. An expression in standard form is shown. The diagram below shows it is equivalent to which expression in factored form?

	x	2
x	x^2	$2x$
9	$9x$	18

$x^2 + 11x + 18$

a. $x(x + 18)$
b. $(x + 2)(x - 9)$
c. $(x + 2)(x + 9)$
d. $(x - 2)(x - 9)$

4. An expression in standard form is shown. The diagram below shows it is equivalent to which expression in factored form?

	x	-5
x	x^2	$-5x$
-7	$-7x$	35

$x^2 - 12x + 35$

a. $(x + 5)(x + 7)$
b. $x(x - 35)$
c. $(x - 5)(x + 7)$
d. $(x - 5)(x - 7)$

5. Find the factored form of $x^2 + 13x + 36$.

a. $(x + 9)(x + 4)$
b. $(x + 18)(x + 2)$
c. $(x + 12)(x + 1)$
d. $(x - 12)(x - 3)$

6. Find the factored form of $x^2 - 14x + 48$.

a. $(x - 6)(x - 8)$
b. $(x - 7)(x - 7)$
c. $(x - 12)(x - 4)$
d. $(x - 10)(x - 4)$

7. Find the factored form of $x^2 + 13x + 30$.

a. $(x + 6)(x + 7)$
b. $(x + 3)(x + 10)$
c. $(x + 6)(x + 5)$
d. $(x + 15)(x + 2)$

8. Find the factored form of $x^2 - 12x + 27$.

a. $(x - 6)(x - 6)$
b. $(x - 27)(x - 1)$
c. $(x - 7)(x - 5)$
d. $(x - 9)(x - 3)$

9. Find the factored form of $x^2 - 13x + 22$.

a. $(x - 6)(x - 7)$
b. $(x - 22)(x - 1)$
c. $(x - 12)(x - 1)$
d. $(x - 11)(x - 2)$

10. Here are pairs of equivalent expressions: one in standard form and the other in factored form. Find the missing numbers.

$x^2 + 20x + 96$ and $(x + __)(x + __)$

a. 14, 6
b. 24, 4
c. 12, 8
d. 10, 10

8.6.6 Lesson Summary

In this lesson, you learned how to:

- Explain how the numbers in a quadratic expression in factored form relate to the numbers in an equivalent expression in standard form.
- Rewrite a quadratic expression in standard form when given the quadratic expression in factored form.
- Rewrite a quadratic expression in factored form when given the quadratic expression in the form of $ax^2 + bx + c$.

Here are the **activities** that helped you reach those goals:

- 8.6.1: Finding and Reasoning Unknown Factors
  - In this activity, you practiced using reasoning to find unknown factors missing from a problem.
- 8.6.2: Using Diagrams to Understand Equivalent Expressions
  - In this activity, you learned how to use a diagram to help visualize the process of rewriting a quadratic expression in standard form into factored form.
  - 8.6.2: Self Check
  - 8.6.2: Additional Resources
- 8.6.3: Rewriting Quadratic Expressions in Standard Form
  - In this activity, you continued learning how to rewrite quadratic expressions in standard form into factored form. This lesson covered expressions in the factored forms $(x + a)(x + b)$ and $(x - a)(x - b)$.
  - 8.6.3: Self Check
  - 8.6.3: Additional Resources
- 8.6.4: Finding the Missing Numbers in Pairs of Equivalent Expressions
  - In this activity, you used the skills you learned rewriting quadratic expressions to find missing values in a conversion from standard form into factored form.

After these activities, you completed the following **practice**:

- 8.6.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

8.7 REWRITING QUADRATIC EXPRESSIONS IN FACTORED FORM, PART 2

8.7.0 Lesson Overview

In this lesson, you will extend your knowledge of transforming quadratic expressions from standard form into factored form. You will continue factoring with expressions that can be rewritten as products of a sum and a difference, $(x + m)(x - n)$.

When you finish this lesson, you will be able to:

- Explain how the numbers and signs in a quadratic expression in factored form relate to the numbers and signs in an equivalent expression in standard form.
- Write an equivalent expression in factored form, given a quadratic expression in standard form with a negative constant term.

Here are the **activities** that will help you reach those goals:

- 8.7.1: Understanding Sums and Products of Integers
- 8.7.2: Interpreting Negative Constant Terms When Factoring Quadratic Expressions
  - 8.7.2: Self Check
  - 8.7.2: Additional Resources
- 8.7.3: Analyzing Factors of 100 and –100
  - 8.7.3: Self Check
  - 8.7.3: Additional Resources
- 8.7.4: Finding the Missing Terms

After that, you'll **practice and review**.

- 8.7.5: Practice
- 8.7.6: Lesson Summary

8.7.1 Understanding Sums and Products of Integers

Warm Up

In the previous lesson, you experienced the role products and sums play in the way that quadratic expressions are converted between standard and factored forms. Now, you will expand your understanding of the concept.

1. List all pairs of integers whose product is –12. For example, 2 and –6.
2. Of the pairs of factors you found, list all pairs that have a positive sum. Explain why they have a positive sum.
3. Of the pairs of factors you found, list all pairs that have a negative sum. Explain why they have a negative sum.

8.7.2 Interpreting Negative Constant Terms When Factoring Quadratic Expressions

Activity

Let's begin by reviewing how to convert between standard and factored form using expressions like the ones we have seen before.

1. Each row of the table below has a pair of equivalent expressions. Complete the table. If you get stuck, consider drawing a diagram.

Factored form	Standard form
$(x+5)(x+6)$	
	$x^2+13x+30$
$(x-3)(x-6)$	
	$x^2-11x+18$

In the warm up, you examined factors that had different signs. Two numbers' signs affect the sum when they are combined and the product when they are multiplied. Use your observations from the warm up to explore these expressions that are in some ways unlike the ones we have seen before.

2. Each row of the table below has a pair of equivalent expressions. Complete the table. If you get stuck, consider drawing a diagram.

Factored form	Standard form
$(x+12)(x-3)$	
	$x^2-9x-36$
	$x^2-35x-36$
	$x^2+35x-36$

3. Name some ways that the expressions in the second table are different from those in the first table (aside from the fact that the expressions use different numbers).

Why Should I Care?



Matteo's mom is a landscape architect. One day, she took Matteo to see the large succulent garden that she had designed for a new hospital. She told him that the garden's area was determined using a single quadratic function.

Then, she introduced Matteo to Lucy, the accountant for the plant store where she bought the succulents. Lucy used quadratic functions to model a demand curve and determine how to maximize profits. This allowed them to find the most fantastic desert plants that would make the plant store the most money.

Both Matteo's mom and Lucy use quadratic functions to excel at their jobs.

Self Check

Find the factored form of the expression, $x^2 - 22x - 48$.

a. $(x - 8)(x - 16)$
b. $(x + 2)(x - 24)$
c. $(x - 2)(x + 24)$
d. $(x + 12)(x - 4)$

Additional Resources

Interpreting Negative Constant Terms When Factoring Quadratic Expressions

Consider the expression $x^2 - 34x + 64$. (Note the leading coefficient is 1.)

We have factored expressions like this before. We know that when the constant term is positive, the signs of the constants in the factored form must be the same.

Since the middle term is negative, we know that both terms will be negative. So, $x^2 - 34x + 64$ factors into $(x - 32)(x - 2)$.

If the middle term was positive in the expression above, then both constants in the factored form would have been positive.

EXAMPLE 1

Now consider the expression $x^2 - 2x - 24$. (Note the leading coefficient is 1.)

The constant term is a negative, –24. To find its factored form, we are searching for two numbers that have a negative product. This means that these two numbers must have opposite signs since multiplying a positive number and a negative number yields a negative product.

Now look at the middle term, –2. We know the sum of these numbers must be –2. This means the factor with the larger absolute value must be negative.

Which factors of –24 have a sum of –2? The answer is –6 and 4. Since we know that the larger value must have a negative sign, this confirms the factors we are searching for are −6 and 4.

This gives a factored form of $(x-6)(x+4)$.

EXAMPLE 2

Find the factored form. (Note the leading coefficient is 1.)

$x^2+7x-18$

The constant term is a negative, –18. To find its factored form, we are searching for two numbers that have a negative product.

The middle term is 7. The same two numbers must have a sum of +7.

Which two numbers have a product of –18 and a sum of +7?

The numbers are 9 and –2.

So, the factored form is $(x+9)(x-2)$.

TRY IT **Interpreting Negative Constant Terms When Factoring Quadratic Expressions**

Find the factored form of each expression.

1. $x^2+7x-44$
2. x^2-x-30

8.7.3 Analyzing Factors of 100 and −100

Activity

For questions 1 – 2, consider the expression $x^2+bx+100$. (Note the leading coefficient is 1.)

1. Create a table with all pairs of factors of +100 that would give positive values of b.
For each pair, state the b value they produce. (Use as many rows as needed.)

Factor 1	Factor 2	b (positive)

2. Create a table with all pairs of factors of +100 that would give negative values of b.
For each pair, state the b value they produce. (Use as many rows as needed.)

Factor 1	Factor 2	b (negative)

For questions 3 – 5, consider the expression $x^2 + bx - 100$. (Note the leading coefficient is 1.)

3. Create a table with all pairs of factors of –100 that would result in positive values of b.
For each pair of factors, state the b value they produce. (Use as many rows as needed.)

Factor 1	Factor 2	b (positive)

4. Create a table with all pairs of factors of –100 that would give negative values of b.
For each pair, state the b value they produce. (Use as many rows as needed.)

Factor 1	Factor 2	b (negative)

5. Create a table with all pairs of factors of –100 that would result in a zero value of b.
For each pair, state the b value they produce. (Use as many rows as needed.)

Factor 1	Factor 2	b (zero)

For problems 6 – 9, write each expression in factored form. Note that the leading coefficient for each expression is 1.

6. $x^2 - 25x + 100$

7. $x^2 + 15x - 100$

8. $x^2 - 15x - 100$

9. $x^2 + 99x - 100$

Are you ready for more?

Extending Your Thinking

1. How many different integers b can you find so that the expression $x^2 + 10x + b$ can be written in factored form?

Video: Analyzing factors of 100 and −100

Watch the following video to learn more about analyzing the factors of 100 and –100 and applying them in factoring quadratic expressions.

Analyzing factors of 100 and -100 (https://openstax.org/r/factors-100)

Self Check

Consider the expression $x^2 + bx - 48$.

Identify all the pairs of factors of –48 that would result in positive values of b.

a. -1 and 48,
-2 and 24,
-3 and 16,
-4 and 12,
-6 and 8
b. -1 and -48,
-2 and -24,
-3 and -16
-4 and -12,
-6 and -8
c. 1 and 48,
2 and 24,
3 and 16,
4 and 12,
6 and 8
d. 1 and -48,
2 and -24,
3 and -16,
4 and -12,
6 and -8

Additional Resources

Analyzing Factors of Constants in Quadratic Expressions

EXAMPLE 1

Let's consider the expression $x^2 + bx + 40$, where the leading coefficient equals 1.

Identify all the factors of 40 that would result in a positive value of b.

We know that the factors of +40 that result in a positive sum are both positive numbers.

Factor 1	Factor 2	b (positive)
1	40	41
2	20	22
4	10	14
5	8	13

Now let's identify all the factors of 40 that would result in a negative value of b.

We know that the factors of +40 that result in a negative sum are both negative numbers.

Factor 1	Factor 2	b (negative)
–1	–40	–41
–2	–20	–22
–4	–10	–14
–5	–8	–13

EXAMPLE 2

Now, let's consider the expression $x^2 + bx - 40$, where the leading coefficient equals 1.

Identify all the factors of –40 that would result in a positive value of b.

Factor 1	Factor 2	b (positive)
–1	40	39
–2	20	18
–4	10	6
–5	8	3

Identify all the factors of –40 that would result in a negative value of b.

Factor 1	Factor 2	b (negative)
1	–40	–39
2	–20	–18
4	–10	–6
5	–8	–3

Notice the similarities between the two tables above. They both contain all of the factors of –40, where one factor is positive and the other is negative.

Notice the difference depends on the value of b, the sum of the factors.

- For positive values of b, the factor with the larger absolute value is positive. The other factor is negative.
- For negative values of b, the factor with the larger absolute value is negative. The other factor is positive.

TRY IT **Analyzing Factors of Constants in Quadratic Expressions**

1. Consider the expression $x^2 + bx - 60$. Identify all of the factors that would result in a positive value of b.

2. Consider the expression $x^2 + bx - 60$. Identify all of the factors that would result in a negative value of b.

8.7.4 Finding the Missing Terms

Cool Down

Here are pairs of equivalent expressions in standard form and factored form. Find the missing symbols and numbers.

1. $x^2 \square 16x \square 17$ and $(x + 1)(x - 17)$

2. $x^2 \square 16x \square 17$ and $(x - 1)(x + 17)$

3. $x^2 + 3x - 28$ and $(x + \square)(x - \square)$

4. $x^2 - 12x - 28$ and $(x \square \square)(x \square \square)$

8.7.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 7, find the factored form of the quadratic expression.

1. $x^2 + 4x - 12$

a. $(x - 2)(x + 6)$
b. $(x + 2)(x - 6)$
c. $(x + 3)(x - 4)$
d. $(x - 2)(x - 6)$

2. $x^2 - 3x - 18$

a. $(x - 7)(x + 3)$
b. $(x - 18)(x + 1)$
c. $(x - 6)(x + 3)$
d. $(x - 9)(x + 2)$

3. $x^2 - 12x - 45$

a. $(x - 15)(x + 3)$
b. $(x - 9)(x + 5)$
c. $(x - 8)(x - 4)$
d. $(x - 45)(x + 1)$

4. $x^2 - x - 20$

a. $(x + 5)(x - 4)$
b. $(x - 20)(x + 1)$
c. $(x - 5)(x + 4)$
d. $(x - 10)(x + 2)$

5. $x^2 + 2x - 35$

a. $(x - 5)(x - 7)$
b. $(x + 5)(x - 7)$
c. $(x + 5)(x + 7)$
d. $(x - 5)(x + 7)$

6. $x^2 - 23x - 50$

a. $(x - 10)(x - 13)$
b. $(x - 25)(x + 2)$
c. $(x - 50)(x + 1)$
d. $(x - 10)(x + 5)$

7. $x^2 + 7x - 60$

a. $(x - 4)(x + 15)$
b. $(x - 10)(x + 3)$
c. $(x - 20)(x + 3)$
d. $(x - 5)(x + 12)$

8. Consider the expression $x^2 + bx - 80$, where the leading coefficient equals 1. Identify all the pairs of factors of –80 that would result in positive values of b.

a. 80 and -1, 40 and -2, 20 and -4, 16 and -5, 10 and -8
b. -80 and -1, -40 and -2, -20 and -4, -16 and -5, -10 and -8
c. -80 and 1, -40 and 2, -20 and 4, -16 and 5, -10 and 8
d. 80 and 1, 40 and 2, 20 and 4, 16 and 5, 10 and 8

9. Consider the expression $x^2 + bx - 80$, where the leading coefficient equals 1. Identify all the pairs of factors of

-80 that would result in negative values of b.

a. 80 and -1, 40 and -2, 20 and -4, 16 and -5, 10 and -8
b. -80 and -1, -40 and -2, -20 and -4, -16 and -5, -10 and -8
c. -80 and 1, -40 and 2, -20 and 4, -16 and 5, -10 and 8
d. 80 and 1, 40 and 2, 20 and 4, 16 and 5, 10 and 8

10. Here are pairs of equivalent expressions: one in standard form and the other in factored form. Identify the missing symbols:

$x^2 \square 15x \square 34$ and $(x+2)(x-17)$

a. – and +
b. + and +
c. + and –
d. – and –

8.7.6 Lesson Summary

In this lesson, you learned how to:

- Explain how the numbers and signs in a quadratic expression in factored form relate to the numbers and signs in an equivalent expression in standard form.
- Write an equivalent expression in factored form, given a quadratic expression in standard form with a negative constant term.

Here are the **activities** that helped you reach those goals:

- 8.7.1: Understanding Sums and Products of Integers
  - In this activity, you identified the factors of various numbers and studied their sums. This built a foundation for understanding how to find the factored forms of different quadratic expressions.
- 8.7.2: Interpreting Negative Constant Terms When Factoring Quadratic Expressions
  - In this activity, you learned how to find the factored form of a quadratic expression that has a negative constant term.
  - 8.7.2: Self Check
  - 8.7.2: Additional Resources
- 8.7.3: Analyzing Factors of 100 and –100
  - In this activity, you continued exploring how to find the factored form of a quadratic expression with a negative constant term. You learned to distinguish the signs of the factors based on the middle, or linear, term in the quadratic expression.
  - 8.7.3: Self Check
  - 8.7.3: Additional Resources
- 8.7.4: Finding the Missing Terms
  - In this activity, you combined all of the skills learned in the lesson to identify missing signs in conversions from quadratic expressions in standard form to factored form.

After these activities, you completed the following **practice**:

- 8.7.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

8.8 REWRITING QUADRATIC EXPRESSIONS IN FACTORED FORM, PART 3

8.8.0 Lesson Overview

In this lesson, you will encounter quadratic expressions without a linear term and consider how to write them in factored form. We will begin by studying numerical examples and noticing that certain expressions are equivalent.

Remember for some digital tools, when you want to enter a number with an exponent, use the ^ symbol. It will look like "x^2" when you want to enter x^2. To make the ^ symbol, push shift and the numeral 6 at the same time.

When you finish this lesson, you will be able to:

- Explain why multiplying binomials that are a sum and a difference, $(x + m)(x - m)$, results in a quadratic expression with no linear term.
- Rewrite a quadratic expression of the form $x^2 - c$ into its factored form.

Here are the **activities** that will help you reach those goals:

- 8.8.1: Evaluating Expressions Using Mental Math
- 8.8.2: Recognizing the Expanded Product of the Difference of Two Squares
  - 8.8.2: Self Check
  - 8.8.2: Additional Resources
- 8.8.3: Factoring Quadratic Equations without a Linear Term
  - 8.8.3: Self Check
  - 8.8.3: Additional Resources
- 8.8.4: Determining if an Expression Can Be Rewritten in Factored Form

After that, you'll **practice and review**.

- 8.8.5: Practice
- 8.8.6: Lesson Summary

8.8.1 Evaluating Expressions Using Mental Math

Warm Up

Finding a product mentally can be done by changing the way you look at a multiplication problem.

For example, $9 \cdot 39$ can be thought of as $9 \cdot (40 - 1)$. For some people, this might be easier to calculate mentally since you can imagine the solution as $360 - 9$, or 351. This method leverages the distributive property of multiplication over addition.

For the following questions, find each product mentally. Then, explain how you changed the way you looked at each multiplication problem to use the distributive property.

1. $9 \cdot 11$

2. $19 \cdot 21$

3. $99 \cdot 101$

4. $109 \cdot 101$

8.8.2 Recognizing the Expanded Product of the Difference of Two Squares

Activity

1. Clare claims that $(10 + 3)(10 - 3)$ is equivalent to $10^2 - 3^2$ and $(20 + 1)(20 - 1)$ is equivalent to $20^2 - 1^2$. Do you agree? Show your reasoning.

2. You can use your observations from the first question to mentally solve an expression such as $105 \cdot 95$.

a. First, think about the problem differently. Evaluate $(100 + 5)(100 - 5)$ using the distributive property, or FOIL. Show

your reasoning.

b. Check your answer by computing $105 \cdot 95$.

3. Is $(x+4)(x-4)$ equivalent to $x^2 - 4^2$?

a. First, support your thinking using a diagram.

	x	4
x		
-4		

b. Now, support your thinking using the distributive property, or FOIL.

4. Is $(x+4)^2$ equivalent to $x^2 + 4^2$? Support your answer, either with or without a diagram.

Are you ready for more?

Extending Your Thinking

1. Explain how your work in the previous questions can help you mentally evaluate $22 \cdot 18$.

2. Explain how your work in the previous questions can help you mentally evaluate $45 \cdot 35$.

3. Here is a shortcut that can be used to mentally square any two-digit number. Let's take 83^2, for example.

- 83 is $80 + 3$.
- Compute 80^2 and 3^2, which gives 6400 and 9. Add these values to get 6409.
- Compute $80 \cdot 3$, which is 240. Double it to get 480.
- Add 6409 and 480 to get 6889.

Try using this method to find the squares of some other two-digit numbers. (With some practice, it is possible to get really fast at this!)
Explain why this method works.

Self Check

Which of the following is equivalent to $9^2 - 6^2$?

a. $(9+6)(9-6)$
b. $(9+6)(9+6)$
c. $(9-6)(9-6)$
d. $(9-6)^2$

Additional Resources

Recognizing the Expanded Product of the Difference of Two Squares

Mathematically, something special happens when two binomials that are conjugates are multiplied. In other words, the product of an expression of the form $(a-b)(a+b)$ results in a unique expression called a difference of two squares. Here are some examples of these special products.

EXAMPLE 1

Let's look at the expression $(12+5)(12-5)$. We want to determine if it is equivalent to $12^2 - 5^2$.

We can use the distributive property of multiplication over addition, or FOIL process, to multiply.

$(12+5)(12-5)$

$= 12^2 - 12(5) + 5(12) - 5^2$

$= 12^2 - 60 + 60 - 5^2$

$= 12^2 - 5^2$

So, $(12+5)(12-5)$ is equivalent to $12^2 - 5^2$.

EXAMPLE 2

Let's look at a different example using a **variable**.

$(x+7)(x-7)$

Is this equivalent to $x^2 - 7^2$?

$(x+7)(x-7)$

$= x^2 - 7x + 7x - 7^2$

$= x^2 - 7^2 = x^2 - 49$

This can also be shown using a diagram.

	x	**–7**
x	x^2	$-7x$
7	$7x$	$-7(7)$

You have studied this process earlier when you learned the difference of squares.

In this lesson, we will reverse this process to rewrite the **standard form** of a **quadratic expression** into its **factored form**.

It is important to remember the **linear term**, or middle term of a quadratic trinomial, equals zero only holds true for the **difference of squares**. It does not work for an expression such as $x^2 + 25$. (In fact, there is no way to multiply two binomials that would result in this expression.)

EXAMPLE 3

Is $(x+5)^2$ equivalent to $x^2 + 25$?

Let's find the equivalent expression for $(x+5)^2$ or $(x+5)(x+5)$.

$(x+5)^2 = (x+5)(x+5)$

$= x^2 + 5x + 5x + 25$

$= x^2 + 10x + 25$

Since $x^2 + 10x + 25$ is not equivalent to $x^2 + 25$, these expressions are not the same.

TRY IT **Recognizing the Expanded Product of the Difference of Two Squares**

1. Is the expression $(8-7)(8+7)$ equivalent to the expression $8^2 - 7^2$? Show your reasoning using

the distributive property, or FOIL.

2. Is $(x+4)(x+4)$ equivalent to x^2+4^2? Support your answer with a diagram.

8.8.3 Factoring Quadratic Equations without a Linear Term

Activity

In questions 1 – 2, each row of the table contains a pair of equivalent expressions.

1. Convert the factored form expression to the standard form.
If you get stuck, consider drawing a diagram.

Factored form	Standard form
$(x-10)(x+10)$	
$(2x+1)(2x-1)$	
$(4-x)(4+x)$	
$(c+\frac{2}{5})(c-\frac{2}{5})$	
$(x+5)(x+5)$	

2. Convert the standard form expression to the factored form.
If you get stuck, consider drawing a diagram.
(HINT: One of the expressions does not convert.)

Factored form	Standard form
	x^2-81
	$49-y^2$
	$9z^2-16$
	$25t^2-81$
	$\frac{49}{16}-d^2$
	x^2-6
	x^2+100

Video: Factoring Quadratic Equations Without a Linear Term

Watch the following video to learn more about factoring a quadratic equation that is written in standard form and has no linear term.

Factoring Quadratic Equations Without a Linear Term (https://openstax.org/r/factor-quadratic-no-linear)

Self Check

Find the factored form of the expression $16p^2-49$.

a. $p(16p-49)$
b. $(4p-7)(4p-7)$
c. $(4p+7)(4p+7)$
d. $(4p+7)(4p-7)$

Additional Resources

Factoring Quadratic Equations Without a Linear Term

We can use what we have learned in the previous lesson to write quadratic equations of a special form into factored form.

Particularly, we can rewrite:

x^2-c^2 into $(x+c)(x-c)$.

Let's look at an example.

EXAMPLE 1

Write $v^2 - 36$ in factored form.

We know that the form we are looking for is $(v - __)(v + __)$.

Since we know the square **root** of v^2 is v and the square root of 36 is 6, the factored form is $(v - 6)(v + 6)$.

We can multiply to check our work.

$$(v - 6)(v + 6)$$
$$= v^2 + 6v - 6v - 36$$
$$= v^2 - 36 \checkmark$$

The process is similar when there is a coefficient on the squared term.

$a^2x^2 - c^2$ factors into $(ax + c)(ax - c)$

Let's look at another example to understand the process.

EXAMPLE 2

Write $16m^2 - 81$ in factored form.

The form we are looking for now is (__ − __)(__ + __).

Taking the square root of the first term gives us $4m$.

So, we have $(4m - __)(4m + __)$.

Now, we are in the same place as in the previous example. The square root of 81 is 9.

So, the factored form is $(4m - 9)(4m + 9)$.

Again, we can multiply to check.

$$(4m - 9)(4m + 9)$$
$$= 4m^2 + 36m - 36m - 81$$
$$= 4m^2 - 81 \checkmark$$

EXAMPLE 3

As we see in the activity, where the variable is located is not important as long as the quadratic binomial is still the difference of two squares.

Write $64 - 9b^2$ in factored form.

We use the same concept to find the factored form.

$64 - 9b^2 = (8 - 3b)(8 + 3b)$

We can multiply to check.

$$(8 - 3b)(8 + 3b)$$
$$= 64 + 24b - 24b - 9b^2$$
$$= 64 - 9b^2 \checkmark$$

Remember that any expression of the form $x^2 + c$ or $ax^2 + c$ does not have a factored form!

- For $x^2 + c$, the c is not squared, and the terms are not being subtracted.
- For $ax^2 + c$, neither the a nor the c is a perfect square, and the terms are not being subtracted.

TRY IT **Factoring Quadratic Equations Without a Linear Term**

1. Write $x^2 - 4$ in factored form.
2. Write $100x^2 - 121$ in factored form.

8.8.4 Determining if an Expression Can Be Rewritten in Factored Form

Cool Down

Write each expression in factored form. If it is not possible, write "not possible."

1. $a^2 - 36$
2. $49 - 25b^2$
3. $c^2 + 9$
4. $\frac{100}{81} - 16d^2$

8.8.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Which of the following is equivalent to $10^2 - 12^2$?

a. $(10 + 12)(10 - 12)$
b. $(10 - 12)^2$
c. $(10 - 12)(10 - 12)$
d. $(10 + 12)(10 + 12)$

For questions 2 – 10, select the expression in factored form.

2. $c^2 - 25$

a. $(c + 5)(c + 5)$
b. $(c - 5)(c + 5)$
c. $(c - 5)(c - 5)$
d. $c(c - 25)$

3. $9w^2 - 121$

a. $(3w - 11)(3w - 11)$
b. $w(9w - 121)$
c. $(3w + 11)(3w + 11)$
d. $(3w - 11)(3w + 11)$

4. $h^2 - 121$

a. $(h + 11)(h + 11)$
b. $(h - 11)(h - 11)$
c. $(h - 11)(h + 11)$
d. $(h - 11)^2$

5. $64 - d^2$

a. $(8 + d)(8 + d)$
b. $(8 - d)(8 + d)$
c. $(8 - d)(8 - d)$
d. Not possible

6. $4b^2 - 25$

a. $(2b + 5)(2b + 5)$
b. $b(4b - 25)$
c. $(2b - 5)(2b - 5)$
d. $(2b - 5)(2b + 5)$

7. $49 - 81z^2$

a. $(7 - 9z)(7 + 9z)$
b. $z(49 - 81z)$
c. $(7 - 9z)(7 - 9z)$
d. $(7 + 9z)(7 + 9z)$

8. $100y^2 - 4$

a. $(10y - 2)(10y - 2)$
b. $(10y - 2)(10y + 2)$
c. $(10y + 2)(10y + 2)$
d. Not possible

9. $x^2 + 9$

a. $(x + 3)(x + 3)$
b. $(x - 3)(x + 3)$

c. $(x-3)(x-3)$
d. Not possible

10. $144-16g^2$

a. $(12-4g)(12-4g)$
b. $(12+4g)(12+4g)$
c. $(12-4g)(12+4g)$
d. Not possible

8.8.6 Lesson Summary

In this lesson, you learned how to:

- Explain why multiplying binomials that are a sum and a difference, $(x + m)(x - m)$, results in a quadratic expression with no linear term.
- Rewrite a quadratic expression of the form $x^2 - c$ into its factored form.

Here are the **activities** that helped you reach those goals:

- 8.8.1: Evaluating Expressions Using Mental Math
  - In this activity, you learned shortcuts to performing mental math that provided insight into strategies for factoring quadratic expressions.
- 8.8.2: Recognizing the Expanded Product of the Difference of Two Squares
  - In this activity, you learned to reverse the process using the difference of squares to find the factored form of particular quadratic expressions.
  - 8.8.2: Self Check
  - 8.8.2: Additional Resources
- 8.8.3: Factoring Quadratic Equations Without a Linear Term
  - In this activity, you continued to practice finding the factored form of quadratic expressions using the difference of squares. You also learned that some binomials of the form $x^2 + c$ do not have a factored form.
  - 8.8.3: Self Check
  - 8.8.3: Additional Resources
- 8.8.4: Determining if an Expression Can Be Rewritten in Factored Form
  - In this activity, you determined whether a quadratic expression was factorable before finding its factored form using the skills learned in this lesson.

After these activities, you completed the following **practice**:

- 8.8.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

8.9 SOLVING QUADRATIC EQUATIONS BY USING FACTORED FORM

8.9.0 Lesson Overview

In this lesson, you will apply what you learned about transforming expressions into factored form to make sense of quadratic equations and persevere in solving them. You will solve equations that were previously only possible to solve using graphing by taking the following steps:

Step 1 - Rearrange equations so that one side of the equal sign is 0.

Step 2 - Rewrite the expression in factored form.

Step 3 - Use the zero product property.

Step 4 - Solve each equation.

When you finish this lesson, you will be able to:

- Rearrange a quadratic equation to be written as *expression in factored form* $= 0$ and find the solutions.
- Recognize quadratic equations that have zero, one, or two solutions when they are written in factored form.

Here are the **activities** that will help you reach those goals:

- 8.9.1: Finding a Solution through Substitution
- 8.9.2: Using Factored Form and the Zero Product Property to Solve Quadratic Equations
  - 8.9.2: Self Check
  - 8.9.2: Additional Resources
- 8.9.3: Writing an Equation to Represent a Quadratic Function with Only One Solution
  - 8.9.3: Self Check
  - 8.9.3: Additional Resources
- 8.9.4: Solving More Quadratic Equations

After that, you'll **practice and review**.

- 8.9.5: Practice
- 8.9.6: Lesson Summary

8.9.1 Finding a Solution through Substitution

Warm Up

Let's try to find at least one solution to $x^2 - 2x - 35 = 0$.

1. Create a table of values. In column 1, enter the whole numbers between 0 and 10 to represent values of x. In column 2, evaluate the expression $x^2 - 2x - 35$ using the values for x.
2. There is another number outside of 0 to 10 that would make the expression $x^2 - 2x - 35$ equal 0. Can you find it?

8.9.2 Using Factored Form and the Zero Product Property to Solve Quadratic Equations

Activity

To solve the equation $n^2 - 2n = 99$, Tyler wrote out the following steps. Analyze Tyler's work by explaining what Tyler did in each step.

Original equation $n^2 - 2n = 99$

1. **Step 1** - $n^2 - 2n - 99 = 0$
2. **Step 2** - $(n - 11)(n + 9) = 0$

3. **Step 3** - $n-11=0$ or $n+9=0$

4. **Step 4** - $n=11$ and $n=-9$

For questions 5 – 9, solve each equation by rewriting it in factored form and using the **zero product property** .

5. $x^2+8x+15=0$

6. $x^2-8x+12=5$

7. $x^2-10x-11=0$

8. $49-x^2=0$

9. $(x+4)(x+5)-30=0$

Are you ready for more?

Extending Your Thinking

1. Solve this equation and explain or show your reasoning.
$(x^2-x-20)(x^2+2x-3)=(x^2+2x-8)(x^2-8x+15)$

Self Check

Solve the equation by rewriting it in factored form and then using the zero product property.

$x^2+3x-59=-5$

a. $x=-9$ and $x=-6$
b. $x=-9$ and $x=6$
c. $x=9$ and $x=6$
d. $x=9$ and $x=-6$

Additional Resources

Using Factored Form and the Zero Product Property to Solve Quadratic Equations

We have seen **quadratic equations** in many different forms. Let's look at an example in which we find the **factored form** and then use the **zero product property** to solve.

EXAMPLE 1

Solve $x^2+3x-50=-10$.

Step 1 - Set the equation equal to 0 by adding 10 to both sides.

$x^2+3x-40=0$

Step 2 - Find the factored form.

Which numbers have a product of –40 and a sum of +3?

$(x+8)(x-5)=0$

Step 3 - Use the zero product property to solve.

Set each factor equal to 0 and solve.

$(x+8)=0 \quad (x-5)=0$

$x=-8 \qquad x=5$

The solution is $x=-8$ and $x=5$.

EXAMPLE 2

Solve $x^2+12x+21=-15$.

Step 1 - Set the equation equal to 0 by adding 15 to both sides.

$x^2+12x+36=0$

Step 2 - Find the factored form.

Which numbers have a product of 36 and a sum of 12?

$(x+6)(x+6)=0$

Step 3 - Use the zero product property to solve.

Set each factor equal to 0 and solve. The factors are the same, so there is one solution.

$(x+6)=0 \quad (x+6)=0$

$x=-6 \qquad x=-6$

The only solution is $x=-6$.

TRY IT **Using Factored Form and the Zero Product Property to Solve Quadratic Equations**

1. Solve $z^2-5z-48=-12$.
2. Solve $p^2-3p-108=-20$.

8.9.3 Writing an Equation to Represent a Quadratic Function with Only One Solution

Activity

In previous lessons, we saw that a **quadratic equation** can have zero, one, or two solutions.

1. Sketch graphs that represent three **quadratic functions**: one that has no zeros, one with one zero, and one with two zeros.
2. Graph the function defined by $f(x)=x^2-2x+1$.
   Examine the x-intercepts of the graph.
   Use the Desmos graphing tool or technology outside the course.
   Access multimedia content (http://openstax.org/books/algebra-1/pages/8-9-3-writing-an-equation-to-represent-a-quadratic-function-with-only-one-solution)
   Explain what the x-intercepts reveal about the function.
3. Solve $x^2-2x+1=0$ by using the factored form and **zero product property**. What solutions do you get?
4. Try to find another quadratic function that you think will only have one zero.
   Use the Desmos graphing tool or technology outside the course.
   Access multimedia content (http://openstax.org/books/algebra-1/pages/8-9-3-writing-an-equation-to-represent-a-quadratic-function-with-only-one-solution)
   Graph it to check your prediction, then enter your final answer.

Video: Quadratic Equations with Only One Solution

Watch the following video to learn more about writing an equation to represent a quadratic function with only one solution.

8.9.3 (https://openstax.org/r/quadratics-one-solution)

Self Check

Without graphing, identify the function whose graph has only one zero.

a. $f(x)=x^2+2x-15$
b. $f(x)=x^2-14x+49$

c. $f(x) = x^2 - 1$
d. $f(x) = x^2 - 7x - 18$

Additional Resources

Writing an Equation to Represent a Quadratic Function with Only One Solution

Identifying the Number of Zeros

How many zeros does the graph of the function $f(x) = x^2 + 5x - 14$ have? How can you find out without graphing the function?

Let's find the factored form of the function and solve it.

Step 1 - Set the function equal to 0.

$x^2 + 5x - 14 = 0$

Step 2 - Find the factored form.

$(x + 7)(x - 2) = 0$

Step 3 - Set each factor equal to 0 and solve.

$x = -7$ and $x = 2$

Since there are two solutions, we know that the function $f(x) = x^2 + 5x - 14$ will have zeros at $x = -7$ and $x = 2$.

In other words, the x-intercepts of the function are $(-7, 0)$ and $(2, 0)$.

For any function, the number of solutions is equal to the number of zeros when graphed.

Writing an Equation to Represent a Quadratic Function with One Solution

We can also find a **quadratic function** if we know the zeros. For example, if we want to write a quadratic function that has one zero or one solution, we can start with a factored form of the function representing one zero or solution.

Any factored form of a function with a factor that repeats itself, such as $(x - 3)(x - 3)$, has only one solution or one zero.

We can graph the function $f(x) = (x - 3)(x - 3)$ to check our prediction.

We know the solution is $x = 3$. Let's rewrite the function in **standard form**.

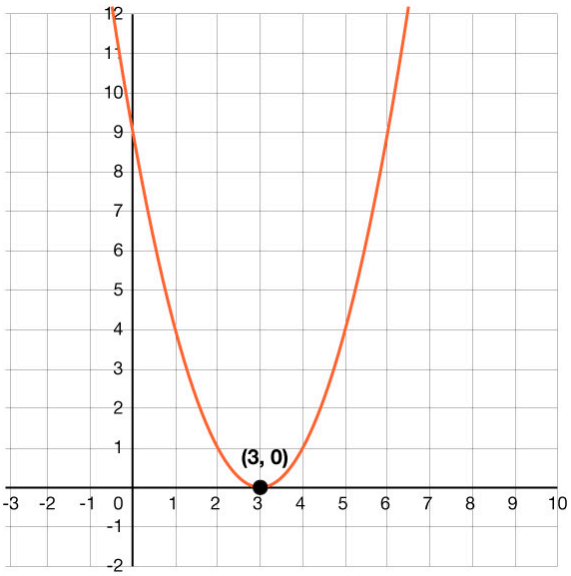
Factored form:

$f(x) = (x - 3)(x - 3)$

Multiply to find standard form:

$f(x) = x^2 - 6x + 9$

Now, we graph the function $f(x) = x^2 - 6x + 9$.



The graph of the function has one zero at $x = 3$. The x-intercept of the function is $(3, 0)$. The equation is said to have a **double root** at this point.

