Intermediate Algebra

Functions and Graphs

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Katherine Yoshiwara
Los Angeles Pierce College

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Preface

Intermediate Algebra is a textbook for students who have some acquaintance with the basic notions of variables and equations, negative numbers, and graphs, although we provide a "Toolkit" to help the reader refresh any skills that may have gotten a little rusty. In this book we journey farther into the subject, to explore a greater variety of topics including graphs and modeling, curve-fitting, variation, exponentials and logarithms, and the conic sections. We use technology to handle data and give some instructions for using a graphing calculator, but these can easily be adapted to any other graphing utility.

We discuss functions and their applications, but not in the detail expected of a precalculus course; our intended audience includes students preparing for the many fields that may not use calculus but nonetheless require facility with quantitative reasoning. We aim to develop that facility in the context of modeling and problem solving.

- Each section of the text includes Examples followed by a similar Practice Exercise that students can try for themselves in WeBWorK.
- There are also "QuickCheck" exercises, usually multiple choice or fill in the blanks, for students to check their understanding of the concepts presented.
- Homework problems come in three groups: a short Skills Warm Up that reviews prerequisite skills for the section, then Skills Practice with new skills, and finally Applications.
- We have included a variety of applied problems that we hope students (and their teachers) will find interesting.
- In addition, each Chapter begins with an Investigation that can be used as a group project or as a guided in-class activity.

Chapter Reviews include a Glossary and a Summary of key concepts as well as Review Problems. There is an Activity booklet available that provides an interactive lesson for each section of the text. The Activities can be completed by students in groups or with guidance from the instructor; or they can be used as support for a lecture format.

Katherine Yoshiwara
Atascadero, CA 2020

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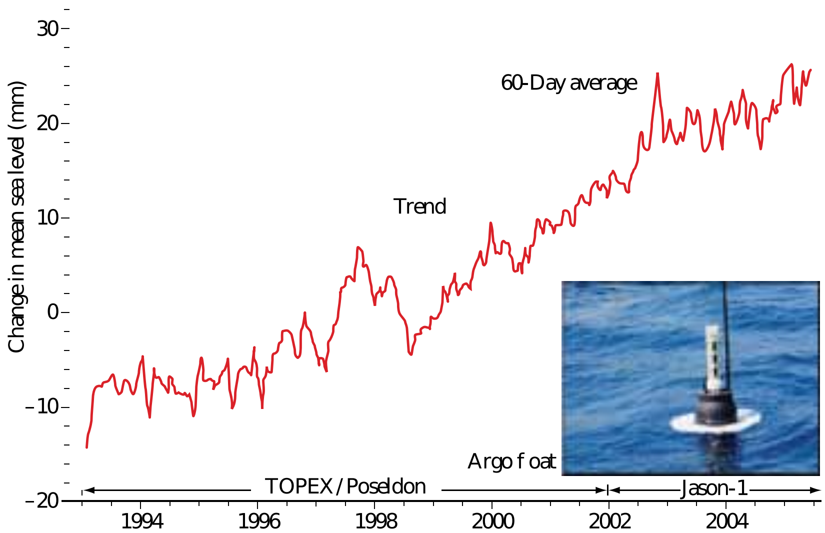
Chapter 1

Linear Models

In the debate over global warming, sea level is a reliable indicator of climate change, because it is affected by both melting glaciers and by the thermal expansion of sea water. During the 20th century, sea level rose by about 15 centimeters, compared to a fairly constant level for the previous 3,000 years.

Not surprisingly, scientists would like to understand the causes of sea-level change as thoroughly as possible, and to measure the rate of sea-level rise as accurately as possible. Using data from satellites and floats (mechanical devices drifting in the ocean), oceanographers at NASA's Jet Propulsion Laboratory have calculated that the sea level rose, on average, 3 millimeters (0.1 inches) per year between 1995 and 2005. The graph below shows the change in mean sea level, measured in millimeters, over that time period.



The Intergovernmental Panel on Climate Change (IPCC) predicts that sea level could rise as much as 1 meter during the 21st century. If that happens, low-lying, densely populated areas in China, Southeast Asia, and the Nile Delta would become uninhabitable, as well as the Gulf Coast and Eastern Seaboard of the United States.

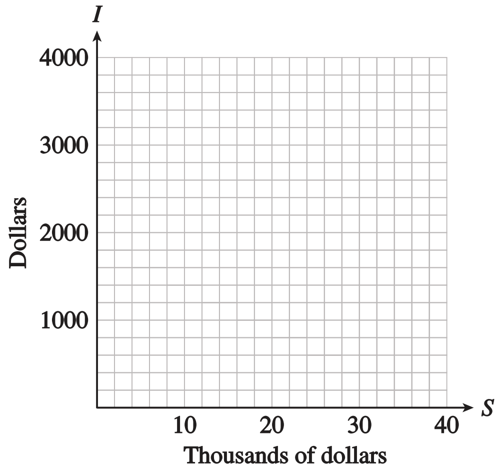
Investigation 1.1 Sales on Commission. Delbert is offered a part-time job selling restaurant equipment. He will be paid \$1000 per month plus a 6% commission on his sales. The sales manager tells Delbert he can expect to sell about \$8000 worth of equipment per month. To help him decide whether to accept the job, Delbert does a few calculations.

1 Based on the sales manager's estimate, what monthly income can Delbert expect from this job? What annual salary would that provide?

2 What would Delbert's monthly salary be if he sold only \$5000 of equipment per month? What would his salary be if he sold \$10,000 worth per month? Compute the monthly incomes for each sales totals shown in the table.

Sales	Income
5000	
10,000	
15,000	
20,000	
25,000	
30,000	

3 Plot your data points on a graph, using sales, S, on the horizontal axis and income, I,on the vertical axis, as shown in the figure. Connect the data points to show Delbert's monthly income for all possible monthly sales totals.



4 Add two new data points to the table by reading values from your graph.

5 Write an algebraic expression for Delbert's monthly income, I, in terms of his monthly sales, S. Use the description in the problem to help you:

He will be paid: \$1000 plus a 6% commission on his sales.

Income = _______________

6 Test your formula from part (5) to see if it gives the same results as those you recorded in the table.

7 Use your formula to find out what monthly sales total Delbert would need in order to have a monthly income of \$2500.

8 Each increase of \$1000 in monthly sales increases Delbert's monthly income by ______

9 Summarize the results of your work: In your own words, describe the relationship between Delbert's monthly sales and his monthly income. Include in your discussion a description of your graph.

Linear Models

Tables, Graphs, and Equations

Mathematical Model.

A **mathematical model** is a simplified description of reality that uses mathematics to help us understand a system or process.

We can use a model to analyze data, identify trends, and predict the effects of change. The first step in creating a model is to describe relationships between the variables involved.

Checkpoint 1.1 QuickCheck 1. What is the first step in creating a mathematical model?

- ⊙ A) Solve the equation.
- ⊙ B) Analyze data, identify trends, and predict the effects of change.
- ⊙ C) Describe the relationships between the variables involved.

⊙ D) Make a table of values.

Answer. C) Describe ... variables involved.

Solution. Describe the relationships between the variables involved.

Starting from a description in words, we can represent the relationship by:

- a table of values
- a graph, or by
- an algebraic equation

Each of these mathematical tools is useful in a different way.

1 A **table of values** lists specific data points with precise numerical values.

2 A **graph** is a visual display of the data. It is easier to spot trends and describe the overall behavior of the variables from a graph.

3 An **algebraic equation** is a compact summary of the model. It can be used to analyze the model and to make predictions.

Checkpoint 1.2 QuickCheck 2. Name three ways to represent a relationship between variables.

⊙ A) By inputs, outputs, or evaluation

⊙ B) By tables, equations, or graphs

⊙ C) By the intercepts, the slope, or the vertex

⊙ D) By numbers, letters, or diagrams

Answer. B) By ... , or graphs

Solution. By tables, equations, or graphs

In the examples that follow, observe the interplay among the three modeling tools and how each contributes to the model.

Example 1.3

In May 2005, the city of Lyons, France, started a bicycle rental program. Over 3,000 bicycles are available at 350 computerized stations around the city. Each of the 52,000 subscribers pays an annual 5 euro fee (about \$7.20) and gets a PIN to access the bicycles. The bicycles rent for 1 euro per hour and can be returned to any station.

Your community decides to set up a similar program, charging a \$5 subscription fee and \$3 an hour for rental. (A fraction of an hour is charged as the corresponding fraction of \$3).

a Make a table of values showing the cost, C, of renting a bike for various lengths of time, t.

b Plot the points on a graph. Draw a curve through the data points.

c Write an equation for C in terms of t.

Solution.

a There is an initial fee of \$5, and a rental fee of \$3 per hour. To find the cost, we multiply the time by \$3 per hour, and add the result to the \$5 subscription fee. For example, the cost of a one-hour bike ride is

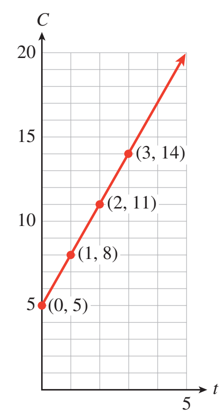
$$\text{Cost} = (\$5 \text{ insurance fee}) + (\$3 \text{ per hour}) \times (\text{one hour})$$
$$C = 5 + 3(\mathbf{1}) = 8$$

A one-hour bike ride costs \$8.

We calculate the cost for the other values of t and record the results in a table as shown below.

Length of rental (hours)	Cost of rental (dollars)		(t, C)
0	5	$C = 5 + 3(\mathbf{0})$	$(0, 5)$
1	8	$C = 5 + 3(\mathbf{1})$	$(1, 8)$
2	11	$C = 5 + 3(\mathbf{2})$	$(2, 11)$
3	14	$C = 5 + 3(\mathbf{3})$	$(3, 14)$

b Each pair of values represents a point on the graph. The first value gives the horizontal coordinate of the point, and the second value gives the vertical coordinate.
The points lie on a straight line, as shown in the figure. The line extends infinitely in only one direction, because negative values of t do not make sense here.



c To write an equation, we let C represent the cost of the rental, and we use t for the number of hours:

$$\text{Cost} = (\$5 \text{ subscription}) + \$3 \cdot (\text{number of hours})$$
$$C = 5 + 3 \cdot t$$

Checkpoint 1.4 Practice 1. Write an equation for the cost C of renting a bicycle for t hours if the subscription fee is \$7 and the rental fee is \$2.50 per hour.

$$\text{Cost} = (\text{subscription fee}) + (\text{hourly rate}) \cdot (\text{number of hours})$$

$C =$ __ $+$ ___ $\cdot\, t$

Hint. Fill in the correct numbers.

Answer 1. 7

Answer 2. 2.5

Solution. $C = 7 + 2.50t$

Equations for Linear Models

The equation we wrote in the first Example is an example of a **linear model**, which describes a variable that increases or decreases at a constant rate.

Definition 1.5 Linear Model.

A **linear model** describes a variable that increases or decreases at a constant rate. It has the form

$$y = (\text{starting value}) + (\text{rate}) \times t$$

Checkpoint 1.6 QuickCheck 3. What sort of variables can be described by a linear model?

- ⊙ A) Increasing variables
- ⊙ B) Variables that change at a constant rate
- ⊙ C) Variables that describe time
- ⊙ D) Variables that can be graphed

Answer. B) ... constant rate

Solution. Variables that change at a constant rate

In the next example we see how a graph and its equation are related.

Example 1.7

Use the equation $C = 5 + 3 \cdot t$ you found in Example 1.3, p. 4 to answer the following questions. Then show how to find the answers by using the graph.

a How much will it cost to rent a bicycle for 6 hours?

b How long can you bicycle for $18.50?

Solution.

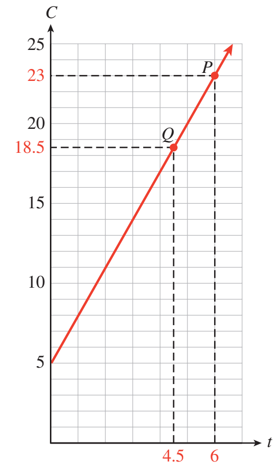
a We substitute $t = \mathbf{6}$ into the equation to find

$$C = 5 + 3(\mathbf{6}) = 23$$

A 6-hour bike ride will cost $23. The point P on the graph in the figure represents the cost of a 6-hour bike ride. The value on the C-axis at the same height as point P is 23, so a 6-hour bike ride costs $23.

We substitute $C = \mathbf{18.50}$ into the equation and solve for t.

$$\mathbf{18.50} = 5 + 3t$$
$$13.50 = 3t$$
$$t = 4.5$$



b

For \$18.50 Annelise can bicycle for $4\frac{1}{2}$ hours. The point Q on the graph represents an \$18.50 bike ride. The value on the t-axis below point Q is 4.5, so \$18.50 will buy a 4.5 hour bike ride.

Checkpoint 1.8 **Practice 2.** In the preceding Example, how long can you bicycle for \$9.50?

Answer: __ hr.

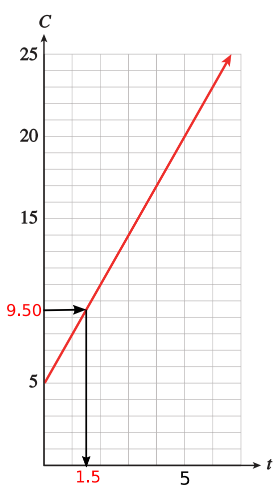
Show this on the graph.

