Euclidean Geometry

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Reviewer: Prof Jeremy Gray, Open University

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Introduction

As part of the work of the **sigma**-funded Fine Art Maths Centre at Central Saint Martins, we have devised a series of geometry workshop courses that make little or no demands as to prerequisites and which are, in most cases, led by practical construction rather than calculation. This booklet and its accompanying resources on Euclidean Geometry represent the first FAMC course to be 'written up'.

We have taught the material in a Fine Art setting, but it could be adapted with little difficulty for Design or Arts and Humanities students; some of it was first tried out in public "drop-in" sessions we ran out of a pub and later a café from 2012 to2014. Our approach is also suitable for those who may previously have had bad experiences with mathematics: algebra and equations are kept to a minimum and could be eliminated.

If you're a maths teacher interested in reaching students whose main interests are artistic, cultural or philosophical we hope you'll find something of value here whether you run the course "as is" or adapt it heavily to suit your own students. Many of the students who follow this one go on to our course *Perspective & the Geometry of Vision*, although it is not a prerequisite.

If you're a student we hope there's enough information here and in the online resources to get you started with Euclidean geometry. Learning almost anything is easier with a good instructor but sometimes we must manage on our own. This book does contain "spoilers" in the form of solutions to problems that are often presented directly after the problems themselves – if possible, try to figure out each problem on your own before peeking.

We're aware that Euclidean geometry isn't a standard part of a mathematics degree, much less any other undergraduate programme, so instructors may need to be reminded about some of the material here, or indeed to learn it for the first time. We've therefore addressed most of our remarks to an intelligent, curious reader who is unfamiliar with the subject.

Experts will, of course, find they can skim over parts that neophytes may need to take slowly. Likewise, some of our remarks are obviously directed to teachers and few of the more wide-reaching examples are boxed off: those with limited mathematical background can safely ignore them.

We assume that students following the course have no formal mathematical training beyond basic arithmetic. The level of prior maths study seems, in our experience, to be a fairly poor predictor of how well a student will cope with their first meeting with Euclidean geometry. Our aim is not to send students away with a large repertoire of theorems, proofs or techniques. Instead we focus persistently on what we think are the important general ideas and skills. In particular, the construction and understanding of careful proofs is given centre stage.

What's in this Booklet?

We begin with some remarks connecting our subject with areas that arts and humanities students probably know about and are interested in. Partly this makes for good motivation, and helps the subject seem less like a "maths course" that stands apart from everything else they're doing; after all, we know it's pedagogically better to connect new learning with prior knowledge and expertise. We would also like to grind our own axe a little here: mathematics can be considered an arts or humanities subject in itself, and its close connections with philosophy and cultural production testify to that. We've included some contentious points that should spark debate.

We then outline six "sessions". When we run the course these are usually run as discrete, two-hour weekly lessons. We cover much of the same material in an extra-mural class "Geometry Through Drawing" in two intensive days. There's more here than we ever cover in a single presentation in either format, so choices must be made based on the students' and presenters' priorities.

We end with an annotated bibliography and a list of online resources.

This booklet and its sister, *Perspective and the Geometry of Vision*, were made possible by a generous grant from **sigma**, the network for excellence in mathematics and statistics support. www.sigma-network.ac.uk

To the Student

Mathematics isn't quite like any other subject you may have studied before. What's more, studying this kind of maths probably isn't going to be similar to the experience you may have had with the subject at school. A short course is, we hope, just the beginning of a long-term process of absorbing, struggling with and thinking about mathematical ideas. Like Sophie Germain, a self-taught French mathematician, you may have to do much of this on your own. Part of what a course like this can do is equip you, at least a little bit, for that journey.

Studying maths at school is mostly about learning to pass a sequence of standardised tests as part of a process of social initiation. Most mathematicians will admit that some useful skills are acquired in the process, but many will also stress that the subject as it's presented in school bears very little resemblance to what excites them, or to what they do all day. Some very successful mathematicians weren't good at the subject at school. Often there's a story about an inspiring teacher or book that led the young student down an alternative path and gave them an insight into the "good stuff" that most teachers aren't really supposed to spend the class's time on. Euclidean geometry can be this "good stuff" if it strikes you in the right way at the right moment.

Maths is a very odd activity. Here's how Andrew Wiles, who proved Fermat's Last Theorem, described the process:

> Perhaps I can best describe my experience of doing mathematics in terms of a journey through a dark unexplored mansion. You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture, but gradually you learn where each piece of furniture is. Finally after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were. Then you move into the next room and spend another six months in the dark. So each of these breakthroughs, while sometimes they're momentary, sometimes over a period of a day or two, they are the culmination of—and couldn't exist without—the many months of stumbling around in the dark that precede them. (Quoted in William Byers, *How Mathematicians Think*, p. 1, and in many other places.)

The key thing is that the experience Wiles is describing is that *most of the time you're confused and lost*. If it feels that way for him, you can expect it to feel that way for you too. Although the maths he's talking about is very high-powered, the process really does feel like that for most of us, at least when we're doing it right, even when we're trying to learn something Wiles can do in his sleep.

Don't be misled by well-meaning people who say learning maths is a linear process that must proceed in tiny steps, in a prescribed order, mastering each one before moving onto the next. On the contrary, we blunder around, confused, reading things out of sequence and going repeatedly back to them, slowly forming a picture of what's going on and how it connects to other things we know. We try things – whenever we can at Central Saint Martins, we *make* or *draw* things! – and gradually the fog of confusion clears, until the thing that once seemed impossible to understand becomes so obvious we've forgotten it was ever difficult.

To the Teacher

The adjective "Euclidean" is supposed to conjure up an attitude or outlook rather than anything more specific: the course is not a course on the *Elements* but a wide-ranging and (we hope) interesting introduction to a selection of topics in synthetic plane geometry, with the construction of the regular pentagon taken as our culminating problem.

We like to teach this material, as far as possible, through practical drawing, on the principle that a construction can be a perfectly good proof in itself. It isn't usually necessary to supplement this by running through the same proof symbolically unless some students find it helps them understood what they've been doing. There are a few proofs, such as Thales' Theorem, that we do "on the board" but we stress that in these cases that following the details of the proof is optional.

Obviously, drawing and making are fun and can be hilariously difficult, which is all to the good. When we say that students should figure something out for themselves, we almost always take this to mean working collaboratively in pairs or in whatever *ad hoc* groupings they like; occasionally a student prefers to work alone, and that's usually OK too. The middle two sessions are deliberately "lighter" and give students a chance to get used to the basic constructions.

This requires some emphasis up-front on what a construction is and how it differs from, for example, a representative drawing or a diagram. It's a good idea to have students explain the steps in their construction when a particular problem has been solved. Having one student give instructions, or explain their thinking, to another is often helpful.

Arts and humanities students can be surprisingly challenging maths students. They often ask very awkward questions or develop misunderstandings that are unexpected and rather profound. This is why we've taken time to explain some apparently simple matters in detail, especially in the early sessions. Your maths and physics students probably won't worry about what a circle is or whether two lines truly intersect at a single point, but your arts and humanities students might.

Toolkit

The main activity in geometry class will be drawing, so we need some equipment. In fact, we won't need much: just a compass, a "straightedge", some pencils and some blank paper. Large pads of paper are good – we like to use A2 although that can be a bit unwieldy to carry around. You can get started with any old school compass, a ruler, a pencil and a scrap of paper but a small investment in better tools makes the whole experience more pleasant.

The compasses we use most often are Rotring Master Bow Compasses like the one pictured on the left; they're quite cheap and work well for most jobs. By spending a little more one can get the kind with an extension bar that allows for bigger circles – helpful but not essential. It's very useful to have a small compass for fiddly work, too. An attachment that will take a pen opens up some options for finishing a drawing or bringing out important elements; only the very bravest will attempt to use a bottle of ink and a ruling pen, though many compasses come with a suitable attachment.

A "straightedge" just means any tool you can use to make a straight line. You may as well use a ruler, although we emphasise from the start that any markings on it (inches, centimetres and so on) are to be ignored. We haven't found an affordable supply of plain metal strips that are well-made enough to provide a good edge: if we did we'd give those out at the start of the course. Metal is nice to work with but plastic can be helpful when you want to be able to see underneath it as you work. An ordinary 30cm (1 foot) one is absolutely fine, although occasionally a 50cm one comes in handy for bigger drawings.

As for pencils, constructions work best when they're precise so sharpness is the most important quality. Since harder pencils take and hold a sharp point better, we recommend having to hand some hard-ish pencils (roughly 3H) and a sharpener. Any pencil, though, will do at a pinch and the common HB ones are fine.

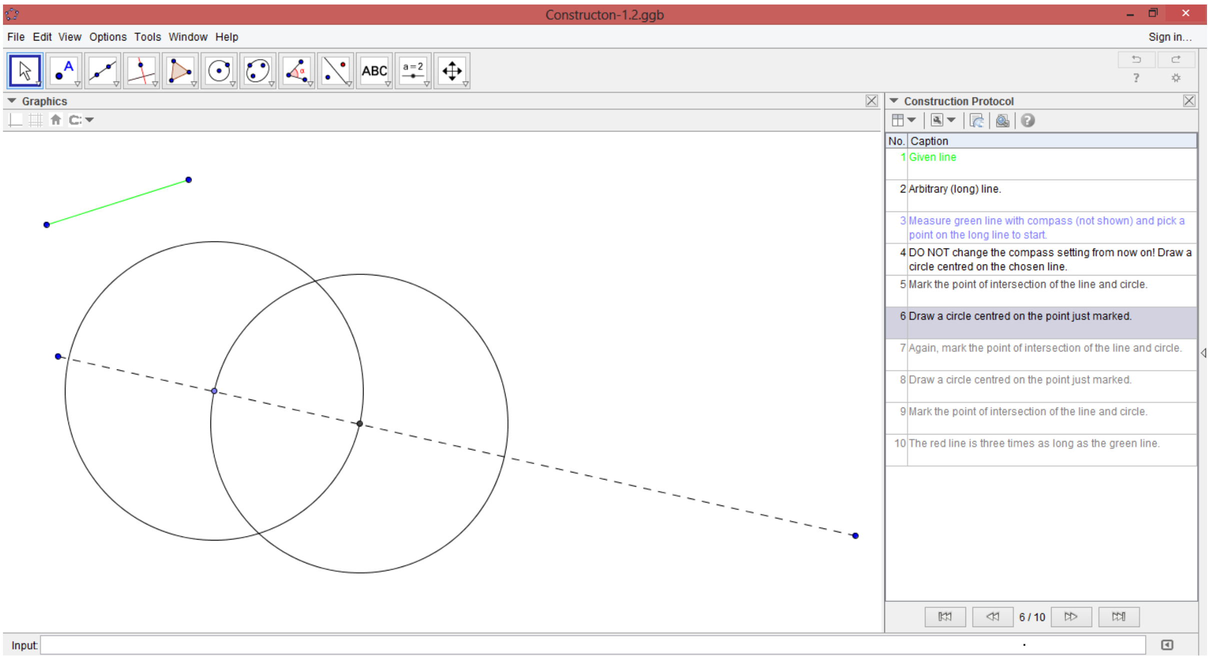
You do not need any of the other things you may find in a "geometry set", such as protractors and set squares and so on. Nor will you need a calculator. In fact, such things will be banned from the geometry classroom. Only the compass, straightedge, pencil and paper are allowed.

On Geogebra

Geogebra is a wonderful piece of free software that makes geometrical constructions quicker and easier. It can be downloaded here: https://www.geogebra.org/. This book is accompanied by some Geogebra files that can be helpful as classroom demonstrations or to experiment with if you're studying alone – we've provided them for all the constructions in the first two sessions and the important or complicated ones in later sessions.

We would always recommend using the latest version, but if using an older one it should be at least version 5. It's possible to run the Geogebra software directly from a USB stick without having to install it on the computer, which can be convenient in some settings.

The main files we have created are named after each construction: for example, Construction 1.02 is found in the file **Construction-1.02.ggb**. When you first open it you should see the Construction Protocol window, as shown on the right of this screenshot:



Notice in particular the buttons at the bottom of the Construction Protocol; these allow you to go back to the start of the construction and then move through it step by step. The caption for each line gives a brief explanation of what to do and should tally with the relevant text in this document. If a caption is only partially displayed, you can adjust the size of the Construction Protocol window until all the relevant text appears.

When first working through a construction we recommend clicking the button to go back to the start, then the button to go through the steps one by one. This works best if you have your drawing kit on hand and can imitate each step on paper as you go. If you cannot see the buttons, this may be because you have an earlier version of Geogebra and need to replace it with a more recent version.

Our files use the following conventions:

- **Green** objects are "given" at the start of the construction. They're the situation we're presented with before we begin, *the problem to be solved*.
- **Red** objects are the "final" result of the construction. The red and green objects together are *the solution to the problem*.

If you need more hints you can see every step in the construction by choosing not to 'Show only Breakpoints' in the Construction Protocol window. These are not annotated though.

Some caveats are in order.

First, to the student: making your own Geogebra resources is far better than using someone else's. An afternoon spent trying to get a construction to work, though it may be frustrating to the point of tears and strong language, and though you might not even succeed in the end, is more educational than anything a teacher can say to you. If you download our resources, you'll have them in two minutes and probably play with them for five or ten, which is not the same thing at all.

Second, also to the student: there is value in physical construction on paper, too. Geogebra can make the construction process feel abstract and indirect. Especially at the beginning you must develop the habit of asking why and how what you're doing works. This is easier when you're using tools that are mechanically simple: Geogebra is a complicated tool and it's not always possible to understand exactly why it does what it does.

Third, again to the student: many of the figures shown in this booklet are complicated-looking. If you understand the steps to produce them, however, they're usually not very difficult to make. We've left in all the construction lines so you don't feel that your end result should be pristine like the illustrations in most textbooks, but if in doubt focus on the coloured lines. Those are usually the main points of interest.

Finally, to the teacher: often enough, demonstrating something on a whiteboard isn't as good as getting your students to work it out for themselves. Showing them the same thing on a computer screen, which as far as anyone can tell works by magic, is even less helpful. You probably already know this, of course, and we appreciate there are plenty of times when a demonstration is the right way to go.

Acknowledgements

Thanks to **sigma** - the network for excellence in mathematics and statistics support - who provided funding both for the establishment of Fine Art Maths Centre at Central Saint Martins and the production of this booklet and its resources.

Professor Jeremy Gray reviewed the contents. Clunie Reid took the photographs. Our thanks to both of them.

Background Material

Outline

The word "geometry" literally means "measurement of the Earth". It's said that the earliest geometers were surveyors in Egypt; every time the Nile's floodwaters receded, they had to go out and re-establish property boundaries and so on in the same way they were before. They also, of course, constructed large buildings that required the precise laying-out of shapes like squares and circles, something that's not easy to do by eye. Wherever such structures were built in the ancient world, in fact, some rough geometric know-how must have been in circulation; there's written evidence of practical geometric knowledge in the Hindu *Sulba Sutras*, which may date back as far as the 8th century BCE, and even earlier in Mesopotamia and Egypt.

Euclid wrote the text known as the *Elements* around 300 BCE, probably summarising and synthesising most of what was known about geometry in the Greek-speaking world at the time. The book is important for two reasons: its contents, which are encyclopaedic, and its unusual method of presentation (to which we return in a moment).

During the medieval period the *Elements* formed the foundation of mathematics education in both Christian and Islamic worlds and its techniques were indispensable to a variety of craftsmen and artisans. A series of Arabic translations appeared during the eighth, ninth and tenth centuries. Arabic thinkers such as ibn al-Haytham (c965-c1040) and Omar Khayyam (1038-1141) examined Euclid's parallel postulate; this work was continued by Nasir al-Din al-Tusi (1201-1274). In the early twelfth century the English scholar Abelard of Bath had made the first translation of the *Elements* into Latin from an Arabic source, and later in the same century notable translations from the Arabic were made by Hermann of Carinthia and Gerard of Cremona. It was not until 1505 that Bartolomeo Zamberti produced the first Latin translation from the original Greek.

A number of (usually partial) vernacular translations suddenly appeared in the second half of the sixteenth century. Tartaglia's Italian version (9 books) appeared in 1543; Forcadel's French in 1566 (9 books); Henry Billingsley's English (13 books) in 1571; Camorano's Spanish (6 books) in 1576; Dou's Dutch version in 1606. Wilhelm Holtzmann made a German approximation of the first six books in 1562, but his version is more of a textbook for merchants than a scholarly translation, with only the more useful results presented and omitting many proofs. Some of these translations rested heavily on existing Latin editions; it is said that Tartaglia made his without any reference to the Greek original at all.

This follows a general trend of the time for the translation of ancient works into regional languages rather than Latin. The central importance of the *Elements* in almost all educational systems remained throughout this period and well into the nineteenth century, and until the early 1600s new discoveries were rare; Euclid and his contemporaries were taken to have solved most of the important problems.

In parallel with this is a fascinating story too tangential to be told here: that of the flourishing of algebra, especially in Italy. The great innovation of the seventeenth century was analytic geometry, which enabled those algebraic techniques to be brought to bear on classical problems.

In a sense the story of geometry from that time to today could be summarized by a famous quote due to Sophie Germain (1776-1831): "Algebra is but written geometry; geometry is but drawn algebra". Our policy on this course is to avoid symbolic manipulations wherever possible, but we'll catch some glimpses of this way of looking at things as we go along. Germain, incidentally, was a mathematical autodidact with no formal training, although she did manage to find some excellent mentors.

The Importance of Method

The arguments of Euclid's *Elements* commence from five "postulates" (axioms), five "common notions" and twenty three "definitions" (some of which are bare statements of meaning, like the definition of a point, and others of which are quite complex, such as the definition of a circle).

In a fine art context we might think of them as the constraints within which we choose to work. The theory of Euclidean geometry is then the artwork produced by attempting to exhaust the potential of these constraints. This, though, is a very modern take on things.

The **common notions** are more like common standards of reasoning that can be used in constructing arguments. They all seek to clear up possible confusion about that "equal" means, and most people seem to find them quite natural and obvious:

- Common notion 1: Things which equal the same thing also equal one another.
- Common notion 2: If equals are added to equals, then the wholes are equal.
- Common notion 3: If equals are subtracted from equals, then the remainders are equal.
- Common notion 4: Things which coincide with one another equal one another.
- Common notion 5: The whole is greater than the part.

The **definitions** are intended to clearly and succinctly capture some geometric object with which we're already familiar – there are rather a lot so we don't list them here, but we do mention a few in Session Three. The first two are: 'A point is that which has no part' and 'A line is breadthless length'.

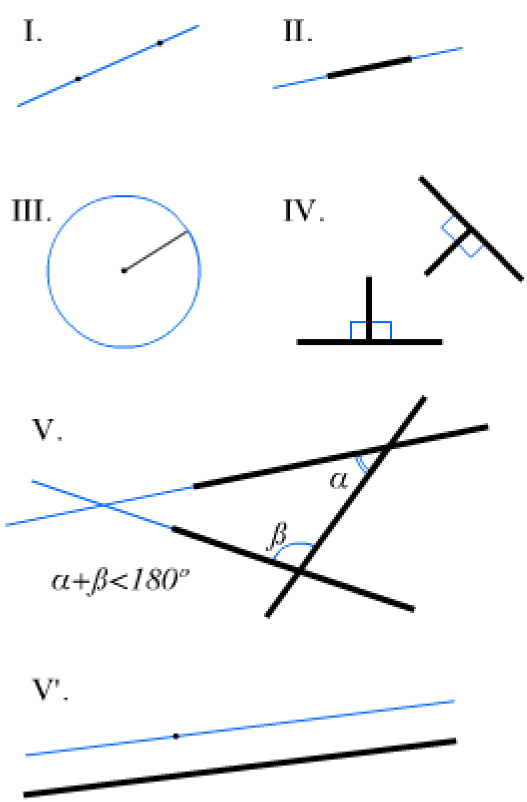
If Euclid's definitions look unsatisfactory, remember it isn't possible to define everything from first principles. They wouldn't help much if you genuinely had no idea what a "point" or a "line" were, but then what you'd need would be a lot more than a one-line definition. If you ask "What is a point?" seriously, you're asking a philosophical question rather than a mathematical one. It's very different to ask "What do you mean by the term 'point' in this text?", given that we both already understand what a point actually is.

The **postulates** are statements about those objects that are claimed to be true without justification. They are the starting-points or "rules of the game", and in fact the first three literally describe the basic operations of our straightedge and compass. Here are the five postulates; we will have a little more to say about the fifth and most complicated-looking in Session Two:

- Postulate 1: To draw a straight line from any point to any point.
- Postulate 2: To produce a finite straight line continuously in a straight line.
- Postulate 3: To describe a circle with any centre and radius.
- Postulate 4: That all right angles equal one another.

- Postulate 5: That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The following is a visual reminder of the five postulates (the fact that there are two images for the fifth postulate will be clarified in Session Two):



(Image from Wikipedia)

Taken together, these common notions, definitions and postulates answer the question: "Where do we start?" We like to provide printed lists of these so that it's clear that there aren't very many of them and most of them aren't very complicated (though a few are weird-looking). We'll have cause to refer to them occasionally, although we don't proceed strictly axiomatically.

The rest of the *Elements* consists mostly of **propositions** and **theorems** and their accompanying **proofs** and **constructions**. Each proposition is a geometrical statement, and its proof is supposed to depend only on these initial definitions, postulates and common notions. Actually, most of the propositions and **theorems** look more like challenges: for example, the very first one is "To construct an equilateral triangle on a given straight line". The proof is then a careful, step-by-step set of instructions for making a drawing that answers the challenge. We do this in Session One.

As the book goes on, each proof can also lean on a growing repertoire of previously-proved propositions. This manner of presentation is still used in maths textbooks today, starting at undergraduate level, and the structure of axiom-theorem-proof is important in formal logic, too. This argumentative style seems to guarantee the validity of its conclusions: it lays its steps out for all to see and check. If we grant the starting-points, it seems we're forced by the proofs to grant the theorems, too.

In the West, the method of Euclid came into especially sharp focus in the decades around 1600, at the same time as philosophers like Francis Bacon (1561-1626) and René Descartes (1596-1650) began to think about the nature of knowledge in a new way. How do we get new knowledge, and how do we assess whether it's the real thing or not? This question arises from doubt, uncertainty, perhaps even anxiety. Questions about the legitimacy of knowledge had appeared on and off for centuries and are not an innovation in the 1600s, but they receive a new pointedness in that context. This will be clearest to students who know about the rise of what we would now call sciences and pseudo-sciences at the time, or who are familiar with the contemporaneous (but largely unrelated) upheavals in the Christian church arising from the Reformation.

Today Descartes is mostly remembered for two things: in mathematics, the invention of analytic geometry, and in philosophy the *Discourse on Method* and the *Meditations*. Euclid's geometry looked to Descartes and his contemporaries like a road map for certainty. Nobody doubted what it contained because it was laid out so plainly: if you understood, you could not doubt. Why could all knowledge not be like this, if it were discovered and set out carefully enough? This thought led to many attempts to copy Euclid's method in other fields. Though influential and often interesting, these were not entirely successful. Perhaps mathematics is a special subject that needs its own methods?

As it turns out, the *Elements* contains a number of examples of dubious or incomplete reasoning – the first appears in Proposition 1 – and Bertrand Russell (1872-1970) was particularly scathing about those who claimed it as an example of perfect rationality. In his *Foundations of Geometry*, David Hilbert (1862-1943) replaced Euclid's postulates, definitions and common notions with a new and more rigorous set of axioms. He was able, using the powerful new tools of formal logic, to fix these problems and to prove that if basic arithmetic contains no contradictions then nor does Euclidean geometry. This is another fruit of the identity of geometry and algebra that Germain's quote makes plain. Philosophy students with a course of logic under their belts may enjoy travelling a little further along this road.

First Session: Methods, Attitudes and Goals

Summary

The purpose of this session is to get students familiar with using the straightedge and compass, to understand what it means to "do a construction" and to have met and understood the following basic constructions:

- Circle with given centre and radius
- Perpendicular bisector
- Angle bisector
- The regular hexagon

We therefore cover **Propositions 1-3, 9 and 10** from **Book 1 of** the *Elements* and a few other topics. Our overriding aim is to get students used to doing constructions, and develop an initial understanding of what they're about. We like to introduce the session with a brief outline of the axiomatic approach and its importance (see Background Material). We find that the bisector is particularly challenging for students to master and repeated practice with the construction of perpendicular lines through given points is crucial; Session Two provides many opportunities for this.

What is a Construction?

A construction is a drawing made by following a sequence of steps in accordance with our postulates. Each step is something simple that we can understand how to reproduce exactly. Because a construction is made of simple elements in a sequence, though, you can communicate it to someone else and if they carry out the steps exactly they'll get the same result you did.

A construction is repeatable: if your steps are written out clearly, someone else can follow them and get exactly the same result you did, at least within a margin of error for the inaccuracies of our tools and hands. This is one reason why we like to ask a student who has a proposed solution to describe the steps to a fellow student one by one: this process tends to reveal any gaps or ambiguities in the solution.

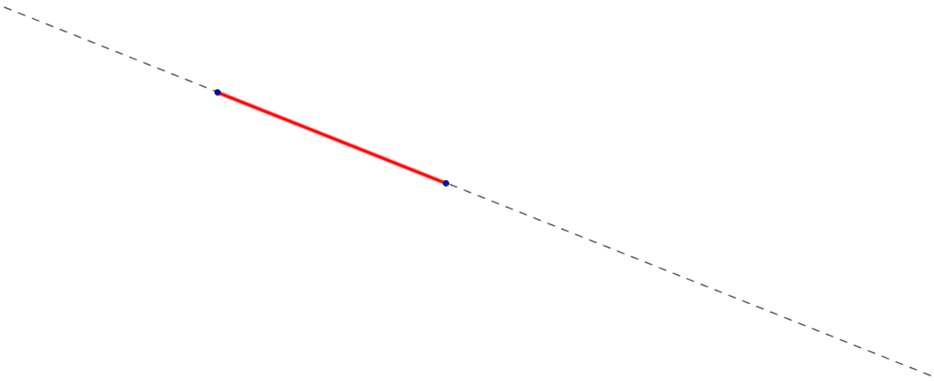
You can imagine how useful this was to ancient architects who wanted a team of builders to lay out, say, a large ground plan in a perfect square. If you think this is easy to do by "eyeballing" it, as you might sketch a small square on paper, you should try it some time.

This means a construction needs to be made with a different kind of rigour and care from a freehand drawing. What you get in return is that the construction is a piece of mathematics all to itself: not that we can explain or describe it with maths, but it *is* maths. In making constructions we're doing maths just as we are (in different ways) when we carry out a calculation. A construction is a proof: its steps form an argument.

A bit of jargon of our own: we use "construction" to mean either the steps taken to draw something or the finished product if, at least to a trained eye, the steps are obvious from it. We use "figure" to mean any picture, including any construction, especially when we want to emphasise how the final result looks rather than the steps taken to get there.

A Note on Lines

Euclidean geometry contains nothing infinitely large; all its objects can be drawn on an ordinary piece of paper. When modern mathematicians refer to a "line" they often have in mind an infinitely extended straight line, but we'll never need this concept. Instead we will work with "line segments": finite straight lines whose end-points are clearly visible. In this image, for example, we can imagine the dotted line going on forever (or as far as we like, anyway) in both directions, and the red line segment is just a finite part of it:



A line segment is a "bit of length" or a "bit of one-dimensional space", if you like, although we'll never really measure length with any standard numerical units, such as centimetres or inches.