So, our prediction was correct.

TRY IT **Writing an Equation to Represent a Quadratic Function with Only One Solution**

1. Write a quadratic function in standard form whose graph has two zeros.
2. Write a quadratic function in standard form whose graph has one zero.

8.9.4 Solving More Quadratic Equations

Cool Down

Solve each equation by rewriting it in **factored form** and using the **zero product property**. Be prepared to explain your reasoning.

1. $x^2 + 12x + 11 = 0$
2. $x^2 - 3 = 1$
3. $x^2 - 6x + 7 = -2$

8.9.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 5, solve each quadratic equation by rewriting it in factored form and using the zero product property.

1. $x^2 + 11x + 35 = 7$

a. $x = -5$ and $x = -7$
b. $x = 14$ and $x = -3$
c. $x = -6$ and $x = -7$
d. $x = -4$ and $x = -7$

2. $x^2 - 14x + 20 = -25$

a. $x = 9$ and $x = 5$
b. $x = -10$ and $x = -4$
c. $x = 10$ and $x = 2$
d. $x = 3$ and $x = 15$

3. $x^2 - 6x - 52 = -12$

a. $x = -8$ and $x = 5$
b. $x = 10$ and $x = -4$
c. $x = 8$ and $x = -8$
d. $x = -16$ and $x = 10$

4. $x^2 - x - 64 = -8$

a. $x = 4$ and $x = -14$
b. $x = 28$ and $x = -2$
c. $x = -7$ and $x = 8$
d. $x = 8$ and $x = -8$

5. $x^2 + 9x + 13 = -5$

a. $x = 3$ and $x = 6$
b. $x = 8$ and $x = 1$
c. $x = -9$ and $x = -2$
d. $x = -3$ and $x = -6$

6. Without graphing, identify the function whose graph has only one zero.

a. $f(x) = x^2 - 8x + 16$
b. $f(x) = x^2 + 6x + 5$
c. $f(x) = x^2 - 13x + 42$
d. $f(x) = x^2 - 9$

7. Without graphing, identify the function whose graph has only one zero.

a. $f(x) = x^2 - 10x + 9$
b. $f(x) = x^2 + 12x + 36$
c. $f(x) = x^2 + x - 30$
d. $f(x) = x^2 + 7x - 30$

8. Without graphing, identify the function whose graph has two zeros.

a. $f(x) = x^2 - 12x + 36$
b. $f(x) = x^2 + 8x + 16$
c. $f(x) = x^2 - 8x + 16$
d. $f(x) = x^2 - 81$

9. Without graphing, identify the function whose graph has two zeros.

a. $f(x) = x^2 - 14x + 49$
b. $f(x) = x^2 + 16x + 64$
c. $f(x) = x^2 + 4x - 21$
d. $f(x) = x^2 - 10x + 25$

10. Solve the equation: $x^2 - 18x + 90 = 9$.

a. $x = 9$
b. $x = -10$ and $x = 8$
c. $x = -9$ and $x = 9$
d. $x = -9$

8.9.6 Lesson Summary

In this lesson, you learned how to:

- Rearrange a quadratic equation to be written as **expression in factored form** $= 0$ and find the solutions.
- Recognize quadratic equations that have zero, one, or two solutions when they are written in factored form.

Here are the **activities** that helped you reach those goals:

- 8.9.1: Finding a Solution through Substitution
  - In this activity, you tried to find the solution to a quadratic equation by guessing and checking. You substituted a value in for x to determine whether the equation was true.
- 8.9.2: Using Factored Form and the Zero Product Property to Solve Quadratic Equations
  - In this activity, you practiced setting an equation equal to 0 before finding its factored form. Then you used the zero product property on the factored form to solve the quadratic equation.
  - 8.9.2: Self Check
  - 8.9.2: Additional Resources
- 8.9.3: Writing an Equation to Represent a Quadratic Function with Only One Solution
  - In this activity, you learned the relationship between the number of solutions found in the factored form of a quadratic function and the number of zeros on the graph of the same quadratic function. You used this relationship to make predictions about the quadratic function.
  - 8.9.3: Self Check
  - 8.9.3: Additional Resources
- 8.9.4: Solving More Quadratic Equations
  - In this activity, you continued to practice solving quadratic equations as you became more familiar with their factored form and the zero product property.

After these activities, you completed the following **practice**:

- 8.9.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

8.10 REWRITING QUADRATIC EXPRESSIONS IN FACTORED FORM, PART 4

8.10.0 Lesson Overview

In this lesson, you will consider how to rewrite expressions in standard form where the leading coefficient is not 1 and the expression is not a difference of two squares.

When you finish this lesson, you will be able to:

- Use the factored form of a quadratic expression or a graph of a quadratic function to answer questions about a situation.
- Write equivalent expressions in factored form when given quadratic expressions of the form $ax^2 + bx + c$ and a is not 1.

Here are the **activities** that will help you reach those goals:

- 8.10.1: Analyzing Various Quadratic Expressions
- 8.10.2: Working with Quadratic Factored Form with Leading Coefficient Other Than One
  - 8.10.2: Self Check
  - 8.10.2: Additional Resources
- 8.10.3: Using Technology to Find Rational Factors
  - 8.10.3: Self Check
  - 8.10.3: Additional Resources
- 8.10.4: Finding the Factors of Quadratic Expressions in Standard Form
  - 8.10.4: Self Check
  - 8.10.4: Additional Resources
- 8.10.5: Solving Quadratic Equations by Any Method

After that, you'll **practice and review**.

- 8.10.6: Practice
- 8.10.7: Lesson Summary

8.10.1 Analyzing Various Quadratic Expressions

Warm Up

1. Examine the four expressions below. Which one doesn't belong?

A.	B.	C.	D.
$(x+4)(x-3)$	$3x^2 - 8x + 5$	$x^2 - 25$	$x^2 + 2x + 3$

   There are many possible answers for which equation does not belong in a group with the others. Determine which equation you think should be eliminated and explain your reasoning.

8.10.2 Working with Quadratic Factored Form with Leading Coefficient Other Than One

Activity

Each row in the given tables contains a pair of equivalent expressions. Complete the tables by converting the given expression into **standard form**. If you get stuck, try drawing a diagram.

1.

Factored form	Standard form
$(3x+1)(x+4)$	

2.

Factored form	Standard form
$(3x+2)(x+2)$	

3.

Factored form	Standard form
$(3x+4)(x+1)$	

4. To find the **factored form** of a **quadratic expression** with a leading **coefficient** other than 1, answer parts a–d.

Factored form	Standard form
	$5x^2+21x+4$

a. This quadratic expression has a leading coefficient of 5. Identify the factors of 5 to determine the leading term of each binomial in factored form.

(___ $x+$ ___)(___ $x+$ ___)

b. The constant term in this quadratic expression is +4. Identify all the factors of +4.

c. The **linear term** in the expression is $21x$. Place each of the factors of +4 you found in part b into the remaining blanks of the factored form structure from part a. Which of these results in a linear term of $21x$ when you use FOIL? (Keep in mind that the order you place these factors into your factored form structure might affect the outcome, so try both orders if needed.)

d. What is the factored form of $5x^2+21x+4$?

5. Use the strategy you developed in question 4 to find the factored form of the quadratic expression shown here in standard form.

Factored form	Standard form
	$3x^2 + 7x + 4$

Are you ready for more?

Extending Your Thinking

1. Find the factored form of the quadratic expression shown.

Factored form	Standard form
	$6x^2 + 11x - 10$

This quadratic expression is different from those in the activity, since its leading coefficient has more than one set of possible factors.
Use the strategy you learned in the activity to factor the expression. If the pair of factors you selected using the leading coefficient, 6, do not give you the correct solution, begin again with another set of factors.
The expression is also different since the constant term is negative, -10. This means when you consider the factors of the constant term, one must be positive and one must be negative.

Self Check

Find the factored form of the quadratic expression $3x^2 + 5x - 2$.

a. $(3x - 1)(x + 2)$
b. $(3x - 2)(x + 1)$
c. $(3x + 2)(x + 1)$
d. $(3x - 2)(x - 1)$

Additional Resources

Working with Expressions of the Form $(px + m)(qx + n)$

You have factored and solved many different types of **quadratic equations**, but the leading **coefficient** of the x^2 term was often 1.

Let's look at the **factored form** of a more complex expression to better understand the process.

EXAMPLE 1

Write $(4x + 1)(x + 3)$ in **standard form**.

To find the **quadratic expression** in standard form, use the distributive property of multiplication over addition, or FOIL.

$$\begin{aligned}(4x + 1)(x + 3) &= 4x^2 + 12x + x + 3 \\ &= 4x^2 + 13x + 3\end{aligned}$$

Now, let's try to go the other direction from standard form to factored form.

EXAMPLE 2

Write $2x^2 + 13x + 15$ in factored form.

Since the leading coefficient is 2, we know the factored form must have the structure:

$(2x + ___)(x + ___)$

Since the constant term is +15, the two missing numbers in the factored form must have a product of 15.

The possibilities include:

+1 and +15

+3 and +5

-1 and -15

-3 and -5

Since the middle term, $13x$, is positive, we know that only the positive factors need to be considered.

Since the two factors are already different inside the parentheses, the order in which we place the numbers is important. We will try both, if necessary.

Let's try 1 and 15 .

$(2x + 15)(x + 1)$

$= 2x^2 + 2x + 15x + 15$

$= 2x^2 + 17x + 15$

not equivalent to the original expression

Try reordering the same numbers.

$(2x + 1)(x + 15)$

$= 2x^2 + 30x + x + 15$

$= 2x^2 + 31x + 15$

not equivalent to the original expression

Let's try 3 and 5 .

$(2x + 5)(x + 3)$

$= 2x^2 + 6x + 5x + 15$

$= 2x^2 + 11x + 15$

not equivalent to the original expression

Try reordering the same numbers.

$(2x + 3)(x + 5)$

$= 2x^2 + 10x + 3x + 15$

$= 2x^2 + 13x + 15$

✓ equivalent to the original expression

So, the factored form of $2x^2 + 13x + 15$ is $(2x + 3)(x + 5)$.

TRY IT **Working with Expressions of the Form $(px + m)(qx + n)$**

Find the factored form of $3x^2 - 11x - 4$.

8.10.3 Using Technology to Find Rational Factors

Activity

An engineer is designing a fountain that shoots out drops of water. The nozzle from which the water is launched is 3 meters above the ground. It shoots out a drop of water at a vertical velocity of 9 meters per second.

Function h models the height in meters, h, of a drop of water t seconds after it is shot out from the nozzle. The function is defined by the equation $h(t) = -5t^2 + 9t + 3$.

How many seconds until the drop of water hits the ground?

1. Write an equation that we could solve to answer the question.
2. Try to solve the equation by writing the expression in **factored form** and using the **zero product property**. If it cannot be factored, explain how you came to this conclusion.
3. Solve the equation by graphing the function. Use the Desmos graphing tool or technology outside the course.
   Access multimedia content (http://openstax.org/books/algebra-1/pages/8-10-3-using-technology-to-find-rational-factors)
   Explain how you found the solution.

Using Technology to Find the Rational Factors

Watch the following video to learn more about using technology to find the rational factors of a **quadratic equation**.

Using Technology to Find the Rational Factors (https://openstax.org/r/rational-factors)

Self Check

A tennis ball is hit by a racket.

The function $h(t) = -3t^2 + 7t + 3$ models the height in meters, h, of the tennis ball t seconds after the impact with the racket.

Use graphing technology to determine the amount of time that has passed from the tennis ball being hit to when it reaches the ground.

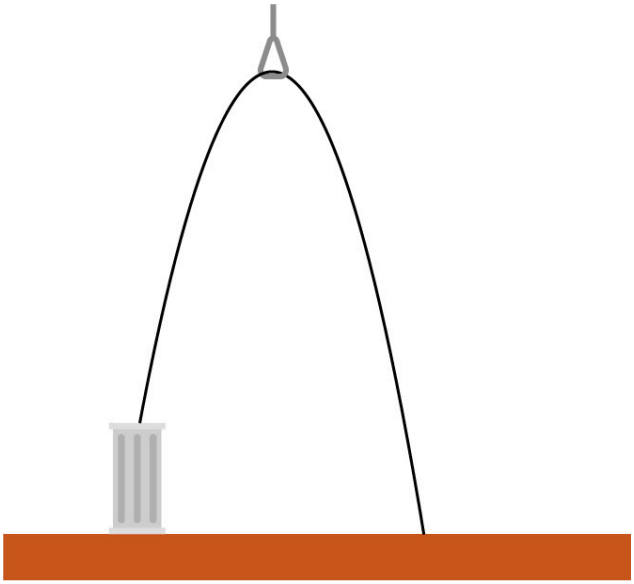
a. 3.124 seconds
b. 0.37 seconds
c. 2.703 seconds
d. 1.167 seconds

Additional Resources

Using Technology to Find Rational Factors

Let's use graphing technology to solve a situation involving a **quadratic function** that cannot be solved using factoring.

A thick steel support cable at a construction site forms an arch-like shape. It starts attached to the top of a column, goes up through a loop at a high point, and then back down to the ground.

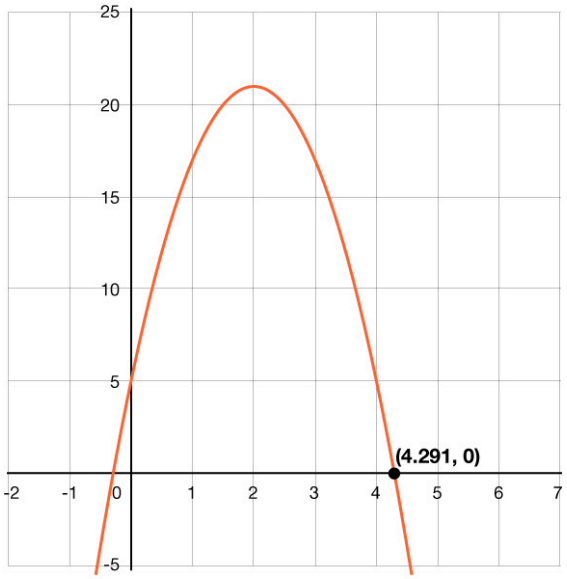


The height of the cable in meters is represented by the function $h(x) = -4x^2 + 16x + 5$ where x is the horizontal distance from the column, in meters.

How far away from the column does the cable attach to the ground?

The first step in solving would be to set the function equal to 0 and try to factor. Since this equation cannot be factored, we will use graphing technology.

Let's graph the function using the equation $y = -4x^2 + 16x + 5$.



We can see the function has a zero at $x = 4.291$. The x-intercept is $(4.291, 0)$.

This means the cable attaches to the ground 4.291 meters away from the column.

What else about the steel cable system do we know from this graph?

We can also tell that the cable attaches to the column 5 meters off the ground at the y-intercept and the loop is approximately 21 meters high at the vertex of the parabola.

TRY IT **Using Technology to Find Rational Factors**

The height, in meters, of a model rocket being launched is modeled by the function

$h(t) = -2t^2 + 14t + 2$ where t is the number of seconds after launch.

How many seconds pass before the rocket reaches the ground?

8.10.4 Finding the Factors of Quadratic Expressions in Standard Form

Activity

This activity is an extension of the expectations in the TEKS.

Here is a clever way to think about **quadratic expressions** that can make it easier to find their **factored form**.

Factor $9x^2 + 21x + 10$.

Step 1 - Notice the square root of the first term, $3x$, is a factor of the second term.

Step 2 - Rewrite in terms of $3x$. The third term is unchanged.

$= (3x)^2 + 7(3x) + 10$

Step 3 - Substitute N for the $3x$.

$= N^2 + 7N + 10$

Step 4 - Factor the new trinomial.

$= (N + 2)(N + 5)$

Step 5 - Replace N with $3x$ to find the final factored form.

$= (3x + 2)(3x + 5)$

Using this substitution method creates a simpler trinomial that we may factor more easily.

1. Use the distributive property, or FOIL, to expand and simplify $(3x + 2)(3x + 5)$. Write the resulting expression in **standard form**. Be prepared to explain if it is equivalent to $9x^2 + 21x + 10$.
2. Study the substitution method and make sense of what was done in each step. Why might the simplified trinomial using N be simpler to factor?

In questions 3 and 4, try this N-substitution method to write each of these expressions in factored form.

3. $4x^2 + 28x + 45$
4. $25x^2 - 35x + 6$

You have probably noticed that the coefficient of the squared term in all of the previous examples is a perfect square. What if that coefficient is not a perfect square?

Here is an example of an expression whose squared term has a coefficient that is not a squared term.

$5x^2 + 17x + 6$

We see that multiplying the leading coefficient by 5 would make the first term into a perfect square, $25x^2$. But, we cannot introduce a new factor that will impact the value of the expression.

We need to find a way to balance the impact of multiplying everything by 5.

Knowing this, we multiply the entire trinomial by $\frac{1}{5}$, since $\frac{1}{5} \cdot 5$ is the same as multiplying by 1.

Step 1 - Multiply by $\frac{1}{5} \cdot 5$.

$= \frac{1}{5} \cdot 5(5x^2 + 17x + 6)$

Step 2 - Distribute the 5 into the trinomial.

$= \frac{1}{5}(25x^2 + 85x + 30)$

Step 3 - Rewrite in terms of the square root of the first term, $5x$.

$= \frac{1}{5}((5x)^2 + 17(5x) + 30)$

Step 4 - Substitute N for $5x$.

$= \frac{1}{5}(N^2 + 17N + 30)$

Step 5 - Factor the trinomial.

$= \frac{1}{5}(N + 15)(N + 2)$

Step 6 - Replace N with $5x$ to find the final factored form.

$= \frac{1}{5}(5x + 15)(5x + 2)$

Step 7 - Factor 5 out of one of the factors.

$= \frac{1}{5} \cdot 5(x + 3)(5x + 2)$

Step 8 - $\frac{1}{5} \cdot 5 = 1$

$= (x + 3)(5x + 2)$

5. Use the distributive property, or FOIL, to expand and simplify $(x + 3)(5x + 2)$. Write the resulting expression in standard form. Be prepared to explain if it is equivalent to $5x^2 + 17x + 6$.

6. Applying the same strategy from the example to a trinomial with a leading term of $6x^2$, what is the multiplication expression equivalent to 1 you would use to rewrite the trinomial? Be prepared to show your reasoning.

For questions 7 and 8, try the N-method to write each of these expressions in factored form.

7. $3x^2 + 16x + 5$

8. $10x^2 - 41x + 4$

Self Check

Find the factored form of the quadratic expression $9x^2 - 6x - 35$.

a. $(3x + 7)(3x - 5)$
b. $(x + 7)(x - 5)$
c. $(3x - 7)(3x + 5)$
d. $(9x + 7)(x - 5)$

Additional Resources

Finding the Factors of Quadratic Expressions in Standard Form

Let's look at a shortcut that can help to factor a complicated **quadratic expression**. This method will be useful only when the **coefficient** of the first term is a perfect square, such as 4 or 9.

EXAMPLE 1

$4x^2 + 4x - 3$

Step 1 - Rewrite the first two terms with $2x$ as the common factor.

$= (2x)^2 + 2(2x) - 3$

Step 2 - Substitute N for $2x$.

$= N^2 + 2N - 3$

Step 3 - Factor the simplified expression.

$= (N + 3)(N - 1)$

Step 4 - Replace N with $2x$.

$= (2x + 3)(2x - 1)$

Here is an example of an expression whose squared term has a coefficient that is not a squared term. A different method can still be applied.

EXAMPLE 2

$5x^2 + 27x - 18$

Step 1 - Multiply by $\frac{1}{5} \cdot 5$ to make a coefficient with a squared term without impacting the overall value of the expression.

$= \frac{1}{5} \cdot 5(5x^2 + 27x - 18)$

Step 2 - Distribute the 5.

$= \frac{1}{5}(25x^2 + 135x - 90)$

Step 3 - Rewrite the first two terms with $5x$ as the common factor.

$= \frac{1}{5}((5x)^2 + 27(5x) - 90)$

Step 4 - Substitute N for $5x$.

$= \frac{1}{5}(N^2 + 27N - 90)$

Step 5 - Factor the simplified expression.

$= \frac{1}{5}(N + 30)(N - 3)$

Step 6 - Replace N with $5x$.

$= \frac{1}{5}(5x + 30)(5x - 3)$

Step 7 - Factor 5 out of one of the expressions.

$= \frac{1}{5} \cdot 5(x + 6)(5x - 3)$

Step 8 - $\frac{1}{5} \cdot 5 = 1$

$= (x + 6)(5x - 3)$

Step 9 - Use the distributive property, or FOIL, to expand $(x + 6)(5x - 3)$. Is it equivalent to $5x^2 + 27x - 18$?

$(x + 6)(5x - 3)$

$= 5x^2 + 30x - 3x - 18$

$= 5x^2 + 27x - 18$

They are equivalent. ✓

TRY IT **Finding the Factors of Quadratic Expressions in Standard Form**

Find the **factored form** of $3x^2 + 25x + 42$.

8.10.5 Solving Quadratic Equations by Any Method

Cool Down

1. Now, combine your new understanding of factoring with using the **zero product property** to solve equations.
   Solve the equation $2x^2 - 7x + 5 = 0$ by any method. Be prepared to explain your reasoning.

Building Character: Curiosity



People typically spend more time learning about things they are curious about and more readily remember the information. Curious people tend to be happier individuals.

Think about your current sense of curiosity. Are the following statements true for you?

- When I didn't know the answer to a question, I couldn't rest until I figured it out.
- I explored a completely new idea or topic—just for the fun of it.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

8.10.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 4, find the factored form of the quadratic expression.

1. $3x^2 + 16x + 5$

a. $(3x + 1)(x + 5)$
b. $(3x - 1)(x - 5)$
c. $(3x + 1)(x - 5)$
d. $(3x - 5)(x - 1)$

2. $2x^2 + 11x - 6$

a. $(2x + 1)(x - 6)$
b. $(2x - 1)(x + 6)$
c. $(2x - 6)(x + 1)$
d. $(2x + 6)(x - 1)$

3. $4x^2 - 13x + 3$

a. $(4x + 1)(x + 3)$
b. $(4x - 3)(x - 1)$
c. $(4x + 3)(x + 1)$
d. $(4x - 1)(x - 3)$

4. $2x^2 - 3x - 2$

a. $(2x - 1)(x + 2)$
b. $(2x - 2)(x + 1)$
c. $(2x + 2)(x - 1)$
d. $(2x + 1)(x - 2)$

5. The height, in meters, of a baseball being hit is modeled by the function $h(x) = -2x^2 + 12x + 3$ where x is the number of seconds since being hit.

Approximately how many seconds pass before the baseball lands on the ground? Use graphing technology if needed.

a. 7.56 seconds
b. 7.26 seconds
c. 6.24 seconds
d. 4.38 seconds

6. The height, in meters, of a roller coaster is modeled by the function $h(x) = -2x^2 + 20x + 5$ where x is the number of seconds after starting.

Approximately how many seconds pass before the roller coaster reaches the ground? Use graphing technology if needed.

a. 10.244 seconds
b. 8.365 seconds
c. 7.963 seconds
d. 5.221 seconds

For questions 7 – 10, find the factored form of the quadratic expression.

7. $3x^2 + 2x - 5$

a. $(3x - 5)(x + 1)$
b. $(3x + 5)(x - 1)$
c. $(x + 5)(3x - 1)$

d. $(3x+5)(x+1)$

8. $4x^2+4x-15$

a. $(2x+3)(2x-5)$
b. $(4x+3)(x-5)$
c. $(2x-3)(2x+5)$
d. $(4x-3)(x+5)$

9. $5x^2-39x-8$

a. $(5x+1)(x-8)$
b. $(5x-8)(x+1)$
c. $(5x+8)(x-1)$
d. $(5x-1)(x+8)$

10. $6x^2+53x-9$

a. $(6x+1)(x-9)$
b. $(2x-1)(3x+9)$
c. $(3x-1)(2x+9)$
d. $(6x-1)(x+9)$

8.10.7 Lesson Summary

In this lesson, you learned how to:

- Use the factored form of a quadratic expression or a graph of a quadratic function to answer questions about a situation.
- Write equivalent expressions in factored form when given quadratic expressions of the form $ax^2 + bx + c$ and a is not 1.

Here are the **activities** that helped you reach those goals:

- 8.10.1: Analyzing Various Quadratic Expressions
  - In this activity, you analyzed different quadratic expressions and found the distinctions that made them unique. Being able to recognize detailed aspects of these expressions helps provide a better understanding that is useful when solving by factoring.
- 8.10.2: Working with Quadratic Factored Form with Leading Coefficient Other Than One
  - In this activity, you learned the process that is involved when attempting to factor a quadratic expression where the leading coefficient is not 1. By using a guess and check method, you were able to solve these expressions.
  - 8.10.2: Self Check
  - 8.10.2: Additional Resources
- 8.10.3: Using Technology to Find Rational Factors
  - In this activity, you applied graphing technology to solve complicated quadratic functions that were not factorable by other means, since the zeros were non-integer rational numbers.
  - 8.10.3: Self Check
  - 8.10.3: Additional Resources
- 8.10.4: Finding the Factors of Quadratic Expressions in Standard Form
  - In this activity, you learned two different methods to factor complicated quadratic equations with leading coefficients greater than 1. You learned to use one method when the leading coefficient was a perfect square and another method when it was not a perfect square.
  - 8.10.4: Self Check
  - 8.10.4: Additional Resources
- 8.10.5: Solving Quadratic Equations by Any Method
  - In this activity, you combined all of the skills you learned in the lesson to factor quadratic equations using any method you preferred.

After these activities, you completed the following **practice**:

- 8.10.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

8.11 WRITING QUADRATIC EQUATIONS GIVEN REAL SOLUTIONS

8.11.0 Lesson Overview

In this lesson, you will reinforce your understanding of the different forms of a quadratic function and translate fluidly between them. When given a function's solutions, factored form, or standard form, you will be able to identify key information about the function.

You will understand what it means for two quadratic functions to be related. Given its solutions and one point, you will be able to specifically identify a specific quadratic function. You will use the skills you have learned to write a quadratic function satisfying the particular parameters given.

When you finish this lesson, you will be able to:

- Find the standard form of a quadratic function given its zeros.
- Find the standard form of a quadratic function given its graph.
- Use the intercept form of a quadratic function to define a specific function given its zeros and a point.

Here are the **activities** that will help you reach those goals:

- 8.11.1: Relating a Quadratic Function and Its Zeros
- 8.11.2: Finding a Quadratic Function from Its Zeros
  - 8.11.2: Self Check
  - 8.11.2: Additional Resources
- 8.11.3: Finding a Quadratic Function from Its Zeros and a Point
  - 8.11.3: Self Check
  - 8.11.3: Additional Resources
- 8.11.4: Writing Quadratics When Given Any Format

After that, you'll **practice and review**.

- 8.11.5: Practice
- 8.11.6: Lesson Summary

8.11.1 Relating a Quadratic Function and Its Zeros

Warm Up

1. Write the quadratic function $f(x) = x^2 - 9x + 20$ in factored form.
2. What are the solutions of the quadratic equation $(x - 4)(x - 5) = 0$?
3. Graph the quadratic function $f(x) = x^2 - 9x + 20$. Use the Desmos graphing tool or technology outside the course.
   Access multimedia content (http://openstax.org/books/algebra-1/pages/8-11-1-relating-a-quadratic-function-and-its-zeros)
   What are the x-intercepts of the graph?
4. Marcus states that the zeros of the function of $f(x) = x^2 - 9x + 20$ are 4 and 5. Do you agree? Be prepared to show your reasoning.

8.11.2 Finding a Quadratic Function from Its Zeros

Activity

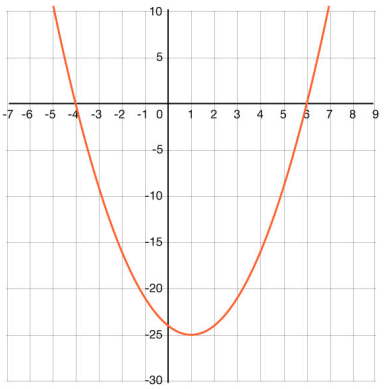
In previous lessons, you learned about the relationship between a quadratic function's zeros and the binomial factors in the factored form of the equation. Use this information to answer the following questions. If needed, review activities 8.9.3 and 8.10.3.

Questions 1 – 3 refer to a quadratic equation that has the zeros –3 and 4.

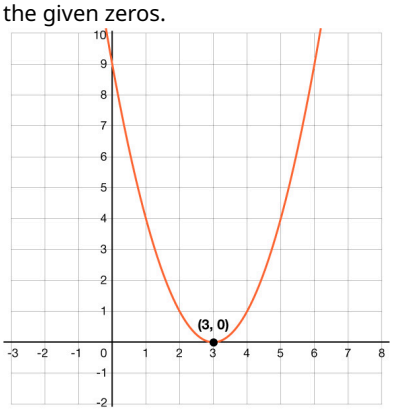
1. Write the factored form of this equation set equal to zero.
2. Now, write the standard form of the same quadratic equation.
3. Write the quadratic equation as the function $f(x)$.

For questions 4 – 8, identify a quadratic function in standard form, $f(x)$, with the given zeros.

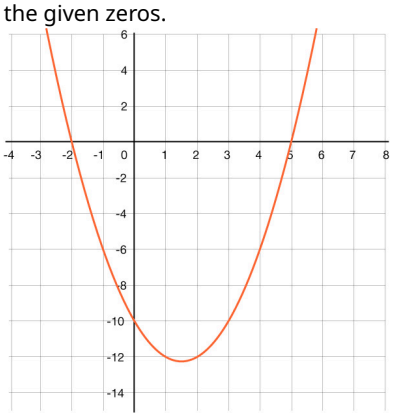
4. 8 and 4
5. −6 and 3
6. $\sqrt{3}$ and $-\sqrt{3}$
7. 5
8. −4
9. Examine the graph below, then identify a quadratic function in standard form, $f(x)$, with the given zeros.



10. Examine the graph below, then identify a quadratic function in standard form, $f(x)$, with the given zeros.



11. Examine the graph below, then identify a quadratic function in standard form, $f(x)$, with the given zeros.



Self Check

Identify a quadratic function, $g(x)$, in standard form, that has the zeros 5 and –7.

a. $g(x) = x^2 - 12x + 35$
b. $g(x) = x^2 + 12x + 35$
c. $g(x) = x^2 - 2x + 35$
d. $g(x) = x^2 + 2x - 35$

Additional Resources

Finding a Quadratic Function from Its Zeros

Given the zeros of a quadratic function, we can write it in standard form.

Let's look at an example.

EXAMPLE 1

Write a function, $h(x)$, that has the zeros 5 and –3.

We know that if the zeros are 5 and –3, then the factor equations that were solved are:

$x - 5 = 0$ and $x + 3 = 0$.

When these equations are solved, they provide the zeros 5 and –3.

We can use these equations to write the function in factored form.

$h(x) = (x - 5)(x + 3)$

By using the distributive property, or FOIL, we can find the standard form of the function.

$h(x) = x^2 - 2x - 15$

Recall that sometimes a function might have only one zero.

EXAMPLE 2

Write a function, $d(x)$, that has one zero at 8.

From previous lessons, we know that when the factor equations were identical, there was only one zero.

$x - 8 = 0$ and $x - 8 = 0$

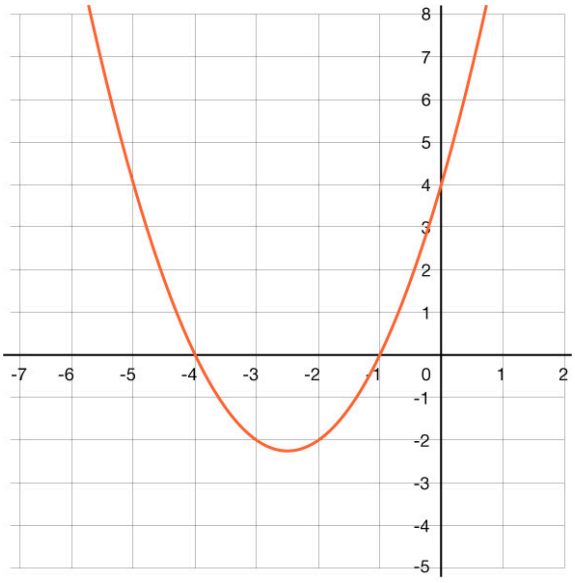
The factored form of this equation is $d(x) = (x - 8)(x - 8)$.

After using the distributive property, or FOIL, the standard form of the function is $d(x) = x^2 - 16x + 64$.

Finding the zeros from a graph follows the same process.

EXAMPLE 3

Write a function, $z(x)$, that corresponds to the graph shown.



The zeros of the graph are –4 and –1. The resulting factor equations are:

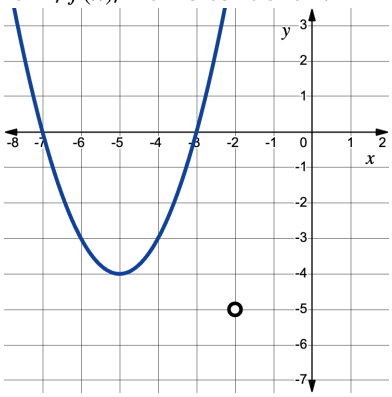
$x + 4 = 0$ and $x + 1 = 0$.

The factored form of this equation is $z(x) = (x + 4)(x + 1)$.

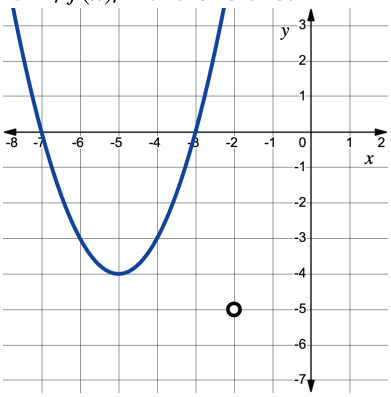
After using FOIL, the standard form of the function is $z(x) = x^2 + 5x + 4$.

TRY IT **Finding a Quadratic Function from Its Zeros**

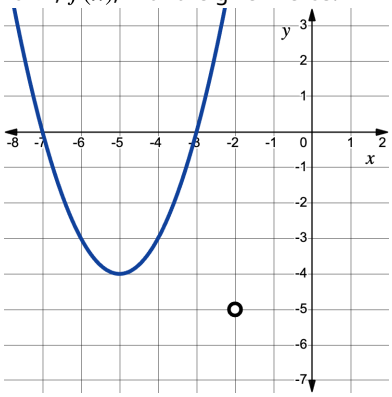
1. Identify a quadratic function in standard form, $f(x)$, with zeros –6 and 1.



2. Identify a quadratic function in standard form, $f(x)$, with the zero –9.



3. Identify a quadratic function in standard form, $f(x)$, with the given zeros.



8.11.3 Finding a Quadratic Function from Its Zeros and a Point

Activity

When finding an exact function that has particular zeros and passes through a certain point, we use the intercept form of a quadratic function:

$$f(x) = a(x - p)(x - q)$$

The variables p and q represent the zeros or roots of the function. The variable a accounts for any constant that might have been factored out when solving.

1. Find a function, $m(x)$, in intercept form, that has the zeros –5 and 8.

Let's use the intercept form of the function above to find the exact function that passes through $(5, -60)$.

2. First, find the value of a. To do this, substitute the values of the point into the intercept form of the function.
3. Substitute the value of a into the intercept form of the function and multiply to find the standard form.

4. Write the quadratic function, $n(x)$, in standard form, that has the zeros 9 and –2 and passes through the point $(0, -36)$.
5. Write the quadratic function, $p(x)$, in standard form, that has the zeros –6 and –4 and passes through the point $(-5, 3)$.
6. Write the quadratic function, $c(x)$, in standard form, that has the zeros 1 and 7 and passes through the point $(9, 16)$.

Video: Finding a Quadratic Function from Its Zeros and a Point

Watch the following video to learn more about finding a quadratic function from its zeros and a point.

Finding a Quadratic Function from Its Zeros and a Point (https://openstax.org/r/quadratic-from-zeros)

Self Check

Write the quadratic function, $z(x)$, in standard form, that has the zeros –1 and 8 and passes through the point $(0, 16)$.

a. $z(x) = x^2 + 7x + 8$
b. $z(x) = 2x^2 - 14x - 16$
c. $z(x) = 3x^2 - 21x - 24$
d. $z(x) = -2x^2 + 14x + 16$

Additional Resources

Finding a Quadratic Function from Its Zeros and a Point

Think about how you would solve the quadratic equation $x^2 - 3x + 2 = 0$.

- The factored form is $(x - 1)(x - 2) = 0$.
- The solutions are $x = 1$ and $x = 2$.

Now think about how you would solve $2x^2 - 6x + 4 = 0$.

- Factor out the GCF, 2.

$2(x^2 - 3x + 2) = 0$

It looks similar to the previous quadratic equation.

- The factored form is $2(x - 1)(x - 2) = 0$.
- Once again, the solutions are $x = 1$ and $x = 2$.

Since they have the same solutions, these two quadratic equations are related. They have the same zeros and the same axis of symmetry.

When we are reversing this process, going from the zeros of a quadratic function to finding its standard form, it is possible there is a multiplier, such as 2 in the example above, that we do not know about.

To account for this, we use the variable a as the multiplier until we find out exactly what it is.

We can incorporate a into the function using the intercept form, $f(x) = a(x - p)(x - q)$. The variables p and q represent the zeros or roots of the function.

Let's look at an example to help us understand.

EXAMPLE 1

Write the quadratic function, $f(x)$, that has the zeros –5 and 4 and passes through the point $(-4, -16)$.

We know the zeros are –5 and 4, so the factored form of the quadratic function is:

$f(x) = (x + 5)(x - 4)$

The intercept form is $f(x) = a(x + 5)(x - 4)$.

We are given one more piece of information that allows us to find the value of a. We know that the function passes

through the point $(-4, -16)$.

We can substitute –4 for x and –16 for the value of the function and solve for a.

$f(x) = a(x + 5)(x - 4)$

$-16 = a((-4) + 5)((-4) - 4)$

$-16 = a(1)(-8)$

$-16 = -8a$

$2 = a$

Since we know $a = 2$, we can substitute 2 for a into the intercept form of the quadratic function.