Hint. Start by finding \$9.50 on the Cost (vertical axis). Then find the point on the graph with C-coordinate \$9.50. Finally, find the t-coordinate of that point.

Answer. 1.5

Solution. 1.5 hours

A figure is below.



In Example 1.7, p. 6, note the different algebraic techniques we used in parts (a) and (b):

- In part (a) we were given a value of t and we **evaluated the expression** $5 + 3t$ to find C.
- In part (b) we were given a value of C and we **solved the equation** $C = 5 + 3t$ to find t.

Checkpoint 1.9 **QuickCheck 4.** Consider the expression $C = 5 + 3t$. Finding a value of C when we know t is called (☐ solving ☐ evaluating) the expression. Finding a value of t when we know C is called (☐ solving

☐ evaluating) the equation.

Answer 1. evaluating

Answer 2. solving

Solution. evaluating; solving

In the examples above, the graph is **increasing** as t increases. In the next Example we consider a **decreasing graph**.

Example 1.10

Leon's camper has a 20-gallon gas tank, and he gets 12 miles to the gallon. (Note that getting 12 miles to the gallon is the same as using $\frac{1}{12}$ gallon of gas per mile.)

a Write an equation for the amount of gasoline, g, left in Leon's tank after he has driven for d miles.

b Make a table of values for the equation.

c Graph the equation.

d If Leon has less than 5 gallons of gas left, how many miles has he driven since his last fill-up? Illustrate on the graph.

Solution.

a We use the form for a linear model,

$$y = (\text{starting value}) + (\text{rate}) \times t$$

However, in this problem, instead of variables y and t, we use g and d. Leon's fuel tank started with 20 gallons, and the amount of gasoline is decreasing at a rate of $\frac{1}{12}$ gallon for every mile that he drives. Thus,

$$g = 20 - \frac{1}{12}d$$

b Every time Leon drives 12 miles, he uses another gallon of gasoline. So, to make the calculations easier, we choose values that are divisible by 12.

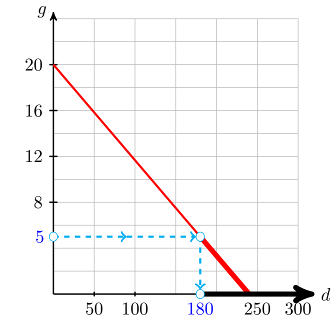
d (miles)	0	48	96	144
g (gallons)	20	16	12	8

c We scale the values of d along the horizontal axis, and the values of g along the vertical axis. Then we plot the points from the table and connect with a line, as shown in the figure below.

d If Leon has less than 5 gallons of gasoline left, then $g < 5$, as shown on the g-axis. Using our model from part (a), we solve the inequality.

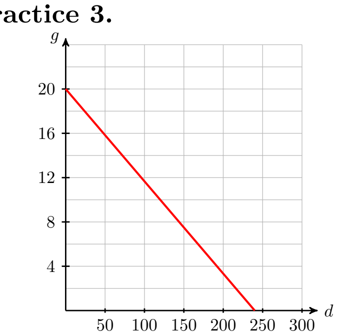
$$\begin{aligned} 20 - \frac{1}{12}d &< 5 \\ -\frac{1}{12}d &< -15 \\ d &> 180 \end{aligned}$$

Leon has driven at least 180 miles. The solution is shown on the graph below.



Note 1.11 In part (d) of the previous Example we used an **inequality** to answer the question. We use inequalities to model English phrases such as ”less than,” ”more than,” ”at least,” and ”at most.”

Checkpoint 1.12 Practice 3.



Leon forgot to reset his odometer after his last fill-up, but he thinks he has driven at least 150 miles. How much gas does he have left?

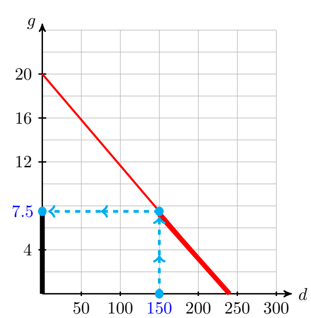
- ⊙ At least 7.5 gallons
- ⊙ At most 7.5 gallons
- ⊙ At least 20 gallons.
- ⊙ At most 240 gallons

Show this on the graph.

Hint. Locate 150 miles on the \(d\)-axis. What part of the axis represents “at least” 150 miles? Find the points on the graph with \(d\)-coordinates at least 150. What are the \(g\)-coordinates of those points?

Answer. At most 7.5 gallons

Solution. At most 7.5 gallon



Problem Set 1.1

Warm Up

1. Solve the equation.

a $8x - 18 = 26$

b $14 - \frac{3}{4}x = 5$

2. Solve the inequality.

a $8x - 18 < 16$

b $14 - \frac{3}{4}x \geq 5$

3. April sells environmentally friendly cleaning products. Her income consists of \$200 per week plus a commission of 9% of her sales. Write an algebraic expression for April's weekly income, I, in terms of her sales, S.

Income	=	Flat Rate	+	Commission
______	=	________	+	__________

4. Trinh is bicycling down a mountain road that loses 500 feet in elevation for each 1 mile of road. She started at an elevation of 6300 feet. Write an expression for Trinh's elevation, h, in terms of the distance she has cycled, d.

Elevation	=	Initial Elevation	−	Decrease
______	=	____________	−	________

Skills Practice

For problems 5 and 6, solve the equation.

5. $3y + 2y = 12 - 5y$

6. $0.8w - 2.6 = 1.4w + 0.3$

For problems 7 and 8, solve the inequality.

7. $3y + 2y > 12 - 5y$

8. $0.8w - 2.6 \leq 1.4w + 0.3$

9. Bruce buys a 50-pound bag of rice and consumes about 0.4 pounds per week. Write an expression for the amount of rice, R, Bruce has left in terms of the number of weeks, w, since he bought the bag.

10. Delbert is offered a job as a salesman. He will be paid \$1000 per month plus a 6% commission on his sales. Write an expression for Delbert's monthly income, I, in terms of his sales, S.

Applications

11. Frank plants a dozen corn seedlings, each 6 inches tall. With plenty of water and sunlight they will grow approximately 2 inches per day. Complete the table of values for the height, h, of the seedlings after t days.

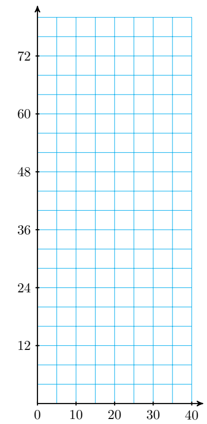
a Complete the table of values for the height, h of the seedlings after t days.

t (days)	0	5	10	15	20
h (inches)					

b Write an equation for the height, h, of the seedlings in terms of the number of days, t, since they were planted.

c Graph the equation on the grid. Label the axes with the correct variables.

To answer questions (d) and (e), read the graph. Show your work on the graph:



d How tall is the corn after 3 weeks?

e When will the corn grow to 6 feet tall? (How many inches is 6 feet?)

f Use algebra to find the answers for parts (d) and (e) above.

g Frank also sets out some tomato plants. The height of the plants ininches after t days is:

$$h = 14 + 1.5t$$

What do the constants in this equation tell us about the tomato plants?

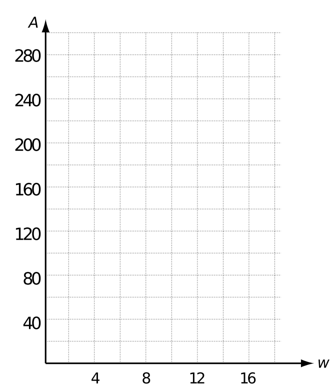
12. On October 31, Betty and Paul fill their 250-gallon oil tank for their heater. Beginning in November they use an average of 15 gallons of oil per week.

a Complete the table of values for the amount of oil, A, left in the tank after w weeks.

w (weeks)	0	4	8	12	16
A (gallons)					

b Write an equation that expresses the amount of oil, A, in terms of the number of weeks, w, since October 31.

c Graph the equation.



d How much did the amount of fuel oil in the tank decrease between the third week and the eighth week? Illustrate this amount on the graph.

e During which weeks will the tank contain more than 175 gallons of fuel oil? Illustrate on the graph.

f Write and solve an inequality to verify your answer to part (e).

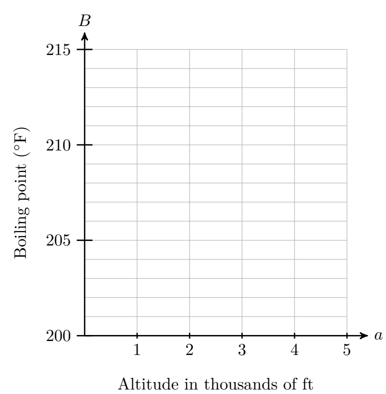
13. The boiling point of water changes with altitude. At sea level, water boils at 212°F, and the boiling point decreases by approximately 2°F for each 1000-foot increase in altitude.

a Write an equation for the boiling point, B, in terms of a, the altitude in thousands of feet.

b Complete the table of values.

Altitude (1000 ft)	0	1	2	3	4	5
Boiling point (°F)						

c Graph the equation.



d How much does the boiling point decrease when the altitude increases from 1000 to 3000 feet? Illustrate this amount on the graph.

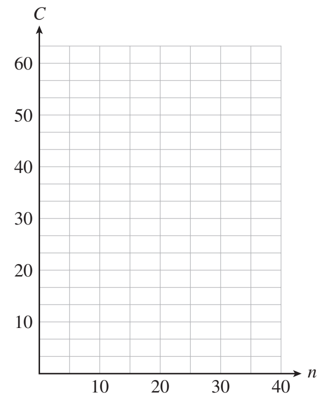
e At what altitudes is the boiling point less than 204°F? Illustrate on the graph.

14. The taxi out of Dulles Airport charges a traveler with one suitcase an initial fee of \$2.00, plus \$1.50 for each mile traveled. Complete the table of values showing the charge, C, for a trip of n miles.

n	0	5	10	15	20	25
C						

a Write an equation for the charge, C, in terms of the number of miles traveled, n.

b Graph the equation.



c What is the charge for a trip to Mount Vernon, 40 miles from the airport? Illustrate the answer on your graph.

d If a ride to the National Institutes of Health (NIH) costs \$39.50, how far is it from the airport to the NIH? Illustrate the answer on your graph.

Graphs and Equations

Equations and Solutions

In this section we review some techniques and terminology related to equations, inequalities, and their graphs.

Definition 1.13 Solution.

A **solution** of an equation is a value of the variable that makes the equation true.

For example, $x = \mathbf{-2}$ is a solution of the equation

$$4x^3 + 5x^2 - 8x = 4$$

because $4(\mathbf{-2})^3 + 5(\mathbf{-2})^2 - 8(\mathbf{-2}) = -2 + 20 + 16 = 4$.

Checkpoint 1.14 QuickCheck 1. Which of the following values are solutions of the equation

$$-2x^3 + x^2 + 16x = 15?$$

☐ \(x=2\) ☐ \(x=-3\) ☐ \(x=3\) ☐ \(x=1\)

Solution. $x = -3$ and $x = 1$ both satisfy the equation.

A **linear equation**, such as the models we studied in the last section, has at most one solution. We find the solution by transforming the equation into a simpler **equivalent equation** whose solution is obvious.

Example 1.15

Solve the equation $3(2x - 5) - 4x = 2x - (6 - 3x)$

Solution. We begin by simplifying each side of the equation.

$3(2x - 5) - 4x = 2x - (6 - 3x)$	Apply the distrubutive law.
$6x - 15 - 4x = 2x - 6 + 3x$	Combine like terms.
$2x - 15 = 5x - 6$	Add $-5x + 15$ to both sides

$$-3x = 9 \qquad \text{Divide both sides by } -3.$$
$$x = -3$$

The solution is -3. You can check that substituting $x = -3$ into the original equation produces a true statement.

Checkpoint 1.16 Practice 1. Find the solution of the equation $16-2(3x-1) = 4x+2(x-3)$.

$x =$___

Answer. 2

Solution. $x = 2$

Linear Inequalities

Although a linear equation can have at most one solution, a linear **inequality** can have many solutions. For example, complete the table of values for the expression $5-2x$:

x	-2	-1	0	1	2	3	4
$5-2x$							

Now use your table to list at least three solutions of the inequality $5-2x < 2$.

Example 1.17

Use algebra to solve the inequality $5 - 2x < 2$.

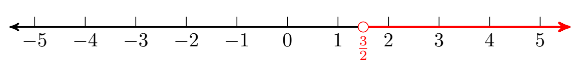
Solution. We begin by isolating the term containing the variable, just as we do when solving a linear equation. We subtract 5 from both sides to obtain

$$-2x < -3$$

Then we divide both sides by -2 to find

$$x > \frac{-3}{-2} = \frac{3}{2} \qquad \text{Reverse the direction of the inequality.}$$

Any value of x greater than $\frac{3}{2}$ is a solution of the inequality. We write the solutions as $x > \frac{3}{2}$. Because we cannot list all of these solutions, we often illustrate them as a graph on a **number line**, as shown below.



In the Example above, we used the following rule for solving linear inequalities.

Solving a Linear Inequality.

If we multiply or divide both sides of an inequality by a negative number, we must reverse the direction of the inequality.

Other than the rule stated in the box above, the rules for solving a linear inequality are the same as the rules for solving a linear equation.