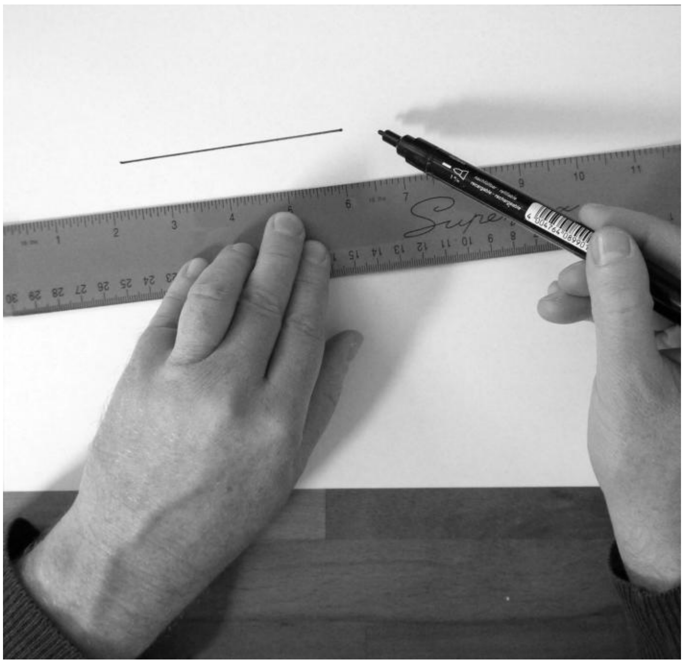
Euclid allows us to "produce" a line segment, which means to extend it by laying the straightedge along it and making it longer in the same direction. Officially we won't need to do this very often but you may find the technique useful in the middle of a construction when you discover a line segment is shorter than you'd like. This is OK because the line segment is really a part of an infinitely long line, and we're just choosing to make it a *bigger* part than it was before.

Reviewer: Jeremy Gray
Open University

Copy a Line segment and Draw a Circle

Our first construction will solve the problem: Given a line segment, copy it, and then construct a circle with a radius of the same length as the line segment. You might be “given” one to work with by your teacher or you can draw a line segment of whatever length you like.

For this first construction and many others there is no constraint on how you start. Our word 'given' will often indicate an arbitrary starting point. So, from now on when we say “a given line segment” we just mean there’s a line segment on the page. It’s fine to draw it yourself, but it might also be “given” to you by a teacher, an examination question or the practical constraints of a real-life problem.


photos: Clunie Reid

Now, the task is to draw a **circle** that has a **radius** of that same length. Drawing circles is the purpose of the compass. Some students may need to be reminded of the following definitions:

- A ***circle*** is the set of points that are all the same distance from a special point, the ***centre***, which isn’t on the circle
- The ***radius*** of a circle is the distance from the centre to any point on the circle.

With these definitions in hand it’s not hard to see how a compass makes circles. First we choose a radius by setting the compass’s legs a certain distance apart – and, to reiterate, “a certain distance” just means qualitatively how far apart they are, independently of any system of measurement. Then we fix a centre for our circle by sticking the spike of the compass into the paper. This point remains fixed while we rotate the drawing tip - at the end of the other leg - around that fixed point. We can do these actions in the other order too: stick the spike, then set the radius.

The circle can be drawn anywhere you like on the page. And you can make multiple copies of that same circle *as long as you don't change the position of the legs between each circle.* This means that the compass is also a device for measuring and storing lengths.

The compass is the only tool we use in this course that has a *memory*. This will be crucial to many of the constructions we do. By virtue of this memory capacity in the compass, we know *by construction* that the circle's radius is equal to the given length.

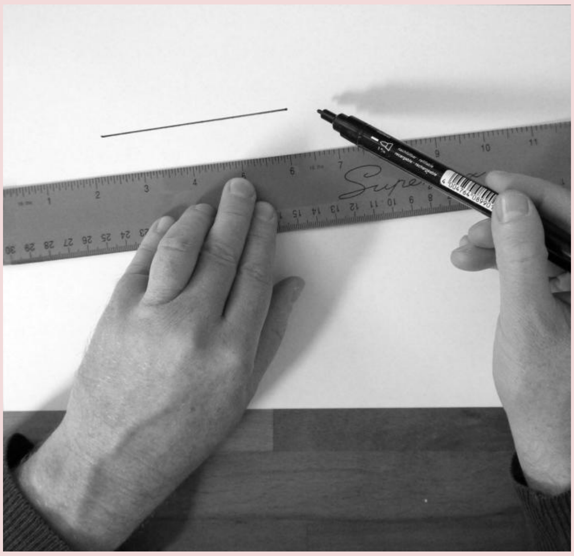
Note, therefore, that when we say "these two line segments have the same *length*", we don't mean the same length measured in centimetres or inches or whatever. We mean the line segments are ***congruent*** in the sense that we could lay them on top of each other and they'd be an exact fit, without one jutting out beyond the other. This idea is drawn from our 'common notions' of 'the same as' or 'less than' or 'more than' (from one perspective, there is no concept of 'length' in Euclid only whether two line segments are equal or 'congruent').

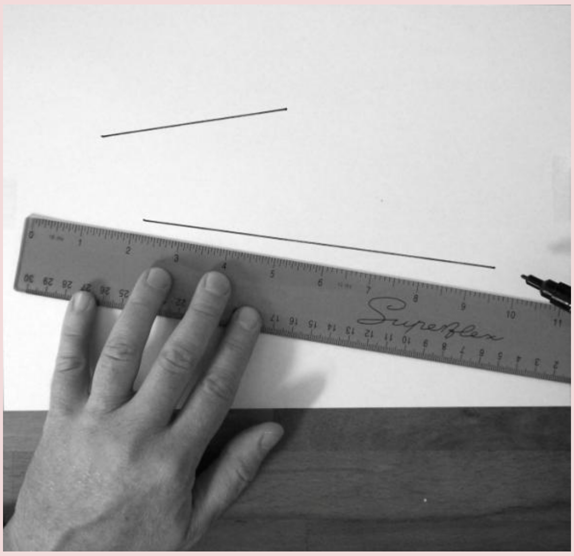
Instead of eyeballing it we'll use the compass to "store" a line segment and copy it somewhere else; perhaps an unexpected use of a tool you might think is only good for drawing circles!

This same technique enables us to copy a given line segment onto another line, as we now show. We've chosen to illustrate this one with photographs to avoid some of the ambiguity and awkwardness inherent in detailed written descriptions of physical movements; all future constructions are illustrated in Geogebra.

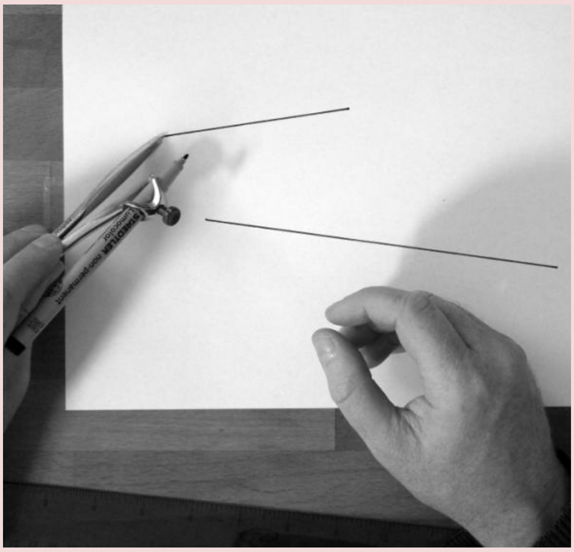
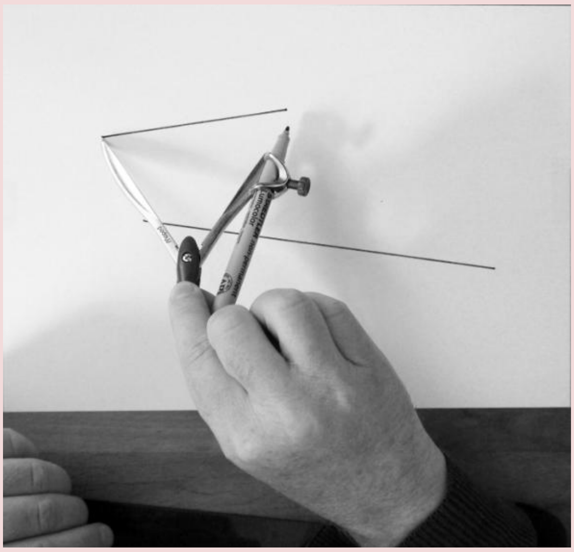
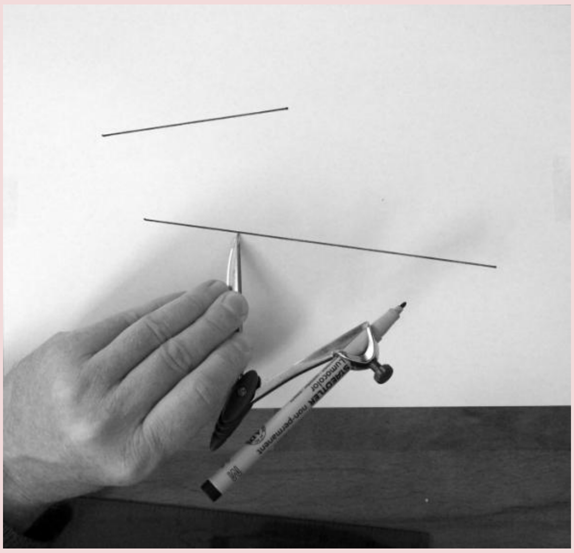
Construction 1.01

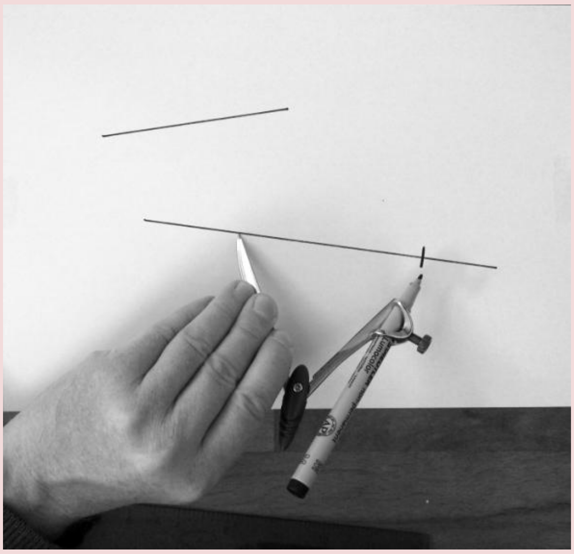
To draw a line segment of the same length as a given one.

<table>
<tr><td></td><td>1. To start us off, a line segment is given.

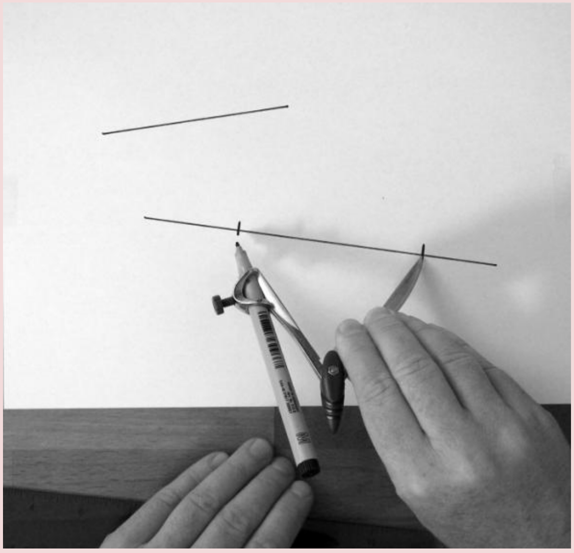
You can just draw any arbitrary line segment if you’re working on your own.</td></tr>
<tr><td></td><td>2. Draw a second arbitrary line segment using a straightedge.

The length isn’t important as long as it’s definitely longer than the first line segment.</td></tr>
</table>

	3. Put your compass spike at one end of the given line segment.
	4. Stretch the pencil point of the compass to touch the other end of the given line segment; this "stores" the length of the line segment in your compass's "memory". **Do not change the width of your compass for the rest of the construction** – otherwise this precious memory will be lost!
	5. Place your compass spike where you like along the line segment you drew in Step 2. It's wise to put it fairly near one end (you'll see why shortly).



6. Find the point along the line segment where the drawing point of the compass intersects the line segment. You can draw a whole circle, the relevant arc of the circle or just place the drawing tip on the point you require.



7. Put the compass spike in the mark you made in Step 6 and make a second mark where the compass spike was placed in Step 6.

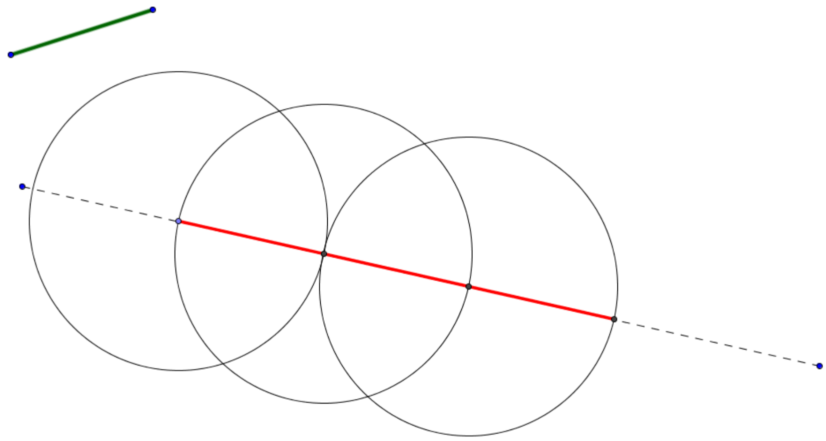
You have now constructed a new line segment – the one that lies between the two marks – which has the same length as the one you were "given".

We can repeat this technique as many times as we like to construct a line segment that is a number of times longer than the one given. In Construction 1.02 we make a line segment three times longer than the one we were given. The first five steps are just a slightly briefer description of the seven-step procedure just described above.

As in future constructions, we show a picture of the finished figure and give a sequence of verbal instructions. These correspond to the steps in the Construction Protocol window in Geogebra, so refer to that if anything is unclear. Remember, the corresponding Geogebra file shares its name with the construction title below.

Construction 1.02

Given a line segment, to create a line segment three times its length.



1. A line segment is given (shown in green).
2. Draw a second arbitrary line segment using a straightedge (shown dashed).
3. “Store” the length of the given (green) line segment using the compass.
4. Place your compass spike anywhere you like along the second line segment. (The one you drew in Step 2.)
5. Find the point along the line segment where the drawing point of the compass intersects the line segment. You can draw a whole circle, as Geogebra does, or just the parts that cross the line segment. You have now constructed a new line segment which has the same length as the one you were 'given'.
6. Repeat the Step 5 twice more. Each time move your compass spike on to the intersection point just created to find the next intersection point.

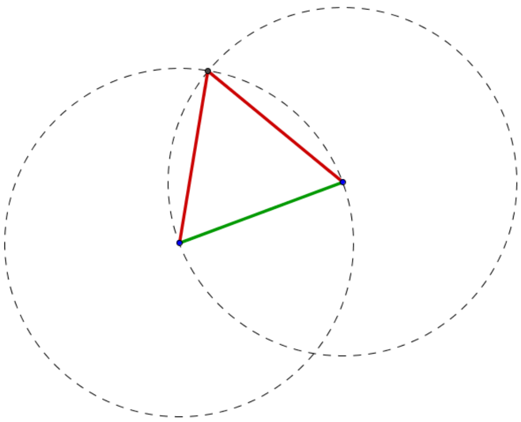
You have now constructed a new line segment which is three times as long as the one you were ‘given’. (The achieved construction will always be shown in red). This is an important practical technique and also the start of a theme we’ll return to several times.

Equilateral Triangle

Having learnt the fundamental techniques of straightedge and compass constructions, Euclid's very first proposition is now open to us. It depends on the properties of the intersections we create. The problem is to construct a triangle whose three sides are all equal lengths – an ***equilateral*** triangle – given the length of one side in the form of a line segment on the page.

Construction 1.03

To construct an equilateral triangle on a given line segment.



1. A line segment is given.
2. "Store" the length of this line segment using the spike and drawing point of the compass.
3. Keep the compass in place and trace out a circle.
4. Without changing length "stored" in the compass, reposition its spike at the other end of the given line segment and trace out a second circle.
5. These two circles cross (intersect) at two points. Choose either of these intersection points to be the third corner of the triangle and join it to the ends of the given line segment to complete the figure.

The corners of the triangle – called ***vertices*** – are the centres of the two circles and a point where the circles cross. Since each of the three sides of the triangle is a radius of circles with the same radius, each side is the same length.

The key idea here – used many times in future constructions – is that any point where two equal-sized circles intersect is the same distance from each of the circles' centres. Since the compass wasn't adjusted between drawing them these radii must all be the same length. In a sense the compass has *already* measured all the line segments and proved they're the same length while the figure was being drawn, since the line segments were all constructed without changing the compass at all.

There's no need to repeat this verification afterwards by measuring (which is error-prone anyway). If we understand how the triangle was constructed, and we believe it was done accurately, that on its own constitutes a proof that the sides are all the same length, at least within a margin of error for accuracy. If the construction were carried out with perfect accuracy, the side-lengths would be exactly equal. Another way to put this is to say that any error can be reduced as much as we need by using better equipment and working more carefully.

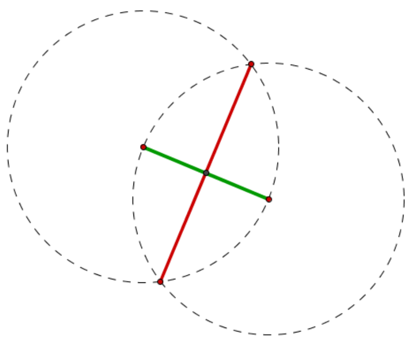
Notice that in Step 5 we did something new. We had two points and we joined them by drawing a straight line segment. This may not seem worthy of consideration, but notice that whenever we have two points, we can always do this with just one unique straight line segment. You can't draw two *different* line segments that join the two points – if you try, they'll overlap exactly or at least one won't be straight. On the other hand, you can't put two dots on a piece of paper that *cannot* be joined by a single line segment. These facts will be used, silently, throughout our constructions in the future.

Perpendicular Bisector

Our next construction also depends on the properties inherent in the construction of the Equilateral triangle. It may appear simple but the ability to cut line segments in half - to bisect - will be fundamental to more complex constructions. It is important to master it thoroughly. It will return in a different context in Session 2. The polygon constructions later in the session give some more opportunities for practice. The problem: given a line segment, construct the point exactly in the middle of the line segment. The construction is almost exactly the same as the one for an equilateral triangle.

Construction 1.04

To construct a line segment that divides a given line segment exactly in half.



1. A line segment is given.
2. "Store" the length of the line segment with the compass.
3. Keep the compass in place and trace out a circle using the line segment's length as the radius.
4. Without changing the length stored by the compass, reposition its spike at the other end of the given line segment and trace out a second circle.
5. The two circles constructed will have intersected at two points.
6. Draw a line segment that connects these two new intersection points.
7. Where this new line segment (red) crosses the given line segment (green) is the exact midpoint of the given line segment.

The symmetry of this construction - which produces two equilateral triangles with a shared side (the given line segment) - means that that line segment connecting the two constructed vertices (neither on the green line segment) must cut their shared side in half.

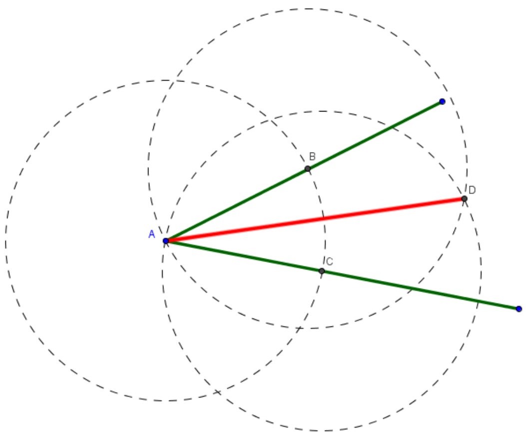
The line segment that defines the midpoint has another very important property that we'll spend more time with in the next session: it's ***perpendicular*** to the original given line segment. In more familiar geometrical jargon we say the line segments cross "at a right angle", like the corner of a square. The way Euclid says it is that the angles on either side are equal: the red line segment isn't "leaning" in one direction or the other relative to the green line segment. The evident symmetry of the figure makes this claim at least plausible.

Angle Bisector

Here the same properties of intersecting circles are used to cut an angle in half. We won't need this construction often but it's easy to do and is sometimes useful (if class time is short this one could be skipped or assigned as homework). Note that just as a length is just given by a line segment, without units of measurement, so an angle is just two line segments crossing, without degrees or suchlike getting involved.

Construction 1.05

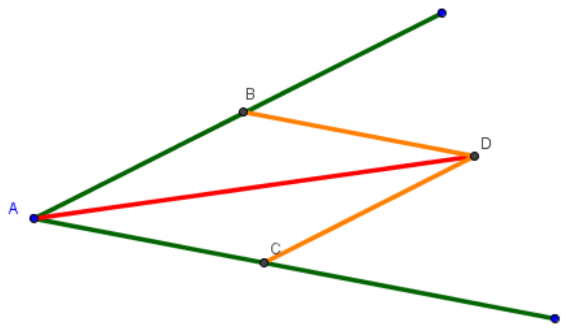
To cut a given angle in half.



1. We are given two line segments which meet at a point. Let us call this point A.
2. Place the compass spike at A.
3. Set the compass to any length shorter than or equal to the shortest of the two line segments.
4. Trace out a circle that crosses both line segments to create two new intersections (call them B and C).
5. Keeping the compass unchanged draw two new circles, one centred on B and one on C.
6. These two new circles will intersect at A where our two given line segments met and at a second new point (call it D).
7. Draw a new line segment that connects A to D.

This new line segment (red) cuts the angle between the two given line segments (green) in half.

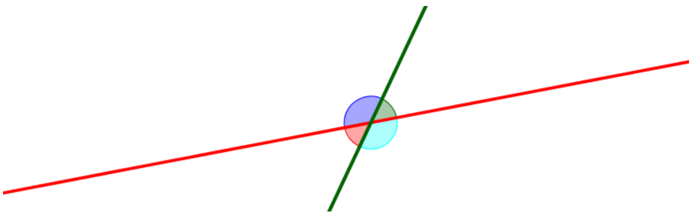
As with the previous construction we can use the symmetry of the figure – guaranteed by its construction – to "see" that the original angle has been divided into two *equal* ones. If we draw in two additional line segments (orange in the image below) we can see that, by construction, triangles ABD and ACD are congruent; in fact each is a "mirror image" of the other along the red line segment. This symmetry means that AD cuts the original angle in half; if it didn't, the triangles wouldn't match there when they were flipped over along the red line segment.



The angle between two line segments can be thought of intuitively as the amount you'd have to turn one of the line segments to get it to point along the same direction as the other one. It's a bit more complicated than this (the intuition works better with "oriented" line segments, which are more like arrows or "vectors") but it's not a terrible way to think of it if you don't push it too hard. In that case, the angle bisector tells you how much you have to turn one line segment to get it halfway to pointing in the same direction as the other one.

Angles Made By Lines

We looked at angles made by two line segments that are joined at their ends, but we could also look at angles made by a pair of lines where they cross:



When the lines cross they make a total of four angles. If we think about lines as something like "directions" and angles as "turnings", we can see that the green and pink angles must be equal. Think about turning the green line until it lies on top of the red line, using the point where they cross as the pivot (like the hinge in a pair of scissors).

If you turn the top part of the green line clockwise down through the green angle, the bottom part of the green line will inevitably rise through the pink angle. Similarly, if you push the top part of the green line anticlockwise through the dark blue angle, the lower part will sweep out the light blue (cyan) one.

We'll show that the green and pink angles are equal – the proof is just the same for the blue and cyan ones.

First, notice that the green and cyan angles must add up to half a turn, since if you turn the green line through both of them it ends up looking exactly the same, but with its top and bottom portions swapped over.

Second, notice that the same goes for the cyan and pink angles; this time imagine turning the red line through both of them, and it'll end up looking as it did only with its left and right portions swapped.

Expressing this in a sort of "algebra", we've said this:

GREEN + CYAN = PINK + CYAN

which implies that

GREEN = PINK

as we hoped.

The Regular Hexagon

The equilateral triangle is an example of a regular polygon. For our purposes a ***polygon*** is a figure that is made by connecting (straight) line segments and which encloses an area: think of demarcating a piece of territory with straight sections of fencing that don't cut across each other. Triangles, squares and hexagons are all polygons. A circle isn't because it's not made of straight line segments. The letter X isn't because it doesn't enclose an area.

A ***regular*** polygon is one that meets two other criteria:

- Its sides are all the same length,
- The angles between its sides are all the same size.

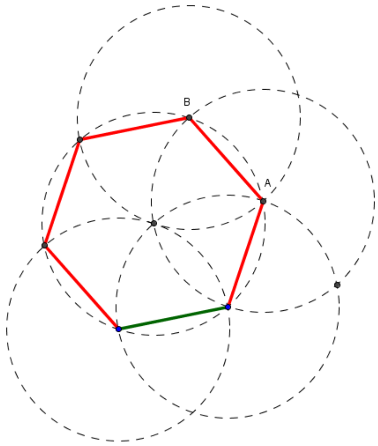
In order of how many sides they have, the first regular polygons are: equilateral triangle (3 sides), square (4 sides), pentagon (5 sides), hexagon (6 sides), heptagon (7 sides), octagon (8 sides) and so on.

Which regular polygons can we construct with our straightedge and compass? It's natural to suppose we'll eventually be able to construct all of them, but we'll see that this is too optimistic. It turn out, though, that we can do quite a few with the constructions we already have in our toolkit.

In particular, on the basis of the equilateral triangle construction we can construct the regular hexagon. This is an example of an important technique of drawing a regular polygon by "walking" around the circumference of a circle with a fixed length (stored by the compass). Note that at steps 7 and 8 the correct intersection point must be chosen, but if you keep the final goal in mind it's quite obvious which is the right one to pick.

Construction 1.06

To construct a regular hexagon on a given line segment.

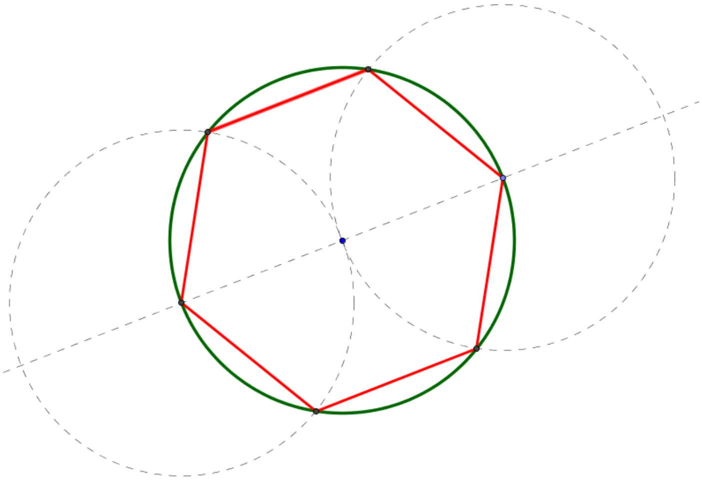


1. A line segment is given.
2. "Store" the length of the line segment using the compass. *Do not change the compass setting from now on!*
3. Keep the compass in place and trace a circle.
4. Reposition the compass spike at the other end of the given line segment and trace a second circle.
5. The two circles constructed will have intersected at two points. Pick either one of them and draw a third circle centred on it.
6. This new circle will cross the other two at two points. Mark them and connect each end of the given line segment to the intersection point nearest to it. We now have the first three sides of a regular hexagon; its sides are the same length as the compass setting.
7. From either of these new intersection points (call it A), draw another circle. This will create a new intersection with one of the existing circles (call this point B); joining A to B gives the fourth side of the hexagon.
8. Repeat the previous step on the other side of the figure; this gives the fifth side.
9. The sixth side is obtained by joining up the end-points of the last two line segments.

We end with another way to construct a regular hexagon. In this case we are not given a side of the hexagon but rather a circle it must fit perfectly inside, with all six vertices just touching it. We say the hexagon is ***inscribed*** in the circle. We assume the centre of the given circle is known – if not, we shall see how to find it in Construction 2.10.

Construction 1.07

To construct a regular hexagon inscribed in a given circle.



1. We are given a circle and its centre.
2. Draw any line segment through the centre of the circle, intersecting it at two opposite points.
3. “Store” the radius of the given circle using the compass and draw a copy of the given circle centred on one of the intersection points.
4. Mark the two points where the circumference of the new circle (copy) crosses the given one. You will now have four of the six vertices needed on the circumference of the given circle.
5. Repeat Step 3 but use the second of the 'opposite points' constructed in Step 2. The new copy of the given circle will generate the final two vertices.
6. This gives six equally-spaced points around the circle; join them up to finish the hexagon.