$f(x) = a(x + 5)(x - 4)$

$f(x) = 2(x + 5)(x - 4)$

$f(x) = 2(x^2 + 5x - 4x - 20)$

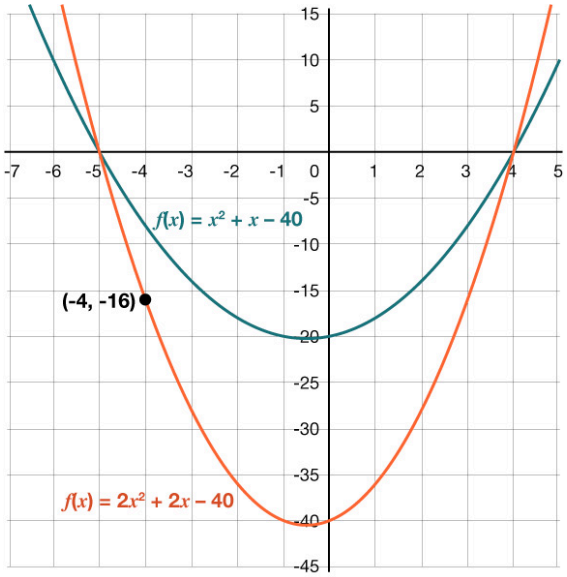
$f(x) = 2(x^2 + x - 20)$

$f(x) = 2x^2 + 2x - 40$

The specific function that has zeros –5 and 4 and passes through the point $(-4, -16)$ is:

$f(x) = 2x^2 + 2x - 40$

Let's graph the functions to check our work.



Notice that $f(x) = x^2 + x - 20$ and $f(x) = 2x^2 + 2x - 40$ have the same zeros. These quadratic functions are related.

Only $f(x) = 2x^2 + 2x - 40$ passes through $(-4, -16)$ and is the specific function we were looking for at the start.

It is very important to note that this is why, in the previous lesson, we were looking for "a function that has zeros at –5 and 4" and not "the function that has zeros at –5 and 4."

This difference in wording is very important, since there are infinitely many functions that are related by their solutions.

We can only determine an exact specific function if we have the additional information of a point through which the function passes.

Let's try one more example.

EXAMPLE 2

Write the quadratic function, $g(x)$, in standard form, that has the zeros –7 and –3 and passes through the point $(-2, -5)$.

Since the zeros are –7 and –3, the factored form of the function is $g(x) = (x + 7)(x + 3)$.

The intercept form of the function is $g(x) = a(x + 7)(x + 3)$.

We substitute values using the point $(-2, -5)$ to find the value of a.

$g(x) = a(x + 7)(x + 3)$

$-5 = a((-2) + 7)((-2) + 3)$

$-5 = a(5)(1)$

$-5 = 5a$

$-1 = a$

Substitute the value of a into the intercept form of the function.

$g(x) = a(x + 7)(x + 3)$

$g(x) = -1(x + 7)(x + 3)$

$g(x) = -1(x^2 + 7x + 3x + 21)$

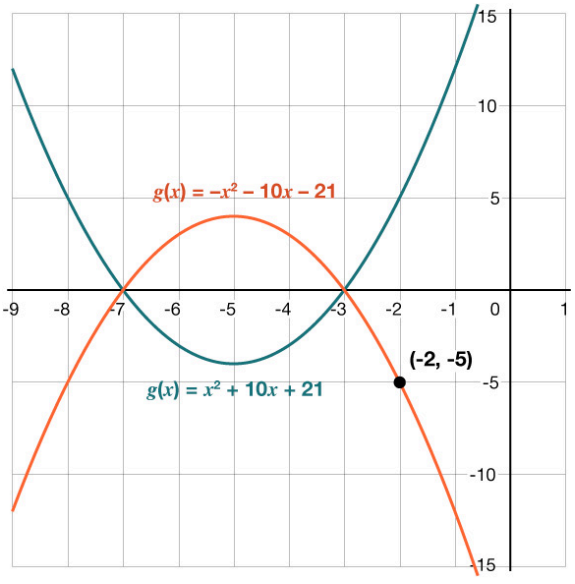
$g(x) = -1(x^2 + 10x + 21)$

$g(x) = -x^2 - 10x - 21$

The specific function that has zeros –7 and –3 and passes through the point $(-2, -5)$ is:

$g(x) = -x^2 - 10x - 21$

Let's graph the functions to check our work.



Again, notice that $g(x) = x^2 + 10x + 21$ and $g(x) = -x^2 - 10x - 21$ have the same zeros. They are related quadratic functions.

Even though the leading coefficient is negative and the function has been reflected over the x-axis, only $g(x) = -x^2 - 10x - 21$ passes through $(-2, -5)$.

TRY IT **Finding a Quadratic Function from Its Zeros and a Point**

Write the quadratic function, $h(x)$, that has the zeros 2 and 6 and passes through the point $(7, 15)$.

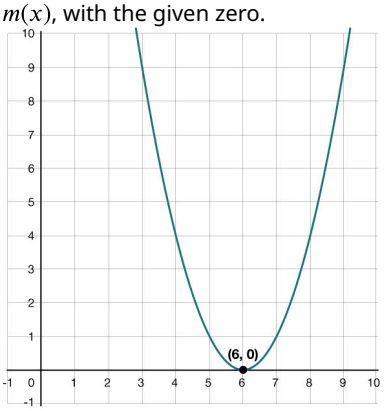
8.11.4 Writing Quadratics When Given Any Format

Cool Down

In this lesson, you have learned how to write a quadratic function from different starting contexts and when provided different information.

1. Identify a quadratic function in standard form, $b(x)$, with the zeros –6 and 8.

2. Identify a quadratic function in standard form, $m(x)$, with the given zero.



3. Write the specific quadratic function, $d(x)$, in standard form, that has the zeros 2 and 5 and passes through the point $(1, 16)$.

8.11.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 – 4, identify a quadratic function in standard form, $f(x)$, with the given zeros.

1. 1 and 9

a. $f(x) = x^2 - 10x + 9$
b. $f(x) = x^2 + 8x + 9$
c. $f(x) = x^2 - 8x + 9$
d. $f(x) = x^2 + 10x + 9$

2. $\sqrt{5}$ and $-\sqrt{5}$

a. $f(x) = x^2 + 5$
b. $f(x) = x^2 - 5$
c. $f(x) = x^2 + 5x - 5$
d. $f(x) = x^2 - 5x - 5$

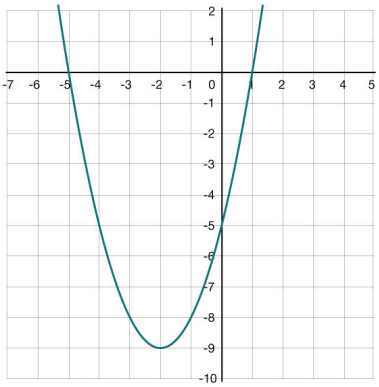
3. -7

a. $f(x) = x^2 - 14x + 49$
b. $f(x) = x^2 - 49$
c. $f(x) = x^2 - 14x - 49$
d. $f(x) = x^2 + 14x + 49$

4. -3

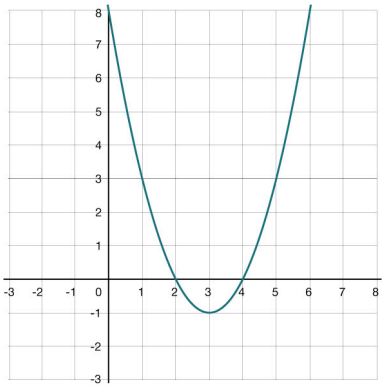
a. $f(x) = x^2 - 9$
b. $f(x) = x^2 + 6x + 9$
c. $f(x) = x^2 + 9$
d. $f(x) = x^2 - 6x + 9$

5. Examine the graph below to identify the zeros of the function. Then, write a quadratic function in standard form with the given zeros.



a. $f(x) = x^2 + 6x - 5$
b. $f(x) = x^2 - 4x - 5$
c. $f(x) = x^2 - 6x - 5$
d. $f(x) = x^2 + 4x - 5$

6. Examine the graph below to identify the zeros of the function. Then, write a quadratic function in standard form with the given zeros.



a. $f(x) = x^2 + 6x + 8$
b. $f(x) = x^2 + 2x - 8$
c. $f(x) = x^2 - 6x + 8$
d. $f(x) = x^2 - 2x - 8$

7. Write the specific quadratic function, $p(x)$, in standard form, that has the zeros –8 and 3 and passes through the point $(2, -20)$.

a. $p(x) = -x^2 - 5x - 6$
b. $p(x) = -x^2 - 5x - 24$
c. $p(x) = 3x^2 + 15x - 72$
d. $p(x) = 2x^2 + 10x - 48$

8. Write the specific quadratic function, $g(x)$, in standard form, that has the zero 2 and passes through the point $(3, -3)$.

a. $g(x) = -3x^2 + 12x - 12$
b. $g(x) = -2x^2 + 8x - 8$
c. $g(x) = 2x^2 - 8x + 3$
d. $g(x) = x^2 - 4x + 4$

9. Write the specific quadratic function, $h(x)$, in standard form, that has the zeros 3 and 5 and passes through the point $(2, 12)$.

a. $h(x) = 3x^2 - 24x + 48$
b. $h(x) = -x^2 + 8x - 15$
c. $h(x) = 4x^2 - 32x + 60$
d. $h(x) = x^2 - 8x + 15$

10. Write the specific quadratic function, $w(x)$, in standard form, that has the zeros 6 and -7 and passes through the point $(5, 12)$.

a. $w(x) = -x^2 - x + 42$
b. $w(x) = -2x^2 - 2x + 84$
c. $w(x) = 2x^2 + 2x - 48$
d. $w(x) = x^2 + x - 42$

8.11.6 Lesson Summary

In this lesson, you learned how to:

- Find the standard form of a quadratic function given its zeros.
- Find the standard form of a quadratic function given its graph.
- Use the intercept form of a quadratic function to define a specific function given its zeros and a point.

Here are the **activities** that helped you reach those goals:

- 8.11.1: Relating a Quadratic Function and Its Zeros
  - In this activity, you reviewed the relationship between a quadratic function's solutions, factored form, and standard form.
- 8.11.2: Finding a Quadratic Function from Its Zeros
  - In this activity, you learned how to find a quadratic function that satisfies the zeros, or solutions, given.
  - 8.11.2: Self Check
  - 8.11.2: Additional Resources
- 8.11.3: Finding a Quadratic Function from Its Zeros and a Point
  - In this activity, you learned to identify a specific quadratic function that has particular solutions and passes through a given point.
  - 8.11.3: Self Check
  - 8.11.3: Additional Resources
- 8.11.4: Writing Quadratics When Given Any Format
  - In this activity, you combined the skills learned in this lesson to identify quadratic functions that meet specific criteria without any additional context.

After these activities, you completed the following **practice**:

- 8.11.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

8.12 USING TECHNOLOGY TO FIND THE QUADRATIC REGRESSION

8.12.0 Lesson Overview

Scientists and mathematicians frequently collect data from their experiments. These data sets can be analyzed to better understand a situation. In this lesson, you will use graphing technology to analyze data sets so that you can identify a curve of best fit for any given data. You will then use the curve of best fit to make accurate predictions about real-world situations.

When you finish this lesson, you will be able to:

- Graph a quadratic data set using technology.
- Find a regression curve, or curve of best fit, using a quadratic data set.
- Use a regression curve to create a quadratic model for making predictions.
- Make predictions about a real-world situation using a quadratic data set.

Here are the **activities** that will help you reach those goals:

- 8.12.1: Graphing a Quadratic Data Set Using Technology
- 8.12.2: Finding the Curve of Best Fit
  - 8.12.2: Self Check
  - 8.12.2: Additional Resources
- 8.12.3: Making Predictions Using a Quadratic Model
  - 8.12.3: Self Check
  - 8.12.3: Additional Resources
- 8.12.4: Finding the Missing Data in the Set

After that, you'll **practice and review**.

- 8.12.5: Practice
- 8.12.6: Lesson Summary

8.12.1 Graphing a Quadratic Data Set Using Technology

Warm Up

1. Click on the "+" sign and choose to add a table in the Desmos graphing tool. This will generate a table for entering a data set.

Desmos: Quadratic Data Set Table

Access multimedia content (http://openstax.org/books/algebra-1/pages/8-12-1-graphing-a-quadratic-data-set-using-technology)

2. Enter the data set shown in the table below into the corresponding x_1 and y_1 columns.

(If preferred, the data can be copied and pasted into Desmos from a spreadsheet using keyboard shortcuts. Windows: Ctrl+C for copy and Ctrl+V for paste. Mac: Command+C for copy and Command+V for paste.)

x_1	y_1
1	2
2	5
3	10

x_1	y_1
4	17
5	26

The window should be appropriately set to display the points in the table, but sometimes it will be necessary to adjust the window of the graph in order to properly view your data set. Use the mouse to drag and change the viewing window. To zoom in or out, use the buttons in the top right or the scrolling wheel on the mouse.

3. Examine the data for a pattern that might fit any quadratic function that you recognize. What equation do you hypothesize would best fit these data points?

In this lesson, we will learn to use technology to analyze a data set, find a curve of fit, and make predictions about a situation given the data that we have.

Why Should I Care?



Safe-Rain produces equipment for some pretty amazing fountains. Some of their fountains shoot jets of water that can form incredible visual displays and can be synchronized to music.

The path of a jet of water can be modeled with a quadratic function. When Safe-Rain designs a new fountain, they use quadratic functions to plan precisely how high and far each jet of water will go. Without quadratic equations to guide it, the water would spray everywhere.

8.12.2 Finding the Curve of Best Fit

Activity

In a previous lesson, you learned how to find the line of best fit. A line of best fit is a line on a graph that best approximates the equation of the points in a data set or scatter plot.

In this activity, we will learn how to find the **curve of best fit**, which is the curve modeled by the equation that best approximates the points in a data set or scatter plot. Since our data set is based on a quadratic function, we must use a curve instead of a line. This process, called **quadratic regression**, is the process of determining the equation of a parabola that best fits a data set.

1. Examine the data set and graph given in the Desmos graphing tool below. Access multimedia content (http://openstax.org/books/algebra-1/pages/8-12-2-finding-the-curve-of-best-fit)
2. Examine the data. Do they appear to you to form a line or a curve?

3. We will use graphing technology to find a curve of best fit. Click the pointer below the table into entry box #2 to enter a new function. The cursor should start blinking. Enter the equation:
$y_1 \sim ax_1^2 + bx_1 + c$
   - Notice that when you type the 1 after y or x, Desmos automatically places it as subscript.
   - Also notice that we are not using an equal sign. The symbol ~ means approximately and tells Desmos to approximate a line or curve of best fit. It should be in the top-left corner of your keyboard, and you might press it using the Shift button. If you are using the Desmos on-screen keyboard, it will be on the ABC keyboard.
   - After pressing the subscript 1 after the y and the first x, use the ^ symbol to tell Desmos you would like to enter an exponent.

4. 1. Once you have entered the curve of best fit, you can press Enter. The curve of best fit will appear. Your screen will look like this:

   Access multimedia content (http://openstax.org/books/algebra-1/pages/8-12-2-finding-the-curve-of-best-fit)
   Look in the lower-left corner of the screen. We will use the a, b, and c values to construct the equation for our curve of best fit.

5. Enter the value of a.
   What is the value of a rounded to the nearest hundredth?

6. Enter the value of b.
   What is the value of b rounded to the nearest hundredth?

7. Enter the value of c.
   What is the value of c rounded to the nearest hundredth?

8. Write an equation in the form $y = ax^2 + bx + c$ for the curve of best fit using the values you found.

You may notice the R^2 value. The closer this value is to 1, the better your curve of best fit matches the data set. The R^2 value for this data set is 0.9972. Since this value is relatively close to 1, this curve of best fit is a reasonably accurate match to the data set.

Self Check

Access multimedia content (http://openstax.org/books/algebra-1/pages/8-12-2-finding-the-curve-of-best-fit)
Find the equation of the curve of best fit. Round each coefficient to the nearest hundredth.

Which is the equation of the curve of best fit?

a. $y = -0.91x^2 + 2.23x + 16.42$
b. $y = -0.94x^2 + 4.51x + 16.96$
c. $y = -0.97x^2 + 7.62x + 19.34$
d. $y = -0.92x^2 + 4.25x + 8.36$

Additional Resources

Finding the Curve of Best Fit

Let's find the curve of best fit for a given data set.

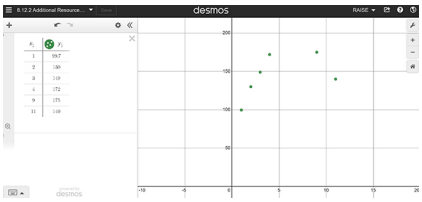
Use the Desmos graphing tool to determine the equation.

Access multimedia content (http://openstax.org/books/algebra-1/pages/8-12-2-finding-the-curve-of-best-fit)
Step 1 - Click the + symbol in the top left and select a table.

Enter the data set as shown. Press Enter.

x_1	y_1
1	99.7
2	130
3	149
4	172
9	175
11	140

The data set will populate the graph as shown here.



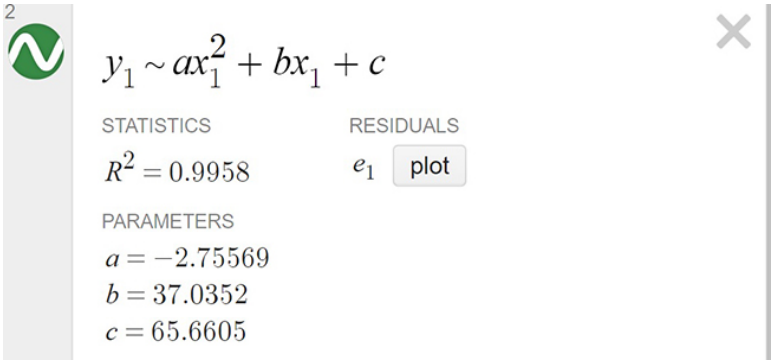
Step 2 - On the left side, click in box #2 for a new item. Enter the instruction line below to generate a curve of best fit.

$y_1 \sim ax_1^2 + bx_1 + c$

- Remember from the activity that Desmos will automatically place the 1 as subscript after an x or y.
- Don't forget to use the symbol ~ that means approximately.
- Use the ^ symbol to enter an exponent.

As soon as you enter the instruction line, Desmos will generate the curve of best fit.

Step 3 - Use the coefficients generated to write the equation for the curve of best fit.



Round each coefficient to the nearest hundredth.

The curve of best fit is $y = -2.76x^2 + 37.04x + 65.66$.

TRY IT **Finding the Curve of Best Fit**

Find the equation of the curve of best fit. Round each coefficient to the nearest hundredth.

Access multimedia content (http://openstax.org/books/algebra-1/pages/8-12-2-finding-the-curve-of-best-fit)

8.12.3 Making Predictions Using a Quadratic Model

Activity

A helium balloon is released and floats up before coming back down to the ground.

The following data set shows the height in feet, y, of the balloon corresponding to the number of minutes passed, x.

x_1 (minutes)	y_1 (height in feet)
1	99.7
1.5	130
3	149
4.5	172
7	108
9	84

1. Use the Desmos below. Click the + symbol in the top left and select a table. Enter the data set into the table.
   Access multimedia content (http://openstax.org/books/algebra-1/pages/8-12-3-making-predictions-using-a-quadratic-model)
2. Use the instruction line, $y_1 \sim ax_1^2 + bx_1 + c$, to find the curve of best fit.
   What is the equation of the curve of best fit? Round coefficients to the nearest hundredth.
3. How high will the balloon be after 4 minutes? (Hint: Graph the line $x = 4$ to help identify where it intersects with the curve of best fit.)
4. How high will the balloon be after 10 minutes?
5. At what time will the balloon first be at a height of 95 feet? (Graph the line $y = 95$.)
6. At what time will the balloon return to a height of 95 feet?
7. When will the balloon reach the ground, or have a height of 0 feet?

Video: Using Technology to Find The Quadratic Regression

Watch the following video to learn more about finding quadratic regression.

Using Technology to Find The Quadratic Regression (https://openstax.org/r/quadratic-regression)

Self Check

Access multimedia content (http://openstax.org/books/algebra-1/pages/8-12-3-making-predictions-using-a-quadratic-model)

A baseball is thrown into the air. The following data set shows the height in feet, y, of the baseball corresponding to the number of seconds passed, x.

Approximately how long will it be before the baseball hits the ground?

a. 7.18 seconds
b. 6.75 seconds
c. 5.74 seconds
d. 2.73 seconds

Additional Resources

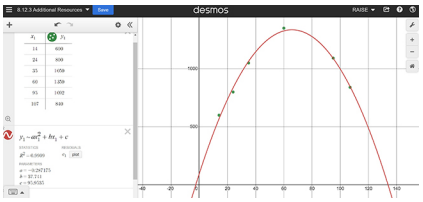
Making Predictions Using a Quadratic Model

A filtration system is being tested in a new aquarium. A pollutant is continuously added to the aquarium, and the filtration system is turned on until it has removed all of the pollutant from the water.

The following data set shows the milliliters (mL) of pollutant present, y, corresponding to the number of minutes that have passed, x, since the experiment started.

Access multimedia content (http://openstax.org/books/algebra-1/pages/8-12-3-making-predictions-using-a-quadratic-model)

Let's enter the instruction line, $y_1 \sim ax_1^2 + bx_1 + c$, to find the curve of best fit.

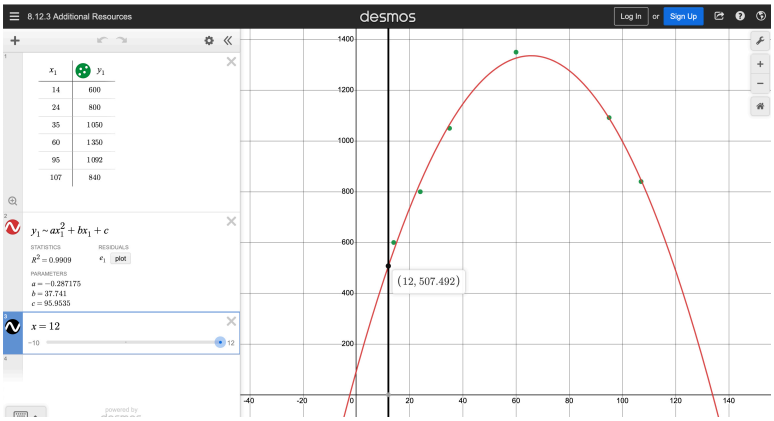


The curve of best fit is determined to be $y = -0.29x^2 + 37.74x + 95.95$.

We can use this curve of best fit to answer questions and make predictions about the amount of pollutant that will be in the water (mL) and the time it takes to completely clean the tank of pollutant.

How many mL of pollutant will be in the water after 12 minutes?

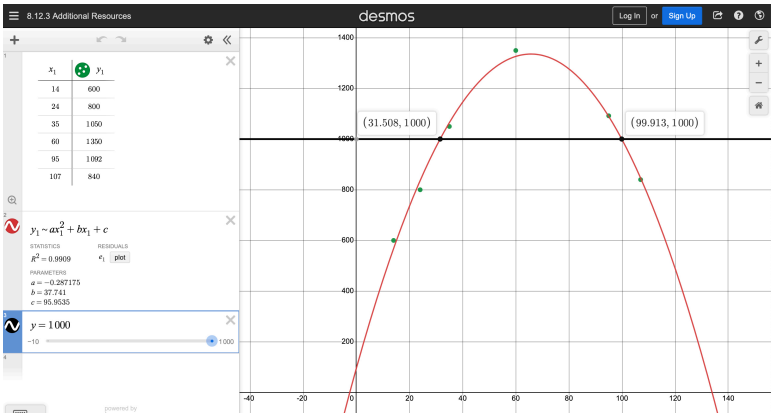
- To answer this question, we want to know the y-value on the curve of best fit when the x-value is 12.
- We can graph the line $x = 12$ and see where it intersects with the curve of best fit.



- By moving the cursor over the intersection point, we see that the y-value is approximately 507.49.
- This means that after 12 minutes, there will be approximately 507.49 mL of pollutant in the water.

At what time(s) will there be 1000 mL of pollutant in the water?

- To answer this question, we want to know the x-value on the curve of best fit when the y-value is 1000.
- We can graph the line $y = 1000$ and see where it intersects with the curve of best fit.



- The line $y = 1000$ intersects with the curve of best fit at two points. This means there are two times when there is 1000 mL of pollutant. We see that the x-values are approximately 31.51 and 99.91.
- In other words, there will be 1000 mL of pollutant at approximately 31.51 minutes and again at approximately 99.91 minutes after the experiment starts.

How long after the experiment has started will the filtration system have removed all of the pollutant from the aquarium?

- To answer this question, we want to know the x-value of the x-intercept of the curve of best fit.
- If we graph $y = 0$, we can click on the intersection point to find its coordinates. The x-intercept of the curve of best fit is $(133.92, 0)$.
- So, approximately 133.92 minutes after the experiment starts, there will be no pollutant left in the water.

TRY IT **Making Predictions Using a Quadratic Model**

Use the same data set from the example in the section above.

Access multimedia content (http://openstax.org/books/algebra-1/pages/8-12-3-making-predictions-using-a-quadratic-model)

The data set shows the milliliters (mL) of pollutant present, y, corresponding to the number of minutes that have passed, x, since the experiment started.

1. How many mL of pollutant will be in the water after 47 minutes?

2. At what time(s) will there be 1215 mL of pollutant in the water?

8.12.4 Finding the Missing Data in the Set

Cool Down

Thomasina used a quadratic data set to create a curve of best fit for her data. Then she accidentally spilled paint on her paper and lost some of the entries in the table.

Access multimedia content (http://openstax.org/books/algebra-1/pages/8-12-4-finding-the-missing-data-in-the-set)

She needs to work backward to fill in the missing entries in her data set. Use the curve of best fit she created to complete the table.

1.

x_1	y_1
6	
	452.06
	632.24
58.2	
	176.12

8.12.5 Practice

Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

Complete the following questions to practice the skills you have learned in this lesson.

Access multimedia content (http://openstax.org/books/algebra-1/pages/8-12-5-practice)

1. Enter the given data into Desmos, then calculate the regression approximation to determine the curve of best fit.

x_1	y_1
2	56
5	116
8	155
11	205
17	230

What is the equation for the curve of best fit that models the data?

a. $y = -0.93x^2 + 22.63x + 10.48$
b. $y = -0.69x^2 + 24.83x + 8.07$
c. $y = -3.11x^2 + 16.32x + 8.94$
d. $y = -0.7x^2 + 35.93x + 8.19$

2. Enter the given data into Desmos, then calculate the regression approximation to determine the curve of best fit.

x_1	y_1
3	105
9	250
14	349
23	400
37	258

What is the equation for the curve of best fit that models the data?

a. $y = -0.74x^2 + 34.32x + 7.44$
b. $y = -0.62x^2 + 17.45x + 9.64$
c. $y = -2.31x^2 + 24.711x + 3.25$
d. $y = -0.56x^2 + 38.74x + 5.31$

Access multimedia content (http://openstax.org/books/algebra-1/pages/8-12-5-practice)

3. Enter the given data into Desmos, then calculate the regression approximation to determine the curve of best fit.

x_1	y_1
0.2	6.5
0.35	6.74
0.66	6.31
1	4.95
1.5	1.2

What is the equation for the curve of best fit that models the data?

a. $y = -5.1x^2 + 9.2x + 7.14$
b. $y = -4.31x^2 + 3.2x + 6.08$
c. $y = -3.1x^2 + 2.6x + 7.93$
d. $y = -6.3x^2 + 8x + 10.6$

Access multimedia content (http://openstax.org/books/algebra-1/pages/8-12-5-practice)

4. Enter the given data into Desmos, then calculate the regression approximation to determine the curve of best fit.

x_1	y_1
0.3	25
0.6	36
1.2	45
1.8	33
2.3	12

What is the equation for the curve of best fit that models the data?

a. $y = -35.78x^2 + 45.21x + 3.74$
b. $y = -3.72x^2 + 7.34x + 8.67$
c. $y = -30.84x^2 + 49.61x + 8.99$
d. $y = -25.41x^2 + 59.28x + 9.64$

Access multimedia content (http://openstax.org/books/algebra-1/pages/8-12-5-practice)
Access multimedia content (http://openstax.org/books/algebra-1/pages/8-12-5-practice)
For questions 5 – 9, use the following information.

The path of a toy rocket after launch is modeled by the following data set. The height of the rocket, in feet, y, corresponds to the number of seconds passed, x.

x_1	y_1
0.4	20
1	41

x_1	y_1
1.5	52
2.3	60
3.5	52

5. What is the equation for the curve of best fit that models the flight of the toy rocket?

a. $y = -9.2x^2 + 46x + 3.51$
b. $y = -4.2x^2 + 34x + 6.34$
c. $y = -11.9x^2 + 30.2x + 3.64$
d. $y = -6.1x^2 + 31x + 1.24$

6. Approximately how high will the rocket be 2 seconds after launch?

a. 61.00 feet
b. 58.71 feet
c. 53.60 feet
d. 45.91 feet

7. At what approximate times after launch will the rocket have a height of 20 feet?

a. 0.82 seconds after launch and 4.18 seconds after launch
b. 0.52 seconds after launch and 4.48 seconds after launch
c. 0.39 seconds after launch and 4.61 seconds after launch
d. 0.26 seconds after launch and 4.74 seconds after launch

8. Approximately how long will it take the rocket to reach its maximum height?

a. 5.08 seconds
b. 3.20 seconds
c. 2.50 seconds
d. 1.73 seconds

9. Approximately how long will it take for the rocket to land on the ground?

a. 6.12 seconds
b. 5.08 seconds
c. 4.63 seconds
d. 2.50 seconds

For question 10, Maria has found a curve of best fit using the equation below.

$$y = -0.46x^2 + 14x + 17$$

10. Maria is asked to predict the y-value on her curve when $x = 10$. To do this, she graphs the function $x = 10$ and analyzes the intersection point of the curve and the line.

What is the y-value Maria is asked to predict?

a. 111
b. 105
c. 93
d. 10

Access multimedia content (http://openstax.org/books/algebra-1/pages/8-12-5-practice)

8.12.6 Lesson Summary

In this lesson, you learned how to:

- Graph a quadratic data set using technology.
- Find a regression curve, or curve of best fit, using a quadratic data set.
- Use a regression curve to create a quadratic model for making predictions.
- Make predictions about a real-world situation using a quadratic data set.

Here are the **activities** that helped you reach those goals:

- 8.12.1: Graphing a Quadratic Data Set Using Technology
  - In this activity, you learned how to represent a data set using Desmos. You looked for a visual pattern in a sample data set.
- 8.12.2: Finding the Curve of Best Fit
  - In this activity, you used graphing technology to generate a curve of best fit when given a data set.
  - 8.12.2: Self Check
  - 8.12.2: Additional Resources
- 8.12.3: Making Predictions Using a Quadratic Model
  - In this activity, you found a curve of best fit from a data set that you used to make predictions about a real-world situation.
  - 8.12.3: Self Check
  - 8.12.3: Additional Resources
- 8.12.4: Finding the Missing Data in the Set
  - In this activity, you populated a sample data set from a curve of best fit.

After these activities, you completed the following **practice**:

- 8.12.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

PROJECT 8: MODELING ROCKET FLIGHT

Project 8 Overview

In this culminating project, you will use your knowledge of quadratic equations in different forms to solve problems.

In the first activity, you will be given a graph of a quadratic function and asked to write the function in factored form. Next, you will determine how many solutions a function has based on an understanding of its factored form. You will also be asked to explain the characteristics of the graph of a quadratic function with no solution.

In the second activity, you will work in pairs to rewrite quadratic expressions in factored and standard form using the strategies learned in the unit.

Finally, in the third activity, you will find the specific equation in standard form of a quadratic function that has certain zeros and passes through a particular point, given a real-world situation. Then, you will use the equation to make a prediction about the situation, given the context of the problem.

Project Activities

1. Relating the Graphs of Quadratic Equations to Their Factored Form by Their Zeros
2. Converting Between Factored Form and Standard Form
3. Modeling Rocket Flight Using a Quadratic Equation

Learning Targets

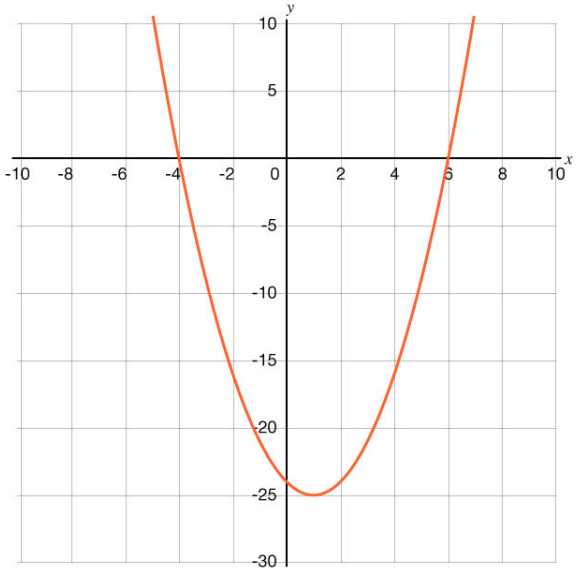
When you finish this project, you will have demonstrated how to:

- Explain why quadratic equations can have no solutions and explain why there are none.
- Recognize quadratic equations that have zero, one, or two solutions when they are written in factored form.
- Write equivalent expressions in factored form when given quadratic expressions of the form ax2 + bx + c and a is not 1.
- Use the intercept form of a quadratic function to define a specific function given its zeros and a point.

Relate Zeros to the Factored Form of Quadratic Equations

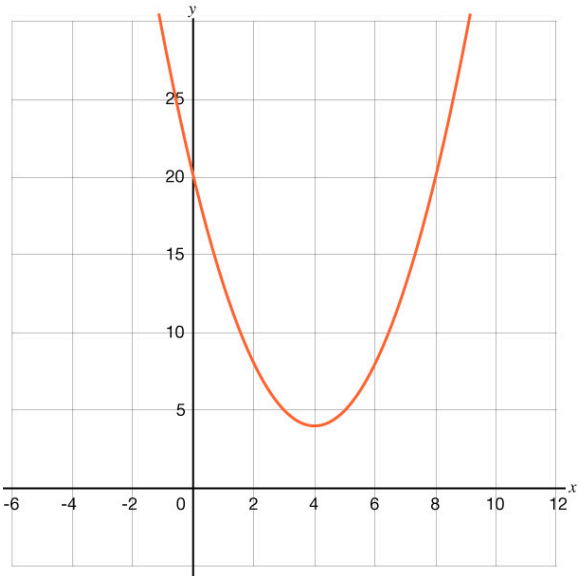
In this unit, you learned about the relationship between the factored form of a quadratic equation, its zeros, and its graph.

1. What is the factored form of the quadratic equation represented by the graph? Explain how you found your answer.



2. How many times does the graph of the equation $(x-5)(x-5)=0$ intersect with the x-axis? Be prepared to show your reasoning.

3. The equation $x^2 - 8x + 20 = 0$ is shown in the graph. How many solutions does the equation have? Be prepared to show your reasoning.



Convert between Factored Form and Standard Form

You have learned strategies for writing a quadratic expression in its factored form and standard form. In each round of this activity, you and your partner will be given equivalent quadratic expressions in different forms.

The goal of each round is to write the quadratic expression as it has been given to your partner. If your expression is in factored form, write it in standard form. If your expression is in standard form, write it in factored form. At the end of each round, compare the expressions with your partner without revealing the expressions for future rounds.

If, in any round, you need help, you may ask your partner to explain the next step without revealing the solution.

Proceed from rounds 1 to 6 while discussing your thinking if there is any disagreement.

Model a Rocket Flight Using a Quadratic Equation

Alejandra lights the fuse on a small toy rocket. The fuse burns down and, after 1 second, the rocket launches into the air. The rocket lands back on the ground after 6 seconds.

After 5 seconds from when Alejandra lit the fuse, the height of the rocket was 8 meters.

1. What is the quadratic equation in standard form that models the flight of the rocket? (Hint: Think of the problem as a quadratic function in which you know the zeros and a point through which the function passes.)
2. Use the equation you found in the previous question to find the height of the rocket, in meters, 3 seconds after Alejandra lit the fuse.

Unit 8 Wrap Up

Throughout this unit, you learned how to solve different quadratic equations. Equations such as these are used to model many different situations, such as the water arcs SafeRain creates using their water jets and fountains.

In this unit, you interpreted, wrote, and solved quadratic equations. You found that writing and solving quadratic equations enabled you to find input values that produce certain output values. You started solving quadratic equations by reasoning and then learned to use the square root property.

You learned that applying the zero product property to equations set equal to zero in their factored form allowed you to find their solutions. You developed fluency in translating quadratic equations between their standard forms and factored forms. You learned to easily convert these equations between their factored form and a list of their solutions or a display of their graph. You also found that not all equations in their standard form can easily be written into factored form.

You learned how to find a quadratic equation in general form when given only its zeros. You explored finding a specific quadratic equation that has particular zeros and passes through a certain point. You discovered that there are related quadratic equations that share the same zeros and axis of symmetry when graphed.

You used graphing technology to perform quadratic regression on a data set to find a curve of best fit. You used the curve of best fit to make predictions about a problem in a real-world context.



Access our Unit 8 Student Self-Assessment (https://openstax.org/r/unit-8-self-assesment) to reflect on the skills you learned during this unit. Congratulations on completing Unit 8!

Building Character: Curiosity

Throughout this unit, you learned that **curiosity** is important for helping you explore different possibilities and seek multiple ways to solve problems. Curiosity will keep you interested in learning and help you find new solutions to difficult problems.

UNIT 9 OVERVIEW AND READINESS

Unit 9 Overview

Why Should I Care?

One of the goals of a business is to make money. But how does a company ensure that it is making a profit?

Watch the video to learn how companies, like The Ensemble Theatre, use quadratic equations on a regular basis to help them be profitable.

Unit 9 Overview (https://openstax.org/r/the-ensemble-theater)

In this unit, you will learn more about quadratic equations. You will see that there are multiple methods to solving quadratic equations. In addition to completing the square, another method is the quadratic formula. The quadratic formula is useful for solving quadratic equations that may not be conveniently solved by completing the square. It also provides a set formula method that can be used to solve any quadratic equation. Understanding how to use quadratic equations is important across many industries, from managing a business to engineering. For example, engineers use quadratic equations for designs that involve curved surfaces such as auto bodies, boats, or other vehicles.