Checkpoint 1.18 QuickCheck 2. Which of the following are correct solutions of the inequality?

- ⊙ $2x > -8$ has solution $x < -4$
- ⊙ $-2x > -8$ has solution $x < 4$
- ⊙ $x + 2 > -8$ has solution $x < -10$
- ⊙ $x - 2 > -8$ has solution $x < -6$

Answer. Choice 2

Solution. $-2x > -8$ has solution $x < 4$

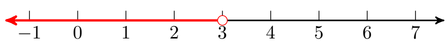
Checkpoint 1.19 Practice 2. Solve the inequality $2x + 1 < 7$, and graph the solutions on a number line.

Solution: ________

Answer. $x < 3$

Solution. $x < 3$

A number line is below.



Equations in Two Variables

An **equation in two variables**, such as

$$-3x + 4y = 24$$

has many solutions. Each **solution** consists of an **ordered pair** of values, one for x and one for y, that together satisfy the equation (make the equation true.)

Checkpoint 1.20 QuickCheck 3. What is the solution of an equation in two variables?

- ⊙ A value of x that makes the equation true.
- ⊙ An ordered pair of values (x, y) that satisfy the equation.
- ⊙ Solving the equation for y in terms of x.
- ⊙ The graph of the equation.

Answer. Choice 2

Solution. An ordered pair of values (x, y) that satisfy the equation.

Example 1.21

a Is $(2, 7.5)$ a solution of the equation $-3x + 4y = 24$?

b Is $(4, 3)$ a solution of the equation $-3x + 4y = 24$?

Solution.

a The ordered pair $(2, 7.5)$ is a solution of the equation above, because it satifies the equation.

$$-3(\mathbf{2}) + 4(\mathbf{7.5}) = -6 + 30 = 24$$

b The ordered pair $(4, 3)$ is not a solution, because it does not satify the equation.

$$-3(\mathbf{4}) + 4(\mathbf{3}) = -12 + 12 = 0 \neq 24$$

Checkpoint 1.22 Practice 3. Find some more solutions of the equation $-3x + 4y = 24$ and complete the table of values:

x	-12	-8	__	0	__
y	__	__	3	__	9

Answer 1. -4

Answer 2. 4

Answer 3. -3

Answer 4. 0

Answer 5. 6

Solution.

x	-12	-8	-4	0	4
y	-3	0	3	6	9

What is the Graph of an Equation?

Definition 1.23 Graph.

The **graph** of an equation in two variables is just a picture of all its solutions.

You might think it would be difficult to find *all* the solutions of an equation, but for a linear equation $Ax + By = C$, we can at least illustrate the solutions: all the solutions lie on a straight line. (Later on we can prove that this is true.)

Checkpoint 1.24 QuickCheck 4. True or false?

a. All the solutions of a linear equation in two variables lie on a straight line. (☐ True ☐ False)

b. The equation $-3x + 4y = 24$ is a linear equation. (☐ True ☐ False)

c. The graph of an equation in two variables is a picture of its solutions. (☐ True ☐ False)

d. If a point lies on the graph of an equation, it is a solution of the equation. (☐ True ☐ False)

Answer 1. True

Answer 2. True

Answer 3. True

Answer 4. True

Solution.

a. True

b. True

c. True

d. True

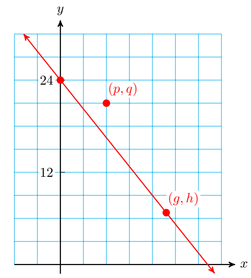
Example 1.25

The figure shows a graph of the equation $Ax + By = C$. Which of the following equations are true?

a $24B = C$

b $Ap + Bq = C$

c $Ag + Bh = C$



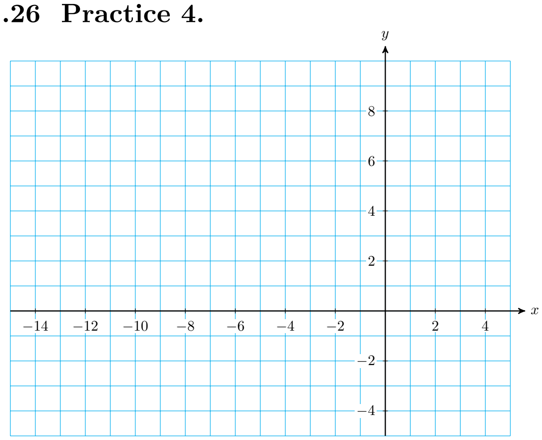
Solution.

a The point $(0, 24)$ lies on the graph, so $x = 0$, $y = 24$ is a solution of the equation. Thus, $A \cdot 0 + B \cdot 24 = C$, or $24B = C$ is a true statement.

b The point (p, q) does not lie on the graph, so $x = p$, $y = q$ does not satisfy the equation, and $Ap + Bq = C$ is a not true.

c The point (g, h) does lie on the graph, so $x = g$, $y = h$ does satisfy the equation, and $Ag + Bh = C$ is a true statement.

Checkpoint 1.26 Practice 4.



On the grid above, plot the points you found in Practice 3. All the points should lie on a straight line; draw the line with a ruler or straightedge. Which of the following points lie on the graph?

- ⊙ $(-13, 4)$
- ⊙ $(-1.6, 4.8)$
- ⊙ $(1.125, 7)$

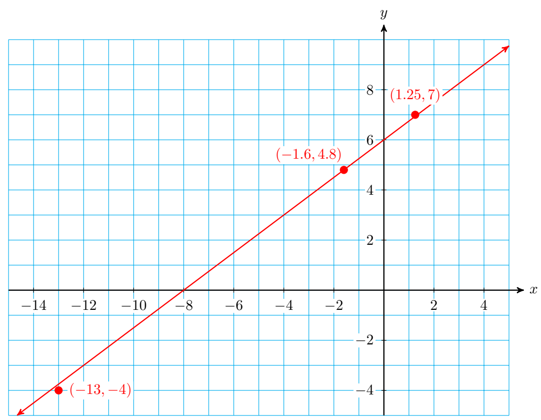
Which of the points above satisfy the equation in Practice 3?

- ⊙ $(-13, 4)$
- ⊙ $(-1.6, 4.8)$
- ⊙ $(1.125, 7)$

Answer 1. Choice 2

Answer 2. Choice 2

Solution. $(-1.6, 4.8)$ lies on the graph and satisfies the equation.



Graphical Solution of Equations

We can use graphs to find solutions to equations in one variable.

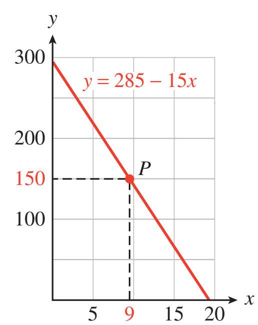
Example 1.27

Use the graph of $y = 285 - 15x$ to solve the equation

$$150 = 285 - 15x$$

Solution.

Compare the two equations in the problem. In the equation we want to solve, y has been replaced by **150**. We begin by locating the point P on the graph for which $y = 150$.
Next we find the x-coordinate of point P by drawing an imaginary line from P straight down to the x-axis. The x-coordinate of P is $x = 9$.
Thus, P is the point $(9, 150)$, and $x = 9$ when $y = 150$. The solution we seek is $x = 9$.



You can verify the solution algebraically by substituting $x = \mathbf{9}$ into the equation:

Does $150 = 285 - 15(\mathbf{9})$?

$$285 - 15(\mathbf{9}) = 285 - 135 = 150 \qquad \text{Yes}$$

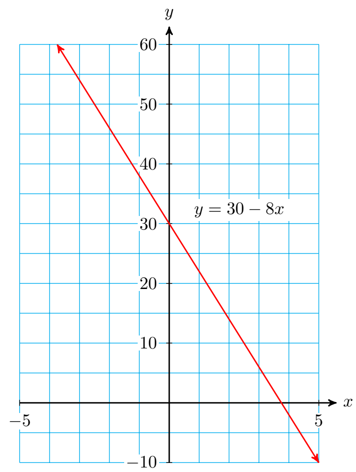
Note 1.28 The relationship between an equation and its graph is an important one. For the previous example, make sure you understand that the following three statements are equivalent.

1 The point $(9, 150)$ lies on the graph of $y = 285 - 15x$.

2 The ordered pair $(9, 150)$ is a solution of the equation $y = 285 - 15x$.

3 $x = 9$ is a solution of the equation $150 = 285 - 15x$.

Checkpoint 1.29 Practice 5.



a. Use the graph of $y = 30 - 8x$ to solve the equation $30 - 8x = 50$.Follow the steps:

- Step 1: Locate the point P on the graph with $y = 50$.
- Step 2: Find the x-coordinate of your point P.

b. Verify your solution algebraically.

$x =$___

Answer. -2.5

Solution. $x = -2.5$

Graphical Solution of Inequalities

We can also use graphs to solve inequalities. Consider the inequality

$$285 - 15x \geq 150$$

To solve this inequality means to find all values of x that make the expression $285 - 15x$ greater than or equal to 150. We could begin by trying some values of x. Here is a table obtained by evaluating $285-15x$.

x	0	2	4	6	8	10	12
$285 - 15x$	285	255	225	195	165	135	105

From the table, we see that values of x less than or equal to 8 are solutions of the inequality, but we have not checked *all* possible x-values. We can get a more complete picture from a graph.

Example 1.30

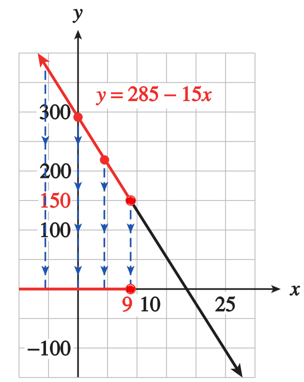
Use the graph of the equation $y = 285 - 15x$ to solve the inequality

$$285 - 15x > 150$$

Solution.

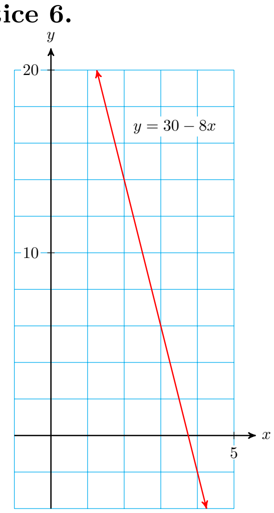
We look for points on the graph with y-coordinates greater than or equal to 150. These points are shown in color. Which x-values produced these points?

We can read the x-coordinates by dropping straight down to the x-axis, as shown by the arows. For example, the x-value corresponding to $y = 150$ is $x = 9$. For larger values of $285-15x$, we must choose x-values less than 9. Thus, all values of x less than or equal to 9 are solutions, as shown on the x-axis.



We write the solutions as $x \leq 9$.

Checkpoint 1.31 Practice 6.



a. Use the graph of $y = 30 - 8x$ to solve the equation $30 - 8x < 14$.Follow the steps:

- Step 1: Locate the point P on the graph with $y = 14$.
- Step 2: Find the x-coordinate of the point P.
- Step 3: Which points on the graph have $y < 14$? Mark them on the graph.
- Step 4: Find the x-coordinates of the points in Step 3. Mark them all on the x-axis.

b. Verify your solution algebraically.

Solution: ___

Answer. $x > 2$

Solution. $x > 2$

Using a Graphing Utility

We can use a graphing utility to graph equations if they are written in the form $y =$ (expression in x) . First, let's review how to solve an equation for y

in terms of x.

Example 1.32

Solve the equation $6x - 5y = 90$ for y in terms of x.

Solution. To begin, we isolate the y-term by subtracting $6x$ from both sides of the equation.

$$\begin{aligned} -5y &= 90 - 6x && \text{Divid both sides by } -5. \\ y &= \frac{90}{-5} - \frac{6x}{-5} && \text{Simplify.} \\ y &= -18 + \frac{6}{5}x \end{aligned}$$

Note that the equation now has the form of the linear models we saw in the last section.

Checkpoint 1.33 **Practice 7.** Solve for y in terms of x: $10y - 15x = 6$

$y =$__________

Answer. $\frac{3}{2}x + \frac{3}{5}$

Solution. $y = 1.5x + 0.6$

Now we are ready to graph an equation with technology. On most calculators, we follow three steps.

To Graph an Equation.

1 Press Y= and enter the equation you wish to graph.

2 Press WINDOW and select a graphing window.

3 Press GRAPH.

Choosing a graphing window corresponds to drawing the x- and y-axes and marking a scale on each axis when we graph by hand. The **standard graphing window** displays values from -10 to 10 on both axes. We can press the ZOOM key and then the number 6 to select this window.

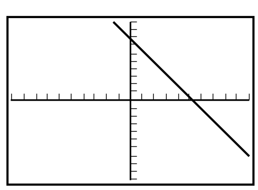
Example 1.34

Use a graphing utility to graph the equation $3x + 2y = 16$.

Solution. First, we solve the equation for y in terms of x.

$$\begin{aligned} 3x + 2y &= 16 && \text{Subtract } 3x \text{ from both sides.} \\ 2y &= -3x + 16 && \text{Divide both sides by 2.} \\ y &= \frac{-3x}{2} + \frac{16}{2} && \text{Simplify.} \\ y &= -1.5x + 8 \end{aligned}$$

Now press Y= and enter $-1.5x + 8$ after $Y_1 =$. (Use the negation key, (-), to enter $-1.5x$; do not use the subtraction key.) We choose the standard graphing window by pressing ZOOM 6. The graph is shown at right.