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Central Saint Martins, UAL

Reviewer: Jeremy Gray
Open University

Second Session: Parallel and Perpendicular

Summary

This session focuses on parallel and perpendicular lines, but we begin by revisiting something from the previous session that wasn't explicit: addition and subtraction of lengths and angles.

We meet two constructions of the square, one very efficient and the other very explicit. We extend this second version to construct general parallelograms. These use two simple and useful extensions of the perpendicular bisector construction.

We then consider division of a line segment into more than two equal parts, largely as a demonstration of the power of parallel lines. The construction is conceptually easy but fiddly; unlike those we've seen so far, however, it's not at all obvious why it works. For this we need Thales' Theorem; in our sessions this is done (along with its proof) as an optional presentation. We include it at the end of this Session.

Note to Teachers

The discussion of parallel lines can be greatly reduced if students aren't interested in it, although defamiliarizing ideas like "parallel" that seem natural is instructive in its own right.

Addition and Subtraction of Lengths

Since it can copy lengths, the compass allows us to do something else: it allows us to add and subtract lengths. So, here are two very easy problems:

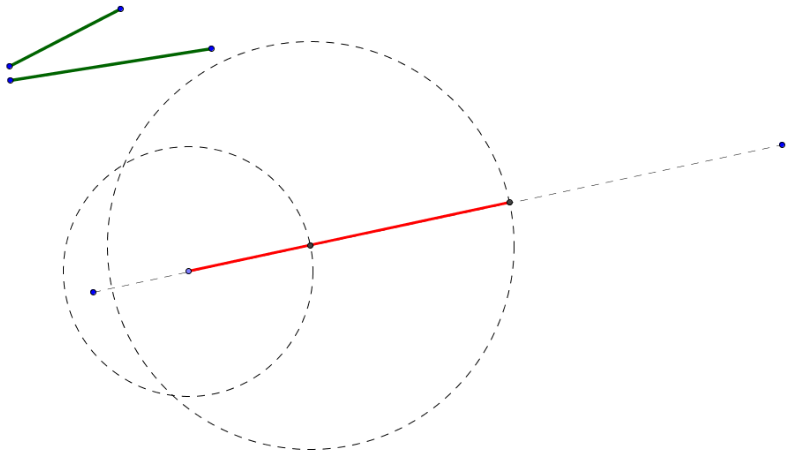
- Given two line segments, construct the line segment whose length is the sum of them.
- Given two line segments, construct the line segment whose length is that of the longer one minus the smaller.

The resulting line segment should be straight, not two line segments joined with a "kink". You should try to find solutions yourself before reading on – if your solutions are different from ours, they might still be correct.

Reviewer: Jeremy Gray
Open University

Construction 2.01

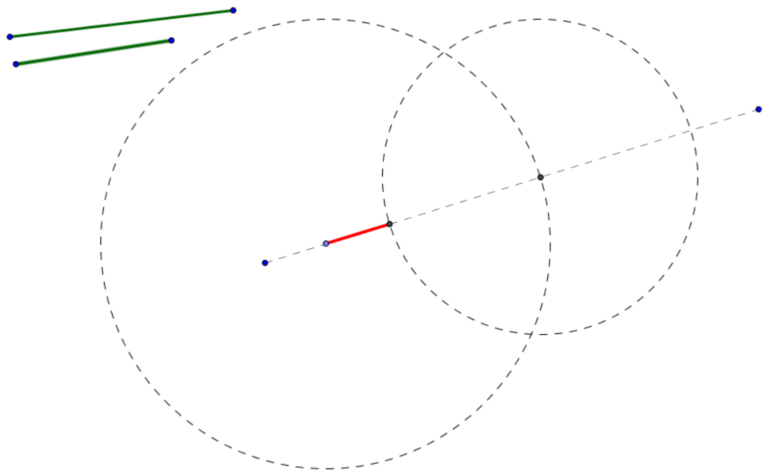
To add two line segments together to make one line segment.



1. Two line segments are given (it's most interesting if they're of unequal lengths).
2. Construct a third, much longer line segment with a straightedge.
3. Use the compass to copy the length of one of the given line segments. Mark off that length on the third line segment you have just constructed.
4. Repeat for the other given line segment, starting from one of the points you created at the previous step.

Construction 2.02

To subtract one line segment from another to make a new line segment.



1. Two line segments are given (they *must* have unequal lengths).
2. Construct a third line segment, much longer line segment with a straightedge.
3. Use the compass to copy the length of the longer of the two given line segments. Mark off that length on the third line segment you have just constructed.
4. Copy the length of the other, shorter given line segment, and again pick one of the end-points you created at the previous step.

5. This time, however, you are looking for the intersection point formed by the circle that goes "back on itself" to “cut off” this length from the last one; the remaining length is the amount left over; the long line segment minus the short one.

Note that with subtraction the order of operations matters, while with addition you can begin with either line segment to reach the desired result.

Maybe these operations don’t seem miraculous but, without introducing any numbers or units of measurement, we can now do arithmetic with line segments. We can even solve simple equations. For example, try this:

- Suppose we are given a line segment from which another length has *already* been subtracted. Given the length that was subtracted, construct the original line segment.

This problem is equivalent to solving the equation $x - y = z$ where

- x is the original, unknown length we need to construct
- y is the amount that was taken away
- z is the length we’re left with

Using algebra we solve this equation by adding y back onto z to retrieve the original length. Note that this is *precisely* how we solve the problem with straightedge and compass. If students have any trouble with this, give them sticks of the appropriate lengths (wooden kebab skewers are suitable) and they will quickly see what to do: put together the new length and the bit that was cut off to get the original length back.

The following problem is, of course, similar and any students who got stuck on the previous one should be able to do this one without any help:

- Suppose we are given a line segment to which another length has already been *added*. Given the length that was added, construct the original line segment.

Of course, this is equivalent to solving the equation $x + y = z$ where y and z are known. We can do something similar with angles; the only difficulty is in copying an angle. The early stages of this construction are identical to those for bisecting an angle.

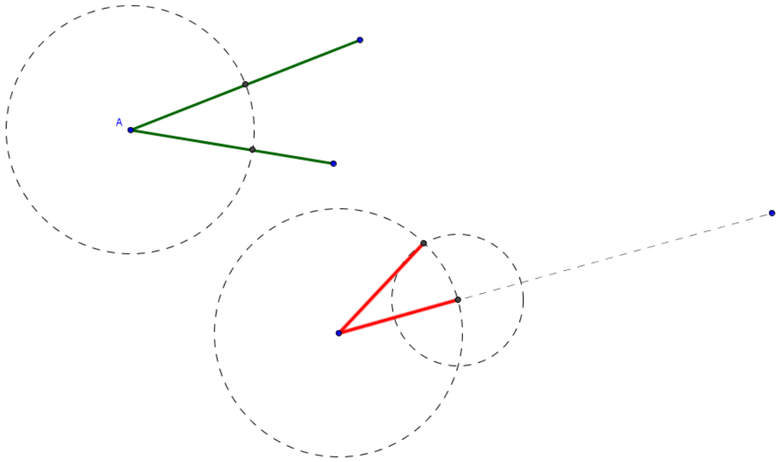
As an optional exercise, take three given line segments, x, y and z (with x longer than y) and construct the line segment that corresponds to $3x – 2y + z$. You’ll need to combine constructions 1.1, 2.1 and 2.2.

Addition and Subtraction of Angles

We can create the same kind of arithmetic for angles, too. The resulting constructions are sometimes handy for creating more complicated figures. We don't use them in later sessions, though, so they can safely be skipped.

Construction 2.03

To copy an angle



1. Two line segments are given that form an angle at point A.
2. Draw a separate long line segment to serve as the starting-point for the construction.
3. Place the compass spike at point A.
4. Set the compass length to any length shorter than the shorter of the two given line segments.
5. Trace out a circle, creating two intersection points.
6. Without altering the compass length, move the spike to one of the end points of the long line segment and draw a new circle with the stored length. This will create one new intersection point on that line segment.
7. Return to the two joined line segments. Place the compass spike on either intersection point and "store" the distance between the two points using the compass.
8. Now back to the long line segment. Place the compass spike on the intersection point there and draw a circle.
9. This construction circle will intersect the circle you drew in Step 6 at two points. Choose either of these.
10. Draw a line segment from the chosen point back to the endpoint of the long line segment. You have copied the angle from the left hand line segments to the right hand line segments.

Some students may be worried by the idea of "arbitrary" choices here and elsewhere, which may seem problematic. One way to work on this is to copy the same angle several times, making different arbitrary choices for the first circle each time. Not only should the angles all "look the same" (which is no guarantee of anything!) but it should become clear through repetition that the construction works independently of this initial choice.

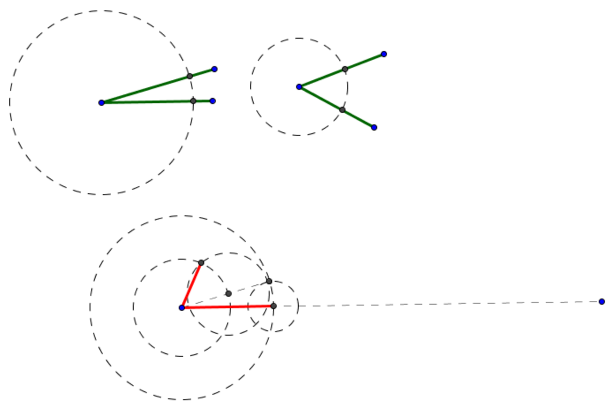
Note to Teachers

The standard proof that this construction works appeals to properties of congruent triangles. We cover those properties in the Third Session.

Now that we can copy an angle, adding and subtracting two angles is quite straightforward. Think of adding angles in the following way. First, face along one line segment and turn by the size of the first angle. Then turn again, this time by the amount of the second angle. The total angle turned is the sum of the two.

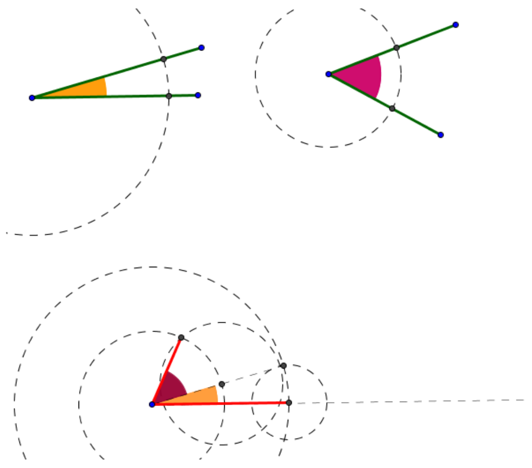
Construction 2.04

To add two angles together



1. We are given two pairs of line segments. Each pair meets at a point.
2. Draw a separate long line segment to act as the base of the construction.
3. Copy one of the angles onto this line segment as in Construction 2.03.
4. Now use the newly-constructed line segment in the previous step as the base, and repeat Construction 2.03 again with the second given angle.

The colour-coding of the angles in the following image may make the idea clearer:



In the final construction, we turn by the amount of the orange angle, then by the amount of the magenta one; the total amount turned is the sum of the two.

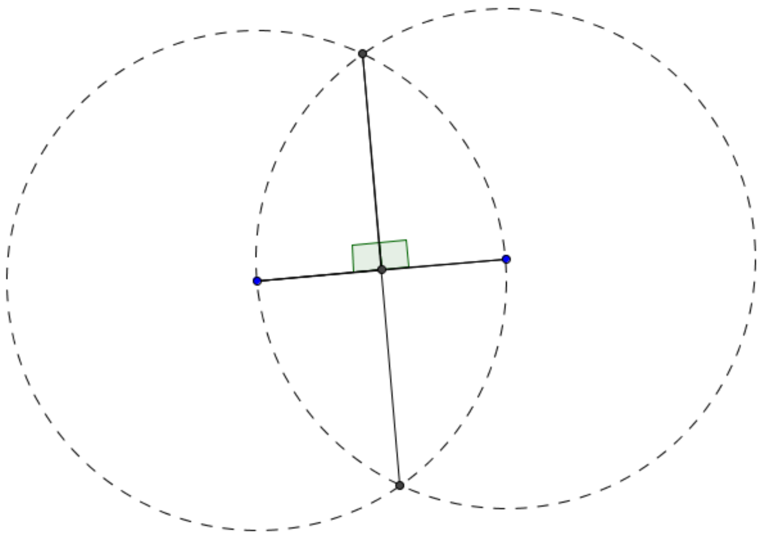
Subtraction of angles is much the same: intuitively speaking, instead of turning further around in the *same* direction we rotate back in the *opposite* direction. You may like to take some time to experiment with adding sequences of angles, especially when they sum to more than a complete circle (360^0), and with subtracting big angles from small ones.

Note to Teachers

One could extend this into various physical activities, and perhaps add in work with bearings from the GCSE syllabus, although this has never seemed relevant and interesting enough to us to merit the time it would take up. If this course were leading, for example, towards work with linear algebra or analytic geometry this kind of diversion might be worth considering.

Perpendicular Lines

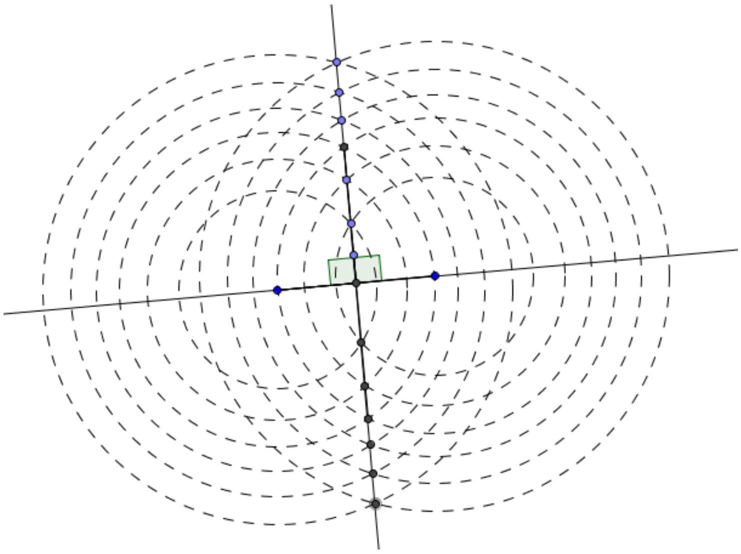
We return now to the bisector (Construction 1.04). We know that it enables us to divide in half but the line segment we construct in the process has another, very important relation to given the line segment: they are ***perpendicular***. The means it crosses the original line segment at the same angles on each side, which we call a ***right angle***:



You can interpret the same drawing as a sum of two right angles making a perfect half-turn or, to put it anachronistically, a turn of 180^0. Consider (and draw, if necessary) what the sum of four right angles is, and hence the sum of two half-turns, if that terminology isn't already too much of a giveaway.

If we imagine extending the perpendicular line segment to an infinitely long *line*, this can also be thought of as the set of all points that are equidistant from the two constructed intersection points. The circles used in the construction give a strong hint about why this is the case – the following diagram is suggestive (see Geogebra file Session-2.1):

Reviewer: Jeremy Gray
Open University



Construction 1.04 shows how to create a perpendicular through the *midpoint* of a given line segment, but we will need to be able to create them at other points, too. We consider now two constraints on the location of the perpendicular:

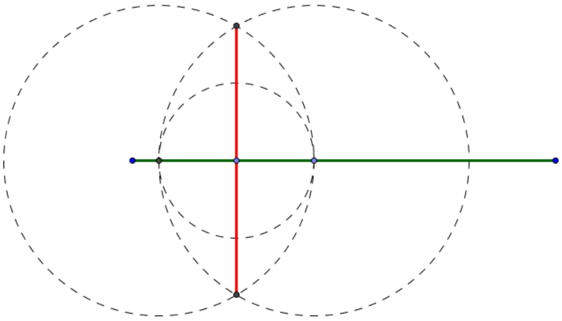
- Given a line segment and a point on it (not the midpoint), construct the perpendicular through that point.
- Given a line segment and a point *not* on the line segment, construct the perpendicular through that point.

If you're feeling confident you could try to work out your own solutions to these problems, although a little ingenuity is needed.

Once you've grasped the idea these are not difficult constructions to carry out, but many find the methods counterintuitive at first. Since they provide the key to constructing parallels we allow plenty of time to practice these two techniques.

Construction 2.05

To construct a perpendicular to a given line segment through a given point *on the given line segment*.



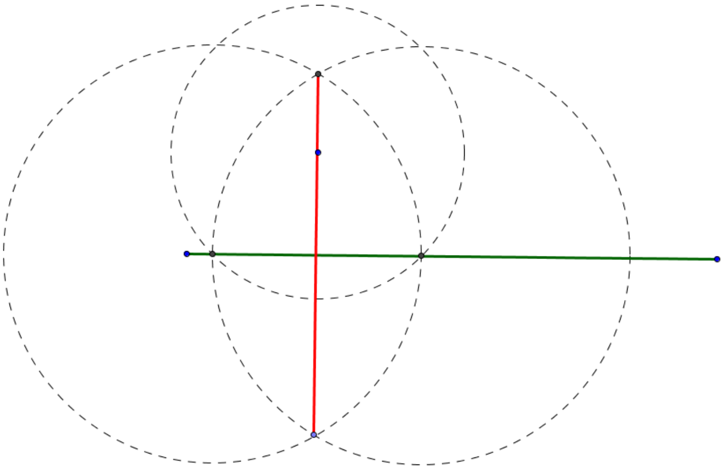
1. A line segment is given, along with a point somewhere on it.
2. Draw any circle centred on the given point that intersects the line segment at two points.
3. Put the compass in one intersection point, put the drawing tip on the other and draw a circle.
4. Repeat for the other intersection point.
5. Mark the two points where these larger circles cross.
6. Join then to obtain a perpendicular to the given line segment through the given point.

Note that steps 3-6 are just the perpendicular bisector construction, but using points that we know (by construction from Step 2!) are equally spaced either side of the point we want our perpendicular to cross.

The same approach can be used to solve the second problem. An arbitrary circle centred on the given point is used to make two intersections on the given line segment; the only difference here is that the "arbitrary" circle must be made large enough to actually intersect the given line segment at two distinct points.

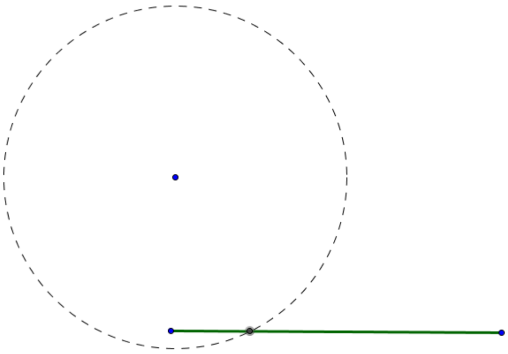
Construction 2.06

To construct a perpendicular to a given line segment through a given point *off the given line segment*.

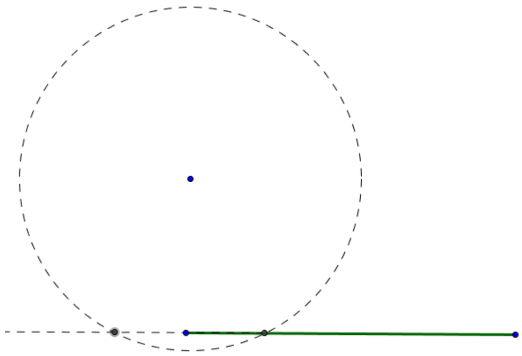


1. A line segment is given, along with a point *not* on it.
2. Draw any circle centred on the given point that intersects the line segment at two points.
3. Put the compass in one intersection point, put the drawing tip on the other and draw a circle.
4. Repeat for the other intersection point.
5. Mark the two points where these larger circles cross.
6. Join then to obtain a perpendicular to the given line segment through the given point.

Note that this construction is essentially identical to 2.5! You may find you can't carry out Step 2 because the circle only crosses the given line segment once:

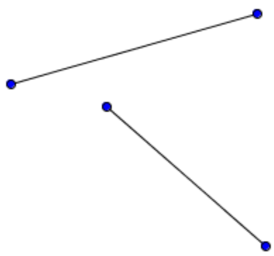


If so, simply extend the given line segment using your straightedge until it's long enough:



Parallel Lines

You may have been told that two lines are parallel if they "never meet", but this definition's no good to us. After all, the lines in our constructions are finite *line segments*. These line segments, for example, never meet but they're not parallel:



Euclid's own Definition 23 puts it this way:

> Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

This doesn't *reliably* help us to decide when two line segments are parallel, since we might produce them for a very long distance and still not be sure whether or not, eventually, they're going to meet.

A better idea is found in the famous Fifth Postulate:

> if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

This is certainly a strange-looking way to define parallel lines, and it's worth spending some time unpacking it and seeing how it relates to the notions of "parallel" the class may be carrying around.

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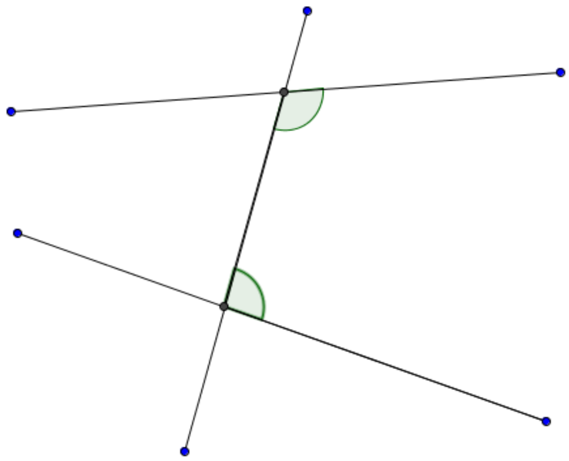
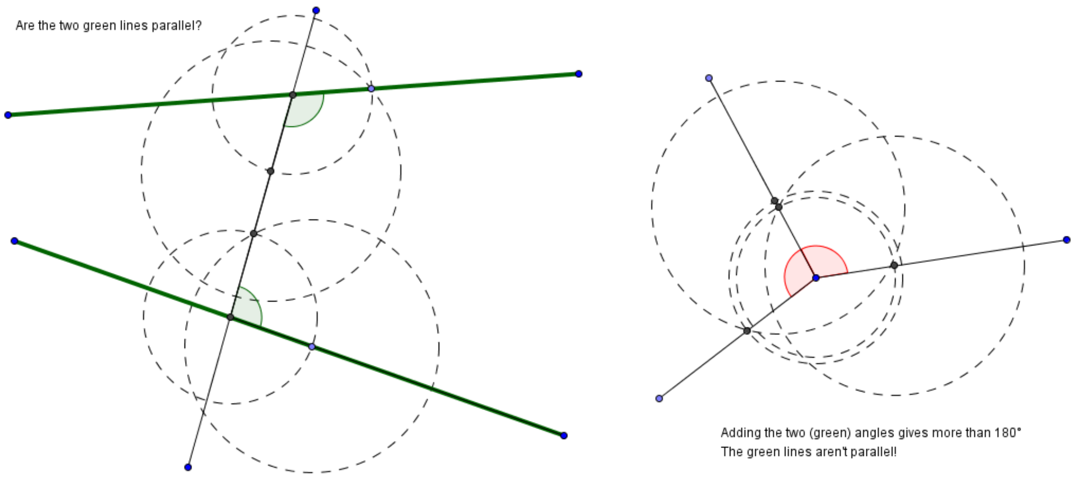
A good way to do this is to draw some examples of the setup Euclid describes: that is, two line segments and another lying across them. We then identify a pair of angles that are "one the same side":

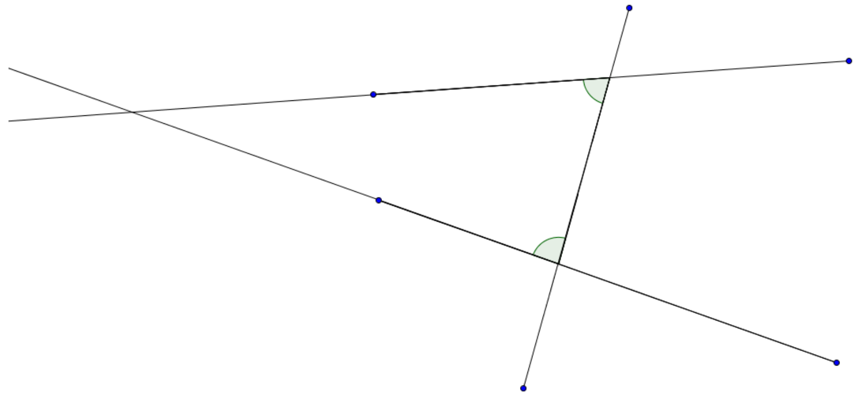


We can copy and add up those angles (Construction 2.04) to see what we get. The original figure is on the left and the resulting sum of the two angles is on the right (see Geogebra file Session-2.2):



Notice the result is more than half a turn, so it's more than two right angles. Is it intuitively obvious that the angles on the other side must add up to *less* than two right angles? Euclid's Fifth Postulate says that if they do, the line segments aren't parallel and will in fact meet on that side if we extend them far enough.

Since the angles on this (left) side of the line segment that cuts the other two add to less than two right angles, Euclid says the two lines should meet. That should eventually happen if we produce the line segments on that side, and we can see that this is indeed true:



Notice that Euclid says the line segments are to be produced “indefinitely” – that is, we don’t necessarily know how far we have to extend them ahead of time. It does *not* mean “infinitely”, because with our straightedge and our finite lives (and pieces of paper) we can’t extend a line segment infinitely far – but luckily we don’t need to. If they’re not perpendicular then the amount we have to extend the lines before they meet is always finite, so we’re still working with line segments rather than (infinite) lines.

Note to Teachers

In fact, Euclid’s use of “infinite” lines is a topic of scholarly debate; Mendell, for example, sifts through the textual evidence and concludes that he was “not bothered” about the issue (Mendell: p. 53). Infinitely extended lines are never needed for any of the geometry we cover on this course, since all of it can be done by practical drawing on a finite sheet of paper.

Euclid does refer to infinite lines explicitly in various places. What isn’t clear is whether he means what Aristotle called a “potential infinity” – a line that can be extended as far as we wish – or an “actual infinity”. Modern mathematicians tend to see no problem with actually infinite lines, since they’re so used to having access to them via much later (especially Cartesian) developments. But we think the distinction is philosophically important.

Constraining ourselves to the potentially infinite – which we call the “indefinite” –at least seems to be consistent with the geometry Euclid develops. We reserve discussion of actual infinities for our follow-up course that includes some projective geometry, *Perspective and the Geometry of Vision*.

Now, if the setup is such that the angles are equal to right angles on both sides, the fifth postulate implies that, however long you produce them, the two line segments won’t ever meet because they cannot meet on one side or the other; since the angles are the same on both sides, symmetry demands that they either meet on both sides or neither. The former is impossible because two different straight lines can have at most one point in common in the plane. An aside: this “fact” is intuitively clear to most people but Euclid never offers any justification for this other than the assumption that “two lines cannot enclose a space” (Proposition 4 of Book 1). (Have a go at drawing some to convince yourself of this!)

Note to Teachers

This is Theorem 1 of Hilbert's *Foundations of Geometry*; he doesn't offer a proof either, but it's clear that it follows immediately from the assumption that two points lie on a *unique* line (see also Hartshorne, p.66).

Incidentally, in elliptic geometry two straight lines *can* cross twice to enclose a space, the 'biangle' – but that's a topic for another course. Think of the lines of longitude on the globe that join at both poles - a space can be enclosed by any pair of longitudinal lines!

This discussion, or parts thereof, should give enough hints to enable you to work out how to construct a line segment parallel to a given line segment by using a perpendicular construction *twice*. Have a think about how to go about it before looking at the recipe below (Construction 2.07).

We give the method to construct a parallel line segment through a *given* point rather than an arbitrary parallel. This combines Constructions 2.05 and 2.06 to create a perpendicular to a perpendicular. In a manner of speaking, we are simply making two 90^0 turns from our original line segment. Notice how our growing "toolkit" of constructions can be used to build up more complicated ones; this is a general theme in a lot of mathematics!

