Yet Another Calculus Text

A Short Introduction with Infinitesimals

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Preface

I intend this book to be, firstly, a introduction to calculus based on the hyperreal number system. In other words, I will use infinitesimal and infinite numbers freely. Just as most beginning calculus books provide no logical justification for the real number system, I will provide none for the hyperreals. The reader interested in questions of foundations should consult books such as Abraham Robinson's *Non-standard Analysis* or Robert Goldblatt's *Lectures on the Hyperreals*.

Secondly, I have aimed the text primarily at readers who already have some familiarity with calculus. Although the book does not explicitly assume any prerequisites beyond basic algebra and trigonometry, in practice the pace is too fast for most of those without some acquaintance with the basic notions of calculus.

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Chapter 1

Derivatives

1.1 The arrow paradox

In his famous arrow paradox, Zeno contends that an arrow cannot move since at every instant of time it is at rest. There are at least two logical problems hidden in this claim.

1.1.1 Zero divided by zero

In one interpretation, Zeno seems to be saying that, since at every instant of time the arrow has a definite position, and hence does not travel any distance during that instant of time, the velocity of the arrow is 0. The question is, if an object travels a distance 0 in time of duration 0, is the velocity of the object 0? That is, is

$$\frac{0}{0} = 0? \tag{1.1.1}$$

To answer this question, we need to examine the meaning of dividing one number by another. If a and b are real numbers, with $b \neq 0$, then

$$\frac{a}{b} = c \tag{1.1.2}$$

means that

$$a = b \times c. \tag{1.1.3}$$

In particular, for any real number $b \neq 0$,

$$\frac{0}{b} = 0 \tag{1.1.4}$$

since $b \times 0 = 0$. Note that if $a \neq 0$, then

$$\frac{a}{0} \tag{1.1.5}$$

is undefined since there does not exist a real number c for which $0 \times c$ is equal to a. We say that division of a non-zero number by zero is *meaningless*. On the other hand,

$$\frac{0}{0} \tag{1.1.6}$$

is undefined because $0 \times c = 0$ for all real numbers c. For this reason, we say that division of zero by zero is *indeterminate*.

The first logical problem exposed by Zeno's arrow paradox is the problem of giving determinate meaning to ratios of quantities with zero magnitude. We shall see that *infinitesimals* give us one way of giving definite meanings to ratios of quantities with zero magnitudes, and these ratios will provide the basis for what we call the *differential calculus*.

1.1.2 Adding up zeroes

Another possible interpretation of the arrow paradox is that if at every instant of time the arrow moves no distance, then the total distance traveled by the arrow is equal to 0 added to itself a large, or even infinite, number of times. Now if n is any positive integer, then, of course,

$$n \times 0 = 0. \tag{1.1.7}$$

That is, zero added to itself a finite number of times is zero. However, if an interval of time is composed of an infinite number of instants, then we are asking for the product of infinity and zero, that is,

$$\infty \times 0. \tag{1.1.8}$$

One might at first think this result should also be zero; however, more careful reasoning is needed.

Note that an interval of time, say the interval $[0, 1]$, is composed of an infinity of instants of no duration. Hence, in this case, the product of infinity and 0 must be 1, the length of the interval. However, the same reasoning applied to the interval $[0, 2]$ would lead us to think that infinity times 0 is 2. Indeed, as with the problem of zero divided by 0, infinity times 0 is indeterminate.

Thus the second logical problem exposed by Zeno's arrow paradox is the problem of giving determinate meaning to infinite sums of zero magnitudes, or, in the simplest cases, to products of infinitesimal and infinite numbers.

Since division is the inverse operation of multiplication we should expect a close connection between these questions. This is in fact the case, as we shall see when we discuss the fundamental theorem of calculus.

1.2 Rates of change

Suppose $x(t)$ gives the position, at some time t, of an object (such as Zeno's arrow) moving along a straight line. The problem we face is that of giving a determinate meaning to the idea of the velocity of the object at a specific instant of time. We first note that we face no logical difficulties in defining an average velocity over an interval of time of non-zero length. That is, if $a < b$, then the object travels a distance

$$\Delta x = x(b) - x(a) \tag{1.2.1}$$

from time $t = a$ to time $t = b$, an interval of time of length $\Delta t = b - a$, and, consequently, the *average velocity* of the object over this interval of time is

$$v_{[a,b]} = \frac{x(b) - x(a)}{b - a} = \frac{\Delta x}{\Delta t}. \tag{1.2.2}$$

Example 1.2.1. Suppose an object, such as a lead ball, is dropped from a height of 100 meters. Ignoring air resistance, the height of the ball above the earth after t seconds is given by

$$x(t) = 100 - 4.9t^2 \text{ meters},$$

a result first discovered by Galileo. Hence, for example, from time $t = 0$ to time $t = 2$ we have

$$\Delta x = x(2) - x(0) = (100 - (4.9)(4)) - 100 = -19.6 \text{ meters},$$

$$\Delta t = 2 - 0 = 2 \text{ seconds},$$

and so

$$v_{[0,2]} = -\frac{19.6}{2} = -9.8 \text{ meters/second}.$$

For another example, from time $t = 1$ to time $t = 4$ we have

$$\Delta x = x(4) - x(1) = 21.6 - 95.1 = -73.5,$$

$$\Delta t = 4 - 1 = 3 \text{ seconds},$$

and so

$$v_{[1,4]} = -\frac{73.5}{3} = -24.5 \text{ meters/second}.$$

Note that both of these average velocities are negative because we have taken the positive direction to be upward from the surface of the earth.

Exercise 1.2.1. Suppose a lead ball is dropped into a well. Ignoring air resistance, the ball will have fallen a distance $x(t) = 16t^2$ feet after t seconds. Find the average velocity of the ball over the intervals (a) $[0, 2]$, (b) $[1, 3]$, and (c) $[1, 1.5]$.

Letting $\Delta t = b - a$, we may rewrite (1.2.2) in the form

$$v_{[a,a+\Delta t]} = \frac{x(a + \Delta t) - x(a)}{\Delta t}. \tag{1.2.3}$$

Using (1.2.3), there are two approaches to generalizing the notion of average velocity over an interval to that of velocity at an instant. The most common approach, at least since the middle of the 19th century, is to consider the effect on $v_{[a,a+\Delta t]}$ as Δt diminishes in magnitude and defining the velocity at time $t = a$ to be the limiting value of these average velocities. The approach we will take in this text is to consider what happens when we take a and b to be, although not equal, immeasurably close to one another.

Example 1.2.2. If we have, as in the previous example,

$$x(t) = 100 - 4.9t^2 \text{ meters},$$

then from time $t = 1$ to time $t = 1 + \Delta t$ we would have

$$\begin{aligned} \Delta x &= x(1 + \Delta t) - x(1) \\ &= (100 - 4.9(1 + \Delta t)^2) - 95.1 \\ &= 4.9 - 4.9(1 + 2\Delta t + (\Delta t)^2) \\ &= -9.8\Delta t - 4.9(\Delta t)^2 \text{ meters.} \end{aligned}$$

Hence the average velocity over the interval $[1, 1 + \Delta t]$ is

$$\begin{aligned} v_{[1,1+\Delta t]} &= \frac{\Delta x}{\Delta t} \\ &= \frac{-9.8\Delta t - 4.9(\Delta t)^2}{\Delta t} \\ &= -9.8 - 4.9\Delta t \text{ meters/second.} \end{aligned}$$

Note that if, for example, $\Delta t = 3$, then we find

$$v_{[1,4]} = -9.8 - (4.9)(3) = -9.8 - 14.7 = -24.5 \text{ meters/second},$$

in agreement with our previous calculations.

Now suppose that the starting time $a = 1$ and the ending time b are different, but the difference is so small that it cannot be measured by any real number. In this case, we call $dt = b - a$ an *infinitesimal*. Similar to our computations above, we have

$$dx = x(1 + dt) - x(1) = -9.8dt - 4.9(dt)^2 \text{ meters},$$

the distance traveled by the object from time $t = 1$ to time $t = 1 + dt$, and

$$v_{[1,1+dt]} = \frac{dx}{dt} = -9.8 - 4.9dt \text{ meters/second},$$

the average velocity of the object over the interval $[1, 1 + dt]$. However, since dt is infinitesimal, so is $4.9dt$. Hence $v_{[1,1+dt]}$ is immeasurably close to -9.8 meters per second. Moreover, this is true no matter what the particular value of dt. Hence we should take the *instantaneous velocity* of the object at time $t = 1$ to be

$$v(1) = -9.8 \text{ meters/second}.$$

Exercise 1.2.2. As in the previous exercise, suppose a lead ball has fallen $x(t) = 16t^2$ feet in t seconds. Find the average velocity of the ball over the interval $[1, 1 + \Delta t]$ and use this result to obtain the answers to parts (b) and (c) of the previous exercise.

Exercise 1.2.3. Find the average velocity of the ball in the previous exercise over the interval $[1, 1 + dt]$, where dt is infinitesimal, and use the result to find the instantaneous velocity of the ball at time $t = 1$.

Example 1.2.3. To find the velocity of the object of the previous examples at time $t = 3$, we compute

$$\begin{aligned} dx &= x(3 + dt) - x(3) \\ &= (100 - 4.9(3 + dt)^2 - 55.9 \\ &= 44.1 - 4.9(9 + 6dt + (dt^2)) \\ &= -29.4dt - 4.9(dt)^2 \text{ meters,} \end{aligned}$$

from which we obtain

$$\frac{dx}{dt} = -29.4 - 4.9dt \text{ meters/second.}$$

As above, we disregard the immeasurable $-4.9dt$ to obtain the velocity of the object at time $t = 3$:

$$v(3) = -29.4\text{meters/second.}$$

Exercise 1.2.4. Find the velocity of the ball in the previous exercise at time $t = 2$.

In general, if $x(t)$ gives the position, at time t, of an object moving along a straight line, then we define the velocity of the object at a time t to be the real number which is infinitesimally close to

$$\frac{x(t + dt) - x(t)}{dt}, \tag{1.2.4}$$

provided there is exactly one such number for any value of the nonzero infinitesimal dt.

Example 1.2.4. For our previous example, we find

$$\begin{aligned} dx &= x(t + dt) - x(t) \\ &= (100 - 4.9(t + dt)^2) - (100 - 4.9t^2) \\ &= -4.9(t + 2tdt + (dt)^2) - 4.9t^2 \\ &= -9.8tdt - 4.9(dt)^2 \text{ meters} \\ &= (-9.8t - 4.9dt)dt. \end{aligned}$$

Hence

$$\frac{dx}{dt} = -9.8t - 4.9dt \text{ meters/second,}$$

and so the velocity of the object at time t is

$$v(t) = -9.8t \text{ meters/second.}$$

In particular,

$$v(1) = -9.8 \text{ meters/second}$$

and

$$v(3) = -9.8(3) = -29.4 \text{ meters/second,}$$

as previously computed.

Exercise 1.2.5. Find the velocity of the ball in the previous exercise at time t. Use your result to verify your previous answers for $v(1)$ and $v(2)$.

Even more generally, we should recognize that velocity is but a particular example of a rate of change, namely, the rate of change of the position of an object with respect to time. In general, given any quantity y as a function of another quantity x, say $y = f(x)$ for some function f, we may ask about the rate of change of y with respect to x. If x changes from $x = a$ to $x = b$ and we let

$$\Delta x = b - a \tag{1.2.5}$$

and

$$\Delta y = f(b) - f(a) = f(a + \Delta x) - f(x), \tag{1.2.6}$$

then

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} \tag{1.2.7}$$

is the *average rate of change* of y with respect to x; if dx is a nonzero infinitesimal, then the real number which is infinitesimally close to

$$\frac{dy}{dx} = \frac{f(x + dx) - f(x)}{dx} \tag{1.2.8}$$

is the *instantaneous rate of change*, or, simply, *rate of change*, of y with respect to x at $x = a$. In subsequent sections we will look at this quantity in more detail, but will consider one more example before delving into technicalities.

Example 1.2.5. Suppose a spherical shaped balloon is being filled with water. If r is the radius of the balloon in centimeters and V is the volume of the balloon, then

$$V = \frac{4}{3}\pi r^3 \text{ centimeters}^3.$$

Since a cubic centimeter of water has a mass of 1 gram, the mass of the water in the balloon is

$$M = \frac{4}{3}\pi r^3 \text{ grams.}$$

To find the rate of change of the mass of the balloon with respect to the radius of the balloon, we first compute

$$\begin{aligned} dM &= \frac{4}{3}\pi(r+dr)^3 - \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi((r^3 + 3r^2 dr + 3r(dr)^2 + (dr)^3) - r^3) \\ &= \frac{4}{3}\pi(3r^2 + 3r dr + (dr)^2) dr \text{ grams,} \end{aligned}$$

from which it follows that

$$\frac{dM}{dr} = \frac{4}{3}\pi(3r^2 + 3r dr + (dr)^2) \text{ grams/centimeter.}$$

Since both $3rdr$ and $(dr)^2$ are infinitesimal, the rate of change of mass of the balloon with respect to the radius of the balloon is

$$\frac{4}{3}\pi(3r^2) = 4\pi r^2 \text{ gams/centimeer.}$$

For example, when the balloon has a radius of 10 centimeters, the mass of the water in the balloon is increasing at a rate of

$$4\pi(10)^2 = 400\pi \text{ grams/centimeter.}$$

It may not be surprising that this is also the surface area of the balloon at that instant.

Exercise 1.2.6. Show that if A is the area of a circle with radius r, then $\frac{dA}{dr} = 2\pi r$.

1.3 The hyperreals

We will let $\mathbb{R}$ denote the set of all real numbers. Intuitively, and historically, we think of these as the numbers sufficient to measure geometric quantities. For example, the set of all rational numbers, that is, numbers expressible as the ratios of integers, is not sufficient for this purpose since, for example, the length of the diagonal of a square with sides of length 1 is the irrational number $\sqrt{2}$. There are numerous technical methods for defining and constructing the real numbers, but, for the purposes of this text, it is sufficient to think of them as the set of all numbers expressible as infinite decimals, repeating if the number is rational and non-repeating otherwise.

A *positive infinitesimal* is any number ϵ with the property that $\epsilon > 0$ and $\epsilon < r$ for any positive real number r. The set of infinitesimals consists of the positive infinitesimals along with their additive inverses and zero. Intuitively, these are the numbers which, except for 0, correspond to quantities which are too small to measure even theoretically. Again, there are technical ways to make the definition and constrution of infinitesimals explicit, but they lie beyond the scope of this text.

The multiplicative inverse of a nonzero infinitesimal is an *infinite* number. That is, for any infinitesimal $\epsilon \neq 0$, the number

$$N = \frac{1}{\epsilon}$$

is an infinite number.

The *finite hyperreal numbers* are numbers of the form $r + \epsilon$, where r is a real number and ϵ is an infinitesimal. The *hyperreal numbers*, which we denote $^*\mathbb{R}$, consist of the finite hyperreal numbers along with all infinite numbers.

For any finite hyperreal number a, there exists a unique real number r for which $a = r + \epsilon$ for some infinitesimal ϵ. In this case, we call r the *shadow* of a and write

$$r = \mathrm{sh}(a). \tag{1.3.1}$$

Alternatively, we may call $\mathrm{sh}(a)$ the *standard part* of a.

We will write $a \simeq b$ to indicate that $a - b$ is an infinitesimal, that is, that a and b are infinitesimally close. In particular, for any finite hyperreal number a, $a \simeq \mathrm{sh}(a)$.

It is important to note that

- if ϵ and δ are infinitesimals, then so is $\epsilon + \delta$,
- if ϵ is an infinitesimal and a is a finite hyperreal number, then $a\epsilon$ is an infinitesimal, and
- if ϵ is a nonzero infinitesimal and a is a hyperreal number with $\mathrm{sh}(a) \neq 0$ (that is, a is not an infinitesimal), then $\frac{a}{\epsilon}$ is infinite.

These are in agreement with our intuition that a finite sum of infinitely small numbers is still infinitely small and that an infinitely small nonzero number will divide into any noninfinitesimal quantity an infinite number of times.

Exercise 1.3.1. Show that $\mathrm{sh}(a + b) = \mathrm{sh}(a) + \mathrm{sh}(b)$ and $\mathrm{sh}(ab) = \mathrm{sh}(a)\mathrm{sh}(b)$, where a and b are any hyperreal numbers.

Exercise 1.3.2. Suppose a is a hyperreal number with $\mathrm{sh}(a) \neq 0$. Show that $\mathrm{sh}\left(\frac{1}{a}\right) = \frac{1}{\mathrm{sh}(a)}$.

1.4 Continuous functions

As (1.2.8) indicates, we would like to define the rate of change of a function $y = f(x)$ with respect to x as the shadow of the ratio of two quantities, $dy = f(x + dx) - f(x)$ and dx, with the latter being a nonzero infinitesimal. From the discussion of the previous section, it follows that we can do this if and only if the numerator dy is also an infinitesimal.

Definition 1.4.1. We say a function f is *continuous* at a real number c if for every infinitesimal ϵ,

$$f(c+\epsilon) \simeq f(c) \tag{1.4.1}$$

Note that $f(c+\epsilon) \simeq f(c)$ is equivalent to $f(c+\epsilon) - f(c) \simeq 0$, that is, $f(c+\epsilon) - f(c)$ is an infinitesimal. In other words, a function f is continuous at a real number c if an infinitesimal change in the value of c results in an infinitesimal change in the value of f.

Example 1.4.1. If $f(x) = x^2$, then, for example, for any infinitesimal ϵ,

$$f(3+\epsilon) = (3+\epsilon)^2 = 9 + 6\epsilon + \epsilon^2 \simeq 9 = f(3).$$

Hence f is continuous at $x = 3$. More generally, for any real number x,

$$f(x+\epsilon) = (x+\epsilon)^2 = x^2 + 2x\epsilon + \epsilon^2 \simeq x^2 = f(x),$$

from which it follows that f is continuous at every real number x.

Exercise 1.4.1. Verify that $f(x) = 3x + 4$ is continuous at $x = 5$.

Exercise 1.4.2. Verify that $g(t) = t^3$ is continuous at $t = 2$.

Given real numbers a and b, we let

$$(a, b) = \{x \mid x \text{ is a real number and } a < x < b\}, \tag{1.4.2}$$

$$(a, \infty) = \{x \mid x \text{ is a real number and } x > a\}, \tag{1.4.3}$$

$$(-\infty, b) = \{x \mid x \text{ is a real number and } x < b\}, \tag{1.4.4}$$

and

$$(-\infty, \infty) = \mathbb{R}. \tag{1.4.5}$$

An *open interval* is any set of one of these forms.

Definition 1.4.2. We say a function f is *continuous* on an open interval I if f is continuous at every real number in I.

Example 1.4.2. From our example above, it follows that $f(x) = x^2$ is continuous on $(-\infty, \infty)$.

Exercise 1.4.3. Verify that $f(x) = 3x + 4$ is continuous on $(-\infty, \infty)$.

Exercise 1.4.4. Verify that $g(t) = t^3$ is continuous on $(-\infty, \infty)$.

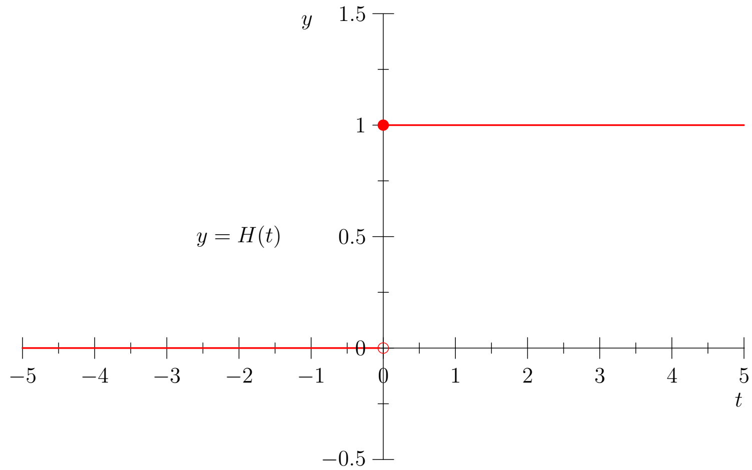

Figure 1.4.1: Graph of the Heaviside function

Example 1.4.3. We call the function

$$H(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geq 0, \end{cases}$$

the *Heaviside function* (see Figure 1.4.1). If ϵ is a positive infinitesimal, then

$$H(0 + \epsilon) = H(\epsilon) = 1 = H(0),$$

whereas

$$H(0 - \epsilon) = H(-\epsilon) = 0.$$

Since 0 is not infinitesimally close to 1, it follows that H is not continuous at 0. However, for any positive real number a and any infinitesimal ϵ (positive or negative),

$$H(a + \epsilon) = 1 = H(a),$$

since $a + \epsilon > 0$, and for any negative real number a and any infinitesimal ϵ,

$$H(a + \epsilon) = 0 = H(a),$$

since $a + \epsilon < 0$. Thus H is continuous on both $(0, \infty)$ and $(-\infty, 0)$.

Note that, in the previous example, the Heaviside function satisfies the condition for continuity at 0 for positive infinitesimals but not for negative infinitesimals. The following definition addresses this situation.

Definition 1.4.3. We say a function f is *continuous from the right* at a real number c if for every infinitesimal $\epsilon > 0$,

$$f(c + \epsilon) \simeq f(c). \tag{1.4.6}$$

Similarly, we say a function f is *continuous from the left* at a real number c if for every infinitesimal $\epsilon > 0$,

$$f(c - \epsilon) \simeq f(c). \tag{1.4.7}$$

Example 1.4.4. In the previous example, H is continuous from the right at $t = 0$, but not from the left.

Of course, if f is continuous both from the left and the right at c, then f is continuous at c.

Example 1.4.5. Suppose

$$f(x) = \begin{cases} 3x + 5, & \text{if } x \leq 1, \\ 10 - 2x, & \text{if } x > 1. \end{cases}$$

If ϵ is a positive infinitesimal, then

$$f(1 + \epsilon) = 3(1 + \epsilon) + 5 = 8 + 3\epsilon \simeq 8 = f(1),$$

so f is continuous from the right at $x = 1$, and

$$f(1 - \epsilon) = 3(1 - \epsilon) + 5 = 8 - 3\epsilon \simeq 8 = f(1),$$

so f is continuous from the left at $x = 1$ as well. Hence f is continuous at $x = 1$.

Exercise 1.4.5. Verify that the function

$$U(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } 0 \leq t \leq 1, \\ 0, & \text{if } t > 1, \end{cases}$$

is continuous from the right at $t = 0$ and continuous from the left at $t = 1$, but not continuous at either $t = 0$ or $t = 1$. See Figure 1.4.2.

Given real numbers a and b, we let

$$[a, b] = \{x \mid x \text{ is a real number and } a \leq x \leq b\}, \tag{1.4.8}$$

$$[a, \infty) = \{x \mid x \text{ is a real number and } x \geq a\}, \tag{1.4.9}$$

and

$$(-\infty, b] = \{x \mid x \text{ is a real number and } x \leq b\}. \tag{1.4.10}$$

A *closed interval* is any set of one of these forms.

Definition 1.4.4. If a and b are real numbers, we say a function f is *continuous* on the closed interval $[a, b]$ if f is continuous on the open interval (a, b), continuous from the right at a, and continuous from the left at b. We say f is *continuous* on the closed interval $[a, \infty)$ if f is continuous on the open interval (a, ∞) and continuous from the right at a. We say f is *continuous* on the closed interval $(-\infty, b]$ if f is continuous on $(-\infty, b)$ and continuous from the left at b.

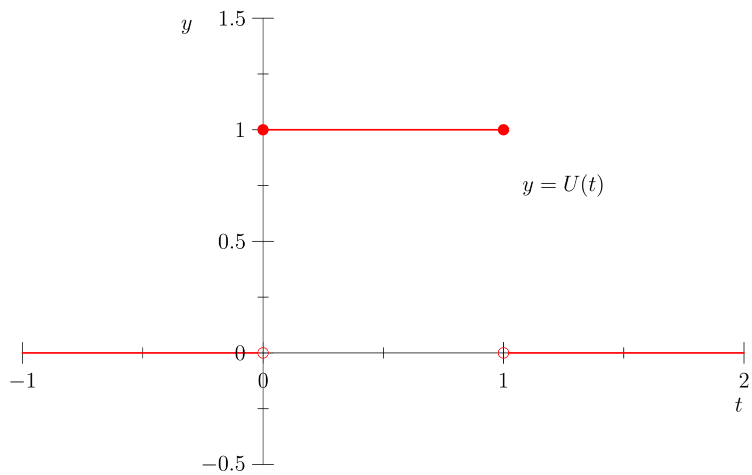

Figure 1.4.2: Graph of $y = U(t)$ from Exercise 1.4.5

Example 1.4.6. We may summarize our results about the Heaviside function as H is continuous on $(-\infty, 0)$ and on $[0, \infty)$.

Exercise 1.4.6. Explain why the function U in the previous exercise is continuous on the intervals $(-\infty, 0)$, $[0, 1]$, and $(1, \infty)$, but not on the interval $(-\infty, \infty)$.

1.5 Properties of continuous functions

Suppose f is continuous at the real number c and k is any fixed real number. If we let $h(x) = kg(x)$, then, for any infinitesimal ϵ,

$$h(c+\epsilon) - h(c) = kf(c+\epsilon) - kf(c) = k(f(c+\epsilon) - f(c)) \tag{1.5.1}$$

is an infinitesimal since, by assumption, $f(c+\epsilon) - f(c)$ is an infinitesimal. Hence $h(c+\epsilon) \simeq h(c)$.

Theorem 1.5.1. If f is continuous at c and k is any fixed real number, then the function $h(x) = kf(x)$ is also continuous at c.

Example 1.5.1. We have seen that $f(x) = x^2$ is continuous on $(-\infty, \infty)$. It now follows that, for example, $g(x) = 5x^2$ is also continuous on $(-\infty, \infty)$.

Suppose that both f and g are continuous at the real number c and we let $s(x) = f(x) + g(x)$. If ϵ is any infinitesimal, then

$$s(c+\epsilon) = f(c+\epsilon) + g(c+\epsilon) \simeq f(c) + g(c) = s(c), \tag{1.5.2}$$

and so s is also continuous at c.

Theorem 1.5.2. If f and g are both continuous at c, then the function

$$s(x) = f(x) + g(x)$$

is also continuous at c.

Example 1.5.2. Since

$$(x+\epsilon)^3 = x^3 + 3x^2\epsilon + 3x\epsilon^2 + \epsilon^3 \simeq x^3$$

for any real number x and any infinitesimal ϵ, it follows that $g(x) = x^3$ is continuous on $(-\infty, \infty)$. From the previous theorems, it then follows that

$$h(x) = 5x^2 + 3x^3$$

is continuous on $(-\infty, \infty)$.

Again, suppose f and g are both continuous at c and let $p(x) = f(x)g(x)$. Then, for any infinitesimal ϵ,

$$\begin{aligned} p(c+\epsilon) - s(c) &= f(c+\epsilon)g(c+\epsilon) - f(c)g(c) \\ &= f(c+\epsilon)g(c+\epsilon) - f(c)g(c+\epsilon) + f(c)g(c+\epsilon) - f(c)g(c) \\ &= g(c+\epsilon)(f(c+\epsilon) - f(c)) + f(c)(g(c+\epsilon) - g(c)), \end{aligned} \tag{1.5.3}$$

which is infinitesimal since both $f(c+\epsilon) - f(c)$ and $g(c+\epsilon) - g(c)$ are. Hence p is continuous at c.

Theorem 1.5.3. If f and g are both continuous at c, then the function

$$p(x) = f(x)g(x)$$

is also continuous at c.

Finally, suppose f and g are continuous at c and $g(c) \neq 0$. Let $q(x) = \frac{f(x)}{g(x)}$. Then, for any infinitesimal ϵ,

$$\begin{aligned} q(c+\epsilon) - q(c) &= \frac{f(c+\epsilon)}{g(c+\epsilon)} - \frac{f(c)}{g(c)} \\ &= \frac{f(c+\epsilon)g(c) - f(c)g(c+\epsilon)}{g(c+\epsilon)g(c)} \\ &= \frac{f(c+\epsilon)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(c+\epsilon)}{g(c+\epsilon)g(c)} \\ &= \frac{g(c)(f(c+\epsilon) - f(c)) - f(c)(g(c+\epsilon) - g(c))}{g(c+\epsilon)g(c)}, \end{aligned} \tag{1.5.4}$$

which is infinitesimal since both $f(c+\epsilon) - f(c)$ and $g(c+\epsilon) - g(c)$ are infinitesimals, and $g(c)g(c+\epsilon)$ is not an infinitesimal. Hence q is continuous at c.

Theorem 1.5.4. If f and g are both continuous at c and $g(c) \neq 0$, then the function

$$q(x) = \frac{f(x)}{g(x)}$$

is continuous at c.

Exercise 1.5.1. Explain why

$$f(x) = \frac{3x+4}{x^2+1}$$

is continuous on $(-\infty, \infty)$.

1.5.1 Polynomials and rational functions

It is now possible to identify two important classes of continuous functions. First, every constant function is continuous: indeed, if $f(x) = k$ for all real values x, and k is any real constant, then for any infinitesimal ϵ,

$$f(x + \epsilon) = k = f(x). \tag{1.5.5}$$

Next, the function $f(x) = x$ is continuous for all real x since, for any infinitesimal ϵ,

$$f(x + \epsilon) = x + \epsilon \simeq x = f(x). \tag{1.5.6}$$

Since the product of continuous functions is continuous, it now follows that, for any nonnegative integer n, $g(x) = x^n$ is continuous on $(-\infty, \infty)$ since it is a constant function if $n = 0$ and a product of $f(x) = x$ by itself n times otherwise.

From this it follows (since constant multiples of continuous functions are again continuous) that all *monomials*, that is, functions of the form $f(x) = ax^n$, where a is a fixed real constant and n is a nonnegative integer, are continuous.

Now a *polynomial* is a function of the form

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, \tag{1.5.7}$$

where a_0, a_1, $\ldots$, a_n are real constants and n is a nonnegative integer. That is, a polynomial is a sum of monomials. Since sums of continuous functions are continuous, we now have the following fundamental result.

Theorem 1.5.5. If P is a polynomial, then P is continuous on (∞, ∞).

Example 1.5.3. The function

$$f(x) = 32 + 14x^5 - 6x^7 + \pi x^{14}$$

is continuous on $(-\infty, \infty)$.

A *rational function* is a ratio of polynomials. That is, if $P(x)$ and $Q(x)$ are polynomials, then

$$R(x) = \frac{P(x)}{Q(x)} \tag{1.5.8}$$

is a rational function. Since ratios of continuous functions are continuous, we have the following.

Theorem 1.5.6. If R is rational function, then R is continuous at every point in its domain.

Example 1.5.4. If

$$f(x) = \frac{3x-4}{x^2-1},$$

then f is a rational function defined for all real x except $x=-1$ and $x=1$. Thus f is continuous on the intervals $(-\infty,-1)$, $(-1,1)$, and $(1,\infty)$.

Exercise 1.5.2. Find the intervals on which

$$f(x) = \frac{3x^2-1}{x^3+1}$$

is continuous.

1.5.2 Trigonometric functions

Recall that if t is a real number and (a,b) is the point in the plane found by traversing the unit circle $x^2+y^2=1$ a distance $|t|$ from $(1,0)$, in the counter-clockwise direction if $t \geq 0$ and in the clockwise direction otherwise, then

$$a = \cos(t) \tag{1.5.9}$$

$$b = \sin(t). \tag{1.5.10}$$

Note that for $0 < t < \pi$, as in Figure 1.5.1, t is greater than the length of the line segment from $A=(1,0)$ to $B=(\cos(t),\sin(t))$. Now the segment from A to B is the hypotenuse of the right triangle with vertices at A, B, and $C=(\cos(t),0)$. Since the distance from C to A is $1-\cos(t))$ and the distance from B to C is $\sin(t)$, it follows from the Pythagorean theorem that

$$\begin{aligned} t^2 &> (1-\cos(t))^2 + \sin^2(t) \\ &= 1 - 2\cos(t) + \cos^2(t) + \sin^2(t) \\ &= 2 - 2\cos(t). \end{aligned} \tag{1.5.11}$$

A similar diagram reveals the same result for $-\pi < t < 0$. Moreover, both t^2 and $2-2\cos(t)$ are 0 when $t=0$, so we have $t^2 \geq 2-2\cos(t)$ for all $-\pi < t < \pi$.

Additionally, $0 \leq 2-2\cos(t) \leq 4$ for all t (since $-1 \leq \cos(t) \leq 1$ for all t), so certainly $t^2 > 2-2\cos(t)$ whenever $|t| > 2$. Hence we have shown that

$$0 \leq 2 - 2\cos(t) \leq t^2 \tag{1.5.12}$$

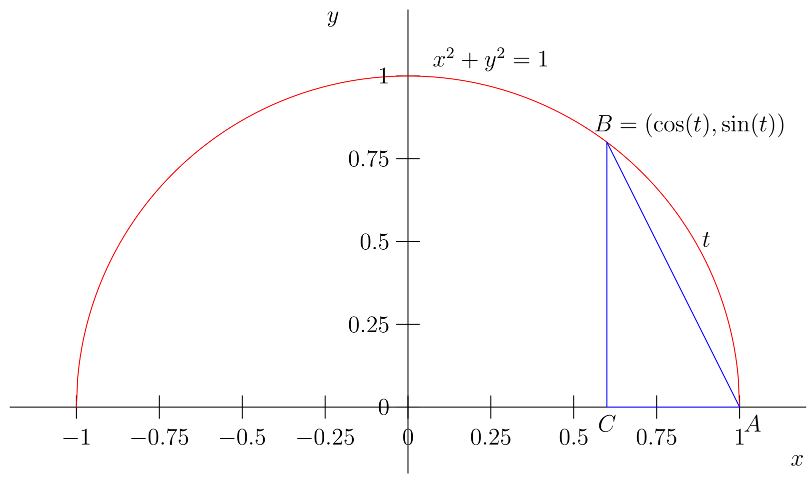

Figure 1.5.1: An arc of length t on the unit circle

for all values of t. Equivalently,

$$0 \le 1 - \cos(t) \le \frac{1}{2}t^2 \tag{1.5.13}$$

Solving for $\cos(t)$, we may also write this as

$$1 - \frac{t^2}{2} \le \cos(t) \le 1 \tag{1.5.14}$$

for all t.

In particular, if ϵ is an infinitesimal, then

$$1 - \frac{\epsilon^2}{2} \le \cos(\epsilon) \le 1 \tag{1.5.15}$$

implies that

$$\cos(0 + \epsilon) = \cos(\epsilon) \simeq 1 = \cos(0). \tag{1.5.16}$$

That is, the function $f(t) = \cos(t)$ is continuous at $t = 0$.