When you want to erase the graph, press Y= and then press CLEAR. This deletes the expression following Y_1.

Checkpoint 1.35 Practice 8.

a. Solve the equation $-3x + 4y = 24$ for y in terms of x.

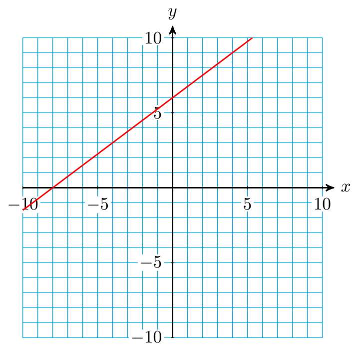
$y =$_____

b. Graph the equation in the standard window.

Answer. $\frac{3}{4}x + 6$

Solution. $y = \dfrac{3}{4}x + 6$

Graph for part (b):



Finding Coordinates with a Graphing Calculator

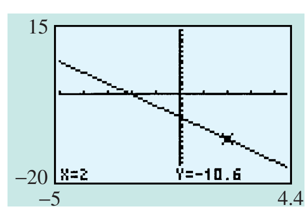
We can use the TRACE feature to find the coordinates of points on a graph.

Example 1.36

Use a graph to find a solution to the equation $y = -2.6x - 5.4$ with y-coordinate -10.6.

Solution. First we graph the equation $y = -2.6x - 5.4$ in the window

Xmin= -5	Xmax= 4.4
Ymin= -20	Ymax= 15



We press TRACE, and a "bug" begins flashing on the display. The coordinates of the bug appear at the bottom of the display, as shown in the figure. We use the left and right arrows to move the bug along the graph. You can check that the coordinates of the point $(2, -10.6)$ do satisfy the equation $y = -2.6x - 5.4$.

Checkpoint 1.37 Practice 9.

a. Graph the equation $y = 32x - 42$ in the window:

Xmin= -4.7	Xmax= 4.7	Xscl= 1
Ymin= -250	Ymax= 50	Yscl= 25

b. Use the Trace feature to find the point that has y-coordinate -122.
   ________ Enter an ordered pair.

c. Verify your answer algebraically by substituting your x-value into the equation.

Answer. $(-2.5, -122)$

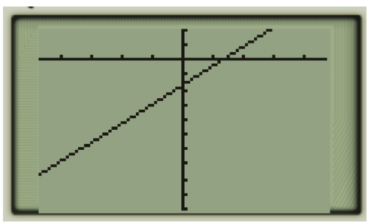
Solution.

a. A graph is below.

b. $(-2.5, -122)$

c. $-122 = 32(\mathbf{-2.5}) - 42$

Graph for part (a):



Problem Set 1.2

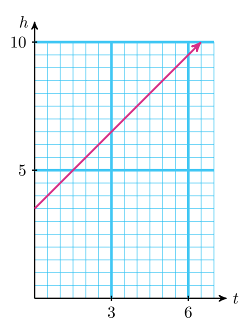
Warm Up

For Problems 1–2, decide whether the ordered pairs are solutions of the equation whose graph is shown.

1.

a $(6.5, 3)$ c $(8, 2)$

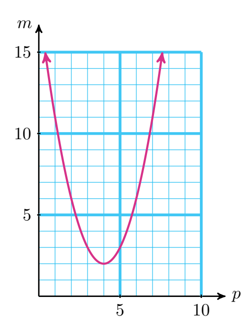
b $(0, 3.5)$ d $(4.5, 1)$



2.

a $(2, 6)$ c $(10, 0)$

b $(4, 2)$ d $(11, 7)$



For Problems 3–6, decide whether the ordered pairs are solutions of the given equation.

3. $y = \frac{3}{4}x$

a $(8, 6)$ c $(2, 3)$

b $(12, 16)$ d $(6, \frac{9}{2})$

4. $y = \frac{x}{2.5}$

a $(1, 2.5)$ c $(5, 2)$

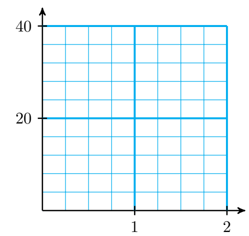
b $(25, 10)$ d $(8, 20)$

5. $w = z - 1.8$
- a $(10, 8.8)$
- b $(6, 7.8)$
- c $(2, \frac{1}{5})$
- d $(9.2, 7.4)$

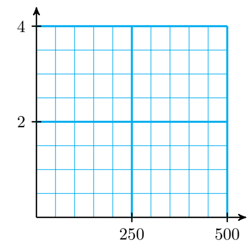
6. $w = 120 - z$
- a $(0, 120)$
- b $(65, 55)$
- c $(150, 30)$
- d $(9.6, 2.4)$

For Problems 7–8, state the interval that each grid line represents on the horizontal and vertical axes.

7.

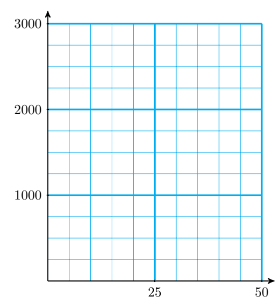


8.



9.

a What interval does each grid line represent on the horizontal axis? On the vertical axis?

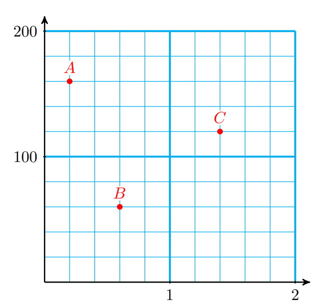


b Plot the following points on the grid:

$$(0, 500),\ (20, 1750),\ (40, 250)$$

10.

a What interval does each grid line represent on the horizontal axis? On the vertical axis?



b Find the coordinates of each point.

Skills Practice

For Problems 11–14, solve.

11. $5 - 2(3x - 4) = 28 - 2x$

11. $0.0048z - 0.12 = -0.08 + 0.0016z$

13. $0.25t + 0.10(t - 4) = 11.85$

14. $0.12t + 0.08(t + 10,000) = 12,000$

For Problems 15–18, solve the inequality and graph the solutions on a number line.

14. $3x - 2 > 1 + 2x$

15. $\dfrac{-2x - 6}{-3} > 2$

17. $\dfrac{-2x - 3}{2} \leq -5$

18. $\dfrac{2x - 3}{3} \leq \dfrac{3x}{-2}$

For Problems 19-24, solve for y in terms of x.

19. $2x - 3y = -72$

20. $4x + 75y = 60,000$

21. $7x = 91 - 13y$

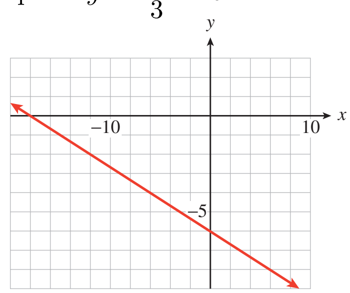
22. $\dfrac{x}{80} + \dfrac{y}{400} = 1$

23. $80x - 360y = 6120$

24. $3x + \dfrac{3}{4}y = \dfrac{1}{2}$

Applications

25. The figure shows a graph of $y = \dfrac{-x}{3} - 6$.



a Use the graph to find all values of x for which

i $y = -4$

ii $y > -4$

iii $y < -4$

b Use the graph to solve

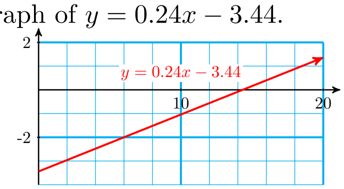
i $\dfrac{-x}{3} - 6 = -4$

ii $\dfrac{-x}{3} - 6 > -4$

iii $\dfrac{-x}{3} - 6 < -4$

c Explain why your answers to parts (a) and (b) are the same.

26. The figure shows a graph of $y = 0.24x - 3.44$.

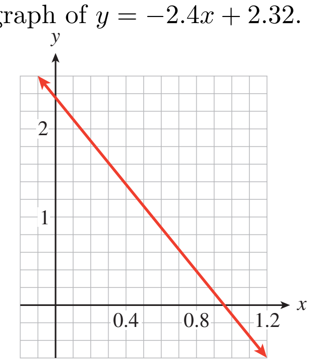


a Use the graph to solve $0.24x - 3.44 = -2$.

b Use the graph to solve $0.24x - 3.44 > -2$.

c Solve the inequality in part (b) algebraically.

27. The figure shows the graph of $y = -2.4x + 2.32$. Use the graph to solve:



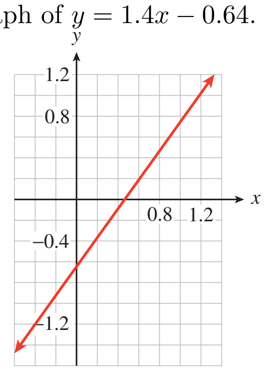
a $1.6 = -2.4x + 2.32$

b $-2.4x + 2.32 = 0.4$

c $-2.4x + 2.32 \geq 1.6$

d $0.4 \geq -2.4x + 2.32$

28. The figure shows the graph of $y = 1.4x - 0.64$. Use the graph to solve:



a $1.4x - 0.64 = 0.2$

b $-1.2 = 1.4x - 0.64$

c $1.4x - 0.64 > 0.2$

d $-1.2 > 1.4x - 0.64$

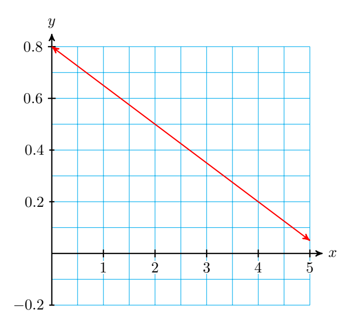
29. Here is a graph of

$$y = 0.8 - kx$$

Use the graph to solve:

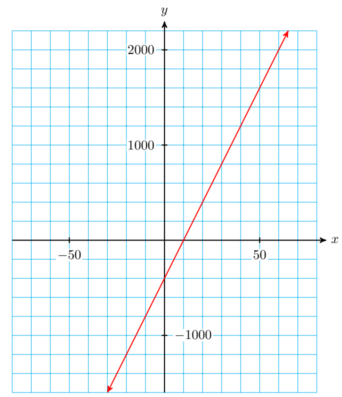
a $0.8 - kx \geq 0.2$

b $0.5 > 0.8 - kx$



30. Here is a graph of

$$y = mx + b$$



Use the graph to solve:

a $mx + b = 1200$

b $-800 = mx + b$

c $mx + b > 400$

d $-1200 \geq mx + b$

In Problems 31 and 32, graph each equation in the window

Xmin= −47	Xmax= 47	Xscl= 10
Ymin= −31	Ymax= 31	Yscl= 10

31. Graph $y = -0.4x + 3.7$. Use the graph to solve the equation or inequality. Then check your answers algebraically.

a Solve $-0.4x + 3.7 = 2.1$

b Solve $-0.4x + 3.7 > -5.1$

32. Graph $y = 6.5 - 18x$. Use the graph to answer the questions. Then check your answers algebraically.

a For what value of x is $y = -13.3$?

b For what value of x is $y = 24.5$?

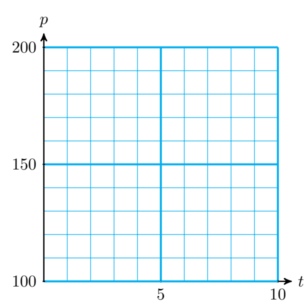
c For what values of x is $y \leq 15.5$?

d For what values of x is $y \geq -7.9$?

33. Kieran's resting blood pressure, in mm Hg, is 120, and it rises by 6 mm for each minute he jogs on a treadmill programmed to increase the level of intensity at a steady rate.

a Find a formula for Kieran's blood pressure, p, in terms of time, t.

b Graph the equation for p for $0 \leq t \leq 10$.



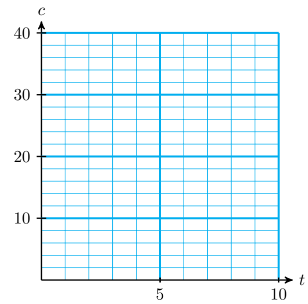
c What is Kieran's blood pressure after 3.5 minutes? Label this point on the graph.

d Kieran's blood pressure should not exceed 165 mm Hg. When will this level be reached? Label this point on the graph.

34. When Francine is at rest, her cardiac output is 5 liters per minute. The output increases by 3 liters per minute for each minute she spends on a cycling machine with increasing intensity.

a Find a formula for Francine's cardiac output, c, in terms of time, t.

b Graph the equation for p for $0 \leq t \leq 10$.



c What will Francine's cardiac output be after 6 minutes? Label this point on the graph.

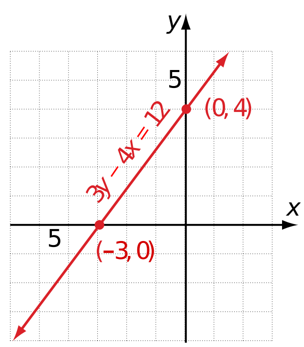
d When will Francine's cardiac output exceed 14.5 liters per minute? Label this point on the graph.

Intercepts

Intercepts of a Graph

Definition 1.38 Intercepts.

The points at which a graph crosses the axes are called the **intercepts** of the graph.



The x-intercept of the graph shown above is $(-3, 0)$, and its y-intercept is $(0, 4)$.

Checkpoint 1.39 QuickCheck 1. What are the intercepts of a graph?