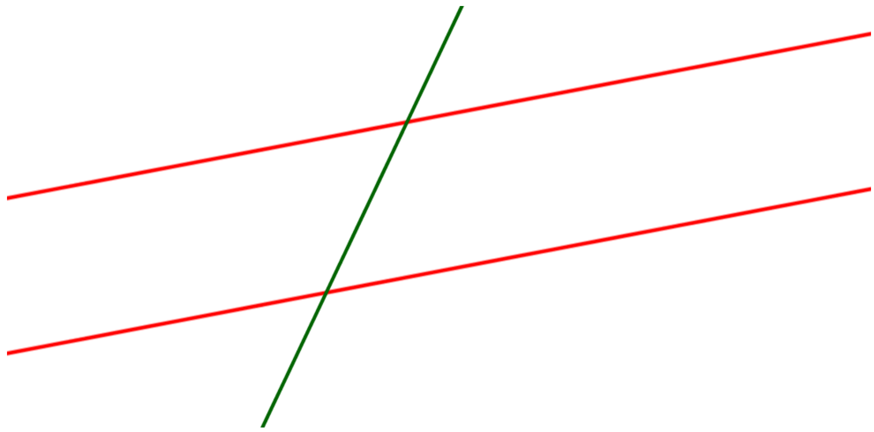
Note to Teachers

It's the existence and uniqueness of parallel lines that makes Euclidean geometry Euclidean – non-Euclidean geometries are usually obtained by making some change to the parallel postulate. In elliptic geometry, there are no (straight) parallels; in hyperbolic geometry there are many through each point.

We don't generally dwell on this in this course because we run a follow-up course that covers this material, but if we didn't we'd want to at least mention this fact here!

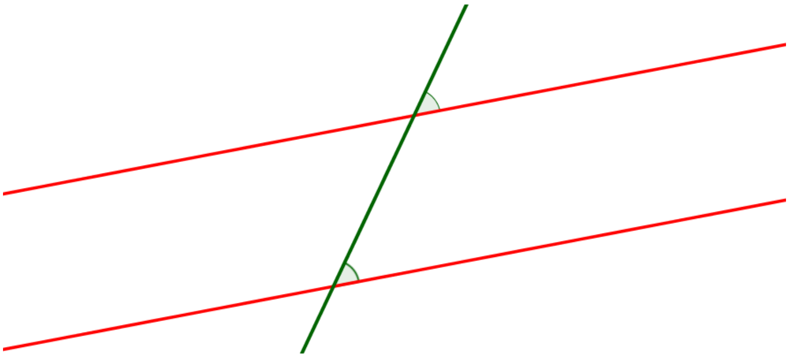
Parallel Lines and Angles

This section describes two very useful theorems about what happens when a pair of parallel lines crosses some other line, as in this diagram:



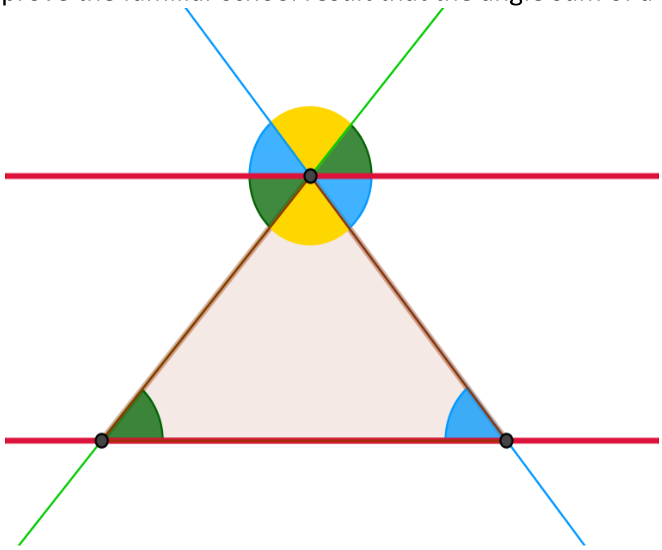
It turns out that we can say some quite interesting things about the way the two parallel (red) lines cut across the green line.

The first thing to notice is that the angle at which the green line cuts the first red line is the same as the one at which it cuts the second red line:



By combining this with the theorem about opposite angles from the end of the previous session we can **find all the angles** in a figure like this if we know just a single one – a common exercise from school geometry. In addition we can prove certain theorems about the internal angles of a triangle.

Using this last insight and those we've seen earlier regarding angles, we are now in a position to prove the familiar school result that the angle sum of a triangle is 180° or π/2 radians.



The above triangle - formed by the lower red line, the blue and green lines - has three angles coloured green, blue and yellow. Using the upper red line and the properties of parallel lines we can identify the congruent angles - all green angles are equal, all blue angles are equal and the two yellow are angles are equal.

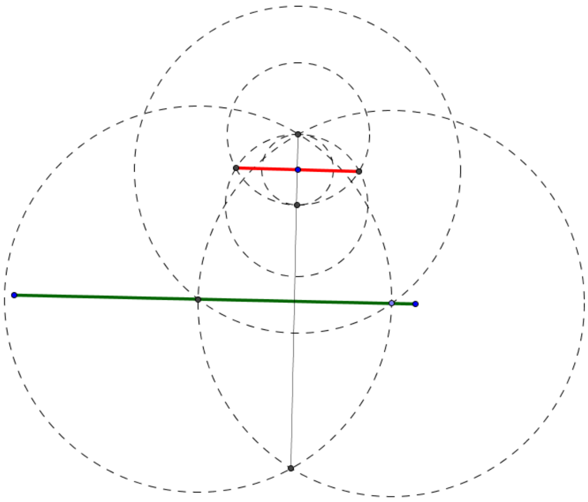
Following the path of the upper red line through the top vertex of the triangle, we can see that angles YELLOW + BLUE + GREEN = a 'half turn' or a straight line. This is therefore the same as the 'internal angle sum' of triangle. By convention there are 180° in both.

Constructing Parallel Lines

We now turn away from the proofs regarding properties to examine the practicalities of construction with straightedge and compass. We'll see though that these theorems and properties associated with parallel lines give us a lot of new deductive power.

Construction 2.07

To construct a line segment through a given point and parallel to a given line segment.



1. We are given a line segment and a point not on the line segment
2. Use Construction 2.06 to construct the perpendicular to the given line segment through the given point, which is not on the first line segment (the result is the thin, vertical, black line segment in the figure).
3. Use Construction 2.05 to construct the perpendicular to this new line segment through the given point, which lies on the first perpendicular (the result is the red line segment, which is indeed parallel to the green line segment and goes through the desired point).

If you require more of a hint for this construction and others, you can adjust the view in Geogebra to show every construction step (not just the 'breakpoints') under the 'Options' menu in the Construction Protocol window.

Squares and other Parallelograms

Having understood the relationship between parallels and perpendiculars, you should now be able to extend the idea to the construction of squares. Producing line segments and copying lengths needs to be done to ensure the end result really is a square, not just a square-ish rectangle. A good way to pose the problem is: given a line segment, construct the square on it (i.e. the square of which the given line segment is a side).

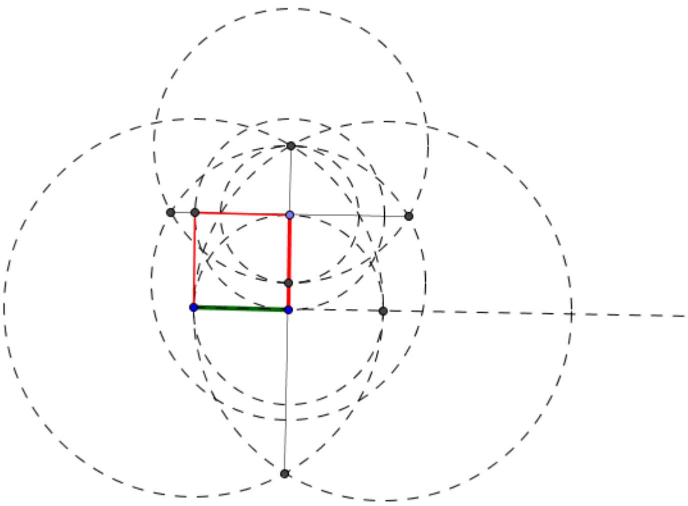
A square is a four-sided polygon that has two important properties:

- It is a rectangle, meaning its sides are at right angles to each other. We now know this means that they are *perpendicular*, and we know how to construct perpendicular line segments.
- It is *regular*, meaning its sides are all the same length. We know how to copy a line segment by storing its length in the compass's "memory".

This may constitute enough hints to work out how to construct a square. Our solution, Construction 2.08, is rather inefficient but it uses only Construction 2.05 and relates very explicitly to these two properties of squares.

Construction 2.08

To construct a square with a given line segment as a side.

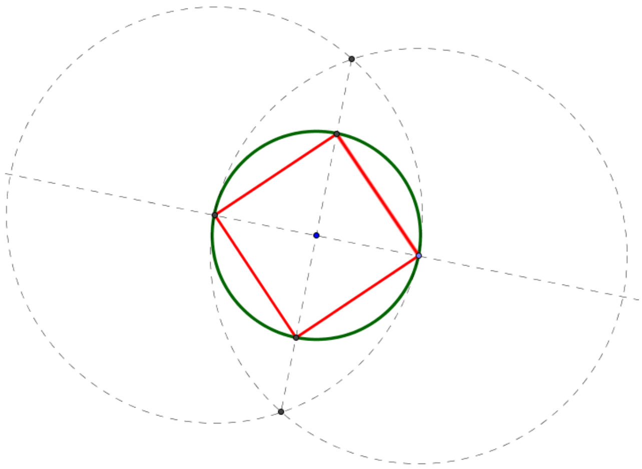


1. We are given a line segment, which will be the first side of the square.
2. Extend the line segment to at least double its original length.
3. Use Construction 2.05 to erect a perpendicular to the given line segment that passes through one of its end-points.
4. Use the compass to copy the length of the given line segment onto this perpendicular (you might need to extend it). That gives the second side of the square (the red side on the right in the figure).
5. Repeat steps 2-4, treating this new line segment exactly like the given one; turning the page 90^o clockwise may help conceptually. The result should be the top red line segment, the third side of the square.
6. Finally, draw the fourth side (red line segment on the left) by joining up the two remaining end points.

It's worth showing another, more efficient way to make a square at this point. It starts with a different problem: given a circle, construct the square that fits exactly inside it, touching it only at the corners. The technical term for this is that the circle is ***inscribed*** in the square. In the following construction we assume the centre of the circle is given along with it. It can be found easily enough if it isn't (Construction 2.10), but we don't usually do the construction in class since we don't need it for anything later.

Construction 2.09

To construct a square inscribed in a given circle.

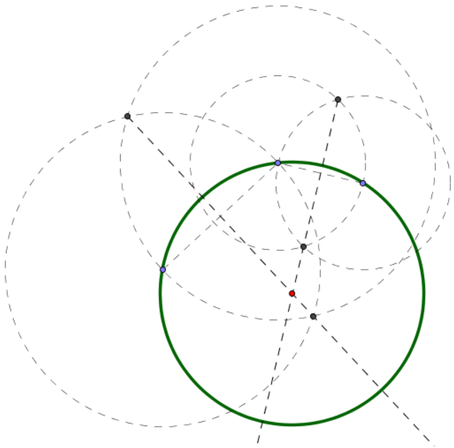


1. We are given a circle and its centre.
2. Draw any line segment through the centre, intersecting the circle at two opposite points.
3. Construct the perpendicular bisector of the part of the line segment that's inside the circle.
4. Identify the two points where the bisector crosses the circle.
5. Join the four points on the circle to form the square.

In case you find yourself working with a circle you didn't draw, it's useful to be able to find the centre. Here is one way; we don't cover enough facts about circle geometry for it to be worth trying to prove it works, or set it as an exercise, but students fresh from (for example) GCSE geometry would make more of it.

Construction 2.10

To find the centre of a given circle.



1. A circle is given but its centre is unknown.
2. Pick any two points on the circle.
3. Connect the two points with a line segment.
4. Construct the perpendicular bisector of this line segment. Make sure it extends well past where the centre of the circle would be.
5. Pick any third point on the circle's circumference.
6. Join it to the closest of the other two points with a line segment
7. Construct the perpendicular bisector of this line segment. Again, ensure the bisector is long enough to go past the centre of the circle.
8. The point where the two bisectors cross is the centre of the circle.

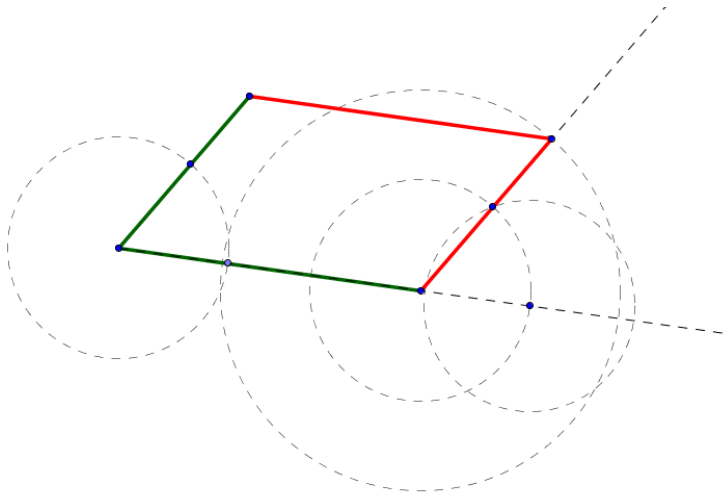
This technique can also be used to prove that given any three points that do not lie on a straight line there is a unique circle where the circumference goes through each point.

The square is just one of a much more general class of polygons called ***parallelograms***, which have four sides and whose pairs of opposite sides are parallel. We'll investigate these shapes a bit further next time. Drawing one is a simple application of copying an angle and a length.

Reviewer: Jeremy Gray
Open University

Construction 2.11

To construct a parallelogram given two adjacent sides.

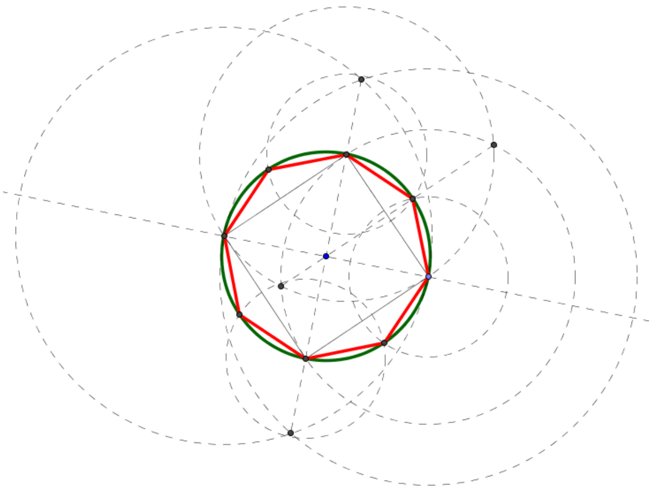


1. We are given two line segments that meet at a point.
2. Extend one of the line segments.
3. Copy the angle between the given line segments at the other end of the extended one (Construction 2.03).
4. "Store" the length of the corresponding given line segment in the compass and duplicate it on the one just constructed.
5. Join up the remaining two endpoints to complete the parallelogram.

Finally, we can also use the square-in-a-circle to construct a regular octagon. The idea is to bisect one of the sides of the square and extend it to the edge of the circle. We then "walk" the new length around the circle. Pay attention to this technique, which will probably be clearer from the Geogebra file than from the verbal instructions – it's used in other places, too.

Construction 2.12

To construct a regular octagon inscribed in a given circle.



1. We are given a circle and its centre.
2. Inscribe a square in the circle (Construction 2.09).
3. Construct the perpendicular bisector of any one of the four sides.
4. Mark the point where the bisector crosses the circle.
5. Join this point to the two nearest vertices of the square, forming the first two corners of the octagon.
6. Put the compass spike on one vertex of the square and the drawing tip on the new vertex of the octagon. Draw a circle at any vertex, creating a new intersection point on the circle.
7. Join this point to its neighbouring vertices to construct the next two sides of the octagon.
8. Repeat this process to get the next two sides...
9. ...and the next two, completing the figure.

We can now construct regular polygons with 3, 4, 6 and 8 sides; you might wonder about the two "missing" polygons, the pentagon (5 sides) and heptagon (7 sides), as well as the "next" polygon, the nonagon. These are all discussed in Session 6.

Division of a Line Segment into Several Parts

We know how to divide a line segment into two equal parts: what about three, four, five or more? The construction that does this requires a clever idea and students will almost certainly have to be shown it rather than being set it as a problem.

We might start by thinking about division into four equal parts by using bisectors or by creating five equal parts by extending the original length by twenty and then halving twice. These approaches work using the methods we have seen so far.

There is though another technique based on Thales's Theorem (see box) which introduces a bit more mathematical proof into the course. It can also serve as a bridge to topics in the Third Session, such as area and Pythagoras.

Note to teachers

We find this proof is a useful diagnostic or differentiation task allowing you to identify students who might enjoy and benefit from more mathematical content beyond the constructions.

This construction is, in our experience, time-consuming and rather tedious to do by hand. It's easy to get confused by the sheer number of lines on the page and their relations of perpendicularity and parallelism. Practice on the earlier techniques pays off here. We find that rotating the construction page sometimes helps with conceptualisation, too.

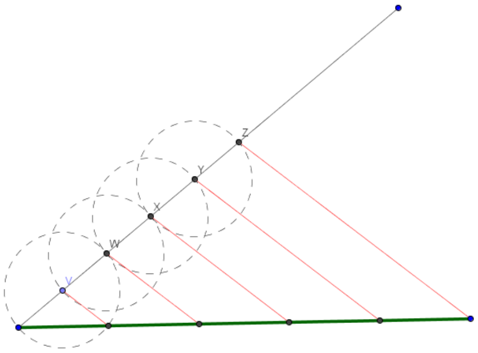
It is also a construction where imprecisions accumulate in drawn constructions and the final result of a first attempt will always fail a simple test of measure. Start with generously-sized lines, use sharp pencils and take your time over accuracy.

Here, it would be worth attempting a second construction in Geogebra for students who remain unconvinced of the method, although seeing a proof of Thales's Theorem will probably help more.

Note that at this stage we usually allow the use of set-squares to speed up the process (once the perpendicular construction is ingrained!). We recommend large, "grown-up" set squares with sharp corners rather than the ones that come in school geometry kits. We make it clear that these are being used as a short-cut only, and that we must always make sure that any construction we do with the set-square could be done by straightedge and compass only, albeit with more labour.

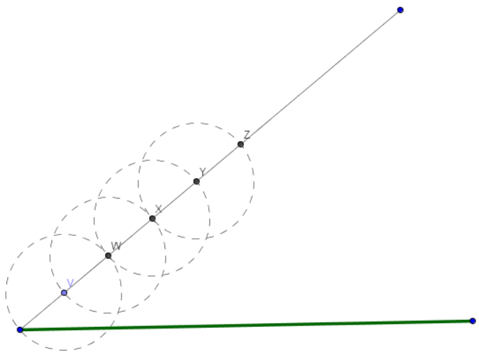
Construction 2.13

To divide a given line segment into five equal parts.



1. A line segment is given.
2. Draw any (long) line segment that shares an endpoint with the given line segment.
3. With the compass, mark out 5 equal (but arbitrary) lengths on the second line segment. You choose your units!

At this point the construction should look like this:



We now have a line segment we *want* to divide into five equal parts, and another line segment that *is* divided into five equal parts. It turns out that we can use a series of parallels to transfer the division of the second line segment onto the first.

4. Join the fifth point (labelled Z in the figure) to the end of the given line segment.
5. Construct a line segment parallel to this one, passing through Y.
6. Repeat this process for the other points (X, W and V).

In the Geogebra file we have used the preset “parallel line segment” tool. This keeps the image reasonably clean but is a bit deceptive; recall that each parallel requires two perpendicular bisector constructions and it’s easy to see how fiddly it can be to do the whole thing by hand.

Reviewer: Jeremy Gray
Open University

Thales' Theorem

Does Construction 2.13 just given "work"? That is, does it really divide the given line segment into the requisite number of *equal* parts, or is it a mere approximation?

The answer is provided by Thales' Theorem, which demonstrates that parallel lines which cut across other lines cross in such a way as to cut the other lines *proportionately*.

We have provided the proof as a separate slide presentation in the online resources as a powerpoint file; it's adapted from Stillwell (p.34-35). This version of the Thales proof depends on arguments relating to the area of triangles, which we cover in the next Session. You may already be familiar with area, in which case feel free to look over the proof now. Otherwise, we indicate in the next Session when the relevant concepts have been introduced.

Note to Teachers

We haven't found a way to do this without talking about ratios of lengths. Fractions are a well-known stumbling-block in school maths education and some students will inevitably find their appearance off-putting.

Given more time it would be nice to allay their fears by showing that they already understand the idea of geometric *proportion*, and that a ratio of lengths is just a symbolic way to express this. For some courses it may even be worth breaking this topic out into a whole separate session, along with the line segment-division construction in the previous section, and adding some more material on geometric interpretations of fractions.

Another way to look at this construction is as a *parallel projection* from one line segment onto the other, but in the context of this course that offers very limited assistance.

Third Session: Making Sense of Area

Summary

So far we've been working in the two-dimensional space of a sheet of paper but only thinking about points, lines and circles. As we'll see in this session, points can be thought of as zero-dimensional and lines as one-dimensional. What's more, although we've constructed some circles, triangles and squares, we've so far only been concerned with their outlines – that is, the one-dimensional line segments that enclose them. In this session we start to work with properly two-dimensional objects directly.

We discover the formula for the area of a triangle and use that to investigate the areas of parallelograms. We then work through Euclid's proof of Pythagoras' Theorem as a presentation. We like to do this rather slowly as it's a powerful and non-obvious result that students are entitled, at this stage, to feel they fully understand.

Note to Teachers

There may be room for more supplementary material in this session than usual, especially of course if the students have seen this proof before. Still, it's worth checking the students have truly understood their prior exposure to it, rather than just absorbed it as a calculating technique, and that they can reproduce all the steps.

Congruence, Measurement and Area

We've already seen the idea of copying and comparing lengths without the need for standard measurement units, and the idea of comparing line segments qualitatively, allowing us to say that one is longer than, shorter than or equal to the other.

This is formalised by the idea of ***congruence***. The word comes direct from Latin *congruere*, "to come together". The intuitive picture to have is of superimposing two figures by sliding them around on the plane: if they can be made to match perfectly, one on top of the other, with no discrepancies, they are congruent.

Sliding figures around on the page is the kind of activity we expect very young children to be able to carry out, and it seems too obvious to be interesting. Euclid uses it without comment; it's usually justified by appealing to a rather specific interpretation of his fourth common notion. The philosopher Arthur Schopenhauer (1788-1860) famously objected to it (in *The World as Will and Representation*, Vol 2 Ch. XIII) and was not the first or the last to do so (see Kline, p.1005-6). Yet something like this underlies many of our ideas of "sameness" in everyday life. After all, when we measure a line segment using a ruler we place the ruler alongside the line segment, saying that the line segment is "the same length" as some portion of the ruler that's marked in a standardized way.

In Euclidean geometry area, like length, is qualitative: we compare the areas of two figures by construction rather than using numbers. The mechanism for this is ***quadrature***. Suppose we want to know whether two given rectangles cover the same area. To answer the question we will, for each rectangle, *construct* a square whose area is the same. This is the rectangle's 'quadrature'. If the quadratures of two figures can be shown to produce congruent squares, the figures are *defined* to have the same area.

Note to Teachers

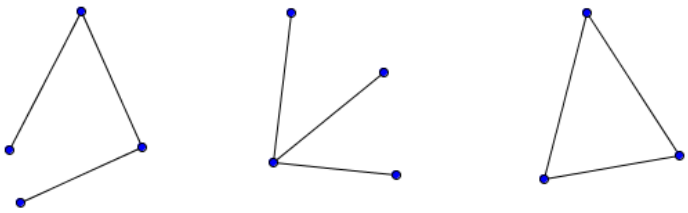
This is a good point at which to introduce the idea of an equivalence relation, if you wish to do so. The point can be made simply by providing a set of card triangles and asking students to sort them into piles of "the same shape". The piles are then equivalence classes.

Zero, One and Two Dimensions

At this point we can pause to consider a few of Euclid's fundamental definitions, from the beginning of Book I, paying particular attention to definitions 3 and 6:

- Definition 1: A point is that which has no part.
- Definition 2: A line is breadthless length.
- **Definition 3: The ends of a line are points.**
- Definition 5: A surface is that which has length and breadth only.
- **Definition 6: The edges of a surface are lines.**

This introduces the idea of a figure having a ***boundary***. In fact, when we constructed a shape like the equilateral triangle we really only constructed its boundary, which consists of three line segments connected together in a certain way. Under what conditions do three line segments make a triangle?



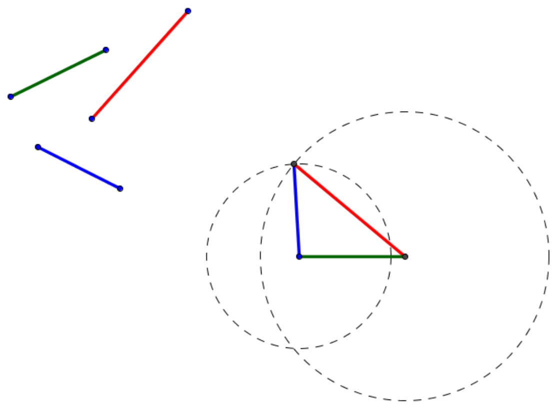
The answer seems to be: when each of their endpoints is shared by exactly *two* line segments. If we travel once around the boundary we must enter and leave each endpoint exactly once.

This idea can be generalized to higher dimensions. It seems that a three-dimensional volume must be bounded by two-dimensional surfaces whose own bounding line segments are all shared in pairs, and this indeed works, producing *polyhedra* including the tetrahedron and cube. These, and the many questions they raise, are topics for another course.

Congruent Triangles

We don't usually do this in class but students interested in or bothered by the idea of congruence may find it interesting. It's also part of the traditional geometry often taught in schools.

Any triangle has three sides joined together to make three internal angles. Our first congruence theorem says that given three segments there's really only one way to assemble them into a triangle.



Experiments with some lengths of wood should make this convincing; the triangle in question can also be constructed by copying lengths, as shown above. Because there are only three of them, there's essentially no other way to put the lengths together. This is the content of Proposition I.8 in the *Elements*, sometimes called by the short name "SSS" for "side-side-side".

Similarly, suppose we are given two segments joined in a fixed angle. It's reasonably clear that there must be only one way to complete the triangle:

This is Proposition I.4 in the *Elements*, sometimes called "SAS" for "side-angle-side". It's important that the angle we're given is the one between the two given segments, otherwise we often have more than one way to complete the triangle.

How does Euclid prove these propositions? The answer is a bit unclear; he uses a method known as "superposition" that isn't explicitly described in the *Elements*. The method seems to involve sliding one of the triangles on top of the other, matching up corresponding sides as you go, and showing that everything "lines up" perfectly. This is such an intuitively obvious way to check that two physical shapes cut out of paper (say) are the same that perhaps it strikes us as not even worth describing.

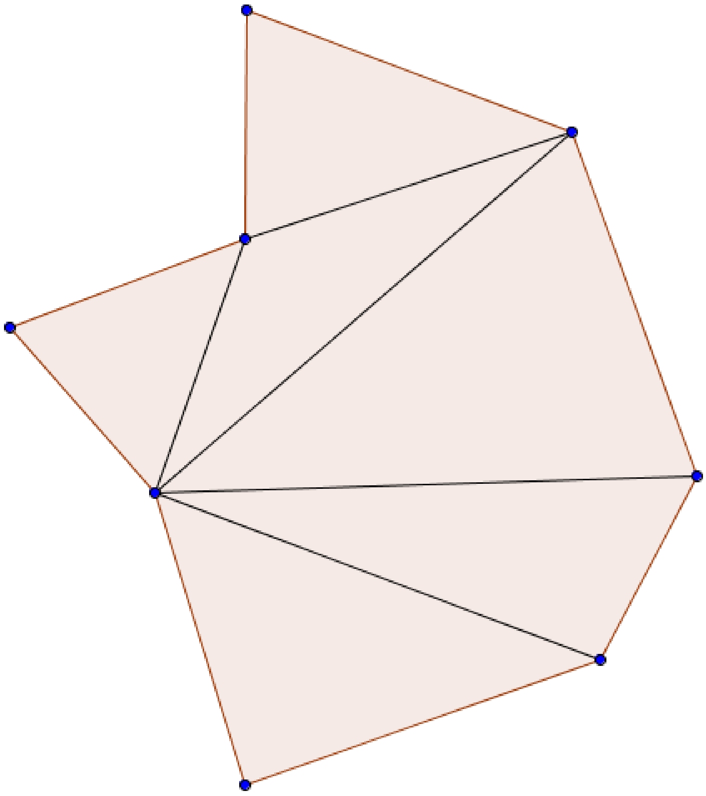
A small complication comes from the fact that you may have to flip the triangle over. This may seem to violate the idea that our geometry is limited to the plane, but it certainly makes sense to ask whether two triangles are congruent when one is on the wall and the other's on the table – we don't have to step outside plane geometry to answer that question.