Moreover, since $0 \le 1 + \cos(t) \le 2$ for all t,

$$\begin{aligned} \sin^2(t) &= 1 - \cos^2(t) \\ &= (1 - \cos(t))(1 + \cos(t)) \\ &\le \frac{t^2}{2}(1 + \cos(t)) \\ &\le t^2, \end{aligned} \tag{1.5.17}$$

from which it follows that

$$|\sin(t)| \le |t| \tag{1.5.18}$$

for any value of t. In particular, for any infinitesimal ϵ,

$$|\sin(0+\epsilon)| = |\sin(\epsilon)| \leq \epsilon, \tag{1.5.19}$$

from which it follows that

$$\sin(\epsilon) \simeq 0 = \sin(0). \tag{1.5.20}$$

That is, the function $g(t) = \sin(t)$ is continuous at $t = 0$.

Using the angle addition formulas for sine and cosine, we see that, for any real number t and infinitesimal ϵ,

$$\cos(t+\epsilon) = \cos(t)\cos(\epsilon) - \sin(t)\sin(\epsilon) \simeq \cos(t) \tag{1.5.21}$$

and

$$\sin(t+\epsilon) = \sin(t)\cos(\epsilon) + \cos(t)\sin(\epsilon) \simeq \sin(t), \tag{1.5.22}$$

since $\cos(\epsilon) \simeq 1$ and $\sin(\epsilon)$ is an infinitesimal. Hence we have the following result.

Theorem 1.5.7. The functions $f(t) = \cos(t)$ and $g(t) = \sin(t)$ are continuous on $(-\infty, \infty)$.

The following theorem now follows from our earlier results about continuous functions.

Theorem 1.5.8. The following functions are continuous at each point in their respective domains:

$$\tan(t) = \frac{\sin(t)}{\cos(t)}, \tag{1.5.23}$$

$$\cot(t) = \frac{\cos(t)}{\sin(t)}, \tag{1.5.24}$$

$$\sec(t) = \frac{1}{\cos(t)}, \tag{1.5.25}$$

$$\csc(t) = \frac{1}{\sin(t)}. \tag{1.5.26}$$

With a little more geometry, we may improve upon the inequalities in (1.5.13) and (1.5.18). Consider an angle $0 < t < \frac{\pi}{2}$, let $A = (1,0)$ and $B = (\cos(t), \sin(t))$ as above, and let D be the point of intersection of the lines tangent to the circle $x^2 + y^2 = 1$ at A and B (see Figure 1.5.2). Note that the triangle with vertices at A, B, and D is isosceles with base of length $\sqrt{2(1-\cos(t))}$ (as derived above) and base angles $\frac{t}{2}$. Moreover, the sum of the lengths of the two legs exceeds t. Since each leg is of length

$$\frac{\frac{1}{2}\sqrt{2(1-\cos(t))}}{\cos\left(\frac{t}{2}\right)}, \tag{1.5.27}$$

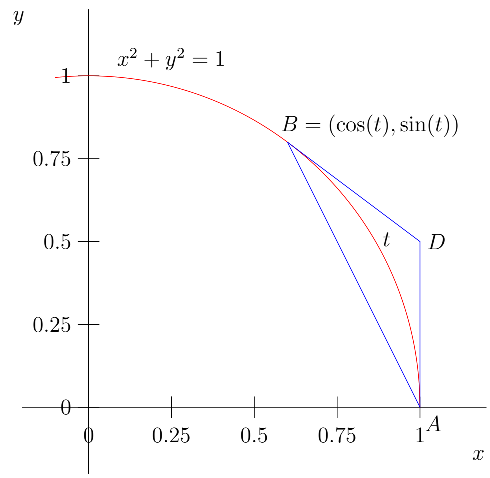

Figure 1.5.2: An upper bound for the arc length t

it follows that

$$t < \frac{\sqrt{2(1-\cos(t))}}{\cos\left(\frac{t}{2}\right)}. \tag{1.5.28}$$

Moreover, both sides of this inequality are 0 when $t = 0$ and we could derive the same inequality for $-\frac{\pi}{2} < t < 0$, so we have

$$t \leq \frac{\sqrt{2(1-\cos(t))}}{\cos\left(\frac{t}{2}\right)} \tag{1.5.29}$$

for all $-\frac{\pi}{2} < t < \frac{\pi}{2}$. It now follows that

$$\frac{1}{2}t^2 \leq \frac{1-\cos(t)}{\cos^2\left(\frac{t}{2}\right)} = \frac{2(1-\cos(t))}{1+\cos(t)} \tag{1.5.30}$$

for all $-\frac{\pi}{2} < t < \frac{\pi}{2}$, where we have used the half-angle identity

$$\cos^2\left(\frac{t}{2}\right) = \frac{1+\cos(t)}{2}. \tag{1.5.31}$$

Combining (1.5.30) with (1.5.13), we have

$$1-\cos(t) \leq \frac{1}{2}t^2 \leq \frac{2(1-\cos(t))}{1+\cos(t)}, \tag{1.5.32}$$

from which we obtain, for all $-\frac{\pi}{2} < t < \frac{\pi}{2}$,

$$\frac{1+\cos(t)}{2} \leq \frac{1-\cos(t)}{\frac{1}{2}t^2} \leq 1. \tag{1.5.33}$$

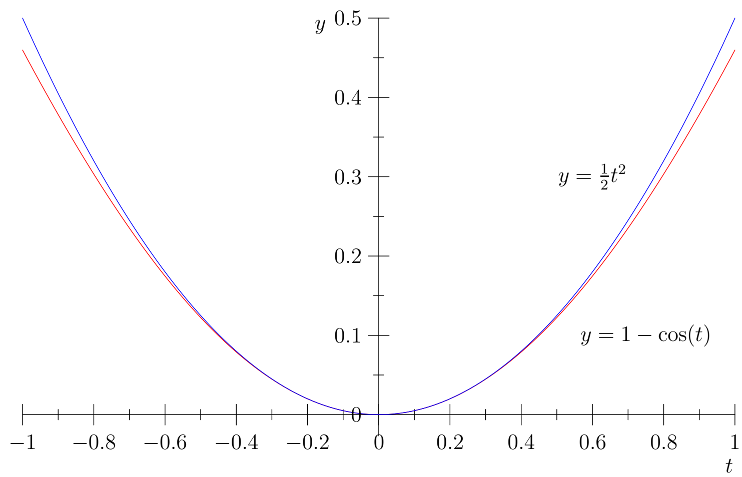

Figure 1.5.3: Comparing $y = 1 - \cos(t)$ with $y = \frac{1}{2}t^2$

Now if ϵ is an infinitesimal, then $1 + \cos(\epsilon) \simeq 2$, and so

$$\frac{1 + \cos(\epsilon)}{2} \simeq 1.$$

Hence, substituting $t = \epsilon$ in (1.5.33), we have

$$\frac{1 - \cos(\epsilon)}{\frac{1}{2}\epsilon^2} \simeq 1. \tag{1.5.34}$$

Moreover, we then have

$$\frac{\sin^2(\epsilon)}{\epsilon^2} = \frac{1 - \cos^2(\epsilon)}{\epsilon^2} = \frac{1 - \cos(\epsilon)}{\epsilon^2}(1 + \cos(t)) \simeq \frac{1}{2}(2) = 1. \tag{1.5.35}$$

Since ϵ and $\sin(\epsilon)$ have the same sign, it follows that

$$\frac{\sin(\epsilon)}{\epsilon} \simeq 1. \tag{1.5.36}$$

For real numbers t, (1.5.34) and (1.5.36) say that, for small values of t,

$$\cos(t) \approx 1 - \frac{t^2}{2} \tag{1.5.37}$$

and

$$\sin(t) \approx t. \tag{1.5.38}$$

Figures 1.5.3 and 1.5.4 graphically display the comparisons in (1.5.37) and (1.5.38).

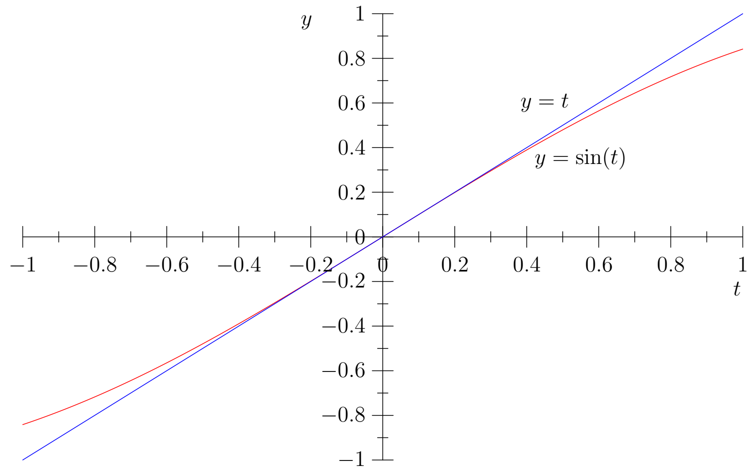

Figure 1.5.4: Comparison of $y = \sin(t)$ with $y = t$

Example 1.5.5. For a numerical comparison, note that for $t = 0.1$, $\cos(t) = 0.9950042$, compared to $1 - \frac{t^2}{2} = 0.995$, and $\sin(t) = 0.0998334$, compared to $t = 0.1$.

Exercise 1.5.3. Verify that the triangle with vertices at A, B, and D in Figure 1.5.2 is an isosceles triangle with base angles of $\frac{t}{2}$ at A and B.

Exercise 1.5.4. Verify the half-angle formula,

$$\cos(\theta) = \frac{1}{2}(1 + \cos(2\theta)),$$

for any angle θ, using the identities $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ (a consequence of the addition formula) and $\sin^2(\theta) + \cos^2(\theta) = 1$.

1.5.3 Compositions

Given functions f and g, we call the function

$$f \circ g(x) = f(g(x)) \tag{1.5.39}$$

the *composition* of f with g. If g is continuous at a real number c, f is continuous at $g(c)$, and ϵ is an infinitesimal, then

$$f \circ g(c + \epsilon) = f(g(c + \epsilon)) \simeq f(g(c)) \tag{1.5.40}$$

since $g(c + \epsilon) \simeq g(c)$.

Theorem 1.5.9. If g is continuous at c and f is continuous at $g(c)$, then $f \circ g$ is continuous at c.

Example 1.5.6. Since $f(t) = \sin(t)$ is continuous for all t and

$$g(t) = \frac{3t^2 + 1}{4t - 8}$$

is continuous at all real numbers except $t = 2$, it follows that

$$h(t) = \sin\left(\frac{3t^2 + 1}{4t - 8}\right)$$

is continuous on the intervals $(-\infty, 2)$ and $(2, \infty)$.

Note that if $f(x) = \sqrt{x}$ and ϵ is an infinitesimal, then, for any $x \neq 0$,

$$\begin{aligned} f(x+\epsilon) - f(x) &= \sqrt{x+\epsilon} - \sqrt{x} \\ &= (\sqrt{x+\epsilon} - \sqrt{x})\left(\frac{\sqrt{x+\epsilon} + \sqrt{x}}{\sqrt{x+\epsilon} + \sqrt{x}}\right) \\ &= \frac{x + \epsilon - x}{\sqrt{x+\epsilon} + \sqrt{x}} \\ &= \frac{\epsilon}{\sqrt{x+\epsilon} + \sqrt{x}}, \end{aligned}$$

which is infinitesimal. Hence f is continuous on $(0, \infty)$. Moreover, if ϵ is a positive infinitesimal, then $\sqrt{\epsilon}$ must be an infinitesimal (since if $a = \sqrt{\epsilon}$ is not an infinitesimal, then $a^2 = \epsilon$ is not an infinitesimal). Hence

$$f(0 + \epsilon) = \sqrt{\epsilon} \simeq 0 = f(0).$$

Thus f is continuous at 0, and so $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Theorem 1.5.10. The function $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Example 1.5.7. It now follows that $f(x) = \sqrt{4x - 2}$ is continuous everywhere it is defined, namely, on $[2, \infty)$.

Exercise 1.5.5. Find the interval or intervals on which $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous.

Exercise 1.5.6. Find the interval or intervals on which

$$g(t) = \sqrt{\frac{1 + t^2}{1 - t^2}}$$

is continuous.

1.5.4 Consequences of continuity

Continuous functions have two important properties that will play key roles in our discussions in the rest of the text: the extreme-value property and the intermediate-value property. Both of these properties rely on technical aspects of the real numbers which lie beyond the scope of this text, and so we will not attempt justifications.

The extreme-value property states that a continuous function on a closed interval $[a, b]$ attains both a maximum and minimum value.

Theorem 1.5.11. If f is continuous on a closed interval $[a, b]$, then there exists a real number c in $[a, b]$ for which $f(c) \leq f(x)$ for all x in $[a, b]$ and a real number d in $[a, b]$ for which $f(d) \geq f(x)$ for all x in $[a, b]$.

The following examples show the necessity of the two conditions of the theorem (that is, the function must be continuous and the interval must be closed in order to ensure the conclusion).

Example 1.5.8. The function $f(x) = x^2$ attains neither a maximum nor a minimum value on the interval $(0, 1)$. Indeed, given any point a in $(0, 1)$, $f(x) > f(a)$ whenever $a < x < 1$ and $f(x) < f(a)$ whenever $0 < x < a$. Of course, this does not contradict the theorem because $(0, 1)$ is not a closed interval. On the closed interval $[0, 1]$, we have $f(1) \geq f(x)$ for all x in $[0, 1]$ and $f(0) \leq f(x)$ for all x in $[0, 1]$, in agreement with the theorem.

In this example the extreme values of f occurred at the endpoints of the interval $[-1, 1]$. This need not be the case. For example, if $g(t) = \sin(t)$, then, on the interval $[0, 2\pi]$, g has a minimum value of -1 at $t = \frac{3\pi}{2}$ and a maximum value of 1 at $t = \frac{\pi}{2}$.

Example 1.5.9. Let

$$f(x) = \begin{cases} \dfrac{1}{x}, & \text{if } -1 \leq x < 0 \text{ or } 0 < x \leq 1, \\ 0, & \text{if } x = 0. \end{cases}$$

See Figure 1.5.5. Then f does not have a maximum value: if $a \leq 0$, then $f(x) > f(a)$ for any $x > 0$, and if $a > 0$, then $f(x) > f(a)$ whenever $0 < x < a$. Similarly, f has no minimum value: if $a \geq 0$, then $f(x) < f(a)$ for any $x < 0$, and if $a < 0$, then $f(x) < f(a)$ whenever $a < x < 0$. The problem this time is that f is not continuous at $x = 0$. Indeed, if ϵ is an infinitesimal, then $f(\epsilon)$ is infinite, and, hence, not infinitesimally close to $f(0) = 0$.

Exercise 1.5.7. Find an example of a continuous function which has both a minimum value and a maximum value on the open interval $(0, 1)$.

Exercise 1.5.8. Find an example of function which has a minimum value and a maximum value on the interval $[0, 1]$, but is not continuous on $[0, 1]$.

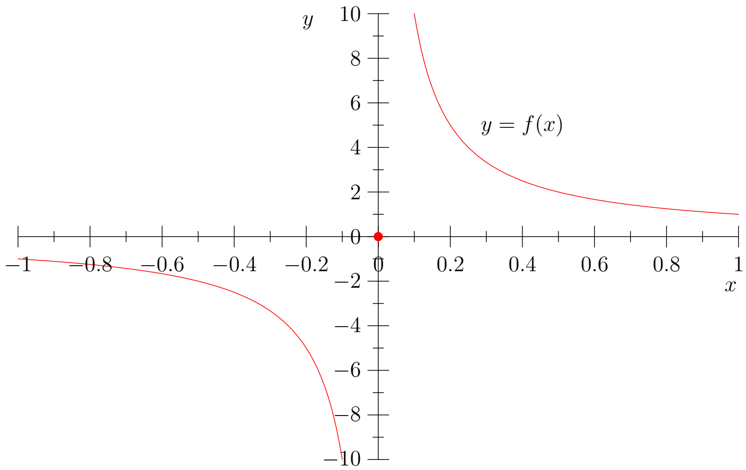

Figure 1.5.5: A function with no minimum or maximum value on $[-1, 1]$

The intermediate-value property states that a continuous function attains all values between any two given values of the function.

Theorem 1.5.12. If f is continuous on the interval $[a, b]$ and m is any value betwen $f(a)$ and $f(b)$, then there exists a real number c in $[a, b]$ for which $f(c) = m$.

The next example shows that a function which is not continuous need not satisfy the intermediate-value property.

Example 1.5.10. If H is the Heaviside function, then $H(-1) = 0$ and $H(1) = 1$, but there does not exist any real number c in $[-1, 1]$ for which $H(c) = \frac{1}{2}$, even though $0 < \frac{1}{2} < 1$.

1.6 The derivative

We now return to the problem of rates of change. Given $y = f(x)$, for any infinitesimal dx we let

$$dy = f(x + dx) - f(x). \tag{1.6.1}$$

If y is a continuous function of x, then dy is infinitesimal and, if $dx \neq 0$, the ratio

$$\frac{dy}{dx} = \frac{f(x + dx) - f(x)}{dx} \tag{1.6.2}$$

is a hyperreal number. If $\frac{dy}{dx}$ is finite, then its shadow, if it is the same for all values of dx, is the rate of change of y with respect to x, which we will call the *derivative* of y with respect to x.

Definition 1.6.1. Given $y = f(x)$, suppose

$$\frac{dy}{dx} = \frac{f(x+dx) - f(x)}{dx} \tag{1.6.3}$$

is finite and has the same shadow for all nonzero infinitesimals dx. Then we call

$$\operatorname{sh}\left(\frac{dy}{dx}\right) \tag{1.6.4}$$

the *derivative* of y with respect to x.

Note that the quotient in (1.6.3) will be infinite if $f(x + dx) - f(x)$ is not an infinitesimal. Hence a function which is not continuous at x cannot have a derivative at x.

There are numerous ways to denote the derivative of a function $y = f(x)$. One is to use $\frac{dy}{dx}$ to denote, depending on the context, both the ratio of the infinitesimals dy and dx and the shadow of this ratio, which is the derivative. Another is to write $f'(x)$ for the derivative of the function f. We will use both of these notations extensively.

Example 1.6.1. If $y = x^2$, then, for any nonzero infinitesimal dx,

$$dy = (x + dx)^2 - x^2 = (x^2 + 2xdx + (dx)^2) - x^2 = (2x + dx)dx.$$

Hence

$$\frac{dy}{dx} = 2x + dx \simeq 2x,$$

and so the derivative of y with respect to x is

$$\frac{dy}{dx} = 2x.$$

Example 1.6.2. If $f(x) = 4x$, then, for any nonzero infinitesimal dx,

$$\frac{f(x+dx) - f(x)}{dx} = \frac{4(x+dx) - 4x}{dx} = \frac{4dx}{dx} = 4.$$

Hence $f'(x) = 4$. Note that this implies that $f(x)$ has a constant rate of change: every change of one unit in x results in a change of 4 units in $f(x)$.

Exercise 1.6.1. Find $\frac{dy}{dx}$ if $y = 5x - 2$.

Exercise 1.6.2. Find $\frac{dy}{dx}$ if $y = x^3$.

Exercise 1.6.3. Find $f'(x)$ if $f(x) = 4x^2$.

To denote the rate of change of y with respect to x at a particular value of x, say, when $x = a$, we write

$$\left.\frac{dy}{dx}\right|_{x=a}. \tag{1.6.5}$$

If $y = f(x)$, then, of course, this is the same as writing $f'(a)$.

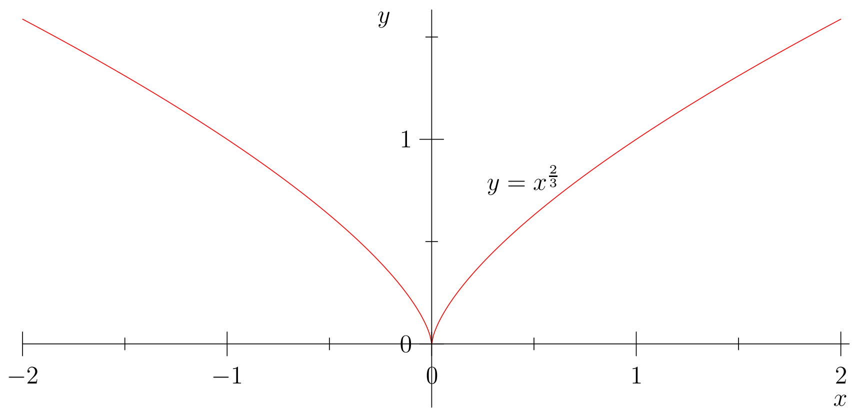

Figure 1.6.1: $y = x^{\frac{2}{3}}$ is continuous, but not differentiable, at $x = 0$

Example 1.6.3. If $y = x^2$, then we saw above that $\frac{dy}{dx} = 2x$. Hence the rate of change of y with respect to x when $x = 3$ is

$$\left.\frac{dy}{dx}\right|_{x=3} = (2)(3) = 6.$$

Example 1.6.4. If $f(x) = x^{\frac{2}{3}}$, then for any infinitesimal dx,

$$f(0 + dx) - f(0) = f(dx) = (dx)^{\frac{2}{3}},$$

which is infinitesimal. Hence f is continuous at $x = 0$. Now if $dx \neq 0$, then

$$\frac{f(0 + dx) - f(0)}{dx} = \frac{(dx)^{\frac{2}{3}}}{dx} = \frac{1}{(dx)^{\frac{1}{3}}}.$$

Since this is infinite, f does not have a derivative at $x = 0$. In particular, this shows that a function may be continuous at a point, but not differentiable at that point. See Figure 1.6.1.

Example 1.6.5. If $f(x) = \sqrt{x}$, then, as we have seen above, for any $x > 0$ and any nonzero infinitesimal dx,

$$\begin{aligned} f(x + dx) - f(x) &= \sqrt{x + dx} - \sqrt{x} \\ &= (\sqrt{x + dx} - \sqrt{x})\frac{\sqrt{x + dx} + \sqrt{x}}{\sqrt{x + dx} + \sqrt{x}} \\ &= \frac{(x + dx) - x}{\sqrt{x + dx} + \sqrt{x}} \\ &= \frac{dx}{\sqrt{x + dx} + \sqrt{x}}. \end{aligned}$$

It now follows that

$$\frac{f(x+dx)-f(x)}{dx} = \frac{1}{\sqrt{x+dx}+\sqrt{x}} \simeq \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Thus

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

For example, the rate of change of y with respect to x when $x = 9$ is

$$f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}.$$

We will sometimes also write

$$\frac{d}{dx}f(x) \tag{1.6.6}$$

for $f'(x)$. With this notation, we could write the result of the previous example as

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Definition 1.6.2. Given a function f, if $f'(a)$ exists we say f is *differentiable* at a. We say f is *differentiable* on an open interval (a, b) if f is differentiable at each point x in (a, b).

Example 1.6.6. The function $y = x^2$ is differentiable on $(-\infty, \infty)$.

Example 1.6.7. The function $f(x) = \sqrt{x}$ is differentiable on $(0, \infty)$. Note that f is not differentiable at $x = 0$ since $f(0 + dx) = f(dx)$ is not defined for all infinitesimals dx.

Example 1.6.8. The function $f(x) = x^{\frac{2}{3}}$ is not differentiable at $x = 0$.

1.7 Properties of derivatives

We will now develop some properties of derivatives with the aim of facilitating their calculation for certain general classes of functions.

To begin, if $f(x) = k$ for all x and some real constant k, then, for any infinitesimal dx,

$$f(x+dx) - f(x) = k - k = 0. \tag{1.7.1}$$

Hence, if $dx \neq 0$,

$$\frac{f(x+dx)-f(x)}{dx} = 0, \tag{1.7.2}$$

and so $f'(x) = 0$. In other words, the derivative of a constant is 0.

Theorem 1.7.1. For any real constant k,

$$\frac{d}{dx}k = 0. \tag{1.7.3}$$

Example 1.7.1. $\frac{d}{dx}4 = 0$.

1.7.1 Sums and differences

Now suppose u and v are both differentiable functions of x. Then, for any infinitesimal dx,

$$\begin{aligned} d(u+v) &= (u(x+dx)+v(x+dx)) - (u(x)-v(x)) \\ &= (u(x+dx)-u(x)) + (v(x+dx)-v(x)) \\ &= du + dv. \end{aligned} \tag{1.7.4}$$

Hence, if $dx \neq 0$,

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}. \tag{1.7.5}$$

In other words, the derivative of a sum is the sum of the derivatives.

Theorem 1.7.2. If f and g are both differentiable and $s(x) = f(x) + g(x)$, then

$$s'(x) = f'(x) + g'(x). \tag{1.7.6}$$

Example 1.7.2. If $y = x^2 + \sqrt{x}$, then, using our results from the previous section,

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sqrt{x}) = 2x + \frac{1}{2\sqrt{x}}.$$

A similar argument shows that

$$\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}. \tag{1.7.7}$$

Exercise 1.7.1. Find the derivative of $y = x^2 + 5$.

Exercise 1.7.2. Find the derivative of $f(x) = \sqrt{x} - x^2 + 3$.

1.7.2 Constant multiples

If c is any real constant and u is a differentiable function of x, then, for any infinitesimal dx,

$$d(cu) = cu(x+dx) - cu(x) = c(u(x+dx) - u(x)) = cdu. \tag{1.7.8}$$

Hence, if $dx \neq 0$,

$$\frac{d(cu)}{dx} = c\frac{du}{dx}. \tag{1.7.9}$$

In other words, the derivative of a constant times a function is the constant times the derivative of the function.

Theorem 1.7.3. If c is a real constant, f is differentiable, and $g(x) = cf(x)$, then

$$g'(x) = cf'(x). \tag{1.7.10}$$

Example 1.7.3. If $y = 5x^2$, then

$$\frac{dy}{dx} = 5\frac{d}{dx}(x^2) = 5(2x) = 10x.$$

Exercise 1.7.3. Find the derivative of $y = 8x^2$.

Exercise 1.7.4. Find the derivative of $f(x) = 4\sqrt{x} + 15$.

1.7.3 Products

Again suppose u and v are differentiable functions of x. Note that, in particular, u and v are continuous, and so both du and dv are infinitesimal for any infinitesimal dx. Moreover, note that

$$u(x + dx) = u(x) + du \text{ and } v(x + dx) = v(x) + dv. \tag{1.7.11}$$

Hence

$$\begin{aligned} d(uv) &= u(x + dx)v(x + dx) - u(x)v(x) \\ &= (u(x) + du)(v(x) + dv) - u(x)v(x) \\ &= (u(x)v(x) + u(x)dv + v(x)du + dudv) - u(x)v(x) \\ &= udv + vdu + dudv, \end{aligned} \tag{1.7.12}$$

and so, if $dx \neq 0$,

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} + du\frac{dv}{dx} \simeq u\frac{dv}{dx} + v\frac{du}{dx} \tag{1.7.13}$$

Thus we have, for any differentiable functions u and v,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}, \tag{1.7.14}$$

which we call the *product rule*.

Theorem 1.7.4. If f and g are both differentiable and $p(x) = f(x)g(x)$, then

$$p'(x) = f(x)g'(x) + g(x)f'(x). \tag{1.7.15}$$

Example 1.7.4. We may use the product rule to find a formula for the derivative of a positive integer power of x. We first note that if $y = x$, then, for any infinitesimal dx,

$$dy = (x + dx) - x = dx, \tag{1.7.16}$$

and so, if $dx \neq 0$,

$$\frac{dy}{dx} = \frac{dx}{dx} = 1. \tag{1.7.17}$$

Thus we have

$$\frac{d}{dx}x = 1, \tag{1.7.18}$$

as we should expect, since $y = x$ implies that y changes at exactly the same rate as x.

Using the product rule, it now follows that

$$\frac{d}{dx}x^2 = \frac{d}{dx}(x \cdot x) = x\frac{d}{dx}x + x\frac{d}{dx}x = x + x = 2x, \tag{1.7.19}$$

in agreement with a previous example. Next, we have

$$\frac{d}{dx}x^3 = x\frac{d}{dx}x^2 + x^2\frac{d}{dx}x = 2x^2 + x^2 = 3x^2 \tag{1.7.20}$$

and

$$\frac{d}{dx}x^4 = x\frac{d}{dx}x^3 + x^3\frac{d}{dx}x = 3x^3 + x^3 = 4x^3. \tag{1.7.21}$$

At this point we might suspect that for any integer $n \geq 1$,

$$\frac{d}{dx}x^n = nx^{n-1}. \tag{1.7.22}$$

This is in fact true, and follows easily from an inductive argument: Suppose we have shown that for any $k < n$,

$$\frac{d}{dx}x^k = kx^{k-1}. \tag{1.7.23}$$

Then

$$\begin{aligned}\frac{d}{dx}x^n &= x\frac{d}{dx}x^{n-1} + x^{n-1}\frac{d}{dx}x \\ &= x((n-1)x^{n-2}) + x^{n-1} \\ &= nx^{n-1}.\end{aligned} \tag{1.7.24}$$

We call this result the *power rule*.

Theorem 1.7.5. For any integer $n \geq 1$,

$$\frac{d}{dx}x^n = nx^{n-1}. \tag{1.7.25}$$

We shall see eventually, in Theorems 1.7.7, 1.7.10, and 2.7.2, that the power rule in fact holds for any real number $n \neq 0$.

Example 1.7.5. When $n = 34$, the power rule shows that

$$\frac{d}{dx}x^{34} = 34x^{33}.$$

Example 1.7.6. If $f(x) = 14x^5$, then, combining the power rule with our result for constant multiples,

$$f'(x) = 14(5x^4) = 70x^4.$$

Exercise 1.7.5. Find the derivative of $y = 13x^5$.

Example 1.7.7. Combining the power rule with our results for constant multiples and differences, we have

$$\frac{d}{dx}(3x^2 - 5x) = 6x - 5.$$

Exercise 1.7.6. Find the derivative of $f(x) = 5x^4 - 3x^2$.

Exercise 1.7.7. Find the derivative of $y = 3x^7 - 3x + 1$.

1.7.4 Polynomials

As the previous examples illustrate, we may put together the above results to easily differentiate any polynomial function. That is, if $n \geq 1$ and a_n, a_{n-1}, $\ldots$, a_0 are any real constants, then

$$\begin{aligned}\frac{d}{dx}&(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0) \\ &= n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 x + a_1. \quad (1.7.26)\end{aligned}$$

Example 1.7.8. If $p(x) = 4x^7 - 13x^3 - x^2 + 21$, then

$$p'(x) = 28x^6 - 39x^2 - 2x.$$

Exercise 1.7.8. Find the derivative of $f(x) = 3x^5 - 6x^4 - 5x^2 + 13$.

1.7.5 Quotients

If u is a differentiable function of x, $u(x) \neq 0$, and dx is an infinitesimal, then

$$\begin{aligned} d\left(\frac{1}{u}\right) &= \frac{1}{u(x+dx)} - \frac{1}{u(x)} \\ &= \frac{1}{u(x)+du} - \frac{1}{u(x)} \\ &= \frac{u - (u+du)}{u(u+du)} \\ &= \frac{-du}{u(u+du)}. \qquad (1.7.27) \end{aligned}$$

Hence, since $u + du \simeq u$, if $dx \neq 0$,

$$\frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{\frac{du}{dx}}{u(u+du)} \simeq -\frac{1}{u^2}\frac{du}{dx}. \qquad (1.7.28)$$

Theorem 1.7.6. If f is differentiable, $f(x) \neq 0$, and

$$g(x) = \frac{1}{f(x)}, \tag{1.7.29}$$

then

$$g'(x) = -\frac{f'(x)}{(f(x))^2}. \tag{1.7.30}$$

Example 1.7.9. If

$$f(x) = \frac{1}{x^2},$$

then

$$f'(x) = -\frac{1}{x^4} \cdot 2x = -\frac{2}{x^3}.$$

Note that the result of the previous example is the same as we would have obtained from applying the power rule with $n = -2$. In fact, we may now show that the power rule holds in general for negative integer powers: If $n < 0$ is an integer, then

$$\frac{d}{dx}x^n = \frac{d}{dx}\left(\frac{1}{x^{-n}}\right) = -\frac{1}{x^{-2n}} \cdot (-nx^{-n-1}) = nx^{n-1}. \tag{1.7.31}$$

Hence we now have our first generalization of the power rule.

Theorem 1.7.7. For any integer $n \neq 0$,

$$\frac{d}{dx}x^n = nx^{n-1}. \tag{1.7.32}$$

Example 1.7.10. If

$$f(x) = 3x^2 - \frac{5}{x^7},$$

then $f(x) = 3x^2 - 5x^{-7}$, and so

$$f'(x) = 6x + 35x^{-8} = 6x + \frac{35}{x^8}.$$

Now suppose u and v are both differentiable functions of x and let

$$y = \frac{u}{v}.$$

Then $u = vy$, so, as we saw above,

$$du = vdy + ydv + dvdy = ydv + (v + dv)dy. \tag{1.7.33}$$

Hence, provided $v(x) \neq 0$,

$$dy = \frac{du - ydv}{v + dv} = \frac{du - \frac{u}{v}dv}{v + dv} = \frac{vdu - udv}{v(v + dv)}. \tag{1.7.34}$$

Thus, for any nonzero infinitesimal dx,

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v(v+dv)} \simeq \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}. \tag{1.7.35}$$

This is the *quotient rule*.

Theorem 1.7.8. If f and g are differentiable, $g(x) \neq 0$, and

$$q(x) = \frac{f(x)}{g(x)}, \tag{1.7.36}$$

then

$$q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}. \tag{1.7.37}$$

One consequence of the quotient rule is that, since we already know how to differentiate polynomials, we may now differentiate any rational function easily.

Example 1.7.11. If

$$f(x) = \frac{3x^2 - 6x + 4}{x^2 + 1},$$

then

$$\begin{aligned} f'(x) &= \frac{(x^2+1)(6x-6) - (3x^2 - 6x + 4)(2x)}{(x^2+1)^2} \\ &= \frac{6x^3 - 6x^2 + 6x - 6 - 6x^3 + 12x^2 - 8x}{(x^2+1)^2} \\ &= \frac{6x^2 - 2x - 6}{(x^2+1)^2}. \end{aligned}$$

Example 1.7.12. We may use either 1.7.30 or 1.7.37 to differentiate

$$y = \frac{5}{x^2+1}. \tag{1.7.38}$$

In either case, we obtain

$$\frac{dy}{dx} = -\frac{5}{(x^2+1)^2}\frac{d}{dx}(x^2+1) = -\frac{10x}{(x^2+1)^2}. \tag{1.7.39}$$

Exercise 1.7.9. Find the derivative of

$$y = \frac{14}{4x^3 - 3x}.$$

Exercise 1.7.10. Find the derivative of

$$f(x) = \frac{4x^3 - 1}{x^2 - 5}.$$

1.7.6 Composition of functions

Suppose y is a differentiable function of u and u is a differentiable function of x. Then y is both a function of u and a function of x, and so we may ask for the derivative of y with respect to x as well as the derivative of y with respect to u. Now if dx is an infinitesimal, then

$$du = u(x + dx) - u(x)$$

is also an infinitesimal (since u is continuous). If $du \neq 0$, then the derivative of y with respect to u is equal to the shadow of $\frac{dy}{du}$. At the same time, if $dx \neq 0$, the derivative of u with respect to x is equal to the shadow of $\frac{du}{dx}$. But

$$\frac{dy}{du}\frac{du}{dx} = \frac{dy}{dx}, \tag{1.7.40}$$

and the shadow of $\frac{dy}{dx}$ is the derivative of y with respect to x. It follows that the derivative of y with respect to x is the product of the derivative of y with respect to u and the derivative of u with respect to x. Of course, du is not necessarily nonzero even if $dx \neq 0$ (for example, if u is a constant function), but the result holds nevertheless, although we will not go into the technical details here.