- A) The variables displayed on the axes
- B) Points where the graphs intersect
- C) The highest and lowest points
- D) Points where the graph intersects the axes

Answer. D) Points ... intersects the axes

Solution. Points where the graph intersects the axes

The coordinates of the intercepts are easy to find.

Intercepts of a Graph.

1 To find the x-intercept, we set $y = 0$ and solve for x.

2 To find the y-intercept, we set $x = 0$ and solve for y.

Example 1.40

In Example 1.10, p. 8 of Section 1.1, p. 3, we graphed an equation, $g = 20 - \frac{1}{12}d$, for the amount of gasoline, g, left in Leon's tank after he has driven for d miles. Find the intercepts of the graph.

Solution. To find the d-intercept, we set $g = 0$ and solve for d.

$$
\begin{aligned}
\mathbf{0} &= 20 - \frac{1}{12}d && \text{Add } \frac{1}{12}d \text{ to both sides.}\\
\frac{1}{12}d &= 20 && \text{Multiply both sides by 12.}\\
d &= 240
\end{aligned}
$$

The d-intercept is $(240, 0)$. To find the g-intercept, we set $d = 0$ and solve for g.

$$g = 20 - \frac{1}{12}(\mathbf{0}) = 20$$

The g-intercept is $(0, 20)$.

Checkpoint 1.41 **Practice 1.** Find the intercepts of the graph of $y = -9 - \dfrac{3}{2}x$. Enter each intercept as an ordered pair.

The x-intercept is ____.

The y-intercept is ____.

Answer 1. $(-6, 0)$

Answer 2. $(0, -9)$

Solution. The x-intercept is $(-6, 0)$.

The y-intercept is $(0, -9)$.

Meaning of the Intercepts

The intercepts of a graph give us information about the situation it models.

Example 1.42

What do the intercepts of the graph in the Example above tell us about the problem situation?

Solution. The d-intercept tells us that when $d = 240$, $g = 0$, or that when Leon has traveled 240 miles, he has 0 gallons of gasoline left; the fuel tank is empty.

The g-intercept tells us that when $d = 0$, $g = 20$, or that when Leon has traveled 0 miles, he has 20 gallons of gasoline left. The fuel tank holds 20 gallons when full.

Checkpoint 1.43 Practice 2. The gas tank in Rosa's Toyota Prius holds 11 gallons, and she gets 48 miles to the gallon.

a. Write an equation for the amount of gasoline, g, left in the tank after Rosa has driven for d miles.

   $g =$________

b. Find the intercepts of the graph.

   The d-intercept is _____.

   The g-intercept is _____.

   What do they tell us about the problem situation?

   The d-intercept gives:

   - ⊙ A) miles Rosa can drive before the tank is empty
   - ⊙ B) miles to the next gas stop
   - ⊙ C) gallons of gasoline before Rosa begins driving
   - ⊙ D) gallons of gasoline per mile of driving

   The g-intercept: gives

   - ⊙ A) miles Rosa can drive before the tank is empty
   - ⊙ B) miles to the next gas stop
   - ⊙ C) gallons of gasoline before Rosa begins driving
   - ⊙ D) gallons of gasoline per mile of driving

Hint. Rewrite the sentence with mathematical symbols:

$$(\text{gasoline left}) = (\text{gallons in full tank}) - (\text{mileage rate}) \times (\text{miles driven})$$

Answer 1. $11 - \frac{1}{48}d$
Answer 2. $(528, 0)$
Answer 3. $(0, 11)$
Answer 4. A) miles ... tank is empty
Answer 5. C) ... begins driving
Solution.

a. $g = 11 - \dfrac{1}{12}d$

b. $(528, 0)$: After Rosa drives 528 miles, the tank will be empty.

   $(0, 11)$: The tank has 11 gallons before Rosa begins driving.

General Form for a Linear Equation

The graphs of the equations we have seen so far are all portions of straight lines. For this reason such equations are called **linear equations**.

The order of the terms in the equation does not matter. For example, the equation in Example 1.3, p. 4 of Section 1.1, p. 3

$$C = 5 + 3t \qquad \text{can be written equivalently as} \qquad -3t + C = 5$$

and the equation in Example 1.10, p. 8 of that section,

$$g = 20 - \frac{1}{12}d \qquad \text{can be written as} \qquad \frac{1}{12}d + g = 20$$

This form of a linear equation, $Ax + By = C$, is called the **general form**.

Definition 1.44 General Form for a Linear Equation.

The **general form** for a linear equation is

$$Ax + By = C$$

(where A and B cannot both be 0).

Checkpoint 1.45 QuickCheck 2. What is the general form of a linear equation?

- ⊙ $y = mx + b$
- ⊙ $Ax + By = C$
- ⊙ Any equation whose graph is a straight line.
- ⊙ Set $x = 0$ and solve for y.

Answer. Choice 2

Solution. The general form of a linear equation is $Ax + By = C$.

Some linear models are easier to use when they are written in the general form.

Example 1.46

The manager at Albert's Appliances has $3000 to spend on advertising for the next fiscal quarter. A 30-second spot on television costs $150 per broadcast, and a 30-second radio ad costs $50.

a The manager decides to buy x television ads and y radio ads. Write an equation relating x and y.

b Make a table of values showing several choices for x and y.

c Plot the points from your table, and graph the equation.

Solution.

a Each television ad costs $150, so x ads will cost $\$150x$. Similarly, y radio ads will cost $\$50y$. The manager has $3000 to spend, so the sum of the costs must be $3000. Thus,

$$150x + 50y = 3000$$

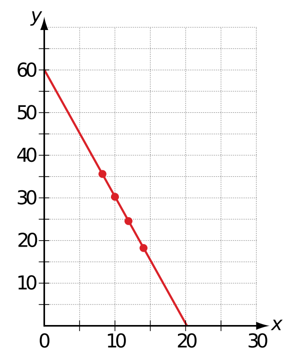
b We choose some values of x, and solve the equation for the corresponding value of y. For example, if $x = \mathbf{10}$ then

$$\begin{aligned} 150(\mathbf{10}) + 50y &= 300 \\ 1500 + 50y &= 3000 \\ 50y &= 1500 \\ y &= 30 \end{aligned}$$

If the manager buys 10 television ads, she can also buy 30 radio ads. You can verify the other entries in the table.

x	8	10	12	14
y	36	30	24	18

c We plot the points from the table. All the solutions lie on a straight line.



Checkpoint 1.47 Practice 3. The manager at Breadbasket Bakery has \$120 to spend on advertising. An ad in the local newspaper costs \$15, and posters cost \$4 each. She decides to buy x ads and y posters. Write an equation relating x and y.

Hint. Use the general form for a linear equation. What is the total amount of money the manager will spend?

Answer. $15x + 4y = 120$

Solution. $15x + 4y = 120$

Intercept Method of Graphing

Because we really need only two points to graph a linear equation, we might as well find the intercepts and use them to draw the graph.

To Graph a Linear Equation by the Intercept Method.

a Find the horizontal and vertical intercepts.

b Plot the intercepts, and draw the line through the two points.

Checkpoint 1.48 QuickCheck 3. Describe the intercept method of graphing a linear equation.

- ⊙ Make a table of values and plot the points.
- ⊙ Extend the line until it crosses both axes.
- ⊙ Solve for y in terms of x.
- ⊙ Plot the points where $x = 0$ and where $y = 0$, then draw the line through them.

Answer. Choice 4

Solution. Plot the intercepts (that is, the points where $x = 0$ and where $y = 0$), then draw the line through them.

Example 1.49

a Find the x- and y-intercepts of the graph of $150x + 50y = 3000$.

b Use the intercepts to graph the equation.

Solution.

a To find the x-intercept, we set $y = \mathbf{0}$.

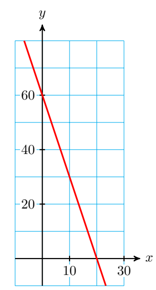
$$150x - 50(\mathbf{0}) = 3000 \quad \text{Simpify.}$$
$$150x = 3000 \quad \text{Divide both sides by 150.}$$
$$x = 20$$

The x-intercept is the point $(20, 0)$. To find the y-intercept, we set $x = \mathbf{0}$.

$$150(\mathbf{0}) - 50y = 3000 \quad \text{Simpify.}$$
$$506 = 3000 \quad \text{Divide both sides by 50.}$$
$$y = 60$$

The y-intercept is the point $(0, 60)$.

b We scale both axes in intervals of 10 and then plot the two intercepts, $(20, 0)$ and $(0, 60)$. We draw the line through them, as shown.



Checkpoint 1.50 Practice 4. Find the x- and y-intercepts of the equation in Practice 3 (about Breadbasket Bakery), and use the intercepts to graph the equation. Enter each intercept as an ordered pair.

The x-intercept is ____.

The y-intercept is ____.

Hint. Choose convenient scales for the x- and y-axes.

Answer 1. $(8, 0)$

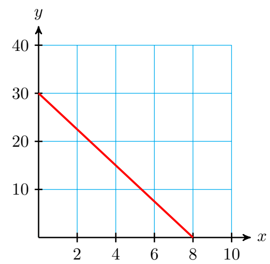
Answer 2. $(0, 30)$

Solution. The x-intercept is $(8, 0)$.

The y-intercept is $(0, 30)$.

A graph is below.

Graph for Breadbasket Bakery:



Two Forms for Linear Equations

We have now seen two forms for linear equations: the general linear form,

$$Ax + By = C$$

and the form for a linear model,

$$y = (\text{starting value}) + (\text{rate}) \times t$$

Sometimes it is useful to convert an equation from one form to the other.

Example 1.51

a Write the equation $4x - 3y = 6$ in the form for a linear model.

b Write the equation $y = -9 - \frac{3}{2}x$ in general linear form.

Solution.

a We would like to solve for y in terms of x. We first isolate the y-term on one side of the equation.

$$\begin{aligned} 4x - 3y &= 6 && \text{Subtract } 4x \text{ from both sides.} \\ -3y &= 6 - 4x && \text{Divide both sides by } -3. \\ \frac{-3y}{-3} &= \frac{6 - 4x}{-3} && \text{Simplify: divide each term by } -3. \\ y - 2 &+ \frac{4}{3}x \end{aligned}$$

b Write the equation in the form $Ax + By = C$.

$$\begin{aligned} y &= -9 - \frac{3}{2}x && \text{Add } \frac{3}{2}x \text{ to both sides.} \\ \frac{3}{2}x + y &= -9 \end{aligned}$$

We can write the equation with integer coefficients by clearing the fractions. We multiply both sides of the equation by **2**.

$$\begin{aligned} \mathbf{2}\left(\frac{3}{2}x + y\right) &= -9(\mathbf{2}) \\ 3x + 2y &= -18 \end{aligned}$$

Caution 1.52 Do not confuse solving for y in terms of x with finding the y-intercept. Compare:

a In Example 5a, we solved $4x{-}3y = 6$ for y in terms of x to get

$$y = -2 + \frac{4}{3}x.$$

This is still an equation in two variables; it is just another (equivalent) form of the original equation.

b To find the y-intercept of the same equation, we first set $x = 0$, then solve for y, as follows:

$$4(0){-}3y = 6$$
$$y = -2$$

This gives us a **particular point** on the graph, namely, $(0, -2)$; the point whose x-coordinate is 0.

Checkpoint 1.53 Practice 5.

a. Write the equation $150x + 50y = 3000$ in the form for a linear model.

$y =$______________

b. Write the equation $y = 0.15x - 3.8$ in general linear form with integer coefficients.

Answer 1. $-3x + 60$

Answer 2. $3x - 20y = 76$

Solution.

a. $y = -3x + 60$

b. $-15x + 100y = -380$ or $3x - 20y = 76$

Problem Set 1.3

Warm Up

1. The owner of a movie theater needs to bring in \$1000 revenue at each screening in order to stay in business. He sells adults' tickets for \$5 each and children's tickets at \$2 each.

a How much revenue does he earn from selling x adults' tickets?

b How much revenue does he earn from selling y children's tickets?

c Write an equation in x and y for the number of tickets he must sell at each screening

2. Karel needs 45 milliliters of a 40% solution of carbolic acid. He plans to mix some 20% solution with some 50% solution.

a How much carbolic acid is in x milliliters of the 20% solution?

b How much carbolic acid is in y milliliters of the 50% solution?

c How much carbolic acid is in the solution Karel needs?

d Write an equation in x and y for the amount of each solution Karel should mix.

For Problems 3 and 4, solve the equation for y in terms of x.

3. $3x + 5y = 16$

4. $20x = 30y - 45,000$

Skills Practice

For Problems 5-8,

a Find the intercepts of the graph.

b Graph the equation by the intercept method.

5. $9x - 12y = 36$

6. $\dfrac{x}{9} - \dfrac{y}{4} = 1$

7. $4y = 20 + 2.5x$

8. $30x = 45y + 60,000$

9. Find the intercepts of the graph for each equation.

a $\dfrac{x}{3} + \dfrac{y}{5} = 1$

b $2x - 4y = 1$

c $\dfrac{2x}{5} - \dfrac{2y}{3} = 1$

d $\dfrac{x}{p} + \dfrac{y}{q} = 1$

e. Why is the equation $\dfrac{x}{a} + \dfrac{y}{b} = 1$ called the **intercept form** for a line?