On this course we help ourselves to the assumption (that is, the axiom) that plane figures can be moved, rotated and reflected and that by doing so we can "line up" their parts in the way Euclid seems to want to. More rigorous modern treatments often take different approaches. After all, to a large extent we get to choose which things we want to make our axioms and which we want to try to prove.

Triangles and Parallelograms

We usually define the area of a rectangle to be its base multiplied by its height (i.e., width multiplied by length). This is almost certainly familiar from previous schooling. We give an explicit construction for the *quadrature* of the rectangle below, which is more in keeping with the spirit of Euclid, but it's a good idea to start with something more familiar.

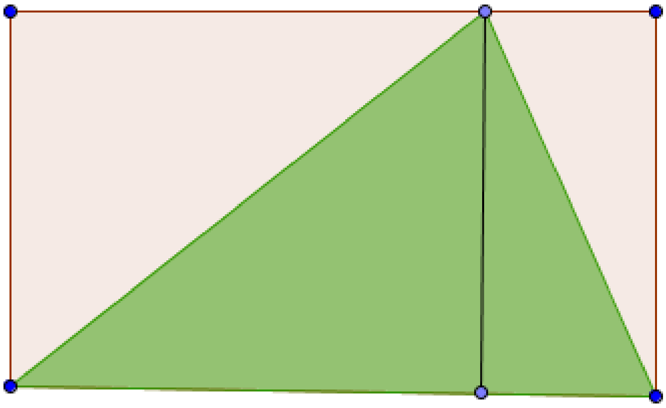
Triangles turn out to be very important for the study of area because any straight-edged polygon can be decomposed into triangles and, as we shall see, we can convert any triangle into a square of the same area.



Note to Teachers

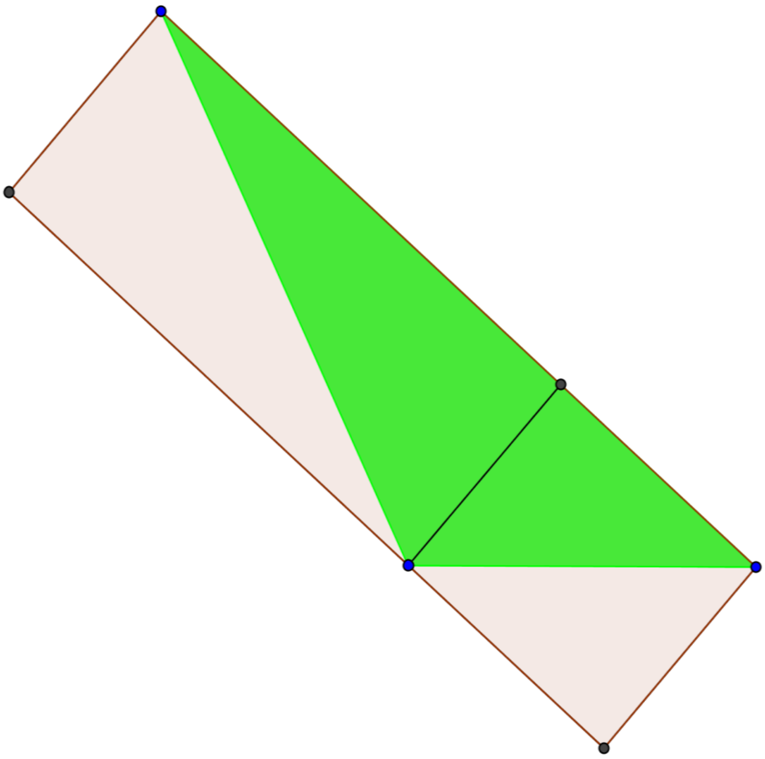
It's easy to prove that convex polygons (those whose internal angles are all less than 180°) can be triangulated in this way, but less easy to extend this to all simple (i.e. non-self-intersecting) polygons. Joseph Malkevitch's article points towards some accessible ways to explore this topic (see the online resources).

It’s a simple matter to see that any triangle can be “surrounded by” a rectangle that it takes up exactly half of the rectangle’s area.



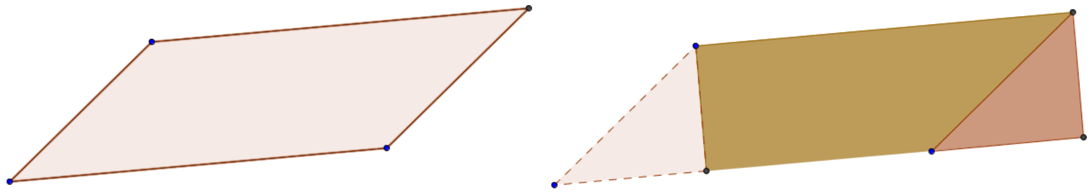
If necessary, this can be made clearer by cutting up rectangles and reassembling the pieces. We define the area of the rectangle to be the long side times the short side, or the base times the height. The area of the triangle is then equal to *half base times height*, in the sense that the triangle takes up half of the area of the rectangle or, to put it more plainly, we can fit two copies of the triangle into the rectangle if we cut them up. Note that we need to ensure that we measure the 'height' as a perpendicular line segment in relation to whichever sides we choose to be the base.

You might sometimes need to rotate your rectangle to make the argument clear.



(You now have the concepts you need to follow the Thales proof we mentioned at the end of the Second Session).

A similar argument can be made for parallelograms: this just involves cutting one triangle off the parallelogram and moving it to make a rectangle. It's well worth constructing some parallelograms (thus reviewing the method from the previous session) and physically cutting them up, then reassembling them into rectangles (parallelograms whose angles are all right angles! this is important!). Slightly more confident students can do the "rearrangement" by construction as well.



The area of the parallelogram is therefore the same as the relevant rectangle.

Quadrature

We've so far been referring to area, which we learned about at school as a product of two lengths (at least in the case of rectangles). This definition isn't very useful, though, when we want to compare the areas of shapes using only straightedge and compass. Fortunately, there's an easy and logical way to do it. The constructions are a bit long-winded but the ideas are quite straightforward.

Given a shape, we want to find its ***quadrature*** – a square that has the same area. We can then directly compare the quadratures of two shapes. It's a way of putting the shape into a "standard form" that allows for judgements to be made about its area without resorting to measurement or, indeed, *numbers* of any kind.

Area is as defined above, while quadrature we take to be a specific technique of constructing a square that has the same area of a given polygon.

Pythagoras' Theorem

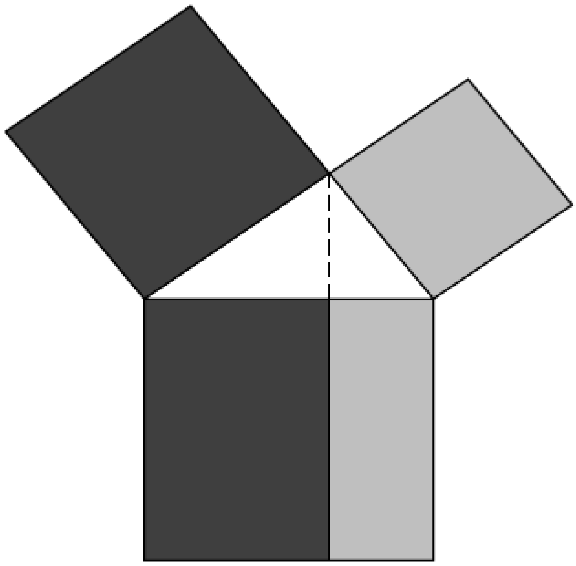
We have what we need at this stage to introduce Euclid's proof of Pythagoras's theorem: the sum of the square constructed on the hypotenuse is equal to the sum of the squares constructed on the two other sides.

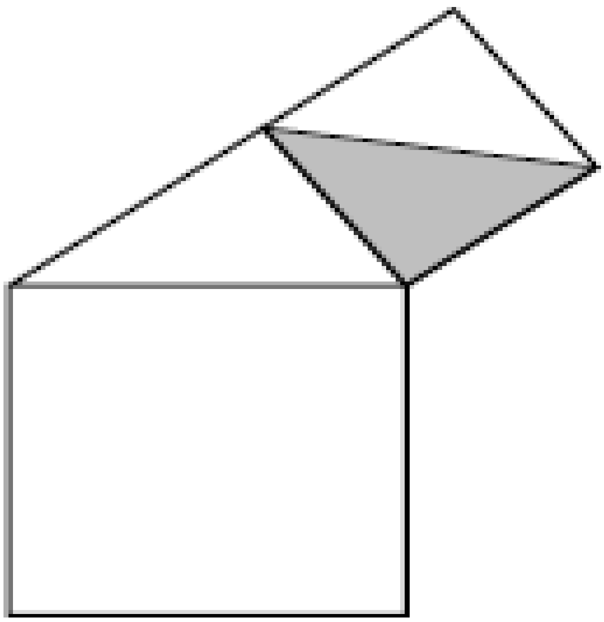
Euclid's specific proof also provides us with the technique we need to construct, for any given rectangle, its quadrature.

We do this as a presentation, using Euclid's own proof that relies only on the conditions under which two triangles have the same area. Our slides are included in the online resources in a powerpoint file called 'Pythagoras Proof'. It is adapted from Stillwell, pp. 32-33.

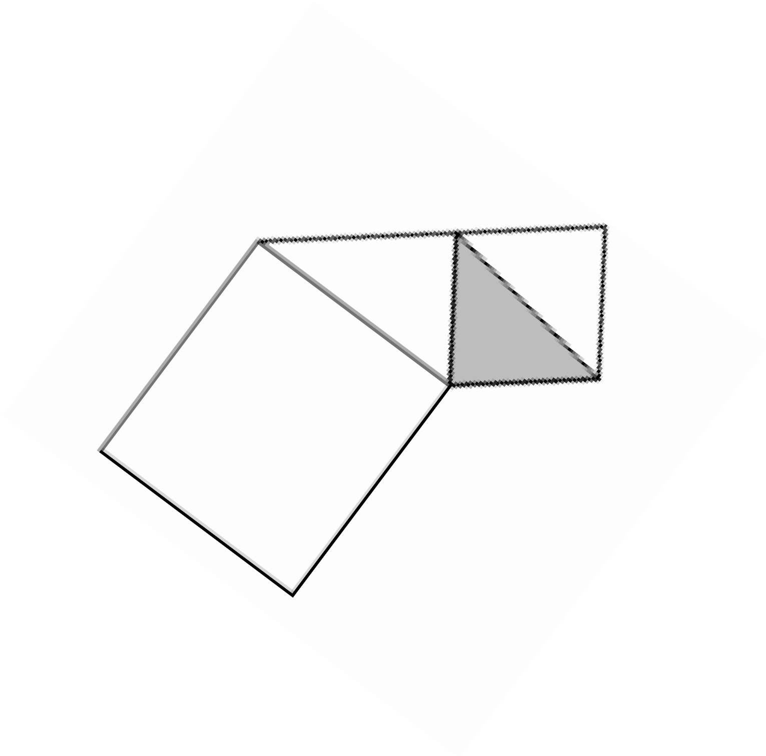
It's important to emphasise that the theorem only works for triangles that have a right angle at one of their vertices. While presenting the proof it's worth pointing out where we use this assumption so it's clear that the same proof won't go through if it's dropped. This is important because, of course, in that case the "theorem" would be false!

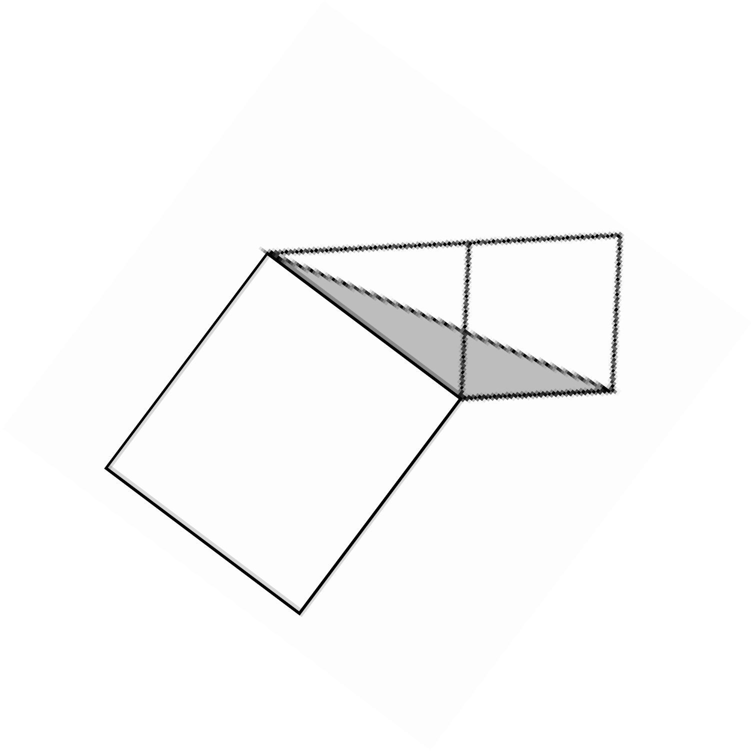
We start with a right-angled triangle with squares constructed on each side. We will prove that each square on the two shorter sides corresponds in area to one of the shaded portions of the square on the hypotenuse.



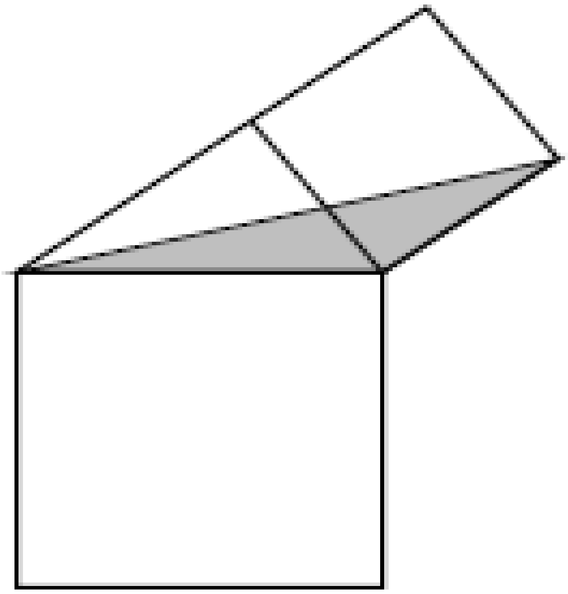


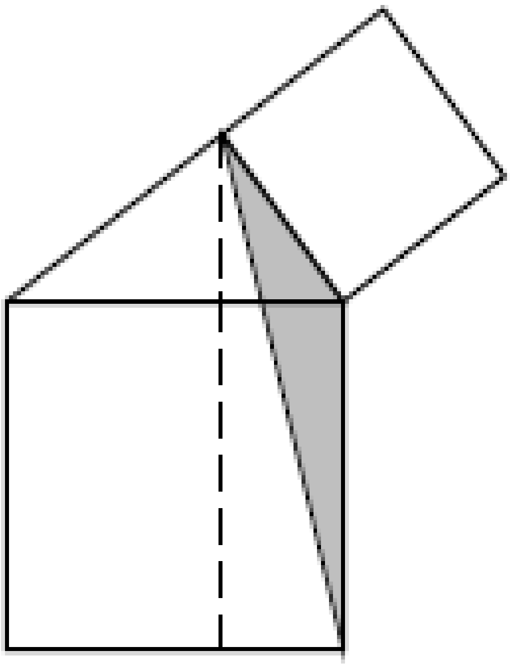
The proof proceeds by using the triangle formed by cutting the square along its diagonal.



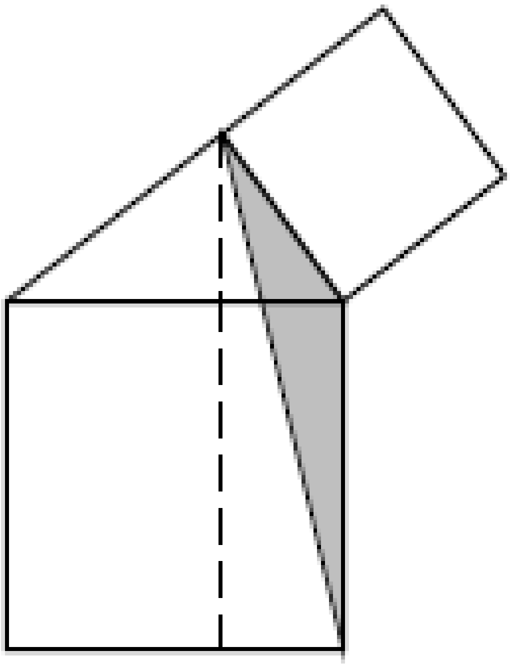


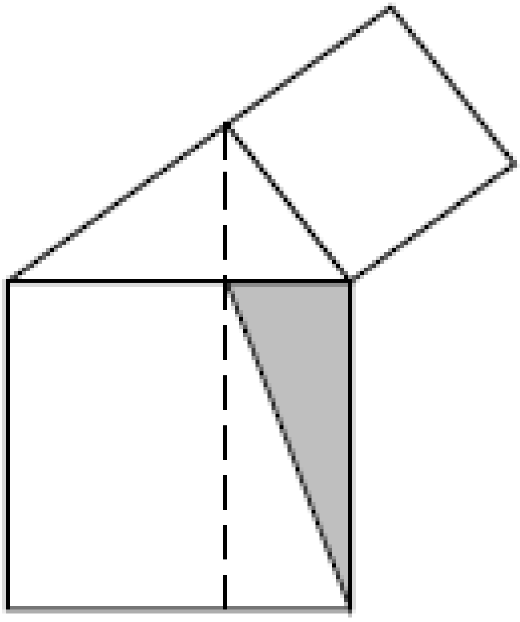
The two triangles above have the same area as they have the same base and height.





The two triangles are the same by rotation given the property that squares have sides of the same length.





The two triangles have the same area because they have the same base and height (as indicated by the dotted line).

Half of the original square is therefore the same area as half of the indicated rectangle. Therefore the whole square has the same area as the associated rectangle.

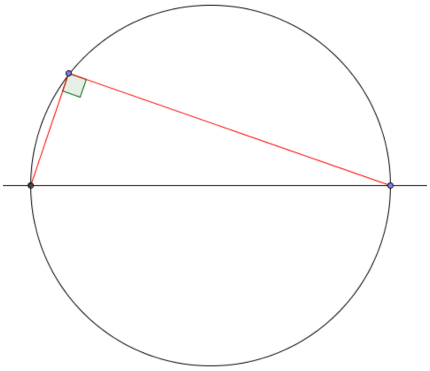
By the same argument, we can repeat for the other shaded rectangle and its corresponding square..

Therefore: The area of the square built on the hypotenuse is equal to the sum of the areas of the two squares built on the other two sides.

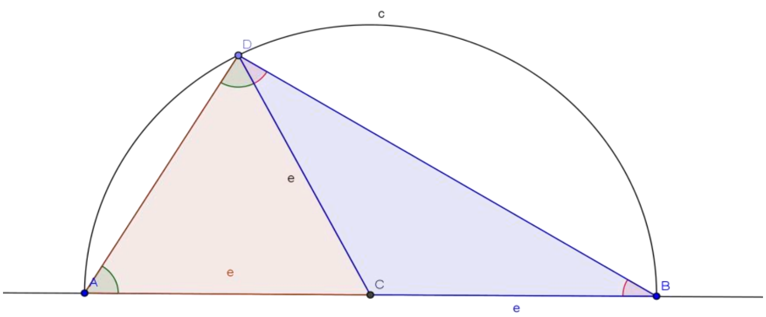
Reviewer: Jeremy Gray
Open University

Note to teachers

One fact we'll use in the quadrature is that the angle in a semicircle in always a right angle. What this means is that if you pick any point on a circle and join it to the points where a line segment through the centre crosses the circle, you get a right angle:



This is not self-evident and students who don't believe it probably deserve to see a proof. To do this, though, several steps are needed. First show that the angles in a triangle are 180^o. Next show that an isosceles triangle always has two equal angles. Finally, prove that the angle in a semicircle is a right angle.



The lines AC, DC and BC are all radii (e) of the semi-circle centred on C and are therefore equal in length. The shaded triangles ACD (orange) and CBD (blue) are both therefore isosceles triangles.
Angle CBD = Angle CDB (green marked in red)

Angle CDA = Angle CAD (both marked in green)

For triangle ABD, its angle sum is CAD + CBD + (CDA +CDB) = 180°

But CDA = CAD & CDB = CBD so CDA + CDB + (CDA + CDB) = 180°
2 x (CDA + CDB) = 180°

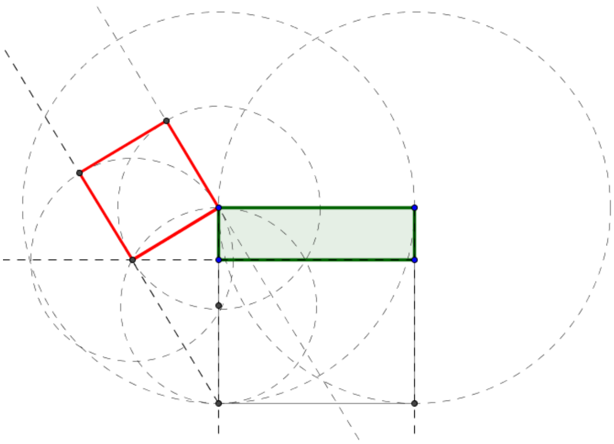
CDA + CDB = 90°
Therefore red + green = right-angle

A Quadrature Construction

We begin with a construction to get the quadrature of any given rectangle. The solution below appears to work by magic, at least until you notice that in fact it's just Euclid's Pythagoras proof in reverse!

Construction 3.01

To construct the quadrature of any given rectangle.

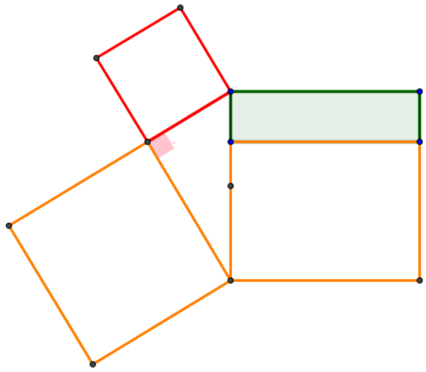


1. We are given a rectangle.
2. Copy the length of a long side along the direction of a short side.
3. Do the same on the other short side and join up the endpoints, so we now have a square.
4. Find the midpoint of one of the extended sides of this square.
5. Draw a circle centred on the midpoint and extending to the two nearest vertices.
6. Extend one of the long edges of the given rectangle so that it cuts through the circle.
7. Draw a line segment from the top vertex of the rectangle to the intersection point just found. This is one side of the square.
8. Complete the construction of the square (see Construction 2.08).

How would anyone think of such a bizarre combination of moves? Why does it work? The answer is that this is simply the famous diagram of the Pythagoras proof: we just did everything backwards compared with how the picture is built up in the proof. We relied on our ability to construct the necessary right-angled triangle.

We started with a rectangle and built up the square on the hypotenuse from it (steps 1-3), then effectively constructed a right-angled triangle with that hypotenuse. As proved above, every triangle constructed from the circumference of the semi-circle will be right-angled (steps 4 & 5). But we can find the one we need by extending the line segment from the given rectangle that will intersect with the semi-circle (steps 6 & 7), and finally we construct the square on its short side (step 8). By Euclid's Pythagoras proof the square constructed has the same area as the given rectangle.

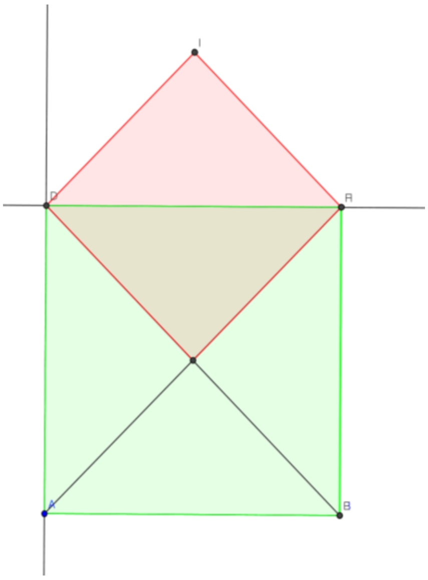
Here is the same Geogebra construction with the construction lines taken out and the other square and rectangle added (orange):



Now that we can find the quadratures of rectangles we can easily extend the technique to cover other shapes. For example, we can find the quadrature of a triangle by (a) turning it into a rectangle, (b) cutting the rectangle in half and then (c) constructing the square corresponding to that new half-sized rectangle.

Construction 3.02

To construct a square half the size of a given square.

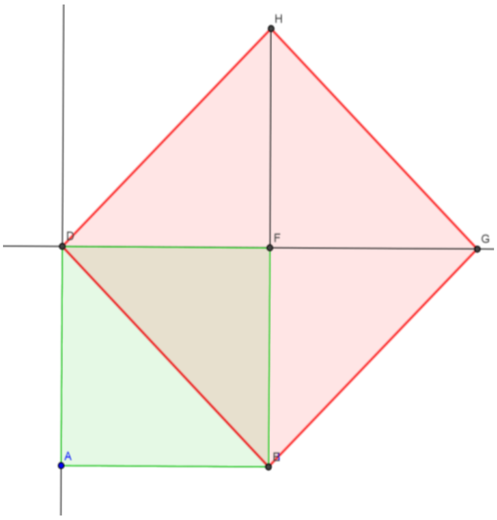


1. We are given a square.
2. Connect two opposite corners with a line segment to form the diagonal.
3. Connect the remaining two corners to form the second diagonal.
4. Construct a new square using one of the four congruent triangles made by the diagonals as a half.

The new square is half the area of the first square as it is composed of two of the congruent triangles while the given square is made of four.

Construction 3.03

To construct a square twice the size of a given square.



1. We are given a square.

2. Connect two opposite corners with a line segment to form the diagonal.

3. Construct a new square using the diagonal as a side.

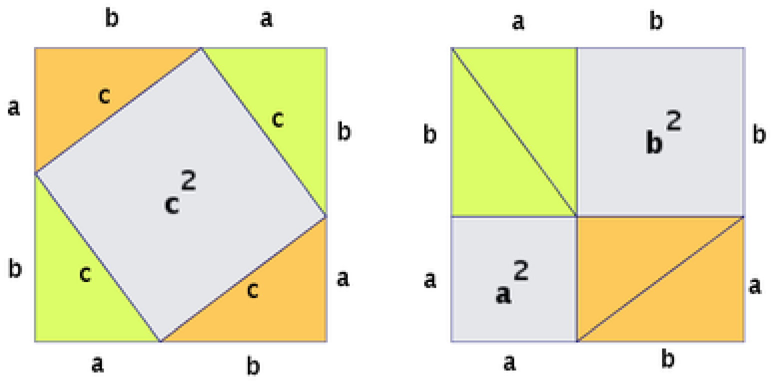
The new square is twice the area of the first square as it is composed of two of the congruent triangles while the given square is made of four.

For those with philosophical interests, closely related proofs are 'drawn out' of the slave boy by Socrates in Plato's dialogue *Meno*.

Furthermore, since any polygon can be dissected into triangles, each of which can be transformed into squares of equal area, the problem of the *whole* areas of polygonal shapes can been solved in a very general way if only we can find a way to *sum up* the separate areas at the end.

Summing the Areas of Squares

Summing the areas of two squares can also be done in a very visual manner, another result derived from Pythagoras's Theorem:



(Image from http://math.stackexchange.com/questions/563359/is-there-a-dissection-proof-of-the-pythagorean-theorem-for-tetrahedra -- adventurous students may like to read around this topic, starting with this link.)

The square 'c' in the left-hand image is the same size as 'a' and 'b' from the right-hand side combined.