We call this result the *chain rule*.

Theorem 1.7.9. If y is a differentiable function of u and u is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}. \tag{1.7.41}$$

Not that if we let $y = f(u)$, $u = g(x)$, and

$$h(x) = f \circ g(x) = f(g(x)),$$

then

$$\frac{dy}{dx} = h'(x), \frac{dy}{du} = f'(g(x)), \text{ and } \frac{du}{dx} = g'(x).$$

Hence we may also express the chain rule in the form

$$h'(x) = f'(g(x))g'(x). \tag{1.7.42}$$

Example 1.7.13. If $y = 3u^2$ and $u = 2x + 1$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (6u)(2) = 12u = 24x + 12.$$

We may verify this result by first finding y directly in terms of x, namely,

$$y = 3u^2 = 3(2x + 1)^2 = 3(4x^2 + 4x + 1) = 12x^2 + 12x + 3,$$

and then differentiating directly:

$$\frac{dy}{dx} = \frac{d}{dx}(12x^2 + 12x + 3) = 24x + 12.$$

Note that if we want to evaluate $\frac{dy}{dx}$ when, for example, $x = 2$, we may either evaluate the final form, that is,

$$\left.\frac{dy}{dx}\right|_{x=2} = (24x + 12)|_{x=2} = 48 + 12 = 60,$$

or, noting that $u = 5$ when $x = 2$, the intermediate form, that is,

$$\left.\frac{dy}{dx}\right|_{x=2} = 12u|_{u=5} = 60.$$

In other words,

$$\left.\frac{dy}{dx}\right|_{x=2} = \left.\frac{dy}{du}\right|_{u=5} \left.\frac{du}{dx}\right|_{x=2}.$$

Exercise 1.7.11. If $y = u^3 + 5$ and $u = x^2 - 1$, find $\left.\frac{dy}{dx}\right|_{x=1}$.

Example 1.7.14. If $h(x) = \sqrt{x^2 + 1}$, then $h(x) = f(g(x))$ where $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$. Since

$$f'(x) = \frac{1}{2\sqrt{x}} \text{ and } g'(x) = 2x,$$

it follows that

$$h'(x) = f'(g(x))g'(x) = \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}}.$$

Exercise 1.7.12. Find the derivative of $f(x) = \sqrt{4x + 6}$.

Exercise 1.7.13. Find the derivative of $y = (x^2 + 5)^{10}$.

Example 1.7.15. As we saw in Example 1.2.5, if M is the mass, in grams, of a spherical balloon being filled with water and r is the radius of the balloon, in centimeters, then

$$M = \frac{4}{3}\pi r^3 \text{ grams}$$

and

$$\frac{dM}{dr} = 4\pi r^2 \text{ grams/centimeter},$$

a result which we may verify easily now using the power rule. Suppose water is being pumped into the balloon so that the radius of the balloon is increasing at the rate of 0.1 centimeters per second when the balloon has a radius of 10 centimeters. Since M is a function of time t, as well as a function of the radius of the balloon r, we might wish to know the rate of change of M with respect to t. Since we are given that

$$\left.\frac{dr}{dt}\right|_{r=10} = 0.1 \text{ centimeters/second},$$

we may use the chain rule to find that

$$\left.\frac{dM}{dt}\right|_{r=10} = \left.\frac{dM}{dr}\right|_{r=10} \left.\frac{dr}{dt}\right|_{r=10} = (400\pi)(0.1) = 40\pi \text{ grams/second}.$$

Exercise 1.7.14. Suppose A is the area and r is the radius of a circular wave at time t. Suppose when $r = 100$ centimeters the radius of the circle is increasing at a rate of 2 centimeters per second. Find the rate at which the area of the circle is growing when $r = 100$ centimeters.

Exercise 1.7.15. If water is being pumped into a spherical balloon at the rate of 100 grams per second, find the rate of change of the radius r of the balloon when the radius of the balloon is $r = 15$ centimeters.

As an important special case of the chain rule, suppose $n \neq 0$ is an integer, g is a differentiable function, and $h(x) = (g(x))^n$. Then h is the composition of $f(x) = x^n$ with g, and so, using the chain rule,

$$h'(x) = f'(g(x))g'(x) = n(g(x))^{n-1}g'(x). \tag{1.7.43}$$

If we let $u = g(x)$, we could also express this result as

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}. \tag{1.7.44}$$

Example 1.7.16. With $n = 10$ and $g(x) = x^2 + 3$, we have

$$\frac{d}{dx}(x^2+3)^{10} = 10(x^2+3)^9(2x) = 20x(x^2+3)^9.$$

Example 1.7.17. If

$$f(x) = \frac{15}{(x^4+5)^2},$$

then we may apply the previous result with $n = -2$ and $g(x) = x^4 + 5$ to obtain

$$f'(x) = -30(x^4+5)^{-3}(4x^3) = -\frac{120x^3}{(x^4+5)^3}.$$

We may use the previous result to derive yet another extension to the power rule. If $n \neq 0$ is an integer and $y = x^{\frac{1}{n}}$, then $y^n = x$, and so, assuming y is differentiable,

$$\frac{d}{dx}y^n = \frac{d}{dx}x. \tag{1.7.45}$$

Hence

$$ny^{n-1}\frac{dy}{dx} = 1, \tag{1.7.46}$$

from which if follows that

$$\frac{dy}{dx} = \frac{1}{ny^{n-1}} = \frac{1}{n}y^{1-n} = \frac{1}{n}\left(x^{\frac{1}{n}}\right)^{1-n} = \frac{1}{n}x^{\frac{1}{n}-1}, \tag{1.7.47}$$

showing that the power rule works for rational powers of the form $\frac{1}{n}$. Note that the above derivation is not complete since we began with the assumption that $y = x^{\frac{1}{n}}$ is differentiable. Although it is beyond the scope of this text, it may be shown that this assumption is justified for $x > 0$ if n is even, and for all $x \neq 0$ if n is odd.

Now if $m \neq 0$ is also an integer, we have, using the chain rule as above,

$$\begin{aligned}\frac{d}{dx}x^{\frac{m}{n}} &= \frac{d}{dx}\left(x^{\frac{1}{n}}\right)^m \\ &= m\left(x^{\frac{1}{n}}\right)^{m-1}\frac{1}{n}x^{\frac{1}{n}-1} \\ &= \frac{m}{n}x^{\frac{m-1}{n}+\frac{1}{n}-1} \\ &= \frac{m}{n}x^{\frac{m}{n}-1}.\end{aligned} \tag{1.7.48}$$

Hence we now see that the power rule holds for any non-zero rational exponent.

Theorem 1.7.10. If $r \neq 0$ is any rational number, then

$$\frac{d}{dx}x^r = rx^{r-1}. \tag{1.7.49}$$

Example 1.7.18. With $r = \frac{1}{2}$ in the previous theorem, we have

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}},$$

in agreement with our earlier direct computation.

Example 1.7.19. If $y = x^{\frac{2}{3}}$, then

$$\frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3x^{\frac{1}{3}}}.$$

Note that $\frac{dy}{dx}$ is not defined at $x = 0$, in agreement with our earlier result showing that y is not differentiable at 0.

Exercise 1.7.16. Find the derivative of $f(x) = 5x^{\frac{4}{5}}$.

We may now generalize 1.7.44 as follows: If u is a differentiable function of x and $r \neq 0$ is a rational number, then

$$\frac{d}{dx}u^r = ru^{r-1}\frac{du}{dx}. \tag{1.7.50}$$

Example 1.7.20. If $f(x) = \sqrt{x^2+1}$, then

$$f'(x) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2+1}}.$$

Example 1.7.21. If

$$g(t) = \frac{1}{t^4+5},$$

then

$$g'(t) = (-1)(t^4+5)^{-2}(4t^3) = -\frac{4t^3}{(t^4+5)^2}.$$

Exercise 1.7.17. Find the derivative of

$$y = \frac{4}{\sqrt{x^2+4}}.$$

Exercise 1.7.18. Find the derivative of $f(x) = (x^2+3x-5)^{10}(3x^4-6x+4)^{12}$.

1.7.7 Trigonometric functions

If $y = \sin(x)$ and $w = \cos(x)$, then, for any infinitesimal dx,

$$\begin{aligned} dy &= \sin(x+dx) - \sin(x) \\ &= \sin(x)\cos(dx) + \sin(dx)\cos(x) - \sin(x) \\ &= \sin(x)(\cos(dx)-1) + \cos(x)\sin(dx) \end{aligned} \tag{1.7.51}$$

and

$$\begin{aligned} dw &= \cos(x+dx) - \cos(x) \\ &= \cos(x)\cos(dx) - \sin(x)\sin(dx) - \cos(x) \\ &= \cos(x)(\cos(dx)-1) - \sin(x)\sin(dx). \end{aligned} \tag{1.7.52}$$

Hence, if $dx \neq 0$,

$$\frac{dy}{dx} = \cos(x)\frac{\sin(dx)}{dx} - \sin(x)\frac{1-\cos(dx)}{dx} \tag{1.7.53}$$

and

$$\frac{dw}{dx} = -\sin(x)\frac{\sin(dx)}{dx} + \cos(x)\frac{1-\cos(dx)}{dx}. \tag{1.7.54}$$

Now from (1.5.13) we know that

$$0 \leq 1 - \cos(dx) \leq \frac{(dx)^2}{2}, \tag{1.7.55}$$

and so

$$0 \leq \frac{1-\cos(x)}{dx} \leq \frac{dx}{2}. \tag{1.7.56}$$

Hence

$$\frac{1-\cos(dx)}{dx} \tag{1.7.57}$$

is an infinitesimal. Moreover, from (1.5.36), we know that

$$\frac{\sin(dx)}{dx} \simeq 1. \tag{1.7.58}$$

Hence

$$\frac{dy}{dx} \simeq \cos(x)(1) - \sin(x)(0) = \cos(x) \tag{1.7.59}$$

and

$$\frac{dw}{dx} \simeq -\sin(x)(1) + \cos(x)(0) = -\sin(x) \tag{1.7.60}$$

That is, we have shown the following.

Theorem 1.7.11. For all real values x,

$$\frac{d}{dx}\sin(x) = \cos(x) \tag{1.7.61}$$

and

$$\frac{d}{dx}\cos(x) = -\sin(x). \tag{1.7.62}$$

Example 1.7.22. Using the chain rule,

$$\frac{d}{dx}\cos(4x) = -\sin(4x)\frac{d}{dt}(4x) = -4\sin(4x).$$

Example 1.7.23. If $f(t) = \sin^2(t)$, then, again using the chain rule,

$$f'(t) = 2\sin(t)\frac{d}{dt}\sin(t) = 2\sin(t)\cos(t).$$

Example 1.7.24. If $g(x) = \cos(x^2)$, then

$$g'(x) = -\sin(x^2)(2x) = -2x\cos(x^2).$$

Example 1.7.25. If $f(x) = \sin^3(4x)$, then, using the chain rule twice,

$$f'(x) = 3\sin^2(4x)\frac{d}{dx}\sin(4x) = 12\sin^2(4x)\cos(4x).$$

Exercise 1.7.19. Find the derivatives of

$$y = \cos(3t+6) \text{ and } w = \sin^2(t)\cos^2(4t).$$

Exercise 1.7.20. Verify the following:

(a) $\dfrac{d}{dt}\tan(t) = \sec^2(t)$ (b) $\dfrac{d}{dt}\cot(t) = -\csc^2(t)$

(c) $\dfrac{d}{dt}\sec(t) = \sec(t)\tan(t)$ (d) $\dfrac{d}{dt}\csc(t) = -\csc(t)\cot(t)$

Exercise 1.7.21. Find the derivative of $y = \sec^2(3t)$.

Exercise 1.7.22. Find the derivative of $f(t) = \tan^2(3t)$.

1.8 A geometric interpretation of the derivative

Recall that if $y = f(x)$, then, for any real number Δx,

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{1.8.1}$$

is the average rate of change of y with respect to x over the interval $[x, x + \Delta x]$ (see (1.2.7)). Now if the graph of y is a straight line, that is, if $f(x) = mx + b$ for some real numbers m and b, then (1.8.1) is m, the slope of the line. In fact, a straight line is characterized by the fact that (1.8.1) is the same for any values of x and Δx. Moreover, (1.8.1) remains the same when Δx is infinitesimal; that is, the derivative of y with respect to x is the slope of the line.

For other differentiable functions f, the value of (1.8.1) depends upon both x and Δx. However, for infinitesimal values of Δx, the shadow of (1.8.1), that is, the derivative $\frac{dy}{dx}$, depends on x alone. Hence it is reasonable to think of $\frac{dy}{dx}$ as the *slope of the curve* $y = f(x)$ at a point x. Whereas the slope of a straight line is constant from point to point, for other differentiable functions the value of the slope of the curve will vary from point to point.

If f is differentiable at a point a, we call the line with slope $f'(a)$ passing through $(a, f(a))$ the *tangent line* to the graph of f at $(a, f(a))$. That is, the tangent line to the graph of $y = f(x)$ at $x = a$ is the line with equation

$$y = f'(a)(x - a) + f(a). \tag{1.8.2}$$

Hence a tangent line to the graph of a function f is a line through a point on the graph of f whose slope is equal to the slope of the graph at that point.

Example 1.8.1. If $f(x) = x^5 - 6x^2 + 5$, then

$$f'(x) = 5x^4 - 12x.$$

In particular, $f'\left(-\frac{1}{2}\right) = \frac{101}{16}$, and so the equation of the line tangent to the graph of f at $x = -\frac{1}{2}$ is

$$y = \frac{101}{6}\left(x + \frac{1}{2}\right) + \frac{111}{32}.$$

See Figure 1.8.1

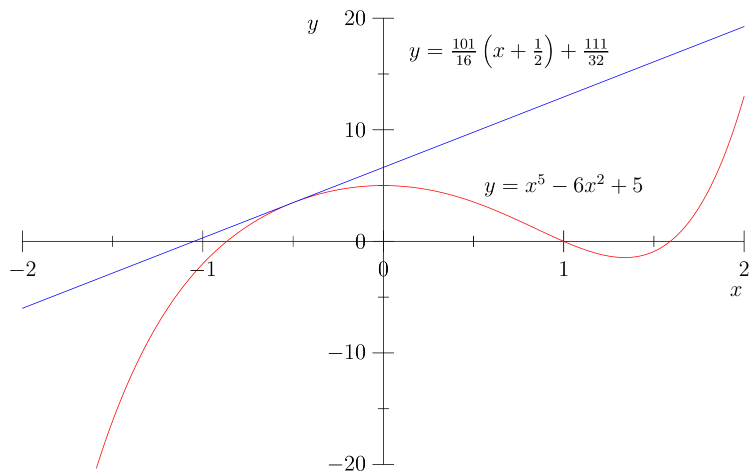

Figure 1.8.1: A tangent line to the graph of $f(x) = x^5 - 6x^2 + 5$

Exercise 1.8.1. Find an equation for the line tangent to the graph of

$$f(x) = 3x^4 - 6x + 3$$

at $x = 2$.

Exercise 1.8.2. Find an equation for the line tangent to the graph of

$$y = 3\sin^2(x)$$

at $x = \frac{\pi}{4}$.

1.9 Increasing, decreasing, and local extrema

Recall that the slope of a line is positive if, and only if, the line rises from left to right. That is, if $m > 0$, $f(x) = mx + b$, and $u < v$, then

$$\begin{aligned} f(v) &= mv + b \\ &= mv - mu + mu + b \\ &= m(v - u) + mu + b \\ &> mu + b \\ &= f(u). \end{aligned} \tag{1.9.1}$$

We should expect that an analogous statement holds for differentiable functions: if f is differentiable and $f'(x) > 0$ for all x in an interval (a, b), then $f(v) > f(u)$ for any $v > u$ in (a, b). This is in fact the case, although the inference requires establishing a direct connection between slope at a point and the average slope over an interval, or, in terms of rates of change, between the instantaneous rate of change at a point and the average rate of change over an interval. The *mean-value theorem* makes this connection.

1.9.1 The mean-value theorem

Recall that the extreme value property tells us that a continuous function on a closed interval must attain both a minimum and a maximum value. Suppose f is continuous on $[a, b]$, differentiable on (a, b), and f attains a maximum value at c with $a < c < b$. In particular, for any infinitesimal dx, $f(c) \geq f(c + dx)$, and so, equivalently, $f(c + dx) - f(c) \leq 0$. It follows that if $dx > 0$,

$$\frac{f(c + dx) - f(c)}{dx} \leq 0, \tag{1.9.2}$$

and if $dx < 0$,

$$\frac{f(c + dx) - f(c)}{dx} \geq 0. \tag{1.9.3}$$

Since both of these values must be infinitesimally close to the same real number, it must be the case that

$$\frac{f(c + dx) - f(c)}{dx} \simeq 0. \tag{1.9.4}$$

That is, we must have $f'(c) = 0$. A similar result holds if f has a minimum at c, and so we have the following basic result.

Theorem 1.9.1. If f is differentiable on (a, b) and attains a maximum, or a minimum, value at c, then $f'(c) = 0$.

Now suppose f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. If f is a constant function, then $f'(c) = 0$ for all c in (a, b). If f is not constant, then there is a point c in (a, b) at which f attains either a maximum or a minimum value, and so $f'(c) = 0$. In either case, we have the following result, known as *Rolle's theorem*.

Theorem 1.9.2. If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there is a real number c in (a, b) for which $f'(c) = 0$.

More generally, suppose f is continuous on $[a, b]$ and differentiable on (a, b). Let

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a). \tag{1.9.5}$$

Note that $g(x)$ is the difference between $f(x)$ and the corresponding y value on the line passing through $(a, f(a))$ and $(b, f(b))$. Moroever, g is continuous on $[a, b]$, differentiable on (a, b), and $g(a) = 0 = g(b)$. Hence Rolle's theorem applies to g, so there must exist a point c in (a, b) for which $g'(c) = 0$. Now

$$g'(c) = f'(x) - \frac{f(b) - f(a)}{b - a}, \tag{1.9.6}$$

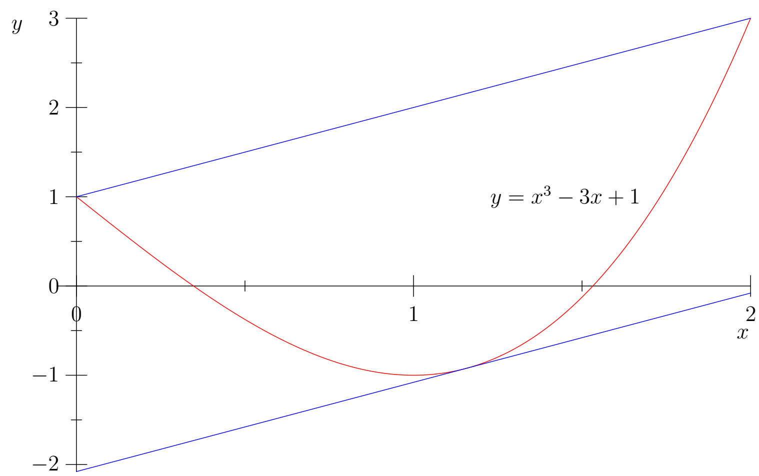

Figure 1.9.1: Graph of $f(x) = x^3 - 3x + 1$ with its tangent line at $x = \sqrt{\frac{4}{3}}$

so we must have

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}. \tag{1.9.7}$$

That is,

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \tag{1.9.8}$$

which is our desired connection between instantaneous and average rates of change, known as the *mean-value theorem.*

Theorem 1.9.3. If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a real number c in (a, b) for which

$$f'(c) = \frac{f(b) - f(a))}{b - a}. \tag{1.9.9}$$

Example 1.9.1. Consider the function $f(x) = x^3 - 3x + 1$ on the interval $[0, 2]$. By the mean-value theorem, there must exist at least one point c in $[0, 2]$ for which

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{3 - 1}{2} = 1.$$

Now $f'(x) = 3x^2 - 3$, so $f'(c) = 1$ implies $3c^2 - 3 = 1$. Hence $c = \sqrt{\frac{4}{3}}$. Note that this implies that the tangent line to the graph of f at $x = \sqrt{\frac{4}{3}}$ is parallel to the line through the endpoints of the graph of f, that is, the points $(0, 1)$ and $(2, 3)$. See Figure 1.9.1.

1.9.2 Increasing and decreasing functions

The preceding discussion leads us to the following definition and theorem.

Definition 1.9.1. We say a function f is *increasing* on an interval I if, whenever $a < b$ are points in I, $f(a) < f(b)$. Similarly, we say f is *decreasing* on I if, whenever $a < b$ are points in I, $f(a) > f(b)$.

Now suppose f is a defined on an interval I and $f'(x) > 0$ for every x in I which is not an endpoint of I. Then given any a and b in I, by the mean-value theorem there exists a point c between a and b for which

$$\frac{f(b) - f(a)}{b - a} = f'(c) > 0. \tag{1.9.10}$$

Since $b - a > 0$, this implies that $f(b) > f(a)$. Hence f is increasing on I. A similar argument shows that f is decreasing on I if $f'(x) < 0$ for every x in I which is not an endpoint of I.

Theorem 1.9.4. Suppose f is defined on an interval I. If $f'(x) > 0$ for every x in I which is not an endpoint of I, then f is increasing on I. If $f'(x) < 0$ for every x in I which is not an endpoint of I, then f is decreasing on I.

Example 1.9.2. Let $f(x) = 2x^3 - 3x^2 - 12x + 1$. Then

$$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1).$$

Hence $f'(x) = 0$ when $x = -1$ and when $x = 2$. Now $x - 2 < 0$ for $x < 2$ and $x - 2 > 0$ for $x > 2$, while $x + 1 < 0$ for $x < -1$ and $x + 1 > 0$ when $x > -1$. Thus $f'(x) > 0$ when $x < -1$ and when $x > 2$, and $f'(x) < 0$ when $-1 < x < 2$. It follows that f is increasing on the intervals $(-\infty, -1)$ and $(2, \infty)$, and decreasing on the interval $(-1, 2)$.

Note that the theorem requires only that we know the sign of f' at points inside a given interval, not at the endpoints. Hence it actually allows us to make the slightly stronger statement that f is increasing on the intervals $(-\infty, -1]$ and $[2, \infty)$, and decreasing on the interval $[-1, 2]$.

Since f is increasing on $(-\infty, -1]$ and decreasing on $[-1, 2]$, the point $(-1, 8)$ must be a high point on the graph of f, although not necessarily the highest point on the graph. We say that f has a *local maximum* of 8 at $x = -1$. Similarly, f is decreasing on $[-1, 2]$ and increasing on $[2, \infty)$, and so the point $(2, -19)$ must be a low point on the graph of f, although, again, not necessarily the lowest point on the graph. We say that f has a *local minimum* of -19 at $x = 2$. From this information, we can begin to see why the graph of f looks as it does in Figure 1.9.2.

Definition 1.9.2. We say f has a *local maximum* at a point c if there exists an interval (a, b) containing c for which $f(c) \geq f(x)$ for all x in (a, b). Similarly, we say f has a *local minimum* at a point c if there exists an interval (a, b) containing c for which $f(c) \leq f(x)$ for all x in (a, b). We say f has a *local extremum* at c if f has either a local maximum or a local minimum at c.

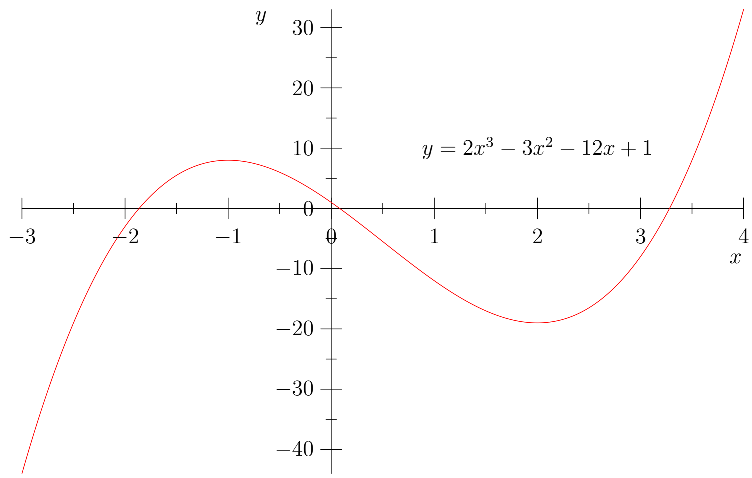

Figure 1.9.2: Graph of $f(x) = 2x^3 - 3x^2 - 12x + 1$

We may now rephrase Theorem 1.9.1 as follows.

Theorem 1.9.5. If f is differentiable at c and has a local extremum at c, then $f'(c) = 0$.

As illustrated in the preceding example, we may identify local minimums of a function f by locating those points at which f changes from decreasing to increasing, and local maximums by locating those points at which f changes from increasing to decreasing.

Example 1.9.3. Let $f(x) = x + 2\sin(x)$. Then $f'(x) = 1 + 2\cos(x)$, and so $f'(x) < 0$ when, and only when,

$$\cos(x) < -\frac{1}{2}.$$

For $0 \leq x \leq 2\pi$, this occurs when, and only when,

$$\frac{2\pi}{3} < x < \frac{4\pi}{3}.$$

Since the cosine function has period 2π, if follows that $f'(x) < 0$ when, and only when, x is in an interval of the form

$$\left(\frac{2\pi}{3} + 2\pi n, \frac{4\pi}{3} + 2\pi n\right)$$

for $n = 0, \pm 1, \pm 2, \ldots$. Hence f is decreasing on these intervals and increasing on intervals of the form

$$\left(-\frac{2\pi}{3} + 2\pi n, \frac{2\pi}{3} + 2\pi n\right),$$

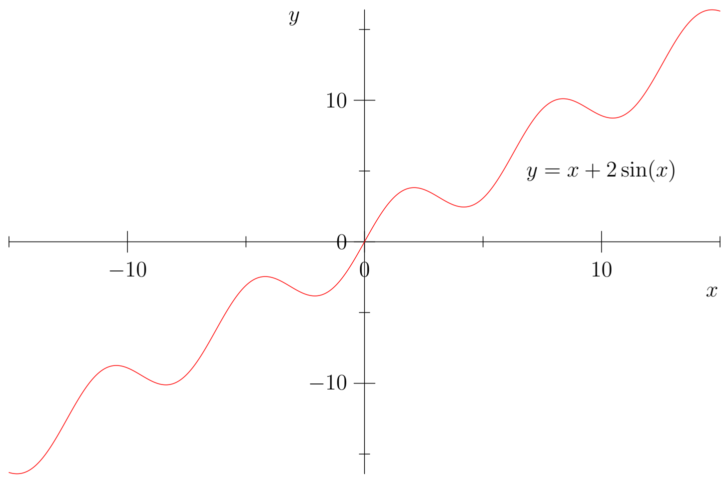

Figure 1.9.3: Graph of $f(x) = x + 2\sin(x)$

$n = 0, \pm 1, \pm 2, \ldots$. It now follows that f has a local maximum at every point of the form

$$x = \frac{2\pi}{3} + 2\pi n$$

and a local minimum at every point of the form

$$x = \frac{4\pi}{3} + 2\pi n.$$

From this information, we can begin to see why the graph of f looks as it does in Figure 1.9.3.

Exercise 1.9.1. Find the intervals where $f(x) = x^3 - 6x$ is increasing and the intervals where f is decreasing. Use this information to identify any local maximums or local minimums of f.

Exercise 1.9.2. Find the intervals where $f(x) = 5x^3 - 3x^5$ is increasing and the intervals where f is decreasing. Use this information to identify any local maximums or local minimums of f.

Exercise 1.9.3. Find the intervals where $f(x) = x + \sin(x)$ is increasing and the intervals where f is decreasing. Use this information to identify any local maximums or local minimums f.

1.10 Optimization

Optimization problems, that is, problems in which we seek to find the greatest or smallest value of some quantity, are common in the applications of mathematics.

Because of the extreme-value property, there is a straightforward algorithm for solving optimization problems involving continuous functions on closed and bounded intervals. Hence we will treat this case first before considering functions on other intervals.

Recall that if $f(c)$ is the maximum, or minimum, value of f on some interval I and f is differentiable at c, then $f'(c) = 0$. Consequently, points at which the derivative vanishes will play an important role in our work on optimization.

Definition 1.10.1. We call a real number c where $f'(c) = 0$ a *stationary point* of f.

1.10.1 Optimization on a closed interval

Suppose f is a continuous function on a closed and bounded interval $[a, b]$. By the extreme-value property, f attains a maximum, as well as a minimum value, on $[a, b]$. In particular, there is a real number c in $[a, b]$ such that $f(c) \geq f(x)$ for all x in $[a, b]$. If c is in (a, b) and f is differentiable at c, then we must have $f'(c) = 0$. The only other possibilities are that f is not differentiable at c, $c = a$, or $c = b$. Similar comments hold for points at which a minimum value occurs.

Definition 1.10.2. We call a real number c a *singular point* of a function f if f is defined on an open interval containing c, but is not differentiable at c.

Theorem 1.10.1. If f is a continuous function on a closed and bounded interval $[a, b]$, then the maximum and minimum values of f occur at either (1) stationary points in the open interval (a, b), (2) singular points in the open interval (a, b), or (3) the endpoints of $[a, b]$.

Hence we have the following procedure for optimizing a continuous function f on an interval $[a, b]$:

(1) Find all stationary and singular points of f in the open interval (a, b).

(2) Evaluate f at all stationary and singular points of (a, b), and at the endpoints a and b.

(3) The maximum value of f is the largest value found in step (2) and the minimum value of f is the smallest value found in step (2).

Example 1.10.1. Consider the function $g(t) = t - 2\cos(t)$ defined on the interval $[0, 2\pi]$. Then

$$g'(t) = 1 + 2\sin(t),$$

and so $g'(t) = 0$ when

$$\sin(t) = -\frac{1}{2}.$$

For t in the open interval $(0, 2\pi)$, this means that either

$$t = \frac{7\pi}{6}$$

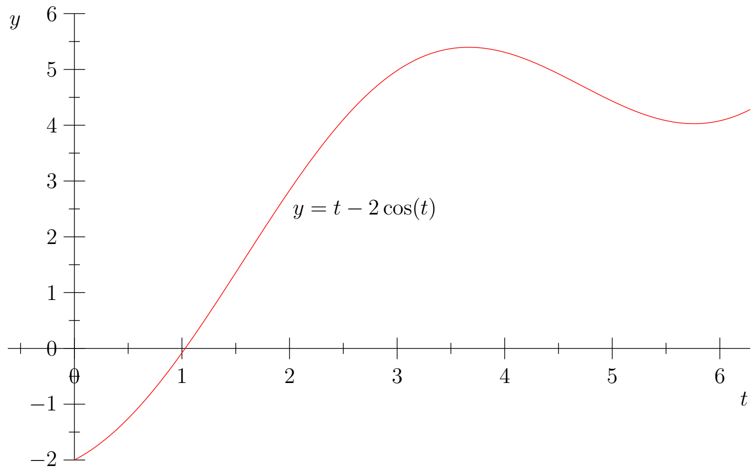

Figure 1.10.1: Graph of $g(t) = t - 2\cos(t)$ on the interval $[0, 2\pi]$

or

$$t = \frac{11\pi}{6}.$$

That is, the stationary points of g in $(0, 2\pi)$ are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. Note that g is differentiable at all points in $(0, 2\pi)$, and so there are no singular points of g in $(0, 2\pi)$. Hence to identify the extreme values of g we need evaluate only

$$g(0) = -2,$$
$$g\left(\frac{7\pi}{6}\right) = \frac{7\pi}{6} + \sqrt{3} \approx 5.39724,$$
$$g\left(\frac{11\pi}{6}\right) = \frac{11\pi}{6} - \sqrt{3} \approx 4.02753,$$

and

$$g(2\pi) = 2\pi - 2 \approx 4.28319.$$

Thus g has a maximum value of 5.39724 at $t = \frac{7\pi}{6}$ and a minimum value of -2 at $t = 0$. See Figure 1.10.1 for the graph of g on $[0, 2\pi]$.

Exercise 1.10.1. Find the maximum and minimum values of

$$f(x) = x^2 + \frac{16}{x}$$

on the interval $[1, 4]$.

Exercise 1.10.2. Find the maximum and minimum values of $g(t) = t - \sin(2t)$ on the interval $[0, \pi]$.

Example 1.10.2. Suppose we inscribe a rectangle R inside the ellipse E with equation

$$4x^2 + y^2 = 16,$$

as shown in Figure 1.10.2. If we let (x, y) be the coordinates of the upper right-hand corner of R, then the area of R is

$$A = (2x)(2y) = 4xy.$$

Since (x, y) is a point on the upper half of the ellipse, we have

$$y = \sqrt{16 - 4x^2} = 2\sqrt{4 - x^2},$$

and so

$$A = 8x\sqrt{4 - x^2}.$$

Now suppose we wish to find the dimensions of R which maximize its area. That is, we want to find the maximum value of A on the interval $[0, 2]$. Now

$$\frac{dA}{dx} = 8x \cdot \frac{-2x}{2\sqrt{4 - x^2}} + 8\sqrt{4 - x^2} = \frac{-8x^2 + 8(4 - x^2)}{\sqrt{4 - x^2}} = \frac{32 - 16x^2}{\sqrt{4 - x^2}}.$$

Hence $\frac{dA}{dx} = 0$, for x in $(0, 2)$, when $32 - 16x^2 = 0$, that is, when $x = \sqrt{2}$. Thus the maximum value of A must occur at $x = 0$, $x = \sqrt{2}$, or $x = 2$. Evaluating, we have

$$A|_{x=0} = 0,$$

$$A|_{x=\sqrt{2}} = 8\sqrt{2}\sqrt{2} = 16,$$

and

$$A|_{x=2} = 0.$$

Hence the rectangle R inscribed in E with the largest area has area 16 when $x = \sqrt{2}$ and $y = 2\sqrt{2}$. That is, R is $2\sqrt{2}$ by $4\sqrt{2}$.