Building Character: Intellectual Humility

In a business, such as The Ensemble Theatre, employees must learn from opposing views and have constructive discussions in their work. Similarly, in this course, you admit when you do not know or understand something and appreciate your classmates' and teachers' insights, and let them know when they raise a point that you hadn't considered. Many jobs require the ability to be willing to change your mind and let people know when you do.

Throughout this unit, you will work on developing your **intellectual humility**. When you approach life with intellectual humility, you open your mind to learning. You are able to learn from opposing views and have more constructive discussions, even when you disagree. No matter how old you are, with intellectual humility you become wiser. It helps you be less judgmental of others, learn more in school, and be a better leader.

Think about yourself. How many of these things are true?

- I question my own opinions, positions, and viewpoints because they could be wrong.
- I reconsider my opinions when presented with new evidence.
- I recognize the value in opinions that are different from my own.
- I accept that my beliefs and attitudes may be wrong.
- In the face of conflicting evidence, I am open to changing my opinions.
- I like finding out new information that differs from what I already think is true.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

Am I Ready to Learn This?

Getting Ready for Unit 9

To be ready for this unit, you need to be able to:

- Simplify radicals.
- Factor perfect square trinomials.
- Use vertex form to name the transformations made to $y = x^2$.

Next, you will answer some questions that will help you see how ready you are for this unit. If you need help, some mini-lessons will help you brush up on anything you need to work on.

Simplifying Radicals: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #1: Simplifying Radicals

Simplify $\sqrt{45}$.

a. $5\sqrt{9}$

b. $9\sqrt{5}$
c. $5\sqrt{3}$
d. $3\sqrt{5}$

Simplifying a Square Root

Some square roots can be simplified using the product property. Here is the product property:

Product Property of n^{th} Roots

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, and $n \geq 2$ is an integer, then

$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ and $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

Here is how the product property can be used to simplify square roots:

How to Simplify a Radical Expression Using the Product Property

Step 1 - Find the largest factor in the radicand that is a perfect power of the index. Use that factor to rewrite the radicand as a product of two factors.

Step 2 - Use the product rule to rewrite the radical as the product of two radicals.

Step 3 - Simplify the root of the perfect power.

EXAMPLE

Find the simplest form of $\sqrt{98}$.

Step 1 - Find the largest factor in the radicand that is a perfect power of the index. Use that factor to rewrite the radicand as a product of two factors. We see that 49 is the largest factor of 98 that has a power of 2. In other words, 49 is the largest perfect square factor of 98.

$98 = 49 \cdot 2$ Always write the perfect square factor first. $\sqrt{49 \cdot 2}$

Step 2 - Use the product rule to rewrite the radical as the product of two radicals.

$\sqrt{49} \cdot \sqrt{2}$

Step 3 - Simplify the root of the perfect power.

$7\sqrt{2}$

TRY IT **Simplifying a Square Root**

Simplify $\sqrt{48}$.

Check Your Understanding

Find the simplest form of $\sqrt{500}$.

a. $10\sqrt{5}$

b. $100\sqrt{5}$

c. $5\sqrt{20}$

d. $2\sqrt{125}$

Video: Simplifying Square Roots Using Perfect Squares

Watch the following video to learn more about how to simplify square roots using perfect squares.

Khan Academy: Simplifying Square Roots

Simplifying square roots | Exponents, radicals, and scientific notation | Pre-Algebra | Khan Academy (https://www.youtube.com/embed/cw3mp8oNASk)

Factoring Perfect Square Trinomials: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #2: Factoring Perfect Square Trinomials

Which of the following expressions is equivalent to $4x^2 - 12x + 9$?

a. $4(x^2 - 3x + 9)$
b. $(2x - 3)(2x + 3)$
c. $(2x + 3)^2$
d. $(2x - 3)^2$

Factor Perfect Square Trinomials

Some trinomials are perfect squares. They result from multiplying a binomial times itself.

$$\begin{pmatrix} a & + & b \\ 3x & + & 4 \end{pmatrix}^2$$

$$a^2 \ + \ 2 \cdot a \cdot b \ + \ b^2$$

$$(3x)^2 + 2(3x \cdot 4) + 4^2$$

$$9x^2 + 24x + 16$$

The trinomial $9x^2 + 24x + 16$ is called a **perfect square trinomial**. It is the square of the binomial $3x + 4$.

If you recognize that the first and last terms are squares and the trinomial fits the perfect square trinomial pattern, you will save yourself a lot of work.

Here is the pattern: the reverse of the binomial squares pattern.

Perfect Square Trinomials Pattern

If a and b are real numbers, then
$a^2 + 2ab + b^2 = (a + b)^2$
$a^2 - 2ab + b^2 = (a - b)^2$

EXAMPLE

Here is an example of how to factor a perfect square trinomial.

Factor $9x^2 + 12x + 4$.

Step 1 - Does the trinomial fit the perfect square trinomials pattern, $a^2 + 2ab + b^2$?

- Is the first term a perfect square? Write it as a square, a^2.
- Is the last term a perfect square? Write it as a square, b^2.
- Check the middle term. Is it $2ab$?

Is $9x^2$ a perfect square? Yes: write it as $(3x)^2$. Is 4 a perfect square? Yes: write it as $(2)^2$. Is $12x$ twice the product of $3x$ and 2? Does it match? Yes, so we have a perfect square trinomial.

$$\begin{array}{ccccc} 9x^2 & + & 12x & + & 4 \\ (3x)^2 & & & & \\ (3x)^2 & & & & (2)^2 \\ (3x)^2 & & & & (2)^2 \\ \searrow & & & & \swarrow \\ & & 2(3x)(2) & & \\ & & 12x & & \end{array}$$

Step 2 - Write the square of the binomial.

$9x^2 + 12x + 4$

$a^2 + 2 \cdot a \cdot b + b^2$

$(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$

$(a + b)^2$

$(3x + 2)^2$

Step 3 - Check.

$(3x)^2 + 2 \cdot 3x + 2 + 2^2$

$(3x)^2 + 2 \cdot 3x + 2 + 2^2$

$9x^2 + 12x + 4$✓

TRY IT **Factor Perfect Square Trinomials**

Factor $81y^2 - 72y + 16$.

Check Your Understanding

Which of the following is equivalent to $36y^2 + 12y + 1$?

a. $(6y + 1)(6y - 1)$

b. $(6y + 1)^2$

c. $(6y - 1)^2$

d. $12(3y^2 + y + 1)$

Videos: Factoring Perfect Squares and Identifying Perfect Square Form

Watch the following videos to learn more about factoring perfect square trinomials and perfect square form.

Khan Academy: Factoring Perfect Squares

Example 4: Factoring quadratics as a perfect square of a difference: (a-b)^2 | Khan Academy (https://www.youtube.com/embed/liRNTieIU_k)

Khan Academy: Identifying Perfect Square Form

Identifying perfect square form | Mathematics II | High School Math | Khan Academy (https://www.youtube.com/embed/4STGvuoBi5U)

Transformations: Mini-Lesson Review

? **MINI LESSON QUESTION**

Question #3: Transformations

Name the transformations performed from $f(x)$ to $g(x) = f(x-3)+4$.

a. Right 3, Down 4
b. Right 4, Down 3
c. Left 3, Up 4
d. Right 3, Up 4

Transforming Functions

A **transformation** is a move to a parent (or original) function. Three types of transformations include those that cause the graph to shift vertically or horizontally or reflect the function over the x- or y-axis.

Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.

Horizontal Shift

Given a function f, a new function $g(x) = f(x-h)$, where h is a constant, is a **horizontal shift** of the function f. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

How to Sketch a Graph Given a Function and Both a Vertical and a Horizontal Shift

Step 1 - Identify the vertical and horizontal shifts from the formula.

Step 2 - The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.

Step 3 - The horizontal shift results from a constant subtracted from the input. Move the graph right for a positive constant and left for a negative constant.

Step 4 - Apply the shifts to the graph in either order.

Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the y-axis.

EXAMPLE

Name the transformations from $f(x)$ to form $g(x) = -f(x + 4) - 8$.

Horizontal Shift
Left 4

Vertical Shift
Down 8

Reflection?
Yes, over the x-axis.

TRY IT **Transforming Functions**

Name the transformations from $f(x)$ to form $g(x) = -f(x + 3) - 2$.

Check Your Understanding

Name all of the transformations from $f(x)$ to form $g(x) = -f(x + 5) - 1$.

a. Left 1, Up 5, Reflect over the x-axis
b. Left 5, Down 1, Reflect over the x-axis
c. Left 5, Down 1, Reflect over the y-axis
d. Right 5, Down 1, Reflect over the x-axis

Video: Identifying Transformations

Watch the following video to see how to identify transformations, including shifts, of functions.

Khan Academy: Shifting Functions Introduction

Shifting functions introduction | Transformations of functions | Algebra 2 | Khan Academy (https://www.youtube.com/embed/RttvubuBhAE)

Unit 9 Overview: Wrap Up

- Simplifying radicals.
- Factoring perfect square trinomials.
- Using vertex form to name the transformations made to $y = x^2$.

If you are still struggling, be sure to reach out to your teacher for additional help. Nobody is good at problems like this the first try. Keep at it and it will get easier.

9.1 WHAT ARE PERFECT SQUARES?

9.1.0 Lesson Overview

In this lesson, you will gain understanding of the structure of perfect-square expressions. This lesson will also help you see that perfect squares can be handy for solving equations because we can find their square roots.

When you finish this lesson, you will be able to:

- Recognize perfect-square expressions written in different forms.
- Recognize quadratic equations that have a perfect-square expression and solve the equations.

Here are the **activities** that will help you reach those goals:

- 9.1.1: Equations with Quadratic Expressions on Both Sides of the Equal Sign
- 9.1.2: Recognizing Structure in Perfect-Square Expressions
  - 9.1.2: Self Check
  - 9.1.2: Additional Resources
- 9.1.3: Solving Multi Step Quadratic Equations Using Perfect Squares
  - 9.1.3: Self Check
  - 9.1.3: Additional Resources
- 9.1.4: Explain Why an Expression Is a Perfect Square

After that, you'll **practice and review**.

- 9.1.5: Practice
- 9.1.6: Lesson Summary

9.1.1 Equations with Quadratic Expressions on Both Sides of the Equal Sign

Warm Up

In each equation, what expression could be substituted for a so the equation is true for all values of x?

1. $x^2 = a^2$

2. $(3x)^2 = a^2$

3. $a^2 = 7x \cdot 7x$

4. $25x^2 = a^2$

5. $a^2 = \frac{1}{4}x^2$

6. $a^2 = (x+1)^2$

7. $(2x-9)(2x-9) = a^2$

9.1.2 Recognizing Structure in Perfect-Square Expressions

Activity

1. Each expression is written as the product of factors. Write an equivalent expression in standard form.

a. $(3x)^2$

b. $7x \cdot 7x$

c. $(x+4)(x+4)$

d. $(x+1)^2$

e. $(x-7)^2$

f. $(x+n)^2$

2. Why do you think the following expressions can be described as **perfect square**s?

$x^2 + 6x + 9$ $x^2 - 16x + 64$ $x^2 + \frac{1}{3}x + \frac{1}{36}$

Are you ready for more?

Extending Your Thinking

1. Write each expression in factored form.
   $x^4 - 30x^2 + 225$
2. $x + 14\sqrt{x} + 49$
3. $5^{2x} + 6 \cdot 5^x + 9$

Self Check

Which of the following is equivalent to $(x - 4)^2$?

a. $x^2 + 8x + 16$
b. $x^2 - 8x + 16$
c. $x^2 + 16$
d. $x^2 - 16$

Additional Resources

Creating Perfect Square Trinomials

A **perfect square trinomial** is created by multiplying a binomial by itself.

EXAMPLE

Write $(x - 2)^2$ as an expanded polynomial in standard form:

Step 1 - Multiply the binomial by itself.

$(x - 2)(x - 2)$

Step 2 - Use the **distributive property** to expand (FOIL).

$x^2 - 2x - 2x + 4$

Step 3 - Combine like terms.

$x^2 - 4x + 4$

TRY IT **Creating Perfect Square Trinomials**

Write $(x + 5)^2$ as an expanded polynomial in standard form.

9.1.3 Solving Multi Step Quadratic Equations Using Perfect Squares

Activity

Han and Jada solved the same **quadratic equation** with different methods. Here they are:

Han's Method	Jada's Method
$(x-6)^2 = 25$	$(x-6)^2 = 25$
$(x-6)(x-6) = 25$	$\sqrt{(x-6)^2} = \sqrt{25}$
$x^2-12x+36 = 25$	$x-6 = \pm 5$
$x^2-12x+11 = 0$	$x-6 = 5$
$(x-11)(x-1) = 0$	$x-6 = -5$
$(x-11) = 0$	$x = 11$ *or* $x = 1$
$(x-1) = 0$	
$x = 11$ *or* $x = 1$	

Work with a partner to solve these equations. For each equation, one partner solves with Han's method, and the other partner solves with Jada's method. Make sure both partners get the same solutions to the same equation. If not, work together to find your mistakes.

1. $(y-5)^2 = 49$

2. $(x+4)^2 = 9$

3. $(z+\frac{1}{3})^2 = 49$

4. $(v-0.1)^2 = 0.36$

Video: Learning About Solving Multi Step Quadratic Equations Using Perfect Squares

Watch the following video to learn more about solving multi step quadratic equations using perfect squares.

Learning About Solving Multi Step Quadratic Equations Using Perfect Squares (https://openstax.org/r/quadratic-perfect-squares)

Self Check

Find all solutions to $(x-3)^2 = 25$.

a. $x = 28$
b. $x = 22, x = 28$
c. $x = -2, x = 8$
d. $x = 8$

Additional Resources

Two Ways to Solve Quadratic Equations

In the activity, Han and Jada used two different methods for solving a quadratic equation, using the **distributive property** and the **square root property**.

Using the Distributive Property to Solve Quadratic Equations

How to Solve a Quadratic Equation Using the Distributive Property:

Step 1 - Isolate the quadratic term and make its coefficient 1.

Step 2 - Rewrite the term so the binomial is multiplied by itself.

Step 3 - Use the distributive property or FOIL.

Step 4 - Combine like terms so that the equation is set equal to 0.

Step 5 - Factor.

Step 6 - Set each factor equal to 0.

Step 7 - Solve.

Step 8 - Check the solutions.

EXAMPLE

Solve $4(x-7)^2 = 36$

Step 1 - Isolate the quadratic term (divide by 4).
$(x-7)^2 = 9$

Step 2 - Rewrite the term so the binomial is multiplied by itself.

$(x-7)(x-7) = 9$

Step 3 - Use the distributive property or FOIL.

$x^2 - 7x - 7x + 49 = 9$

Step 4 - Combine like terms so that the equation is set equal to 0.

$x^2 - 14x + 40 = 0$

Step 5 - Factor.

$(x-10)(x-4) = 0$

Step 6 - Set each factor equal to 0.

$x - 10 = 0,\ x - 4 = 0$

Step 7 - Solve.

$x - 10 = 0 \qquad x - 4 = 0$

$x = 10 \qquad x = 4$

Step 8 - Check the solutions.

$4(10-7)^2 = 36 \qquad 4(4-7)^2 = 36$

$4(3)^2 = 36 \qquad 4(-3)^2 = 36$

$36 = 36 \qquad 36 = 36$

TRY IT **Using the Distributive Property to Solve Quadratic Equations**
Use the distributive property to solve $3(x-4)^2 = 48$.

If $x^2 = k$, then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k} \quad \text{or} \quad x = \pm\sqrt{k}$$

How to Solve a Quadratic Equation Using the Square Root Property:

Step 1 - Isolate the quadratic term and make its coefficient one.

Step 2 - Use the square root property. Take the square root of both sides of the equation and remember the "plus or minus."

Step 3 - Rewrite the equation as two equations.

Step 4 - Solve each equation.

Step 5 - Check the solutions.

EXAMPLE

Solve $4(x-7)^2 = 36$

Step 1 - Isolate the quadratic term (divide by 4).

$(x-7)^2 = 9$

Step 2 - Use the square root property.

$\sqrt{(x-7)^2} = \sqrt{9}$

$x - 7 = \pm 3$

Step 3 - Rewrite the equation as two equations.

$x - 7 = -3$

$x - 7 = 3$

Step 4 - Solve each equation.

$x - 7 = -3$ $\quad$ $x - 7 = 3$

$x = 4$ $\quad$ $x = 10$

Step 5 - Check the solutions.

$4(4-7)^2 = 36$ $\quad$ $4(10-7)^2 = 36$

$4(-3)^2 = 36$ $\quad$ $4(3)^2 = 36$

$36 = 36$ $\quad$ $36 = 36$

TRY IT **Using the Square Root Property to Solve Quadratic Equations**

Use the square root property to solve $3(x-4)^2 = 48$.

9.1.4 Explain Why an Expression Is a Perfect Square

Cool Down

1. Explain why it makes sense to call the expression $x^2 + 20x + 100$ a **perfect square**.
2. Solve $x^2 + 20x + 100 = 81$.

9.1.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. **Select three** expressions that are perfect squares.

a. $(x+5)(x+5)$
b. $(-9+c)(c-9)$
c. $(y-10)(10-y)$
d. $(2x-1)(2x+1)$
e. $(4-3x)(3-4x)$
f. $(a+b)(b+a)$

2. Each diagram represents the square of an expression or a perfect square. Complete the cells in the last table by answering questions a–d.

Diagram 1

$(n+7)^2$

	n	7
n	n^2	$7n$
7	$7n$	7^2

Diagram 2

$(5-m)^2$

	5	$-m$
5	5^2	$-5m$
$-m$	$-5m$	$(-m)^2$

Diagram 3

$(h+\frac{1}{3})^2$

	h	$\frac{1}{3}$
h	a	b
$\frac{1}{3}$	c	d

a. What is the answer for Cell a in Diagram 3?
   a. $(\frac{1}{3})^2$
   b. $(\frac{1}{3}h)^2$
   c. h^2
   d. $\frac{1}{3}h$

b. What is the answer for Cell b in Diagram 3?
   a. $(\frac{1}{3})^2$
   b. $(\frac{1}{3}h)^2$
   c. h^2
   d. $\frac{1}{3}h$

c. What is the answer for Cell c in Diagram 3?
   a. $(\frac{1}{3})^2$
   b. $(\frac{1}{3}h)^2$
   c. h^2
   d. $\frac{1}{3}h$

d. What is the answer for Cell d in Diagram 3?
   a. $(\frac{1}{3})^2$
   b. $(\frac{1}{3}h)^2$
   c. h^2
   d. $\frac{1}{3}h$

3. Rewrite the perfect-square expression $(h + \frac{1}{3})^2$ in standard form $ax^2 + bx + c$.

a. $h^2 + \frac{1}{9}h + \frac{2}{3}$
b. $h^2 + \frac{2}{6}h + \frac{2}{6}$
c. $h^2 + \frac{2}{3}h + \frac{1}{9}$
d. $2h + \frac{2}{3}h + \frac{1}{9}$

4. Rewrite the perfect-square expression $(5 - m)^2$ in standard form $ax^2 + bx + c$.

a. $25m^2 - 10$
b. $m^2 - 10m + 25$
c. $m^2 + 10m - 25$
d. $-12m + 10$

5. Rewrite the perfect-square expression $(n + 7)^2$ in standard form $ax^2 + bx + c$.

a. $n^2 + 14n + 49$
b. $n^2 + 49n + 14$
c. $n^2 + 14n + 14$
d. $16n + 49$

For 6 - 9, solve each equation, then choose the **two** solutions.

6. $(x - 1)^2 = 4$

a. $x = -1$
b. $x = 5$
c. $x = 2$
d. $x = 3$
e. $x = 6$

7. $(x + 5)^2 = 81$

a. $x = 76$
b. $x = -14$
c. $x = 86$
d. $x = 3$
e. $x = 4$

8. $(x + 11)^2 = 121$

a. $x = 132$
b. $x = 110$
c. $x = 0$
d. $x = -22$
e. $x = 4$

9. $(x - 7)^2 = \frac{64}{49}$

a. $x = 2$
b. $x = -\frac{279}{79}$
c. $x = \frac{41}{7}$
d. $x = \frac{407}{49}$
e. $x = \frac{57}{7}$

10. Solve the equation $(x - 2)^2 = 0$.

9.1.6 Lesson Summary

In this lesson, you learned how to:

- Recognize perfect-square expressions written in different forms.
- Recognize quadratic equations that have a perfect-square expression and solve the equations.

Here are the **activities** that helped you reach those goals:

- 9.1.1: Equations with Quadratic Expressions on Both Sides of the Equal Sign
  - In this activity, you determined what needed to be substituted for a in order to create a true equation.
- 9.1.2: Recognizing Structure in Perfect-Square Expressions
  - In this activity, you multiplied binomial expressions to create perfect squares.
  - 9.1.2: Self Check
  - 9.1.2: Additional Resources
- 9.1.3: Solving Multi Step Quadratic Equations Using Perfect Squares
  - In this activity, you solved quadratic equations using two different methods. One method involved taking the square root of both sides of the equation. The other method involved multiplying two binomials and then factoring.
  - 9.1.3: Self Check
  - 9.1.3: Additional Resources
- 9.1.4: Explain Why an Expression Is a Perfect Square
  - In this activity, you explained why we call an expression a perfect square and you solved a factored form of a quadratic equation.

After these activities, you completed the following **practice**:

- 9.1.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

9.2 COMPLETING THE SQUARE, PART 1

9.2.0 Lesson Overview

In this lesson, you will learn that if we rearrange and rewrite the expression on one side of a quadratic equation to be a perfect square (that is, if we complete the square) we can find the solutions of the equation.

When you finish this lesson, you will be able to:

- Explain what it means to "complete the square" and describe how to do it.
- Solve quadratic equations by completing the square and finding square roots.

Here are the **activities** that will help you reach those goals:

- 9.2.1: Different Forms of Perfect Squares
- 9.2.2: Standard and Factored Forms of Perfect Squares
  - 9.2.2: Self Check
  - 9.2.2: Additional Resources
- 9.2.3: Completing the Square
  - 9.2.3: Self Check
  - 9.2.3: Additional Resources
- 9.2.4: Making a Perfect Square

After that, you'll **practice and review**.

- 9.2.5: Practice
- 9.2.6: Lesson Summary

9.2.1 Different Forms of Perfect Squares

Warm Up

1. **Select three** expressions that are perfect squares.
   a. $(x+5)(5+x)$
   b. $(x+5)(x-5)$
   c. $(x-3)^2$
   d. $x-3^2$
   e. $x^2+8x+16$
   f. $x^2+10x+20$
2. Explain how you know the expressions you selected are perfect squares.

9.2.2 Standard and Factored Forms of Perfect Squares

Activity

1. What is the factored form of x^2+6x+9?
2. What is the factored form of $x^2-10x+25$?
3. What is the standard form of $(x-7)^2$?
4. What is the value of c in the expression $x^2-20x+c$?
5. If $x^2-20x+c$ is factored to be $(x-\ \ )^2$, what is the missing value of the factored form of this expression?
6. What is the value of c in the expression $x^2+16x+c$?
7. If $x^2+16x+c$ is factored to be $(x-\ \ )^2$, what is the missing value of the factored form of this expression?
8. What is the value of c in the expression x^2+7x+c?

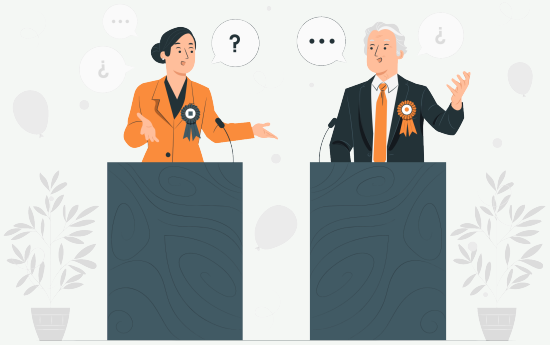
9. If $x^2 + 7x + c$ is factored to be $(x - \quad)^2$, what is the missing value of the factored form of this expression?

10. What is the value of c in the expression $x^2 + bx + c$?

11. If $x^2 + bx + c$ is factored to be $(x - \quad)^2$, what is the missing value of the factored form of this expression?

Notice that finding the value of c helped to create a **perfect square trinomial**. This is called **completing the square**.

Building Character: Intellectual Humility



When you approach life with intellectual humility, you open your mind to learning. You are able to learn from opposing views and have more constructive discussions, even when you disagree.

Think about yourself. Are these statements true for you?

- I question my own opinions, positions, and viewpoints because they could be wrong.
- In the face of conflicting evidence, I am open to changing my opinions.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

Self Check

Which value will create a perfect square trinomial when added to $x^2 - 14x +$ _____?

a. 196
b. 49
c. -49
d. -7

Additional Resources

Completing the Square

When a number is added to a **quadratic expression** to make a **perfect square trinomial**, that is called **completing the square**.

Here is how to complete the square of a quadratic expression, $ax^2 + bx + c$.

How to Complete a Square of $x^2 + bx$

Step 1 - Identify b, the coefficient of x.

Step 2 - Find half of b and then square it. In other words, $(\frac{b}{2})^2$.

Step 3 - Add the $(\frac{b}{2})^2$ to $x^2 + bx$.

Step 4 - Factor the perfect square trinomial, writing it as a binomial squared.

EXAMPLE

Complete the square to make a **perfect square trinomial**. Then write the result as a binomial squared.

$x^2 - 26x$

Step 1 - Identify b, the coefficient of x.

The coefficient of x is -26.

Step 2 - Find half of b and then square it. In other words, $(\frac{b}{2})^2$.

$(-\frac{26}{2})^2 = (-13)^2 = 169$

Step 3 - Add the $(\frac{b}{2})^2$ to $x^2 + bx$.

$x^2 - 26x + 169$

Step 4 - Factor the perfect square trinomial, writing it as a binomial squared.

$(x - 13)^2$

TRY IT **Completing the Square**

Complete the square to make a **perfect square trinomial**. Then, factor the trinomial.

$x^2 + 8x + ____$

9.2.3 Completing the Square

Activity

One technique for solving **quadratic equation**s is called **completing the square**. Here are two examples of how Diego

and Mai completed the square to solve the same equation.

Diego	Mai
$x^2+10x+9 = 0$	$x^2+10x+9 = 0$
$x^2+10x = -9$	$x^2+10x+9+16 = 16$
$x^2+10x+25 = -9+25$	$x^2+10x+25 = 16$
$x^2+10x+25 = 16$	$(x+5)^2 = 16$
$(x+5)^2 = 16$	$\sqrt{(x+5)^2} = \sqrt{16}$
$\sqrt{(x+5)^2} = \sqrt{16}$	$(x+5) = \pm 4$
$x+5 = \pm 4$	$x+5=4$ and $x+5=-4$
$x+5=4$ or $x+5=-4$	$x=-1$ or $x=-9$
$x=-1$ and $x=-9$	

Study the worked examples. Then, try solving these equations by completing the square for questions 1 – 5.

1. $x^2+6x+8=0$

2. $x^2+12x=13$

3. $0=x^2-10x+21$

4. $x^2-2x+3=83$

5. $x^2+40=14x$

Video: Solve Equations by Completing the Square

Watch the following video to learn more about solving equations by completing the square.

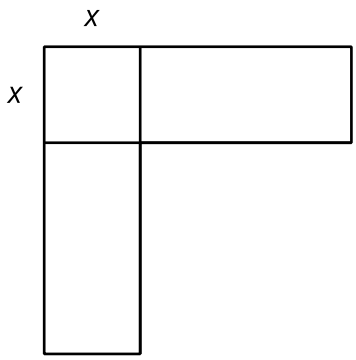
Solve Equations by Completing the Square (https://openstax.org/r/complete-square)

Are you ready for more?

Extending Your Thinking

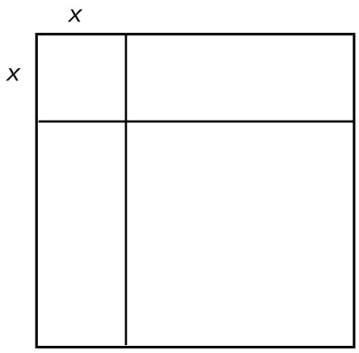
Here is a diagram made of a square and two congruent rectangles. Its total area is x^2+35x square units.

Diagram A



1. What is the length of the unlabeled side of each of the two rectangles in Diagram A?

2. In Diagram B, we add lines to make the figure a square. The area of the square becomes a quadratic equation.



Calculate the equation, then put that equation in the standard form ax^2+bx+c to answer questions a and b.

a. Using the standard form ax^2+bx+c to represent the area of the square in Diagram B, what is the value of the coefficient b?

b. Using the standard form $ax^2 + bx + c$ to represent the area of the square in Diagram B, what is the value of the constant c?

3. How is the process of finding the area of the entire figure like the process of building perfect squares for expressions like $x^2 + bx$?

Self Check

Find the solutions to $x^2 - 10x = -9$ by completing the square.

a. $x = 9$
b. $x = 21$
c. $x = 1, x = 9$
d. $x = 25$

Additional Resources

Solving Quadratic Equations by Completing the Square

EXAMPLE

Solve $x^2 + 8x = 48$ by **completing the square**.

Step 1 - Isolate the variable terms on one side and place the constant term on the other. This equation has all the variables on the left.

$ax^2 + bx = c$

$x^2 + 8x = 48$

Step 2 - Find $(\frac{b}{2})^2$, the number to complete the square. Add to both sides of the equation. Take half of 8 and square it.

$4^2 = 16$

Add 16 to BOTH sides of the equation.

$x^2 + 8x + ____ = 48$

$(\frac{b}{2})^2$

$x^2 + 8x + 16 = 48 + 16$

Step 3 - Factor the **perfect square trinomial** as a binomial square.

$x^2 + 8x + 16 = (x + 4)^2$

Substitute the binomial square term for the trinomial.

$(x + 4)^2 = 64$

Step 4 - Use the square root property.

$\sqrt{(x + 4)^2} = \sqrt{64}$

$x + 4 = \pm 8$

Step 5 - Simplify the radical and then solve the two resulting equations.

$x + 4 = \pm 8$ $x + 4 = 8$

$x + 4 = -8$ $x = 4$

$x = -12$

Step 6 - Check the solutions. Put each answer in the original equation to check.

Substitute $x = 4$.

$x^2 + 8x = 48$

$(4)^2 + 8(4) \stackrel{?}{=} 48$

$16 + 32 \stackrel{?}{=} 48$

$48 = 48$✔

Substitute $x = -12$.

$x^2 + 8x = 48$

$(-12)^2 + 8(-12) \stackrel{?}{=} 48$

$144 - 96 \stackrel{?}{=} 48$

$48 = 48$✔

TRY IT **Solving Quadratic Equations by Completing the Square**

Solve $x^2 + 4x = 5$ by completing the square.

9.2.4 Making a Perfect Square

Cool Down

For questions 1 – 3, what could be added to each expression to make it a perfect square?

1. What could be added to make the expression a perfect square?
   $x^2 + 12x$
2. What could be added to make the expression a perfect square?
   $x^2 - 6x + 1$
3. What could be added to make the expression a perfect square?
   $x^2 + 14x - 10$
4. Solve the equation $x^2 - 16x = -60$ by **completing the square**. Show your reasoning.

9.2.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For questions 1 - 5, the standard form of an equation is $ax^2 + bx + c$. For each expression, find the missing constant c that would make the expression a perfect square.

1. $x^2 - 6x + __c__$
2. $x^2 + 2x + __c__$
3. $x^2 + 14x + __c__$
4. $x^2 - 4x + __c__$
5. $x^2 + 24x + __c__$

6. If $x^2 - 6x + c$ is factored to be $(x_____)^2$, what is the missing value of the factored form of this expression? Remember to include the sign.

7. If $x^2 + 2x + c$ is factored to be $(x_____)^2$, what is the missing value of the factored form of this expression? Remember to include the sign.

8. If $x^2 + 14x + c$ is factored to be $(x_____)^2$, what is the missing value of the factored form of this expression? Remember to enter the sign.

9. If $x^2 - 4x + c$ is factored to be $(x_____)^2$, what is the missing value of the factored form of this expression? Remember to enter the sign.

10. If $x^2 + 24x + c$ is factored to be $(x_____)^2$, what is the missing value of the factored form of this expression? Remember to enter the sign.

For each equation in questions 11 - 14, use the following scenario.

Jada looks at Mai's work and is confused. She doesn't see how Mai got her answer.

Complete Mai's missing steps by answering the questions to help Jada see how Mai solved the equation.

$x^2 + 12x = 13$

Step 1 - $x^2 + 12x + _____ = 13 + _____$

Step 2 - $(x + 6)^2 = 49$

Step 3 - $\sqrt{(x + 6)^2} = \sqrt{49}$

Step 4 - $x + 6 = \pm _____$

Step 5 - $x + 6 = _____$

Step 6 - $x + 6 = - _____$

$x = 1$ or $x = -13$

11. What is the missing term in Step 1?
12. What is the missing term in Steps 4, 5, and 6?
13. What is the value of x in Step 5?
14. What is the value of x in Step 6?

For questions 15 - 20, solve choose the equivalent equation with a perfect square on one side.

15. $x^2 + 8x = 2$

a. $(x+2)^2 = 9$
b. $(x+1)^2 = 1$
c. $(x+4)^2 = 18$
d. $(x-10)^2 = 91$
e. $(x+5)^2 = 12$
f. $(x-7)^2 = 54$

16. $x^2 + 10x = -13$

a. $(x+2)^2 = 9$
b. $(x+1)^2 = 1$
c. $(x+4)^2 = 18$
d. $(x-10)^2 = 91$
e. $(x+5)^2 = 12$
f. $(x-7)^2 = 54$

17. $x^2 - 14x = 5$

a. $(x+2)^2 = 9$
b. $(x+1)^2 = 1$
c. $(x+4)^2 = 18$
d. $(x-10)^2 = 91$
e. $(x+5)^2 = 12$
f. $(x-7)^2 = 54$

18. $x^2 + 2x = 0$

a. $(x+2)^2 = 9$
b. $(x+1)^2 = 1$
c. $(x+4)^2 = 18$
d. $(x-10)^2 = 91$
e. $(x+5)^2 = 12$
f. $(x-7)^2 = 54$

19. $x^2 + 4x - 5 = 0$

a. $(x+2)^2 = 9$
b. $(x+1)^2 = 1$
c. $(x+4)^2 = 18$
d. $(x-10)^2 = 91$
e. $(x+5)^2 = 12$
f. $(x-7)^2 = 54$

20. $x^2 - 20x = -9$

a. $(x+2)^2 = 9$
b. $(x+1)^2 = 1$
c. $(x+4)^2 = 18$
d. $(x-10)^2 = 91$
e. $(x+5)^2 = 12$
f. $(x-7)^2 = 54$

For questions 21–23, solve each equation by completing the square, then choose the **two** roots.

21. $x^2 - 6x + 5 = 12$

a. $x = -1$
b. $x = 5$
c. $x = 2$
d. $x = 7$

e. $x = 6$

22. $x^2 - 2x = 8$

a. $x = 9$
b. $x = -2$
c. $x = 8$
d. $x = 3$
e. $x = 4$

23. $11 = x^2 + 4x - 1$

a. $x = 14$
b. $x = 110$
c. $x = -6$
d. $x = 2$
e. $x = 4$

24. Solve the equation $x^2 - 18x + 60 = -21$ by completing the square.

a. $x = -39$
b. $x = 78$
c. $x = 39$
d. $x = 9$
e. $x = -21$

9.2.6 Lesson Summary

In this lesson, you learned how to:

- Explain what it means to "complete the square" and describe how to do it.
- Solve quadratic equations by completing the square and finding square roots.

Here are the **activities** that helped you reach those goals:

- 9.2.1: Different Forms of Perfect Squares
  - In this activity, you identified which expressions were perfect squares.
- 9.2.2: Standard and Factored Forms of Perfect Squares
  - In this activity, you rewrote perfect squares given in standard form into factored form and vice versa.
  - 9.2.2: Self Check
  - 9.2.2: Additional Resources
- 9.2.3: Completing the Square
  - In this activity, you solved quadratic equations using completing the square strategy.
  - 9.2.3: Self Check
  - 9.2.3: Additional Resources
- 9.2.4: Making a Perfect Square
  - In this activity, you determined what needed to be added to a polynomial to make a perfect square trinomial.

After these activities, you completed the following **practice**:

- 9.2.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

9.3 COMPLETING THE SQUARE, PART 2

9.3.0 Lesson Overview

In this lesson, you will learn that completing the square can be used to solve any quadratic equation, including equations that involve rational numbers that are not integers.

When you finish this lesson, you will be able to:

- Solve by completing the square when given a quadratic equation in which the coefficient of the squared term is 1.

Here are the **activities** that will help you reach those goals:

- 9.3.1: Solving Equations with Fractions
- 9.3.2: Using Completing the Square to Solve Equations
  - 9.3.2: Self Check
  - 9.3.2: Additional Resources
- 9.3.3: Solve by Completing the Square
  - 9.3.3: Self Check
  - 9.3.3: Additional Resources
- 9.3.4: Completing the Square with Fractions

After that, you'll **practice and review**.

- 9.3.5: Practice
- 9.3.6: Lesson Summary

9.3.1 Solving Equations with Fractions

Warm Up

Solve each equation mentally.

1. $x + x = \frac{1}{4}$

2. $(\frac{3}{2})^2 = x$

3. $\frac{3}{5} + x = \frac{9}{5}$

4. $\frac{1}{12} + x = \frac{1}{4}$

9.3.2 Using Completing the Square to Solve Equations

Activity

Completing the square can be used to solve quadratic equations.

EXAMPLE

Solve $x^2 + 8x - 2 = 0$.

To complete the square to solve an equation, follow these steps:

Step 1 - Write the equation in the form $ax^2 + bx = c$.

$x^2 + 8x = -2$

Step 2 - Complete the square by finding $(\frac{b}{2})^2$.

$(\frac{b}{2})^2 = (\frac{8}{2})^2 = 16$

Step 3 - Add the value to both sides of the equation.

$x^2 + 8x + 16 = -2 + 16$

Step 4 - Factor the **perfect square** trinomial and simplify the constants.