10. Write an equation in intercept form (see Problem 9) for the line with the given intercepts. Then write the equation in general form.

a $(6, 0), (0, 2)$

b $(-3, 0), (0, 8)$

c $\left(\dfrac{3}{4}, 0\right), \left(0, \dfrac{-1}{4}\right)$

d $(v, 0), (0, -w)$

e $\left(\dfrac{1}{H}, 0\right), \left(0, \dfrac{1}{T}\right)$

11.

a Find the y-intercept of the line $y = mx + b$.

b Find the x-intercept of the line $y = mx + b$.

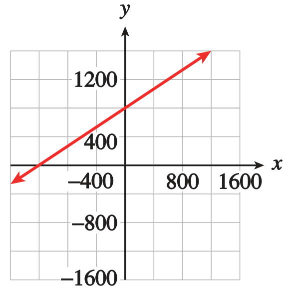
12.

a Find the y-intercept of the line $Ax + By = C$.

b Find the x-intercept of the line $Ax + By = C$.

For Problems 13-16, write an equation in general form for the line.

13.



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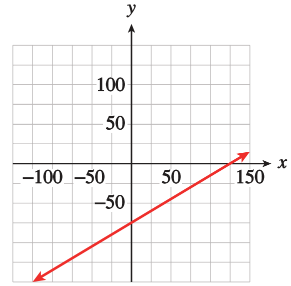

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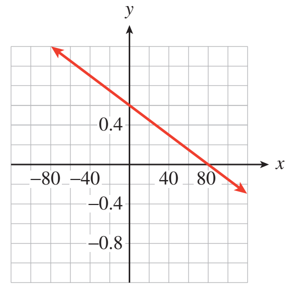

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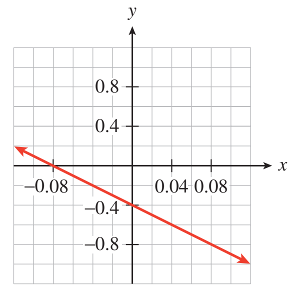


For Problems 17-20, write the equation in two standard forms:

a the general linear form, $Ax + By = C$, with integer coefficients, and

b the form for a linear model, $y =$ (starting value) + (rate) $\times x$

17. $\dfrac{2x}{3} + \dfrac{3y}{11} = 1$

18. $\dfrac{8x}{7} + \dfrac{2y}{7} = 1$

19. $0.4x = 4.8 - 1.2y$

20. $-0.8y = 12.8 - 3.2x$

Applications

21. Delbert must increase his daily potassium intake by 1800 mg. He decides to eat a combination of figs and bananas. One gram of fig contains 9 mg of potassium, and one gram of banana contains 4 mg of potassium.

a How many mg of potassium are in x grams of figs?

b How many mg of potassium are in y grams of bananas?

c Write an equation for the number of grams of fig, x, and the number of grams of banana, y, that Delbert needs to eat daily.

d Find the intercepts of the graph. What do the intercepts tell us about Delbert's diet?

22. Five pounds of body fat is equivalent to approximately 16,000 calories. Carol can burn 600 calories per hour bicycling and 400 calories per hour swimming.

a How many calories will Carol burn in x hours of cycling?

b How many calories will she burn in y hours of swimming?

c Write an equation that relates the number of hours, x, of cycling and y, of swimming Carol needs to perform in order to lose 5 pounds.

d Find the intercepts of the graph. What do the intercepts tell us about Carol's exercise program?

23. A deep-sea diver is taking some readings at a depth of 400 feet. He begins rising at a rate of 20 feet per minute.

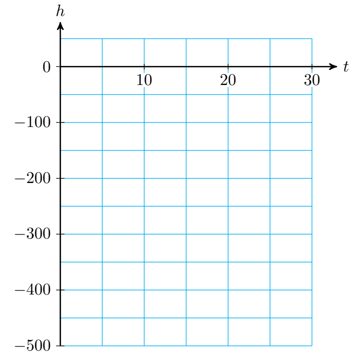
a Complete the table of values for the diver's altitude h after t minutes. (A depth of 400 feet is the same as an altitude of -400 feet.)

t	0	5	10	15	20
h					

b Write an equation for the diver's altitude, h, in terms of the number of minutes, t, elapsed.

c Find the intercepts and sketch the graph.

t	h
0	
	0



d Explain what each intercept tells us about this problem.

24. In central Nebraska, each acre of corn requires 25 acre-inches of water per year, and each acre of winter wheat requires 18 acre-inches of water. (An acre-inch is the amount of water needed to cover one acre of land to a depth of one inch.) A farmer can count on 9000 acre-inches of water for the coming year. (Source: Institute of Agriculture and Natural Resources, University ofNebraska)

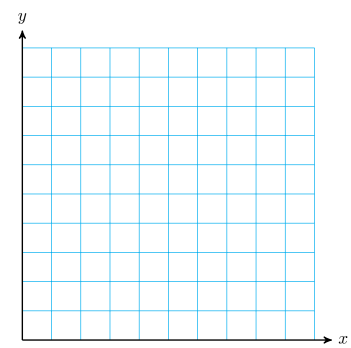
a Write an equation relating the number of acres of corn, x, and the number of acres of wheat, y, that the farmer can plant.

b Complete the table.

x	50	100	150	200
y				

c Find the intercepts of the graph.

x	y
0	
	0



d Use the intercepts to help you choose appropriate scales for the axes, and graph the equation.

e What do the intercepts tell us about the problem?

f What does the point $(288, 100)$ mean in this context?

25. The owner of a gas station has \$19,200 to spend on unleaded gas this month. Regular unleaded costs him \$2.40 per gallon, and premium unleaded costs \$3.20 per gallon.

a How much do x gallons of regular cost? How much do y gallons of premium cost?

b Write an equation in general form that relates the amount of regular unleaded gasoline, x, the owner can buy and the amount of premium unleaded, y.

c Find the intercepts and sketch the graph.

d What do the intercepts tell us about the amount of gasoline the owner can purchase?

26. Leslie plans to invest some money in two CD accounts. The first account pays 3.6% interest per year, and the second account pays 2.8% interest per year. Leslie would like to earn \$500 per year on her investment.

a If Leslie invests x dollars in the first account, how much interest will she earn? How much interest will she earn if she invests y dollars in the second account?

b Write an equation in general form that relates x and y if Leslie earns \$500 interest.

c Find the intercepts and sketch the graph.

d What do the intercepts tell us about Leslie's investments?

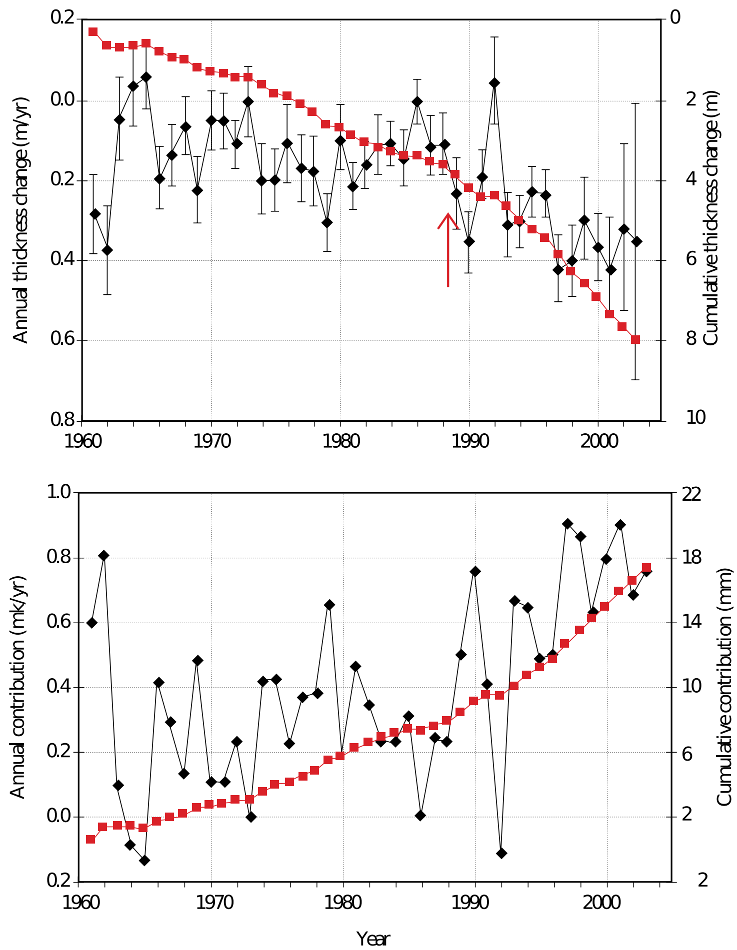
Slope

Glacier Melt

Glaciers around the world are retreating at accelerating rates, and many lower-latitude mountain glaciers may disappear entirely within decades. The meltwater from these smaller glaciers contributed as much as 40% to the total rise in sea level over the 1990s.

By measuring the change in ice and snow height at fixed points and multiplying by the surface area of the glacier, scientists calculate the total volume of water lost from land-based glaciers. Dividing this volume by the surface area of the world's oceans gives the resulting change in sea level.

The graphs below show the change in thickness of the land-based glaciers overthe past 5 years, and the rise in sea level attributed to their melting.



Rate of Change

Rate.

A **rate** is a type of ratio that compares two quantities with different units.

Checkpoint 1.54 QuickCheck 1. A ratio that compares two quantities with different units is called a (☐ rate ☐ integer ☐ price) .

Answer. rate

Solution. rate

In the example below, notice that each rate has units of the form $\frac{\text{something}}{\text{something else}}$, which we read as "something per something else."

Example 1.55

Caryn bought 7 used paperback novels for a total of \$2.45. Use a ratio to calculate the price per book.

Solution.

$$\frac{\text{total cost in dollars}}{\text{number of books}} = \frac{2.45 \text{ dollars}}{7 \text{ books}} = 0.35 \text{ dollars/book}$$

The novels were priced at a rate of \$0.35 per book.

Checkpoint 1.56 Practice 1. Delbert biked 34 miles in 5 hours. Use a ratio to calculate his average speed.

$$\frac{\text{distance in miles}}{\text{time in hours}} =$$

Delbert biked at a rate of ___ (☐ miles ☐ hours ☐ miles per hour ☐ hours per mile)

Answer 1. 6.8

Answer 2. miles per hour

Solution. 6.8 miles per hour

Definition 1.57 Rate of Change.

A **rate of change** is a special kind of ratio that compares the change in two quantities or variables.

Example 1.58

Gregor is driving across Montana. At 1 pm his trip odometer reads 189 miles, and at 4 pm it reads 360 miles. Calculate Gregor's average speed as a rate of change.

Solution. We have two variables: time, t, and distance, d, and the following data points:

t	d
1	189
4	360

Gregor's speed is the ratio of the distance he traveled to the time it took. The distance he traveled is the change in his odometer reading (from 189 miles to 360 miles), and the time it took is the change in the clock reading (from 1 pm to 4 pm).The units of this ratio are miles per hour.

In mathematics, we use the symbol Δ (delta) for **change in**. Thus

$$\text{distance traveled} = \Delta d = 360 - 189 = 171 \text{ miles}$$
$$\text{time elapsed} = \Delta t = 4 - 1 = 3 \text{ hours}$$

Gregor's average speed is the ratio $\dfrac{\text{distance traveled}}{\text{time elapsed}} = \dfrac{\Delta d}{\Delta t}$, so

$$\text{Speed} = \frac{\Delta d}{\Delta t} = \frac{171 \text{ miles}}{3 \text{ hours}} = 57 \text{ miles/hour}$$

Checkpoint 1.59 QuickCheck 2. What does the symbol Δ stand for?

- ⊙ A triangle
- ⊙ Speed
- ⊙ Change in
- ⊙ Coordinate

Answer. Change in

Solution. Change in

Checkpoint 1.60 Practice 2. Nelson is a long-distance truck driver. On a recent trip through the Midwest,he noted these odometer readings:

4 am	127 miles
10 am	421 miles

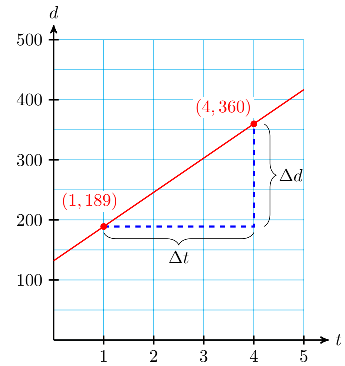
What was Nelson's average speed?
___ (☐ miles ☐ hours ☐ miles per hour ☐ hours per mile)

Answer 1. 49

Answer 2. miles per hour

Solution. 49 miles per hour

How do we see the rate of change on a graph? Let's consider the example above. The graph shows Gregor's distance, d, at time t. We plot the two data points, $(1, 189)$ and $(4, 360)$, and draw a straight line joining them. We can illustrate Δd and Δt by vertical and horizontal line segments, as shown on the graph.



The rate of change of distance with respect to time, or speed, is the ratio of Δd to Δt. It measures how much d changes for each unit increase in t, or how far Gregor travels in each hour. This quantity, the ratio $\dfrac{\Delta d}{\Delta t}$, is the slope of the line.

Checkpoint 1.61 QuickCheck 3. What feature of a graph illustrates rate of change?

- ⊙ The y-axis
- ⊙ Slope
- ⊙ Scales on the axes
- ⊙ The doordinate of a point

Answer. Choice 2

Solution. Slope

Review of Slope

Slope.