Exercise 1.10.3. Find the dimensions of the rectangle R with largest area which may be inscribed in the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are positive real numbers.

Exercise 1.10.4. A piece of wire, 100 centimeters in length, is cut into two pieces, one of which is used to form a square and the other a circle. Find the lengths of the pieces so that sum of the areas of the square and the circle are (a) maximum and (b) minimum.

Exercise 1.10.5. Show that of all rectangles of a given perimeter P, the square is the one with the largest area.

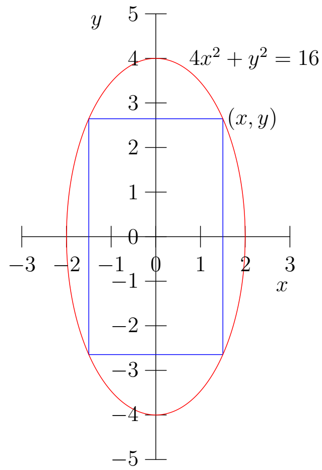

Figure 1.10.2: A rectangle inscribed in the ellipse $4x^2 + y^2 = 16$

1.10.2 Optimization on other intervals

We now consider the case of a continuous function f on an interval I which is either not closed or not bounded. The extreme-value property does not apply in this case, and, as we have seen, we have no guarantee that f has an extreme value on the interval. Hence, in general, this situation requires more careful analysis than that of the previous section.

However, there is one case which arises frequently and which is capable of a simple analysis. Suppose that c is a point in I which is either a stationary or singular point of f, and that f is differentiable at all other points of I. If $f'(x) < 0$ for all x in I with $x < c$ and $f'(x) > 0$ for all x in I with $c < x$, then f is decreasing before c and increasing after c, and so must have a minimum value at c. Similarly, if $f'(x) > 0$ for all x in I with $x < c$ and $f'(x) < 0$ for all x in I with $c < x$, then f is increasing before c and decreasing after c, and so must have a maximum value at c. The next examples will illustrate.

Example 1.10.3. Consider the problem of finding the extreme values of

$$y = 4x^2 + \frac{1}{x}$$

on the interval $(0, \infty)$. Since

$$\frac{dy}{dx} = 8x - \frac{1}{x^2},$$

we see that $\frac{dy}{dx} < 0$ when, and only when,

$$8x < \frac{1}{x^2}.$$

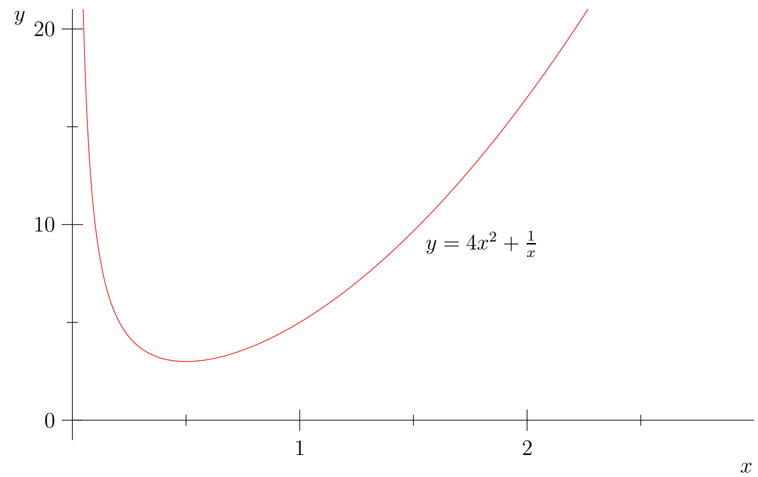

Figure 1.10.3: Graph of $y = 4x^2 + \frac{1}{x}$

This is equivalent to

$$x^3 < \frac{1}{8},$$

so $\frac{dy}{dx} < 0$ on $(0, \infty)$ when, and only when, $0 < x < \frac{1}{2}$. Similarly, we see that $\frac{dy}{dx} > 0$ when, and only when, $x > \frac{1}{2}$. Thus y is a decreasing function of x on the interval $\left(0, \frac{1}{2}\right)$ and an increasing function of x on the interval $\left(\frac{1}{2}, \infty\right)$, and so must have an minimum value at $x = \frac{1}{2}$. Note, however, that y does not have a maximum value: given any $x = c$, if $c < \frac{1}{2}$ we may find a larger value for y by using any $0 < x < c$, and if $c > \frac{1}{2}$ we may find a larger value for y by using any $x > c$. Thus we conclude that y has a minimum value of 3 at $x = \frac{1}{2}$, but does not have a maximum value. See Figure 1.10.3.

Example 1.10.4. Consider the problem of finding the shortest distance from the point $A = (0, 1)$ to the parabola P with equation $y = x^2$. If (x, y) is a point on P (see Figure 1.10.4), then the distance from A to (x, y) is

$$D = \sqrt{(x-0)^2 + (y-1)^2} = \sqrt{x^2 + (x^2-1)^2}.$$

Our problem then is to find the minimum value of D on the interval $(-\infty, \infty)$. However, to make the problem somewhat easier to work with, we note that, since D is always a positive value, finding the minimum value of D is equivalent to finding the minimum value of D^2. So letting

$$z = D^2 = x^2 + (x^2-1)^2 = x^2 + x^4 - 2x^2 + 1 = x^4 - x^2 + 1,$$

our problem becomes that of finding the minimum value of z on $(-\infty, \infty)$. Now

$$\frac{dz}{dx} = 4x^3 - 2x,$$

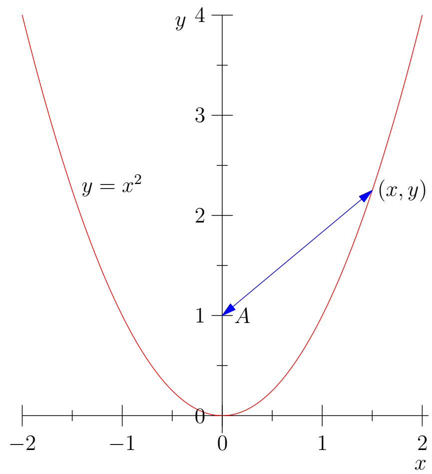

Figure 1.10.4: Distance from $A = (0, 1)$ to a point (x, y) on the graph of $y = x^2$

so $\frac{dz}{dx} = 0$ when, and only when,

$$0 = 4x^3 - 2x = 2x(2x^2 - 1),$$

that is, when, and only when, $x = -\frac{1}{\sqrt{2}}$, $x = 0$, or $x = \frac{1}{\sqrt{2}}$. Now $2x < 0$ when $-\infty < x < 0$ and $2x > 0$ when $0 < x < \infty$, whereas $2x^2 - 1 < 0$ when $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and $2x^2 - 1 > 0$ either when $x < -\frac{1}{\sqrt{2}}$ or when $x > \frac{1}{\sqrt{2}}$. Taking the product of $2x$ and $2x^2 - 1$, we see that $\frac{dz}{dx} < 0$ when $x < -\frac{1}{\sqrt{2}}$ and when $0 < x < \frac{1}{\sqrt{2}}$, and $\frac{dz}{dx} > 0$ when $-\frac{1}{\sqrt{2}} < x < 0$ and when $x > \frac{1}{\sqrt{2}}$. It follows that z is a decreasing function of x on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and on $\left(0, \frac{1}{\sqrt{2}}\right)$, and is an increasing function of x on $\left(-\frac{1}{\sqrt{2}}, 0\right)$ and on $\left(\frac{1}{\sqrt{2}}, \infty\right)$.

It now follows that z has a local minimum of $\frac{3}{4}$ at $x = -\frac{1}{\sqrt{2}}$, a local maximum of 1 at $x = 0$, and another local minimum of $\frac{3}{4}$ at $x = \frac{1}{\sqrt{2}}$. Note that $\frac{3}{4}$ is the minimum value of z both on the the interval $(-\infty, 0)$ and on the interval $(0, \infty)$; since z has a local maximum of 1 at $x = 0$, it follows that $\frac{3}{4}$ is in fact the minimum value of z on $(-\infty, \infty)$. Hence we may conclude that the minimum distance from A to P is $\frac{\sqrt{3}}{2}$, and the points on P closest to A are $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$. Note, however, that z does not have a maximum value, even though it has a local maximum value at $x = 0$. See Figure 1.10.5 for the graph of z.

Exercise 1.10.6. Find the point on the parabola $y = x^2$ which is closest to the point $(3, 0)$.

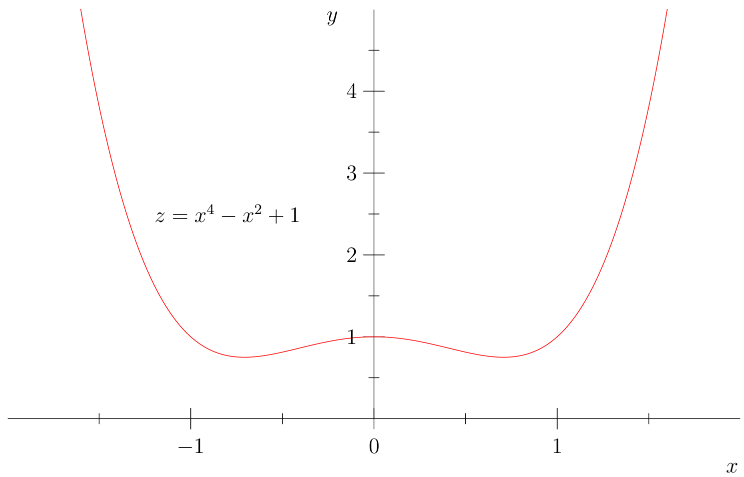

Figure 1.10.5: Graph of $z = x^4 - x^2 + 1$

Exercise 1.10.7. Show that of all rectangles of a given area A, the square is the one with the shortest perimeter.

Exercise 1.10.8. Show that of all right circular cylinders with a fixed volume V, the one with with height and diameter equal has the minimum surface area.

Exercise 1.10.9. Find the points on the ellipse $4x^2 + y^2 = 16$ which are (a) closest to and (b) farthest from the point $(0, 1)$.

1.11 Implicit differentiation and rates of change

Many curves of interest are not the graphs of functions. For example, for constants $a > 0$ and $b > 0$, the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1.11.1}$$

describes an ellipse E which intersects the x-axis at $(-a, 0)$ and $(a, 0)$ and the y-axis at $(-b, 0)$ and $(b, 0)$ (see Figure 1.11.1). The ellipse E is not the graph of a function since, for any $-a < x < a$, both

$$\left(x, -\frac{b}{a}\sqrt{a^2 - x^2}\right) \tag{1.11.2}$$

and

$$\left(x, \frac{b}{a}\sqrt{a^2 - x^2}\right) \tag{1.11.3}$$

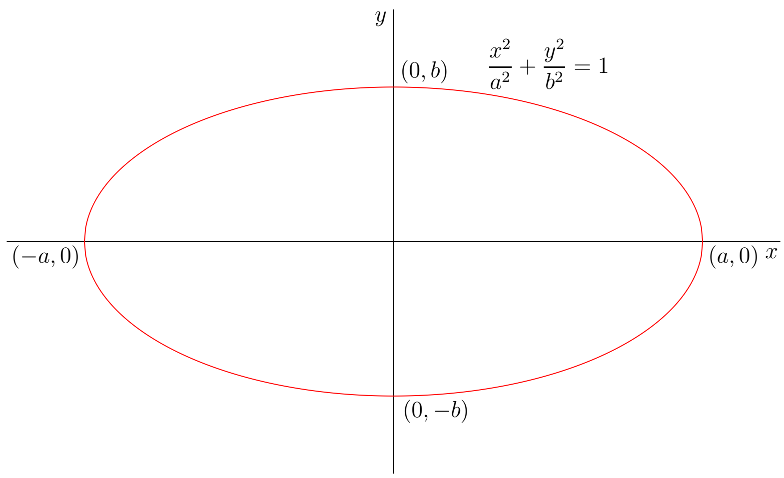

Figure 1.11.1: The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

lie on E. Nevertheless, we expect that E will have a tangent line at every point. Note, however, that the tangent lines at $(-a, 0)$ and $(a, 0)$ are vertical lines, and so do not have a slope. At all other points, we may find the slope of the tangent line by treating y in (1.11.1) as a function of x, differentiating both sides of the equation with respect to x, and solving for $\frac{dy}{dx}$. In general, given a function f of x and y and a constant c, this technique will work to find the slope of the curve defined by an equation $f(x, y) = c$. Note that in applying this technique we are assuming that y is differentiable. This is in fact true for a wide range of relationships defined by f, but the technical details are beyond the scope of this text.

Example 1.11.1. Using $a = 2$ and $b = 1$, (1.11.1) becomes, after multiplying both sides of the equation by 4,

$$x^2 + 4y^2 = 4.$$

Differentiating both sides of this equation by x, and remembering to use the chain rule when differentiating y^2, we obtain

$$2x + 8y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = -\frac{x}{4y},$$

which is defined whenever $y \neq 0$ (corresponding to the points $(-2, 0)$ and $(2, 0)$, at which, as we saw above, the slope of the tangent lines is undefined). For

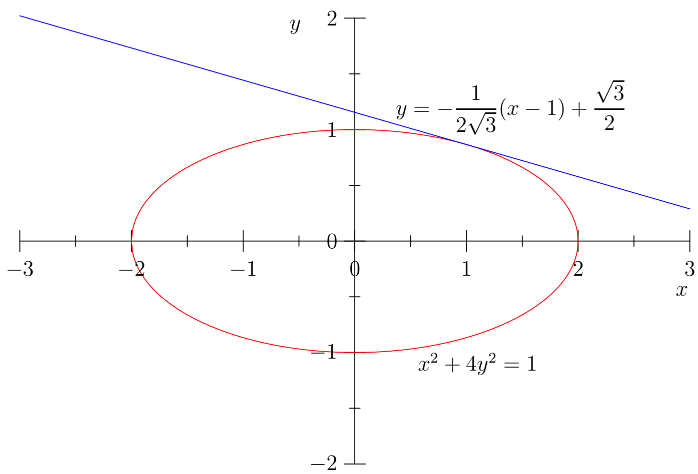

Figure 1.11.2: The ellipse $x^2 + 4y^2 = 4$ with tangent line at $\left(1, \frac{\sqrt{3}}{2}\right)$

example, we have

$$\left.\frac{dy}{dx}\right|_{(x,y)=\left(1,\frac{\sqrt{3}}{2}\right)} = -\frac{1}{2\sqrt{3}},$$

and so the equation of the line tangent to the ellipse at the point $\left(1, \frac{\sqrt{3}}{2}\right)$ is

$$y = -\frac{1}{2\sqrt{3}}(x-1) + \frac{\sqrt{3}}{2}.$$

See Figure 1.11.2.

Example 1.11.2. Consider the hyperbola H with equation

$$x^2 - 4xy + y^2 = 4.$$

Differentiating both sides of the equation, remembering to treat y as a function of x, we have

$$2x - 4x\frac{dy}{dx} - 4y + 2y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$, we see that

$$\frac{dy}{dx} = \frac{4y - 2x}{2y - 4x} = \frac{2y - x}{y - 2x}.$$

For example,

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,0)} = \frac{2}{4} = \frac{1}{2}.$$

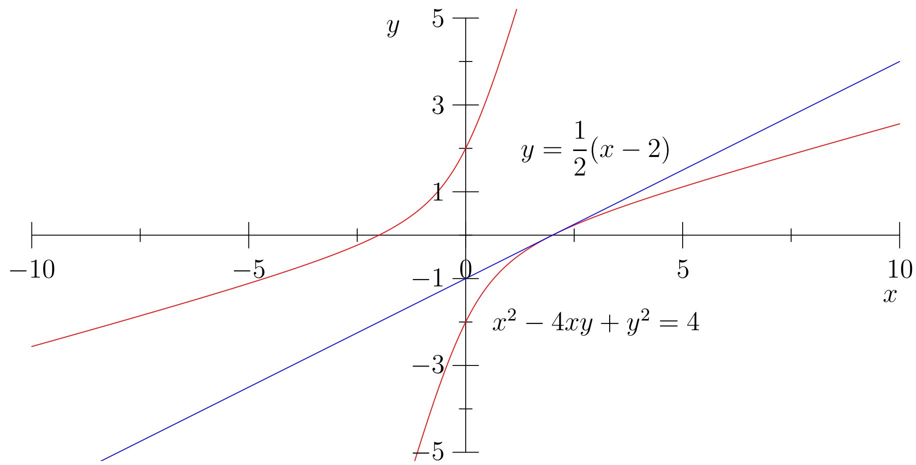

Figure 1.11.3: The hyperbola $x^2 - 4xy + y^2 = 4$ with tangent line at $(2, 0)$

Hence the equation of the line tangent to H at $(2, 0)$ is

$$y = \frac{1}{2}(x-2).$$

See Figure 1.11.3.

Exercise 1.11.1. Find $\frac{dy}{dx}$ if $y^2 + 8xy - x^2 = 10$.

Exercise 1.11.2. Find $\left.\frac{dy}{dx}\right|_{(x,y)=(2,-1)}$ if $x^2y + 3xy - 12y = 2$.

Exercise 1.11.3. Find the equation of the line tangent to the circle with equation $x^2 + y^2 = 25$ at the point $(3, 4)$.

Exercise 1.11.4. Find the equation of the line tangent to the ellipse with equation $x^2 + xy + y^2 = 19$ at the point $(2, 3)$.

The technique described above, known as *implicit differentiation*, is also useful in finding rates of change for variables related by an equation. The next examples illustrate this idea, with the first being similar to examples we saw earlier while discussing the chain rule.

Example 1.11.3. Suppose oil is being poured onto the surface of a calm body of water. As the oil spreads out, it forms a right circular cylinder whose volume is

$$V = \pi r^2 h,$$

where r and h are, respectively, the radius and height of the cylinder. Now suppose the oil is being poured out at a rate of 10 cubic centimeters per second

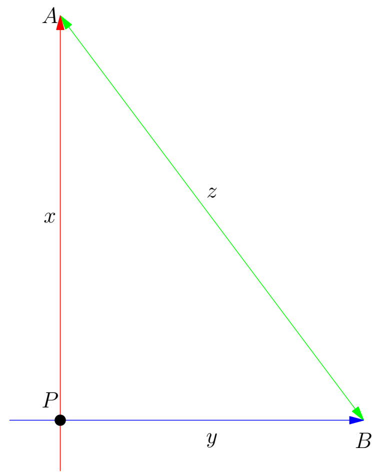

Figure 1.11.4: Ships A and B passing a point P

and that the height remains a constant 0.25 centimeters. Then the volume of the cylinder is increasing at a rate of 10 cubic centimeters per second, so

$$\frac{dV}{dt} = 10 \text{ cm}^3/\text{sec}$$

at any time t. Now with $h = 0.25$,

$$V = 0.25\pi r^2,$$

so

$$\frac{dV}{dt} = \frac{1}{2}\pi r \frac{dr}{dt}.$$

Hence

$$\frac{dr}{dt} = \frac{2}{\pi r}\frac{dV}{dt} = \frac{20}{\pi r} \text{ cm/sec}.$$

For example, if $r = 10$ centimeters at some time $t = t_0$, then

$$\left.\frac{dr}{dt}\right|_{t=t_0} = \frac{20}{10\pi} = \frac{2}{\pi} \approx 0.6366 \text{ cm/sec}.$$

Example 1.11.4. Suppose ship A, headed due north at 20 miles per hour, and ship B, headed due east at 30 miles per hour, both pass through the same point P in the ocean, ship A at noon and ship B two hours later (see Figure 1.11.4). If we let x denote the distance from A to P t hours after noon, y denote the distance from B to P t hours after noon, and z denote the distance from A to B t hours after noon, then, by the Pythagorean theorem,

$$z^2 = x^2 + y^2.$$

Differentiating this equation with respect to t, we find

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt},$$

or

$$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$

For example, at 4 in the afternoon, that is, when $t = 4$, we know that

$$x = (4)(20) = 80 \text{ miles},$$

$$y = (2)(30) = 60 \text{ miles},$$

and

$$z = \sqrt{80^2 + 60^2} = 100 \text{ miles},$$

so

$$100\frac{dz}{dt} = 80\frac{dx}{dt} + 60\frac{dy}{dt} \text{ miles/hour}.$$

Since at any time t,

$$\frac{dx}{dt} = 20 \text{ miles/hour}$$

and

$$\frac{dy}{dt} = 30 \text{ miles/hour},$$

we have

$$\left.\frac{dz}{dt}\right|_{t=4} = \frac{(80)(20) + (60)(30)}{100} = 34 \text{ miles/hour}.$$

Exercise 1.11.5. Suppose the volume of a cube is growing at a rate of 150 cubic centimeters per second. Find the rate at which the length of a side of the cube is growing when each side of the cube is 10 centimeters.

Exercise 1.11.6. A plane flies over a point P on the surface of the earth at a height of 4 miles. Find the rate of change of the distance between P and the plane one minute later if the plane is traveling at 300 miles per hour.

Exercise 1.11.7. Suppose the length of a rectangle is growing at a rate of 2 centimeters per second and its width is growing at a rate of 4 centimeters per second. Find the rate of change of the area of the rectangle when the length is 10 centimeters and the width is 12 centimeters.

1.12 Higher-order derivatives

Given two quantities, y and x, with y a function of x, we know that the derivative $\frac{dy}{dx}$ is the rate of change of y with respect to x. Since $\frac{dy}{dx}$ is then itself a function of x, we may ask for its rate of change with respect to x, which we call the *second-order derivative* of y with respect to x and denote $\frac{d^2y}{dx^2}$.

Example 1.12.1. If $y = 4x^5 - 3x^2 + 4$, then

$$\frac{dy}{dx} = 20x^4 - 6x,$$

and so

$$\frac{d^2y}{dx^2} = 80x^3 - 6.$$

Of course, we could continue to differentiate: the *third derivative* of y with respect to x is

$$\frac{d^3y}{dx^3} = 240x^2,$$

the *fourth derivative* of y with respect to x is

$$\frac{d^4y}{dx^4} = 480x,$$

and so on.

If y is a function of x with $y = f(x)$, then we may also denote the second derivative of y with respect to x by $f''(x)$, the third derivative by $f'''(x)$, and so on. The prime notation becomes cumbersome after awhile, and so we may replace the primes with the corresponding number in parentheses; that is, we may write, for example, $f''''(x)$ as $f^{(4)}(x)$.

Example 1.12.2. If

$$f(x) = \frac{1}{x},$$

then

$$f'(x) = -\frac{1}{x^2},$$

$$f''(x) = \frac{2}{x^3},$$

$$f'''(x) = -\frac{6}{x^4},$$

and

$$f^{(4)}(x) = \frac{24}{x^5}.$$

Exercise 1.12.1. Find the first, second, and third-order derivatives of $y = \sin(2x)$.

Exercise 1.12.2. Find the first, second, and third-order derivatives of $f(x) = \sqrt{4x+1}$

1.12.1 Acceleration

If x is the position, at time t, of an object moving along a straight line, then we know that

$$v = \frac{dx}{dt} \tag{1.12.1}$$

is the velocity of the object at time t. Since acceleration is the rate of change of velocity, it follows that the acceleration of the object is

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}. \tag{1.12.2}$$

Example 1.12.3. Suppose an object, such as a lead ball, is dropped from a height of 100 meters. Ignoring air resistance, the height of the ball above the earth after t seconds is given by

$$x(t) = 100 - 4.9t^2 \text{ meters},$$

as we discussed in Section 1.2. Hence the velocity of the object after t seconds is

$$v(t) = -9.8t \text{ meters/second}$$

and the acceleration of the object is

$$a(t) = -9.8 \text{ meters/second}^2.$$

Thus the acceleration of an object in free-fall near the surface of the earth, ignoring air resistance, is constant. Historically, Galileo started with this observation about acceleration of objects in free-fall and worked in the other direction to discover the formulas for velocity and position.

Exercise 1.12.3. Suppose an object oscillating at the end of a spring has position $x = 10\cos(\pi t)$ (measured in centimeters from the equilibrium position) at time t seconds. Find the acceleration of the object at time $t = 1.25$.

1.12.2 Concavity

The second derivative of a function f tells us the rate at which the slope of the graph of f is changing. Geometrically, this translates into measuring the *concavity* of the graph of the function.

Definition 1.12.1. We say the graph of a function f is *concave upward* on an open interval (a, b) if f' is an increasing function on (a, b). We say the graph of a function f is *concave downward* on an open interval (a, b) if f' is a decreasing function on (a, b).

To determine the concavity of the graph of a function f, we need to determine the intervals on which f' is increasing and the intervals on which f' is decreasing. Hence, from our earlier work, we need identify when the derivative of f' is positive and when it is negative.

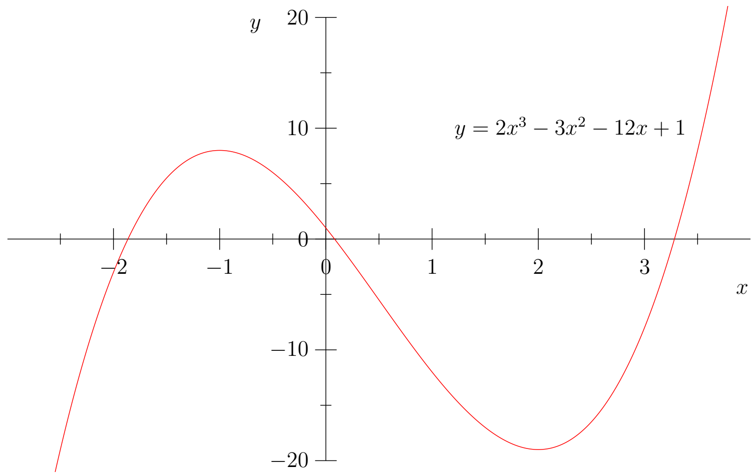

Figure 1.12.1: Graph of $f(x) = 2x^3 - 3x^2 - 12x + 1$

Theorem 1.12.1. If f is twice differentiable on (a, b), then the graph of f is concave upward on (a, b) if $f''(x) > 0$ for all x in (a, b), and concave downward on (a, b) if $f''(x) < 0$ for all x in (a, b).

Example 1.12.4. If $f(x) = 2x^3 - 3x^2 - 12x + 1$, then

$$f'(x) = 6x^2 - 6x - 12$$

and

$$f''(x) = 12x - 6.$$

Hence $f''(x) < 0$ when $x < \frac{1}{2}$ and $f''(x) > 0$ when $x > \frac{1}{2}$, and so the graph of f is concave downward on the interval $\left(-\infty, \frac{1}{2}\right)$ and concave upward on the interval $\left(\frac{1}{2}, \infty\right)$. One may see the distinction between concave downward and concave upward very clearly in the graph of f shown in Figure 1.12.1.

We call a point on the graph of a function f at which the concavity changes, either from upward to downward or from downward to upward, a *point of inflection*. In the previous example, $\left(\frac{1}{2}, -\frac{11}{2}\right)$ is a point of inflection.

Exercise 1.12.4. Find the intervals on which the graph of $f(x) = 5x^3 - 3x^5$ is concave upward and the intervals on which the graph is concave downward. What are the points of inflection?

1.12.3 The second-derivative test

Suppose c is a stationary point of f and $f''(c) > 0$. Then, since f'' is the derivative of f' and $f'(c) = 0$, for any infinitesimal $dx \neq 0$,

$$\frac{f'(c + dx) - f'(c)}{dx} = \frac{f'(c + dx)}{dx} > 0. \tag{1.12.3}$$

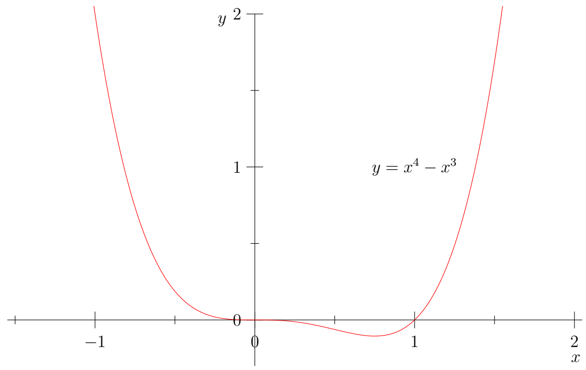

Figure 1.12.2: Graph of $f(x) = x^4 - x^3$

It follows that $f'(c + dx) > 0$ when $dx > 0$ and $f'(c + dx) < 0$ when $dx < 0$. Hence f is decreasing to the left of c and increasing to the right of c, and so f has a local minimum at c. Similarly, if $f''(c) < 0$ at a stationary point c, then f has a local maximum at c. This result is the *second-derivative test*.

Example 1.12.5. If $f(x) = x^4 - x^3$, then

$$f'(x) = 4x^3 - 3x^2 = x^2(4x - 3)$$

and

$$f''(x) = 12x^2 - 6x = 6x(2x - 1).$$

Hence f has stationary points $x = 0$ and $x = \frac{3}{4}$. Since

$$f''(0) = 0$$

and

$$f''\left(\frac{3}{4}\right) = \frac{9}{4} > 0,$$

we see that f has a local minimum at $x = \frac{3}{4}$. Although the second derivative test tells us nothing about the nature of the critical point $x = 0$, we know, since f has a local minimum at $x = \frac{3}{4}$, that f is decreasing on $\left(0, \frac{3}{4}\right)$ and increasing on $\left(\frac{3}{4}, \infty\right)$. Moreover, since $4x - 3 < 0$ for all $x < 0$, it follows that $f'(x) < 0$ for all $x < 0$, and so f is also decreasing on $(-\infty, 0)$. Hence f has neither a local maximum nor a local minimum at $x = 0$. Finally, since $f''(x) < 0$ for $0 < x < \frac{1}{2}$ and $f''(x) > 0$ for all other x, we see that the graph of f is concave downward on the interval $\left(0, \frac{1}{2}\right)$ and concave upward on the intervals $(-\infty, 0)$ and $\left(\frac{1}{2}, \infty\right)$. See Figure 1.12.2.

Exercise 1.12.5. Use the second-derivative test to find all local maximums and minimums of

$$f(x) = x + \frac{1}{x}.$$

Exercise 1.12.6. Find all local maximums and minimums of $g(t) = 5t^7 - 7t^5$.

Chapter 2

Integrals

2.1 Integrals

We now turn our attention to the other side of Zeno's arrow paradox. In the previous chapter we began with the problem of finding the velocity of an object given a function which defined the position of the object at every instant of time. We now suppose that we are given a function which specifies the velocity v of an object, moving along a straight line, at every instant of time t, and we wish to find the position x of the object at time t. There are two approaches to finding x; we will investigate both, leading us to the fundamental theorem of calculus.

First, from our earlier work we know that v is the derivative of x. That is,

$$\frac{dx}{dt} = v. \tag{2.1.1}$$

Hence to find x we need to find a function which has v for its derivative.

Definition 2.1.1. Given a function f defined on an open interval (a, b), we call a function F an *integral* of f if $F'(x) = f(x)$ for all x in (a, b).

Example 2.1.1. If $f(x) = 3x^2$, then $F(x) = x^3$ is an integral of f on $(-\infty, \infty)$ since $F'(x) = 3x^2$ for all x. However, note that F is not the only integral of f: for other examples, both $G(x) = x^3 + 4$ and $H(x) = x^3 + 15$ are integrals of f as well. Indeed, since the derivative of a constant is 0 the function $L(x) = x^3 + c$ is an integral of f for any constant c.

In general, if F is an integral of f, then $G(x) = F(x) + c$ is also an integral of f for any constant c. Are there any other integrals of f? That is, if we start with both F and G being integrals of f, does it follow that $G(x) - F(x) = c$ for some constant c and for all x? To answer this question, first note that if we let $H(x) = G(x) - F(x)$, then

$$H'(x) = G'(x) - F'(x) = f(x) - f(x) = 0 \tag{2.1.2}$$

for all x. Hence H is an integral of the constant function $g(x) = 0$ for all x. So our question becomes: If $H'(x) = 0$ for all x, does it follow that $H(x) = c$ for some constant c and all x? If it does, then

$$c = H(x) = F(x) - G(x), \tag{2.1.3}$$

and indeed F and G differ by only a constant. So suppose we are given $H'(x) = 0$ for all x in an open interval (a, b). Then, for any two points $u < v$ in (a, b), it follows from the mean-value theorem that

$$H(v) - H(u) = H'(d)(u - v) \tag{2.1.4}$$

for some d in (a, b). But then $H'(d) = 0$, so $H(v) - H(u) = 0$, that is, $H(u) = H(v)$. Since this is true for any arbitrary points u and v in (a, b), it follows that H must be constant on (a, b).

Theorem 2.1.1. If $F'(x) = G'(x)$ for all x in (a, b), then there exists a constant c such that $G(x) = F(x) + c$ for all x in (a, b).

In particular, if $F'(x) = 0$ for all x in (a, b), then F is constant on (a, b).

Example 2.1.2. Since

$$\frac{d}{dx}\left(\frac{3}{2}x^2 + 4x\right) = 3x + 4,$$

any integral of $f(x) = 3x + 4$ must be of the form

$$F(x) = \frac{3}{2}x^2 + 4x + c$$

for some constant c.

We denote an integral of a function f by

$$\int f(x)dx. \tag{2.1.5}$$

The motivation for this notation will be more evident once we discuss the fundamental theorem of calculus.

Example 2.1.3. Since

$$\frac{d}{dx}(4x^3 - \sin(x)) = 12x^2 - \cos(x),$$

it follows that

$$\int (12x^2 - \cos(x))dx = 4x^3 - \sin(x) + c,$$

where, as before, c is some constant.