$(x+4)^2 = 14$

Step 5 - Take the square root of each side (remember the plus-minus sign with the constant).

$\sqrt{(x+4)^2} = \sqrt{14}$

$x+4 = \pm\sqrt{14}$

Step 6 - Write two separate equations and solve each.

$x+4 = \sqrt{14}$ or $x+4 = -\sqrt{14}$ $x = -4+\sqrt{14}$ or $x = -4-\sqrt{14}$

Solve each equation by completing the square. In part a, select the value of $(\frac{b}{2})^2$ that will complete the square. In part b, choose the **two** roots.

1. $(x-3)(x+1) = 5$

a. What is the correct value of $(\frac{b}{2})^2$ that will complete the square?

b. **Select two** roots that solve the quadratic equation.

a. $x = -2$
b. $x = 1$
c. $x = 8$
d. $x = 4$
e. $x = 9$

2. $x^2 + \frac{1}{2}x = \frac{3}{16}$

a. What is the correct value of $(\frac{b}{2})^2$ that will complete the square?

b. **Select two** roots that solve the quadratic equation.

a. $x = \frac{1}{2}$
b. $x = \frac{1}{4}$
c. $x = -\frac{3}{4}$
d. $x = -\frac{1}{4}$
e. $x = 1$

3. $x^2 + 3x + \frac{8}{4} = 0$

a. What is the correct value of $(\frac{b}{2})^2$ that will complete the square?

b. **Select two** roots that solve the quadratic equation.

a. $x = -2$
b. $x = \frac{1}{16}$
c. $x = 9$
d. $x = \frac{1}{4}$
e. $x = -1$

4. $(7-x)(3-x)+3 = 0$

a. What is the correct value of $(\frac{b}{2})^2$ that will complete the square?

b. Select two roots that solve the quadratic equation.

a. $x = -2$
b. $x = 6$
c. $x = 21$
d. $x = 4$
e. $x = 9$

5. $x^2 + 1.6x + 0.63 = 0$

a. What is the correct value of $(\frac{b}{2})^2$ that will complete the square?

b. **Select two** roots that solve the quadratic equation.

a. $x = -0.7$
b. $x = 0.4$
c. $x = 0.27$
d. $x = 0.7$
e. $x = -0.9$

Video: Solve by Completing the Square

Watch the following video to learn more about solving to complete the square.

Solve by Completing the Square (https://openstax.org/r/complete-square-2)

Self Check

After solving using completing the square, what are the zeros of the quadratic equation $x^2 + 5x + \frac{9}{4} = 0$?

a. $x = \frac{1}{2}$
b. $x = -\frac{1}{2}, x = -\frac{9}{2}$
c. $x = \frac{25}{4}$
d. $x = \frac{1}{4}, x = -\frac{19}{4}$

Additional Resources

Completing the Square with Fractions

Completing the square can be a useful method for solving quadratic equations in cases in which it is not easy to rewrite an expression in factored form. For example, let's solve this equation:

$$x^2 + 5x - \frac{75}{4} = 0$$

First, we'll add $\frac{75}{4}$ to each side to make things easier on ourselves.

$$x^2 + 5x - \frac{75}{4} + \frac{75}{4} = 0 + \frac{75}{4}$$

$$x^2 + 5x = \frac{75}{4}$$

To complete the square, take $\frac{1}{2}$ of the coefficient of the linear term 5, which is $\frac{5}{2}$, and square it, which is $\frac{25}{4}$. Add this to each side:

$$x^2 + 5x + \frac{25}{4} = \frac{75}{4} + \frac{25}{4}$$

$$x^2 + 5x + \frac{25}{4} = \frac{100}{4}$$

Notice that $\frac{100}{4}$ is equal to 25 and rewrite it:

$$x^2 + 5x + \frac{25}{4} = 25$$

Since the left side is now a perfect square, let's rewrite it:

$$(x + \frac{5}{2})^2 = 25$$

For this equation to be true, one of these equations must be true:

$$x + \frac{5}{2} = 5 \text{ or } x + \frac{5}{2} = -5$$

To finish up, we can subtract $\frac{5}{2}$ from each side of the equal sign in each equation.

$$x = 5 - \frac{5}{2} \text{ or } x = -5 - \frac{5}{2}$$

$$x = \frac{5}{2} \text{ or } x = -\frac{15}{2}$$

$x = 2\frac{1}{2}$ or $x = -7\frac{1}{2}$

It takes some practice to become proficient at completing the square, but it makes it possible to solve many more equations than you could by methods you learned previously.

TRY IT **Completing the Square with Fractions**

Solve $x^2 + 3x - 4 = 0$ using **completing the square**, then find the zeros of the quadratic equation.

9.3.3 Solve by Completing the Square

Activity

Here are four equations followed by worked solutions of the equations. Each solution has at least one error.

1. Solve the equation $x^2 + 14x = -24$. Then look at the following worked solution (with errors) and find and describe the error or errors in the worked solution.

$$\begin{aligned} x^2 + 14x &= -24 && \text{Step 1} \\ x^2 + 14x + 28 &= 4 && \text{Step 2} \\ (x+7)^2 &= 4 && \text{Step 3} \\ x + 7 &= 2 && \text{Step 4} \\ x + 7 &= -2 && \text{Step 5} \\ x &= -5 \text{ and } x = 9 && \text{Step 6} \end{aligned}$$

Here are four equations followed by worked solutions of the equations. Each solution has at least one error.

a. Where was the error made in the worked solution above and what was the error?

2. Solve the equation $x^2 - 10x + 16 = 0$. Then look at the following worked solution (with errors) and find and describe the error or errors in the worked solution.

$$\begin{aligned} x^2 - 10x + 16 &= 0 && \text{Step 1} \\ x^2 - 10x + 25 &= 9 && \text{Step 2} \\ (x-5)^2 &= 9 && \text{Step 3} \\ x - 5 &= 9 && \text{Step 4} \\ x - 5 &= -9 && \text{Step 5} \\ x = 14 &\text{ and } x = -4 && \text{Step 6} \end{aligned}$$

a. Where was the error made in the worked solution above and what was the error?

3. Solve the equation $x^2 + 2.4x = -0.8$. Then look at the following worked solution (with errors) and find and describe the error or errors in the worked solution.

$$\begin{aligned} x^2 + 2.4x &= -0.8 && \text{Step 1} \\ x^2 + 2.4x + 1.44 &= 0.64 && \text{Step 2} \\ (x+1.2)^2 &= 0.64 && \text{Step 3} \\ x + 1.2 &= 0.8 && \text{Step 4} \end{aligned}$$

a. Where was the error made in the worked solution above and what was the error?

4. Solve the equation $x^2 - \frac{6}{5}x + \frac{1}{5} = 0$. Then look at the following worked solution (with errors) and find and describe the error or errors in the worked solution.

$$\begin{aligned}
x^2 - \tfrac{6}{5}x + \tfrac{1}{5} &= 0 && \text{Step 1}\\
x^2 - \tfrac{6}{5}x + \tfrac{9}{25} &= \tfrac{9}{25} && \text{Step 2}\\
\left(x - \tfrac{3}{5}\right)^2 &= \tfrac{9}{25} && \text{Step 3}\\
x - \tfrac{3}{5} &= \tfrac{3}{5} && \text{Step 4}\\
x - \tfrac{3}{5} &= -\tfrac{3}{5} && \text{Step 5}\\
x = \tfrac{6}{5} &\text{ and } x = 0 && \text{Step 6}
\end{aligned}$$

a. Where was the error made in the worked solution above and what was the error?

Self Check

Examine the work shown. Which of the following is the error that took place when the equation was solved by completing the square?

$x^2 - 6x - 7 = 0$

$x^2 - 6x = 7$

$x^2 - 6x + 9 = 16$

$(x - 3)^2 = 16$

$x - 3 = 4$

$x = 7$

a. This equation cannot be solved by completing the square.
b. When completing the square, the wrong number was found for $(\frac{b}{2})^2$.
c. When taking the square root, only the positive solution was included.
d. The 7 should have been subtracted from each side.

Additional Resources

Spotting the Error When Solving Quadratics

To find the error when solving quadratics, it is helpful to first solve the problem on your own.

Solve the quadratic equation by **completing the square** and finding its zeros. Identify the error in the worked solution below.

$$\begin{aligned}
x^2 + 10x - 34 &= 0 && \text{Step 1}\\
x^2 + 10x &= 34 && \text{Step 2}\\
x^2 + 10x + 25 &= 34 - 25 && \text{Step 3}\\
(x + 5)^2 &= 9 && \text{Step 4}\\
x + 5 &= \pm\ 3 && \text{Step 5}\\
x &= -8, x = -2 && \text{Step 6}
\end{aligned}$$

Now, find the error by working through the problem and comparing steps:

$$\begin{aligned}
x^2 + 10x - 34 &= 0 && \text{Step 1}\\
x^2 + 10x &= 34 && \text{Step 2}\\
x^2 + 10x + 25 &= 34 + 25 && \text{Step 3}
\end{aligned}$$

In Step 3, 25 was added to the left side of the equation, but 25 was SUBTRACTED from the right side of the equation. Equations must remain balanced, and 25 should be added to each side of the equation.

TRY IT **Spotting the Error When Solving Quadratics**

Examine the steps below and identify the error in solving the quadratic equation by completing the square:

$$\begin{aligned} x^2 + 8x + 7 &= 0 && \text{Step 1} \\ x^2 + 8x &= -7 && \text{Step 2} \\ x^2 + 8x + 16 &= -7 + 16 && \text{Step 3} \\ (x+4)^2 &= 9 && \text{Step 4} \\ x + 4 &= 3 && \text{Step 5} \\ x &= -1 && \text{Step 6} \end{aligned}$$

9.3.4 Completing the Square with Fractions

Cool Down

1. The solutions to this equation are $x = \frac{3}{4}$ and $x = \frac{1}{4}$. Show how to find those solutions by **completing the square**. $x^2 - x + \frac{3}{16} = 0$

9.3.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For the expressions in questions 1 – 4, choose which number would make the expression a perfect square. Then, choose an equivalent expression in factored form.

1. $x^2 + 3x$

a. What number can be added to make $x^2 + 3x$ a perfect square?

a. $-\frac{2}{3}$
b. $\frac{9}{4}$
c. $\frac{4}{9}$
d. $\frac{2}{3}$

b. Which expression is equivalent to $x^2 + 3x$?

a. $3(x + 1)$
b. $(x + 3)(x + 3)$
c. $x(3x)$
d. $x(x + 3)$

2. $x^2 + 0.6x$

a. What number can be added to make $x^2 + 0.6x$ a perfect square?

a. 0.03
b. 0.36
c. 0.81
d. 0.09

b. Which expression is equivalent to $x^2 + 0.6x$?

a. $0.6(x + 1)$
b. $x(0.6x)$
c. $x(x + 0.6)$
d. $x(x + 0.6x)$

3. $x^2 - 11x$

a. What number can be added to make $x^2 - 11x$ a perfect square?

a. $\frac{121}{4}$
b. $\frac{5}{2}$
c. $\frac{25}{2}$
d. $\frac{11}{2}$

b. Which expression is equivalent to $x^2 - 11x$?

a. $11x(x - 1)$
b. $11(x - 1)$
c. $x(x - 11)$
d. $x(x - 11x)$

4. $x^2 - \frac{5}{2}x$

a. What number can be added to make $x^2 - \frac{5}{2}x$ a perfect square?

a. $\frac{6}{8}$
b. $\frac{5}{8}$
c. $\frac{25}{16}$

d. $\frac{25}{4}$

b. Which expression is equivalent to $x^2 - \frac{5}{2}x$?

a. $\frac{5}{2}(x-1)$
b. $\frac{5}{2}x(x-1)$
c. $x(x-\frac{5}{2}x)$
d. $x(x-\frac{5}{2})$

5. What number can be added to make $x^2 + x$ a perfect square?

a. 0
b. 1
c. $\frac{1}{2}$
d. $\frac{1}{4}$

6. What number can be added to make $x^2 - \frac{3}{2}x$ a perfect square?

a. $-\frac{2}{3}$
b. $\frac{9}{16}$
c. $\frac{4}{9}$
d. $\frac{2}{3}$

7. Noah is solving the equation $x^2 + 8x + 15 = 3$. He begins by rewriting the expression on the left in factored form and writes $(x+3)(x+5) = 3$. He does not know what to do next.

a. Do you think using the factored form is the best way for Noah to solve this equation?

a. No
b. Yes

b. Solve the original equation, then choose the **two** roots.

a. $x = -4$
b. $x = 12$
c. $x = -2$
d. $x = -6$
e. $x = 4$

8. An equation and its solutions are given. Solve the equation by completing the square, then choose whether you agree or disagree with the given solutions.

a. $x^2 + 20x + 50 = 14$. The solutions are $x = -18$ and $x = -2$.

a. Agree
b. Disagree

b. $x^2 + 1.6x = 0.36$. The solutions are $x = -1.8$ and $x = 0.2$.

a. Agree
b. Disagree

c. $x^2 - 5x = \frac{11}{4}$. The solutions are $x = \frac{11}{4}$ and $x = -12$.

a. Agree
b. Disagree

9. Solve each equation, then choose the **two** roots.

a. $x^2 - 0.5x = 0.5$

a. $x = -0.5$

b. $x = 1$
c. $x = 0$
d. $x = 2$
e. $x = -1$

b. $x^2 + 0.8x = 0.09$

   a. $x = -0.9$
   b. $x = 0.8$
   c. $x = 0.4$
   d. $x = 0.1$
   e. $x = 0.81$

c. $x^2 + \frac{13}{3}x = \frac{56}{36}$

   a. $x = \frac{28}{3}$
   b. $x = \frac{1}{9}$
   c. $x = \frac{112}{36}$
   d. $x = -\frac{14}{3}$
   e. $x = \frac{1}{3}$

9.3.6 Lesson Summary

In this lesson, you learned how to:

- Solve by completing the square when given a quadratic equation in which the coefficient of the squared term is 1.

Here are the **activities** that helped you reach those goals:

- 9.3.1: Solving Equations with Fractions
  - In this activity, you mentally solved equations with fractions and exponents.
- 9.3.2: Using Completing the Square to Solve Equations
  - In this activity, you solved quadratic equations involving rational numbers including decimals and fractions.
  - 9.3.2: Self Check
  - 9.3.2: Additional Resources
- 9.3.3: Solve by Completing the Square
  - In this activity, you examined the steps to solving quadratic equations by completing the square and identified the step where an error occurred.
  - 9.3.3: Self Check
  - 9.3.3: Additional Resources
- 9.3.4: Completing the Square with Fractions
  - In this activity, you showed how to get the given zeros of the quadratic by completing the square.

After these activities, you completed the following **practice**:

- 9.3.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

9.4 COMPLETING THE SQUARE, PART 3

9.4.0 Lesson Overview

In this lesson, you will learn to complete the square to solve quadratic equations, in which the squared term has a coefficient other than 1.

When you finish this lesson, you will be able to:

- Complete the square for quadratic expressions of the form $ax^2 + bx + c$ when a is not 1 and explain the process.
- Solve quadratic equations in which the squared term coefficient is not 1 by completing the square.

Here are the **activities** that will help you reach those goals:

- 9.4.1: Perfect Squares with Coefficients Other than 1
- 9.4.2: Rewrite Squared Expressions
  - 9.4.2: Self Check
  - 9.4.2: Additional Resources
- 9.4.3: Standard Form and Squared Factors
  - 9.4.3: Self Check
  - 9.4.3: Additional Resources
- 9.4.4: Three Different Methods for Solving an Equation
- 9.4.5: Perfect Square and Squared Factors

After that, you'll **practice and review**.

- 9.4.6: Practice
- 9.4.7: Lesson Summary

9.4.1 Perfect Squares with Coefficients Other than 1

Warm Up

Elena says, "$(x + 3)^2$ can be expanded into $x^2 + 6x + 9$. Likewise, $(2x + 3)^2$ can be expanded into $4x^2 + 6x + 9$."

Find an error in Elena's statement and correct the error. Show your reasoning.

9.4.2 Rewrite Squared Expressions

Activity

1. Write each expression in standard form:

a. $(4x + 1)^2$

b. $(5x - 2)^2$

c. $(\frac{1}{2}x + 7)^2$

d. $(3x + n)^2$

e. $(kx + m)^2$

2. Decide if each expression is a perfect square. If so, write an equivalent expression of the form $(kx + m)^2$. If not, suggest one change to turn it into a perfect square.

f. $4x^2 + 12x + 9$

g. $4x^2 + 8x + 25$

Video: Rewrite Squared Expressions

Watch the following video to learn more about rewriting squared expressions.

Rewrite Squared Expressions (https://openstax.org/r/squared-expressions)

Self Check

Find the equivalent factored expression for $4x^2 + 12x + 9$.

a. $(2x + 3)(2x - 3)$
b. $(2x + 3)^2$
c. $(2x - 3)^2$
d. $4(x^2 + 3x + 9)$

Factoring Perfect Square Trinomials

To factor **perfect square trinomial**s, follow the formulas:

$a^2 + 2ab + b^2 = (a + b)^2$

$a^2 - 2ab + b^2 = (a - b)^2$

Let's look at an example of how to factor perfect square trinomials.

EXAMPLE

Factor $9x^2 + 12x + 4$.

Step 1 - Does the trinomial fit the perfect square trinomial pattern, $a^2 + 2ab + b^2$? If so, write the first term as a^2 and last constant as b^2.

- Is the first term a perfect square? Write it as a square, a^2. Is $9x^2$ a perfect square? Yes — write it as $(3x)^2$.
  $9x^2 + 12x + 4$
  $(3x)^2$
- Is the last term a perfect square? Write it as a square, b^2. Is 4 a perfect square? Yes — write it as $(2)^2$.
  $9x^2 + 12x + 4$
  $(3x)^2 \qquad (2)^2$
- Check the middle term. Is it $2ab$? Is $12x$ twice the product of $3x$ and 2? Does it match? Yes, so we have a perfect trinomial.

$9x^2 + 12x + 4$
$(3x)^2 \qquad (2)^2$
$(3x)^2 \qquad (2)^2$
$\searrow \qquad \swarrow$
$2(3x)(2)$
$12x$

Step 2 - Factor the binomial into: $a^2 + 2ab + b^2 = (a + b)^2$.

$9x^2 + 12x + 4$
$a^2 + 2 \cdot a \cdot b + b^2$
$(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$
$(a + b)^2$
$(3x + 2)^2$

Step 3 - Check.

$(3x+2)^2$

$(3x)^2+2\cdot 3x\cdot 2+2^2$

$9x^2+12x+4$✔

TRY IT **Factoring Perfect Square Trinomials**

Factor $9x^2-30x+25$.

9.4.3 Standard Form and Squared Factors

Activity

1. For each expression, find the value of c to make a perfect square in standard form.

a. $100x^2+80x+c$

b. $36x^2-60x+c$

c. $25x^2+40x+c$

d. $0.25x^2-14x+c$

2. For each expression, refer to your responses in question 1 to help write an equivalent expression in the form of squared **factors**.

a. $100x^2+80x+c$

b. $36x^2-60x+c$

c. $25x^2+40x+c$

d. $0.25x^2-14x+c$

3. Write your own pair of equivalent expressions.

a. Create an expression that is a perfect square trinomial in standard form (ax^2+bx+c).

b. Now write the same equivalent expression in the form of squared factors $(kx+m)^2$.

4. Solve each equation by completing the square.

a. $25x^2+40x=-12$

b. $36x^2-60x+10=-6$

Self Check

Solve by completing the square: $25x^2+30x=7$.

a. $x=-\frac{3}{5}$

b. $x=\frac{13}{5}, x=-\frac{19}{5}$

c. $x=\frac{1}{5}, x=-\frac{7}{5}$

d. $x=9$

Find the Value to Make a Perfect Square and Solve

In earlier lessons, we worked with perfect squares such as $(x+1)^2$ and $(x-5)(x-5)$. We learned that their equivalent expressions in standard form follow a predictable pattern:

- In general, $(x+m)^2$ can be written as $x^2+2mx+m^2$.

- If a quadratic expression of the form $ax^2 + bx + c$ is a perfect square, and the value of a is 1, then the value of b is $2m$, and the value of c is m^2 for some value of m.

In this lesson, the variable in the **factors** being squared had a coefficient other than 1, for example $(3x + 1)^2$ and $(2x - 5)(2x - 5)$. Their equivalent expression in standard form also followed the same pattern we saw earlier.

Squared factors	Standard form
$(3x + 1)^2$	$(3x)^2 + 2(3x)(1) + 1^2$ or $9x^2 + 6x + 1$
$(2x - 5)^2$	$(2x)^2 + 2(2x)(-5) + (-5)^2$ or $4x^2 - 20x + 25$

In general, $(kx + m)^2$ can be written as:

$(kx)^2 + 2(kx)(m) + m^2$ or $k^2x^2 + 2kmx + m^2$

If a quadratic expression is of the form $ax^2 + bx + c$, then:

- the value of a is k^2
- the value of b is $2km$
- the value of c is m^2

We can use this pattern to help us complete the square and solve equations when the squared term x^2 has a coefficient other than 1—for example: $16x^2 + 40x = 11$.

What constant term c can we add to make the expression on the left of the equal sign a perfect square? And how do we write this expression as squared factors?

- 16 is 4^2, so the squared factors could be $(4x + m)^2$.
- 40 is equal to $2(4m)$, so $2(4m) = 40$ or $8m = 40$. This means that $m = 5$.
- If c is m^2, then $c = 5^2$ or $c = 25$.
- So the expression $16x^2 + 40x + 25$ is a perfect square and is equivalent to $(4x + 5)^2$.

Let's solve the equation $16x^2 + 40x = 11$ by completing the square!

Step 1 -

$16x^2 + 40x = 11$

To be a perfect square, $(4x + m)^2$ is needed.

We found the $4x$ by determining the square root of the first term, $16x^2$.

Step 2 -

Now, find the value of m to make a perfect square.

Since $b = 2km$, and we know that $k = 4$ (from step 1), and that $b = 40$ (from the coefficient of the linear term), we substitute the values and solve.

$40 = 2(4m)$

$40 = 8m,$

so $m = 5.$

Step 3 -

Find $c = m^2$.

$c = m^2 = 5^2 = 25$

Step 4 -

Add c to both sides of the equation.

$16x^2 + 40x + 25 = 11 + 25$

$16x^2 + 40x + 25 = 36$

Step 5 -

Factor and simplify.

$(4x + 5)^2 = 6^2$

Step 6 -

Take the square root of both sides. Remember the plus or minus.

$(4x + 5)^2 = 6^2$

$\sqrt{(4x + 5)^2} = \sqrt{6^2}$

$4x + 5 = \pm 6$

Step 7 -

Write two equations and solve.

$4x + 5 = 6$ or $4x + 5 = -6$

$4x = 1$ or $4x = -11$

$x = \frac{1}{4}$ or $x = -\frac{11}{4}$

TRY IT **Find the Value to Make a Perfect Square and Solve**

Solve $4x^2 + 28x = -13$.

9.4.4 Three Different Methods for Solving an Equation

Activity

Here are three methods for solving $3x^2 + 8x + 5 = 0$.

Carefully examine each method.

Method 1:	Method 2:	Method 3:
$3x^2 + 8x + 5 = 0$	$3x^2 + 8x + 5 = 0$	$3x^2 + 8x + 5 = 0$
$(3x + 5)(x + 1) = 0$	$9x^2 + 24x + 15 = 0$	$9x^2 + 24x + 15 = 0$
$x + 1 = 0$	$(3x)^2 + 8(3x) + 15 = 0$	$9x^2 + 24x + 16 = 1$
$3x + 5 = 0$	$U^2 + 8U + 15 = 0$	$(3x + 4)^2 = 1$
$x = -1$	$(U + 5)(U + 3) = 0$	$3x + 4 = 1$
$3x = -5$	$U + 5 = 0$	$3x + 4 = -1$
$x = -\frac{5}{3}$	$U + 3 = 0$	$x = -\frac{5}{3}$ and $x = -1$
$x = -\frac{5}{3}$ and $x = -1$	$U = -5$	
	$U = -3$	
	$3x = -5$	
	$3x = -3$	
	$x = -\frac{5}{3}$ and $x = -1$	

Once you understand the methods, use each method at least one time to solve these equations.

1. $5x^2 + 17x + 6 = 0$

2. $6x^2 + 19x = -10$

3. $8x^2 - 33x + 4 = 0$
4. $8x^2 - 26x = -21$
5. $10x^2 + 37x = 36$
6. $12x^2 + 20x - 77 = 0$

Are you ready for more?

Extending Your Thinking

1. Solve the equation $2x^2 - 3x + 1 = 0$, then choose the **two** roots.
   A. $x = -\frac{3}{2}$
   B. $x = -\frac{5}{2}$
   C. $x = 1$
   D. $x = \frac{5}{2}$
   E. $x = \frac{1}{2}$
2. Be prepared to show your reasoning for your solution.

9.4.5 Perfect Square and Squared Factors

Cool Down

1. Answer the following questions for expression $4x^2 - 28x + c$.

a. Find the value of c to make the expression a perfect square.

b. Write an equivalent expression in the form of squared factors.

2. Solve the equation $4x^2 - 28x = -33$ by completing the square. Show your reasoning.

9.4.6 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. **Select three** expressions that are perfect squares.

   a. $9x^2 + 24x + 16$
   b. $2x^2 + 20x + 100$
   c. $(7 - 3x)^2$
   d. $(5x + 4)(5x - 4)$
   e. $(1 - 2x)(-2x + 1)$
   f. $4x^2 + 6x + 94$

2. Find the missing number that makes the expression a perfect square.

$49x^2 - _____x + 16$
$36x^2 + _____x + 4$
$4x^2 - _____x + 25$
$9x^2 + _____x + 9$
$121x^2 + _____x + 9$

3. Choose the completely factored form of the expression.

   a. $49x^2 - 56x + 16$

      a. $(28x - 11)^2$
      b. $(7x - 4)^2$
      c. $(49x - 16)^2$
      d. $(7x + 4)^2$

   b. $36x^2 + 24x + 4$

      a. $4(3x + 1)^2$
      b. $(6x + 2)^2$
      c. $(36x + 4)^2$
      d. $(6x - 2)^2$

   c. $4x^2 - 20x + 25$

      a. $(2x + 5)^2$
      b. $(4x - 25)^2$
      c. $(2x - 5)^2$
      d. $(10x + 7)^2$

   d. $121x^2 + 66x + 9$

      a. $(11x - 3)^2$
      b. $(3x - 3)^2$
      c. $(121x - 9)^2$
      d. $(11x + 3)^2$

4. Find the missing number that makes the expression a perfect square.

$9x^2 + 42x + _____$
$49x^2 - 28x + _____$
$25x^2 + 110x + _____$
$64x^2 - 144x + _____$
$4x^2 + 24x + _____$

5. Based on your responses in question 4 above, choose the factored form of the expression.

   a. $9x^2 + 42x + _____$

a. $(9x+3)^2$
b. $(3x-7)^2$
c. $(9x-21)^2$
d. $(3x+7)^2$

b. $49x^2-28x+$ _____

a. $(7x+7)^2$
b. $(7x-2)^2$
c. $(7x+2)^2$
d. $(4x+7)^2$

c. $25x^2+110x+$ _____

a. $(5x+11)^2$
b. $(11x-5)^2$
c. $(11x+5)^2$
d. $(5x-11)^2$

d. $64x^2-144x+$ _____

a. $(8x+9)^2$
b. $(64x-9)^2$
c. $(8x-9)^2$
d. $(72x-17)^2$

e. $4x^2+24x+$ _____

a. $(x+6)^2$
b. $(2x+6)^2$
c. $(4x-26)^2$
d. $(2x-6)^2$

6. For each expression, find the value of c to make a perfect square in standard form (ax^2+bx+c).

$4x^2+4x+c$
$25x^2-30x+c$

7. Use the value of c needed to make the expression a perfect square (from the previous question). Then, solve each equation.

a. $4x^2+4x+c=0$

a. $x=\frac{1}{6}$
b. $x=2$
c. $x=-\frac{3}{2}$
d. $x=\frac{3}{2}$
e. $x=-\frac{1}{2}$
f. $x=\frac{1}{2}$

b. $25x^2-30x+c=0$

a. $x=\frac{3}{5}$
b. $x=5$
c. $x=4$
d. $x=\frac{2}{5}$
e. $x=-\frac{4}{5}$
f. $x=\frac{1}{2}$

9.4.7 Lesson Summary

In this lesson, you learned how to:

- Complete the square for quadratic expressions of the form $ax^2 + bx + c$ when a is not 1 and explain the process.
- Solve quadratic equations in which the squared term coefficient is not 1 by completing the square.

Here are the **activities** that helped you reach those goals:

- 9.4.1: Perfect Squares with Coefficients Other than 1
  - In this activity, you found the error when completing the square when the leading coefficient is not 1.
- 9.4.2: Rewrite Squared Expressions
  - In this activity, you determined if a quadratic expression was a perfect square. If it was not a perfect square, you made a suggestion that could make the expression a perfect square.
  - 9.4.2: Self Check
  - 9.4.2: Additional Resources
- 9.4.3: Standard Form and Squared Factors
  - In this activity, you found the value of c to complete the square of the quadratic then wrote the quadratic in factored form.
  - 9.4.3: Self Check
  - 9.4.3: Additional Resources
- 9.4.4: Three Different Methods for Solving an Equation
  - In this activity, you explored three different methods for solving quadratic equations then used each method to solve an equation.
- 9.4.5: Perfect Square and Squared Factors
  - In this activity, you completed the square then solved the quadratic equation.

After these activities, you completed the following **practice**:

- 9.4.6: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

9.5 QUADRATIC EQUATIONS WITH IRRATIONAL SOLUTIONS

9.5.0 Lesson Overview

In this lesson, you will continue to learn that some solutions to quadratic equations are irrational and can also include expressions that involve a rational number and an irrational number.

When you finish this lesson, you will be able to:

- Use the radical and "plus-minus" symbols to represent solutions to quadratic equations.
- Explain why the plus-minus symbol is used when solving quadratic equations by finding square roots.

Here are the **activities** that will help you reach those goals:

- 9.5.1: Roots of Squares
- 9.5.2: Solutions Written as Square Roots
  - 9.5.2: Self Check
  - 9.5.2: Additional Resources
- 9.5.3: Finding Irrational Solutions by Completing the Square
  - 9.5.3: Self Check
  - 9.5.3: Additional Resources
- 9.5.4: Finding Exact Solutions

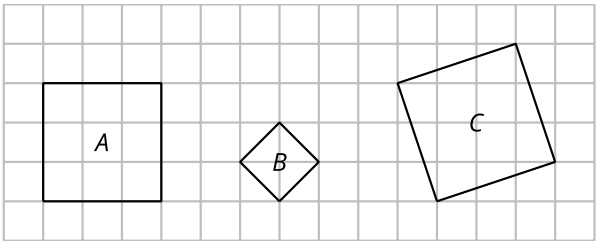
After that, you'll **practice and review**.

- 9.5.5: Practice
- 9.5.6: Lesson Summary

9.5.1 Roots of Squares

Warm Up

Here are some squares whose vertices are on a grid.



Find the area and the side length of each square.

1. What is the area of Graph A?

2. What is the side length of Graph A?

3. What is the area of Graph B?

4. What is the side length of Graph B?

5. What is the area of Graph C?

6. What is the side length of Graph C?

9.5.2 Solutions Written as Square Roots

Activity

Solve each equation. Use the notation when appropriate and express radicals in simplest form.

1. $x^2 - 13 = -12$

2. $(x-6)^2 = 0$

3. $x^2 + 9 = 0$

So far in this activity you encountered radical solutions that were easy to simplify. Sometimes you will get radical solutions like $\sqrt{12}$. Let's review how to simplify more difficult radicals using perfect squares.

Step 1 - Look at the ways to factor the number under the radical.

For example, factor 12 from $\sqrt{12}$.

$12 = 6 * 2$ and $12 = 4 * 3$

Step 2 - Choose the factorization of the number that includes a perfect square. Rewrite the number under the radical using that factorization.

$\sqrt{12} = \sqrt{4 * 3}$

Step 3 - Recall that $\sqrt{a * b} = \sqrt{a} * \sqrt{b}$.

$\sqrt{4 * 3} = \sqrt{4} * \sqrt{3}$

Step 4 - Simplify the perfect squares.

$\sqrt{4} * \sqrt{3} = 2\sqrt{3}$

So $\sqrt{12} = 2\sqrt{3}$

4. $x^2 = 18$

5. $x^2 + 1 = 18$

6. $(x + 1)^2 = 18$

Video: Solving Quadratics with Square Roots

Watch the following video to learn more about equations with solutions that are square roots.

Solving Quadratics with Square Roots (https://openstax.org/r/quadratic-square-roots)

Self Check

Solve $x^2 - 3 = 19$.

a. $x = \pm 22$
b. $x = \pm\sqrt{22}$
c. $x = \pm\sqrt{19} + 3$
d. $x = -4, x = 4$

Additional Resources

Solving Quadratics with Radical Solutions

When solving quadratic equations, it is important to remember that:

- Any positive number has two square roots, one positive and one negative, because there are two numbers that can be squared to make that number. (For example, 6^2 and $(-6)^2$ both equal 36, so 6 and -6 are both square roots of 36.)
- The square root symbol ($\sqrt{\ }$) can be used to express the positive square root of a number. For example, the square root of 36 is 6, but it can also be written as $\sqrt{36}$ because $\sqrt{36} \cdot \sqrt{36} = 36$.
- To express the negative square root of a number, say 36, we can write -6 or $-\sqrt{36}$.
- When a number is not a perfect square (for example, 40), we can express its square roots by writing $\sqrt{40}$ and $-\sqrt{40}$.

How could we write the solutions to an equation like $(x + 4)^2 = 11$? This equation is saying, "something squared is 11." To make the equation true, that something must be $\sqrt{11}$ or $-\sqrt{11}$.

We can write:

$x + 4 = \sqrt{11}$ or $x + 4 = -\sqrt{11}$

$x = -4 + \sqrt{11}$ or $x = -4 + -\sqrt{11}$

A more compact way to write the two solutions to the equation is: $x = -4 \pm \sqrt{11}$.

TRY IT **Solving Quadratics with Radical Solutions**

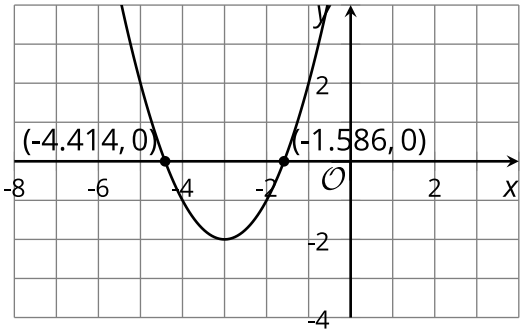
Solve $(x-2)^2 = 17$.

9.5.3 Finding Irrational Solutions by Completing the Square

Activity

Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

Here is an example of an equation being solved by graphing and by **completing the square**.



$$\begin{aligned} x^2 + 6x + 7 &= 0 \\ x^2 + 6x + 9 &= 2 \\ (x+3)^2 &= 2 \\ x + 3 &= \pm\sqrt{2} \\ x &= -3 \pm \sqrt{2} \end{aligned}$$

Verify: $\sqrt{2}$ is approximately 1.414. So $-3 + \sqrt{2} \approx -1.586$ and $-3 - \sqrt{2} \approx -4.414$.

For each equation, find the exact solutions by completing the square and the approximate solutions by graphing. Then, verify that the solutions found using the two methods are close. If you get stuck, refer back to the example.

1. $x^2 + 4x + 1 = 0$

a. Find the exact solutions by completing the square.

b. Graph the equation to find approximate solutions.

Use the Desmos graphing tool or technology outside the course.

Access multimedia content (http://openstax.org/books/algebra-1/pages/9-5-3-finding-irrational-solutions-by-completing-the-square)

c. Are the solutions you found using the two methods close?

2. $x^2 - 10x + 18 = 0$

a. Find the exact solutions by completing the square.

b. Graph the equation to find approximate solutions.

Use the Desmos graphing tool or technology outside the course.

Access multimedia content (http://openstax.org/books/algebra-1/pages/9-5-3-finding-irrational-solutions-by-completing-the-square)

c. Are the solutions you found using the two methods close?

3. $x^2 + 5x + \frac{1}{4} = 0$

a. Find the exact solutions by completing the square.

b. Graph the equation to find approximate solutions.

Use the Desmos graphing tool or technology outside the course.

Access multimedia content (http://openstax.org/books/algebra-1/pages/9-5-3-finding-irrational-solutions-by-completing-the-square)

c. Are the solutions you found using the two methods close?

4. $x^2 + \frac{8}{3}x + \frac{14}{9} = 0$

a. Find the exact solutions by completing the square.

b. Graph the equation to find approximate solutions.

Use the Desmos graphing tool or technology outside the course.

Access multimedia content (http://openstax.org/books/algebra-1/pages/9-5-3-finding-irrational-solutions-by-completing-the-square)

c. Are the solutions you found using the two methods close?

Are you ready for more?

Extending Your Thinking

1. What quadratic equation of the form $ax^2 + bx + c = 0$ would have the solutions $x = 5 - \sqrt{2}$ and $x = 5 + \sqrt{2}$?

Self Check

Which of the following are the solutions to $(x - 2)^2 = 12$, rounded to the nearest thousandth?

a. $x \approx 3.162$
b. $x = 14$
c. $x \approx -1.464, x \approx 5.464$
d. $x \approx -3.162, x \approx 3.162$

Additional Resources

Using the Calculator for Approximate Solutions

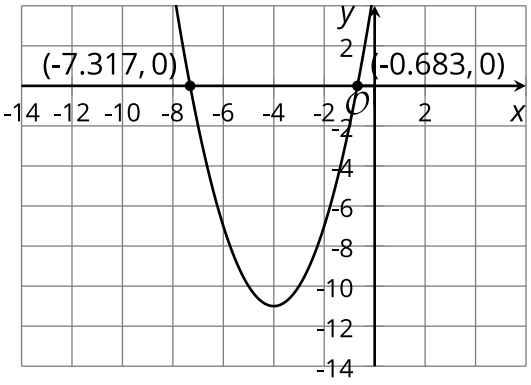
Let's look at the solutions to the equations from 9.5.2: Additional Resources:

A more compact way to write the two solutions to the equation $(x + 4)^2 = 11$ is: $x = -4 \pm \sqrt{11}$.