The **slope** of a line is a rate of change that measures the steepness of the line.

The slope tells us how much the y-coordinate changes for each unit of increase in the x-coordinate, as we move from one point to another along the line.

Checkpoint 1.62 QuickCheck 4. The slope of a line tells us how much the y-coordinate changes

- ⊙ From one end of the graph to the other.

- ⊙ As we move up the y-axis.
- ⊙ From one point to the next point.
- ⊙ For each unit of increase in the x-coordinate.

Answer. Choice 4

Solution. For each unit of increase in the x-coordinate.

Definition 1.63 Slope of a Line.

$$\text{slope} = \frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}$$

or, in symbols

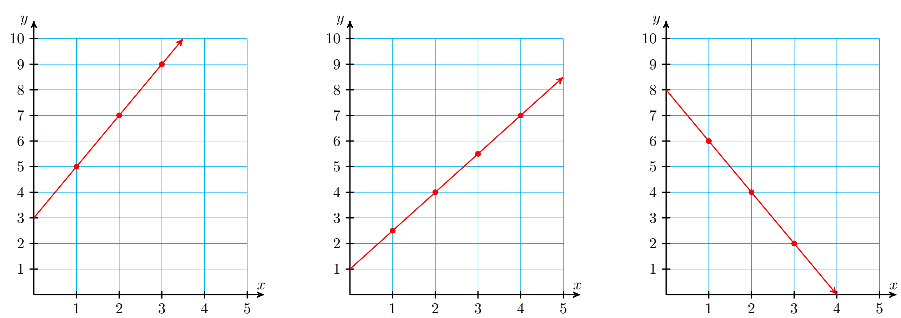
$$m = \frac{\Delta y}{\Delta x}$$

In the definition of slope,

$$\Delta x \text{ is } \begin{cases} \text{positive if } x \text{ increases} & \text{(we move to the right)} \\ \text{negative if } x \text{ decreases} & \text{(we move to the left)} \end{cases}$$

$$\Delta y \text{ is } \begin{cases} \text{positive if } y \text{ increases} & \text{(we move up)} \\ \text{negative if } y \text{ decreases} & \text{(we move down)} \end{cases}$$

How does slope measure the steepness of a line? Study the three examples below and notice that for each 1 unit increase in x:



A: y increases by 2 units

$$m = \frac{\Delta y}{\Delta x} = \frac{2}{1} = 2$$

B: y increases by $\frac{3}{2}$ units

$$m = \frac{\Delta y}{\Delta x} = \frac{3}{2}$$

C: y decreases by 2 units

$$m = \frac{\Delta y}{\Delta x} = \frac{-2}{1} = -2$$

From these examples, we can make the following observations:

Slope and Steepness.

a For positive slopes, the larger the value of m, the more the y-value increases for each unit increase in x, and the more we climb up as our location changes from left to right. (So graph A is steeper than graph B.)

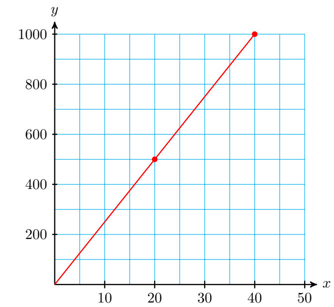
b If y decreases as we move from left to right, then Δy is negative when Δx is positive, so their ratio (the slope) is negative. (See graph C.)

Example 1.64

a Compute the slope of the line.

b Illustrate the slope on the graph by drawing a vertical segment of length Δy and a horizontal segment of length Δx.

c If $\Delta x = 1$, what is the length of the vertical segment?

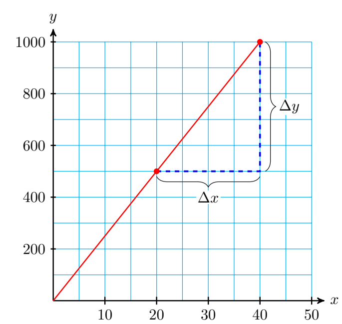


Solution.

a The points $(20, 500)$ and $(40, 1000)$ lie on the graph. As we move from the first point to the second point, x increases by 20 units, so $\Delta x = 20$, and y increases by 500 units, so $\Delta y = 500$. Thus

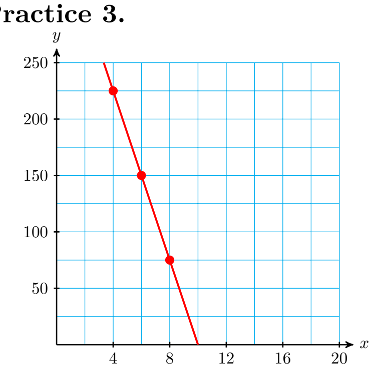
$$\frac{\Delta y}{\Delta x} = \frac{500}{20} = 25$$

b The segments are shown in the graph below.



c The slope is 25, which means that y increases 25 units for each 1-unit increase in x. So, if $\Delta x = 1$, then $\Delta y = 25$.

Checkpoint 1.65 Practice 3.



a. Compute the slope of the line.

$m =$___

b. Illustrate the slope on the graph by drawing a vertical segment for Δy and a horizontal segment for Δx.

c. If $\Delta x = 1$, what is the length of the vertical segment?

Th length is ___

Answer 1. -37.5

Answer 2. 37.5

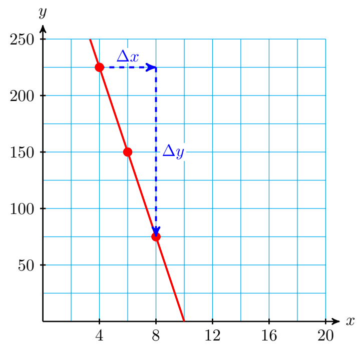
Solution.

a. -37.5

b. A figure is below.

c. 37.5

Graph for part (b):



Interpreting Slope as a Rate

The slope of a line measures the rate of change of y with respect to x.

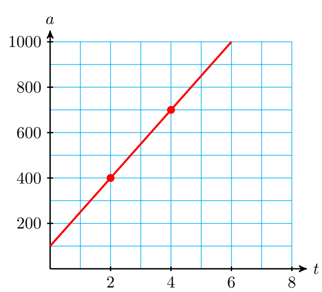
Example 1.66

The graph shows the altitude, a (in feet), of a skier t minutes after getting on a ski lift.

a Choose two points on the graph and compute the slope, including units.

b Explain what the slope measures in this problem.

c Write a linear model for a in terms of t.



Solution.

a We choose the points $(2, 400)$ and $(4, 700)$, as shown on the graph. Then

$$m = \frac{\Delta a}{\Delta t} = \frac{300 \text{ feet}}{2 \text{ minutes}} = 150 \text{ feet/minute}$$

b The slope gives the rate of change of altitude with respect to time. The skier rises at a rate of 150 feet per minute.

c A linear model has the form

$$y = (\text{starting value}) + (\text{rate}) \times t$$

From the graph, we see that at $t = 0$ the skier's altitude was $a = 100$, and we calculated the rate of change of altitude as 150 feet per minute. Substituting these values into the formula, we find

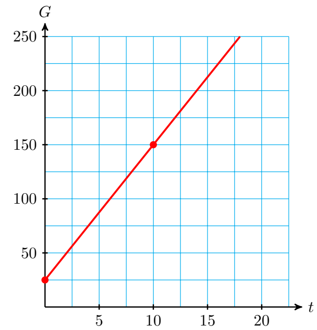
$$a = 100 + 150t$$

Slope as a Rate of Change.

1 The slope of a line measures the **rate of change** of y with respect to x.

2 The units of Δy and Δx can help us interpret the slope as a rate.

Checkpoint 1.67 Practice 4.



The graph shows the amount of garbage G (in tons) that has been deposited at a dumpsite t years after new regulations go into effect.

a. Choose two points on the graph and compute the slope, including units.

The slope is ___ (☐ tons ☐ years ☐ tons per year ☐ years per ton)

b. Explain what the slope measures in this problem.

⊙ A) The rate at which garbage accumulates

⊙ B) The steepness of the garbage pile

⊙ C) The volume of the garbage

⊙ D) The cost to use the dump site

c. Choose two different points and compute the slope again. Do you get the same value as before?

The slope is ___ (☐ tons ☐ years ☐ tons per year ☐ years per ton)

Answer 1. 12.5

Answer 2. tons per year

Answer 3. A) ... garbage accumulates

Answer 4. 12.5

Answer 5. tons per year

Solution.

a. 12.5 tons per year

b. The slope gives the rate at which garbage is accumulatig.

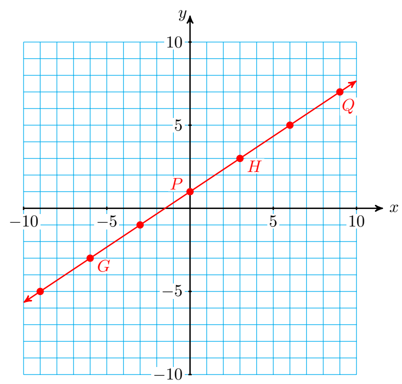
c. The slope is the same using any two points on the line.

Lines Have Constant Slope

Lines Have Constant Slope.

The slope of a line is constant: no matter which two points you pick to compute the slope, you will always get the same value.

For the line shown in the figure, try computing the slope using the points P and Q, and then using the points G and H. In each case, you should find that the slope is $\frac{3}{2}$.



Here is another way to look at slopes. If we start at any point on the line shown and move 9 units to the right, what value of Δy will bring us back to the line? We can use the slope formula with $\Delta x = 9$.

$$m = \frac{\Delta y}{\Delta x} \qquad \text{Substitute the known values.}$$

$$\frac{2}{3} = \frac{\Delta y}{9} \qquad \text{Solve for } \Delta y.$$

$$\Delta y = 9\left(\frac{2}{3}\right) = 6$$

You can use the graph to check this result for yourself; try starting at the point $(-6, -3)$.

The fact that lines have constant slope has two important consequences. First because m is constant for a given line, we can use the formula $m = \dfrac{\Delta y}{\Delta x}$ to find Δy when we know Δx, or to find Δx when we know Δy.

Checkpoint 1.68 QuickCheck 5. What formula can we use to find Δy when we know Δx, or Δx when we know Δy?

- ⊙ The equation of a line
- ⊙ The general linear formula
- ⊙ The slope formula
- ⊙ $D = RT$

Answer. Choice 3

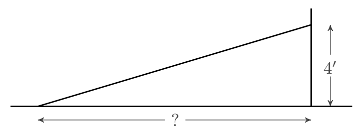
Solution. The slope formula

Example 1.69

A wheelchair ramp can have a slope of no more than 24%, or 0.24. What horizontal distance is needed if the ramp must climb an elevation of 4 feet?

Solution.

We first draw a sketch of the wheelchair ramp and label Δx and Δy. We are given that $\Delta y = 4$ feet, and we are looking for Δx. We substitute the known values into the slope formula, and solve for Δx.



$$0.24 = \frac{4}{\Delta x} \qquad \text{Multiply both sides by } \Delta x.$$

$$0.24\Delta x = 4 \qquad \text{Divide both sides by } 0.24.$$

$$\Delta x = \frac{4}{0.24} = 16.\overline{6}$$

The wheelchair ramp must have a horizontal length of $16\frac{2}{3}$ feet, or 16 feet 8 inches.

Checkpoint 1.70 **Practice 5.** A wheelchair ramp can have a slope of no more than 24%, or 0.24. What height can the wheelchair ramp climb over a horizontal distance of 10 feet?

Answer: _____

Hint. Do we know the value of Δy or of Δx?

Answer. 2.4 ft

Solution. 2.4 ft

Here is a second consequence of the fact that lines have constant slope: We can tell whether a collection of data points lies on a straight line by computing slopes. If the slopes between pairs of data points are all the same, the points lie on a straight line.

Checkpoint 1.71 **QuickCheck 6.** How can we can tell whether a collection of data points lies on a straight line?

- ⊙ A) Plot them and look at the graph.
- ⊙ B) Calculate the slopes between points.
- ⊙ C) Find an equation for the line.
- ⊙ D) It depends on the scales on the axes.

Answer. B) Calculate ... between points.

Solution. Calculate the slopes between points.

Example 1.72

Could this table represent a linear equation? Explain why or why not.

x	-6	-3	0	3	8
y	20	18	16	14	12

Solution. We compute the slope between each consecutive pair of points. In each case

$$m = \frac{\Delta y}{\Delta x} = \frac{-2}{3}$$

Because the slope is the same for all pairs of points, the table could be linear.

Checkpoint 1.73 **Practice 6.** Could this table represent a linear equation? (☐ Yes ☐ No)

t	5	10	15	20	25
P	0	3	6	12	24

Explain why or why not.

⊙ A) The slope is the same berween all pairs of points.

⊙ B) The slope between points is not constant.

Hint. Calculate the slopes between points.

Answer 1. No

Answer 2. B) The ... not constant.

Solution. No, the slope between points is not constant.

Problem Set 1.4

Warm Up

Compute ratios to answer the questions in Problems 1–4.

1. Carl runs 100 meters in 10 seconds. Anthony runs 200 meters in 19.6 seconds. Who has the faster average speed?

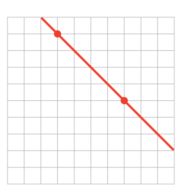
2. On his 512-mile round trip to Las Vegas and back, Corey needed 16 gallons of gasoline. He used 13 gallons of gasoline on a 429-mile trip to Los Angeles. On which trip did he get better fuel economy?