Treating this famous pair of images as a hint towards a *construction* allows us to solve the problem: given two squares, construct the square whose area is the sum of their areas. Or, expressing the same thing algebraically, we're solving the equation

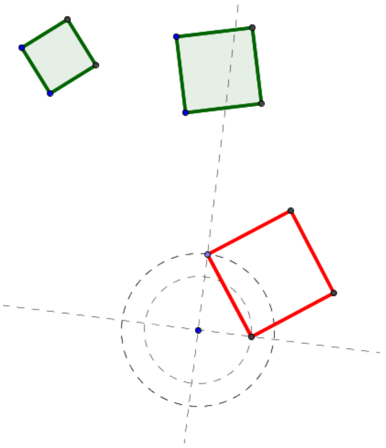
$$c = \sqrt{a^2 + b^2}$$

by construction.

Through repeated use of this method any number of quadrature results can be combined to one final quadrature. This gives us a general-purpose method for "squaring" any shape that we can divide into rectangles and/or triangles. The solution is suggested by the pictures above and depends only on Pythagoras's Theorem, so try to devise it on your own before reading on.

Construction 3.04

To construct a square whose area is the sum of the areas of two given squares.



1. Two squares are given.
2. Draw an arbitrary straight line segment.
3. Pick a starting-point on your line segment, “store” the length of a side of the first square with your compass and draw a circle centred on the starting point.
4. Do the same with the second square.
5. Construct a perpendicular to your original line segment, passing through the starting point.
6. Mark where the SECOND circle crosses this perpendicular.
7. Join the two intersection points.
8. Draw the square using this line segment as one side.

Reviewer: Jeremy Gray
Open University

Fourth Session: Tilings

Summary

This is a lighter and more hands-on session than the preceding ones. We allow plenty of time for drawing. It gives students a chance to practice the constructions they've learned so far and to think a bit more carefully about the idea of area.

We cover the three regular tilings of the plane, then some Islamic patterns. This is a good opportunity to revisit the idea of symmetry that's been mentioned a few times and to bring it into focus in a slightly more modern way.

It's also the occasion on which more creative practices begin to come to the fore and students get a sense of the possibilities associated with the techniques they have begun to master. The prevalence of the pentagon in Islamic tilings is a motivator for learning the final fundamental technique of this course - the square root constructions of Session 5.

The Idea of a Tiling

From the beginning of the course we've worked with a sheet of paper as our medium. In a sense the paper started out invisible: it was just the substrate for our lines and circles, a stage on which they could be presented. In looking at area in the last session we began to "see" the surface of the paper as something interesting in itself. In this session we take this a few steps further, looking at ways to "find our way around" this surface.

For us, a tiling is a construction that divides up the page. Since the page may be as big as we like, such a construction must be a procedure we could carry out as many times as we need to: indefinitely often, even if not really infinitely. A tiling is a way to "tame" the smooth open space of the page.

We won't worry about the tiling filling the page entirely or exactly. If we're able to tile a part of the page, and we see how to extend the process to cover a larger portion, that's sufficient. The exact size and proportions of the sheet of paper are irrelevant for us.

These tilings are attractive and interesting in themselves, and have great importance in the history of visual culture in many parts of the world. They return if and when students meet other kinds of space besides the flat plane, where other tilings may be possible and the ones we look at here impossible. They may also be revisited in studies of symmetry or various other mathematical contexts; adventurous readers may want to look at Grünbaum & Shephard or at Conway et al. (see the Bibliography).

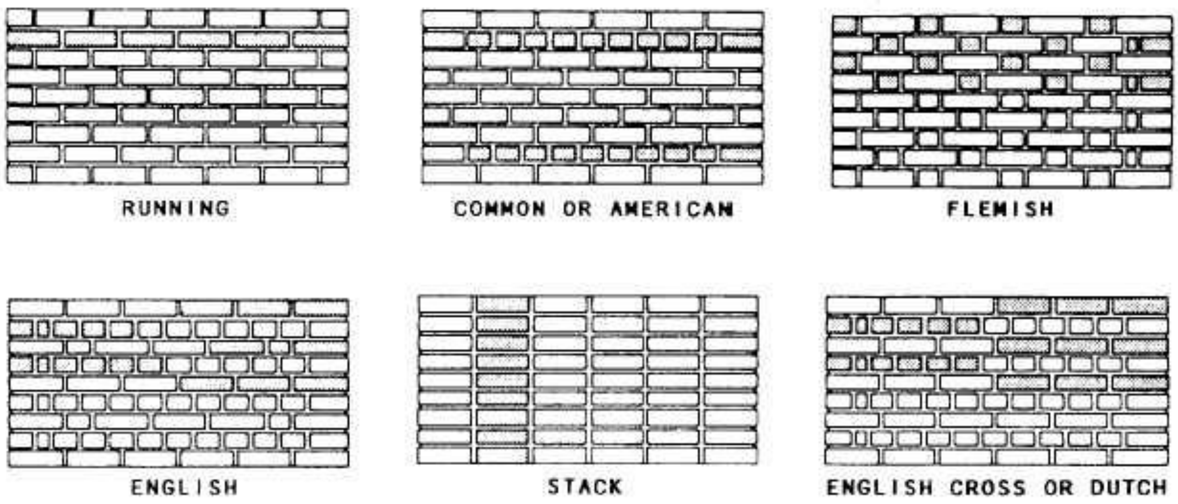
Euclidean and Related Tilings

To keep things simple at first, we limit ourselves to tiling with a single polygon. Everybody "knows" we can tile an area with squares. With luck, such a tiling might even be visible on the floor, walls or ceiling of the classroom itself. Constructing them rigorously is a time-consuming exercise in repeated perpendicular constructions and isn't very enlightening.

Note to Teachers

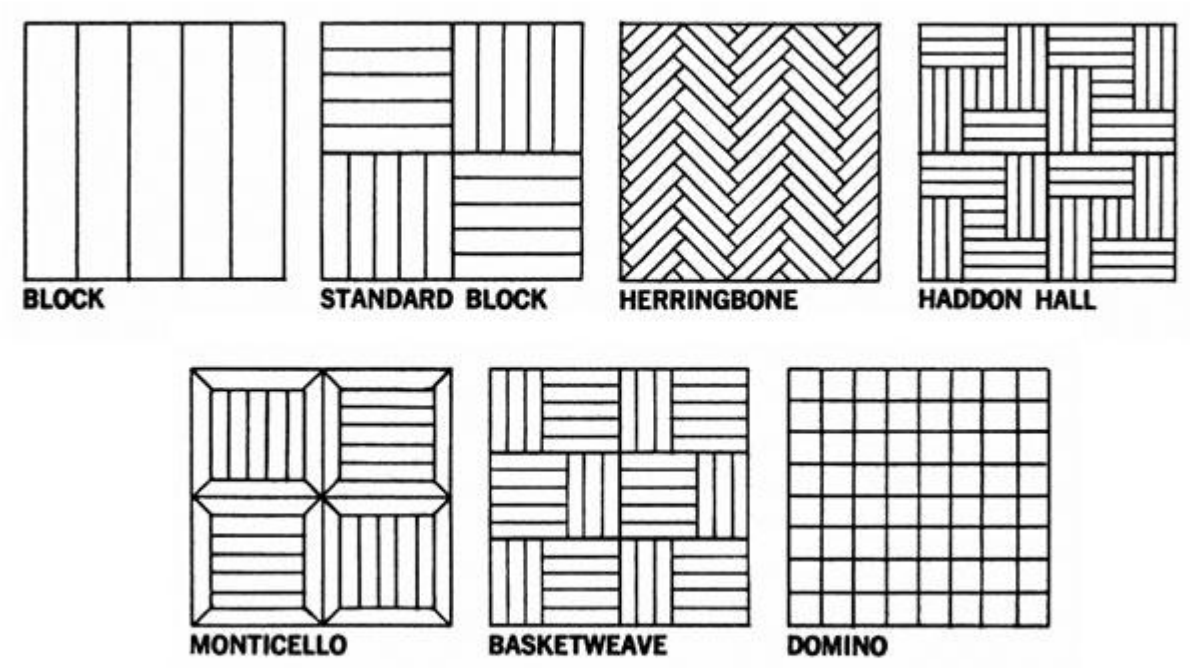
Students may be encouraged to think about tilings by rectangles and parallelograms, too, and to sketch them. This opens the way for an intuitive discussion of getting from one to another by stretching, squeezing and shearing transformations. This can open a way to a discussion of linear transformations if that's appropriate for your group; since we don't assume familiarity with analytical geometry we don't follow that route.

You may also enjoy discussing the tilings that appear in brickwork, but note that many standard "bonds" used by bricklayers use the bricks both end-on and side-on, producing tilings with two distinct shapes:



(Image from tpub.com)

A better source is the similar patterns used in parquet flooring:

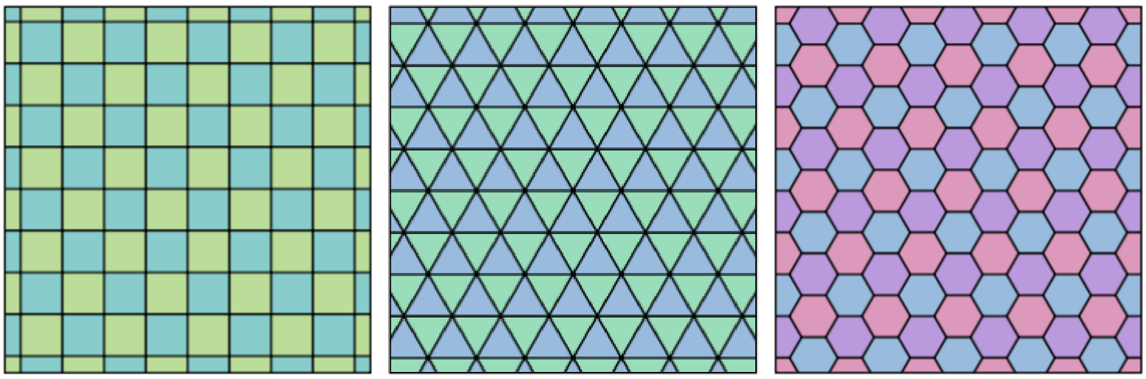


(Image from Victoriana.com).

Note to Teachers

For a group of students who were not interested in drawing activities, these patterns can be used to lead into puzzles about dominoes and their generalizations, polyominoes. We have spent an enjoyable two hours with adult learners working on problems from Golomb's famous book on this subject (see Bibliography), including chessboard dissections and similar tiling-related topics.

With the idea of a tiling fixed, we move to tiling with equilateral triangles, regular pentagons and regular hexagons. Are they possible? Experiment, again by sketching or, if they're available, manipulating some actual physical tiles. Soon enough you should come up with the three "uniform" tilings, and will perhaps feel there's something especially simple or regular about them (the colours in these images have no significance):

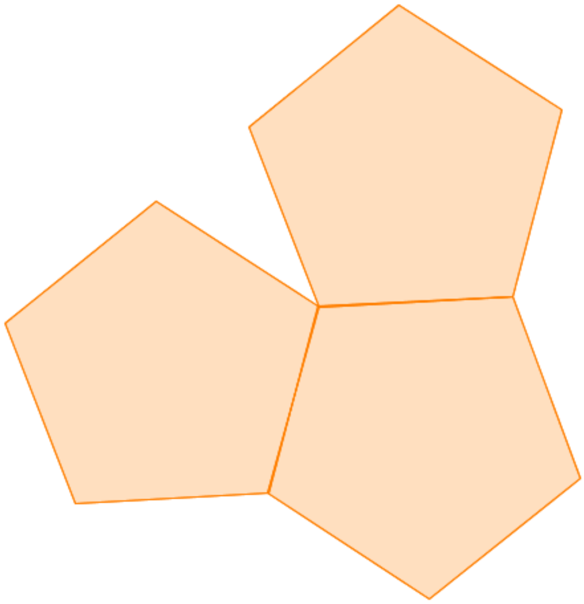


(Images from Wikipedia)

Part of their regularity comes from the fact that the tiles are all identical regular polygons. Additionally, the vertices of the tiles always meet together (a vertex never abuts the middle of an edge, as in a running bond brick wall), and the same number of tiles always meet at each vertex.

We can interpret each meeting of vertices as a sum of the angles at the vertices. In the square tiling, four corners are added to produce a single full turn. With the triangles, six corners add to the same amount. With the hexagons, three add to again give the same result. Note that we are still merely counting angles rather than appealing to numerical measures of their sizes.

Though it falls short of a proof, the impossibility of using regular pentagons in this way can be convincingly shown by the following figure, which shows that having three pentagons meet at a point isn't enough, but doesn't leave room for a fourth:



A more general question: Are there any other regular polygons, with more than six sides, that can make a uniform tiling? Try to answer this question before reading on.

The key observation is that when we add a side to a regular polygon the angle between the sides gets larger. So when we take a square and add a side to make a regular pentagon the angle at each vertex increases to 'make room'. Hence, we must be able to fit *fewer* of them around a single point than before. With the hexagonal tiling we have three angles meeting at a point. It follows that if we're to make a uniform tiling with any regular polygon with more than six sides, fewer than three of them will have to meet at a point. Yet it's intuitively clear that a polygon can't have angles that are an entire half-turn (180^o) – such a "polygon" would be a straight line segment, and couldn't "enclose an area" on its own. So no uniform tiling on the plane is possible with regular polygons having more than six sides.

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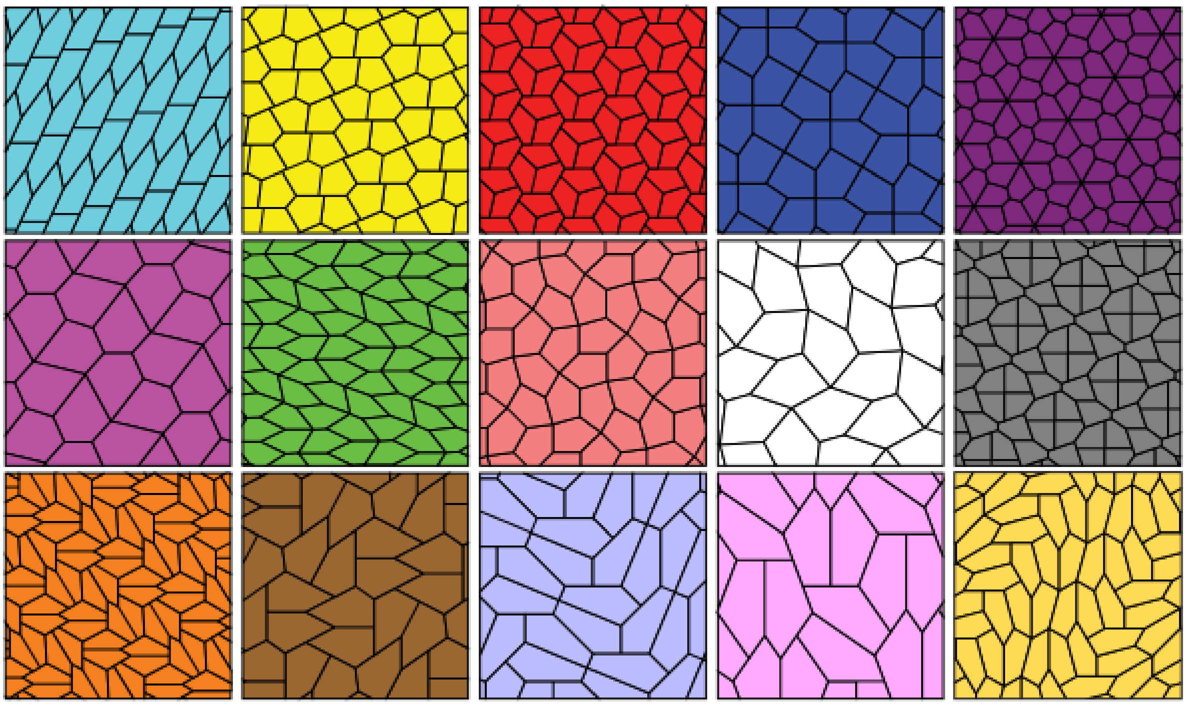
Note that all these statements so far have concerned regular polygons, whose sides and angles are all identical. In fact you can tile the plane with any triangle and any quadrilateral (four-sided polygon), but only with certain irregular pentagons – see Alex Bellos's article in the online resources for more information.

In the image below, each tiling is done with one irregular pentagon. There are only fifteen known to work, with the most recent (yellow, bottom right corner) only discovered in 2015.

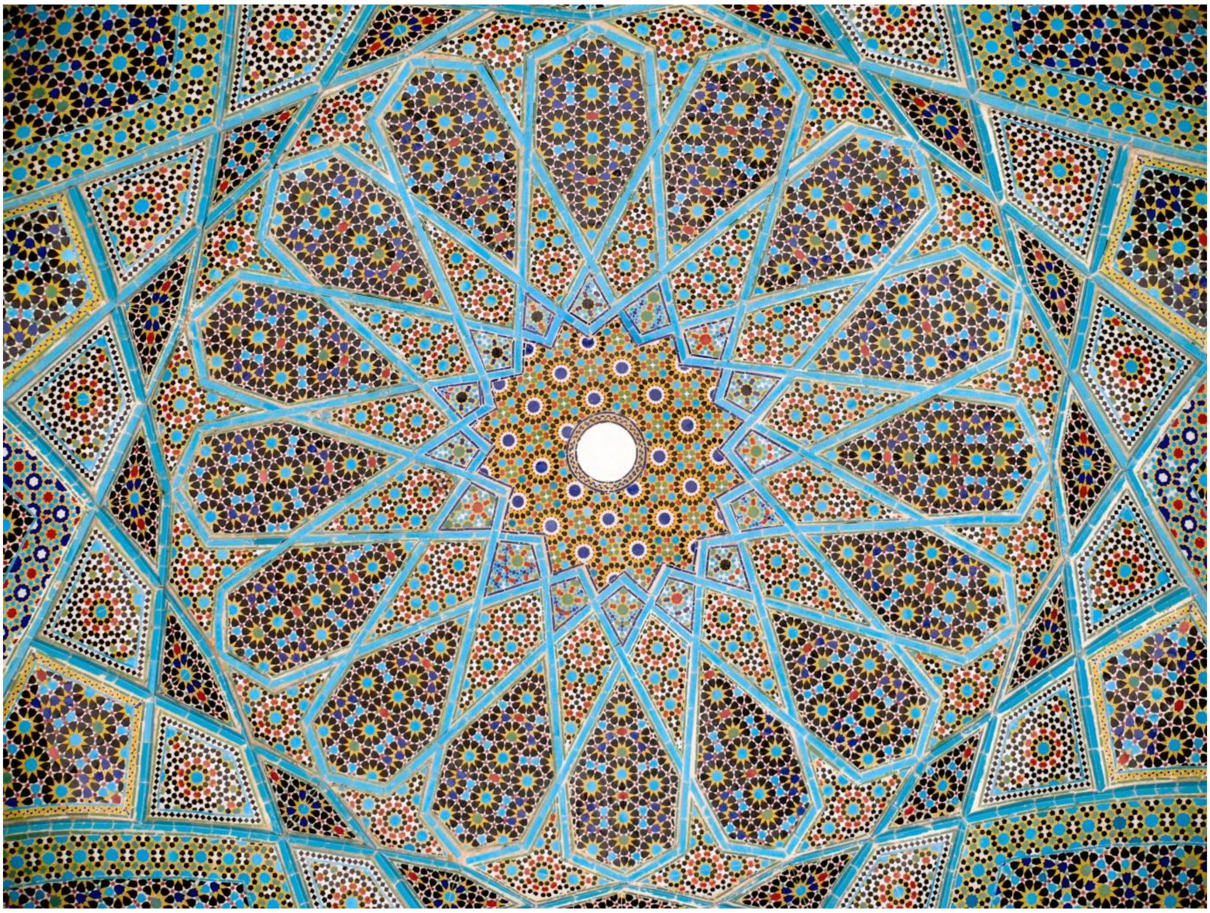


(Image from Wikipedia)

Islamic Tilings

A huge number of tiling variations could now be explored: Grünbaum& Shephard offers a splendid collection while Conway et al. present colourful images and a rather sophisticated discussion of their symmetries (see the Bibliography). We choose to focus on patterns from the Islamic tradition for a number of reasons.

First, their constructions are relatively easy to understand. Second, there are good resources available online. Third, they look cool. Finally, they remind us that these techniques were developed and preserved by artisans and artists in the middle ages, and that geometric knowledge can take this form as well as the more abstract form we find in textbooks.



(Image from Wikipedia)

For this part of the class we make no claim at all to originality: we use some of the activities from this PDF:

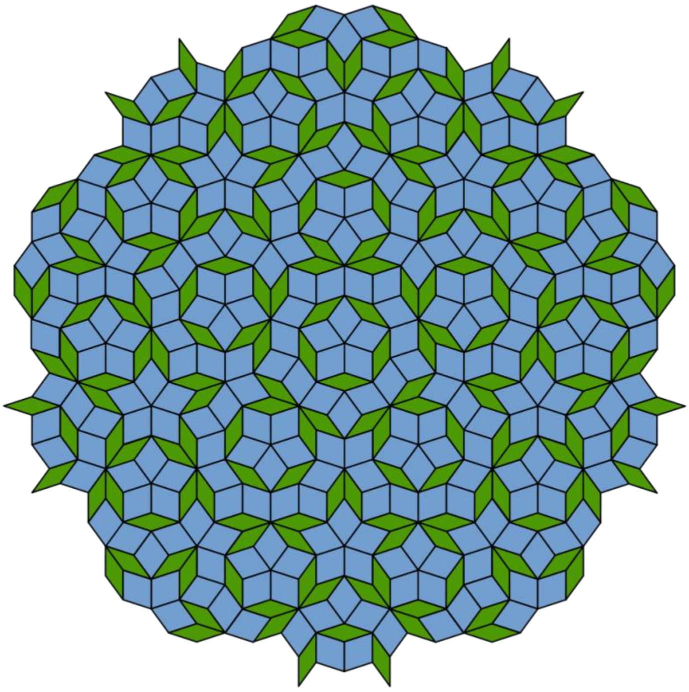
http://britton.disted.camosun.bc.ca/Islamic_Art_and_Geometric_Design.pdf

This includes some sheets with prepared grids of different kinds which enable you to pursue more intricate large-scale patterns without the need to construct the basic tilings each time.

Note that we follow the convention of using "Islamic" as shorthand for "coming from the traditions of the Islamic world"; although often found in mosques these patterns are also used in secular settings and similar ideas can be found in other religious traditions too. A few examples are included in the online resources.

Further Tilings

We often end this session with a brief discussion of Penrose tilings, which use only two distinct tile shapes but the pattern generated never repeats. The construction of the two classic Penrose tiles (as in the image) requires a special angle, the Golden Angle. This is related to the Golden Section and the pentagon, the topics of the next two sessions. Students interested in constructing Penrose tilings by hand will be well-equipped to do so after that (see the online resources for instructions). Geogebra is recommended; it makes the process much less laborious.



(Image from Wikipedia)

Note to Teachers

The idea of a "tiling" makes sense in three dimensions, too, except the tiles must now be three-dimensional objects; the Wikipedia page on Bravais lattices is a good place to start with this. Since this takes us outside the two dimensions of the page, however, such things must be deferred to another course, or to students' private study.

Hilbert and Cohn-Vossen (see Bibliography) has more ideas on this topic, with a discussion that's partially comprehensible to someone who's completed this course. With patience Geogebra's 3D mode can be coaxed into producing three-dimensional Bravais tilings.

Of course, topics concerning the symmetries of tilings, including the famous classification theorems for frieze and wallpaper groups, would be wonderful to cover given time. Grünbaum& Shephard and Conway et al. are both good resources.

Fifth Session: Square Roots, Spirals and the Golden Ratio

Summary

This session sets the groundwork for the construction of the regular pentagon. In the process it introduces some very famous (and pretty) constructions:

- Square roots and the Spiral of Theodorus
- The Golden Ratio, Rectangle and Spiral

Those who are comfortable with square roots as an algebraic operation may need more supplementary material; those for whom the “tick symbol” √ has always indicated some kind of occult mathematical trickery should be on more friendly terms with it by the end of the session.

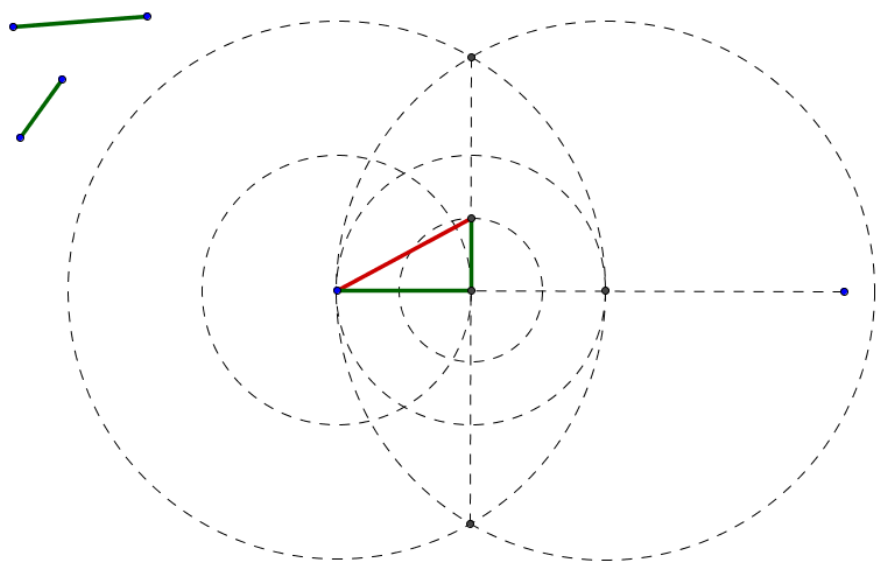
The Idea of a Square Root

We are getting close to our goal of constructing the regular pentagon, but we’ll need to know how to carry out one more arithmetical operation by straightedge-and-compass construction: taking square roots.

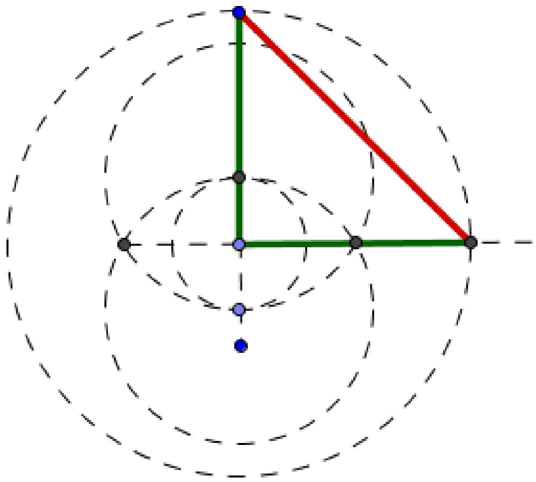
Back in the Second Session we constructed a square on a given line segment: that is, a line segment that was to be one of its sides. So we have a mechanism for turning a given length into a square area. The square root is just the inverse operation: given an area, its square root is the length of one of the sides of its quadrature.

We already know the techniques we need to construct the line segments we desire thanks to what we learnt about quadrature through Pythagoras's theorem. In this context a square root is rather uninteresting: given a square, its square root is just any line segment that’s congruent to one of its sides. In this session and the next we introduce a more modern way of looking at things.

Suppose we’re given the following problem: from two given lengths, construct a right-angled triangle. This is easy: place the two given lengths at right angles and add a third line segment to complete the triangle.



Consider the following figure, in which two congruent line segments (green) are set up at a right angle, using the familiar perpendicular construction, and a third (in red) is added to complete the triangle:



Units of measurement are as before entirely arbitrary, so we can choose to set our unit length to equal our two congruent line segments. Let those line segments equal '1'.