How large or small are those numbers? Are they positive or negative? We can use a calculator to compute the approximate values of both expressions:

$-4 + \sqrt{11} \approx -0.683$ or $-4 - \sqrt{11} \approx -7.317$

We can also approximate the solutions by graphing. The equation $(x + 4)^2 = 11$ is equivalent to $(x + 4)^2 - 11 = 0$, so we can graph the function $y = (x + 4)^2 - 11$ and find its zeros by locating the x-intercepts of the graph.



TRY IT **Using the Calculator for Approximate Solutions**

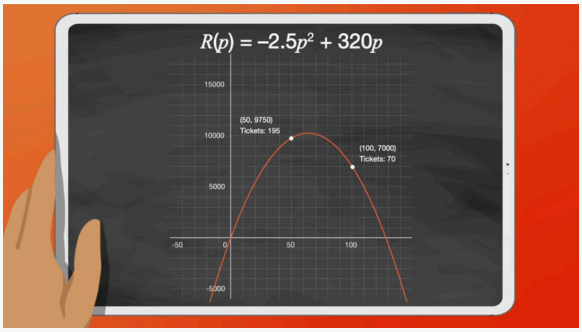
Approximate the solutions to $x = 3 \pm \sqrt{19}$.

9.5.4 Finding Exact Solutions

Cool Down

1. For the equation $x^2 + 14x = -46$, the approximate solutions are $x \approx -8.732$ and $x \approx -5.268$. Find the exact solutions by **completing the square**. Show your reasoning.

Why Should I Care?



The Ensemble Theatre is exploiting two ancient practices in new and exciting ways. Those ancient practices are quadratic equations and live theatre. Running a live theatre company requires many things, and one of the most important is the old tool called the quadratic function.

The theatre can use quadratic functions to determine ticket price to maximize the profit. Ticket prices need to be set high enough to recoup the cost of expenses and generate revenue while still being a price point that attracts customers.

The vertex of the parabola that represents the Ensemble Theatre's ticket price can tell us at what ticket price they can sell the most tickets and maximize the revenue.

9.5.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Solve each equation and write the solutions using the ± notation, then choose the correct solution from the list.

a. $x^2 = 144$

   a. $x = \pm 24$
   b. $x = \pm 12$
   c. $x = \pm 72$
   d. $x = \pm 144$

b. $x^2 = 5$

   a. $x = \pm \sqrt{10}$
   b. $x = \pm \frac{5}{2}$
   c. $x = \pm 5$
   d. $x = \pm \sqrt{5}$

c. $4x^2 = 28$

   a. $x = \pm \sqrt{7}$
   b. $x = \pm \frac{7}{2}$
   c. $x = \pm 7$
   d. $x = \pm \sqrt{\frac{7}{2}}$

d. $x^2 = \frac{25}{4}$

   a. $x = \pm \frac{5}{4}$
   b. $x = \pm \frac{10}{2}$
   c. $x = \pm \frac{5}{2}$
   d. $x = \pm \frac{25}{4}$

e. $2x^2 = 22$

   a. $x = \pm \sqrt{11}$
   b. $x = \pm \frac{11}{2}$
   c. $x = \pm 11$
   d. $x = \pm \sqrt{\frac{11}{2}}$

f. $7x^2 = 16$

   a. $x = \pm \frac{\sqrt{16}}{7}$
   b. $x = \pm 8$
   c. $x = \pm 16$
   d. $x = \pm \sqrt{\frac{16}{7}}$

2. For each expression, choose an equivalent expression.

a. $10 \pm \sqrt{4}$

   a. $4 + \sqrt{10}$ and $4 - \sqrt{10}$
   b. 3 and 5
   c. 8 and 12
   d. $4 + \sqrt{2}$ and $4 - \sqrt{2}$
   e. -17 and 5

b. $4 \pm \sqrt{10}$

a. $4 + \sqrt{10}$ and $4 - \sqrt{10}$
b. 3 and 5
c. 8 and 12
d. $4 + \sqrt{2}$ and $4 - \sqrt{2}$
e. -17 and 5

c. $\sqrt{16} \pm \sqrt{2}$

a. $4 + \sqrt{10}$ and $4 - \sqrt{10}$
b. 3 and 5
c. 8 and 12
d. $4 + \sqrt{2}$ and $4 - \sqrt{2}$
e. -17 and 5

3. What is the difference between $\sqrt{9}$ and the solutions to $x^2 = 9$?

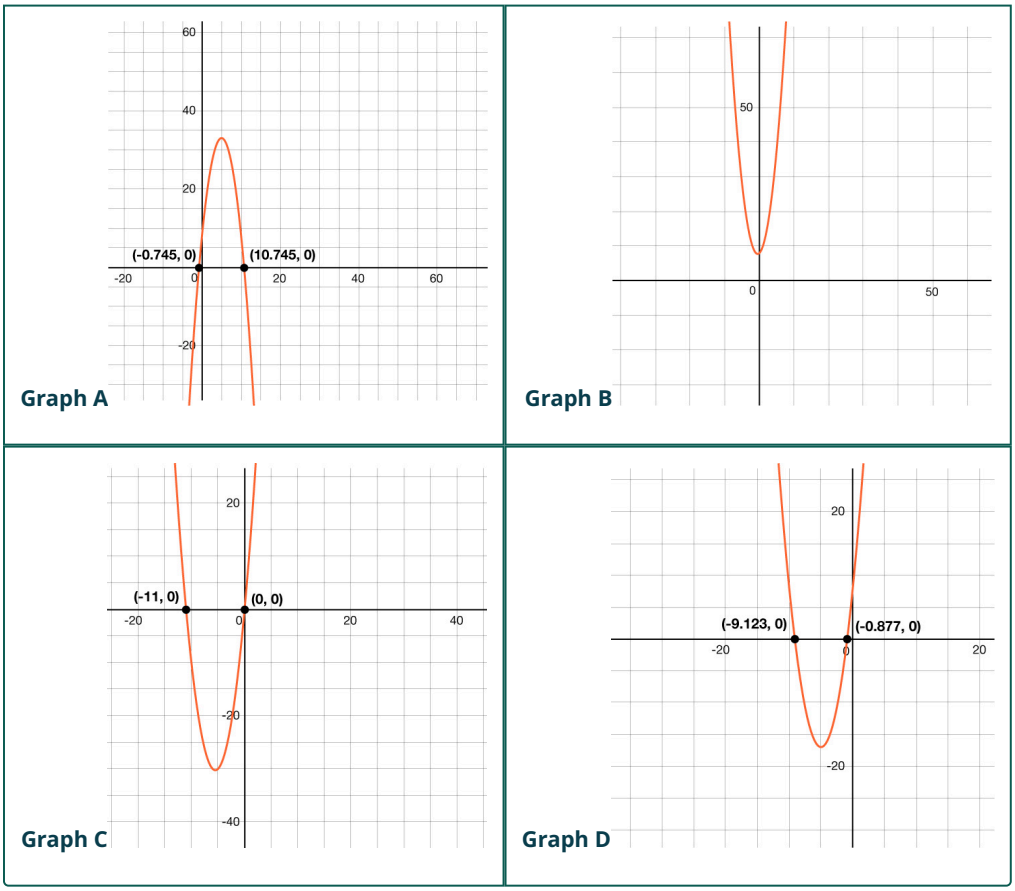
a. There is no difference; they are equal.
b. $\sqrt{9}$ is equal to 3, while there are two numbers that make $x^2 = 9$ true: 3 and -3.
c. $\sqrt{9}$ is equal to -3, while there are two numbers that make $x^2 = 9$ true: 3 and -3.
d. $\sqrt{9}$ is equal to 3, while $x^2 = 9$ is equal to -3.

4. Technology required. For each equation, find the exact solutions by completing the square and the approximate solutions by graphing. Then, verify that the solutions found using the two methods are close.

a. Solve the equation $x^2 + 10x + 8 = 0$ by completing the square. **Select two** exact solutions.

a. $x = -5 + \sqrt{17}$
b. $x = 5 + \sqrt{17}$
c. $x = 5 - \sqrt{17}$
d. $x = -5 - \sqrt{17}$

b. Choose the graph that best represents the equation $x^2 + 10x + 8 = 0$.



a. A
b. B
c. C
d. D

c. Are the solutions and the graph for the equation $x^2 + 10x + 8 = 0$ similar?

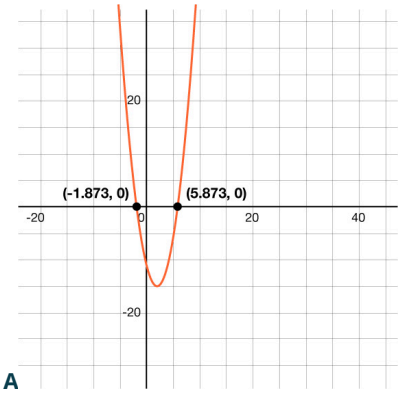
a. They are similar because the square root of a number is always positive.
b. They are not similar because the square root of a number is always positive.
c. They are similar because the solutions are close.
d. They are not similar.

5. Answer the following questions about $x^2 - 4x - 11 = 0$.

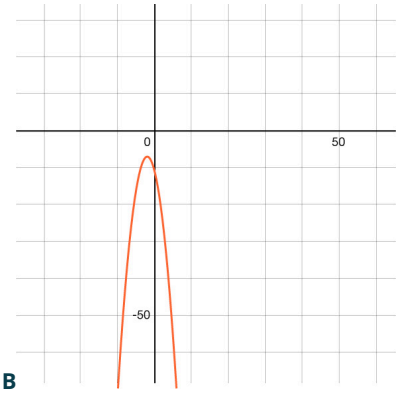
a. Solve the equation $x^2 - 4x - 11 = 0$ by completing the square. **Select two** exact solutions.

a. $x = -2 + \sqrt{15}$
b. $x = 2 + \sqrt{15}$
c. $x = 2 - \sqrt{15}$
d. $x = -2 - \sqrt{15}$

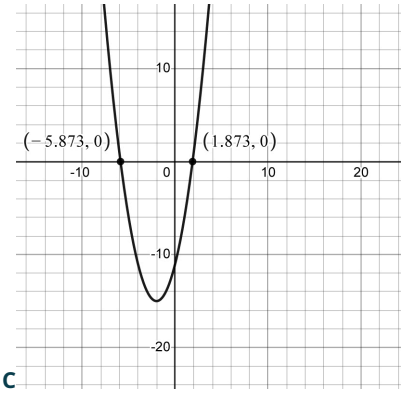
b. Choose the graph that best represents the equation $x^2 - 4x - 11 = 0$.



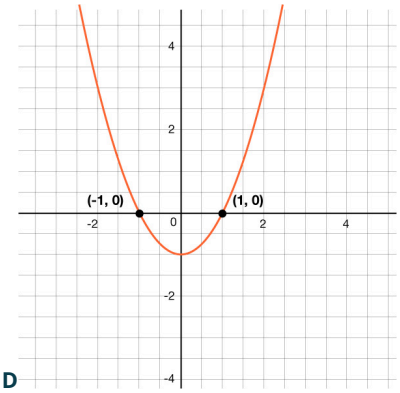
Graph A



Graph B



Graph C



Graph D

a. A
b. B
c. C
d. D

c. Are the solutions and the graph for the equation $x^2 - 4x - 11 = 0$ similar?

a. They are similar because the square root of a number is always positive.
b. They are not similar.
c. They are not similar because the square root of a number is always positive.
d. They are similar because the solutions are close

9.5.6 Lesson Summary

In this lesson, you learned how to:

- Use the radical and "plus-minus" symbols to represent solutions to quadratic equations.
- Explain why the plus-minus symbol is used when solving quadratic equations by finding square roots.

Here are the **activities** that helped you reach those goals:

- 9.5.1: Roots of Squares
  - In this activity, you found the area and side lengths of squares to explore square roots.
- 9.5.2: Solutions Written as Square Roots
  - In this activity, you solved quadratic equations that had radicals left in the solutions.
  - 9.5.2: Self Check
  - 9.5.2: Additional Resources
- 9.5.3: Finding Irrational Solutions by Completing the Square
  - In this activity, you solved quadratic equations by completing the square. You graphed the equations to find the approximate solutions using a graphing calculator.
  - 9.5.3: Self Check
  - 9.5.3: Additional Resources
- 9.5.4: Finding Exact Solutions
  - In this activity, you solved quadratic equations and used a calculator to approximate the actual solution(s).

After these activities, you completed the following **practice**:

- 9.5.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

9.6 THE QUADRATIC FORMULA

9.6.0 Lesson Overview

In this lesson, you will encounter the quadratic formula and learn that it can be used to solve any quadratic equation. You can use the formula and verify that it produces the same solutions as those found using other methods, but it can be much more practical for certain equations.

When you finish this lesson, you will be able to:

- Use the quadratic formula to solve quadratic equations.
- Explain when some methods for solving quadratic equations can be more convenient than others.

Here are the **activities** that will help you reach those goals:

- 9.6.1: Evaluating Expressions
- 9.6.2: Choosing the Best Method to Solve an Equation
  - 9.6.2: Self Check
  - 9.6.2: Additional Resources
- 9.6.3: The Quadratic Formula
  - 9.6.3: Self Check
  - 9.6.3: Additional Resources
- 9.6.4: Solving Equations Using the Quadratic Formula

After that, you'll **practice and review**.

- 9.6.5: Practice
- 9.6.6: Lesson Summary

9.6.1 Evaluating Expressions

Warm Up

Each expression represents two numbers. Evaluate the expressions and find the **two** numbers.

1. $1 \pm \sqrt{49}$

2. $\frac{8 \pm 2}{5}$

3. $\pm \sqrt{(-5)^2 - 4 \cdot 4 \cdot 1}$

4. $\frac{-18 \pm \sqrt{36}}{2 \cdot 3}$

9.6.2 Choosing the Best Method to Solve an Equation

Activity

Write down your answer for how to solve each equation. Find the two solutions by rewriting it in factored form or by **completing the square**. Be prepared to explain your choice of method.

1. **Select two** solutions for $x^2 - 2x - 1.25 = 0$.

a. 2.3
b. 3
c. –1
d. –0.5

2. **Select two** solutions for $5x^2 + 9x - 44 = 0$.

a. 2.2
b. –20
c. –4
d. 11

3. **Select two** solutions for $x^2 + 1.25x = 0.375$.
   a. 0.75
   b. 0.25
   c. –1.5
   d. 2.5

4. **Select two** solutions for $4x^2 - 28x + 29 = 0$.
   a. $-7 + \sqrt{2}$
   b. $7 - \sqrt{2}$
   c. $\frac{7}{2} + \sqrt{5}$
   d. $\frac{7}{2} - \sqrt{5}$

Self Check

Use completing the square to solve $2x^2 + 10x + 8 = 0$.
a. $x = -5 \pm \sqrt{17}$
b. $x = -4, x = -1$
c. $x = 1, x = 4$
d. $x = \frac{25}{4}$

Additional Resources

Solve by Using Perfect Squares and Completing the Square

Solve $2x^2 + 5x + 2 = 0$ by **completing the square**.

Step 1 - Divide all terms by the leading coefficient.

$\frac{2x^2+5x+2}{2} = \frac{0}{2}\ x^2 + \frac{5}{2}x + 1 = 0$

Step 2 - Move the constant to the right side of the equation.

$x^2 + \frac{5}{2}x = -1$

Step 3 - Complete the square and add to each side: $(\frac{5}{2} \times \frac{1}{2})^2 = \frac{25}{16}$.

$x^2 + \frac{5}{2}x + \frac{25}{16} = -1 + \frac{25}{16}$

Step 4 - Simplify. (Hint: create a common denominator before adding.)

$x^2 + \frac{5}{2}x + \frac{25}{16} = -\frac{16}{16} + \frac{25}{16}$

$(x + \frac{5}{4})^2 = \frac{9}{16}$

Step 5 - Take the square root of each side.

$\sqrt{(x + \frac{5}{4})^2} = \sqrt{\frac{9}{16}}$

$x + \frac{5}{4} = \pm\frac{3}{4}$

Step 6 - Write two equations and solve.

$x + \frac{5}{4} = \frac{3}{4} \qquad x + \frac{5}{4} = -\frac{3}{4}$

$x = -\frac{2}{4} = -\frac{1}{2} \quad x = -\frac{8}{4} = -2$

TRY IT **Solve by Using Perfect Squares and Completing the Square**

Solve $4x^2 + 36x + 37 = 0$.

9.6.3 The Quadratic Formula

Activity

Here is a formula called the **quadratic formula**.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The formula can be used to find the solutions to any quadratic equation in the form of $ax^2 + bx + c = 0$, where a, b, and c are numbers and a is not 0.

This example shows how it is used to solve $x^2 - 8x + 15 = 0$, in which $a = 1$, $b = -8$, and $c = 15$.

Step 1 - State the original equation.

$x^2 - 8x + 15 = 0$

Step 2 - Identify the values of a, b, and c.

$a = 1$, $b = -8$, and $c = 15$

Step 3 - Substitute the values of a, b, and c into the formula.

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - (4)(1)(15)}}{2(1)}$$

Step 4 - Evaluate each part of the expression.

$$x = \frac{8 \pm \sqrt{64 - 60}}{2}$$

$$x = \frac{8 \pm \sqrt{4}}{2}$$

$$x = \frac{8 \pm 2}{2}$$

Step 5 - Write each equation separately and solve.

$x = \frac{8+2}{2}$ and $x = \frac{8-2}{2}$

Step 6 - Simplify.

$x = \frac{10}{2}$ and $x = \frac{6}{2}$

Step 7 - Find the solutions.

$x = 5$ and $x = 3$

Here are some quadratic equations and their solutions. Use the quadratic formula to show that the solutions are correct.

1. $x^2 + 4x - 5 = 0$. The solutions are $x = -5$ and $x = 1$.
2. $x^2 + 7x + 12 = 0$. The solutions are $x = -3$ and $x = -4$.
3. $x^2 + 10x + 18 = 0$. The solution is $x = -5 \pm \sqrt{7}$.
4. $x^2 - 8x + 11 = 0$. The solution is $x = 4 \pm \sqrt{5}$.
5. $9x^2 - 6x + 1 = 0$. The solution is $x = \frac{1}{3}$.
6. $6x^2 + 9x - 15 = 0$. The solutions are $x = -\frac{5}{2}$ and $x = 1$.

Video: Solving Quadratics With the Quadratic Formula

Watch the following video to learn more about solving quadratics with the quadratic formula.

Solving Quadratics With the Quadratic Formula (https://openstax.org/r/quadratic-formula-solve)

Self Check

Solve $2x^2 - 6x + 3 = 0$ using the quadratic formula.

a. $x = \frac{6 \pm \sqrt{-12}}{4}$

b. $x = \frac{6 \pm \sqrt{60}}{4}$

c. $x = \frac{6 \pm \sqrt{12}}{4}$

d. $x = \frac{-6 \pm \sqrt{12}}{4}$

Additional Resources

Solving With the Quadratic Formula

Quadratic Formula

The solutions to a quadratic equation of the form $ax^2 + bx + c$, where $a \neq 0$, are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To use the **quadratic formula**, we substitute the values of a, b, and c from the standard form into the expression on the right side of the formula. Then we simplify the expression. The result is the pair of solutions to the quadratic equation.

Notice the formula is an equation. Make sure you use both sides of the equation.

How to Solve a Quadratic Equation Using the Quadratic Formula

Solve by using the quadratic formula: $2x^2 + 9x - 5 = 0$.

Step 1 - Write the quadratic equation in standard form.

This equation is in standard form.

$$ax^2 + bx + c = 0$$

$$2x^2 + 9x - 5 = 0$$

Step 2 - Identify the a, b, and c values.

$a = 2, b = 9, c = -5$

Step 3 - Write the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Step 4 - Then substitute in the values of a, b, and c.

$$a = 2, b = 9, c = -5$$

$$x = \frac{-9 \pm \sqrt{9^2 - 4 \cdot 2 \cdot (-5)}}{2 \cdot 2}$$

Step 5 - Simplify the fraction and solve for x.

$$x = \frac{-9 \pm \sqrt{81 - (-40)}}{4}$$

$$x = \frac{-9 \pm \sqrt{121}}{4}$$

$$x = \frac{-9 + 11}{4}$$

$x = \frac{-9-11}{4}$

$x = \frac{2}{4}$

$x = \frac{-20}{4}$

$x = \frac{1}{2}$

$x = -5$

Step 6 - Check the solutions. Put each answer in the original equation to check. Substitute $x = \frac{1}{2}$.

$2\left(\frac{1}{2}\right)^2 + 9\cdot\frac{1}{2} - 5 \stackrel{?}{=} 0$

$2\left(\frac{1}{4}\right) + 9\cdot\frac{1}{2} - 5 \stackrel{?}{=} 0$

$\frac{2}{4} + \frac{9}{2} - 5 \stackrel{?}{=} 0$

$\frac{1}{2} + \frac{9}{2} - 5 \stackrel{?}{=} 0$

$\frac{10}{2} - 5 \stackrel{?}{=} 0$

$5 - 5 \stackrel{?}{=} 0$

$0 = 0$ ✔

Substitute $x = -5$.

$2(-5)^2 + 9(-5) - 5 \stackrel{?}{=} 0$

$2(25) - 45 - 5 \stackrel{?}{=} 0$

$50 - 45 - 5 \stackrel{?}{=} 0$

$0 = 0$ ✔

TRY IT **Solving With the Quadratic Formula**

Solve $2x^2 + 10x + 11 = 0$ using the quadratic formula.

9.6.4 Solving Equations Using the Quadratic Formula

Cool Down

Here is the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

1. Use the formula to solve the equation $x^2 + 9x + 8 = 0$.

9.6.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

For each equation in questions 1 – 5, identify the values of a, b, and c that you would substitute into the quadratic formula to solve the equation.

1. $3x^2 + 8x + 4 = 0$

a. What is the value of a?

b. What is the value of b?

c. What is the value of c?

2. $2x^2 - 5x + 2 = 0$

a. What is the value of a?

b. What is the value of b?

c. What is the value of c?

3. $-9x^2 + 13x - 1 = 0$

a. What is the value of a?

b. What is the value of b?

c. What is the value of c?

4. $x^2 + x - 11 = 0$

a. What is the value of a?

b. What is the value of b?

c. What is the value of c?

5. $-x^2 + 16x + 64 = 0$

a. What is the value of a?

b. What is the value of b?

c. What is the value of c?

6. For each equation, use the quadratic formula to solve the equation, then choose the correct **two** solutions for x.

a. $x^2 + 9x + 20 = 0$

a. 4
b. -4
c. -5
d. 5
e. 1
f. 9
g. 20

b. $x^2 - 10x + 21 = 0$

a. 3
b. -7
c. 21
d. -10
e. 1
f. -4
g. 7

c. $3x^2 - 5x + 1 = 0$

a. $\frac{13}{6} + \frac{\sqrt{5}}{6}$
b. -1
c. $\frac{5}{6} - \frac{\sqrt{13}}{6}$
d. $\frac{13}{6} - \frac{\sqrt{5}}{6}$
e. $-\frac{18}{6} - 3$
f. $\frac{5}{6} + \frac{\sqrt{13}}{6}$
g. $\frac{18}{6} - 3$

For each equation for questions 7 – 12, identify the values of a, b, and c that you would substitute into the quadratic formula to solve the equation.

7. $x^2 + 9x + 18 = 0$

a. What is the value of a?

b. What is the value of b?

c. What is the value of c?

8. $4x^2 - 3x + 11 = 0$

a. What is the value of a?

b. What is the value of b?

c. What is the value of c?

9. $81 - x + 5x^2 = 0$

a. What is the value of a?

b. What is the value of b?

c. What is the value of c?

10. $45x^2 + 3x = \frac{1}{3}$

a. What is the value of a?

b. What is the value of b?

c. What is the value of c?

a. 0
b. 1
c. $\frac{1}{3}$
d. $-\frac{1}{3}$

11. $121 = x^2$

a. What is the value of a?

b. What is the value of b?

c. What is the value of c?

12. $7x + 14x^2 = 42$

a. What is the value of a?

b. What is the value of b?

c. What is the value of c?

9.6.6 Lesson Summary

In this lesson, you learned how to:

- Use the quadratic formula to solve quadratic equations.
- Explain when some methods for solving quadratic equations can be more convenient than others.

Here are the **activities** that helped you reach those goals:

- 9.6.1: Evaluating Expressions
  - In this activity, you evaluated expressions using order of operations, square roots, the plus or minus sign, and more.
- 9.6.2: Choosing the Best Method to Solve an Equation
  - In this activity, you chose between writing a quadratic equation in factored form or completing the square to solve an equation.
  - 9.6.2: Self Check
  - 9.6.2: Additional Resources
- 9.6.3: The Quadratic Formula
  - In this activity, you used the quadratic formula to show that the solutions of the given quadratic equations were correct.
  - 9.6.3: Self Check
  - 9.6.3: Additional Resources
- 9.6.4: Solving Equations Using the Quadratic Formula
  - In this activity, you used the quadratic formula to solve the given equation.

After these activities, you completed the following **practice**:

- 9.6.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

9.7 APPLYING THE QUADRATIC FORMULA

9.7.0 Lesson Overview

In this lesson, you will learn how to make decisions about the quadratic formula. We will pay close attention to each step in the solving process and analyze errors commonly made when applying the formula. You will need to consider how to verify that the solutions you obtained with the quadratic formula are correct.

When you finish this lesson, you will be able to:

- Identify common errors when using the quadratic formula.
- Describe some ways to tell if a number is a solution to a quadratic equation.

Here are the **activities** that will help you reach those goals:

- 9.7.1: Evaluating Variable Expressions
- 9.7.2: Common Calculation Errors When Using the Quadratic Formula
  - 9.7.2: Self Check
  - 9.7.2: Additional Resources
- 9.7.3: Different Methods of Checking Solutions of Quadratic Equations
  - 9.7.3: Self Check
  - 9.7.3: Additional Resources
- 9.7.4: Practice Spotting Calculation Errors

After that, you'll **practice and review**.

- 9.7.5: Practice
- 9.7.6: Lesson Summary

9.7.1 Evaluating Variable Expressions

Warm Up

Evaluate each expression for $a = 9$, $b = -5$, and $c = -2$.

1. What is the value of $-b$?

2. What is the value of b^2?

3. What is the value of $b^2 - 4ac$?

4. **Select two** answers for $-b \pm \sqrt{a}$.

a. –14
b. 8
c. 2
d. 4

9.7.2 Common Calculation Errors When Using the Quadratic Formula

Activity

Here are four equations, followed by attempts to solve them using the **quadratic formula**. Each attempt contains at least one error.

Solve the equations by using the quadratic formula. Then, find and describe the error(s) in the worked solutions of the same equations as the ones you solved.

1. Equation 1: $2x^2 + 3 = 8x$
Here is the worked solution with errors:
Step 1 - $2x^2 - 8x + 3 = 0$
Step 2 - $a = 2$, $b = -8$, $c = 3$
Step 3 - $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
Step 4 - $x = \frac{-(-8) \pm \sqrt{(-8)^2-(4)(2)(3)}}{2(2)}$
Step 5 - $x = \frac{8 \pm \sqrt{64-24}}{4}$
Step 6 - $x = \frac{8 \pm \sqrt{40}}{4}$
Step 7 - $x = 2 \pm \sqrt{10}$

2. Equation 2: $x^2 + 3x = 10$
Here is the worked solution with errors:
Step 1 - $a = 1$, $b = 3$, $c = 10$
Step 2 - $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
Step 3 - $x = \frac{-(3) \pm \sqrt{3^2-(4)(1)(10)}}{2(1)}$
Step 4 - $x = \frac{-3 \pm \sqrt{9-40}}{2}$
Step 5 - $x = \frac{-3 \pm \sqrt{-31}}{2}$
No Solutions

3. Equation 3: $9x^2 - 2x - 1 = 0$
Here is the worked solution with errors:
Step 1 - $a = 9$, $b = -2$, $c = -1$
Step 2 - $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
Step 3 - $x = \frac{2 \pm \sqrt{(-2)^2-(4)(9)(-1)}}{2}$
Step 4 - $x = \frac{2 \pm \sqrt{4+36}}{2}$
Step 5 - $x = \frac{2 \pm \sqrt{40}}{2}$

4. Equation 4: $x^2 - 10x + 23 = 0$
Here is the worked solution with errors:
Step 1 - $a = 1$, $b = -10$, $c = 23$
Step 2 - $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
Step 3 - $x = \frac{-10 \pm \sqrt{(-10)^2-(4)(1)(23)}}{2}$
Step 4 - $x = \frac{-10 \pm \sqrt{-100-92}}{2}$
Step 5 - $x = \frac{-10 \pm \sqrt{-192}}{2}$
No Solutions

Self Check

When solving $x^2 - 30x + 26 = 0$ using the quadratic formula, Sloane's work showed the formula as this:

$x = \frac{-30 \pm \sqrt{900-104}}{2}$.

Which of the following describes the error in her work?

a. The discriminant should be $-900 - 104$, not $900 - 104$; $x = \frac{-30 \pm \sqrt{900-104}}{2}$.
b. The first term in the numerator should be 30, not -30; $x = \frac{30 \pm \sqrt{900-104}}{2}$.
c. The discriminant should be $900 + 104$, not $900 - 104$; $x = \frac{-30 \pm \sqrt{900+104}}{2}$.
d. The denominator should be 4, not 2; $x = \frac{-30 \pm \sqrt{900-104}}{4}$.

Additional Resources

Common Errors When Using the Quadratic Formula

The quadratic formula has many parts in it. A small error in any one part can lead to incorrect solutions.

Suppose we are solving $2x^2 - 6 = 11x$. To use the formula, let's rewrite it in the form of $ax^2 + bx + c = 0$, which gives: $2x^2 - 11x - 6 = 0$.

Here are some common errors to avoid:

Using the wrong values for a, b, and c in the formula.

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \frac{-11 \pm \sqrt{(11)^2-4(2)(-6)}}{2(2)}$

This is incorrect! b is -11, so $-b$ is -(-11), which is 11, not -11.

The correct formula is:
$x = \frac{11 \pm \sqrt{(11)^2 - 4(2)(-6)}}{2(2)}$

Forgetting to multiply 2 by a for the denominator in the formula.

$x = \frac{11 \pm \sqrt{(11)^2 - 4(2)(-6)}}{2}$

This is incorrect! The denominator is $2a$, which is $2(2)$ or 4.

The correct formula is:
$x = \frac{11 \pm \sqrt{(11)^2 - 4(2)(-6)}}{2(2)}$

Forgetting that squaring a negative number produces a positive number.

$x = \frac{11 \pm \sqrt{-121 - 4(2)(-6)}}{4}$

This is incorrect! $(-11)^2$ is 121, not -121.

The correct formula is:
$x = \frac{11 \pm \sqrt{121 - 4(2)(-6)}}{4}$

Forgetting that a negative number times a positive number is a negative number.

$x = \frac{11 \pm \sqrt{121 - 48}}{4}$

This is incorrect! $4(2)(-6) = -48$ and $121 - (-48)$ is $121 + 48$.

The correct formula is:
$x = \frac{11 \pm \sqrt{121 + 48}}{4}$

Making calculation errors or not following the properties of algebra.

$x = \frac{11 \pm \sqrt{169}}{4}$

$x = 11 \pm \sqrt{42.25}$

This is inBoth parts of the numerator, the 11 and the $\sqrt{169}$, get divided by 4. Also, $\frac{\sqrt{169}}{4}$ is not $\sqrt{42.25}$.

The correct formula is:
$x = \frac{11 \pm 13}{4}$

Let's finish by evaluating $\frac{11 \pm 13}{4}$ correctly:
$x = \frac{11+13}{4}$ or $x = \frac{11-13}{4}$
$x = \frac{24}{4}$ or $x = -\frac{2}{4}$
$x = 6$ or $x = -\frac{1}{2}$

To make sure our solutions are indeed correct, we can substitute the solutions back into the original equations and see whether each solution keeps the equation true.

Checking 6 as a solution:
$2x^2 - 6 = 11x$
$2(6)^2 - 6 = 11(6)$
$2(36) - 6 = 66$
$72 - 6 = 66$ $66 = 66$ ✓

Checking $-\frac{1}{2}$ as a solution:

$2x^2 - 6 = 11x$

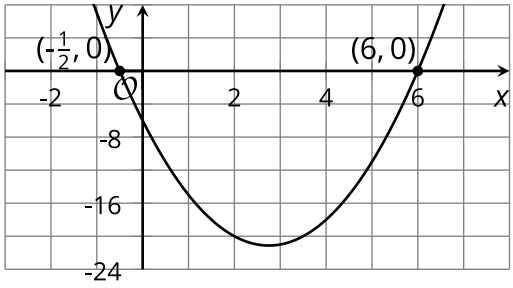
$2(-\frac{1}{2})^2 - 6 = 11(-\frac{1}{2})$

$2(\frac{1}{4}) - 6 = -\frac{11}{2}$

$\frac{1}{2} - 6 = -5\frac{1}{2}$

$-5\frac{1}{2} = -5\frac{1}{2}$ ✓

We can also graph the equation $y = 2x^2 - 11x - 6$ and find its x-intercepts to see whether our solutions to $2x^2 - 11x - 6 = 0$ are accurate (or close to accurate).



TRY IT **Common Errors When Using the Quadratic Formula**

Zariah was solving the equation $3x^2 - 8x + 4 = 0$ with the quadratic formula and used the steps below. She then had to stop because she realized there was a problem. Locate and describe her error.

Step 1 - $x = \frac{-(-8) \pm \sqrt{-8^2 - 4(3)(4)}}{6}$

Step 2 - $x = \frac{8 \pm \sqrt{-64 - 48}}{6}$

9.7.3 Different Methods of Checking Solutions of Quadratic Equations

Activity

1. The equation $h(t) = 2 + 30t - 5t^2$ represents the height, as a function of time, of a pumpkin that was catapulted up in the air. Height is measured in meters and time is measured in seconds.

a. The pumpkin reached a maximum height of 47 meters. How many seconds after launch did that happen?

b. Show your reasoning.

c. Suppose someone was unconvinced by your solution. Find another way (besides the steps you already took) to show your solution is correct.

2. The equation $r(p) = 80p - p^2$ models the revenue a band expects to collect as a function of the price of one concert ticket. Ticket prices and revenues are in dollars.

A band member says that a ticket price of either \$15.50 or \$74.50 would generate approximately \$1000 in revenue. Do you agree? Show your reasoning.

Are you ready for more?

Extending Your Thinking

1. Function g is defined by the equation $g(t) = 2 + 30t - 5t^2 - 47$. Its graph opens downward.
Find the zeros of function g without graphing.

2. Show your reasoning.

3. Explain or show how the zeros you found can tell us the vertex of the graph of g.
4. Recall that $g(t) = 2 + 30t - 5t^2 - 47$ and $h(t) = 2 + 30t - 5t^2$. Examine these functions g and h (which defined the height of the pumpkin). Explain how the maximum of function h, once we know it, can tell us the maximum of g.

Video: Explaining Your Solutions to Quadratic Situations

Watch the following video to learn more about proving quadratic solutions.

Explaining Solutions to Quadratic Situations (https://openstax.org/r/quadratic-solutions)

Self Check

The equation $g(t) = 2 + 23.7t - 4.9t^2$ models the height, in meters, of a pumpkin t seconds after it has been launched from a catapult. Which of the following substitutions into the quadratic formula is correct to find the value of t where the pumpkin hit the ground?

a. $t = \frac{-23.7 \pm \sqrt{600.89}}{4}$
b. $t = \frac{-23.7 \pm \sqrt{600.89}}{-4.9}$
c. $t = \frac{-23.7 \pm \sqrt{600.89}}{-9.8}$
d. $t = \frac{23.7 \pm \sqrt{600.89}}{-9.8}$

Additional Resources

Answering Questions About Situations With the Quadratic Formula

Here is an example of someone solving a **quadratic equation** that has no solutions:

Step 1 - $(x + 3)^2 + 9 = 0$

Step 2 - $(x + 3)^2 = -9$

Step 3 - $x + 3 = \pm\sqrt{-9}$

Study the example. At what point did you realize the equation had no solutions?

- For two numbers to add up to 0, they have to be opposites. Since the number 9 is positive, the other number must be negative, which a square cannot be.

Explain how you know the equation $49 + x^2 = 0$ has no solutions.

- A square is always positive, so a square plus a positive number must also be positive and can never equal zero.

TRY IT **Answering Questions About Situations With the Quadratic Formula**

Function h models the height of an object, in meters, t seconds after it is launched into the air. It is defined by $h(t) = -5t^2 + 25t$. How much time will it take the object to reach 15 meters?

9.7.4 Practice Spotting Calculation Errors

Cool Down

1. Here is someone's solution to the equation $3x^2 - 4x = 20$.

Step 1 - $3x^2 - 4x = 20$

Step 2 - $a = 3$, $b = -4$, $c = 20$

Step 3 - $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Step 4 - $x = \frac{-4 \pm \sqrt{(-4)^2 - (4)(3)(20)}}{2(3)}$

Step 5 - $x = \frac{-4 \pm \sqrt{-16 - 240}}{6}$

Step 6 - $x = \frac{-4 \pm \sqrt{-256}}{6}$

No Solutions

Identify as many errors as you can and briefly explain each error.

9.7.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Mai and Jada are solving the equation $2x^2 - 7x = 15$ using the quadratic formula.

Answer the following questions about the values of a, b, and c if this equation is written in standard form, $ax^2 + bx + c = 0$.

a. What is the value of a?

b. What is the value of b?

c. What is the value of c?

d. Mai and Jada are solving the equation $2x^2 - 7x = 15$ using the quadratic formula but found different solutions.