From our rules for differentiation, it follows easily that

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c \tag{2.1.6}$$

for every rational $n \neq -1$, and

$$\int \sin(x)dx = -\cos(x) + c, \tag{2.1.7}$$

$$\int \cos(x)dx = \sin(x) + c, \tag{2.1.8}$$

$$\int \sec^2(x)dx = \tan(x) + c, \tag{2.1.9}$$

$$\int \csc^2(x)dx = -\csc(x) + c, \tag{2.1.10}$$

$$\int \sec(x)\tan(x)dx = \sec(x) + c, \tag{2.1.11}$$

and

$$\int \csc(x)\cot(x)dx = -\csc(x) + c, \tag{2.1.12}$$

where in each case c represents an arbitrary constant. Note that differentiation of the right-hand side of each of the above verifies these statements. Moreover, if follows from our work with derivatives that for any functions f and g and any constant k,

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx, \tag{2.1.13}$$

$$\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx, \tag{2.1.14}$$

and

$$\int kf(x)dx = k\int f(x)dx. \tag{2.1.15}$$

Example 2.1.4.

$$\int \left(5x^3 - 6x + 8\right) dx = \frac{5}{4}x^4 - 3x^2 + 8x + c.$$

Example 2.1.5.

$$\int (\sin(x) - 4\cos(x))dx = -\cos(x) - 4\sin(x) + c.$$

Example 2.1.6. Making an adjustment for the chain rule, we see that

$$\int \sin(5x)dx = -\frac{1}{5}\cos(5x) + c.$$

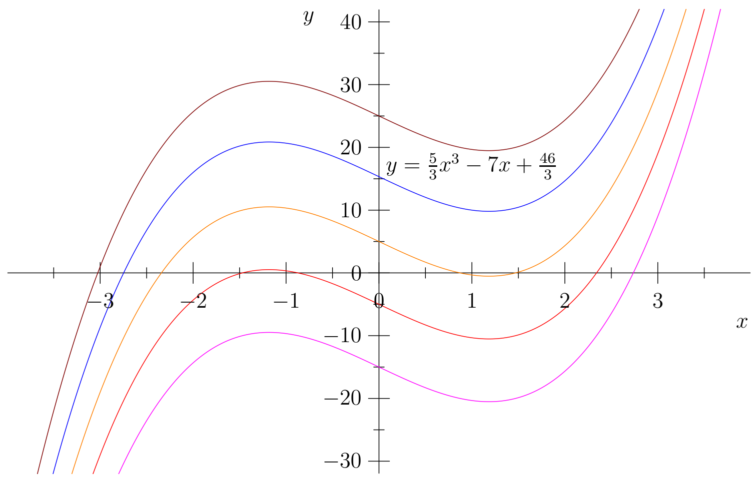

Figure 2.1.1: Parallel curves $y = \frac{5}{3}x^3 - 7x + c$

Example 2.1.7. Suppose we wish to find the integral $F(x)$ of $f(x) = 5x^2 - 7$ for which $F(1) = 10$. Now

$$\int (5x^2 - 7)dx = \frac{5}{3}x^3 - 7x + c,$$

so

$$F(x) = \frac{5}{3}x^3 - 7x + c$$

for some constant c. Now we want

$$10 = F(1) = \frac{5}{3} - 7 + c,$$

so we must have

$$c = 10 + 7 - \frac{5}{3} = \frac{46}{3}.$$

Hence the desired integral is

$$F(x) = \frac{5}{3}x^3 - 7x + \frac{46}{3}.$$

Note that, geometrically, from the family of parallel curves with equations of the form $y = \frac{5}{3}x^3 - 7x + c$, we are finding the one that passes through the point $(1, 10)$. Figure 2.1.1 shows five such curves, with the graph of F in blue.

Exercise 2.1.1. Evaluate each of the following:

(a) $\int (x^2 + 3)dx$

(b) $\int \frac{1}{x^2}\, dx$

(c) $\int (3\sin(x) - 5\sec(x)\tan(x))dx$

(d) $\int 4\sqrt{x}\, dx$

Exercise 2.1.2. Find an integral F of $f(x) = 5x^4 - 4x$ which satisfies $F(2) = 12$.

Returning to our original problem, we can now say that if $v(t)$ is, at time t, the velocity of an object moving along a straight line and $x(t)$ is the object's position at time t, then

$$x(t) = \int v(t)dt. \tag{2.1.16}$$

However, note that (2.1.16) is little more than a restatement of (2.1.1) with new notation.

Example 2.1.8. Suppose the velocity of an object oscillating at the end of a spring is

$$v(t) = -20\sin(5t) \text{ centimeters/second.}$$

If $x(t)$ is the position of the object at time t, then

$$x(t) = -\int 20\sin(5t)dt = 4\cos(5t) + c \text{ centimeters}$$

for some constant c. If in addition we know that the object was initially 4 centimeters from the origin, that is, that $x(0) = 4$, then we would have

$$4 = x(0) = 4 + c.$$

Hence we would have $c = 0$, and so

$$x(t) = 4\cos(5t) \text{ centimeters}$$

completely specifies the position of the object at time t.

Exercise 2.1.3. Suppose the velocity of an object at time t is $v(t) = 10\sin(t)$ centimeters per second. Find $x(t)$, the position of the object at time t, if $x(0) = 10$ centimeters.

2.1.1 The case of constant acceleration

Galileo was the first to notice that, ignoring the effects of air resistance, objects in free fall near the surface of the earth fall with constant acceleration. Suppose that $x(t)$, $v(t)$, and $a(t)$ specify, at time t, the position, velocity, and acceleration of an object moving along a straight line, and, moreover, suppose

$$a(t) = g \tag{2.1.17}$$

for some constant g and all values of t. Since acceleration is the derivative of velocity, it follows that

$$v(t) = \int a(t)dt = \int g dt = gt + c \tag{2.1.18}$$

for some constant c. Now if we let $v_0 = v(0)$, the velocity of the object at time $t = 0$, then

$$v_0 = v(0) = c. \tag{2.1.19}$$

Hence

$$v(t) = gt + v_0. \tag{2.1.20}$$

Next we see that

$$x(t) = \int v(t)dt = \int (gt + v_0)dt = \frac{1}{2}gt^2 + v_0 t + c \tag{2.1.21}$$

for some constant c. If we let $x_0 = x(0)$, the position of the object at time $t = 0$, then

$$x_0 = x(0) = c. \tag{2.1.22}$$

Hence we have

$$x(t) = \frac{1}{2}gt^2 + v_0 t + x_0. \tag{2.1.23}$$

In the important case of an object in free fall near the surface of the earth, g is the constant acceleration due to gravity. When working in units of meters and seconds, and taking up as the positive direction, we have $g = -9.8$ meters per second per second, and when working with units of feet and seconds $g = -32$ feet per second per second. Hence, in the former case, (2.1.23) becomes

$$x(t) = -4.9t^2 + v_0 t + x_0 \tag{2.1.24}$$

and, in the latter case, (2.1.23) becomes

$$x(t) = -16t^2 + v_0 t + x_0. \tag{2.1.25}$$

Example 2.1.9. Suppose an object is thrown upward from atop a 10 meter tall tower with an initial velocity of 20 meters per second. Then, using (2.1.24), the position of the object after t seconds is

$$x(t) = -4.9t^2 + 20t + 10 \text{ meters.}$$

Hence, for example, since the object will reach its maximum height when its velocity is 0, we see that the object reaches its maximum height when

$$-9.8t + 20 = 0,$$

that is, when

$$t = \frac{20}{9.8} \approx 2.04 \text{ seconds.}$$

Thus the object will reach a maximum height of

$$x(2.04) \approx -4.9(2.04)^2 + 20(2.04) + 10 \approx 30.41 \text{ meters.}$$

Exercise 2.1.4. For an object in free-fall near the surface of Mars, $g = -3.69$ meters per second per second. Find the maximum height reached by an object thrown vertically into the air from atop a 10 meter tall tower on Mars with an initial velocity of 20 meters per second.

2.2 Definite integrals

We will now consider another approach to solving the problem of finding the position function given a velocity function. As above, suppose $v(t)$ specifies, at time t, the velocity of an object moving along a straight line, starting at time $t = a$ and ending at time $t = b$. Let $x(t)$ be the position of the object at time t and let $x_0 = x(a)$ be the initial position of the object.

Recall that if an object travels at constant velocity r for a time T, then its change in position is simply rT (to the right if $r > 0$ and to the left if $r < 0$). It follows that if $v(t)$ were constant, say $v(t) = r$ for some fixed real number r and all t in $[a, b]$, then

$$x(t) = x(a) + r(t - a) = x_0 + r(t - a) \tag{2.2.1}$$

since the object will have been traveling, after starting at x_0, at a velocity r for a time period of length $t - a$.

If $v(t)$ is not constant, but doesn't vary by much, then (2.2.1) will give a good approximation to $x(t)$. In general, v may change significantly over the interval, but, as long as v is a continuous function, we can subdivide $[a, b]$ into small intervals for which $v(t)$ does not change by much over any given subinterval. That is, if we choose points t_0, t_1, t_2, …, t_n such that

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b, \tag{2.2.2}$$

let $\Delta t_i = t_i - t_{i-1}$ for $i = 1, 2, 3, \ldots, n$, and choose real numbers t_1^*, t_2^*, t_3^*, …, t_n^* so that $t_{i-1} \leq t_i^* \leq t_i$, then

$$v(t_i^*)\Delta t_i \tag{2.2.3}$$

will approximate well the change of position of the object from time $t = t_{i-1}$ to time $t = t_i$, provided v is continuous and Δt_i is small. It follows that we may approximate the position of the object at time $t = b$ by adding together all the approximate changes in position over the subintervals. That is,

$$x(b) \approx x(a) + v(t_1^*)\Delta t_1 + v(t_2^*)\Delta t_2 + \cdots + v(t_n^*)\Delta t_n. \tag{2.2.4}$$

Example 2.2.1. In an earlier example, we had $v(t) = -20\sin(5t)$ centimeters per second and $x(0) = 4$ centimeters. To approximate $x(2)$, we will divide $[0, 2]$ into four equal subintervals, each of length 0.5. That is, we will take

$$t_0 = 0.0, t_1 = 0.5, t_2 = 1, t_3 = 1.5, t_4 = 2,$$

and

$$\Delta t_1 = 0.5, \Delta t_2 = 0.5, \Delta t_3 = 0.5, \Delta t_4 = 0.5.$$

Good choices for points to evaluate $v(t)$ are the midpoints of the subintervals. In this case, that means we should take

$$t_1^* = 0.25, t_2^* = 0.75, t_3^* = 1.25, t_4^* = 1.75.$$

Then we have

$$\begin{aligned} x(2) &\approx x(0) + v(0.25)\Delta t_1 + v(0.75)\Delta t_2 + v(1.25)\Delta t_3 + v(1.75)\Delta t_4 \\ &= 4 - 20\sin(1.25)(0.5) - 20\sin(3.75)(0.5) - 20\sin(6.25)(0.5) \\ &\qquad - 20\sin(8.75)(0.5) \\ &\approx -5.6897, \end{aligned}$$

Note that, from our earlier work, we know that the exact answer is

$$x(2) = 4\cos(10) \approx -3.3563.$$

We may improve upon our approximation by using smaller subintervals. For example, if we divide $[0, 2]$ into 10 equal subintervals, each of length 0.2, then we would have

$$\begin{aligned} &t_0 = 0.0, t_1 = 0.2, t_2 = 0.4, t_3 = 0.6, t_4 = 0.8, t_5 = 1.0, \\ &t_6 = 1.2, t_7 = 1.4, t_8 = 1.6, t_9 = 1.8, t_{10} = 2.0 \end{aligned}$$

and

$$\Delta t_1 = \Delta t_2 = \cdots = \Delta t_{10} = 0.2.$$

If we evaluate $v(t)$ at the midpoints again, then we take

$$\begin{aligned} &t_1^* = 0.1, t_2^* = 0.3, t_3^* = 0.5, t_4^* = 0.7, t_5^* = 0.9, \\ &t_6^* = 1.1, t_7^* = 1.3, t_8^* = 1.5, t_9^* = 1.7, t_{10}^* = 1.9. \end{aligned}$$

Hence we have

$$\begin{aligned} x(2) &\approx x(0) + v(0.1)\Delta t_1 + v(0.3)\Delta t_2 + v(0.5)\Delta t_3 + \cdots + v(1.9)\Delta t_{10} \\ &= 4 + 0.2(v(0.1) + v(0.3) + v(0.5) + \cdots + v(1.9)) \\ &= 4 + 0.2(-20\sin(0.5) - 20\sin(1.5) - 20\sin(2.5) - \cdots - 20\sin(9.5)) \\ &\approx -3.6720, \end{aligned}$$

a significant improvement over our first approximation.

Exercise 2.2.1. Suppose the velocity of an object at time t is $v(t) = 10\sin(t)$ centimeters per second. Let $x(t)$ be the position of the object at time t. If $x(0) = 10$ centimeters, use the technique of the previous example to approximate $x(3)$ using (a) $n = 6$ and (b) $n = 12$ subintervals.

One question which arises immediately is why we would want to find approximations to the position function when we already know how to find the position function exactly using an integral. There are two answers. First, it is not always possible to find an integral for a given function, even when an integral does exist. For example, if the velocity function in the previous example were $v(t) = -20\sin(5t^2)$, then it would not be possible to write an expression for the integral of v in terms of the elementary functions of calculus. In this case, the best we could do is look for good approximations for the position function x.

Second, this approach also leads to an exact expression for the position function. If we let N be an infinitely large positive integer and divide $[a, b]$ into an infinite number of equal subintervals of infinitesimal length

$$dt = \frac{b-a}{N}, \tag{2.2.5}$$

then we should expect that

$$x(b) \simeq x(a) + v(t_1^*)dt + v(t_2^*)dt + \cdots + v(t_N^*)dt, \tag{2.2.6}$$

where, similar to the work above, t_i^* is a hyperreal number in the ith subinterval. Rewriting this as

$$x(b) - x(a) \simeq v(t_1^*)dt + v(t_2^*)dt + \cdots + v(t_N^*)dt, \tag{2.2.7}$$

we are saying that the change in position of the object from time $t = a$ to time $t = b$ is equal to an infinite sum of infinitesimal changes. Although Zeno was correct in saying that an infinite sum of zeros is still zero, an infinite sum of infinitesimal values need not be infinitesimal.

The right-hand side of (2.2.7) provides the motivation for the following definition.

Definition 2.2.1. Suppose f is a continuous function on a closed bounded interval $[a, b]$. Given a positive integer N, finite or infinite, we call a set of numbers $\{t_0, t_1, t_2, \cdots, t_n\}$ a *partition* of $[a, b]$ if

$$a = t_0 < t_1 < t_2 < \cdots < t_N = b. \tag{2.2.8}$$

Given such a partition of $[a, b]$, let t_i^* denote a number with $t_{i-1} \leq t_i^* \leq t_i$ and let $\Delta t_i = t_i - t_{i-1}$, where $i = 1, 2, 3, \ldots, N$. We call a sum of the form

$$\sum_{i=1}^{N} f(t_i^*)\Delta t_i = f(t_1^*)\Delta t_1 + f(t_2^*)\Delta t_2 + f(t_3^*)\Delta t_3 + \cdots + f(t_N^*)\Delta t_N \tag{2.2.9}$$

a *Riemann sum*. If N is infinite and the partition forms subintervals of equal length

$$dt = \Delta t_1 = \Delta t_2 = \cdots = \Delta t_n, \tag{2.2.10}$$

then we call the shadow of the Riemann sum the *definite integral* of f from a to b, which we denote

$$\int_a^b f(t)dt. \tag{2.2.11}$$

That is,

$$\int_a^b f(t)dt = \text{sh}\left(\sum_{i=1}^{N} f(t_i^*)dt\right). \tag{2.2.12}$$

Note that in order for (2.2.12) to make sense, we need the Riemann sum on the right-hand side to be finite (so it has a shadow) and to have the same shadow for all choices of infinite integers N and evaluation points t_i^* (so the definite integral has a unique value). These are both true for continuous functions on closed bounded intervals, but the verifications would take us into subtle properties of continuous functions beyond the scope of this text.

In terms of velocity and position functions, if $v(t)$ is the velocity and $x(t)$ the position of an object moving along a straight line from time $t = a$ to time $t = b$, we may now write (combining (2.2.7) with (2.2.12))

$$x(b) - x(a) = \int_a^b v(t)dt. \tag{2.2.13}$$

Since v is the derivative of x, we might ask if (2.2.13) is true for any differentiable function. That is, if f is differentiable on an interval which contains the real numbers a and b, is it always the case that

$$f(b) - f(a) = \int_a^b f'(t)dt? \tag{2.2.14}$$

This is in fact true, but requires more explanation than just the intuitive approach of our example with positions and velocities. We will come back to this important result, which we call the *fundamental theorem of calculus*, after developing some properties of definite integrals.

2.3 Properties of definite integrals

Suppose f is a continuous function on a closed interval $[a, b]$. Let N be a positive infinite integer, $dx = \frac{b-a}{N}$, and, for $i = 1, 2, \cdots, N$, let x_i^* a number in the ith subinterval of $[a, b]$ when it is partitioned into N intervals of equal length dx.

We first note that if $f(x) = 1$ for all x in $[a, b]$, then

$$\sum_{i=1}^{N} f(x_i^*)dx = \sum_{i=1}^{N} dx = b - a \tag{2.3.1}$$

since the sum of the lengths of the subintervals must be the length of the interval. Hence

$$\int_a^b f(x)dx = \int_a^b dx = b - a. \tag{2.3.2}$$

More generally, if $f(x) = k$ for all x in $[a, b]$, where k is a fixed real number, then

$$\sum_{i=1}^{N} f(x_i^*)dx = \sum_{i=1}^{N} kdx = k\sum_{i=1}^{N} dx = k(b - a), \tag{2.3.3}$$

and so

$$\int_a^b f(x)dx = \int_a^b kdx = k(b - a). \tag{2.3.4}$$

That is, the definite integral of a constant is the constant times the length of the interval. In particular, the integral of 1 over an interval is simply the length of the interval.

If f is an arbitrary continuous function and k is a fixed constant, then

$$\sum_{i=1}^{N} kf(x_i^*)dx = k\sum_{i=1}^{N} f(x_i*)dx, \tag{2.3.5}$$

and so

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx. \tag{2.3.6}$$

That is, the definite integral of a constant times f is the constant times the definite integral of f.

If g is also a continuous function on $[a, b]$, then

$$\sum_{i=1}^{N}(f(x_i^*) + g(x_i^*))dx = \sum_{i=1}^{N} f(x_i^*)dx + \sum_{i=1}^{N} g(x_i^*)dx, \tag{2.3.7}$$

and so

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx. \tag{2.3.8}$$

Now suppose c is another real number with $a < c < b$. If the closed interval $[a, c]$ is divisible into M intervals of length dx, where M is a positive infinite integer less than N, then

$$\sum_{i=1}^{N} f(x_i^*)dx = \sum_{i=1}^{M} f(x_i^*)dx + \sum_{i=M+1}^{N} f(x_i^*)dx \tag{2.3.9}$$

implies that

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx. \tag{2.3.10}$$

This is a reflection of our intuition that, for an object moving along a straight line, the change in position from time $t = a$ to time $t = b$ is equal to the change in position from time $t = a$ to time $t = c$ plus the change in position from time $t = c$ to time $t = b$. Although we assumed that $[a, c]$ was divisible into an integer number of subintervals of length dx, the result holds in general.

The final properties which we will consider revolve around a basic inequality. If f and g are both continuous on $[a, b]$ with $f(x) \leq g(x)$ for all x in $[a, b]$, then

$$\sum_{i=1}^{N} f(x_i^*)dx \leq \sum_{i=1}^{N} g(x_i^*)dx, \tag{2.3.11}$$

from which it follows that

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx. \tag{2.3.12}$$

For example, if m and M are constants with $m \leq f(x) \leq M$ for all x in $[a, b]$, then

$$\int_a^b m dx \leq \int_a^b f(x) dx \leq \int_a^b M dx, \tag{2.3.13}$$

and so

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a). \tag{2.3.14}$$

Note, in particular, that if $f(x) \geq 0$ for all x in $[a, b]$, then

$$\int_a^b f(x) dx \geq 0. \tag{2.3.15}$$

Example 2.3.1. From the observation that

$$f(x) = \frac{1}{1+x^2}$$

is increasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$, it is easy to see that

$$\frac{1}{2} \leq \frac{1}{1+x^2} \leq 1$$

for all x in $[-1, 1]$. Hence

$$1 \leq \int_{-1}^1 \frac{1}{1+x^2} dx \leq 2.$$

We will eventually see, in Example 2.6.20, that

$$\int_{-1}^1 \frac{1}{1+x^2} dx = \frac{\pi}{2} \approx 1.5708.$$

Since for any real number a, $-|a| \leq a \leq |a|$ (indeed, either $a = |a|$ or $a = -|a|$), we have

$$-|f(x)| \leq f(x) \leq |f(x)| \tag{2.3.16}$$

for all x in $[a, b]$. Hence

$$-\int_a^b |f(x)| dx \leq \int_a^b f(x) dx \leq \int_a^b |f(x)| dx, \tag{2.3.17}$$

or, equivalently,

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx. \tag{2.3.18}$$

Notice that, since the definite integral is just a generalized version of summation, this result is a generalization of the triangle inequality: Given any real numbers a and b,

$$|a+b| \leq |a| + |b|. \tag{2.3.19}$$

The next theorem summarizes the properties of definite integrals that we have discussed above.

Theorem 2.3.1. Suppose f and g are continuous functions on $[a, b]$, c is any real number with $a < c < b$, and k is a fixed real number. Then

(a) $\int_a^b k dx = k(b-a)$,

(b) $\int_a^b kf(x)dx = k \int_a^b f(x)dx$,

(c) $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$,

(d) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$,

(e) if $f(x) \leq g(x)$ for all x in $[a, b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$,

(f) if $m \leq f(x) \leq M$ for all x in $[a, b]$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$,

(g) $\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$.

Exercise 2.3.1. Show that

$$\frac{1}{2} \leq \int_1^2 \frac{1}{x}\, dx \leq 1.$$

2.4 The fundamental theorem of integrals

The main theorem of this section is key to understanding the importance of definite integrals. In particular, we will invoke it in developing new applications for definite integrals. Moreover, we will use it to verify the fundamental theorem of calculus.

We first need some new notation and terminology. Suppose ϵ is a nonzero infinitesimal. Intuitively, ϵ is infinitely smaller than any nonzero real number. One way to express this is to note that for any nonzero real number r,

$$\frac{\epsilon}{r} \simeq 0, \tag{2.4.1}$$

that is, the ratio of ϵ to r is an infinitesimal. Now we also have

$$\frac{\epsilon^2}{\epsilon} = \epsilon \simeq 0, \tag{2.4.2}$$

that is, the ratio of ϵ^2 to ϵ is an infinitesimal. Intuitively, this means that ϵ^2 is infinitely smaller than ϵ itself. This is related to a fact about real numbers:

For any real number r with $0 < r < 1$, r^2 is smaller than r. For example, if $r = 0.01$, then $r^2 = 0.0001$.

Definition 2.4.1. Given a nonzero hyperreal number ϵ, we say another hyperreal number δ is of an *order* less than ϵ if $\frac{\delta}{\epsilon}$ is an infinitesimal, in which case we write $\delta \sim o(\epsilon)$.

In other words, we have

$$\delta \sim o(\epsilon) \text{ if and only if } \frac{\delta}{\epsilon} \simeq 0. \tag{2.4.3}$$

Example 2.4.1. If α is any infinitesimal, than $\alpha \sim o(1)$ since $\frac{\alpha}{1} = \alpha$ is an infinitesimal.

Example 2.4.2. If ϵ is any nonzero infinitesimal, then $\epsilon^2 \sim o(\epsilon)$ since

$$\frac{\epsilon^2}{\epsilon} = \epsilon \simeq 0.$$

Now suppose N is a positive infinite integer, $\epsilon = \frac{1}{N}$, and $\delta_i \sim o(\epsilon)$ for $i = 1, 2, \ldots, N$. Then, for any positive real number r,

$$\frac{|\delta_i|}{\epsilon} < r, \tag{2.4.4}$$

and so

$$\sum_{i=1}^{N} \frac{|\delta_i|}{\epsilon} < rN. \tag{2.4.5}$$

Multiplying both sides by ϵ, we have

$$\sum_{i=1}^{N} |\delta_i| < rN\epsilon = rN\frac{1}{N} = r. \tag{2.4.6}$$

Since this holds for all positive real numbers r, it follows that $\sum_{i=1}^{N} |\delta_i|$ is an infinitesimal. Now

$$\left|\sum_{i=1}^{N} \delta_i\right| \leq \sum_{i=1}^{N} |\delta_i|, \tag{2.4.7}$$

and so we may conclude that $\sum_{i=1}^{n} \delta_i$ is an infinitesimal. In words, the sum of N infinitesimals of order less than $\frac{1}{N}$ is still an infinitesimal.

Now suppose B is a function that for any real numbers $a < b$ in an open interval I assigns a value $B(a, b)$. Moreover, suppose B has the following two properties:

- for any $a < c < b$ in I, $B(a, b) = B(a, c) + B(c, b)$, and

- for some continuous function h and any nonzero infinitesimal dx,

$$B(x, x + dx) - h(x)dx \sim o(dx) \tag{2.4.8}$$

  for any x in I.

For a positive infinite integer N, let $dx = \frac{b-a}{N}$ and let

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b \tag{2.4.9}$$

be a partition of $[a, b]$ into N equal intervals of length dx. Then

$$\begin{aligned} B(a,b) &= B(x_0,x_1) + B(x_1,x_2) + B(x_2,x_3) + \cdots + B(x_{N-1},x_N) \\ &= \sum_{i=1}^{N} B(x_{i-1}, x_{i-1} + dx) \\ &= \sum_{i=1}^{N} ((B(x_{i-1}, x_{i-1} + dx) - h(x_{i-1})dx) + h(x_{i-1})dx) \\ &= \sum_{i=1}^{N} (B(x_{i-1}, x_{i-1} + dx) - h(x_{i-1})dx) + \sum_{i=1}^{N} h(x_{i-1})dx \\ &\simeq \sum_{i=1}^{N} (B(x_{i-1}, x_{i-1} + dx) - h(x_{i-1})dx) + \int_a^b h(x)dx. \end{aligned} \tag{2.4.10}$$

Since the final sum on the right is the sum of N infinitesimals of order less than $\frac{1}{N}$, it follows that

$$B(a,b) = \int_a^b h(x)dx. \tag{2.4.11}$$

This result is basic to understanding both the computation of definite integrals and their applications. We call it the *fundamental theorem of integrals*.

Theorem 2.4.1. Suppose B is a function that for any real numbers $a < b$ in an open interval I assigns a value $B(a, b)$ and satisfies

- for any $a < c < b$ in I, $B(a, b) = B(a, c) + B(c, b)$, and
- for some continuous function h and any nonzero infinitesimal dx,

$$B(x, x + dx) - h(x)dx \sim o(dx) \tag{2.4.12}$$

  for any x in I.

Then

$$B(a,b) = \int_a^b h(x)dx \tag{2.4.13}$$

for any real numbers a and b in I.

We will look at several applications of definite integrals in the next section. For now, we note how this theorem provides a method for evaluating integrals. Namely, given a function f which is differentiable on an open interval I, define, for every $a < b$ in I,

$$B(a,b) = f(b) - f(a). \tag{2.4.14}$$

Then, for any a, b, and c in I with $a < c < b$,

$$\begin{aligned} B(a,b) &= f(b) - f(a) \\ &= (f(b) - f(c)) + (f(c) - f(a)) \\ &= B(a,c) + B(c,b). \end{aligned} \tag{2.4.15}$$

Moreover, for any infinitesimal dx and any x in I,

$$\frac{B(x, x+dx)}{dx} = \frac{f(x+dx) - f(x)}{dx} \simeq f'(x), \tag{2.4.16}$$

from which it follows that

$$\frac{B(x, x+dx) - f'(x)dx}{dx} \tag{2.4.17}$$

is an infinitesimal. Hence

$$B(x, dx) - f'(x)dx \sim o(dx), \tag{2.4.18}$$

and so it follows from Theorem 2.4.1 that

$$f(b) - f(a) = B(a,b) = \int_a^b f'(x)dx. \tag{2.4.19}$$

This is the *fundamental theorem of calculus*.

Theorem 2.4.2. If f is differentiable on an open interval I, then for every $a < b$ in I,

$$\int_a^b f'(x)dx = f(b) - f(a). \tag{2.4.20}$$

Example 2.4.3. To evaluate

$$\int_0^1 xdx,$$

we first note that $g(x) = x$ is the derivative of $f(x) = \frac{1}{2}x^2$. Hence, by Theorem 2.4.2,

$$\int_0^1 xdx = f(1) - f(0) = \frac{1}{2} - 0 = \frac{1}{2}$$

We will write

$$f(x)\big|_a^b = f(b) - f(a) \tag{2.4.21}$$

to simplify the notation for evaluating an integral using Theorem 2.4.2. With this notation, the previous example becomes

$$\int_0^1 xdx = \frac{1}{2}x^2\bigg|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

Example 2.4.4. Since

$$\int x^2 dx = \frac{1}{3}x^3 + c,$$

we have

$$\int_1^2 x^2 dx = \frac{1}{3}x^3\Big|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

Example 2.4.5. Since

$$\int -20\sin(5x)dx = 4\cos(5x) + c,$$

we have

$$\int_0^{2\pi} -20\sin(5t)dt = 4\cos(5t)|_0^{2\pi} = 4 - 4 = 0.$$

Note that if we consider an object moving along a straight line with velocity $v(t) = -20\sin(5t)$, then this definite integral computes the change in position of the object from time $t = 0$ to time $t = 2\pi$. In this case, the object, although always in motion, is in the same position at time $t = 2\pi$ as it was at time $t = 2\pi$.

Exercise 2.4.1. Evaluate $\displaystyle\int_0^1 x^4 dx$.

Exercise 2.4.2. Evaluate $\displaystyle\int_0^\pi \sin(x)dx$.

Exercise 2.4.3. Suppose the velocity of an object moving along a straight line is $v(t) = 10\sin(t)$ centimeters per second. Find the change in position of the object from time $t = 0$ to time $t = \pi$.

2.5 Applications of definite integrals

In this section we will look at several examples of applications for definite integrals.

2.5.1 Area between curves

Consider two continuous functions f and g on an open interval I with $f(x) \leq g(x)$ for all x in I. For any $a < b$ in I, let $R(a, b)$ be the region in the plane consisting of the points (x, y) for which $a \leq x \leq b$ and $f(x) \leq y \leq g(x)$. That is, $R(a, b)$ is bounded above by the curve $y = g(x)$, below by the curve $y = f(x)$, on the left by the vertical line $x = a$, and on the right by the vertical line $x = b$, as in Figure 2.5.1. Let

$$A(a, b) = \text{ area of } R(a, b). \tag{2.5.1}$$

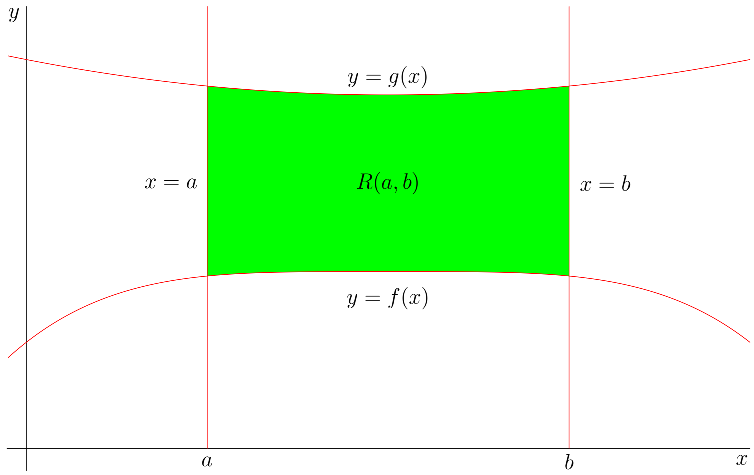

Figure 2.5.1: Region $R(a,b)$ between the graphs of $y = g(x)$ and $y = f(x)$

Clearly, for any $a \le c \le b$,

$$A(a,b) = A(a,c) + A(c,b). \tag{2.5.2}$$

Now for an x in I and a positive infinitesimal dx, let c be the point at which $g(u) - f(u)$ attains its minimum value for $x \le u \le x + dx$ and let d be the point at which $g(u) - f(u)$ attains its maximum value for $x \le u \le x + dx$. Then

$$(g(c) - f(c))dx \le A(x, x + dx) \le (g(d) - f(d))dx. \tag{2.5.3}$$

Moreover, since

$$g(c) - f(c) \le g(x) - f(x) \le g(d) - f(d), \tag{2.5.4}$$

we also have

$$(g(c) - f(c))dx \le (g(x) - f(x))dx \le (g(d) - f(d))dx. \tag{2.5.5}$$

Putting (2.5.3) and (2.5.5) together, we have

$$|A(x, dx) - (g(x) - f(x))dx| \le ((g(d) - f(d)) - (f(c) - g(c)))dx \tag{2.5.6}$$

or

$$\frac{|A(x, dx) - (g(x) - f(x))dx|}{dx} \le (g(d) - f(d)) - (f(c) - g(c)) \tag{2.5.7}$$

Now since $c \simeq x$ and $d \simeq x$,

$$(g(d) - f(d)) - (g(c) - f(c)) = (g(d) - g(c)) + (f(c) - f(d)) \simeq 0. \tag{2.5.8}$$

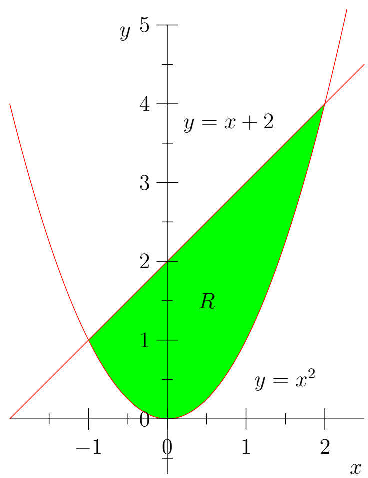

Figure 2.5.2: The region bounded by the curves $y = x + 2$ and $y = x^2$

Hence

$$A(x, dx) - (g(x) - f(x))dx \sim o(dx). \tag{2.5.9}$$

It now follows from Theorem 2.4.1 that

$$A(a, b) = \int_a^b (g(x) - f(x))dx. \tag{2.5.10}$$

Example 2.5.1. Let A be the area of the region R bounded by the curves with equations $y = x^2$ and $y = x + 2$. Note that these curves intersect when $x^2 = x + 2$, that is when

$$0 = x^2 - x - 2 = (x + 1)(x - 2).$$

Hence they intersect at the points $(-1, 1)$ and $(2, 4)$, and so R is the region in the plane bounded above by the curve $y = x + 2$, below by the curve $y = x^2$, on the right by $x = -1$, and on the left by $x = 2$. See Figure 2.5.2. Thus we have

$$\begin{aligned} A &= \int_{-1}^{2} (x + 2 - x^2)dx \\ &= \left(\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3 \right) \Bigg|_{-1}^{2} \\ &= \left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) \\ &= \frac{9}{2}. \end{aligned}$$

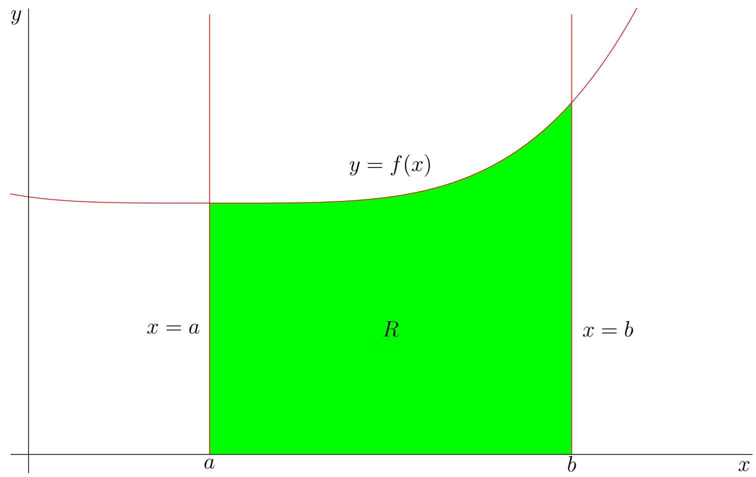

Figure 2.5.3: Area beneath the graph of a function f

Exercise 2.5.1. Find the area of the region bounded by the curves $y = x$ and $y = x^2$.