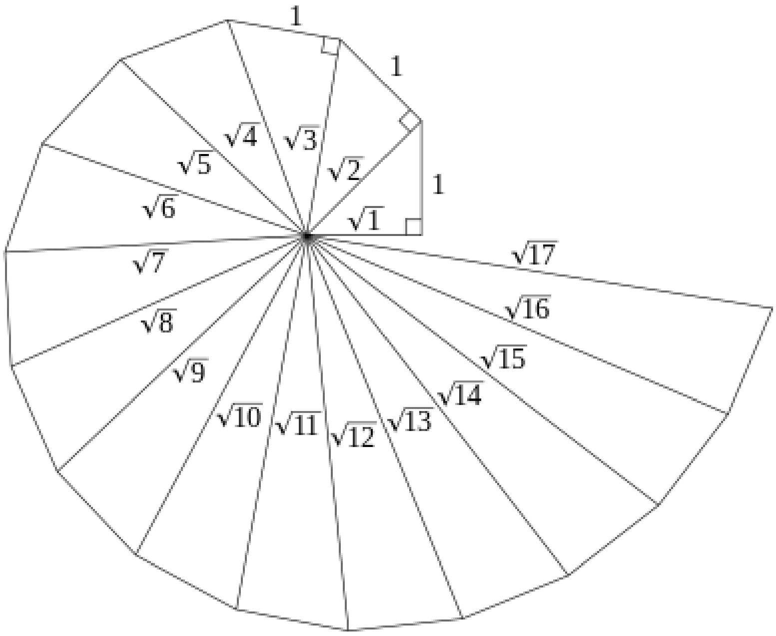
Suppose we also agree that a square with sides of this length is said to have an area equal to 1 *area unit* (note that length and area are quite different things and there's no compulsion on us to measure them in compatible units; it does make life easier, though). Then Pythagoras's Theorem tells us that the red line segment must be the side of a square whose area is 2 area units. Refer back to the images used for Constructions 3.02 and 3.03 when we halved and doubled given squares for a simple idea of using congruent triangles as units of area. Here, the triangle in the image represents half a unit - put two together to make a square of area 1.

It makes pretty good sense to define the length of the red side as the square root of two area units. In shorthand, we can call this length $\sqrt{2}$, remembering that like a lot of mathematical notation this compresses a lot of information into a small symbol. All it means is 'if a square's area is 2 area units, this is how long each of its sides must be in matching length units'.

Incidentally, this 'tick' symbol probably dates from the sixteenth century, although its exact origin and how it got its shape aren't known for certain. Some writers believe it's a stylised 'r', for *radix*, the Latin word for 'root'. Incidentally, the tick that's used by teachers to mark schoolwork is a different symbol entirely – its origins are similarly obscure, but it probably comes from the first letter of the Latin *veritas*, meaning 'truth'.

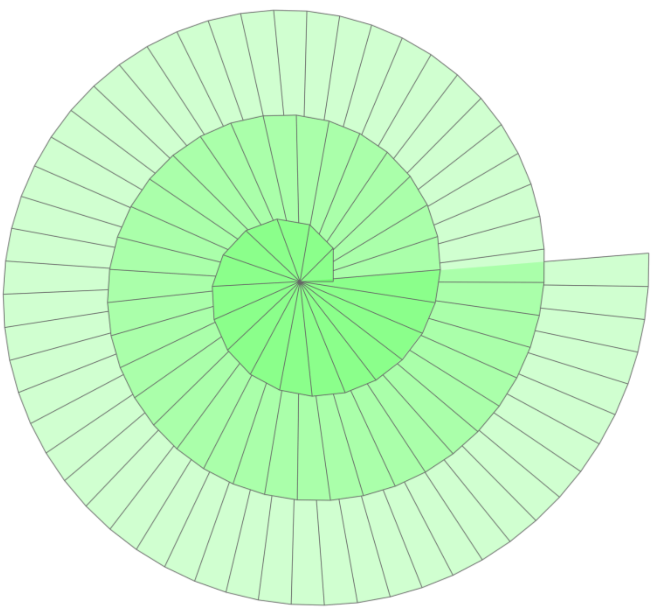
The Spiral of Theodorus

The idea of this construction is to start with the figure that ended the previous section and keep adding single-unit lengths to produce a spiral of right-angled triangles. Once the principle is understood the construction is conceptually easy. If you allow set-squares in your classroom, remind the students they should be grateful for this as it greatly reduces the amount of labour involved.



(Image from Wikipedia)

Incidentally, if you like you can continue the spiral by starting with the length $\sqrt{17}$ on a fresh sheet of paper. The resulting spirals can then be cut out, taped together and "stretched" upwards into a sort of spiral staircase that can be continued until you run out of patience (or paper).



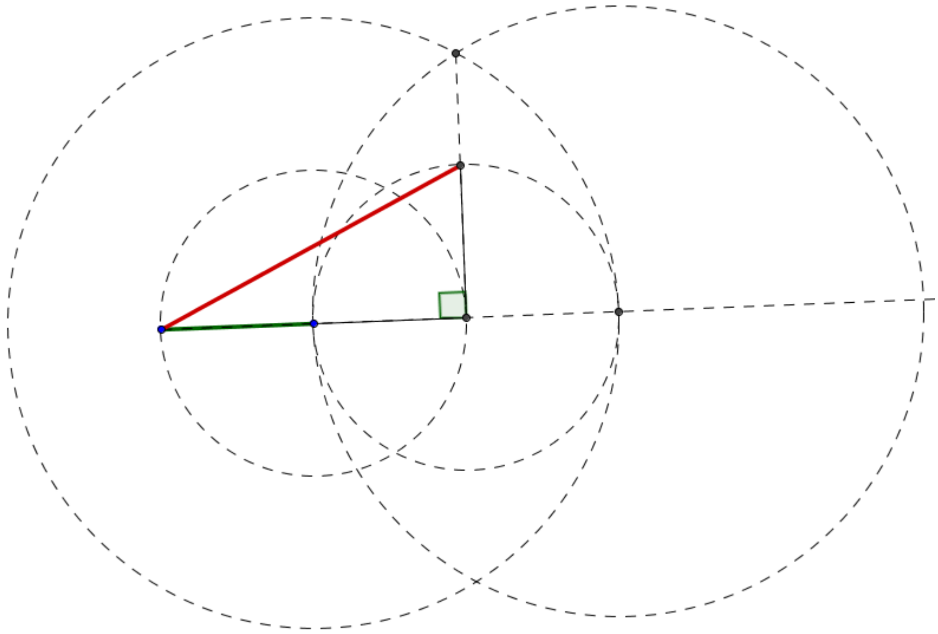
(Image from Wikipedia)

Reviewer: Jeremy Gray
Open University

Note that the outer layers of the spiral have their central point obscured by the inner layers - each segment-shape is still a right-angled triangle.

The spiral gives us a recipe for constructing a length equal to the square root of any multiple of a given length – for example, given a line segment and a unit length we can construct $\sqrt{5}$ “of that line segment”, literally meaning the side of a square whose area is five times the area of the square on the unit line segment.

We can do this by constructing the whole spiral from the start. Often we can take a shortcut, as in this example where we notice that the triangle involving $\sqrt{5}$ in the original spiral has a right-angle formed by two of its sides, one of length 1 and another of length 2 (i.e., $\sqrt{4}$):



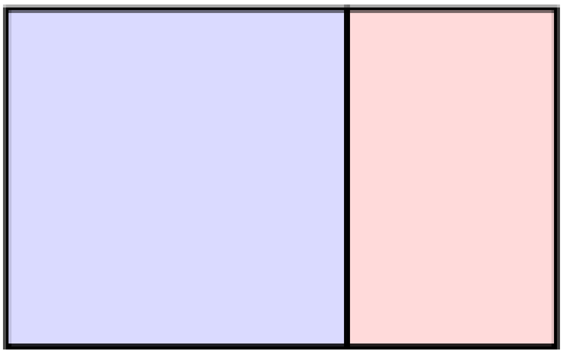
Note to Teachers

This is a good moment to present the proof that square roots that aren’t whole numbers are irrational. The standard proof is very famous and very ancient (see the online resources). Students who are allergic to algebraic reasoning, however, will likely find it unsatisfying. We generally save such things for another course unless there seems to be a genuine appetite for the topic. If presenting it, it would help to have covered the material in the section ‘Geometry as Algebra’ (below) first.

The Golden Rectangle and Spiral

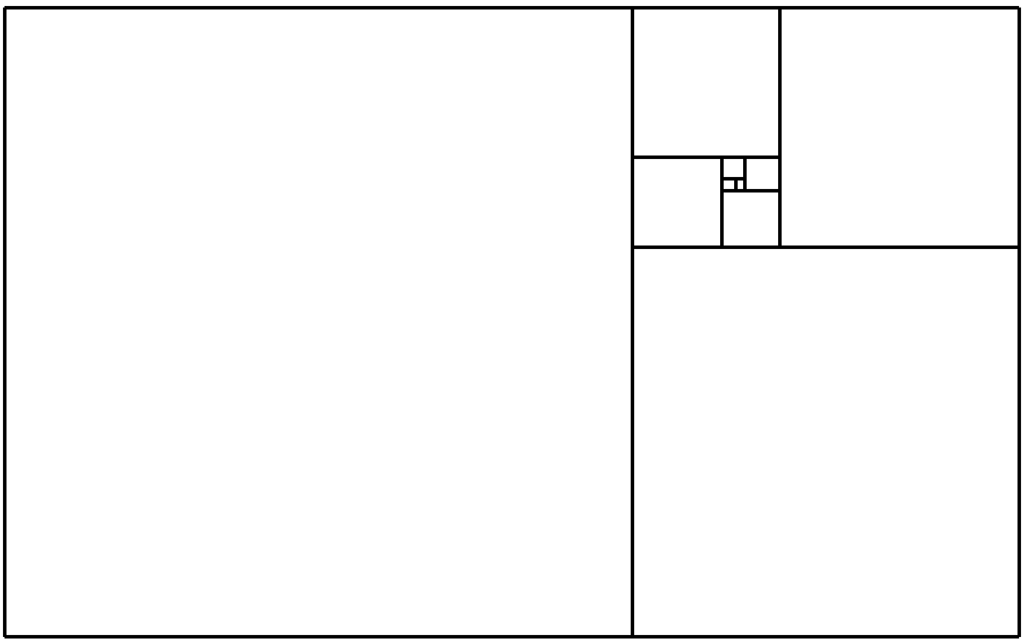
Here is a new problem that seems to have nothing to do with square roots and also leads us into the mystical realms of sacred geometry and some grand (and controversial) claims about beauty and proportion. Most of our students have heard of the Golden Ratio in the art-historical context, and some have come across the idea of Golden Spirals in nature. We tend to caution against some of the over-enthusiastic claims one finds about both topics; we refer interested students to Keith Devlin's lecture on the subject (see the online resources). Our interest will, however, lead to a result and a new construction whose 'magic' is impressive in itself.

Let's start with the Golden Rectangle.



(Image adapted from Wikipedia)

Let us imagine a rectangle with very particular proportions. Its length and width stand in such a relation that you can cut the square constructed from its shorter side from the rectangle and be left with a new rectangle whose proportions are exactly the same as the one you started with. That means you can repeat the process - cut another square from the rectangle to produce yet another rectangle with the same proportions. We can continue this process indefinitely:



(Image source: http://www.math.harvard.edu/archive/101_spring_00/www/gallery/gold/index.html)

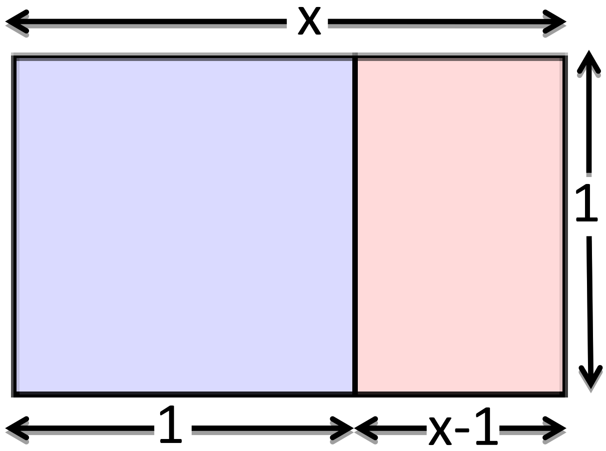
Yet a few attempts show that simply drawing any old rectangle will not yield this effect at all, leading to the question of how to construct one that does work. At this point some algebra is unavoidable, remembering that we get to 'play God' by choosing appropriate units.

Let's take our short side to have the unit length 1, representing the longer side as ‘x’ because we don’t know how long it should be yet. We want to express the length x in a way that will lead to something we can construct, and this is what the algebra helps us with.

The ratio of long side to short side will be:

$$\frac{x}{1}$$

Once we have cut the square from the first rectangle then we have two new sides: one of length 1 - now the longer side - and a new 'short side' which will measure x-1, because we have cut the length '1' from the original long side.



The ratio of long to short side in the new rectangle will be:

$$\frac{1}{x-1}$$

To solve our problem and construct the appropriate rectangle we need these ratios to be equal, so

$$\frac{x}{1}=\frac{1}{x-1}$$

This is the first and only time an equation (with an equals sign) appears in the course. We’ve seen equality of lengths and areas defined by congruence but this is a slightly different sort of thing. Still, we hope that students can at least see that proportion is a *geometrical* quantity just like those are and that we’re not just shuffling symbols around.

The next steps are where we temporarily lose those in the class who don’t remember their GCSE maths. We rearrange as follows:

$$\frac{x}{1}=\frac{1}{x-1}$$

$$x(x-1)=1$$

Reviewer: Jeremy Gray
Open University

$$x(x-1)-1=0$$

$$x^2-x-1=0$$

In the interests of time, we can solve this equation with the quadratic formula (see below) to reach the conclusion

$$x=\frac{1\pm\sqrt{5}}{2}$$

Sadly, the solution of the quadratic by this method will most likely appear to be magic to anyone who hasn't seen the formula before. In the following section we give a practical construction of the solution; this is optional but offers a useful way to avoid either going through the algebra or leaving a sense that something crucial has been missed out.

You might expect us only to be interested in positive solution to the formula

$$x=\frac{1+\sqrt{5}}{2}$$

And for the golden rectangle, we do only need this solution, but we will show that the negative solution appears in a particular construction of the pentagon and decagon (Session 6):

$$x=\frac{1-\sqrt{5}}{2}$$

So returning to our problem we need to create a rectangle who sides express the ratio

$$1:\frac{1+\sqrt{5}}{2}$$

or

$$2:1+\sqrt{5}$$

We are now in a position to construct the required length in the right-hand side of the equation.

Note to Very Persistent Students

For those who haven't seen it before, the algebra above is not magic. It relies on a simple formula for solving equations of the form

$$ax^2+bx+c=0$$

Here a, b and c are specified numbers, and we would like to find the values of x that make this equation come out true.

The formula, which was known in a different form to the ancient Babylonians, is

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

where choosing whether to use "+" or "-" in the top line usually gives two different solutions, each of which satisfies the equation just as well as the other.

Our equation is

$$x^2 - x - 1 = 0$$

So we have a = 1, b = -1, c = -1. Plugging these into the formula is how we get the answer.

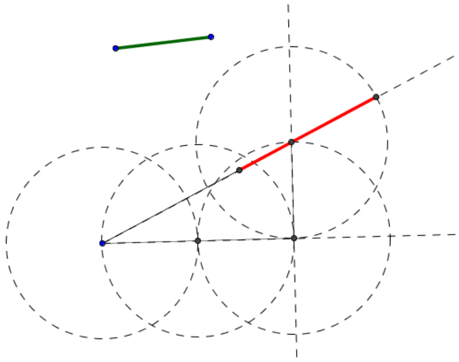
Incidentally, the standard derivation of the formula isn't very enlightening but the principle of solving equations using arithmetic (including roots) is profound. We'll see a glimpse of this, though sadly not more than that, in Session 6.

Note to Teachers

Teachers who are feeling a little guilty about soft-pedalling the derivation will enjoy seeing their students ably figuring out this rather complicated construction for themselves, bringing together various techniques from the course. The final section of this chapter serves as a reminder of Sophie Germain's sentiment that geometry and algebra are really two sides of the same coin; the algebra is merely a short-cut route to the construction, but the construction and the algebraic conclusion are really one and the same thing.

Construction 5.01

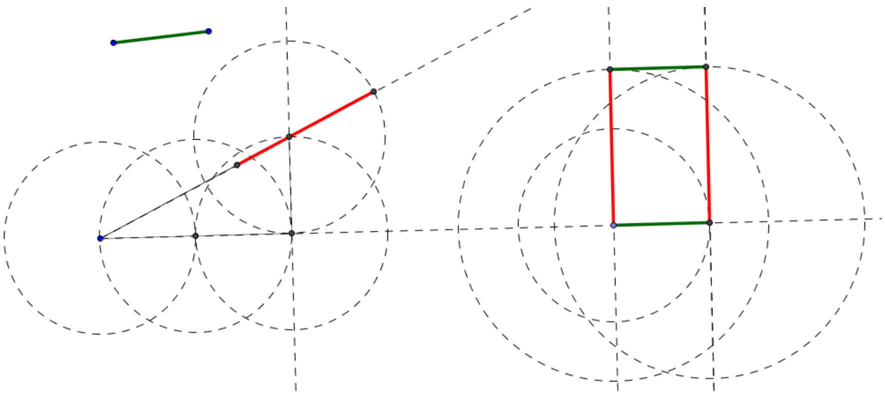
To construct the longer side of the Golden Rectangle, given the shorter side.



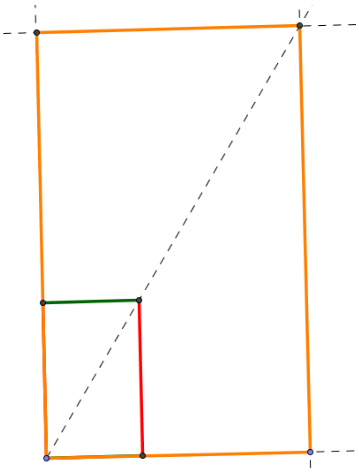
1. A line segment is given. We take its length to be 1 unit.
2. Construct a line segment twice as long as the given line segment.
3. Construct a perpendicular through one end of this line segment.
4. Copy the given line segment onto the perpendicular.
5. Join the ends to create a right-angled triangle. The longest side of the triangle, which we've just drawn is $\sqrt{5}$ units long.
6. Add the given length to the line segment that was just constructed; this gives us the length $\sqrt{5}+1$.
7. Bisect this line segment to obtain the length needed, which is $\frac{\sqrt{5}+1}{2}$

Now we can construct a Golden rectangle using the constructed length and the unit length; this is just a simple matter of constructing perpendiculars and copying the appropriate lengths.

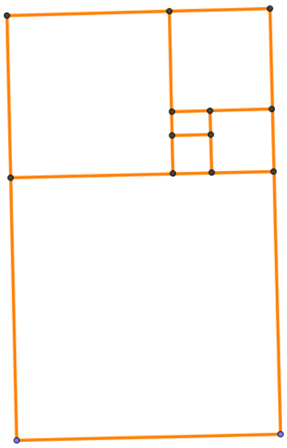
If you have grasped the procedure then you will realise that the right-angled triangle achieved in step 5 has sides 1, 2 & $\sqrt{5}$. This is all you need to make the rectangle with sides 2 & $\sqrt{5}+1$ which is obviously also a Golden rectangle.



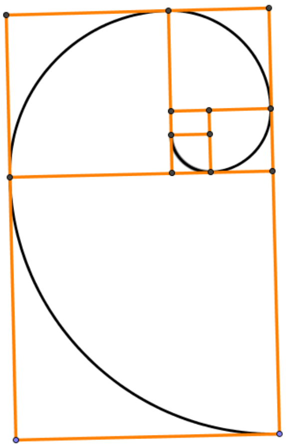
The diagonal of the golden rectangle expresses the golden ratio (short side:long side) and so this can be extended indefinitely to make a Golden rectangle of arbitrary size. The construction is actually familiar to many people from computer drawing packages, in which objects can be resized while retaining their proportions by dragging one corner along an appropriate diagonal. This is useful for the following part because a large rectangle makes for a nicer final effect.



Having arrived at a suitably large Golden Rectangle, cut off the appropriate square and see that the process can be continued indefinitely.



It's then an easy matter - using the compass and readjusting to the length of each square's sides - to add a quarter-circle to each square to produce the famous Golden Spiral:



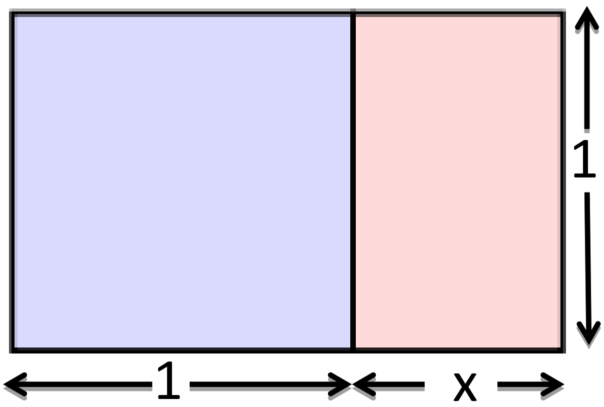
Solving the Equation by Construction

In this optional section we show an alternative way to solve the Golden Ratio equation with rather more transparent and well-motivated use of symbolic manipulation and no explicit dependence on the quadratic formula. This rather ingenious argument was suggested to us by Prof Jeremy Gray.

We do need one small trick, which is to re-express the Golden Ratio in this way:

$$\frac{x+1}{1} = \frac{1}{x}$$

All we've done here is start from the pink rectangle - instead of the whole rectangle - and swap the roles of the long and short sides, making "1" the long side and "x" the short one.

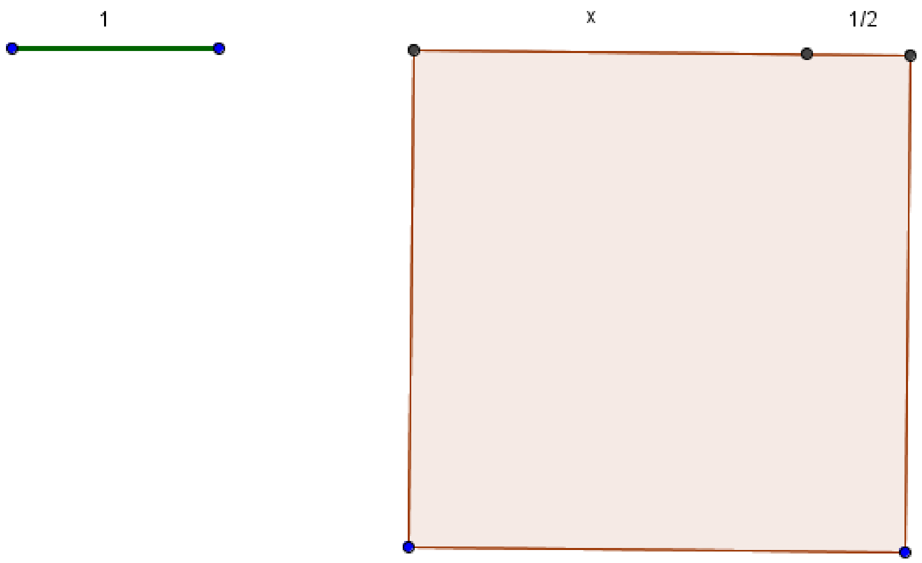


This yields the equation:

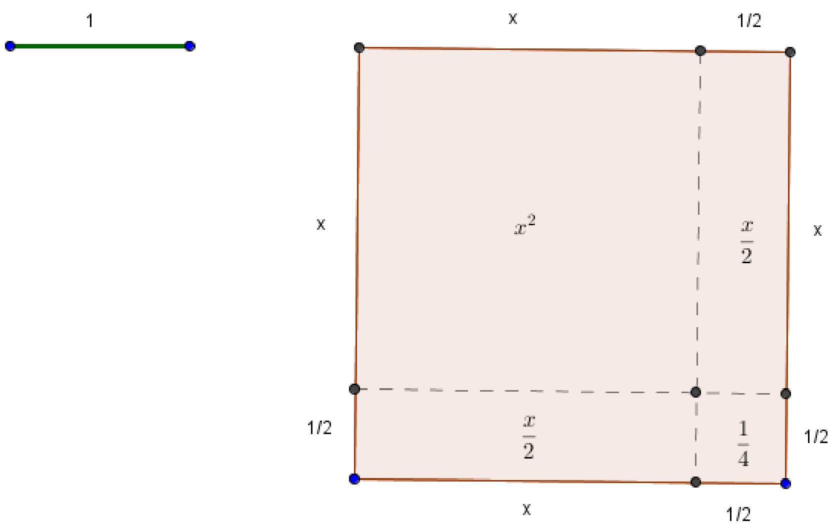
$$x^2 + x = 1$$

We'll use this relationship to find the right value of x to complete a simple construction.

We begin by defining a unit length and imagining a square that we *define* to have sides of length x + ½ (we can't construct this square yet because we don't know how long x should be – this is what we hope to discover):



The square can be cut into four regions and their areas can be calculated:



By adding up the four separate areas we can see that the total area of the square is

$$x^2 + x + \frac{1}{4}$$

Now, recall that the Golden Ratio equation (in the slightly different form we're using in this section) is

$$x^2 + x = 1$$

Substituting the new Golden ratio equation into the other expression allows us to replace the $x^2 + x$ with 1 to leave the simple expression of the square's area as

$$1 + \frac{1}{4} = \frac{5}{4}$$

If that's the area, one side of the square must be the square root of this number, which is

$$\frac{\sqrt{5}}{2}$$

Finally, remember that we already have another expression for the side of the square: x + ½. Equating the two expressions for the length of the side gives us

$$x + \frac{1}{2} = \frac{\sqrt{5}}{2}$$

Evidently, then

$$x = \frac{\sqrt{5}}{2} - \frac{1}{2} = \frac{\sqrt{5} - 1}{2}$$

which is the positive solution to the Golden Ratio equation.

Geometry as Algebra

This is always a good point to take a breather and to reinforce the big conceptual point: we can do arithmetic without numbers with our straightedge and compass. In fact we've seen a number of ways to carry out arithmetic operations with straightedge and compass:

- Multiply a length by a whole number (Session 1)
- Add two lengths (Session 2)
- Subtract one length from another (Session 2)
- Divide a length by a whole number (Session 2)
- Take the square root of a length (Session 5)

We now have the tools to address the construction of the regular pentagon in the next session. And there is a surprise there!

Note to Teachers

If time permits one might be tempted to show the "missing" constructions that enable us to multiply or divide a length by another length. They can be found in Stillwell (p. 10-11). This could be done closer to the beginning of this session if desired, or even at the end of Session 2.

These also work thanks to Thales' Theorem, but there's something odd about them. They're decidedly not in the Euclidean "spirit", according to which a product of two line segments ought to be an area (a parallelogram), not another line segment. What's more, they only work relative to a "unit" length; different choices of unit result in different answers, which may lead to well-justified perplexity among some students! Such considerations, interesting as they are, would take us too far off-track.

Sixth Session: Constructability

Summary

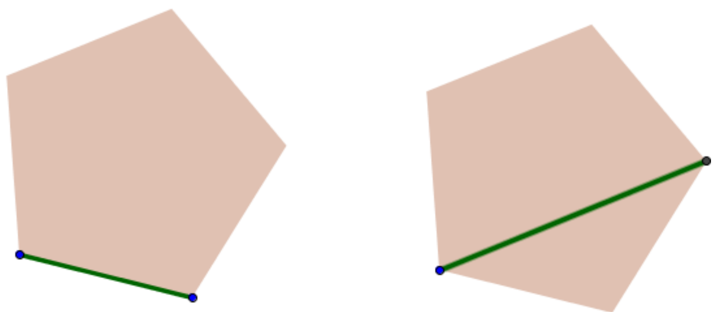
Our goal in this session is the construction of the regular pentagon. This requires us to bring together several of the techniques we've learned so far and represents a genuine achievement. We don't follow Euclid's approach, which is both hard to justify and fiddly to carry out. Instead we offer several alternative solutions, one of which is very easy to understand.

We generally find that with the recap of material from earlier sessions and our insistence on each step being fully justified and understood, this single topic consumes most of the session. We also like to leave time at the end of this final session for broader discussion of the themes of the course and how they intersect with the students' interests and artistic practices. We usually spend some of that time discussing constructability and the problem of the heptagon, although of course this can only be described in outline.

Construction of the Pentagon

So far we have constructed regular polygons with three, four, six and eight sides. The methods used have been quite miscellaneous, but the constructions weren't difficult. We saved the pentagon for this last session because it's quite a bit more subtle – it requires us to discover and construct a certain proportion (ratio) of lengths and it requires square roots (which in turn depended on an investigation of the concept of area). Let's look at why this is.