Mai's Method	Jada's Method
$x = \frac{-7 \pm \sqrt{7^2 - 4(2)(-15)}}{2(2)}$ $x = \frac{-7 \pm \sqrt{49 - (-120)}}{4}$ $x = \frac{-7 \pm \sqrt{169}}{4}$ $x = \frac{-7 \pm 13}{4}$ $x = \frac{-7+13}{4} = \frac{6}{4} = \frac{3}{2}$ $x = \frac{-7-13}{4} = \frac{-20}{4} = -5$	$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(-15)}}{2(2)}$ $x = \frac{7 \pm \sqrt{49 - (-120)}}{4}$ $x = \frac{7 \pm \sqrt{71}}{4}$

Who has the correct solution?

a. Mai
b. Jada
c. Both of them
d. Neither one

2. The equation $h(t) = -16t^2 + 80t + 64$ represents the height, in feet, of a potato t seconds after it was launched from a mechanical device.

a. Choose the equation that would allow us to find the time the potato hits the ground.

a. $0 = t^2 - 5t - 4$
b. $0 = -16t^2 + 80t + 64$
c. $0 = -t^2 - 5t + 4$
d. $0 = 16t^2 + 80t + 64$

b. Solve the equation without graphing and write it down to find the approximate time the potato hits the ground rounded to one decimal place.

3. Priya found $x = 3$ and $x = -1$ as solutions to $3x^2 - 6x - 9 = 0$.

a. If you check Priya's answer by substitution, is the answer correct?

a. No
b. Yes

b. If you check Priya's answer by graphing, is the answer correct?

a. No
b. Yes

c. If you check Priya's answer by using the quadratic formula, is the answer correct?

a. No
b. Yes

4. A frog jumps in the air. The height, in inches, of the frog is modeled by the function $h(t) = 60t - 75t^2$, where t is the time after it jumped, measured in seconds.

a. **Select two** solutions to the equation $60t - 75t^2 = 0$.

a. 0
b. $\frac{4}{5}$
c. 5
d. 1
e. $-\frac{1}{5}$

b. What do the solutions tell us about the jumping frog? **Select two** that apply.

a. The frog landed 1 seconds after it jumped and was on the ground again.
b. The frog landed $\frac{4}{5}$ seconds after it jumped and was on the ground again.
c. The height of the frog was 5 inches before it jumped.
d. The negative solution shows how far down the frog was before it jumped.
e. The height of the frog was 0 inches before it jumped.

5. A tennis ball is hit straight up in the air, and its height, in feet above the ground, is modeled by the equation $f(t) = 4 + 12t - 16t^2$, where t is measured in seconds since the ball was thrown.

a. **Select two** solutions to the equation $0 = 4 + 12t - 16t^2$.

a. $\frac{1}{4}$
b. 4
c. 16
d. 1
e. $-\frac{1}{4}$

b. What do the solutions tell us about the tennis ball? **Select two** that apply.

a. The ball hits the ground $\frac{1}{4}$ second after it is hit.
b. The negative solution does not have meaning because a negative value for time does not make sense.
c. The ball hits the ground 1 second after it is hit.
d. The negative solution is a result of the bounce of the ball.
e. The ball hits the ground $-\frac{1}{4}$ second after it is hit.

9.7.6 Lesson Summary

In this lesson, you learned how to:

- Identify common errors when using the quadratic formula.
- Describe some ways to tell if a number is a solution to a quadratic equation.

Here are the **activities** that helped you reach those goals:

- 9.7.1: Evaluating Variable Expressions
  - In this activity, you evaluated expressions using substitution.
- 9.7.2: Common Calculation Errors When Using the Quadratic Formula
  - In this activity, you explored how four different quadratic equations were solved by the quadratic formula. With each one, you identified the error that occurred and explained why it was an incorrect step.
  - 9.7.2: Self Check
  - 9.7.3: Additional Resources
- 9.7.3: Different Methods of Checking Solutions of Quadratic Equations
  - In this activity, you looked at an application of quadratic equations and used the quadratic formula to prove your answer, and justify if you agreed with the answers given by a sample student.
  - 9.7.3: Self Check
  - 9.7.3: Additional Resources
- 9.7.4: Practice Spotting Calculation Errors
  - In this activity, you examined an attempt to solve a quadratic equation using the quadratic formula and identified and explained all errors in the solution.

After these activities, you completed the following **practice**:

- 9.7.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

9.8 DERIVING THE QUADRATIC FORMULA

9.8.0 Lesson Overview

In this lesson, you will again be working with the quadratic formula. You will begin to analyze how it produces solutions to any quadratic equation and how it relates to completing the square.

When you finish this lesson, you will be able to:

- Explain the steps and complete some missing steps for deriving the quadratic formula.
- Explain how the quadratic formula is related to the process of completing the square for a quadratic equation $ax^2 + bx + c = 0$.

Here are the **activities** that will help you reach those goals:

- 9.8.1: Deriving the Quadratic Formula, Part 1
  - 9.8.1: Self Check
  - 9.8.1: Additional Resources
- 9.8.2: Deriving the Quadratic Formula, Part 2
  - 9.8.2: Self Check
  - 9.8.2: Additional Resources
- 9.8.3: Understanding That the Quadratic Formula Is the Combined Steps of Completing the Square

After that, you'll **practice and review**.

- 9.8.4: Practice
- 9.8.5: Lesson Summary

9.8.1 Deriving the Quadratic Formula, Part 1

Activity

1. One way to solve the quadratic equation $x^2 + 5x + 3 = 0$ is by **completing the square**. A partially solved equation is shown here. Study Steps 1 – 4.
   Original equation $x^2 + 5x + 3 = 0$
   Step 1 - Multiply each side by 4. This helps us avoid some tricky fractions. (In general, multiply each side of the equation by 4 times the coefficient of the x^2 term. In this case, that is just 4 times 1, which is 4.)
   $4x^2 + 20x + 12 = 0$
   Step 2 - Subtract 12 from each side.
   $4x^2 + 20x = -12$
   Step 3 - Rewrite
   $4x^2$ as $(2x)^2$ and $20x$ as $10(2x)$.
   $(2x)^2 + 10(2x) = -12$
   Step 4 - Use P as a placeholder for $2x$.
   $P^2 + 10P = -12$
   For steps 5 – 11, let P serve as a placeholder for $2x$, continue to solve for x but without evaluating any part of the expression. For each step, explain the mathematical operation(s) being performed.
   Step 5 - _____
   $P^2 + 10P + ____^2 = -12 + ____^2$
   Step 6 - _____
   $(P + ____)^2 = -12 + ____^2$
   Step 7 - _____
   $\sqrt{(P + ____)^2} = \sqrt{-12 + ____^2}$
   $P + ____ = \pm\sqrt{-12 + ____^2}$
   Step 8 - _____
   $P = ____ \pm \sqrt{-12 + ____^2}$
   Step 9 - _____
   $P = ____ \pm \sqrt{____^2 - 12}$
   Step 10 -
   _____
   $2x = ____ \pm \sqrt{____^2 - 12}$
   Step 11
   _____
   $x =$
2. Explain how the solution is related to the **quadratic formula**.

Self Check

After completing the square, Odem used the following steps when deriving the quadratic formula.

$x^2 + 8x + 16 = -11 + 16$
$x^2 + 8x + 16 = 5$

What should he do next?

a. Complete the square again.
b. Factor the perfect square trinomial.

c. Take the square root of both sides.
d. Subtract 16 from both sides.

Additional Resources

Deriving the Quadratic Formula Using an Example

Here is another example that goes through the process of deriving the **quadratic formula**:

Original equation $x^2 + 7x + 4 = 0$

Step 1 - Multiply each side by 4. This helps us avoid some tricky fractions. (In general, multiply each side of the equation by 4 times the coefficient of the x^2 term. In this case, that is just 4 times 1, which is 4.)

$4x^2 + 28x + 16 = 0$

Step 2 - Subtract 16 from each side.

$4x^2 + 28x = -16$

Step 3 - Rewrite.

$4x^2$ as $(2x)^2$ and $28x$ as $14(2x)$

$(2x)^2 + 14(2x) = -16$

Step 4 - Use P as a placeholder for $2x$.

$P^2 + 14P = -16$

Step 5 - Add to complete the square.

$P^2 + 14P + 7^2 = -16 + 7^2$

Step 6 - Write the left side as a squared factor.

$(P + 7)^2 = -16 + 7^2$

Step 7 - Find the square root of the expression on the right.

$P + 7 = \pm\sqrt{-16 + 7^2}$

Step 8 - Subtract 7 from both sides to isolate.

$P = -7 \pm \sqrt{-16 + 7^2}$

Step 9 - Rearrange the expression under the square root sign.

$P = -7 \pm \sqrt{7^2 - 16}$

Step 10 - Re-substitute $2x$ for P.

$2x = -7 \pm \sqrt{7^2 - 16}$

Step 11 - Divide each side by 2 to isolate x.

$x = \frac{-7 \pm \sqrt{7^2 - 16}}{2}$

Notice the original equation, $y = x^2 + 7x + 4$, has $a = 1$, $b = 7$, $c = 4$, and the last step is of the form $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, which is the quadratic formula.

TRY IT **Deriving the Quadratic Formula Using an Example**

Jaiden was deriving the quadratic formula from the quadratic $x^2 + 11x + 5 = 0$. What is the step after the step shown below?

$$(x+\frac{11}{2})^2 = \frac{11^2-4(1)(5)}{4(1)}$$

$$(x+\frac{11}{2})^2 = \frac{121-20}{4}$$

9.8.2 Deriving the Quadratic Formula, Part 2

Activity

1. Here is one way to make sense of how the **quadratic formula** came about. Study the derivation until you can explain what happened in each step. Write down your explanation for each step.

Original equation

$ax^2+bx+c=0$

Step 1 - $4a^2x^2+4abx+4ac=0$

Step 2 - $4a^2x^2+4abx=-4ac$

Step 3 - $(2ax)^2+2b(2ax)=-4ac$

Step 4 - $M^2+2bM=-4ac$

Step 5 - $M^2+2bM+b^2=-4ac+b^2$

Step 6 - $(M+b)^2=b^2-4ac$

Step 7 - $\sqrt{(M+b)^2}=\sqrt{b^2-4ac}$

$M+b=\pm\sqrt{b^2-4ac}$

Step 8 - $M=-b\pm\sqrt{b^2-4ac}$

Step 9 - $2ax=-b\pm\sqrt{b^2-4ac}$

Step 10 -

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Are you ready for more?

Extending Your Thinking

1. Here is another way to derive the quadratic formula by **completing the square**.

- First, divide each side of the equation $ax^2+bx+c=0$ by a to get $x^2+\frac{b}{a}x+\frac{c}{a}=0$.
- Then, complete the square for $x^2+\frac{b}{a}x+\frac{c}{a}=0$.

In problems 1 – 10, briefly explain what happens in each step by answering the question.

Original equation

$x^2+\frac{b}{a}x+\frac{c}{a}=0$

Step 1

What constant should be subtracted from both sides?

2. Step 2

What constant do we need to add to both sides to complete the square?

3. Step 3

Now we can complete the square. What is the binomial that is squared on the left side?

4. Step 4

Next, we add the fractions on the right side of the equation. What will the common denominator be?

5. Step 5

Once the fractions on the right side of the equation are added together, what will the numerator of that new fraction be?

6. Step 6

What is the next step necessary to isolate x on the left side?

7. Step 7
How can the square root be simplified?

8. Step 8
What is the denominator of the fraction on the right side after simplification?

9. Step 9
What is the next step I need to take to isolate x on the left side of the equation?

10. Step 10
What do you notice about the denominator of both fractions on the right side?

Video: Deriving the Quadratic Formula

Watch the following video to learn more about deriving the quadratic formula.

Deriving the Quadratic Formula (https://openstax.org/r/quadratic-formula-derive)

Self Check

Looking at the steps of deriving the quadratic formula, which step should follow the one below?

$(x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2}$

a. Subtract $\frac{b}{2a}$ from both sides.
b. Multiply both sides by $4a^2$.
c. Take the square root of both sides.
d. Complete the square.

Additional Resources

A General Way to Derive the Quadratic Formula

Here is another way to look at how to derive the **quadratic formula**:

We can derive the quadratic formula by **completing the square**. We will assume that the leading coefficient is positive; if it is negative, we can multiply the equation by −1 and obtain a positive a. Given $ax^2+bx+c=0$, $a\neq 0$, we will complete the square as follows:

Step 1 - First, move the constant term to the right side of the equal sign:

$ax^2+bx=-c$

Step 2 - As we want the leading coefficient to equal 1, divide through by a:

$x^2+\frac{b}{a}x=-\frac{c}{a}$

Step 3 - Then, find $\frac{1}{2}$ of the middle term, and add $(\frac{1}{2}\cdot\frac{b}{a})^2=\frac{b^2}{4a^2}$ to both sides of the equal sign:

$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$

Step 4 - Next, write the left side as a **perfect square**. Find the common denominator of the right side and write it as a single fraction:

$(x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2}$

Step 5 - Now, use the square root property, which gives:

$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$

$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$

Step 6 - Finally, add $-\frac{b}{2a}$ to both sides of the equation and combine the terms on the right side. Thus:

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

TRY IT **A General Way to Derive the Quadratic Formula**

Using the general way to derive the quadratic formula, find the step after

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

9.8.3 Understanding That the Quadratic Formula Is the Combined Steps of Completing the Square

Cool Down

Here is one solution for solving $x^2 + 3x + 8 = 0$ by **completing the square**, where each step is shown, but numerical expressions are not evaluated.

Original equation $x^2 + 3x + 8 = 0$

Step 1 - $x^2 + 3x = -8$

Step 2 - $4x^2 + 4(3x) = 4(-8)$

Step 3 - $(2x)^2 + 6(2x) = -32$

Step 4 - $P^2 + 6P = -32$

Step 5 - $P^2 + 6P + 3^2 = -32 + 3^2$

Step 6 - $(P + 3)^2 = 3^2 - 32$

Step 7 - $P + 3 = \pm\sqrt{3^2 - 32}$

Step 8 - $P = -3 \pm \sqrt{3^2 - 32}$

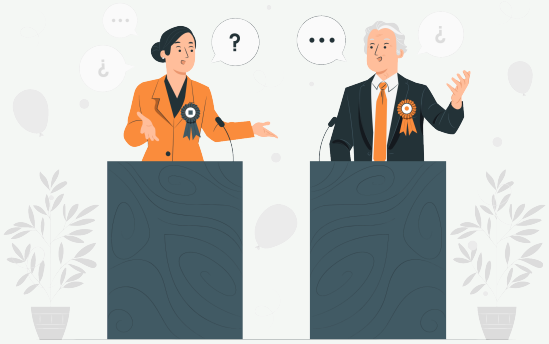
Step 9 - $2x = -3 \pm \sqrt{3^2 - 32}$

Step 10 -

$$x = \frac{-3 \pm \sqrt{3^2-32}}{2}$$

1. In Step 2, the equation is multiplied by 4. Why might that be?
2. In Step 5, 3^2 is added to each side. Why might that be?
3. What happened between Step 5 and Step 6?

Building Character: Intellectual Humility



No matter how old you are, with intellectual humility you become wiser. It helps you be less judgmental of others, learn more in school, and be a better leader.

Think about yourself. Are these statements true for you?

- I recognize the value in opinions that are different from my own.
- I accept that my beliefs and attitudes may be wrong.

Don't worry if none of these statements are true for you. Developing this trait takes time. Your first step starts today!

9.8.4 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. The quadratic equation $x^2 + 7x + 10 = 0$ is in the form of $ax^2 + bx + c = 0$.

a. What is the value of a?

b. What is the value of b?

c. What is the value of c?

Examine the solution method that has been started below and use it to answer the remaining three parts of question 1.

Original equation
$x^2 + 7x + 10 = 0$

Step 1 - Subtract 10 from each side.
$x^2 + 7x = -10$

Step 2 - Multiply each side by 4.
$4x^2 + 4(7x) = -4(10)$

Step 3 - Rewrite $4x^2$ as $(2x)^2$ and $4(7x)$ as $2(7)2x$.
$(2x)^2 + 2(7)(2x) + ____^2 = ____^2 - 4(10)$

Step 4 - _____
$(2x + ____)^2 = ____^2 - 4(10)$

Step 5 - _____
$2x + ____ = \pm\sqrt{____^2 - 4(10)}$

Step 6 - _____
$2x = ____ \pm \sqrt{____^2 - 4(10)}$

Step 7 - _____
$x = ____$

d. In Step 2, what might be a good reason for multiplying each side of the equation by 4?

a. Multiplying by 4 makes the coefficient of the squared term a perfect square, which makes it easier to complete the square.
b. Multiplying by 4 makes the coefficient of the b term a perfect square, which makes it easier to complete the square.
c. If the value of c is positive and you multiply by 4, it will be easier to divide the b term by 3 before you square it.
d. You must always multiply both sides by 4 when completing the square.

e. Complete the unfinished steps, then **select two** solutions.

a. -10
b. -5
c. -2
d. -4

f. Substitute the values of a, b, and c into the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, but do not evaluate any of the expressions. Explain how this expression is related to solving $x^2 + 7x + 10 = 0$ by completing the square.

a. For this equation, the quadratic formula and completing the square are not the same and therefore not related.
b. They are related because neither one can have a negative c value.
c. Rather than evaluating at each step, the calculation is done all at once, at the end.
d. In both forms of solving the equation, you need to list the values of a, b, and c to determine the solutions.

2. Consider the standard form of the equation $x^2 - 39 = 0$.

a. What is the value of a?

b. What is the value of b?

c. What is the value of c?

d. Can you use the quadratic formula to solve this equation?

   a. No
   b. Yes

e. Can you solve this equation using square roots?

   a. No, this equation can only be solved by the quadratic formula.
   b. Yes, the solutions are $\pm\sqrt{39}$.
   c. No, there is no solution.
   d. Yes, the solutions are ± 39.

3. Clare is deriving the quadratic formula by solving $ax^2 + bx + c = 0$ by completing the square.

She arrived at this equation: $(2ax + b)^2 = b^2 - 4ac$.

Choose the best description of what she needs to do to finish solving for x.

   a. Find the square roots of each side. Divide each side by 2. Subtract b from each side.
   b. Divide each side by 2. Subtract b from each side. Then find the square roots of each side.
   c. Find the square roots of each side. Subtract b from each side. Then divide each side by $2a$.
   d. Subtract b^2 from each side. Find the square roots of each side. Then divide each side by 2.

9.8.5 Lesson Summary

In this lesson, you learned how to:

- Explain the steps and complete some missing steps for deriving the quadratic formula.
- Explain how the quadratic formula is related to the process of completing the square for a quadratic equation $ax^2 + bx + c = 0$.

Here are the **activities** that helped you reach those goals:

- 9.8.1: Deriving the Quadratic Formula, Part 1
  - In this activity, you learned how to derive the quadratic formula by completing the square for a quadratic equation without evaluating some of the numerical expressions along the way.
  - 9.8.1: Self Check
  - 9.8.1: Additional Resources
- 9.8.2: Deriving the Quadratic Formula, Part 2
  - In this activity, you studied a series of steps to solve $ax^2 + bx + c = 0$ by completing the square and making sense of how it leads to the quadratic formula.
  - 9.8.2: Self Check
  - 9.8.2: Additional Resources
- 9.8.3: Understanding That the Quadratic Formula Is the Combined Steps of Completing the Square
  - In this activity, you had to explain parts of a solution for solving an equation by completing the square, where each step was shown, but numerical expressions were not evaluated.

After these activities, you completed the following **practice**:

- 9.8.4: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

9.9 WRITING QUADRATICS IN DIFFERENT FORMS

9.9.0 Lesson Overview

In this lesson, you will practice identifying parts of quadratic functions from different forms. You will also work to rewrite quadratic functions from one form to another.

When you finish this lesson, you will be able to:

- Identify the vertex of a quadratic function from an equation.
- Write quadratics in vertex, factored, and standard forms.
- Write quadratic functions in vertex form given a vertex and another point.

Here are the **activities** that will help you reach those goals:

- 9.9.1: Finding the Vertex in Vertex Form
- 9.9.2: Different Forms of Quadratics
  - 9.9.2: Self Check
  - 9.9.2: Additional Resources
- 9.9.3: Vertex Form Given a Vertex and Point
  - 9.9.3: Self Check
  - 9.9.3: Additional Resources
- 9.9.4: Rewriting Quadratics

After that, you'll **practice and review**.

- 9.9.5: Practice
- 9.9.6: Lesson Summary

9.9.1 Finding the Vertex in Vertex Form

Warm Up

1. Write the quadratic function $f(x) = x^2 + 6x + 13$ in vertex form.
2. Using the vertex form of the quadratic found in question 1, what is the vertex?
3. Graph both equations from question 1 (in standard form and vertex form).
   Access multimedia content (http://openstax.org/books/algebra-1/pages/9-9-1-finding-the-vertex-in-vertex-form)
4. Determine if the vertex for the quadratic function is a maximum or minimum.
5. What is the axis of symmetry?

9.9.2 Different Forms of Quadratics

Activity

For questions 1 – 5, write each function in vertex form.

1. $y = x^2 - 4x + 7$
2. $y = (x - 1)(x + 3)$
3. $y = (x - 2)(x + 2)$
4. $y = -x^2 - 2x - 6$
5. $y = 2x^2 - 12x + 22$

For questions 6 and 7, write each function in standard form.

6. $y = (x - 2)^2 + 7$
7. $y = (x + 3)^2 - 2$

Are you ready for more?

Extending Your Thinking

1. A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$. What is the maximum height?

Self Check

Which of the following is $f(x) = (x-2)^2 + 15$ written in standard form?

a. $f(x) = x^2 + 4x + 19$
b. $f(x) = x^2 - 4x + 19$
c. $f(x) = x^2 - 4x + 11$
d. $f(x) = x^2 + 19$

Additional Resources

Writing Quadratics in Different Forms

Three Common Quadratic Forms:

The standard form of a quadratic function is $f(x) = ax^2 + bx + c$ where a, b, c, are real numbers and $a \neq 0$.

The vertex form of a quadratic function is $f(x) = a(x-h)^2 + k$ where (h, k) is the vertex.

The factored form of the quadratic function is $f(x) = (x-a)(x-b)$ where a and b are roots of $f(x)$.

EXAMPLE 1

Write the function $f(x) = (x-2)(x+8)$ in standard form.

Step 1 - multiply the binomials using distribution (FOIL).

$f(x) = (x-2)(x+8)$

$f(x) = x^2 - 2x + 8x - 16$

Step 2 - Simplify.

$f(x) = x^2 + 6x - 16$

EXAMPLE 2

Write the function $f(x) = (x-2)(x+8)$ in vertex form.

Hint: Start by using the answer from example 1.

Step 1 - Complete the square.

$(\frac{b}{2})^2 = (\frac{6}{2})^2 = 9$

Step 2 - Balance the equation. Add the constant to the quadratic to complete the square and subtract it from the original constant. $f(x) = x^2 + 6x + 9 - 16 - 9$

Finally, factor the trinomial and simplify.
$f(x) = (x+3)^2 - 25$

EXAMPLE 3

Rewrite the function in standard form:
$f(x) = 6(x-2)^2 + 4$

Step 1 - Expand the square.

$6(x-2)^2 + 4\ 6(x^2 - 4x + 4) + 4$

Step 2 - Distribute the coefficient.

$6x^2 - 24x + 24 + 4$

Step 3 - Simplify like terms
$f(x) = 6x^2 - 24x + 28$

EXAMPLE 4

Rewrite the function in standard form:
$f(x) = 5(x-3)^2 + 2$

Step 1 - Expand the square.

$5(x-3)^2 + 2$

$5(x^2 - 6x + 9) + 2$

Step 2 - Distribute the coefficient.

$5x^2 - 30x + 45 + 2$

Step 3 - Simplify like terms
$f(x) = 5x^2 - 30x + 47$

TRY IT **Writing Quadratic Functions in Equivalent Forms**
Write the function $f(x) = (x+5)(x+3)$ in vertex form.

9.9.3 Vertex Form Given a Vertex and Point

Access the Desmos guide PDF (https://openstax.org/r/desmos-guide) for tips on solving problems with the Desmos graphing calculator.

Activity

For questions 1 – 2, write an equation in vertex form of the quadratic function that has a vertex at (5, 8) and also goes through the point $(-3, 6)$.

1. Find the value of a for vertex form.

2. Write the equation in vertex form.

For questions 3 and 4, write an equation in vertex form of the quadratic function that has a vertex at $(-4, 3)$ and also goes through the point $(2, 5)$.

3. Find the value of a for vertex form.

4. Write the equation in vertex form.

For questions 5 – 6, write an equation in vertex form of the quadratic function that has a vertex at $(6, -2)$ and also goes through the point $(3, 1)$.

5. Find the value of a for vertex form.

6. Write the equation in vertex form.

Video: Writing an Quadratic Equation from a Vertex and a Point

Watch the following video to learn more about writing a quadratic equation from a vertex and a point.

9 9 3 Worked Solution FINAL (https://www.youtube.com/embed/1IUEVwIjywc?si=lbcWBG5YBlTAOW1k)

Self Check

Which of the following is the quadratic equation of the quadratic with a vertex of $(4, -2)$ that also goes through the point $(6, 5)$?

a. $y = \frac{4}{7}(x-6)^2 - 5$
b. $y = \frac{7}{4}(x-6)^2 - 5$
c. $y = \frac{7}{4}(x-4)^2 - 2$
d. $y = \frac{4}{7}(x-4)^2 - 2$

Additional Resources

Creating Vertex Form from a Vertex and a Point

Write the quadratic in vertex form that has a vertex at $(3, 4)$ and goes through the point $(1, 2)$.

First start with the vertex form of a quadratic function.

$y = a(x-h)^2 + k$

Step 1 - Substitute the vertex in at (h, k).

$y = a(x-3)^2 + 4$

Step 2 - Substitute the other point at (x, y).

$2 = a(1-3)^2 + 4$

Step 3 - Solve for a.

$2 = a(1-3)^2 + 4$

$2 = a(-2)^2 + 4$

$2 = a(4) + 4$

$2 - 4 = a(4) + 4 - 4$

$-2 = a(4)$

$-\frac{2}{4} = \frac{a(4)}{4}$

$a = -\frac{1}{2}$

Step 4 - Substitute the value of a value and the vertex (h, k) into the vertex form.

$y = -\frac{1}{2}(x-3)^2 + 4$

TRY IT **Creating Vertex Form from a Vertex and a Point**

Write the quadratic in vertex form that has a vertex at $(2, 5)$ and goes through the point $(3, 7)$.

9.9.4 Rewriting Quadratics

Cool Down

For questions 1 – 2, use the function $f(x) = x^2$ that is shifted to create the function $g(x)$ that has a vertex at $(3, -2)$.

1. Write $g(x)$ in vertex form.

2. Write $g(x)$ in standard form.

9.9.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Which of the following points is the vertex of the quadratic function $f(x) = x^2 + 8x + 22$?

a. $(4, 6)$
b. $(-4, 6)$
c. $(0, 22)$
d. $(8, 22)$

For questions 2 – 4, use the function $f(x) = (x - 4)(x + 2)$.

2. Write the function in standard form.

3. Write the function from question 2 in vertex form.

4. Identify the vertex of the quadratic function $f(x) = (x - 1)^2 - 9$.

a. $(-9, -1)$
b. $(-1, 9)$
c. $(1, 9)$
d. $(1, -9)$

For questions 5 - 6, write an equation in vertex form of the quadratic function that has a vertex at $(-4, 1)$ and also goes through the point $(6, -1)$.

5. Find the value of a for vertex form in simplest form.

6. Write the equation in vertex form. Hint: Be sure to use $y =$ to denote your equation.

For questions 7 and 8, write an equation in vertex form of the quadratic function that has a vertex at $(-2, -3)$ and also goes through the point $(4, 2)$.

7. Find the value of a for vertex form in simplest form.

8. Write the equation in vertex form. Hint: Be sure to use $y =$ to denote your equation.

For questions 9 - 10, use the function $f(x) = x^2$ that is shifted to create the function $g(x)$ that has a vertex at $(-4, 6)$

9. Write $g(x)$ in vertex form.

10. Write $g(x)$ in standard form.

9.9.6 Lesson Summary

In this lesson, you learned how to:

- Identify the vertex of a quadratic function from an equation.
- Write quadratics in vertex, factored, and standard forms.
- Write quadratic functions in vertex form given a vertex and another point.

Here are the **activities** that helped you reach those goals:

- 9.9.1: Finding the Vertex in Vertex Form
  - In this activity, you identified the vertex from the quadratic function that was written in vertex form.
- 9.9.2: Different Forms of Quadratics
  - In this activity, you practiced writing quadratic functions in the different forms: vertex form, factored form, and standard form.
  - 9.9.2: Self Check
  - 9.9.2: Additional Resources
- 9.9.3: Vertex Form Given a Vertex and Point
  - In this activity, you continued working with quadratic functions and their forms by writing a quadratic function in vertex form given its vertex and a point on the quadratic graph.
  - 9.9.3: Self Check
  - 9.9.3: Additional Resources
- 9.9.4: Rewriting Quadratics
  - In this activity, you wrote the equation for a quadratic function in vertex form when given a transformation of the parent function.

After these activities, you completed the following **practice**:

- 9.9.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

9.10 REWRITING QUADRATIC EXPRESSIONS IN VERTEX FORM

9.10.0 Lesson Overview

In this lesson, you will review the vertex form of a quadratic expression, its advantage, and its connections to the graph. You will also review how to transform expressions in factored form to standard form, and expressions in vertex form into standard form. You will then experiment with transforming the same expressions in standard form back to vertex form.

When you finish this lesson, you will be able to:

- Identify the vertex of the graph of a quadratic function when the expression that defines it is written in vertex form.
- Explain the meaning of the term "vertex form" and recognize examples of quadratic expressions written in this form.
- When given a quadratic expression in factored form, rewrite it in standard form.
- When given a quadratic expression in standard form, rewrite it in vertex form.

Here are the **activities** that will help you reach those goals:

- 9.10.1: The Vertex and Intercepts of a Function
- 9.10.2: Expanding from Factored Form to Standard Form
- 9.10.3: Converting from Standard Form to Vertex Form
  - 9.10.3: Self Check
  - 9.10.3: Additional Resources
- 9.10.4: Rewriting Expressions in Vertex Form
  - 9.10.4: Self Check
  - 9.10.4: Additional Resources
- 9.10.5: Different Forms of Quadratic Expressions
- 9.10.6: Vertex Form and Coordinates of the Vertex

After that, you'll **practice and review**.

- 9.10.7: Practice
- 9.10.8: Lesson Summary

9.10.1 The Vertex and Intercepts of a Function

Warm Up

These expressions each define the same function.

- $x^2 + 6x + 8$
- $(x + 2)(x + 4)$
- $(x + 3)^2 - 1$

Without graphing or doing any calculations, determine where the following features would be on a graph that represents the function.

1. What is the x-coordinate of the vertex (of the graph)?
   The x-coordinate of the **vertex (of a graph)**.

2. What is the y-coordinate of the vertex (of the graph)?
   The y-coordinate of the vertex (of a graph).

3. **Select both x-intercepts** .
   a. $(-2, 0)$
   b. $(-6, 0)$
   c. $(-4, 0)$
   d. $(-8, 0)$

4. What is the y-coordinate of the y-intercept?
The y-coordinate of the **y-intercept** $(0, _____)$

9.10.2 Expanding from Factored Form to Standard Form

Warm Up

Recall learning how to identify and factor perfect square trinomials in Unit 6. Here are some perfect squares in factored and standard forms, and an expression showing how the two forms are related.

Study the first few examples in rows 1 and 2. Create a similar table to the one modeled below. Copy the table (or use a copy provided by your teacher) and then complete the missing numbers in rows 3 – 6 in the rest of the table.

Row	Factored Form		Standard Form
1	$(x+4)^2$	$(1x)^2+2(1x)(4)+4^2$	$x^2+8x+16$
2	$(2x+5)^2$	$(2x)^2+2(2x)(5)+5^2$	$4x^2+20x+25$
3	$(3x-4)^2$	$(3x)^2+2(___\,x)(___)+(___)^2$	$9x^2-24x+16$
4	$(5x+___)^2$	$(___\,x)^2+2(___\,x)(___)+(___)^2$	$25x^2+30x+___$
5	$(kx+m)^2$	$(___\,x)^2+2(___\,x)(___)+(___)^2$	$(___)^2x^2+___\,x+(___)^2$

1. Enter the missing values from row 3.

Row	Factored Form		Standard Form
3	$(3x-4)^2$	$(3x)^2+2(___\,x)(___)+(___)^2$ $(3x)^2+2(\ \mathbf{a.}\ x)(\ \mathbf{b.}\ )+(\ \mathbf{c.}\ )^2$	$9x^2-24x+16$

a. What is the missing value of a?

b. What is the missing value of b?

c. What is the missing value of c?

2. Enter the missing values from row 4.

Row	Factored Form		Standard Form
4	$(5x+___)^2$ $(5x+\ \mathbf{a.}\ )^2$	$(___\,x)^2+2(___\,x)(___)+(___)^2$ $(\ \mathbf{b.}\ x)^2+2(\ \mathbf{c.}\ x)(\ \mathbf{d.}\ )+(\ \mathbf{e.}\ )^2$	$25x^2+30x+___$ $25x^2+30x+\ \mathbf{f.}$

a. What is the missing value of a?

b. What is the missing value of b?

c. What is the missing value of c?

d. What is the missing value of d?

e. What is the missing value of e?

f. What is the missing value of f?

3. Enter the missing values from row 5.

Row	Factored Form		Standard Form
5	$(kx+m)^2$	$(___\ x)^2+2(___\ x)(___)+(___)^2$ $(\underline{\ \mathbf{a.}\ }\ x)^2+2(\underline{\ \mathbf{b.}\ }\ x)(\underline{\ \mathbf{c.}\ })+(\underline{\ \mathbf{d.}\ })^2$	$(___)^2x^2+___\ x+(___)^2$ $(\underline{\ \mathbf{e.}\ })^2x^2+\underline{\ \mathbf{f.}\ }x+(\underline{\ \mathbf{g.}\ })^2$

a. What is the missing value of a?

b. What is the missing value of b?

c. What is the missing value of c?

d. What is the missing value of d?

e. What is the missing value of e?

f. What is the missing value of f?

g. What is the missing value of g?

4. Look at the expression in the last row of the table. If ax^2+bx+c is equivalent to $(kx+m)^2$, how are a, b, and c related to k and m?

9.10.3 Converting from Standard Form to Vertex Form

Activity

You've expanded from factored form to standard form. To learn how to convert from standard form to **vertex form (of a quadratic expression)**, start by going backward and convert a quadratic from vertex form to standard form.

1. Rewrite $(x+5)^2+1$ in standard form.

2. Rewrite $(x-3)^2-7$ in standard form.

3. Think about the steps you took, and about reversing them. Try converting one or both of the expressions in standard form back into vertex form. Explain how you go about converting the expressions.

4. Test your strategy by rewriting $x^2+10x+9$ in vertex form.

5. Let's check the expression you rewrote in vertex form. Use the Desmos graphing tool or technology outside the course to graph both $x^2+10x+9$ and your new expression.

Access multimedia content (http://openstax.org/books/algebra-1/pages/9-10-3-converting-from-standard-form-to-vertex-form)

a. Does it appear that both expressions define the same function?

b. If you convert your expression in vertex form back into standard form, do you get $x^2+10x+9$?

Self Check

Which of the following is the equivalent vertex form of the expression $x^2+14x-3$?

a. $(x+7)^2+46$
b. $(x+7)^2-52$
c. $(x+7)^2-3$
d. $(x+7)^2$

Additional Resources

Standard to Vertex Form

Different forms are useful for finding different characteristics of quadratic functions.

Practice writing $x^2 + 8x - 7$ into **vertex form (of a quadratic expression)**.

Step 1 - Work to complete the square:

$b = 8$, so $(\frac{b}{2})^2 = (\frac{8}{2})^2 = 16$.

Step 2 - Add the value to create the perfect square, and subtract the value from the original constant (this is like adding 0).

$x^2 + 8x + 16 - 7 - 16$

Step 3 - Factor the trinomial and simplify.

$(x + 4)^2 - 23$

TRY IT **Standard to Vertex Form**

Convert the quadratic expression $x^2 + 16x - 10$ to vertex form.

9.10.4 Rewriting Expressions in Vertex Form

Activity

1. Here is one way to rewrite $3x^2 + 12x + 9$ in **vertex form (of a quadratic expression)**.

a. Study the steps and write a brief explanation of what is happening at each step.

Original expression $3x^2 + 12x + 9$

Step 1 - $3(x^2 + 4x + 3)$

Step 2 - $3(x^2 + 4x + 4 + 3 - 4)$

Step 3 - $3(x^2 + 4x + 4 - 1)$

Step 4 - $3((x + 2)^2 - 1)$

Step 5 - $3(x + 2)^2 - 3$

b. What is the x-coordinate of the **vertex (of a graph)** that represents this expression?

c. What is the y-coordinate of the vertex of the graph that represents this expression?

d. Does the graph open upward or downward? Explain how you know.

2. Rewrite $-2x^2 - 4x + 6$ in vertex form.

3. Rewrite $4x^2 + 24x + 20$ in vertex form.

4. Rewrite $-x^2 + 20x$ in vertex form.

Are you ready for more?

Extending Your Thinking

1. Fill in the missing parts of the function $f(x) = 2(x - 3)(x - 9)$ in vertex form $f(x) =$ **a.** $(x$ **b.** **c.** $)^2$ **d.** **e.** without completing the square. (Hint: Think about finding the zeros of the function.)

a. What is the value of a?

b. What is the sign of b?

c. What is the value of c?

d. What is the sign of d?

e. What is the value of e?

f. Be prepared to show your reasoning for your answers.

2. Fill in the missing parts of the function $g(x) = 2(x-3)(x-9)+21$ in vertex form $g(x) =$ **a.** $(x$ **b.** **c.** $)^2$ **d.** **e.** without completing the square.

a. What is the value of a.

b. What is the sign of b.

c. What is the value of c.

d. What is the sign of d.

e. What is the value of e.

f. Be prepared to show your reasoning for your answers.

Video: Rewriting Expressions in Vertex Form

Watch the following video to learn more about rewriting expressions in vertex form.

Rewriting Expressions in Vertex Form (https://openstax.org/r/vertex-form-rewrite)

Self Check

Write $2x^2+20x+20$ in vertex form.

a. $2(x+5)^2-15$
b. $2(x^2+10x+10)$
c. $2(x+5)^2-30$
d. $2(x+5)^2$

Additional Resources

Vertex Form with a Leading Coefficient Besides 1

Write $2x^2+8x+4$ in **vertex form (of a quadratic expression)**.