Exercise 2.5.2. Find the area of the region bounded by the curves $y = x^2$ and $y = 2 - x^2$.

Exercise 2.5.3. Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.

Now consider a continuous function f on an interval $[a, b]$ with $f(x) \geq 0$ for all x in $[a, b]$. If A is the area of the region R bounded above by the graph of $y = f(x)$, below by the graph of $y = 0$ (that is, the x-axis), on the right by the vertical line $x = a$, and on the left by the graph of $x = b$ (see Figure 2.5.3), then

$$A = \int_a^b (f(x) - 0)dx = \int_a^b f(x)dx. \tag{2.5.11}$$

This gives us a geometric interpretation for a the definite integral of a nonnegative function f over an interval $[a, b]$ as the area beneath the graph of f and above the x-axis.

Example 2.5.2. If A is the area of the region R beneath the graph of $y = \sin(x)$ over the interval $[0, \pi]$, as in Figure 2.5.4, then

$$A = \int_0^\pi \sin(x)dx = -\cos(x)|_0^\pi = 1 + 1 = 2.$$

On the other hand, if f is continuous on $[a, b]$ with $f(x) \leq 0$ for all x in $[a, b]$, and A is the area of the region R above the graph of $y = f(x)$ and below

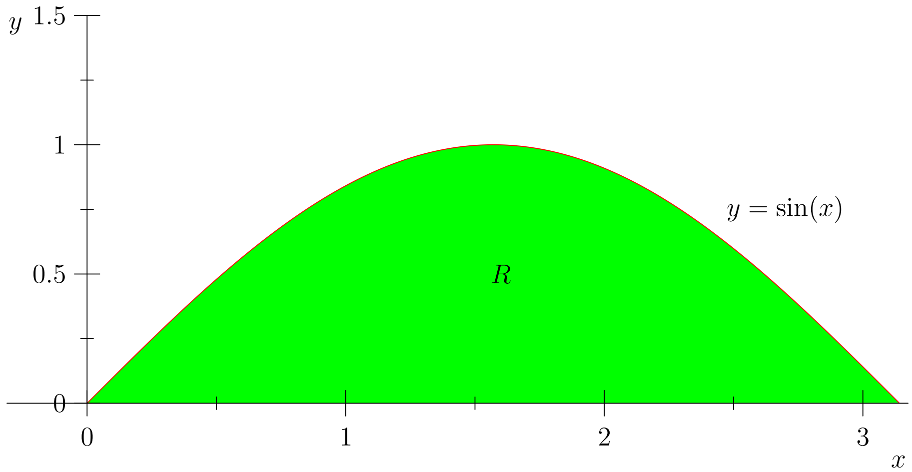

Figure 2.5.4: Area beneath the graph of $y = \sin(x)$

the x axis, then

$$A = \int_a^b (0 - f(x))dx = -\int_a^b f(x)dx. \tag{2.5.12}$$

That is, the definite integral of a non-positive function f over an interval $[a, b]$ is the negative of the area above the graph of f and beneath the x-axis.

In general, given a continuous function f on an interval let R be the region bounded by the x-axis and the graph of $y = f(x)$. If A^+ is the area of the part of R which lies above the x-axis and A^- is the area of the part of R which lies below the x-axis, then

$$\int_a^b f(x)dx = A^+ - A^-. \tag{2.5.13}$$

Example 2.5.3. Note that

$$\int_0^{2\pi} \sin(x)dx = -\cos(x)|_0^{2\pi} = -1 + 1 = 0.$$

Geometrically, we can see this result in Figure 2.5.5. If R^+ is the the region beneath the graph of $y = \sin(x)$ over the interval $[0, \pi]$ and R^- is the region above the graph of $y = \sin(x)$ over the interval $[0, 2\pi]$, then these two regions have the same area. Hence the integral, which is the area of R^+ minus the area of R^-, is 0.

Exercise 2.5.4. Evaluate

$$\int_{-1}^1 x^3 dx$$

and explain the result geometrically.

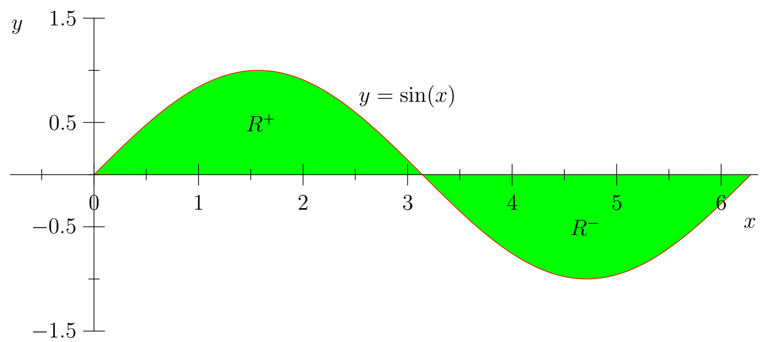

Figure 2.5.5: Area of R^+ is the same as the area of R^-

Exercise 2.5.5. Evaluate

$$\int_{-1}^{2} x dx$$

and explain the result geometrically.

Exercise 2.5.6. Explain, geometrically, why

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{\pi}{2}.$$

2.5.2 Volumes

Consider a three-dimensional body B. Given a line, which we will call the z-axis, let $V(a, b)$ be the volume of B which lies between planes which are perpendicular to the z-axis and pass through $z = a$ and $z = b$. Clearly, for any $a < c < b$,

$$V(a, b) = V(a, c) + V(c, b). \tag{2.5.14}$$

Now suppose that, for any $a \leq x \leq b$, $R(z)$ is a cross section of B perpendicular to the z-axis. See Figure 2.5.6. Let $A(z)$ be the area of $R(z)$. We assume A is a continuous function of z. For a positive infinitesimal dz, let A have, on the interval $[z, z + dz]$, a minimum value at c and a maximum value at d. Then

$$A(c)dz \leq V(z, z + dz) \leq A(d)dz. \tag{2.5.15}$$

Since we also have $A(c) \leq A(z) \leq A(d)$, it follows that

$$|V(z, z + dz) - A(z)dz| \leq (A(d) - A(c))dz. \tag{2.5.16}$$

Thus

$$\frac{|V(z, z + dz) - A(z)dz|}{dz} \leq A(d) - A(c). \tag{2.5.17}$$

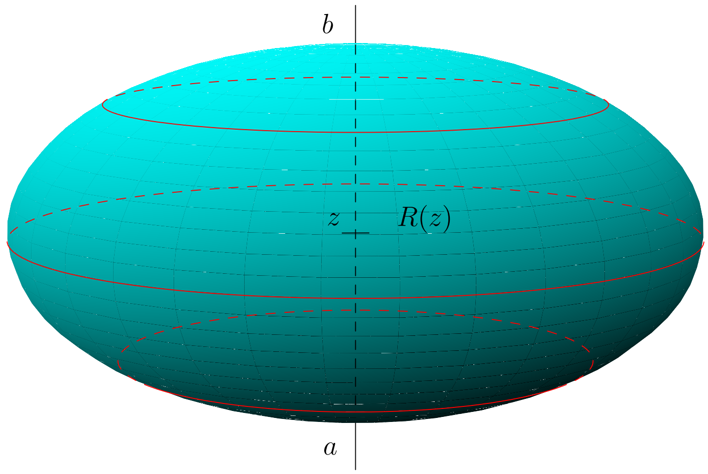

Figure 2.5.6: Cross sections of a body perpendicular to the z-axis

Since A is continuous and $d \simeq z$ and $c \simeq z$, $A(d) - A(c)$ is infinitesimal. Hence

$$V(z, z + dz) - A(z)dz \sim o(dz), \tag{2.5.18}$$

from which it follows, by Theorem 2.4.1, that

$$V(a, b) = \int_a^b A(z)dz. \tag{2.5.19}$$

Example 2.5.4. The unit sphere S, with center at the origin, is the set of all points (x, y, z) satisfying $x^2 + y^2 + z^2 = 1$ (see Figure 2.5.7). For a fixed value of z between -1 and 1, the cross section $R(z)$ of S perpendicular to the z-axis is the set of points (x, y) satisfying the equation $x^2 + y^2 = 1 - z^2$. That is, $R(z)$ is a circle with radius $\sqrt{1 - z^2}$. Hence $R(z)$ has area

$$A(z) = \pi(1 - z^2).$$

If V is the volume of S, it now follows that

$$\begin{aligned} V &= \int_{-1}^{1} \pi(1 - z^2)dx \\ &= \pi \left(z - \frac{1}{3} z^3 \right) \Big|_{-1}^{1} \\ &= \pi \left(\frac{2}{3} + \frac{2}{3} \right) \\ &= \frac{4\pi}{3}. \end{aligned}$$

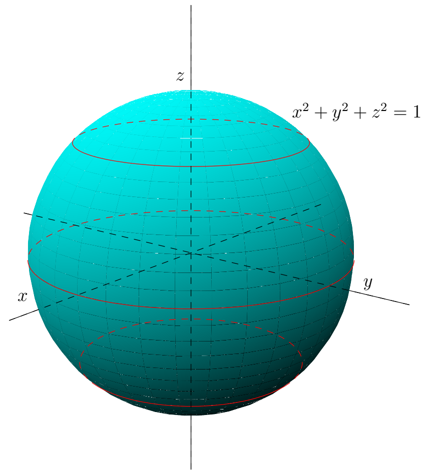

Figure 2.5.7: The unit sphere $x^2 + y^2 + z^2 = 1$

Exercise 2.5.7. For $r > 0$, the equation of a sphere S of radius r is $x^2 + y^2 + z^2 = r^2$. Show that the volume of S is $\frac{4}{3}\pi r^3$.

Exercise 2.5.8. Let P be a pyramid with a square base having corners at $(1, 1, 0)$, $(1, -1, 0)$, $(-1, -1, 0)$, and $(-1, 1, 0)$ in the xy-plane and top vertex at $(0, 0, 1)$ on the z-axis. Show that the volume of P is $\frac{4}{3}$.

Example 2.5.5. Let T be the region bounded by the z-axis and the graph of $z = x^2$ for $0 \leq x \leq 1$. Let B be the three-dimensional body created by rotating T about the z-axis. See Figure 2.5.8. If $R(z)$ is a cross section of B perpendicular to the z-axis, then $R(z)$ is a circle with radius $\sqrt{z}$. Thus, if $A(z)$ is the area of $R(z)$, we have

$$A(z) = \pi z.$$

If V is the volume of B, then

$$V = \int_0^1 \pi z dz = \pi \frac{z^2}{2}\bigg|_0^1 = \frac{\pi}{2}.$$

Exercise 2.5.9. Let T be the region bounded by z-axis and the graph of $z = x$ for $0 \leq x \leq 2$. Find the volume of the solid B obtained by rotating T about the z-axis.

Exercise 2.5.10. Let T be the region bounded by z-axis and the graph of $z = x^4$ for $0 \leq x \leq 1$. Find the volume of the solid B obtained by rotating T about the z-axis.

Example 2.5.6. Let T be the region bounded by the graphs of $z = x^4$ and $x = x^2$ for $0 \leq x \leq 1$. Let B be the three-dimensional body created by rotating

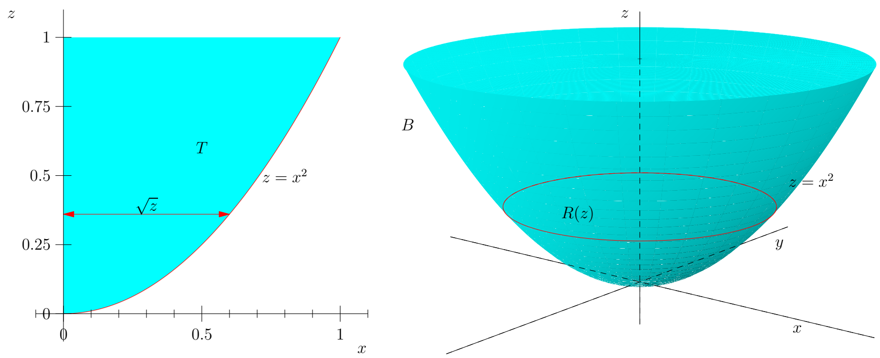

Figure 2.5.8: Region T rotated about z-axis to create solid body B

T about the z-axis. See Figure 2.5.9. If $R(z)$ is a cross section of B perpendicular to the z-axis, then $R(z)$ is the region between the circles with radii $z^{\frac{1}{4}}$ and $\sqrt{z}$, an annulus. Hence if $A(z)$ is the area of $R(z)$, then

$$A(z) = \pi \left(z^{\frac{1}{4}}\right)^2 - \pi \left(\sqrt{z}\right)^2 = \pi(\sqrt{z} - z).$$

If V is the volume of B, then

$$V = \int_0^1 \pi(\sqrt{z} - z)dz = \pi \left(\frac{2}{3}z^{\frac{3}{2}} - \frac{1}{2}z^2\right)\Big|_0^1 = \pi \left(\frac{2}{3} - \frac{1}{2}\right) = \frac{\pi}{6}.$$

Exercise 2.5.11. Let T be the region bounded by the curves $z = x$ and $z = x^2$. Find the volume of the solid B obtained by rotating T about the z-axis.

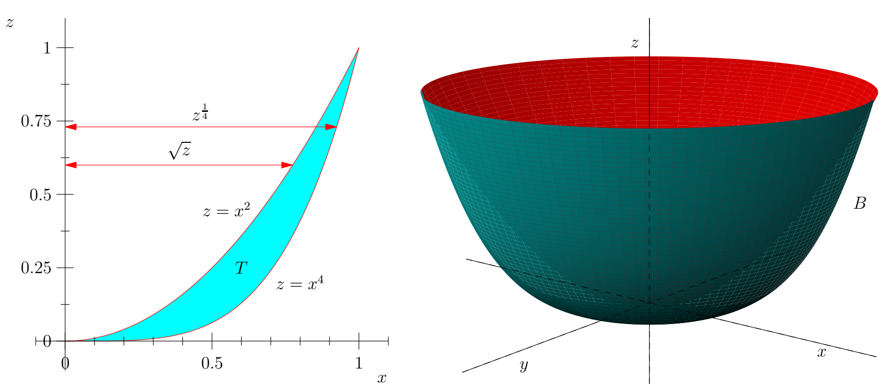

Figure 2.5.9: Region T rotated about z-axis to create solid body B

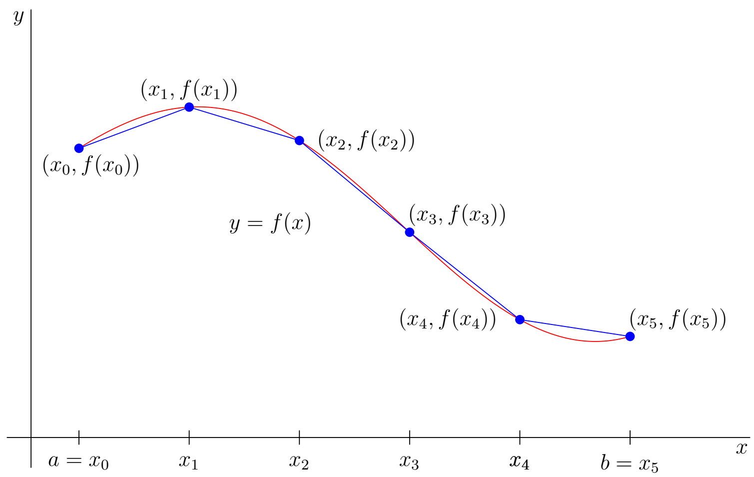

Figure 2.5.10: Approximating the length of a curve with $N = 5$ subintervals

2.5.3 Arc length

Consider a function f which is continuous on the closed interval $[a, b]$. Let C be the graph of f over $[a, b]$ and let L be the length of C. To approximate L, we first divide $[a, b]$ into N equal subintervals, each of length

$$\Delta x = \frac{b-a}{N}, \tag{2.5.20}$$

and let $x_0 = a, x_1, x_2, \ldots, x_N = b$ be the endpoints of the subintervals. If L_i is the length of the line from $(x_{i-1}, f(x_{i-1}))$ to $(x_i, f(x_i))$, for $i = 1, 2, \ldots, N$ (see Figure 2.5.10), then

$$L \approx L_1 + L_2 + \cdots + L_N = \sum_{i=1}^{N} L_i. \tag{2.5.21}$$

Since

$$L_i = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}, \tag{2.5.22}$$

where

$$\Delta y_i = f(x_i) - f(x_{i-1}) = f(x_{i-1} + \Delta x) - f(x_{i-1}), \tag{2.5.23}$$

we have

$$L \approx \sum_{i=1}^{N} \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sum_{i=1}^{N} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\, \Delta x. \tag{2.5.24}$$

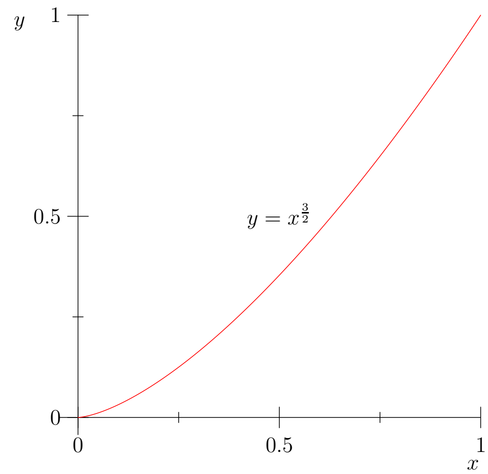

Figure 2.5.11: Graph of $y = x^{\frac{3}{2}}$ over $[0, 1]$

Now we should expect the approximation in (2.5.24) to become exact when N is infinite. That is, for N a positive infinite integer, let

$$dx = \frac{b-a}{N}, \tag{2.5.25}$$

and

$$dy = f(x + dx) - f(x). \tag{2.5.26}$$

If the shadow of

$$\sum_{i=1}^{N} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{2.5.27}$$

is the same for any choice of N, then we call (2.5.27) the *arc length* of C. Now if f is differentiable on an open interval containing $[a, b]$, and f' is continuous on $[a, b]$, then (2.5.27) becomes the definite integral of $\sqrt{1 + (f'(x))^2}$. That is, the arc length of C is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx. \tag{2.5.28}$$

Example 2.5.7. Let C be the graph of $f(x) = x^{\frac{3}{2}}$ over the interval $[0, 1]$ (see Figure 2.5.11) and let L be the length of C. Since $f'(x) = \frac{3}{2}\sqrt{x}$, we have

$$L = \int_0^1 \sqrt{1 + \frac{9}{4}x}\, dx.$$

Now

$$\int \sqrt{x}\, dx = \frac{2}{3}x^{\frac{3}{2}} + c,$$

so we might expect an integral of $\sqrt{1+\frac{9}{4}}$ to be

$$\frac{2}{3}\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}+c.$$

However,

$$\frac{d}{dx}\frac{2}{3}\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}=\frac{9}{4}\sqrt{1+\frac{9}{4}x},$$

and so, dividing our original guess by $\frac{9}{4}$, we have

$$\int\sqrt{1+\frac{9}{4}x}\;dx=\frac{8}{27}\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}+c,$$

which may be verified by differentiation. Hence

$$\begin{aligned}L&=\int_0^1\sqrt{1+\frac{9}{4}x}\;dx\\&=\frac{8}{27}\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}\Bigg|_0^1\\&=\frac{8}{27}\left(\frac{13\sqrt{13}}{8}-1\right)\\&=\frac{13\sqrt{13}-8}{27}\\&\approx 1.4397.\end{aligned}$$

Example 2.5.8. Let C be the graph of $f(x)=x^2$ over the interval $[0,1]$ (see Figure 2.5.12) and let L be the length of C. Since $f'(x)=2x$,

$$L=\int_0^1\sqrt{1+4x^2}\;dx.$$

However, we do not have the tools at this time to evaluate this definite integral exactly. Still, we may use (2.5.24) to find an approximation for L. For example, if we take $N=100$ in (2.5.24), then $\Delta x=0.01$ and

$$\begin{aligned}\Delta y_i&=f(0.01i)-f(0.01(i-1))\\&=(0.01i)^2-(0.01(i-1))^2\\&=0.0001(i^2-(i^2-2i+1))\\&=0.0001(2i-1)\end{aligned}\tag{2.5.29}$$

for $i=1,2,\ldots,N$, and so

$$L\approx\sum_{i=1}^{100}\sqrt{(\Delta x)^2+(\Delta y_i)^2}\approx 1.4789.$$

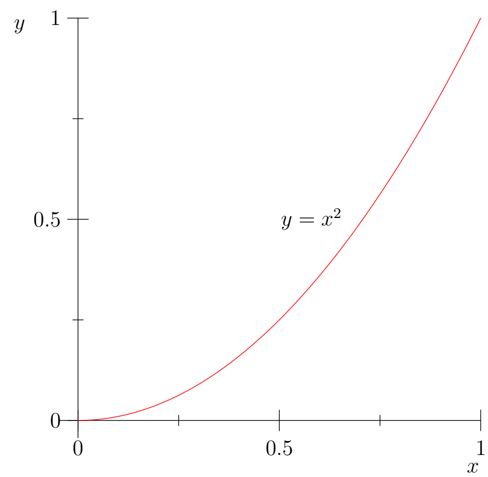

Figure 2.5.12: Graph of $y = x^2$ over $[0, 1]$

We will return to this example in Examples 2.6.19 and 2.7.9 to find an exact expression for L.

Exercise 2.5.12. Let C be the graph of $y = \frac{2}{3}x^{\frac{3}{2}}$ over the interval $[1, 3]$. Find the length of C.

Exercise 2.5.13. Let C be the graph of $y = \sin(x)$ over the interval $[0, \pi]$. Use (2.5.24) with $N = 10$ to approximate the length of C.

2.6 Some techniques for evaluating integrals

2.6.1 Change of variable

If F is an integral of f and φ is a differentiable function, then, using the chain rule,

$$\frac{d}{dx}F(\varphi(x)) = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x). \tag{2.6.1}$$

Written in terms of integrals, we have

$$\int f(\varphi(x))\varphi'(x)dx = F(\varphi(x)) + c. \tag{2.6.2}$$

If we let $u = \varphi(x)$ and note that

$$\int f(u)du = F(u) + c, \tag{2.6.3}$$

we may express (2.6.2) as

$$\int f(\varphi(x))\varphi'(x)dx = \int f(u)du. \tag{2.6.4}$$

That is, we may evaluate

$$\int f(\varphi(x))\varphi'(x)dx,$$

by changing the variable to $u = \varphi(x)$, with $\varphi'(x)dx$ becoming du since

$$\frac{du}{dx} = \varphi'(x).$$

Example 2.6.1. To evaluate

$$\int 2x\sqrt{1+x^2}\ dx,$$

let

$$u = 1 + x^2$$
$$du = 2xdx.$$

Then

$$\int 2x\sqrt{1+x^2}\ dx = \int \sqrt{u}\ du = \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + c.$$

Example 2.6.2. To evaluate

$$\int x\sin(x^2)dx,$$

let

$$u = x^2$$
$$du = 2xdx$$

Note that in this case we cannot make a direct substitution of u and du since $du = 2xdx$ does not appear as part of the integral. However, du differs from xdx by only a constant factor, and we may rewrite $du = 2xdx$ as

$$\frac{1}{2}du = xdx.$$

Now we may perform the change of variable:

$$\int x\sin(u)dx = \frac{1}{2}\int \sin(u)du = -\frac{1}{2}\cos(u) + c = -\frac{1}{2}\cos(x^2) + c.$$

Example 2.6.3. Note that we could evaluate the integral

$$\int \cos(4x)dx$$

using the substitution

$$\begin{aligned} u &= 4x \\ du &= 4dx, \end{aligned}$$

which gives us

$$\int \cos(4x)dx = \frac{1}{4}\int \cos(u)du = \frac{1}{4}\sin(u) + c = \frac{1}{4}\sin(4x) + c.$$

However, it is probably quicker, and easier, to guess that the integral of $\cos(4x)$ should be close to $\sin(4x)$, and then correcting this guess appropriately after noting that

$$\frac{d}{dx}\sin(4x) = 4\cos(4x).$$

Example 2.6.4. To evaluate

$$\int \cos^2(5x)\sin(5x)dx,$$

let

$$\begin{aligned} u &= \cos(5x) \\ du &= -5\sin(5x)dx. \end{aligned}$$

Then

$$\int \cos^2(5x)\sin(5x)dx = -\frac{1}{5}\int u^2du = -\frac{1}{15}u^3 + c = -\frac{1}{15}\cos^3(5x) + c.$$

Now consider the definite integral

$$\int_a^b f(\varphi(x))\varphi'(x)dx.$$

If F is an integral of f, $c = \varphi(a)$, and $d = \varphi(b)$, then

$$\begin{aligned} \int_a^b f(\varphi(x))\varphi'(x)dx &= F(\varphi(x))\big|_a^b \\ &= F(\varphi(b)) - F(\varphi(a)) \\ &= F(d) - F(c) \\ &= F(u)\big|_c^d \\ &= \int_c^d f(u)du. \end{aligned} \tag{2.6.5}$$

That is, we may use a change of variable to evaluate a definite integral in the same manner as above, the only difference being that we must change the limits of integration to reflect the values of the new variable u.

Example 2.6.5. To evaluate

$$\int_0^1 \frac{x}{\sqrt{1+x^2}}\, dx,$$

let

$$\begin{aligned} u &= 1 + x^2 \\ du &= 2x dx. \end{aligned}$$

Note that when $x = 0$, $u = 1$, and when $x = 1$, $u = 2$. Hence

$$\int_0^1 \frac{x}{\sqrt{1+x^2}}\, dx = \frac{1}{2}\int_1^2 \frac{1}{\sqrt{u}}\, du = \sqrt{u}\Big|_1^2 = \sqrt{2} - 1.$$

Example 2.6.6. To evaluate

$$\int_0^{\frac{\pi}{4}} \cos^2(2x)\sin(2x)dx,$$

let

$$\begin{aligned} u &= \cos(2x) \\ du &= -2\sin(2x)dx. \end{aligned}$$

Then $u = 1$ when $x = 0$ and $u = 0$ when $x = \frac{\pi}{4}$, so

$$\int_0^{\frac{\pi}{4}} \cos^2(2x)\sin(2x)dx = -\frac{1}{2}\int_1^0 u^2 du.$$

Note that, after making the change of variable, the upper limit of integration is less than the lower limit of integration, a situation not covered by our definition of the definite integral or our statement of the fundamental theorem of calculus. However, the result on substitutions above shows that we will obtain the correct result if we apply the fundamental theorem as usual. Moreover, this points toward an extension of our definition: if $b < a$, then we should have

$$\int_a^b f(x)dx = -\int_b^a f(x)dx, \tag{2.6.6}$$

which is consistent with both the fundamental theorem of calculus and with the definition of the definite integral (since, if $b < a$, $dx = \frac{b-a}{N} < 0$ for any positive infinite integer N). With this, we may finish the evaluation:

$$\int_0^{\frac{\pi}{4}} \cos^2(2x)\sin(2x)dx = -\frac{1}{2}\int_1^0 u^2 du = \frac{1}{2}\int_0^1 u^2 du = \frac{u^3}{6}\Bigg|_0^1 = \frac{1}{6}.$$

Exercise 2.6.1. Evaluate $\int 3x^2\sqrt{1+x^3}\,dx$.

Exercise 2.6.2. Evaluate $\int x\sqrt{4+3x^2}\,dx$.

Exercise 2.6.3. Evaluate $\int \sec^2(3x)\tan^2(3x)dx$.

Exercise 2.6.4. Evaluate $\int_0^2 \frac{x}{\sqrt{4+x^2}}\,dx$.

Exercise 2.6.5. Evaluate $\int_0^{\frac{\pi}{6}} \sin(3x)dx$.

Exercise 2.6.6. Evaluate $\int_0^{\frac{\pi}{2}} \sin^4(2x)\cos(2x)dx$.

2.6.2 Integration by parts

Suppose u and v are both differentiable functions of x. Since, by the product rule,

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}, \tag{2.6.7}$$

we have

$$u\frac{dv}{dx} = \frac{d}{dx}uv - v\frac{du}{dx}. \tag{2.6.8}$$

Hence, integrating both sides with respect to x,

$$\int u\frac{dv}{dx}dx = \int \frac{d}{dx}uv - \int v\frac{du}{dx} = uv - \int v\frac{du}{dx}dx, \tag{2.6.9}$$

which we may write as

$$\int udv = uv - \int vdu. \tag{2.6.10}$$

This last formulation, known as *integration by parts*, is useful whenever the integral on the right of (2.6.10) is in someway simpler than the integral on the left. The next examples will illustrate some typical cases.

Example 2.6.7. Consider the integral

$$\int x\cos(x)dx.$$

If we let $u = x$ and $dv = \cos(x)dx$, then $du = dx$ and we may let $v = \sin(x)$. Note that we have some choice for v since the only requirement is that it is an integral of $\cos(x)$. Using (2.6.10), we have

$$\int x\sin(x)dx = uv - \int vdu = x\sin(x) - \int \sin(x)dx = x\sin(x) + \cos(x) + c.$$

In evaluating a definite integral using integration by parts, we must remember to evaluate each piece of the integral. That is,

$$\int u dv = uv|_a^b - \int_a^b v du. \tag{2.6.11}$$

Example 2.6.8. To evaluate

$$\int_0^\pi x^2 \sin(x) dx,$$

let

$$\begin{aligned} u &= x^2 & dv &= \sin(x) dx \\ du &= 2x dx & v &= -\cos(x). \end{aligned}$$

Then, using (2.6.11),

$$\int_0^\pi x^2 \sin(x) dx = -x^2 \cos(x)|_0^\pi + \int_0^\pi 2x \cos(x) dx = \pi^2 + \int_0^\pi 2x \cos(x) dx.$$

Note that the final integral is simpler than the integral with which we started, but still requires another integration by parts to finish the evaluation. Namely, if we now let

$$\begin{aligned} u &= 2x & dv &= \cos(x) \\ du &= 2dx & v &= \sin(x), \end{aligned}$$

we have

$$\begin{aligned} \int_0^\pi x^2 \sin(x) dx &= \pi^2 + 2x \sin(x)|_0^\pi - \int_0^\pi 2 \sin(x) dx \\ &= \pi^2 + (0 - 0) + 2 \cos(x)|_0^\pi \\ &= \pi^2 - 2 - 2 \\ &= \pi^2 - 4. \end{aligned}$$

Example 2.6.9. To evaluate

$$\int_0^1 x\sqrt{1+x}\ dx,$$

let

$$\begin{aligned} u &= x & dv &= \sqrt{1+x}\ dx \\ du &= dx & v &= \frac{2}{3}(1+x)^{\frac{3}{2}}. \end{aligned}$$

Then

$$\begin{aligned}\int_0^1 x\sqrt{1+x}\ dx &= \frac{2}{3}x(1+x)^{\frac{3}{2}}\bigg|_0^1 - \frac{2}{3}\int_0^1 (1+x)^{\frac{3}{2}}dx \\ &= \frac{4\sqrt{2}}{3} - \frac{4}{15}(1+x)^{\frac{5}{2}}\bigg|_0^1 \\ &= \frac{4\sqrt{2}}{3} - \frac{16\sqrt{2}-4}{15} \\ &= \frac{4\sqrt{2}+4}{15}.\end{aligned}$$

Exercise 2.6.7. Evaluate $\int x\sin(2x)dx$.

Exercise 2.6.8. Evaluate $\int x^2\cos(3x)dx$.

Exercise 2.6.9. Evaluate $\int_0^{\pi} x\cos\left(\frac{1}{2}x\right)dx$.

Exercise 2.6.10. Evaluate $\int_0^{\frac{\pi}{2}} 3x^2\cos(x^2)dx$.

Exercise 2.6.11. Evaluate $\int_0^2 x^2\sqrt{1+x}\ dx$.

2.6.3 Some integrals involving trigonometric functions

The next examples will illustrate how various identities are useful in simplifying some integrals involving trigonometric functions.

Example 2.6.10. To evaluate the integral

$$\int_0^{\pi} \sin^2(x)dx,$$

we will use the half-angle formula:

$$\sin^2(x) = \frac{1-\cos(2x)}{2}. \tag{2.6.12}$$

Then

$$\begin{aligned}\int_0^{\pi} \sin^2(x)dx &= \frac{1}{2}\int_0^{\pi}(1-\cos(2x))dx \\ &= \frac{1}{2}x\bigg|_0^{\pi} - \frac{1}{4}\sin(2x)\bigg|_0^{\pi} \\ &= \frac{\pi}{2}.\end{aligned}$$

There is also a half-angle formula for cosine, namely,

$$\cos^2(x) = \frac{1+\cos(2x)}{2}. \tag{2.6.13}$$

As illustrated in the next example, we may use the half-angle formulas recursively to evaluate the integral of any even power of sine or cosine.

Example 2.6.11. Using 2.6.13 twice, we have

$$\begin{aligned}
\int_0^\pi \cos^4(3x)dx &= \int_0^\pi \left(\cos^2(3x)\right)^2 dx \\
&= \int_0^\pi \left(\frac{1}{2}(1+\cos(6x))\right)^2 dx \\
&= \frac{1}{4}\int_0^\pi (1+2\cos(6x)+\cos^2(6x))dx \\
&= \frac{1}{4}x\Big|_0^\pi + \frac{1}{12}\sin(6x)\Big|_0^\pi + \frac{1}{8}\int_0^\pi (1+\cos(12x))dx \\
&= \frac{\pi}{4} + \frac{1}{8}x\Big|_0^\pi + \frac{1}{96}\sin(12x)\Big|_0^\pi \\
&= \frac{3\pi}{8}.
\end{aligned}$$

Exercise 2.6.12. Evaluate $\displaystyle\int_0^\pi \sin^2(2x)dx$.

Exercise 2.6.13. Evaluate $\displaystyle\int_0^\pi \cos^2(3x)dx$.

Exercise 2.6.14. Evaluate $\displaystyle\int \cos^4(x)dx$.

The next example illustrates a *reduction formula*.