The problem of constructing the regular pentagon can be illustrated by a little experimentation. Firstly, we cannot easily find the angles we need to put together the sides with the techniques we have learnt so far. Secondly, suppose we are given two lengths, one of which is the distance between any two adjacent vertices of a pentagon and the other between any two non-adjacent vertices.



Can we get a regular pentagon from any two such line segments? A little experimentation suggests not. The line segments don't join up or, if the second line's too short, they cross each other.

It's easy to construct a pentagon of a given side-length (the green line segment on the left) if we can somehow find out the length of the diagonal (the green line segment on the right). Try "cheating" in the following way: "store" the two lengths in the diagrams above – having two compasses is handy here – and use them to construct a pentagon. It may take a little experimentation but once you hit on the basic idea the whole process becomes fairly straightforward.

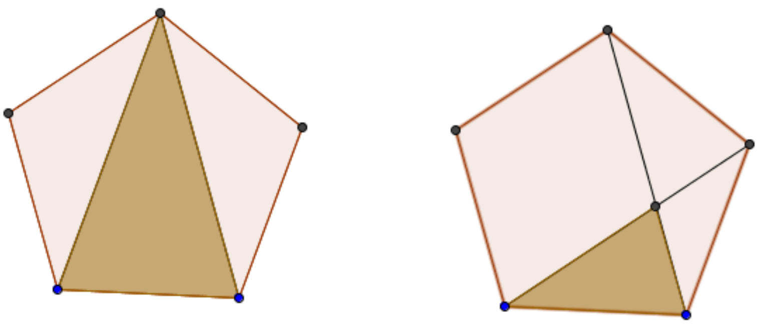
It's also clear enough that the secret must be the proportion – the ratio – between the two lengths. After all, if we make the line segment on the left 3 times bigger, doing the same with the line segment on the left should enable us to construct a pentagon again. What is that ratio, and can we use it to construct the second line segment if we're given the first one?

Note to Teachers

We don't always include a careful justification of the ratio in class. The argument that follows raises plenty of questions and would take some time to go through in detail.

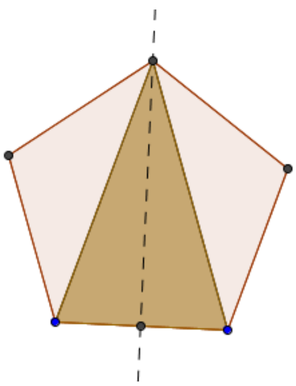
In a more practically-oriented presentation it might even be acceptable to simply present the solution and move on to the problem of using it in a construction.

Consider these two triangles in a regular pentagon:

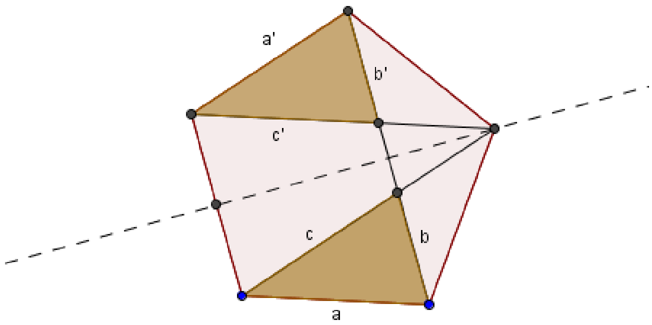


We will argue step-by-step that the triangle on the right is a scaled-down copy of the one on the left, with exactly the same proportions. We then figure out the factor by which the big triangle must have been scaled down to produce the small one. This leads us to an explicit formula for the ratio of lengths that makes the construction possible.

Any regular pentagon has five lines of mirror symmetry; one of them shows that the big triangle is symmetrical, too:

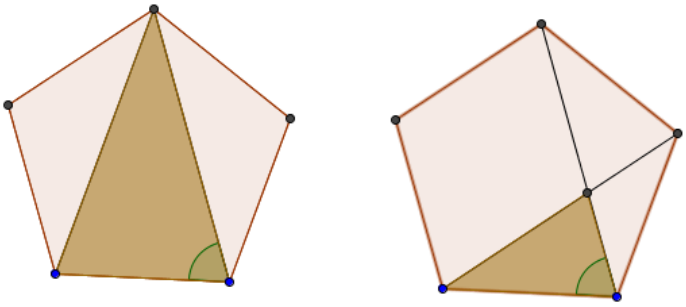


A similar approach works for the small triangle, but this time a mirror line flips it over and moves it to another part of the pentagon – the original triangle's sides are labelled a,b and c while the corresponding sides after the reflection are labelled a', b' and c':



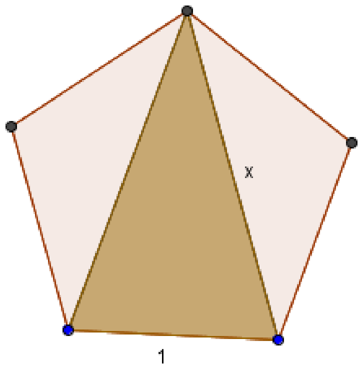
We know that the angle a’b’ is the same as the angle ab, from the fact that reflections don’t change the sizes of angles. Now note that the two red line segments a and a’ are parallel; this is a consequence of the regularity of the pentagon. It follows (from properties of parallel lines!) that the angles ab and c’b’ are equal. Since a’b’ = ab = c’b’, it must be that a’b’ = c’b’. But then it follows that ab = cb.

So, like the bigger triangle, this one has two equal angles. Call the pair of matching angles the “base angles” and the other one the “apex angle” (it may help to think of the smaller angle as lying down on its side). We can see right away that the base angles of the two triangles are equal, since one of them is shared:

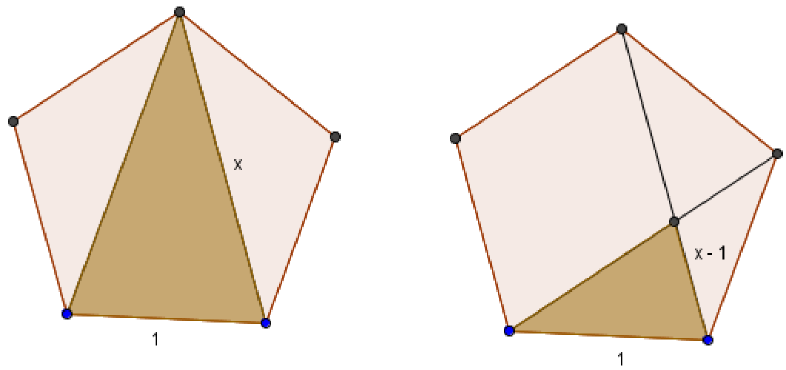


This is enough to prove that the small triangle on the right is just a “scaled down replica” of the bigger one (*Elements* Book VI Definition 1). Our task is to find out what the “scale factor” is.

To do this we pass from intuitive, pictorial reasoning to a tiny bit of algebra. We label the sides in the following way: the long side on the left will be x, since it’s the length we want to find. The smaller side on the left, equal to the longer side on the right, will be our unit length so we’ll write it as 1.



What about the other, shortest side? Well, by symmetry we can see that it's actually equal to x – 1 (in the sense of ordinary subtraction of lengths by construction).



So we can now express the ratios of the side lengths in the two triangles as follows:

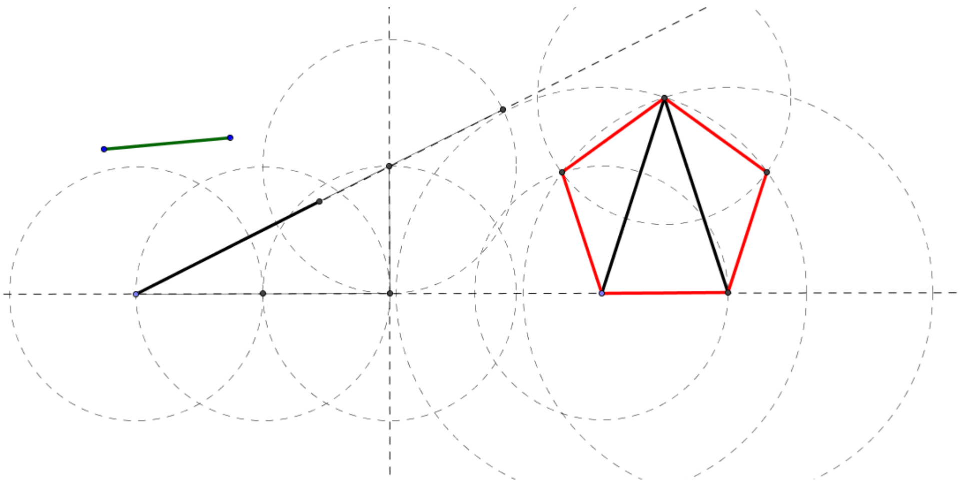
$$\frac{x}{1} = \frac{1}{x-1}$$

This tells us (remember from last session!) that the ratio we seek is the Golden Ratio. Hence the task of constructing the regular pentagon is reduced to something much simpler: constructing two line segments in Golden Ratio. This should be quite enough information to enable the class to carry out the construction for themselves.

The method this exposition implies is rather mechanical and not very elegant. We choose a side-length, construct the Golden Ratio based on it, then use these two to build a pentagon point-by-point. This is easy to understand but hardly pretty.

Construction 6.01

To construct a regular pentagon on a given line segment.



The diagram shows the given green line segment (left), the construction of the corresponding Golden Ratio length (middle) and the use of these two to build up the finished pentagon (right). The diagram looks scary but the individual steps are easy.

1. A line segment is given.
2. Build the Golden Ratio length corresponding to this line segment; that is, taking the given line segment to be 1, construct the length $\frac{1+\sqrt{5}}{2}$.
3. Construct a triangle that has the given line segment as one side and the golden ratio length for each of its two others.
4. On one of the longer sides of this triangle, construct another triangle whose other two sides are the original given length.
5. Repeat Step 4 for the other side.

The following method is less obvious but enables us to solve a different problem that has applications in design.

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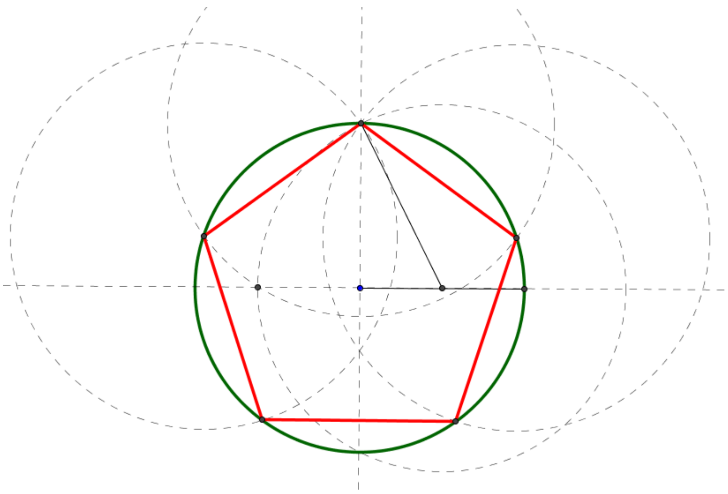
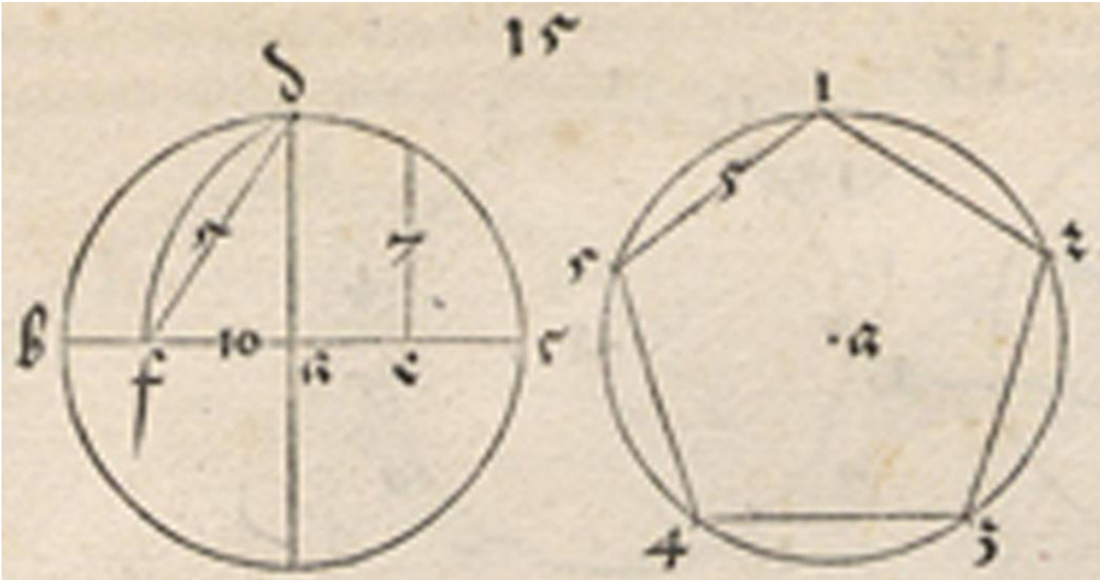
Construction 6.02

To inscribe a regular pentagon in a given circle.



1. A circle and its centre are given (if the centre is not known, use Construction 2.10).
2. Draw any line segment through the centre along with its perpendicular bisector, also through the centre, creating four intersection points.
3. Choose any of the four line segments between the centre and an intersection point on the circle; find its midpoint.
4. Join this midpoint to one of the other two closest intersection points. The length of this line segment is $\frac{\sqrt{5}}{2}$ in relation to the unit length.
5. Draw a circle with this radius, centred on the midpoint you found in Step 3. Mark where it intersects the horizontal line segment. (Relative to the unit length, the length from the centre of the circle to this new intersection point is $\frac{1}{2} - \frac{\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2}$;the "other" solution of the Golden Ratio's quadratic equation. It is also the length needed to inscribe the regular decagon into the circle.)
6. Put the compass spike in the top intersection point and the drawing tip in the point just marked. This length is what you need to inscribe the pentagon into the circle, it is twice the length of the length constructed in Step 5.
7. "Walk" this length around the circle, starting and ending at the top intersection point. (There are various shortcut ways to do this).
8. Join the resulting intersections to complete the pentagon.

Here the same method from Construction 6.02 is reprinted from artist Albrecht Dürer's *Four Books of Measurement* of 1525:



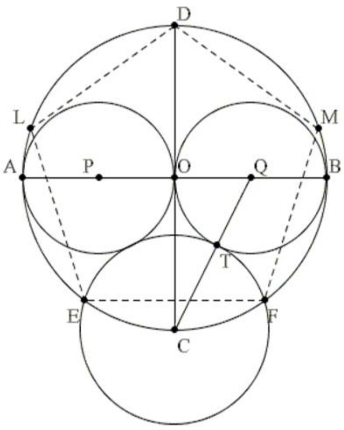
The figure on the left shows how to construct three different lengths to inscribe three different regular polygons inside the given circle:

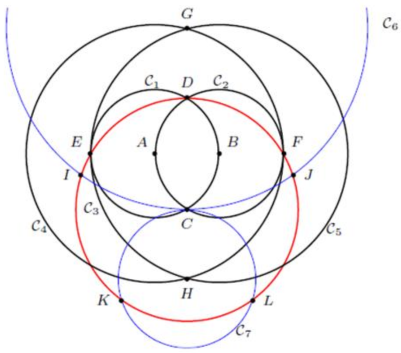
- the pentagon (line δf, marked '5');
- heptagon (the perpendicular line segment above point e marked '7');
- and decagon (line line segment af, marked 10).

The right-hand image selects the length, '5', from the left to make the pentagon.

We will have more to say about the heptagon below!

There are many alternative constructions of the pentagon, and you may like to try some of them. We especially like the extremely quick one on the left below, which was invented in the nineteenth century by Yosifusa Hirano. On the right is a "Mascheroni" construction (one that uses only compass - no straightedge!) invented in 2008 by Kurt Hofstetter (see the [online resources]); the pentagon is formed by connecting points DIJKL on the red circle. In both cases a little frowning at the diagram is sufficient to figure out the steps required, although justifying them is more difficult.





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We have now reached the culmination of the technical side of the course. From a practical perspective you have hopefully mastered the techniques of straightedge & compass; understanding the construction of the pentagon is a significant milestone.

Which Regular Polygons can we Construct?

At this stage it's good to recap. We can construct - with straightedge and compass - regular polygons with the following numbers of sides: 3 (triangle), 4 (square), 5 (pentagon), 6 (hexagon) and 8 (octagon). We can also extend the idea of bisecting sides from Session 1 to the pentagon, giving a regular 10-sided polygon (a decagon). We can also apply this again to the hexagon to get the 12-sided dodecagon, and so on.

Yet this leaves some gaps: the 7-sided heptagon, 9-sided nonagon and more. These regular polygons exist, but *we cannot construct them with straightedge and compass*.? That is a surprising answer , but sadly a proof is beyond the scope of this course.

Dürer's method for constructing the heptagon, given immediately above, is not a proper construction - it gives us a very good approximation but cannot solve the problem of inscribing the regular heptagon inside a circle. Experiment in Geogebra to see the small, but significant margin of error!

Impossible constructions have a long a rather strange history. Three famous ones date from Classical antiquity: squaring a given circle, trisecting a given angle and doubling a given cube.

The first of these problems involves returning to the quadrature techniques of Session 3. We can cut any straight-edged polygon into a number of triangles - this means we can construct a square of the equivalent area. Can we do the same with the circle?

Leonardo da Vinci is one of many who became obsessed with the problem: from 1504 onwards he produced hundreds of pages of notes in the attempt to solve it. The number that trips us up is the number π (pi), which you might recall from school. The proof that π can't be constructed using our methods wasn't achieved until 1882, and only then did reputable people stop trying to square circles. We're not aware of any elementary proof; the most common approach uses complex numbers and a little calculus (see the online resources).

The second and third problems are similar to the problem of constructing the regular pentagon, except that in the case of the pentagon we had to construct a number involving a *square* root whereas in these new cases we must construct a *cube* root. It turns out that this can't be done in general within Euclid's constraints, though the reasons are different than the case of π.

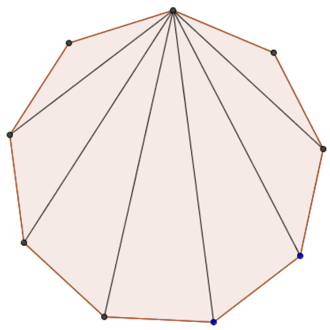
To prove any of these facts requires us to step outside the world we've been occupying, since it's pretty difficult to prove something can't be done with constructions *by construction*! The impossibility of these constructions, though long-suspected, wasn't proved until the nineteenth century and properly belongs to what is now called *abstract algebra*.

That sounds more forbidding than it is but it does require us to go through a fairly long process of setting up our tools, just as we had to learn quite a lot about straightedge-and-compass constructions before we could do anything really interesting. Because the proofs are difficult to communicate to someone without the requisite background, some people persist in trying to accomplish these feats even today.

The cases of the heptagon and nonagon are exactly similar to these classical problems. Constructing the regular heptagon is equivalent to constructing a length that solves the cubic equation

$$x^3 + x^2 - 2x - 1 = 0$$

which yields a length that can't be constructed with straightedge and compass. As for the nonagon, dividing it into 7 triangles shows that its internal angles are (180 x 7)/9 = 140^o:



Suppose for a moment we *could* construct it. Then we could go on to construct an angle of 20^o by subtracting one of its internal angles from 180^o (a straight line) and bisecting. But an angle of 20^o isn't constructible; in fact it turns out that if we could do this we would have effectively constructed a root of the (again, cubic) equation

$$8x^3 - 6x - 1 = 0$$

which is, again, provably impossible. You can't do something impossible just by going about it in a roundabout fashion!

Similar arguments work for all the other non-constructible polygons. The great surprise was the 17-sided case, which Carl Friedrich Gauss proved *could* be constructed in 1796. He did this by a similar line of reasoning to the one we've been gesturing towards: showing that the task was equivalent to solving a particular equation, then showing that a solution exists that can be broken down into a combination of steps each of which can be carried out by straightedge and compass.

This in itself was pretty complicated; and it wasn't even the worst one:

> From [Gauss's equation] one can design a construction for the regular 17-gon, though it is not very efficient. A more elegant construction can be found in [Stewart, Ch. 17] [see Bibliography], along with a reference for Richelot's construction of the regular 257-gon in 1832. There is also the story of Professor Hermes of Lingen, who late in the nineteenth century worked 10 years on the construction of the regular 65537-gon. (Cox p.272)

It should be pointed out that Euclid's constraints are very severe. If we're allowed to add a third tool to our kit alongside the straightedge and compass, we can often achieve additional things that are impossible without them. The most startling fact is that making two scratch-marks on your straightedge is enough to open out a whole world of so-called *neusis* constructions. Apart from squaring the circle, all the impossible constructions we just discussed become possible when we make this small adjustment. The marked straightedge is therefore a much more powerful tool than the unmarked one – Roger Alperin's paper gives some details (see the online resources). You might also enjoy exploring constructions that use paper-folding, another powerful technique (again, see the online resources).

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Online Resources

- Geogebra: http://www.geogebra.org/
- Euclid's *Elements*
    - A great electronic edition with interactive animations (though many browsers now block these): http://aleph0.clarku.edu/~djoyce/java/elements/elements.html
    - Oliver Byrne's beautiful visual edition from 1847: https://archive.org/details/firstsixbooksofe00eucl or a slightly more accessible form here: http://www.math.ubc.ca/~cass/Euclid/byrne.html
    - Useful links to historic editions and similar information: http://www.math.ubc.ca/~cass/Euclid
- Handy sources for straightedge-and-compass constructions:
    - http://www.mathopenref.com/tocs/constructionstoc.html
    - http://www.susqu.edu/brakke/constructions/constructions.htm
- Regular polygons construction video (we like to show this at the start of Session One, when it seems quite mysterious, then again at the start of Session Six when students are well-positioned to interpret what's going on): http://www.youtube.com/watch?v=LBgIWQcC6lM
- Constructions or approximations used in architecture, design etc:
    - Islamic tilings workbook: http://britton.disted.camosun.bc.ca/Islamic_Art_and_Geometric_Design.pdf
    - http://www.stonecarvingcourses.com/the-geometry-of-gothic-architecture
    - http://www.romeofthewest.com/2011/08/build-your-own-gothic-cathedral.html
    - http://www.codesmiths.com/shed/workshop/techniques/arches.htm (clearer on quinto arcuto)
    - http://www.mfdabbs.pwp.blueyonder.co.uk/Maths_Pages/SketchPad_Files/Gothic_Arch_Tangencies/Gothic_Arches.html (nice examples of combinations)
    - http://www.thisiscarpentry.com/2012/01/06/circular-based-arches-part-1/, http://www.thisiscarpentry.com/2012/02/03/circular-based-arches-part-2/, http://www.thisiscarpentry.com/2012/03/02/circular-based-arches-part-3/
    - http://www.sbebuilders.com/tools/geometry/treatise/
    - C. J. Dudley's PhD thesis on the geometry of Peterborough Cathedral (includes many interesting construction ideas): http://medievalarchitecturalgeometry.com/Index.htm
    - Jim Wilson's ideas for decorative constructions in circles: http://jwilson.coe.uga.edu/emt725/Circle%20Patterns/Patterns.html
- Some unusual or historically interesting constructions
    - Dürer's straightedge-and-compass constructions: http://divisbyzero.com/2011/03/22/albrecht-durers-ruler-and-compass-constructions/
    - Van Schooten's straightedge constructions: http://mathdl.maa.org/mathDL/46/?pa=content&sa=viewDocument&nodeId=268&bodyId=153
    - "Rusty compass" constructions: http://www.cut-the-knot.org/do_you_know/GoldenRatioByRustyCompass.shtml

    - The Thébault Problems: http://www.cut-the-knot.org/Curriculum/Geometry/Thebault1.shtml
- Triangles
    - Joseph Malkevitch's article "Diagonals" discusses dividing polygons into triangles: http://www.ams.org/samplings/feature-column/fcarc-diagonals1
    - Angle in a semicircle proof: http://www.algebra.com/algebra/homework/Circles/Theorems-on-Triangle-and-Circle.lesson
- Tilings
    - Metropolitan Museum of Art booklet on Islamic tiling methods: http://www.metmuseum.org/~/media/Files/Learn/For%20Educators/Publications%20for%20Educators/Islamic_Art_and_Geometric_Design.pdf
    - Alex Bellos article on the pentagonal tilings: http://www.theguardian.com/science/alexs-adventures-in-numberland/2015/aug/10/attack-on-the-pentagon-results-in-discovery-of-new-mathematical-tile
    - Penrose tilings (with sufficiently detailed instructions for students to construct their own): http://jwilson.coe.uga.edu/emat6680fa05/schultz/penrose/penrose_main.html
- Irrational Numbers:
    - Standard proof of the irrationality of $\sqrt{2}$: http://www.math.utah.edu/~pa/math/q1.html
    - More (mostly elementary) proofs and the relevant quote from Plato: http://www.cut-the-knot.org/proofs/sq_root.shtml. Note especially Proof 18, which is done by construction, though care would be needed to present it effectively.
    - Students who are upset by the whole business may find Timothy Gowers's essays helpful, especially this one, though they're aimed at a different audience: https://www.dpmms.cam.ac.uk/~wtg10/reals.html
- Golden Ratio
    - http://www.cut-the-knot.org/do_you_know/GoldenRatio.shtml
    - http://www.cut-the-knot.org/pythagoras/pentagon.shtml
    - http://www.cut-the-knot.org/pythagoras/cos36.shtml
    - Keith Devlin's entertaining take-down of a number of popular fallacies about the Golden Ratio: https://www.dpmms.cam.ac.uk/~wtg10/reals.html
- Pentagon constructions
    - An account of Hofstetter's 2008 construction: http://www.cut-the-knot.org/pythagoras/MascheroniPentagon.shtml#H
- Resources on the constructability of the 17-gon:
    - David Eisenbud makes a brave attempt to construct a 17-gon by hand: https://www.youtube.com/watch?v=87uo2TPrsl8
    - http://mathworld.wolfram.com/TrigonometryAnglesPi17.html
    - http://mathworld.wolfram.com/ConstructiblePolygon.html
    - http://mathworld.wolfram.com/Heptadecagon.html

  - http://demonstrations.wolfram.com/NGonPolynomials/
- Impossible Constructions
  - Dudley Underwood's deliciously snarky account of attempts to trisect the angle: http://web.mst.edu/~lmhall/WhatToDoWhenTrisectorComes.pdf
  - A short, hand-waving account of why cube roots aren't constructible: https://www.math.toronto.edu/mathnet/questionCorner/impossconstruct.html
  - A dissertation on polygon construction, including a detailed proof of the constructability of the 17-gon: http://www.math.iastate.edu/thesisarchive/MSM/EekhoffMSMSS07.pdf
  - Proof that π is transcendental (uses calculus): http://sixthform.info/maths/files/pitrans.pdf
  - Origami constructions: http://www.langorigami.com/science/math/hja/hja.php
  - A more technical paper by Roger Alperin touching on neusis and origami constructions: http://www.math.sjsu.edu/~alperin/TRFin.pdf
  - Hobbes's false method for squaring the circle:
    - http://www-history.mcs.st-and.ac.uk/Biographies/Hobbes.html
    - http://archive.org/stream/englishworkstho21hobbgoog#page/n312/mode/2up

Bibliography

The first section lists books explicitly referred to in the text; not all are at an appropriate level for students on this course. The second list, on the other hand, collects some books we frequently recommend to our students.

We do sometimes refer to physical editions of Euclid's *Elements* but the online version is generally more accessible and convenient (see Online Resources).

Works Referred to in this Text

Conway, J. H., Burgiel, H. & Goodman-Strauss, C. (2008) *The Symmetries of Things*. Wellesley, MA: A. K. Peters Ltd.

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Mendell., H. (2015) What's location got to do with it? Place, space and the infinite in Classical Greek mathematics. In de Risi (ed.) *Mathematizing Space: The Objects of Geometry from Antiquity to the Early Modern Age*. New York: Springer.

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Stillwell, J. (2005) *The Four Pillars of Geometry*. New York: Springer.

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Suggested Wider Reading

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Walser, H. (2006) *99 Points of Intersection*. Washington: Mathematical Association of America.