Original expression $2x^2+8x+4$

Step 1 - Rewrite the expression with 2 as a factor. $2(x^2+4x+2)$

Step 2 - Add 4 to complete the square. Subtract 4 to keep the value of the expression the same. $2(x^2+4x+4+2-4)$

Step 3 - Combine like terms so we can see the perfect square, x^2+4x+4. $2(x^2+4x+4-2)$

Step 4 - Rewrite x^2+4x+4 as a squared expression. $2((x+2)^2-2)$

Step 5 - Apply the distributive property to put the expression in vertex form. $2(x+2)^2-4$

> **TRY IT** **Vertex Form with a Leading Coefficient Besides 1**
> Write $4x^2+24x+12$ in vertex form.

9.10.5 Different Forms of Quadratic Expressions

Activity

Your teacher will give you either a problem card or a data card. Do not show or read your card to your partner.

If your teacher gives you the data card:

1. Silently read the information on your card.
2. Ask your partner, "What specific information do you need?" and wait for your partner to ask for information. Only give information that is on your card. (Do not figure out anything for your partner!)

3. Before telling your partner the information, ask, "Why do you need to know (that piece of information)?"
4. Read the problem card and solve the problem independently.
5. Share the data card and discuss your reasoning.

If your teacher gives you the problem card:

1. Silently read your card and think about what information you need to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain to your partner how you are using the information to solve the problem.
4. When you have enough information, share the problem card with your partner, and solve the problem independently.
5. Read the data card and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

1. Use the information about functions f and g to answer the questions.

- Function f is defined by $(x + 3)(x + 9)$.
- Function g is defined by $(x + 6)^2 - 1$.

a. Write an expression in **vertex form (of a quadratic expression)** that could define a quadratic function, f.

b. Write an expression in factored form that could define a quadratic function, g.

c. Show that f and g do not define the same function.

2. Functions a and b are quadratic functions.

Function a is defined by $(x - 5)^2 - 4$.

Function b is defined by $(x + 1)(x - 5)$.

a. What are the zeros of function a? **Select two.**

b. What is the x-coordinate of the **vertex (of a graph)** representing function b?

c. What is the y-coordinate of the vertex of the graph representing function b?

d. Show that a and b do not define the same function.

9.10.6 Vertex Form and Coordinates of the Vertex

Cool Down

1. A quadratic function is defined by the expression $x^2 - 20x + 19$.
Rewrite the expression in **vertex form (of a quadratic expression)** and give the coordinates of the **vertex (of a graph)**. Show your reasoning.

9.10.7 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. The following quadratic expressions all define the same function.

- $(x+5)(x+3)$
- $x^2+8x+15$
- $(x+4)^2-1$

Select three statements that are true about the graph of this function.

a. The y-intercept is $(0,-15)$.
b. The vertex is $(-4,-1)$.
c. The x-intercepts are $(-5,0)$ and $(-3,0)$.
d. The x-intercepts are $(0,5)$ and $(0,3)$.
e. The x-intercept is $(0,15)$.
f. The y-intercept is $(0,15)$.
g. The vertex is $(4,-1)$.

2. The following expressions all define the same quadratic function.

- $(x-4)(x+6)$
- $x^2+2x-24$
- $(x+1)^2-25$

a. What is the value of the y-coordinate of the y-intercept $(0,___)$ of the graph of the function?

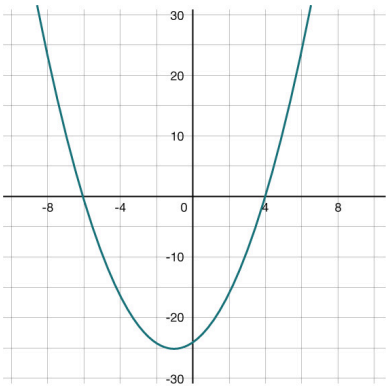
b. **Select two** x-intercepts of the graph.

a. $(6,0)$
b. $(-4,0)$
c. $(-6,0)$
d. $(5,0)$
e. $(4,0)$

c. What is the x-coordinate of the vertex of the graph?

d. What is the y-coordinate of the vertex of the graph?

e. Is this an accurate graph of the function?



a. No
b. Yes

3. Here is one way an expression in standard form is rewritten into vertex form.

Original expression
x^2-7x+6

Step 1 - $x^2 - 7x + (-\frac{7}{2})^2 + 6 - (-\frac{7}{2})^2$

Step 2 - $(x - \frac{7}{2})^2 + 6 - \frac{49}{4}$

Step 3 - $(x - \frac{7}{2})^2 + \frac{24}{4} - \frac{49}{4}$

Step 4 - $(x - \frac{7}{2})^2 - \frac{25}{4}$

a. In Step 1, where did the number $-\frac{7}{2}$ come from?

   a. Only the non-linear term was divided by 2.
   b. It is half of -7 being squared.
   c. Every term in the expression was divided by 2.
   d. It is the substitute value for x.

b. In Step 1, why was $(-\frac{7}{2})^2$ added and then subtracted?

   a. It is subtracted to complete the square. It is added to change the value of the expression.
   b. It is added to complete the square. It is subtracted to change the value of the expression.
   c. It is subtracted to complete the square. It is added to keep the value of the expression unchanged.
   d. It is added to complete the square. It is subtracted to keep the value of the expression unchanged.

c. What happened in Step 2?

   a. The expression is added to keep the value of the expression unchanged.
   b. The expression, which is not a perfect square, is written as a squared expression.
   c. The expression, which is a perfect square, is written as a squared expression.
   d. The expression is subtracted to keep the value of the expression unchanged.

d. What happened in Step 3?

   a. The constant term is written as a fraction so that it has a common denominator, which makes it easier to subtract.
   b. The linear term is written as a fraction so that it has a common denominator, which makes it easier to subtract.
   c. A random term was added to complete the square.
   d. The quadratic term is written as a fraction so that it has a common denominator, which makes it easier to subtract.

e. What does the last expression tell us about the vertex of the graph of a function defined by this expression?

   a. The vertex of the graph is at $(\frac{5}{2}, -6)$.
   b. The vertex of the graph is at $(\frac{7}{2}, -6\frac{1}{4})$.
   c. The vertex of the graph is at $(\frac{5}{2}, -\frac{5}{2})$.
   d. The vertex of the graph is at $(-6\frac{1}{4}, \frac{7}{2})$.

4. Enter the missing number, including the sign, for the expression when $d(x) = x^2 + 12x + 36$ is written in vertex form.

$d(x) = (x___)^2$.

5. Enter the missing numbers, including the signs, for the expression when $f(x) = x^2 + 10x + 21$ is written in vertex form. $f(x) = (x + \underline{\textbf{a.}})^2 \underline{\textbf{b.}}$

a. What is the value of the missing number, including the sign, in the place of the letter a?

b. What is the value of the missing number, including the sign, for the letter b?

6. Enter the missing numbers, including the signs, for the expression when $g(x) = 2x^2 - 20x + 32$ is written in vertex form. $g(x) = \underline{\textbf{a.}}\,(x + \underline{\textbf{b.}})^2 \underline{\textbf{c.}}$

a. What is the value of the missing number, including the sign, in place of letter a?

b. What is the value of the missing number, including the sign, in place of letter b?

c. What is the value of the missing number, including the sign, in place of letter c?

7. Enter the missing numbers, including the signs, for the expression when $h(x) = (4x - 3)^2$ is expanded to standard form. $h(x) = (\underline{\ \textbf{a.}\ }x)^2 + 2(\underline{\ \textbf{b.}\ }x)(-3) + (\underline{\ \textbf{c.}\ })^2$

a. What is the value of the missing number, including the sign, in place of letter a?

b. What is the value of the missing number, including the sign, in place of letter b?

c. What is the value of the missing number, including the sign, in place of letter c?

9.10.8 Lesson Summary

In this lesson, you learned how to:

- Identify the vertex of the graph of a quadratic function when the expression that defines it is written in vertex form.
- Explain the meaning of the term "vertex form" and recognize examples of quadratic expressions written in this form.
- When given a quadratic expression in factored form, rewrite it in standard form.
- When given a quadratic expression in standard form, rewrite it in vertex form.

Here are the **activities** that helped you reach those goals:

- 9.10.1: The Vertex and Intercepts of a Function
  - In this activity, you determined the vertex, the x-intercepts, and the y-intercept for three different expressions that represent the same function.
- 9.10.2: Expanding from Factored Form to Standard Form
  - In this activity, you rewrote expressions that were in factored form into standard form.
- 9.10.3: Converting from Standard Form to Vertex Form
  - In this activity, you rewrote two expressions that were in vertex form into standard form.
  - 9.10.3: Self Check
  - 9.10.3: Additional Resources
- 9.10.4: Rewriting Expressions in Vertex Form
  - In this activity, you transformed expressions, whose squared term had a coefficient other than 1, into vertex form.
  - 9.10.4: Self Check
  - 9.10.4: Additional Resources
- 9.10.5: Different Forms of Quadratic Expressions
  - In this activity, you wrote expressions in vertex form and factored form that could define a quadratic function.
- 9.10.6: Vertex Form and Coordinates of the Vertex
  - In this activity, you wrote an expression in vertex form and found the coordinates of the vertex.

After these activities, you completed the following **practice**:

- 9.10.7: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

9.11 USING QUADRATIC EXPRESSIONS IN VERTEX FORM TO SOLVE PROBLEMS

9.11.0 Lesson Overview

In this lesson, you will learn how to use the vertex form of a quadratic to determine the maximum or minimum value of a function and to solve problems.

When you finish this lesson, you will be able to:

- Find the maximum or minimum of a function by writing the quadratic expression that defines it in vertex form.
- Explain why the vertex is a maximum or minimum when given a quadratic function in vertex form.

Here are the **activities** that will help you reach those goals:

- 9.11.1: Maximum and Minimum Value of a Function
- 9.11.2: Does the Vertex Represent the Minimum or Maximum Value?
  - 9.11.2: Self Check
  - 9.11.2: Additional Resources
- 9.11.3: Comparing Maximums between Quadratics
  - 9.11.3: Self Check
  - 9.11.3: Additional Resources
- 9.11.4: Maximum, Minimum, and Analyzing the Vertex

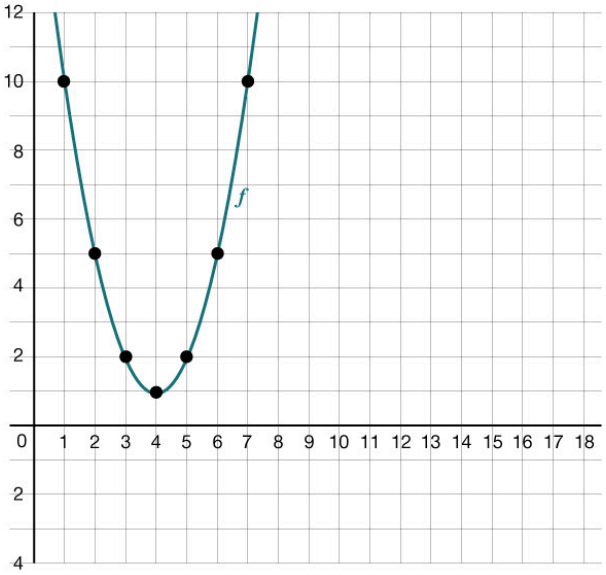
After that, you'll **practice and review**.

- 9.11.5: Practice
- 9.11.6: Lesson Summary

9.11.1 Maximum and Minimum Value of a Function

Warm Up

Examine the graph that represents the function, f, defined by $f(x) = (x-4)^2 + 1$.



Use this graph to answer questions 1 – 2.

1. $f(1)$ can be expressed in words as "the value of f when x is 1." Find or compute:

a. The value of f when x is 1.

b. Find the value of $f(3)$.

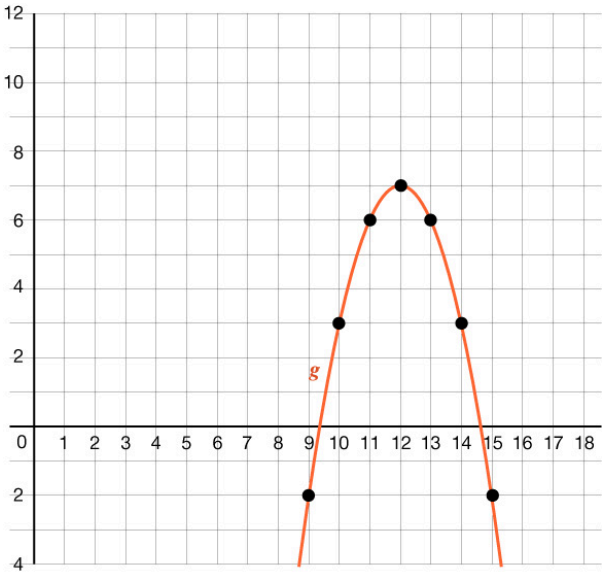
c. Find the value of $f(10)$.

2. Can you find an x-value that would make $f(x)$:

a. Less than 1?

b. Greater than 10,000?

Examine the graph that represents the function, g, defined by $g(x) = -(x-12)^2 + 7$.



Use this graph to answer questions 3 and 4.

3. $g(9)$ can be expressed in words as "the value of g when x is 9." Find or compute:

a. Find the value of g when x is 9

b. Find the value of $g(13)$.

c. Find the value of $g(2)$.

4. Can you find an x-value that would make $g(x)$:

a. Greater than 7?

b. Less than -10,000?

9.11.2 Does the Vertex Represent the Minimum or Maximum Value?

Activity

1. The graph that represents $p(x) = (x-8)^2 + 1$ has its **vertex (of a graph)** at $(8, 1)$. Here is one way to show, without graphing, that $(8, 1)$ corresponds to the **minimum** value of p.

- When $x = 8$, the value of $(x-8)^2$ is 0 because $(8-8)^2 = 0^2 = 0$.
- Squaring any number always results in a positive number, so when x is any value other than 8, $(x-8)$ will be a number other than 0, and when squared, $(x-8)^2$ will be positive.
- Any positive number is greater than 0, so when $x \neq 8$, the value of $(x-8)^2$ will be greater than when $x = 8$. In other words, p has the least value when $x = 8$.

Use similar reasoning to explain why the point $(4, 1)$ corresponds to the **maximum** value of q, defined by $q(x) = -2(x-4)^2 + 1$.

The greatest value of a function is the maximum value. The lowest value of a function is the minimum value. Since infinity cannot be reached or touched, it is not considered a maximum or minimum.

In quadratic functions, the maximum or minimum value occurs at the vertex of the **parabola**.

2. Here are some quadratic functions and the coordinates of the vertex of the graph of each. Determine if the vertex corresponds to the maximum or the minimum value of the function.

	Equation	Coordinates of the vertex
a.	$f(x) = -(x-4)^2 + 6$	$(4, 6)$
b.	$g(x) = (x+7)^2 - 3$	$(-7, -3)$
c.	$h(x) = 4(x+5)^2 + 7$	$(-5, 7)$
d.	$k(x) = x^2 - 6x - 3$	$(3, -12)$
e.	$m(x) = -x^2 + 8x$	$(4, 16)$

a. Determine if the vertex corresponds to the maximum or the minimum value of the function $f(x) = -(x-4)^2 + 6$.

b. Determine if the vertex corresponds to the maximum or the minimum value of the function $g(x) = (x+7)^2 - 3$.

c. Determine if the vertex corresponds to the maximum or the minimum value of the function $h(x) = 4(x+5)^2 + 7$.

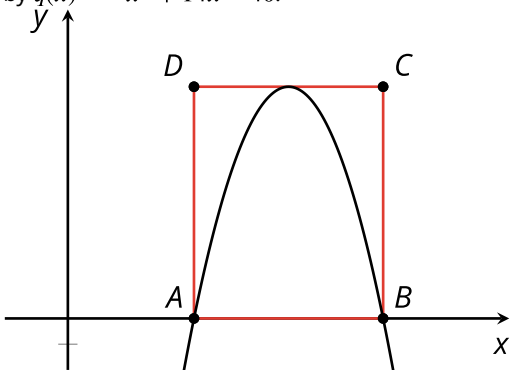
d. Determine if the vertex corresponds to the maximum or the minimum value of the function $k(x) = x^2 - 6x - 3$.

e. Determine if the vertex corresponds to the maximum or the minimum value of the function $m(x) = -x^2 + 8x$.

Are you ready for more?

Extending Your Thinking

1. Here is a portion of the graph of function q, defined by $q(x) = -x^2 + 14x - 40$.



$ABCD$ is a rectangle. Points A and B coincide with the x-intercepts of the graph, and segment CD just touches the vertex of the graph.
Find the area of $ABCD$. Show your reasoning.

Self Check

Which of the following is the correct vertex for $h(x) = -3(x-2)^2 + 5$? Describe the vertex as a maximum or minimum.

a. $(-2, -5)$, minimum
b. $(2, 5)$, maximum
c. $(2, 5)$, minimum

d. $(-2, -5)$, maximum

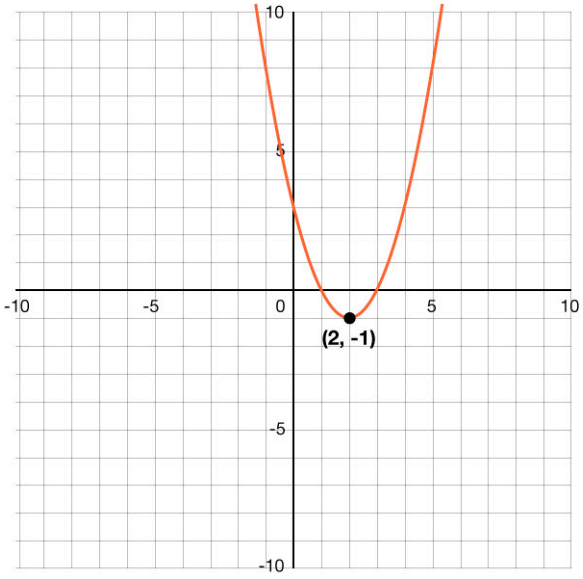
Additional Resources

Identify a Maximum or Minimum

Recall that the **maximum** is the highest point of the graph and the **minimum** is the lowest point of the graph. A quadratic function shape is called a **parabola**, and a parabola will only have a maximum or a minimum.

EXAMPLE 1

Graph $f(x) = (x-2)^2 - 1$.



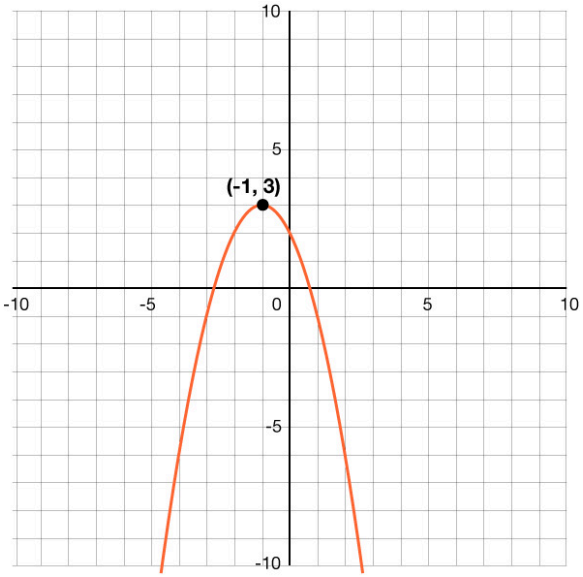
In the **vertex form (of a quadratic expression)**, $f(x) = a(x-h)^2 + k$, tthe **vertex (of a graph)** is located at (h, k).

For the function graphed, the vertex is at $(2, -1)$.

Since the leading coefficient, a, is positive, the parabola opens up. This means the vertex is a minimum.

EXAMPLE 2

Graph $f(x) = -(x+1)^2 + 3$.



The vertex is located at $(-1, 3)$.

Since the leading coefficient is negative, the parabola opens down and the vertex is a maximum.

TRY IT **Identify a Maximum or Minimum**

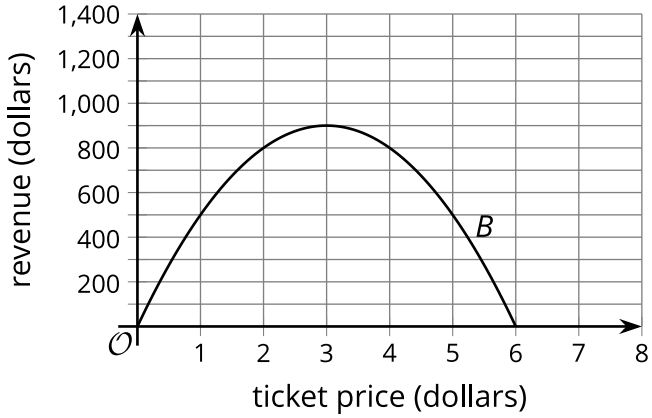
For the function $f(x) = (x + 4)^2 - 2$, identify the vertex and tell if it is a maximum or minimum.

9.11.3 Comparing Maximums between Quadratics

Activity

A function A, defined by $p(600 - 75p)$, describes the revenue collected from the sales of tickets for Performance A, a musical.

The graph represents a function B that models the revenue collected from the sales of tickets for Performance B, a Shakespearean comedy.



In both functions, p represents the price of one ticket, and both revenues and prices are measured in dollars.

1. Without creating a graph of A, determine which performance gives the greater **maximum** revenue when tickets are p dollars each. Explain or show your reasoning.

Video: Comparing Maximums between Quadratics

Watch the following video to learn more about comparing maximums between quadratics.

Comparing Maximums between Quadratics (https://openstax.org/r/quadratic-maximums)

Self Check

A function, defined by $p(600 - 60p)$, describes the revenue collected from the sales of tickets for a concert where p is the price of one ticket. Find the maximum revenue, in dollars.

a. 1500
b. 10
c. 5
d. 600

Additional Resources

Finding Maximum Revenue

The revenue brought in by a newspaper is modeled by $R = p(-2500p + 159,000)$. Find the maximum revenue where R is revenue and p is price per newspaper subscription.

Step 1 - Find the horizontal intercepts or x-intercepts (where $p = 0$).

First intercept -

This occurs at $p = 0$ or $(0, 0)$.

Second intercept -

$$\begin{aligned} -2500p + 159,000 &= 0 \\ -2500p &= -159,000 \\ p &= 63.6 \end{aligned}$$

Step 2 - Find the x-coordinate of the vertex halfway through the horizontal intercepts.

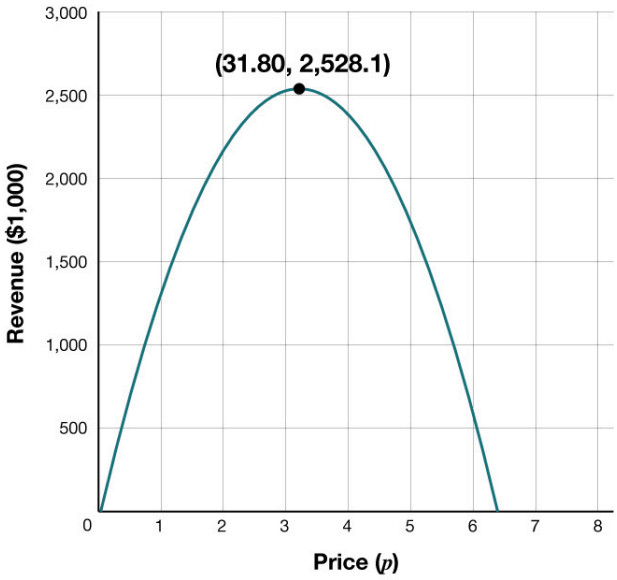
The vertex (of a graph) will occur halfway between $p = 0$ and $p = 63.6$. This occurs when $p = 31.8$.

Step 3 - Write in the function in standard form and substitute $p = 31.8$.

$R = -2500p^2 + 159,000p$

$R = -2500(31.8)^2 + 159,000(31.8)$

The maximum revenue is at $2,528,100 with a $31.80 subscription.



TRY IT **Finding Maximum Revenue**

Suppose that the price per unit in dollars of a cell phone production is modeled by $p = \$45 - 0.0125x$, where x is in thousands of phones produced, and the revenue represented by thousands of dollars is $R = x \cdot p$. Find the production level that will maximize revenue and the maximum revenue.

9.11.4 Maximum, Minimum, and Analyzing the Vertex

Cool Down

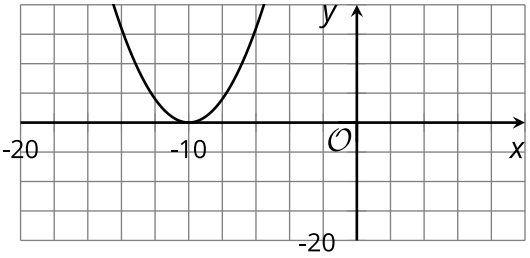
1. Without graphing, find the **vertex (of a graph)** of the graph of a quadratic function defined by $f(x) = -x^2 - 14x - 60$. Show your reasoning.
2. Does the y-coordinate of the vertex correspond to a **maximum** or a **minimum** value of the function? Explain how you know.

Note: If the vertex isn't asked for and the "value of the maximum or minimum" is asked for, the maximum or minimum value occurs at the y-value of the vertex because that is the value of the coordinate that represents the vertical height of the point.

9.11.5 Practice

Complete the following questions to practice the skills you have learned in this lesson.

1. Here is a graph of a quadratic function $f(x)$. What is the minimum value of $f(x)$?



2. The graph that represents $f(x) = (x + 1)^2 - 4$ has its vertex at $(-1, -4)$.

Explain how we can tell from the expression $(x + 1)^2 - 4$ that -4 is the minimum value of f rather than the maximum value.

a. The coefficient of the quadratic term is positive; therefore, the vertex will be a minimum value.
b. We cannot tell from the expression; we have to graph it.
c. The expression $(x + 1)^2$ is a squared expression, so its value will always be positive or zero.
d. The constant is always the minimum value.

3. $g(x) = (x - 5)^2 + 6$ defines a quadratic function. Find the vertex of the graph of the function. Then, state whether the vertex corresponds to the maximum or the minimum value of the function.

a. What is the x-coordinate of the vertex?

b. What is the y-coordinate of the vertex?

c. Does the vertex correspond to the maximum or the minimum value of the function?

a. Minimum
b. Maximum

4. $h(x) = (x + 5)^2 - 1$ defines a quadratic function. Find the vertex of the graph of the function. Then, state whether the vertex corresponds to the maximum or the minimum value of the function.

a. What is the x-coordinate of the vertex?

b. What is the y-coordinate of the vertex?

c. Does the vertex correspond to the maximum or the minimum value of the function?

a. Minimum
b. Maximum

5. $f(x) = -2(x + 3)^2 - 10$ defines a quadratic function. Find the vertex of the graph of the function. Then, state whether the vertex corresponds to the maximum or the minimum value of the function.

a. What is the x-coordinate of the vertex?

b. What is the y-coordinate of the vertex?

c. Does the vertex correspond to the maximum or the minimum value of the function?

a. Minimum
b. Maximum

6. $p(x) = 3(x - 7)^2 + 11$ defines a quadratic function. Find the vertex of the graph of the function. Then, state whether the vertex corresponds to the maximum or the minimum value of the function.

a. What is the x-coordinate of the vertex?

b. What is the y-coordinate of the vertex?

c. Does the vertex correspond to the maximum or the minimum value of the function?

a. Minimum
b. Maximum

7. $g(x) = -(x-2)^2 - 2$ defines a quadratic function. Find the vertex of the graph of the function. Then, state whether the vertex corresponds to the maximum or the minimum value of the function.

a. What is the x-coordinate of the vertex?

b. What is the y-coordinate of the vertex?

c. Does the vertex correspond to the maximum or the minimum value of the function?

a. Minimum
b. Maximum

8. $h(x) = (x+1)^2$ defines a quadratic function. Find the vertex of the graph of the function. Then, state whether the vertex corresponds to the maximum or the minimum value of the function.

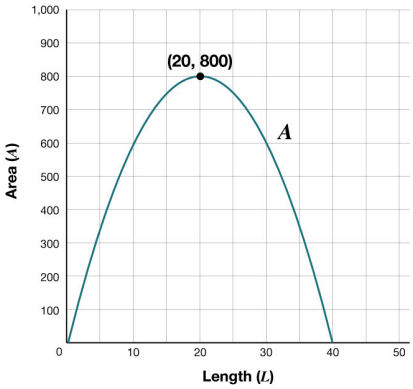
a. What is the x-coordinate of the vertex?

b. What is the y-coordinate of the vertex?

c. Does the vertex correspond to the maximum or the minimum value of the function?

a. Minimum
b. Maximum

9. A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side. The function is graphed below. What is the maximum area, in square feet, that can be fenced in?



9.11.6 Lesson Summary

In this lesson, you learned how to:

- Find the maximum or minimum of a function by writing the quadratic expression that defines it in vertex form.
- Explain why the vertex is a maximum or minimum when given a quadratic function in vertex form.

Here are the **activities** that helped you reach those goals:

- 9.11.1: Maximum and Minimum Value of a Function
  - In this activity, you reviewed the meaning of maximum or minimum value of a function. Then you practiced interpreting the language related to maximum and minimum values of functions.
- 9.11.2: Does the Vertex Represent the Minimum or Maximum Value?
  - In this activity, you used the vertex form to find out if a vertex represents the minimum or the maximum value of each function.
  - 9.11.2: Self Check
  - 9.11.2: Additional Resources
- 9.11.3: Comparing Maximums between Quadratics
  - In this activity, you were given two functions in different forms. One was graphed, and the other was in factored form. You used these two pieces of information, without creating a graph of the function in factored form, to determine the maximum value of each one.
  - 9.11.3: Self Check
  - 9.11.3: Additional Resources
- 9.11.4: Maximum, Minimum, and Analyzing the Vertex
  - In this activity, you found the vertex of the graph of a quadratic function and then determined if the y-coordinate of the vertex corresponded to a maximum or a minimum value of the function.

After these activities, you completed the following **practice**:

- 9.11.5: Practice

Checking In

On a scale of 1 to 5, how confident do you feel about the learning goals of this lesson?

PROJECT 9: USING QUADRATIC EQUATIONS TO MODEL SITUATIONS AND SOLVE PROBLEMS

Project 9 Overview

In this culminating project, you will synthesize methods of solving quadratic equations and graphing quadratic functions to answer questions about within a context. You will use tools learned throughout the unit to grapple with solving problems about a quadratic function that represents a context. Since this work requires using what you know to tackle an unfamiliar problem, you will need to make sense of problems and demonstrate perseverance.

Technology isn't required for this lesson, but there are opportunities for you to choose to use appropriate technology to solve problems.

Project Activities

1. Equations of Two Lines and a Curve
2. Analyzing a Dive
3. A Linear Function and a Quadratic Function
4. Profit from a River Cruise

Learning Targets

When you finish this project, you will have demonstrated how to:

- Interpret information about a quadratic function given its equation or a graph.
- Rewrite quadratic functions in different but equivalent forms of my choosing and use that form to solve problems.
- Decide which quadratic form to use depending on the questions being asked.

Equations of Two Lines and a Curve

1. Write an equation representing the line that passes through each pair of points.
   a. $(3, 3)$ and $(5, 5)$
   b. $(0, 4)$ and $(-4, 0)$
2. Solve this equation: $x + 1 = (x - 2)^2 - 3$. Show your reasoning.

Analyze a Dive

The function h, defined by $h(t) = -5t^2 + 10t + 7.5$, models the height of a diver above the water (in meters) t seconds after the diver leaves the board. For each question, explain your reasoning.

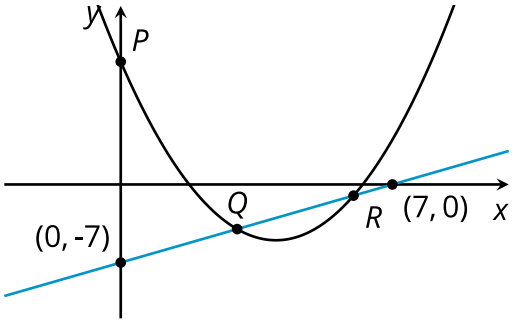
1. How high above the water is the diving board?
2. When does the diver hit the water?
3. At what point during her descent toward the water is the diver at the same height as the diving board?
4. When does the diver reach the maximum height of the dive?
5. What is the maximum height the diver reaches during the dive?

A Linear Function and a Quadratic Function

Here are graphs of a linear function and a quadratic function. The quadratic function is defined by the expression

$(x - 4)^2 - 5$

Find the coordinates of P, Q, and R without using graphing technology. Show your reasoning.



Profit from a River Cruise

A travel company uses a quadratic function to model the profit, in dollars, that it expects to earn from selling tickets to a river cruise at d dollars per person. The expression $-d^2 + 100d - 900$ defines this function.

Without graphing, find the price that would generate the maximum profit. Then, determine that maximum profit.

Unit 9 Wrap Up

In this unit, you learned how companies like The Ensemble Theatre use quadratic equations to solve problems. Business managers use quadratics to represent revenue as a product of the price of a ticket and the quantity sold. Quadratic equations can also be used to predict profit and loss. Other companies use quadratics to determine the best materials to use or to determine which dimensions will be best to maximize a given space.

Building Character: Intellectual Humility

Throughout this unit, you learned how to solve quadratic equations using the methods of completing the square and the quadratic formula. You also learned how to use graphs and other methods to check your solutions.

Finally, you wrote quadratic expressions in vertex form to identify the vertex as well as the maximum or minimum value of the quadratic function. This value was then used to solve real-life problems involving the maximum revenue.



Access our Unit 9 Student Self-Assessment (https://openstax.org/r/unit-9-self-assesment) to reflect on the skills you learned during this unit. Congratulations on completing Unit 9!

Throughout this unit, you learned that **intellectual humility** is important for recognizing the limitations of your knowledge. You learned that when you approach life with intellectual humility, you open your mind to learning. You are able to learn from opposing views and have more constructive discussions, even when you disagree. You learned that no matter how old you are, with intellectual humility, you become wiser. It helps you be less judgmental of others, learn more in school, and be a better leader.

Index

A

Addition Property of Equality 226
arithmetic sequence 700, 707
asymptote 852
average rate of change 590
axis of symmetry 0
axis of symmetry 1220

B

binomial 945

C

causal relationship 481
coefficient 1344, 1345, 1350
common difference 700, 771
common ratio 693
completing the square 1400, 1400, 1401, 1403, 1404, 1409, 1412, 1413, 1414, 1431, 1433, 1439, 1440, 1458, 1460, 1461, 1462
conjugate pair 967
constant ratio 771, 783
constraint 20, 21, 50
continue off the coordinate plane 791
continuous function 677
correlation 481
correlation coefficient 449
curve of best fit 1370

D

degree of a polynomial 945
dependent variable 512, 513
difference of squares 1327
difference of squares 967
discrete function 677
discriminant 0
distributive property 1139, 1392, 1393, 0
domain 659, 662, 1119, 1122
double root 1273, 1294, 1339

E

Elimination 226
equation 17, 30, 48, 58, 628
exponential functions 861
expression 18
extrapolation 392

F

factored form 1141
factored form 1345
factored form 1210, 1327, 1336, 1339, 1344, 1347, 1349, 1351
factoring 992
factors 1421, 1422
function 510, 614, 615, 0, 1295
function notation 522
functions 628, 630

G

geometric sequence 693, 707
greatest common factor 992
growth factor 783, 785

H

horizontal asymptote 852
horizontal compression 647
Horizontal dilations 646
horizontal reflection 647
horizontal stretch 647

I

identity 80
independent variable 512, 513
index 760, 761
initial velocity 1108
input 512, 533
interpolation 392

L

line of best fit 412
linear function 553, 636
linear inequality 302
linear term 1327, 1344

M

mapping 512
maximum 1486, 1488, 1490, 1491
minimum 1486, 1488, 1491
model 392
monomial 945

O

output 512, 533

P

parabola 1109, 1487, 1488
parent function 1167
perfect square 1391, 1395, 1410, 0, 0, 0, 1461, 0
perfect square trinomial 1387, 1392, 1400, 1400, 1401, 1401, 1403, 1420
point-slope form 582
polynomial 945
polynomial function 950
Power Property for Exponents 751
Product Property for Exponents 745
Product to a Power Property for Exponents 752
Properties of Negative Exponents 749

Q

quadratic equations 1336
quadratic expression 1345
quadratic function 1338
quadratic equations 1283
quadratic equation 1260, 1337, 1347, 0, 1393, 1401, 1451
quadratic equations 1345
quadratic expression 1072, 1295, 1327, 1344, 1350, 1400
quadratic expressions 1349
quadratic formula 1441, 1442, 1443, 1447, 0, 0, 1458, 1459, 1460, 1461
quadratic function 1295, 1347
quadratic functions 1337
quadratic regression 1370
Quotient to a Negative Power Property 751
Quotient to a Power Property for Exponents 752
Quotient Property for Exponents 746

R

radical expression 0
radicals 760, 761
radicand 760
range 661, 662, 1119, 1122
recursive definition 707
relation 510, 511
revenue 1117
root 1329
roots 1286, 1294

S

scientific notation 819
sequence 685
slope 577
slope-intercept form 582
slope-intercept form 582
solution of a system of linear inequalities 351
solution of an equation 48, 75
Solutions 50, 1294
square root of a number 1273
square root property 1296, 1393
standard form 1141, 1142, 1210, 1327, 1338, 1343, 1345, 1349
Standard form of a linear equation 582
substitution 215

system of linear inequalities 351
system of equations 227
system of inequalities 350
system of linear equations 351

T

term 719
terms 685
transformation 643, 1389
transformations 1167
trinomial 945

V

variable 48, 1327
variables 29, 58
vertex 1109
vertex (of a graph) 1119, 1122, 1475, 1478, 1480, 1480, 1486, 1488, 0, 1491
vertex form 1210
vertex form (of a quadratic expression) 1477, 1478, 1478, 1479, 1480, 1480, 1488, 0
vertical compression 643
vertical dilation 643
vertical reflection 643
vertical stretch 643

X

x -intercepts 1475
x -intercepts 1294

Y

y -intercept 1476

Z

zero 1109
zero product property 1283
Zero Exponent Property 748
zero exponent rule 784
zero of the function 1119, 1122
zero product property 1284, 1284, 1285, 1286, 1296, 1336, 1336, 1337, 1339, 1347, 1351
zeros 1294