Example 2.6.12. Suppose $n \geq 2$ is an integer and we wish to evaluate

$$\int_0^\pi \sin^n(x)dx.$$

We begin with an integration by parts: if we let

$$\begin{aligned}
u &= \sin^{n-1}(x) & dv &= \sin(x)dx \\
du &= (n-1)\sin^{n-2}(x)\cos(x)dx & v &= -\cos(x),
\end{aligned}$$

then

$$\begin{aligned}
\int_0^\pi \sin^n(x)dx &= -\sin^{n-1}(x)\cos(x)\big|_0^\pi + (n-1)\int_0^\pi \sin^{n-2}(x)\cos^2(x)dx \\
&= (n-1)\int_0^\pi \sin^{n-2}(x)\cos^2(x)dx.
\end{aligned}$$

Now $\cos^2(x) = 1 - \sin^2(x)$, so we have

$$\begin{aligned}\int_0^\pi \sin^n(x)dx &= (n-1)\int_0^\pi \sin^{n-2}(x)(1-\sin^2(x))dx \\ &= (n-1)\int_0^\pi \sin^{n-2}(x)dx - (n-1)\int_0^\pi \sin^n(x)dx.\end{aligned}$$

Notice that $\int_0^\pi \sin^n(x)dx$ occurs on both sides of this equation. Hence we may solve for this quantity, first obtaining

$$n\int_0^\pi \sin^n(x)dx = (n-1)\int_0^\pi \sin^{n-2}(x)dx,$$

and then

$$\int_0^\pi \sin^n(x)dx = \frac{n-1}{n}\int_0^\pi \sin^{n-2}(x)dx. \tag{2.6.14}$$

Note that, although we have not yet found the value of our integral, we have reduced the power of $\sin(x)$ in the integral. We may now use (2.6.14) repeatedly to reduce the power of $\sin(x)$ until we can easily evaluate the resulting integral. For example, if $n = 6$ we have

$$\begin{aligned}\int_0^\pi \sin^6(x)dx &= \frac{5}{6}\int_0^\pi \sin^4(x)dx \\ &= \frac{5}{6}\frac{3}{4}\int_0^\pi \sin^2(x)dx \\ &= \frac{5}{6}\frac{3}{4}\frac{1}{2}\int_0^\pi dx \\ &= \frac{5\pi}{16}.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_0^\pi \sin^5(x)dx &= \frac{4}{5}\int_0^\pi \sin^3(x)dx \\ &= \frac{4}{5}\frac{2}{3}\int_0^\pi \sin(x)dx \\ &= -\frac{8}{15}\cos(x)\Big|_0^\pi \\ &= \frac{16}{15}.\end{aligned}$$

Exercise 2.6.15. Use the reduction formula (2.6.14) to evaluate

$$\int_0^\pi \sin^8(x)dx.$$

Exercise 2.6.16. Use the reduction formula (2.6.14) to evaluate

$$\int_0^\pi \sin^7(x)dx.$$

Exercise 2.6.17. Derive the reduction formula

$$\int_0^\pi \cos^n(x) = \frac{n-1}{n}\int_0^\pi \cos^{n-2}(x)dx,$$

where $n \geq 2$ is an integer.

Exercise 2.6.18. Use the reduction formula of the previous exercise to evaluate

$$\int_0^\pi \cos^6(x)dx.$$

Exercise 2.6.19. Derive the reduction formulas

$$\int \sin^n(x)dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)dx$$

and

$$\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)dx,$$

where $n \geq 2$ is an integer.

Example 2.6.13. An alternative to using a reduction formula in the last example begins with noting that

$$\begin{aligned}
\int_0^\pi \sin^5(x)dx &= \int_0^\pi \sin^4(x)\sin(x)dx \\
&= \int_0^\pi (\sin^2(x))^2\sin(x)dx \\
&= \int_0^\pi (1-\cos^2(x))^2\sin(x)dx \\
&= \int_0^\pi (1-2\cos^2(x)+\cos^4(x))\sin(x)dx.
\end{aligned}$$

The latter integral may now be evaluated using the change of variable

$$\begin{aligned}
u &= \cos(x) \\
du &= -\sin(x)dx,
\end{aligned}$$

giving us

$$\begin{aligned}
\int_0^\pi \sin^5(x)dx &= -\int_1^{-1}(1-2u^2+u^4)du \\
&= \int_{-1}^{1}(1-2u^2+u^4)du \\
&= \left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right)\Big|_{-1}^{1} \\
&= \left(1-\frac{2}{3}+\frac{1}{5}\right) - \left(-1+\frac{2}{3}-\frac{1}{5}\right) \\
&= \frac{16}{15},
\end{aligned}$$

as we saw above.

Exercise 2.6.20. Evaluate $\int_0^{\frac{\pi}{4}} \cos^5(2x)dx$.

Our next example relies on a trigonometric identity for $\sin(ax)\cos(bx)$, where a and b are both real numbers. We first note that, using the angle addition and subtraction formulas for sine,

$$\sin((a+b)x) = \sin(ax)\cos(bx) + \sin(bx)\cos(ax) \tag{2.6.15}$$
$$\sin((a-b)x) = \sin(ax)\cos(bx) - \sin(bx)\cos(bx). \tag{2.6.16}$$

Adding these together, we have

$$2\sin(ax)\cos(bx) = \sin((a+b)x) + \sin((a-b)x), \tag{2.6.17}$$

and so

$$\sin(ax)\cos(bx) = \frac{1}{2}(\sin((a+b)x) + \sin((a-b)x)). \tag{2.6.18}$$

Example 2.6.14. To evaluate

$$\int_0^\pi \sin(2x)\cos(3x)dx,$$

we first note that, using (2.6.18) with $a=2$ and $b=3$,

$$\sin(2x)\cos(3x) = \frac{1}{2}(\sin(5x)+\sin(-x)) = \frac{1}{2}(\sin(5x)-\sin(x)).$$

Hence

$$\begin{aligned}\int_0^\pi \sin(2x)\cos(3x)dx &= \frac{1}{2}\int_0^\pi \sin(5x)dx - \frac{1}{2}\int_0^\pi \sin(x)dx \\ &= -\frac{1}{10}\cos(5x)\Big|_0^\pi + \frac{1}{2}\cos(x)\Big|_0^\pi \\ &= \left(\frac{1}{10}+\frac{1}{10}\right) + \left(-\frac{1}{2}-\frac{1}{2}\right) \\ &= -\frac{4}{5}.\end{aligned}$$

For integrals involving $\sin(ax)\sin(bx)$, we begin with the the angle addition and subtraction formulas for cosine,

$$\cos((a+b)x) = \cos(ax)\cos(bx) - \sin(bx)\sin(ax) \tag{2.6.19}$$

$$\cos((a-b)x) = \cos(ax)\cos(bx) + \sin(bx)\sin(bx). \tag{2.6.20}$$

Subtracting the first of these from the second, we have

$$2\sin(ax)\sin(bx) = \cos((a-b)x) - \cos((a+b)x), \tag{2.6.21}$$

and so

$$\sin(ax)\sin(bx) = \frac{1}{2}(\cos((a-b)x) - \cos((a+b)x)). \tag{2.6.22}$$

Example 2.6.15. To evaluate

$$\int_0^\pi \sin(3x)\sin(5x)dx,$$

we first note that, using (2.6.22) with $a = 3$ and $b = 5$,

$$\sin(3x)\sin(5x) = \frac{1}{2}(\cos(-2x) - \cos(8x)) = \frac{1}{2}(\cos(2x) - \cos(8x)).$$

Note that we would have the same identity if we had chosen $a = 5$ and $b = 3$. Then

$$\begin{aligned}\int_0^\pi \sin(3x)\sin(5x)dx &= \frac{1}{2}\int_0^\pi \cos(2x)dx - \frac{1}{2}\int_0^\pi \cos(8x)dx \\ &= \frac{1}{4}\sin(2x)\Big|_0^\pi - \frac{1}{16}\sin(8x)\Big|_0^\pi \\ &= 0.\end{aligned}$$

For integrals involving $\cos(ax)\cos(bx)$, we add (2.6.19) to (2.6.20) to obtain

$$2\cos(ax)\cos(bx) = \cos((a+b)x) + \cos((a-b)x), \tag{2.6.23}$$

which leads to

$$\cos(ax)\cos(bx) = \frac{1}{2}(\cos((a+b)x) + \cos((a-b)x)). \tag{2.6.24}$$

Example 2.6.16. To evaluate

$$\int_0^{\frac{\pi}{2}} \cos(3x)\cos(5x)dx,$$

we note that, using (2.6.24) with $a = 3$ and $b = 5$,

$$\cos(3x)\cos(5x) = \frac{1}{2}(\cos(8x) + \cos(-2x)) = \frac{1}{2}(\cos(8x) + \cos(2x)).$$

Hence

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos(3x)\cos(5x)dx &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(8x)dx + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2x)dx \\ &= \frac{1}{16}\sin(8x)\bigg|_0^{\frac{\pi}{2}} - \frac{1}{4}\sin(2x)\bigg|_0^{\frac{\pi}{2}} \\ &= 0.\end{aligned}$$

Exercise 2.6.21. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin(x)\cos(2x)dx$.

Exercise 2.6.22. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin(x)\sin(2x)dx$.

Exercise 2.6.23. Evaluate

$$\int_0^{\frac{\pi}{2}} \sin(3x)\cos(3x)dx.$$

Note: This may be evaluated with a substitution.

Exercise 2.6.24. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\cos(2x)dx$.

Exercise 2.6.25. For any positive integers m and n, show that

$$\int_0^{2\pi} \sin(mx)\cos(nx)dx = 0,$$

$$\int_0^{2\pi} \sin(mx)\sin(nx)dx = \begin{cases} 0, & \text{if } m \neq n, \\ \pi & \text{if } m = n, \end{cases}$$

and

$$\int_0^{2\pi} \cos(mx)\cos(nx)dx = \begin{cases} 0, & \text{if } m \neq n, \\ \pi, & \text{if } m = n. \end{cases}$$

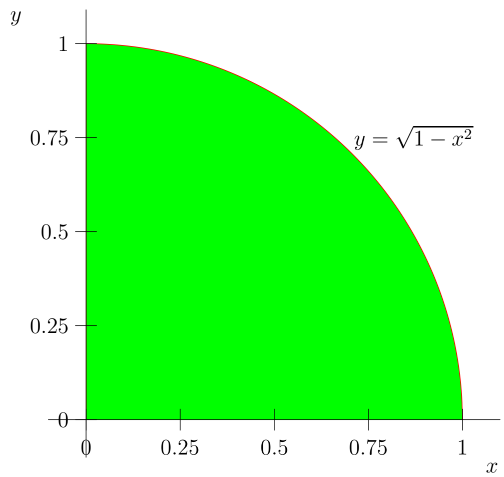

Figure 2.6.1: Region beneath $y = \sqrt{1-x^2}$ over the interval $[0, 1]$

2.6.4 Change of variable revisited

Suppose f is a continuous function on the interval $[a, b]$ and φ is either an increasing function defined on an interval $[c, d]$ with $\varphi(c) = a$ and $\varphi(d) = b$, or a decreasing function defined on $[c, d]$ with $\varphi(c) = b$ and $\varphi(d) = a$. Then, by (2.6.5), changing the notation as necessary,

$$\int_a^b f(x)dx = \int_c^d f(\varphi(z))\varphi'(z)dz. \tag{2.6.25}$$

Earlier we used (2.6.25) to simplify integrals in the form of the right-hand side; in this section we will look at some examples which simplify in the other direction.

Example 2.6.17. Since the graph of $y = \sqrt{1-x^2}$ for $0 \le x \le 1$ is one-quarter of the circle $x^2 + y^2 = 1$ (see Figure 2.6.1), we know that

$$\int_0^1 \sqrt{1-x^2}dx = \frac{\pi}{4}.$$

We will now see how to use a change of variable to evaluate this integral using the fundamental theorem. The idea is to make use of the trigonometric identity $1 - \sin^2(z) = \cos^2(z)$. That is, suppose we let $x = \sin(z)$ for $0 \le z \le \frac{\pi}{2}$. Then

$$\sqrt{1-x^2} = \sqrt{1-\sin^2(z)} = \sqrt{\cos^2(z)} = |\cos(z)| = \cos(z),$$

where the final equality follows since $\cos(z) \ge 0$ for $0 \le z \le \frac{\pi}{2}$. Now

$$dx = \cos(z)dz,$$

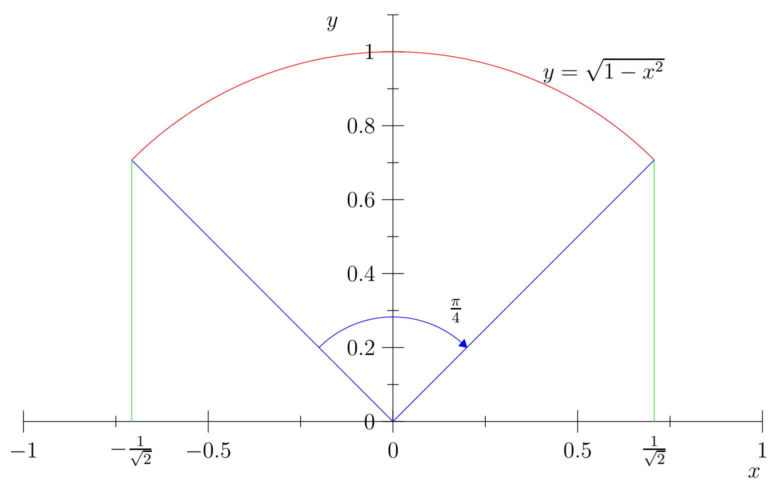

Figure 2.6.2: Arc of $x^2 + y^2 = 1$ between $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

so we have

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}dx &= \int_0^{\frac{\pi}{2}} \cos(z)\cos(z)dz \\ &= \int_0^{\frac{\pi}{2}} \cos^2(z)dz \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos(2z))dz \\ &= \frac{1}{2}z\Big|_0^{\frac{\pi}{2}} + \frac{1}{4}\sin(2z)\Big|_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4},\end{aligned}$$

as we expected.

Example 2.6.18. Let C be the circle with equation $x^2 + y^2 = 1$ and let L be the length of the shorter arc of C between $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (see Figure 2.6.2). Since the circumference of C is 2π and this arc is one-fourth of the circumference of C, we should have $L = \frac{\pi}{2}$. We will now show that this agrees with (2.5.28), the formula we derived for computing arc length. Now $y = \sqrt{1-x^2}$, so

$$\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{1-x^2}}.$$

Hence

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{1-x^2}} = \sqrt{\frac{1-x^2+x^2}{1-x^2}} = \frac{1}{\sqrt{1-x^2}}.$$

Hence, by (2.5.28),

$$L = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^2}}dx.$$

If we let

$$\begin{aligned} x &= \sin(z) \\ dx &= \cos(z)dz, \end{aligned}$$

then

$$\begin{aligned} L &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\sqrt{1-\sin^2(z)}} \cos(z)dz \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos(z)}{\sqrt{\cos^2(z)}}dz \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos(z)}{\cos(z)}dz \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dz \\ &= \frac{\pi}{2}. \end{aligned}$$

Exercise 2.6.26. Use the change of variable $x = 2\sin(z)$ to evaluate

$$\int_{-2}^{2} \sqrt{4-x^2}dx.$$

Exercise 2.6.27. Evaluate $\displaystyle\int_{-2}^{1} \frac{2}{\sqrt{16-x^2}}dx.$

Example 2.6.19. In Example 2.5.8 we saw that the arc length L of the parabola $y = x^2$ over the interval $[0, 1]$ is

$$L = \int_0^1 \sqrt{1+4x^2}dx.$$

However, at that point we did not have the means to evaluate this integral. We now have most, although not all, of the necessary tools. To begin, we will first make the change of variable

$$u = 2x$$
$$du = 2dx,$$

which gives us

$$L = \frac{1}{2}\int_0^2 \sqrt{1+u^2}du.$$

Next, we recall the trigonometric identity

$$1 + \tan^2(t) = \sec^2(t) \tag{2.6.26}$$

(a consequence of dividing each term of the identity $\cos^2(t) + \sin^2(t) = 1$ by $\cos^2(t)$), which is a hint that the change of variable

$$x = \tan(z)$$
$$dx = \sec^2(z)$$

might be of use. If we let α be the angle for which $\tan(\alpha) = 2$, with $0 < \alpha < \frac{\pi}{2}$, and note that $\tan(0) = 0$ and

$$\sqrt{1 + \tan^2(z)} = \sqrt{\sec^2(z)} = |\sec(z)| = \sec(z)$$

(note that $\sec(z) > 0$ since $0 \leq z \leq \frac{\pi}{2}$), then

$$L = \frac{1}{2}\int_0^\alpha \sec(z)\sec^2(z)dz = \frac{1}{2}\int_0^\alpha \sec^3(z)dz.$$

We may reduce the integral on the right using an integration by parts: Letting

$$u = \sec(z)dz \qquad dv = \sec^2(z)dx$$
$$du = \sec(z)\tan(z)dz \qquad v = \tan(z),$$

we have

$$\begin{aligned}\int_0^\alpha \sec^3(z)dz &= \sec(z)\tan(z)|_0^\alpha - \int_0^\alpha \sec(z)\tan^2(z)dz\\ &= \sec(\alpha)\tan(\alpha) - \int_0^\alpha \sec(z)(\sec^2(z)-1)dz\\ &= 2\sqrt{5} - \int_0^\alpha \sec^3(z)dz + \int_0^\alpha \sec(z)dz,\end{aligned}$$

where we have used the fact that $\tan(\alpha) = 2$ and $1 + \tan^2(\alpha) = \sec^2(z)$ to find that $\sec(\alpha) = \sqrt{5}$. It now follows that

$$2\int_0^\alpha \sec^3(z)dz = 2\sqrt{5} + \int_0^\alpha \sec(z)dz,$$

and so

$$\int_0^\alpha \sec^3(z)dz = \sqrt{5} + \frac{1}{2}\int_0^\alpha \sec(z)dz.$$

Hence

$$L = \frac{\sqrt{5}}{2} + \frac{1}{4}\int_0^\alpha \sec(z)dz.$$

For this reduced integral, we notice that

$$\int_0^\alpha \sec(z)dz = \int_0^\alpha \sec(z)\frac{\sec(z)+\tan(z)}{\sec(z)+\tan(z)}dz = \int_0^\alpha \frac{\sec^2(z)+\sec(z)\tan(z)}{\sec(z)+\tan(z)}dz,$$

and so the change of variable

$$\begin{aligned} w &= \sec(z) + \tan(z) \\ dw &= (\sec(z)\tan(z) + \sec^2(z))dz \end{aligned}$$

gives us

$$\int_0^\alpha \sec(z)dz = \int_1^{2+\sqrt{5}} \frac{1}{w}dw.$$

Thus we now have

$$L = \frac{\sqrt{5}}{2} + \frac{1}{4}\int_1^{2+\sqrt{5}} \frac{1}{w}dw.$$

Although greatly simplified from the integral with which we started, nevertheless we cannot evaluate the remaining integral with our current tools. Indeed, we may use the fundamental theorem of calculus to evaluate, for any rational number n, any definite integral involving w^n, except in the very case we are facing now, that is, when $n = -1$. We will fill in this gap in the next section, and finish this example at that time (see Example 2.7.9).

Example 2.6.20. For a simpler example of the change of variable used in the previous example, consider the integral

$$\int_{-1}^1 \frac{1}{1+x^2}dx,$$

the area under the curve

$$y = \frac{1}{1+x^2}$$

over the interval $[-1, 1]$ (see Figure 2.6.3). If we let

$$\begin{aligned} x &= \tan(z) \\ dx &= \sec^2(z)dz, \end{aligned}$$

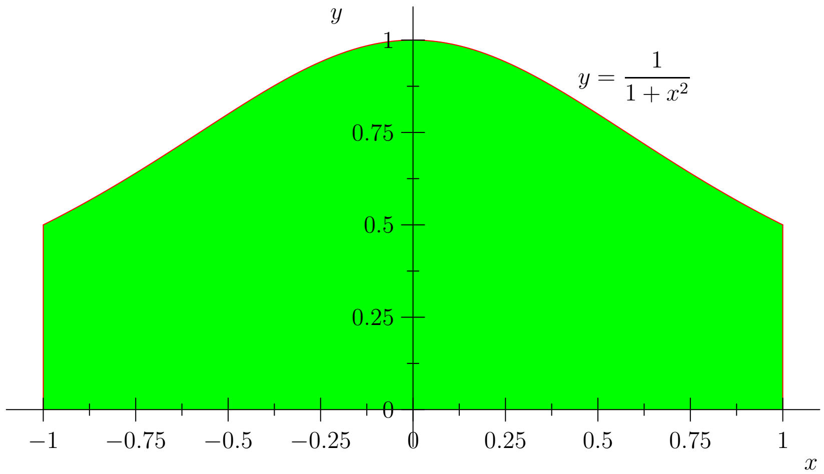

Figure 2.6.3: Region beneath $y = \frac{1}{1+x^2}$ over the interval $[-1, 1]$

and note that $\tan\left(-\frac{\pi}{4}\right) = -1$ and $\tan\left(\frac{\pi}{4}\right) = 1$, then

$$\begin{aligned}\int_{-1}^{1} \frac{1}{1+x^2} dx &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\tan^2(z)} \sec^2(z) dz \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2(z)}{\sec^2(z)} dz \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dz \\ &= \frac{\pi}{2}.\end{aligned}$$

You should compare this with the simple approximation we saw in Example 2.3.1.

Exercise 2.6.28. Evaluate $\displaystyle\int_{-3}^{3} \frac{6}{9+x^2} dx$.

Exercise 2.6.29. Evaluate $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1+4x^2} dx$.

Exercise 2.6.30. Show that for any positive integer $n > 2$,

$$\int \sec^n(x) dx = \frac{1}{n-1} \sec^{n-2}(x) \tan(x) + \frac{n-2}{n-1} \int \sec^{n-2}(x) dx.$$

2.7 The exponential and logarithm functions

There are many applications in which it is necessary to find a function y of a variable t which has the property that

$$\frac{dy}{dt} = ky \tag{2.7.1}$$

for some constant real number k. Examples include modeling the growth of certain animal populations, where y is the size of the population at time t and $k > 0$ depends on the rate at which the population is growing, and describing the decay of a radioactive substance, where y is the amount of a radioactive material present at time t and $k < 0$ depends on the rate at which the element decays. We will first consider that case $k = 1$; that is, we will look for a function $y = f(t)$ with the property that $f'(t) = f(t)$.

2.7.1 The exponential function

Suppose f is a differential function on $(-\infty, \infty)$ with the property that $f'(t) = f(t)$ for all t (one may show that such a function does indeed exist, although we will not go into the details here). Now $f'(t) = f(t)$ implies, by the fundamental theorem of calculus, that

$$\int_0^t f(x)dx = \int_0^t f'(x)dx = f(t) - f(0) \tag{2.7.2}$$

for all t. The value of $f(0$ is arbitrary; we will find it convenient to take $f(0) = 1$. That is, we are now looking for a function f which satisfies

$$f(t) = 1 + \int_0^t f(x)dx. \tag{2.7.3}$$

Suppose we divide $[0, t]$ into N subintervals of equal length $\Delta x = \frac{t}{N}$, where N is a positive integer, and let $x_0, x_1, x_2, \ldots, x_N$ be the endpoints of these intervals. Now for any $i = 1, 2, \ldots, N$, using (2.7.3),

$$\begin{aligned} f(x_i) &= 1 + \int_0^{x_i} f(x)dx \\ &= 1 + \int_0^{x_{i-1}} f(x)dx + \int_{x_{1-1}}^{x_i} f(x)dx \\ &= f(x_{i-1}) + \int_{x_{1-1}}^{x_i} f(x)dx. \end{aligned} \tag{2.7.4}$$

Moreover, for small Δx,

$$\int_{x_{1-1}}^{x_i} f(x)dx \approx f(x_{i-1})\Delta x, \tag{2.7.5}$$

and so we have

$$f(x_i) \approx f(x_{i-1}) + f(x_{i-1})\Delta x = f(x_{i-1})(1 + \Delta x). \tag{2.7.6}$$

Hence we have

$$f(x_0) = f(0) = 1, \tag{2.7.7}$$
$$f(x_1) \approx f(x_0)(1 + \Delta x) = 1 + \Delta x, \tag{2.7.8}$$
$$f(x_2) \approx f(x_1)(1 + \Delta x) \approx (1 + \Delta x)^2, \tag{2.7.9}$$
$$f(x_3) \approx f(x_2)(1 + \Delta x) \approx (1 + \Delta x)^3, \tag{2.7.10}$$
$$f(x_4) \approx f(x_3)(1 + \Delta x) \approx (1 + \Delta x)^4, \tag{2.7.11}$$
$$\vdots \qquad \vdots \qquad \vdots \tag{2.7.12}$$
$$f(x_N) \approx f(x_{N-1})(1 + \Delta x) \approx (1 + \Delta x)^N \tag{2.7.13}$$

Now $x_N = t$ and $\Delta x = \frac{t}{N}$, so we have

$$f(t) \approx \left(1 + \frac{t}{N}\right)^N. \tag{2.7.14}$$

Moreover, if we followed the same procedure with N infinite and $dx = \frac{t}{N}$, we should expect (although we have not proved)

$$f(t) \simeq \left(1 + \frac{t}{N}\right)^N. \tag{2.7.15}$$

We will let $e = f(1)$. That is,

$$e = \operatorname{sh}\left(1 + \frac{1}{N}\right)^N, \tag{2.7.16}$$

where N is any positive infinite integer. We call e *Euler's number*. Now if t is any real number, then

$$f(t) = \left(1 + \frac{t}{N}\right)^N = \left(\left(1 + \frac{1}{\frac{N}{t}}\right)^{\frac{N}{t}}\right)^t = e^t, \tag{2.7.17}$$

where we have used the fact that $\frac{N}{t}$ is infinite since N is infinite and t is finite. (Note, however, that $\frac{N}{t}$ is not an integer, as required in (2.7.16). The statement is nevertheless true, but this is a detail which we will not pursue here.) Thus the function

$$f(t) = e^t \tag{2.7.18}$$

has the property that $f'(t) = f(t)$, that is,

$$\frac{d}{dt}e^t = e^t. \tag{2.7.19}$$

In fact, one may show that $f(t) = e^t$ is the only function for which $f(0) = 1$ and $f'(t) = f(t)$. We call this function the *exponential function*, and sometimes write $\exp(t)$ for e^t.

It has been shown that e is an irrational number. Although, like π, we may not express e exactly in decimal notation, we may use (2.7.16) to find approximations, replacing the infinite N with a large finite value for N. For example, with $N = 200,000$, we find that

$$e \approx \left(1 + \frac{1}{200000}\right)^{200000} \approx 2.71828, \tag{2.7.20}$$

which is correct to 5 decimal places.

Example 2.7.1. If $f(t) = e^{5t}$, then f is the composition of $h(t) = 5t$ and $g(u) = e^u$. Hence, using the chain rule,

$$f'(t) = g'(h(t))h'(t) = e^{5t} \cdot 5 = 5e^{5t}.$$

In general, if $h(t)$ is differentiable, then, by the chain rule,

$$\frac{d}{dt}e^{h(t)} = h'(t)e^{h(t)}. \tag{2.7.21}$$

Example 2.7.2. If $f(x) = 6e^{-x^2}$, then

$$f'(x) = -12xe^{-x^2}.$$

Exercise 2.7.1. Find the derivative of $g(x) = 12e^{-7x}$.

Exercise 2.7.2. Find the derivative of $f(t) = 3t^2e^{-t}$.

Example 2.7.3. Let $f(t) = e^t$ and $g(t) = e^{-t}$. Since $e^t > 0$ for all t, we have $f'(t) = e^t > 0$ and $f''(t) = e^t > 0$ for all t, and so f is increasing on $(-\infty, \infty)$ and the graph of f is concave upward on $(-\infty, \infty)$. On the other hand, $g'(t) = -e^{-t} < 0$ and $g''(t) = e^{-t} > 0$ for all t, so g is decreasing on $(-\infty, \infty)$ and the graph of g is concave upward on $(-\infty, \infty)$. See Figure 2.7.1.

Of course, it follows from (2.7.19) that

$$\int e^t dt = e^t + c. \tag{2.7.22}$$

Example 2.7.4. From what we have seen with the examples of derivatives above, we have

$$\int_0^1 e^{-t} dt = -e^{-t}\Big|_0^1 = -e^{-1} + e^0 = 1 - e^{-1} \approx 0.6321.$$

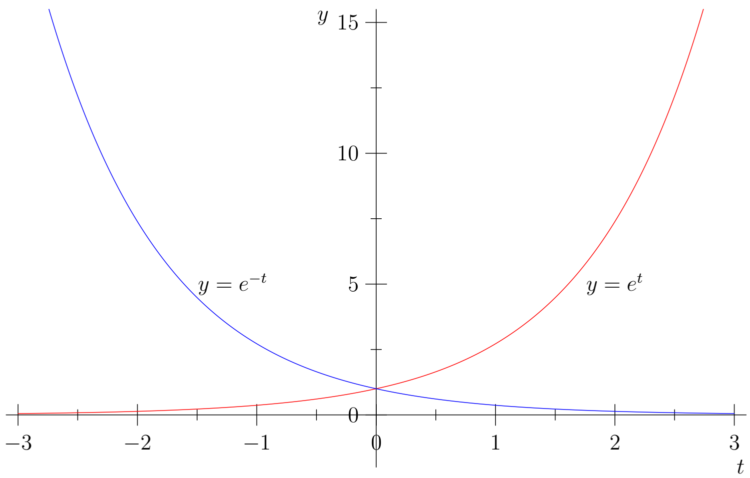

Figure 2.7.1: Graphs of $y = e^t$ and $y = e^{-t}$

Example 2.7.5. To evaluate

$$\int xe^{-x^2}dx,$$

we will use the change of variable

$$u = -x^2$$
$$du = -2xdx.$$

Then

$$\int xe^{-^2}dx = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + c = -\frac{1}{2}e^{-x^2} + c.$$

Example 2.7.6. To evaluate

$$\int xe^{-2x}dx,$$

we will use integration by parts:

$$u = x \qquad dv = e^{-2x}dx$$
$$du = dx \qquad v = -\frac{1}{2}e^{-2x}.$$

Then

$$\int xe^{-2x}dx = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c.$$

Exercise 2.7.3. Evaluate $\int_0^4 5e^{-2x}dx$.

Exercise 2.7.4. Evaluate $\int_0^1 x^2 e^{-x^3} dx$.

Exercise 2.7.5. Evaluate $\int x^2 e^{-x} dx$.

Note that if $y = ae^{kt}$, where a and k are any real constants, then

$$\frac{dy}{dt} = kae^{kt} = ky. \tag{2.7.23}$$

That is, y satisfies the *differential equation* (2.7.1) with which we began this section. We will consider example applications of this equation after a discussion of the *logarithm function*, the inverse of the exponential function.

2.7.2 The logarithm function

The *logarithm function* is the inverse of the exponential function. That is, for a positive real number x, $y = \log(x)$, read y is the logarithm of x, if and only if $e^y = x$. In particular, note that for any positive real number x,

$$e^{\log(x)} = x, \tag{2.7.24}$$

and for any real number x,

$$\log\left(e^x\right) = x. \tag{2.7.25}$$

Also, since $e^0 = 1$, it follows that $\log(1) = 0$.

Since $\log(x)$ is the power to which one must raise e in order to obtain x, logarithms inherit their basic properties from the properties of exponents. For example, for any positive real numbers x and y,

$$\log(xy) = \log(x) + \log(y) \tag{2.7.26}$$

since

$$e^{\log(x)+\log(y)} = e^{\log(x)}e^{\log(y)} = xy. \tag{2.7.27}$$

Similarly, for any positive real number x and any real number a,

$$\log\left(x^a\right) = a\log(x) \tag{2.7.28}$$

since

$$e^{a\log(x)} = \left(e^{\log(x)}\right)^a = x^a. \tag{2.7.29}$$

Exercise 2.7.6. Verify that for any positive real numbers x and y,

$$\log\left(\frac{x}{y}\right) = \log(x) - \log(y). \tag{2.7.30}$$

Note in particular that this implies that

$$\log\left(\frac{1}{y}\right) = -\log(y). \tag{2.7.31}$$

To find the derivative of the logarithm function, we first note that if $y = \log(x)$, then $e^y = x$, and so

$$\frac{d}{dx}e^y = \frac{d}{dx}x. \tag{2.7.32}$$

Applying the chain rule, it follows that

$$e^y\frac{dy}{dx} = 1. \tag{2.7.33}$$

Hence

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}. \tag{2.7.34}$$

Theorem 2.7.1. For any real number $x > 0$,

$$\frac{d}{dx}\log(x) = \frac{1}{x}. \tag{2.7.35}$$

Example 2.7.7. Since, for all $x > 0$

$$\frac{d}{dx}\log(x) = \frac{1}{x} > 0$$

and

$$\frac{d^2}{dx^2}\log(x) = -\frac{1}{x^2} < 0,$$

the function $y = \log(x)$ is increasing on $(0, \infty)$ and its graph is concave downward on $(0, \infty)$. See Figure 2.7.2.

Example 2.7.8. If $f(x) = \log(x^2 + 1)$, then, using the chain rule,

$$f'(x) = \frac{1}{x^2+1}\frac{d}{dx}(x^2+1) = \frac{2x}{x^2+1}.$$

Exercise 2.7.7. Find the derivative of $f(x) = \log(3x + 4)$.

Exercise 2.7.8. Find the derivative of $y = (x + 1)\log(x + 1)$.

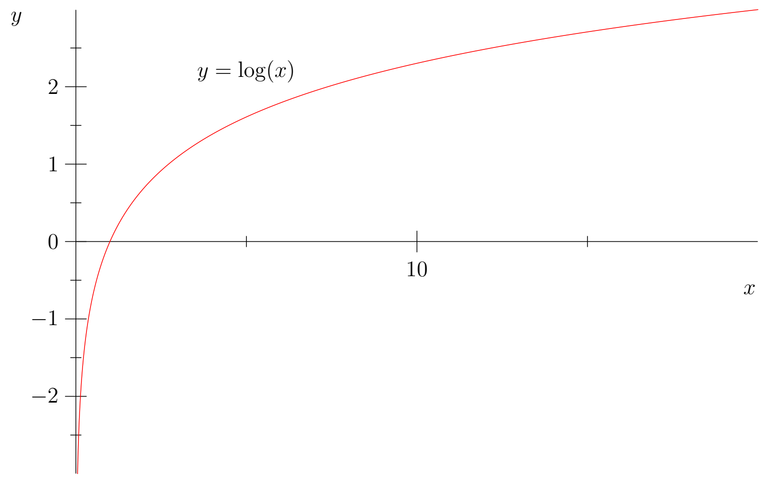

Figure 2.7.2: Graph of $y = \log(x)$

Using the fundamental theorem of calculus, it now follows that, for any $x > 0$,

$$\int_1^x \frac{1}{t} dt = \log(t)|_1^x = \log(x) - \log(1) = \log(x). \tag{2.7.36}$$

This provides a geometric interpretation of $\log(x)$ as the area under the graph of $y = \frac{1}{t}$ from 1 to x. For example, $\log(10)$ is the area under the graph of $y = \frac{1}{t}$ from 1 to 10 (see Figure 2.7.3).

Example 2.7.9. We may now complete the example, discussed in Example 2.5.8 and continued in Example 2.6.19, of finding the length L of the graph of $y = x^2$ over the interval $[0, 1]$. In those examples we found that

$$L = \int_0^1 \sqrt{1 + 4x^2} dx = \frac{\sqrt{5}}{2} + \frac{1}{4} \int_1^{2+\sqrt{5}} \frac{1}{w} dw.$$

Now we see that

$$\int_1^{2+\sqrt{5}} \frac{1}{w} dw = \log(w)|_1^{2+\sqrt{5}} = \log(2 + \sqrt{5}),$$

and so

$$L = \frac{\sqrt{5}}{2} + \frac{1}{4} \log(2 + \sqrt{5}).$$

Rounding to four decimal places, this gives us $L \approx 1.4789$, the same approximation we obtained in Example 2.5.8. Note, however, the advantage of having an exact expression for the answer: We may use the exact expression to easily approximate L to however many digits we desire, whereas we are unsure of the

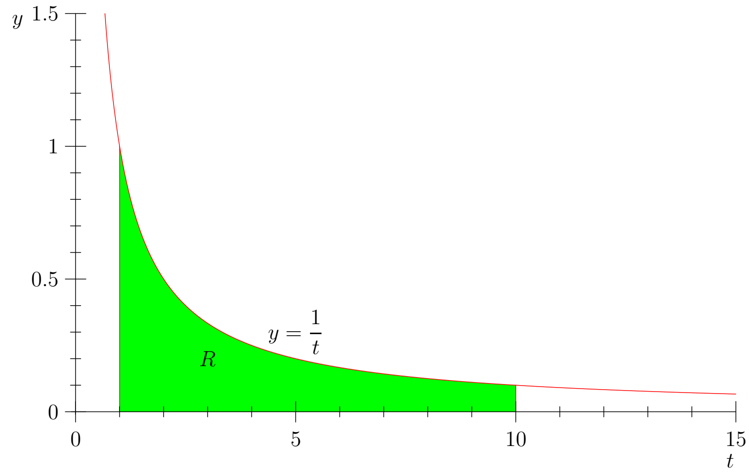

Figure 2.7.3: Area under the graph of $y = \dfrac{1}{t}$

precision of our original approximate result, and would need to recalculate the approximating sum whenever we wanted to try to improve upon our accuracy. Moreover, our procedure for finding the exact expression for L may be extended easily to find an expression for the length of any segment of the parabola $y = x^2$.

Example 2.7.10. To evaluate

$$\int_0^1 \frac{x}{1+x^2}dx,$$

we first make the change of variable

$$\begin{aligned} u &= 1 + x^2 \\ du &= 2xdx. \end{aligned}$$

Then

$$\int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\int_1^2 \frac{1}{u}du = \frac{1}{2}\log(u)\Big|_1^2 = \frac{\log(2)}{2}.$$

Example 2.7.11. Evaluating

$$\int_1^{10} \log(x)dx$$

provides an interesting application of integration by parts. If we let

$$\begin{aligned} u &= \log(x) & dv &= dx \\ du &= \frac{1}{x}dx & v &= x, \end{aligned}$$

then

$$\int_1^{10} \log(x)dx = x\log(x)|_1^{10} - \int_1^{10} dx = 10\log(10) - 9.$$

Example 2.7.12. We could use the change of variable $u = 3x + 2$ to evaluate

$$\int_0^5 \frac{4}{3x+2}dx,$$

or just make the appropriate correction for the chain rule:

$$\int_0^5 \frac{4}{3x+2}dx = \frac{4}{3}\log(3x+2)\Big|_0^5 = \frac{4}{3}(\log(17) - \log(2)) = \frac{4}{3}\log\left(\frac{17}{2}\right).$$

Exercise 2.7.9. Evaluate $\displaystyle\int_0^2 \frac{1}{x+1}dx.$

Exercise 2.7.10. Evaluate $\displaystyle\int_{-1}^2 \frac{x}{3x^2+4}dx.$

Exercise 2.7.11. Evaluate $\displaystyle\int_1^2 x\log(x)dx.$

Exercise 2.7.12. Evaluate $\displaystyle\int_{-1}^1 \sqrt{1+x^2}dx.$

Exercise 2.7.13. Evaluate $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{1+x^2}}dx.$

It is now possible to extend the power rule for differentiating x^n. Suppose $n \neq 0$ is a real number and note that, for $x > 0$,

$$x^n = e^{\log(x^n)} = e^{n\log(x)}. \tag{2.7.37}$$

Then

$$\begin{aligned}
\frac{d}{dx}x^n &= \frac{d}{dx}e^{n\log(x)} \\
&= e^{n\log(x)}\frac{d}{dx}n\log(x) \\
&= \frac{n}{x}e^{n\log(x)} \\
&= \frac{n}{x}x^n \\
&= nx^{n-1}.
\end{aligned} \tag{2.7.38}$$

Thus we have our final form of the power rule.

Theorem 2.7.2. For any real number $n \neq 0$, if $x > 0$,

$$\frac{d}{dx}x^n = nx^{n-1}. \tag{2.7.39}$$

Example 2.7.13. If $f(x) = x^\pi$, then $f'(x) = \pi x^{\pi-1}$

Exercise 2.7.14. Find

$$\frac{d}{dx}\pi^x$$

by first writing $\pi^x = e^{x\log(\pi)}$. How does this result compare with the result of the previous example?

2.7.3 Some applications

As mentioned at the beginning of this section, there are many applications in which one desires to find a function y which, for some constant k, satisfies the differential equation

$$\frac{dy}{dt} = ky. \tag{2.7.40}$$

Such an equation arises whenever the desired quantity grows, or decreases, at a rate which is proportional to its current value. As we saw above, a function of the form

$$y = \alpha e^{kt} \tag{2.7.41}$$

satisfies this equation for any real constant α. Moreover, it may be shown that any solution must be of this form.

For example, (2.7.40) is used to model radioactive decay. That is, if one begins with y_0 grams of a radioactive element and y is the amount of the element which remains after t years, then there is some constant k (which depends on the particular element being considered) for which

$$\frac{dy}{dt} = ky. \tag{2.7.42}$$

It follows that, for some real number α,

$$y = \alpha e^{kt}. \tag{2.7.43}$$

Since we are given that $y = y_0$ when $t = 0$, it follows that

$$y_0 = y(0) = \alpha e^0 = \alpha. \tag{2.7.44}$$

Hence

$$y = y_0 e^{kt}. \tag{2.7.45}$$

Now suppose $t_1 < t_2$ are such that y at time t_2 is one-half of y at time t_1. Then

$$\frac{1}{2}y_0 e^{kt_1} = y_0 e^{kt_2}. \tag{2.7.46}$$

Thus

$$e^{kt_1} = 2e^{kt_2}, \tag{2.7.47}$$

and so

$$2 = \frac{e^{kt_1}}{e^{kt_2}} = e^{k(t_1 - t_2)}. \tag{2.7.48}$$

Hence

$$\log(2) = \log\left(e^{k(t_1 - t_2)}\right) = k(t_1 - t_2), \tag{2.7.49}$$

from which it follows that

$$t_2 - t_1 = -\frac{\log(2)}{k}. \tag{2.7.50}$$

Note that the right-hand side of (2.7.50) does not depend on t_1, and so the time required for one-half of a radioactive element to decay does not depend on the initial amount of the element. We call this time the *half-life* of the element.

Typically the rate of decay of a radioactive element is expressed in terms of its half-life. From (2.7.50), we see that if the half-life of a particular element is T, then the decay rate of the element is

$$k = -\frac{\log 2}{T}. \tag{2.7.51}$$

Example 2.7.14. Carbon-14 is a naturally occurring radioactive isotope of carbon with a half-life of 5730 years. A living organism will maintain constant levels of carbon-14, which will begin to decay once the organism dies and is buried. Because of this, the amount of carbon-14 in the remains of an organism may be used to estimate its age. For example, suppose a piece of wood found buried at an archaeological site has 14% of its original carbon-14. If T is the number of years since the wood was buried and y_0 is the original amount of carbon-14 in the wood, then

$$0.14y_0 = y_0 e^{kT},$$

where, from (2.7.51),

$$k = -\frac{\log(2)}{5730}.$$

It follows that $0.14 = e^{kT}$, and so

$$kT = \log(0.14).$$

Thus

$$T = \frac{\log(0.14)}{k} = -\frac{5730 \log(0.14)}{\log(2)} \approx 16,253 \text{ years.}$$

Exercise 2.7.15. Suppose a piece of wood buried at an archaeological site has 23% of its original carbon-14. For how many years has the wood been buried?

Exercise 2.7.16. Suppose a radioactive element has a half-life of 24,065 years. How many years will it take for a given sample to decay to the point that only 10% of the original amount remains?

The differential equation (2.7.40) also serves in some situations as a simple model for population growth. Suppose, for example, that y is the size of a population of a certain species of animal over a specified habitat. In the absence of any extraneous limits on the size of the population (such as a limitation on the food supply), we would expect the rate of growth of the population to be proportional to the current population; that is, we would expect y to satisfy (2.7.40) for some constant k. In particular, if y_0 is the size of the population at some initial time and y is the size of the population t year later, then

$$y = y_0 e^{kt}. \tag{2.7.52}$$

Now suppose we know the population is y_1 at some time $t_1 > 0$. Then

$$y_1 = y_0 e^{kt_1}, \tag{2.7.53}$$

and so, after dividing through by y_0 and taking the logarithm of both sides,

$$k = \frac{1}{t_1} \log\left(\frac{y_1}{y_0}\right). \tag{2.7.54}$$

Example 2.7.15. Suppose that a certain habitat initially holds a population of 1000 deer, and that five years later the population has grown to 1200 deer. If we let y be the size of the population after t years, and assuming no constraints on the growth of the population, we would have

$$y = 1000e^{kt},$$

where

$$k = \frac{1}{5} \log\left(\frac{1200}{1000}\right) = \frac{\log(1.2)}{5}.$$

Then, for example, this model would predict a population of

$$\begin{aligned} y(10) &= 1000e^{10k} \\ &= 1000e^{2\log(1.2)} \\ &= 1000e^{\log\left(1.2^2\right)} \\ &= (1000)(1.2)^2 \\ &= 1440 \text{ deer} \end{aligned}$$

after five more years.

If y is the size of a population of animals, we call the differential equation

$$\frac{dy}{dt} = ky, \tag{2.7.55}$$

and the resulting solution,

$$y = y_0 e^{kt}, \tag{2.7.56}$$

the *natural growth model*. Although often relatively accurate over small time intervals, this model is clearly unrealistic for any extended period of time as it predicts an unbounded growth. Even in the best of situations, other factors, such as availability of food and shelter, will eventually come into play.

The *logistic model* introduces a variation in the basic model (2.7.40) which factors in the limiting effect of space and food. In this case, we suppose that there is an upper bound, say M, to the size of the population which the habitat can sustain and that the rate of growth of the population decreases as the size of the population approaches this limiting value. More precisely, let k be the natural rate of growth of the population, that is, the rate of growth if there are no constraining factors, or when the size of the population is small compared with M, and let y be the size of the population at time t. Then we suppose k is decreased by a factor of $1 - \frac{y}{M}$, that is, the proportion of room left for growth. The resulting differential equation for the logistic model is

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right) = \frac{k}{M}y(M-y) = \beta y(M-y), \tag{2.7.57}$$

where $\beta = \frac{k}{M}$.

To solve (2.7.57), we first rewrite it using s for the independent variable, that is, as

$$\frac{dy}{ds} = \beta y(M-y), \tag{2.7.58}$$

and then divide through both sides by $y(M-y)$ to obtain

$$\frac{1}{y(M-y)}\frac{dy}{ds} = \beta. \tag{2.7.59}$$

Next, we integrate both sides of this equation from 0 to some fixed time t:

$$\int_0^t \frac{1}{y(M-y)}\frac{dy}{ds}ds = \int_0^t \beta ds. \tag{2.7.60}$$

For the right hand side of (2.7.60), we have simply

$$\int_0^t \beta ds = \beta t. \tag{2.7.61}$$

For the left-hand side, we begin with the change of variable

$$\begin{aligned} u &= y \\ du &= \frac{dy}{ds}ds, \end{aligned} \tag{2.7.62}$$

from which we obtain

$$\int_0^t \frac{1}{y(M-y)}\frac{dy}{ds}ds = \int_{y_0}^{y} \frac{1}{u(M-u)}du, \tag{2.7.63}$$

where we have again used y_0 to denote the size of the population at time $s = 0$ (and noting that y is the size of the population when $s = t$). We assume that $0 < y_0 < M$ and $0 < y < M$; that is, we assume that the populations involved are positive and do not exceed the maximum sustainable population.

To evaluate (2.7.63), we rely upon a result involving what are known as *partial fraction decompositions*: There exist real numbers A and B such that

$$\frac{1}{u(M-u)} = \frac{A}{u} + \frac{B}{M-u}. \tag{2.7.64}$$

To find A and B, we note that (2.7.64) implies that

$$\frac{1}{u(M-u)} = \frac{A(M-u)}{u(M-u)} + \frac{Bu}{u(M-u)} = \frac{A(M-u)+Bu}{u(M-u)}. \tag{2.7.65}$$

It follows that, for all values of u,

$$1 = A(M-u) + Bu. \tag{2.7.66}$$

In particular, when $u = 0$ we have $1 = AM$, and when $u = M$ we have $1 = BM$. Hence

$$A = \frac{1}{M} \text{ and } B = \frac{1}{M}, \tag{2.7.67}$$

and so

$$\frac{1}{u(M-u)} = \frac{1}{M}\frac{1}{u} + \frac{1}{M}\frac{1}{M-u}. \tag{2.7.68}$$

Hence we now have

$$\begin{aligned}
\int_{y_0}^{y} \frac{1}{u(M-u)}du &= \frac{1}{M}\int_{y_0}^{y}\frac{1}{u}du + \frac{1}{M}\int_{y_0}^{y}\frac{1}{M-u}du \\
&= \frac{1}{M}\log(u)\Big|_{y_0}^{y} - \frac{1}{M}\log(M-u)\Big|_{y_0}^{y} \\
&= \frac{1}{M}(\log(y) - \log(y_0) - \log(M-y) \\
&\qquad + \log(M-y_0)) \\
&= \frac{1}{M}\log\left(\frac{y(M-y_0)}{y_0(M-y)}\right).
\end{aligned} \tag{2.7.69}$$

Combining (2.7.60), (2.7.61), and (2.7.69), we have

$$\beta M t = \log\left(\frac{y(M-y_0)}{y_0(M-y)}\right), \tag{2.7.70}$$

which now need to solve for y. To begin, exponentiate both sides to obtain

$$e^{\beta M t} = \frac{y(M-y_0)}{y_0(M-y)}. \tag{2.7.71}$$

It follows that

$$y(M-y_0)=e^{\beta M t}y_0(M-y)=y_0Me^{\beta M t}-y_0ye^{\beta M t}, \tag{2.7.72}$$

and so

$$y_0Me^{\beta M t}=y(M-y_0)+y_0ye^{\beta M t}=(M+y_0(e^{\beta M t}-1))y. \tag{2.7.73}$$

Thus

$$y=\frac{y_0Me^{\beta M t}}{(M+y_0(e^{\beta M t}-1))}. \tag{2.7.74}$$

If we divide the numerator and denominator of the right-hand side by $e^{\beta M t}$, we have

$$y=\frac{y_0M}{Me^{-\beta M t}+y_0-y_0e^{-\beta M t}}, \tag{2.7.75}$$

from which we obtain our final form

$$y=\frac{y_0M}{y_0+(M-y_0)e^{-\beta M t}}. \tag{2.7.76}$$

Note that when $t=0$, (2.7.76) reduces to $y=y_0$, as it should given our initial condition, and when t is infinite, $e^{-\beta M t}\simeq 0$, and so $y=M$. Hence $y_0\le y<M$ for all t, with y approaching M as t grows.

Recalling that $\beta=\frac{k}{M}$, where k is the natural rate of growth of the population, we may rewrite (2.7.76) as

$$y=\frac{y_0M}{y_0+(M-y_0)e^{-kt}}. \tag{2.7.77}$$

Example 2.7.16. In our previous example, where y represented the number of deer in a certain habitat after t years, we found

$$k=\frac{\log(1.2)}{5}.$$

Now suppose the habitat can support no more than $20,000$ deer. Then the logistic model would give us

$$z=\frac{(1000)(20000)}{1000+(20000-1000)e^{kt}}=\frac{20000}{1+19e^{-kt}}$$

for the number of deer after t years. After ten years, this model would predict a population of

$$z(10)=\frac{20000}{1+19e^{-10k}}\approx 1409 \text{ deer},$$

only slightly less than the 1440 predicted by the natural growth model. However, the differences between the two models become more pronounced with time. For example, after forty years, the natural growth model predicts

$$y(40)=1000e^{40k}\approx 4300 \text{ deer},$$

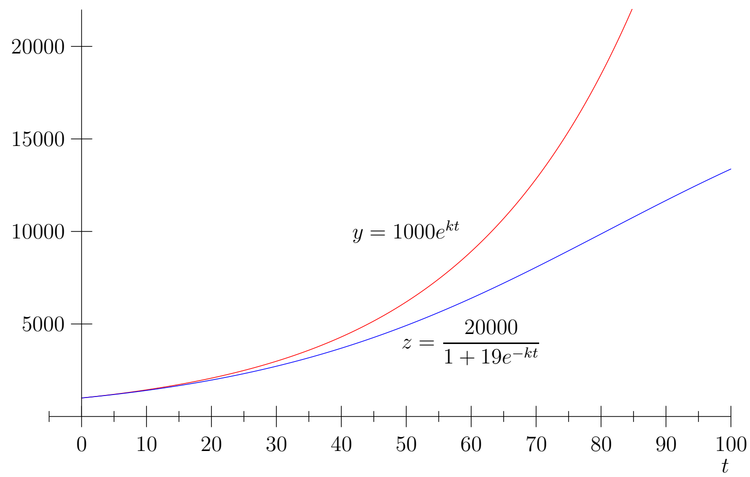

Figure 2.7.4: Comparison of natural and logistic growth models

while the logistic model predicts

$$z(40) = \frac{20000}{1 + 19e^{-40k}} \approx 3691 \text{ deer}$$

and in 100 years, the natural growth model predicts

$$y(100) = 1000e^{100k} \approx 38338\text{deer},$$

while the logistic model predicts only

$$z(100) = \frac{20000}{1 + 19e^{-100k}} \approx 13373 \text{ deer}.$$

Of course, over time the natural growth model predicts a population which grows without any bound, whereas the logistic model predicts that the population, while always increasing, will never surpass $20,000$. See Figure 2.7.4.

Exercise 2.7.17. Suppose a certain population of otters grows from 500 to 600 in 4 years. Using a natural growth model, predict how many years it will take for the population to double.

Exercise 2.7.18. Suppose habitat of the otters in the previous exercise can support no more than 1500 otters. Using a logistic model, and the natural rate of growth found in the previous exercise, predict how many years it will take for the population to double.

In evaluating (2.7.63) we made use of a partial fraction decomposition. More generally, suppose p and q are polynomials, the degree of p is less than the degree of q, and q factors completely into distinct linear factors, say,

$$q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n). \tag{2.7.78}$$

Then it may be shown that there exist consants A_1, A_2, ..., A_n for which

$$\frac{p(x)}{q(x)} = \frac{A_1}{a_1x + b_2} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nx + b_n}. \tag{2.7.79}$$

The evaluation of

$$\int_a^b \frac{p(x)}{q(x)}dx,$$

for any real numbers a and b for which $q(x) \neq 0$ for all x in $[a, b]$, then follows easily.

Example 2.7.17. To evaluate

$$\int_0^1 \frac{x}{x^2 - 4}dx,$$

we first note that, since $x^2 - 4 = (x - 2)(x + 2)$, there exist constants A and B for which

$$\frac{x}{x^2 - 4} = \frac{A}{x - 2} + \frac{B}{x + 2}.$$

It follows that

$$\frac{x}{x^2 - 4} = \frac{A(x + 2) + B(x - 2)}{x^2 - 4},$$

and so

$$x = A(x + 2) + B(x - 2)$$

for all values of x. In particular, when $x = 2$ we have $2 = 4A$, and when $x = -2$ we have $-2 = -4B$. Hence $A = \frac{1}{2}$ and $B = \frac{1}{2}$, and so

$$\frac{x}{x^2 - 4} = \frac{1}{2}\frac{1}{x - 2} + \frac{1}{2}\frac{1}{x + 2}.$$

And so we have

$$\int_0^1 \frac{x}{x^2 - 4}dx = \frac{1}{2}\int_0^1 \frac{1}{x - 2}dx + \frac{1}{2}\int_0^1 \frac{1}{x + 2}dx.$$

Now

$$\frac{1}{2}\int_0^1 \frac{1}{x + 2}dx = \frac{1}{2}\log(x + 2)\bigg|_0^1 = \frac{1}{2}(\log(3) - \log(2)),$$

but the first integral requires a bit more care because $x - 2 < 0$ for $0 \leq x \leq 1$. If we make the change of variables,

$$\begin{aligned} u &= -(x - 2) \\ du &= -dx, \end{aligned}$$

then, since $x - 2 = -u$,

$$\begin{aligned}\frac{1}{2}\int_0^1 \frac{1}{x-2}dx &= \frac{1}{2}\int_2^1 \frac{1}{u}du \\ &= -\frac{1}{2}\int_1^2 \frac{1}{u}du \\ &= -\left.\frac{1}{2}\log(u)\right|_1^2 \\ &= -\frac{1}{2}\log(2).\end{aligned}$$

Hence

$$\int_0^1 \frac{x}{x^2-1}dx = \frac{1}{2}(\log(3) - \log(2)) - \frac{1}{2}\log(2) = \frac{1}{2}\log(3) - \log(2).$$

Exercise 2.7.19. Evaluate $\displaystyle\int_{-2}^2 \frac{1}{9-x^2}dx$.

Exercise 2.7.20. Evaluate $\displaystyle\int_0^1 \frac{x+4}{x^2+3x+2}dx$.

Answers to Exercises

1.2.1. (a) 32 feet per second, (b) 64 feet per second, (c) 40 feet per second

1.2.2. $v_{[1,1+\Delta t]} = 32 + 16\Delta t$ feet per second

1.2.3. $v_{[1,1+dt]} = 32 + 16dt$ feet per second, $v(1) = 32$ feet per second

1.2.4. $v(2) = 64$ feet per second

1.2.5. $v(t) = 32t$ feet per second

1.3.1. Let $a = r_1 + \epsilon_1$ and $b = r_2 + \epsilon_2$, where r_1 and r_2 are real numbers and ϵ_1 and ϵ_2 are infinitesimals. Note that $a + b = (r_1 + r_2) + (\epsilon_1 + \epsilon_2)$ and $ab = r_1 r_2 + (r_1\epsilon_2 + r_2\epsilon_1 + \epsilon_1\epsilon_2)$.

1.3.2. Let $a = r + \epsilon$, where $r \neq 0$ is a real number and ϵ is an infinitesimal. Note that

$$\frac{1}{a} = \frac{1}{r} + \frac{\epsilon}{r(r+\epsilon)}.$$

1.5.2. $(-\infty, -1)$ and $(-1, \infty)$

1.5.5. $(-\infty, 0)$ and $(0, \infty)$

1.5.6. $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$

1.6.1. $\dfrac{dy}{dx} = 5$

1.6.2. $\dfrac{dy}{dx} = 3x^2$

1.6.3. $f'(x) = 8x$

1.7.1. $\dfrac{dy}{dx} = 2x$

1.7.2. $f'(x) = \dfrac{1}{2\sqrt{x}} - 2x$

1.7.3. $\dfrac{dy}{dx} = 16x$

1.7.4. $f'(x) = \dfrac{2}{\sqrt{x}}$

1.7.5. $\dfrac{dy}{dx} = 65x^4$

1.7.6. $f'(x) = 20x^3 - 6x$

1.7.7. $\dfrac{dy}{dx} = 21x^6 - 3$

1.7.8. $f'(x) = 15x^4 - 24x^3 - 10x$

1.7.9. $\dfrac{dy}{dx} = \dfrac{42 - 168x^2}{(4x^3 - 3x)^2}.$

1.7.10. $f'(x)\dfrac{4x^4 - 60x^2 + 2x}{(x^2 - 5)^2}$

1.7.11. $\left.\frac{dy}{dx}\right|_{x=1} = 216$

1.7.12. $f'(x) = \frac{2}{\sqrt{4x+6}}$

1.7.13. $\frac{dy}{dx} = 20x(x^2+5)^9$

1.7.14. $\left.\frac{dA}{dt}\right|_{r=100} = 400\pi \text{ cm}^2/\text{sec}$

1.7.15. $\left.\frac{dr}{dt}\right|_{r=15} = \frac{1}{9\pi}$ centimeters per second

1.7.16. $f'(x) = \frac{4}{x^{\frac{1}{5}}}$

1.7.17. $\frac{dy}{dx} = -\frac{4x}{(x^2+4)^{\frac{3}{2}}}$

1.7.18.

$$f'(x) = 12(x^2+3x-5)^{10}(3x^4-6x+4)^{11}(12x^3-6)+ \\ 10(x^2+3x-5)^9(3x^4-6x+4)^{12}(2x+3)$$

1.7.19. $\frac{dy}{dt} = -3\sin(3t+6)$
$\frac{dw}{dt} = -8\sin^2(t)\cos(4t)\sin(4t) + 2\sin(t)\cos(t)\cos^2(4t).$

1.7.21. $\frac{dy}{dx} = 6\sec^2(3t)\tan(3t)$

1.7.22. $f'(t) = 6\tan(3t)\sec^2(3t)$

1.8.1. $y = 90(x-2) + 39$

1.8.2. $y = 3\left(t - \frac{\pi}{4}\right) + \frac{3}{2}$

1.9.1. f is increasing on $(-\infty, -\sqrt{2}]$ and on $[\sqrt{2}, \infty]$, decreasing on $[-\sqrt{2}, \sqrt{2}]$; f has a local maximum of $4\sqrt{2}$ at $x = -\sqrt{2}$ and a local minimum of $-4\sqrt{2}$ at $x = \sqrt{2}$.

1.9.2. f is increasing on $[-1, 0]$ and on $[0, 1]$ (or, simply, $[-1, 1]$; f is decreasing on $(-\infty, -1]$ and on $[1, \infty)$; f has a local maximum of 2 at $x = 1$ and a local minimum of -2 at $x = -1$.

1.9.3. f is increasing on all intervals of the form $[-n\pi, n\pi]$, where $n = 1, 2, \ldots$; f has no local maximums or minimums (indeed, f is increasing on $(-\infty, \infty)$).

1.10.1. Maximum value of 20 at $x = 4$; minimum value of 12 at $x = 2$

1.10.2. Maximum value of 3.4840 at $t = \frac{5\pi}{6}$; minimum value of -0.3424 at $t = \frac{\pi}{6}$

1.10.3. R is $\sqrt{2}a$ by $\sqrt{2}b$

1.10.4. (a) all the wire is used for the circle; (b) 43.99 cm use for the square, 56.01 cm used for the circle

1.10.6. $(1, 1)$

1.10.9. (a) $\left(\frac{4\sqrt{2}}{3}, \frac{4}{3}\right)$ and $\left(-\frac{4\sqrt{2}}{3}, \frac{4}{3}\right)$ (b) $(0, -4)$

1.11.1. $\frac{dy}{dx} = \frac{x - 4y}{4x + y}$

1.11.2. $\left.\dfrac{dy}{dx}\right|_{(x,y)=(2,-1)} = -\dfrac{7}{2}$

1.11.3. $y = -\dfrac{3}{4}(x-3)+4.$

1.11.4. $y = -\dfrac{7}{8}(x-2)+3$

1.11.5. $\dfrac{1}{2}$ cm/sec

1.11.6. 234.26 miles/hour

1.11.7. 64 cm^2/sec

1.12.1. $\dfrac{dy}{dx} = 2\cos(2x),\ \dfrac{d^2y}{dx^2} = -4\sin(2x),\ \dfrac{d^3y}{dx^3} = -8\cos(2x)$

1.12.2. $f'(x) = \dfrac{2}{\sqrt{4x+1}},\ f''(x) = -\dfrac{4}{(4x+1)^{\frac{3}{2}}},\ f'''(x) = \dfrac{6}{(4x+1)^{\frac{5}{2}}}$

1.12.3. 69.79 cm/sec

1.12.4. Concave upward on $(-\infty,-1)$ and $(0,1)$; concave downward on $(-1,0)$ and $(1,\infty)$; Points of inflection: $(-1,-2)$, $(0,0)$, $(1,2)$

1.12.5. Local maximum of -2 at $x=-1$; local minimum of 2 at $x=1$

1.12.6. Local maximum of 2 at $t=-1$; local minimum of -2 at $t=1$

2.1.1. (a) $\displaystyle\int (x^2+3)dx = \frac{1}{3}x^3+3x+c$

(b) $\displaystyle\int \frac{1}{x^2}\,dx = -\frac{1}{x}+c$

(c) $\int (3\sin(x) - 5\sec(x)\tan(x))dx = -3\cos(x) - 5\sec(x) + c$

(d) $\int 4\sqrt{x}\ dx = 6x^{\frac{3}{2}} + c$

2.1.2. $F(x) = x^5 - 2x^2 - 12$

2.1.3. $x(t) = -10\cos(t) + 20$

2.1.4. 64.20 meters

2.2.1. (a) 30.11 centimeters, (b) 29.95 centimeters

2.4.1. $\int_0^1 x^4 dx = \frac{1}{5}$

2.4.2. $\int_0^\pi \sin(x) dx = 2$

2.4.3. 20 centimeters

2.5.1. $\frac{1}{6}$

2.5.2. $\frac{8}{3}$

2.5.3. $\frac{1}{6}$

2.5.4. 0

2.5.5. $\frac{3}{2}$

2.5.9. $\frac{8\pi}{3}$

2.5.10. $\frac{2\pi}{3}$

2.5.11. $\frac{\pi}{6}$

2.5.12. $\frac{16-4\sqrt{2}}{3}$

2.5.13. 3.8153

2.6.1. $\int 3x^2\sqrt{1+x^3}\ dx = \frac{2}{3}(1+x^3)^{\frac{3}{2}} + c$

2.6.2. $\int x\sqrt{4+3x^2}\ dx = \frac{1}{9}(4+3x^2)^{\frac{3}{2}} + c$

2.6.3. $\int \sec^2(4x)\tan^2(4x)dx = \frac{1}{12}\tan^3(4x) + c$

2.6.4. $\int_0^2 \frac{x}{\sqrt{4+x^2}}\ dx = 2\sqrt{2} - 2$

2.6.5. $\int_0^{\frac{\pi}{3}} \sin(3x)dx = \frac{2}{3}$

2.6.6. $\int_0^{\frac{\pi}{4}} \sin^4(2x)\cos(2x)dx = \frac{1}{10}$

2.6.7. $\int x\sin(2x)dx = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + c$

2.6.8. $\int x^2 \cos(3x)dx = \frac{1}{3}x^2 \sin(3x) + \frac{2}{9}x\cos(3x) - \frac{2}{27}\sin(3x) + c$

2.6.9. $\int_0^{\pi} x\cos\left(\frac{1}{2}x\right) dx = 2\pi - 4$

2.6.10. $\int_0^{\frac{\pi}{2}} 3x^2 \cos(x^2)dx = -6\pi$

2.6.11. $\int_0^2 x^2\sqrt{1+x}\ dx = \frac{264\sqrt{3}-16}{105}$

2.6.12. $\int_0^{\pi} \sin^2(2x)dx = \frac{\pi}{2}$

2.6.13. $\int_0^{\pi} \cos^2(3x)dx = \frac{\pi}{2}$

2.6.14. $\int \cos^4(x)dx = \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c$

2.6.15. $\int_0^{\pi} \sin^8(x)dx = \frac{35\pi}{128}$

2.6.16. $\int_0^{\pi} \sin^7(x)dx = \frac{32}{35}$

2.6.18. $\int_0^{\pi} \cos^6(x)dx = \frac{5\pi}{16}$

2.6.20. $\int_0^{\frac{\pi}{4}} \cos^5(2x)dx = \frac{4}{15}$

2.6.21. $\int_0^{\frac{\pi}{2}} \sin(2x)\sin(x)dx = -\frac{1}{3}$

2.6.22. $\int_0^{\frac{\pi}{2}} \sin(x)\sin(2x)dx = \frac{2}{3}$

2.6.23. $\int_0^{\frac{\pi}{2}} \sin(3x)\cos(3x)dx = \frac{1}{6}$

2.6.24. $\int_0^{\frac{\pi}{2}} \cos(x)\cos(2x)dx = \frac{1}{3}$

2.6.26. $\int_{-2}^{2} \sqrt{4-x^2}dx = 2\pi$

2.6.27. $\int_{-2}^{2} \frac{2}{\sqrt{16-x^2}}dx = \frac{\pi}{6}$

2.6.28. $\int_{-3}^{3} \frac{6}{9+x^2}dx = \pi$

2.6.29. $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1+4x^2}dx = \frac{\pi}{4}$

2.7.1. $g'(x) = -84e^{-7x}$.

2.7.2. $f'(t) = (6t-3t^2)e^{-t}$

2.7.3. $\int_0^4 5e^{-2x}dx = \frac{5}{2} - \frac{5}{2}e^{-8} \approx 2.4992$

2.7.4. $\int_0^1 x^2e^{-x^3}dx = \frac{1}{3}(1-e) \approx 0.2107$

2.7.5. $\int x^2e^{-x}dx = -2e^{-x} - 2xe^{-x} - x^2e^{-x} + c$

2.7.7. $f'(x) = \frac{3}{3x+4}$

2.7.8. $\frac{dy}{dx} = 1 + \log(x+1)$

2.7.9. $\int_0^2 \frac{1}{x+1} dx = \log(3)$

2.7.10. $\int_{-1}^2 \frac{x}{3x^2+4} dx = \frac{1}{6} \log\left(\frac{16}{7}\right)$

2.7.11. $\int_1^2 x \log(x) dx = 2\log(2) - \frac{3}{4}$

2.7.12. $\int_{-1}^1 \sqrt{1+x^2} dx = \sqrt{2} + \log(1+\sqrt{2})$

2.7.13. $\int_{-1}^1 \frac{1}{\sqrt{1+x^2}} dx = 2\log(1+\sqrt{2})$

2.7.14. $\frac{d}{dx}\pi^x = (\log(\pi))\pi^x$

2.7.15. $12,149$ years

2.7.16. $79,942$ years

2.7.17. 15 years

2.7.18. 31 years

2.7.19. $\int_{-2}^2 \frac{1}{9-x^2} dx = \frac{1}{3} \log(5)$

2.7.20. $\int_0^1 \frac{x+14}{x^2+3x+2} dx = 5\log(2) - 2\log(3)$

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