3. Grimy Gulch Pass rises 0.6 miles over a horizontal distance of 26 miles. Bob's driveway rises 12 feet over a horizontal distance of 150 feet. Which is steeper?

4. Which is steeper, the truck ramp for Acme Movers, which rises 4 feet over a horizontal distance of 9 feet, or a toy truck ramp, which rises 3 centimeters over a horizontal distance of 7 centimeters?

In Problems 5-8, compute the slope of the line through the indicated points. On both axes, one square represents one unit.

5.



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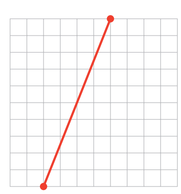

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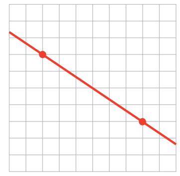

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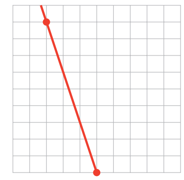
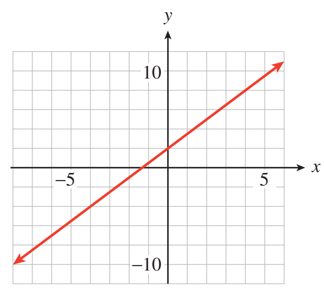


Skills Practice

9.

a Compute the slope of the line.



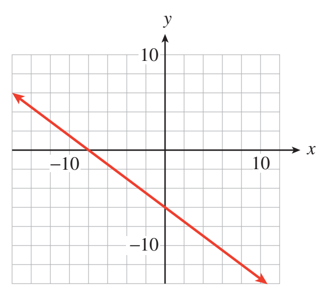
b Start at point $(0, 2)$ and move 4 units in the positive x-direction. How many units must you move in the y-direction to get back to the line? What is the ratio of Δy to Δx?

c Start at point $(0, 2)$ and move -6 units in the positive x-direction. How many units must you move in the y-direction to get back to the line? What is the ratio of Δy to Δx?

d Suppose you start at any point on the line and move 18 units in the x-direction. How many units must you move in the y-direction to get back to the line? Use the equation $m = \dfrac{\Delta y}{\Delta x}$ to calculate your answer.

10.

a Compute the slope of the line.



b Start at point $(0, -6)$ and move -6 units in the y-direction (down). How many units must you move in the x-direction to get back to the line? What is the ratio of Δy to Δx?

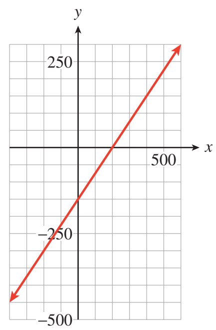
c Start at point $(0, -6)$ and move 9 units in the positive y-direction. How many units must you move in the x-direction to get back to the line? What is the ratio of Δy to Δx?

d Suppose you start at any point on the line and move 20 units in the

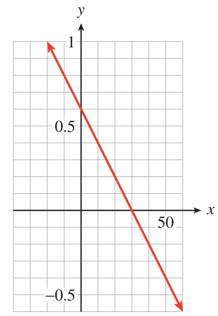
y-direction. How many units must you move in the y-direction to get back to the line? Use the equation $m = \dfrac{\Delta y}{\Delta x}$ to calculate your answer.

For Problems 11-14, compute the slope of the line. Note the scales on the axes.

11.



12.



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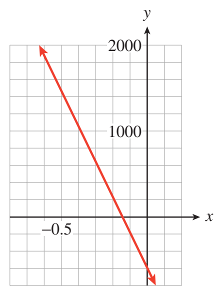
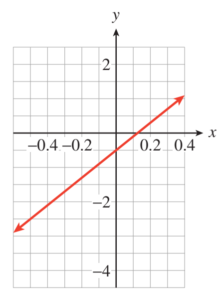


14.



For Problems 15 and 16,

a Graph the line by the intercept method.

b Use the intercepts to compute the slope.

c Use the intercepts to illustrate the slope on each graph. Put arrows on Δx and Δy to indicate the direction of motion.

15. $9x + 12y = 36$

16. $\dfrac{x}{7} - \dfrac{y}{4} = 1$

17. Residential staircaises are usutally built with a slope of 70%, or $\dfrac{7}{10}$. If the vertical distance between stories is 10 feet, how much horizontal space does the staircase require?

18. A line has slope $m = \dfrac{-4}{5}$. Use the equation $m = \dfrac{\Delta y}{\Delta x}$ to find the horizontal or vertical change along the line.

a $\Delta x = -10$

b $\Delta y = 2$

c $\Delta x = 12$

d $\Delta y = -6$

For Problems 19 and 20, which tables represent variables that are related by a linear equation? (Hint: which relationships have constant slope?)

19.

a

x	y
2	12
3	17
4	22
5	27
6	32

b

t	P
2	12
3	9
4	16
5	25
6	36

20.

a

h	w
−6	20
−3	18
0	16
3	14
6	12

b

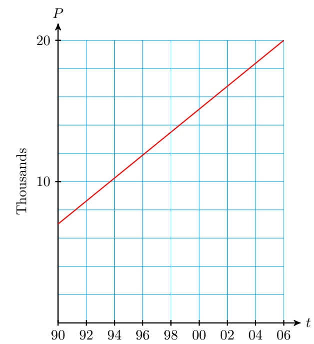
t	d
5	0
10	3
15	6
20	12
25	24

Applications

21. The population of Smallville grew from 7000 people in 1990 to 16,600 in 2002.

a Use a rate of change to calculate the town's average rate of growth, in people per year. First, complete the table of values:

t	P



change in population $= \Delta P =$

time elapsed $= \Delta t =$

rate of growth $= \dfrac{\Delta P}{\Delta t} =$

b Illustrate ΔP and Δt by line segments on the graph. (Note that the P-axis is labeled in thousands.)

c How much did the town grow each year?

d Write a linear model for P in terms of t.

22. A traditional first experiment for chemistry students is to make 90 observations about a burning candle. Delbert records the height, h, of the candle in inches at various times t minutes after he lit it.

t	0	10	30	45
h	12	11.5	10.5	9.75

a Choose appropriate scales for the axes and plot the data. Do the points lie on a straight line?

b Compute the slope of the graph, including units, and explain what the slope tells us about the candle.

c Write a linear model for h in terms of t.

For Problems 23 and 24,

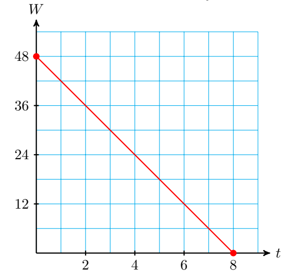
a Choose two points and compute the slope, including units.

b Write the slope as a rate of change, including units.

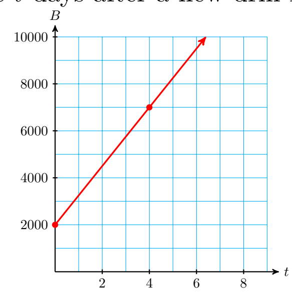
c Illustrate the slope on the graph.

d Write a linear model for the variables.

23. The graph shows the number of liters of emergency water W remaining in a southern California household t days after an earthquake.



24. The graph shows the number of barrels of oil, B, that have been pumped at a drill site t days after a new drill is installed.



25. A spring is suspended from the ceiling. The table shows the length of the spring, in centimeters, as it is stretched by hanging various weights from it.

Weight, kg	3	4	8	10	12	15	22
Length, cm	25.87	25.88	26.36	26.6	26.84	27.2	28.04

a If you plot the data, will the points lie on a straight line? Why or why not?

b Interpret the slope as a rate of change. Include units in your answer

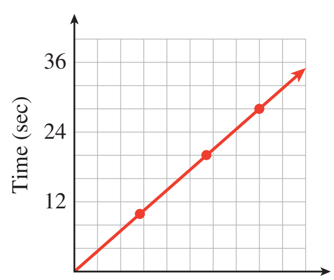
26. The table gives the radius and circumference of various circles, rounded to three decimal places.

r	C
4	25.133
6	37.699
10	62.832
15	94.248

a If we plot the data, will the points lie on a straight line? Why or why not?

b What familiar number does the slope turn out to be? (Hint: Recall a formula from geometry.) What does the slope tell us about circles?

27. Geologists calculate the speed of seismic waves by plotting the travel times for waves to reach seismometers at known distances from the epicenter. The speed of the wave can help them determine the nature of the material it passes through. The graph shows a travel time graph for P-waves from a shallow earthquake.



a Why do you think the graph is plotted with distance as the independent variable?

b Use the graph to calculate the speed of the wave.

28. Niagara Falls was discovered by Father Louis Hennepin in 1682. In 1952, much of the water of the Niagara River was diverted for hydroelectric power, but until that time erosion caused the Falls to recede upstream by 3 feet per year.

a How far did the Falls recede from 1682 to 1952?

b The Falls were formed about 12,000 years ago during the end of the last ice age. How far downstream from their current position were they then? (Give your answer in miles.)

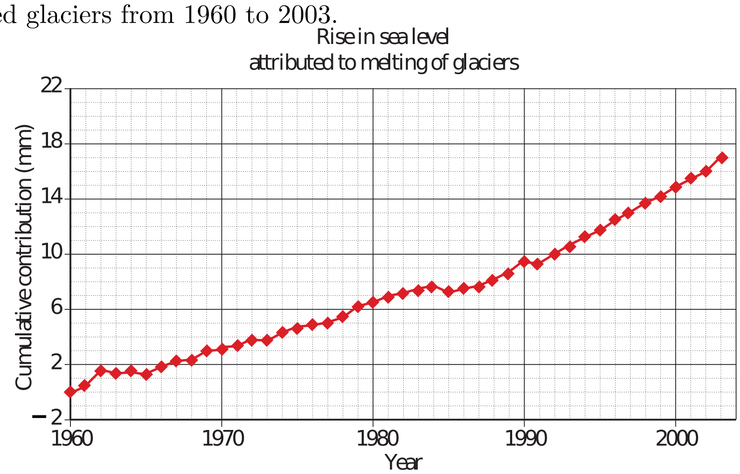
29. Naismith's Rule is used by runners and walkers to estimate journey times in hilly terrain. In 1892 Naismith wrote in the Scottish Mountaineering Club Journal that a person "in fair condition should allow for easy expeditions an hour for every three miles on the map, with an additional hour for every 2000 feet of ascent." (Source: Scarf, 1998.)

a According to Naismith, 1 unit of ascent requires the same travel time as how many units of horizontall travel? (This is called Naismith's number.) Round your answer to one decimal place.

b A walk in the Brecon Beacons in Wales covers 3.75 kilometers horizontally and climbs 582 meters. What is the equivalent flat distance?

c If you can walk at a pace of 15 minutes per kilometer over flat ground, how long will the walk in the Brecon Beacons take you?

30. The graph shows the rise in sea level attributed to the melting of land-based glaciers from 1960 to 2003.



(a) The graph appears to be almost linear from 1992 to 2002. Read the graph to complete the table, then compute the slope of the graph over that time interval, including units. What does the slope mean in this situation?

Year	Sea level
1992	
2002	

(b) What was the total change in sea level from land-based glaciers over the time period from 1960 to 2003?

(c) Calculate the average rate of change of sea level from land-based glaciers from 1960 to 2003.

(d) From 1960 to 2003, the land-based glaciers decreased in thickness by about 8 meters (or 0.008 km). The total area of those glaciers is 785,000 square kilometers. Calculate the total volume of water released by melting. (Hint: Volume = area × thickness)

(e) The surface area of the world's oceans is 361.6 million square kilometers. When the meltwater from the land-based glaciers (that's the volume you calculated in part (d)) enters the oceans, how much will the sea level rise, in kilometers? Use the formula in part (d). Convert your answer to millimeters, and check your answer against your answer to part (b).

Equations of Lines

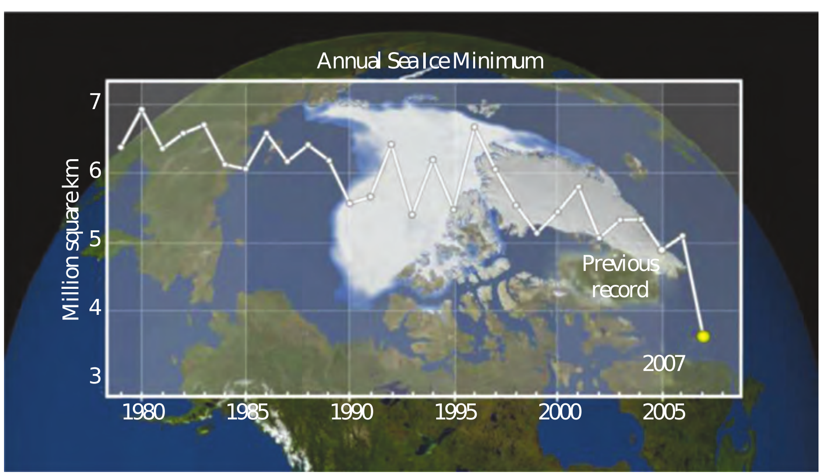
Polar Ice

Three main factors influence the energy balance of the Earth and its temperature:

- The total energy influx from the sun
- The chemical composition of the atmosphere
- The ability of the Earth's surface to reflect light, or **albedo**

Because polar ice reflects light from the sun, the radiation balance over an ice-covered ocean is very different from the balance over an open ocean. The ice component of the climate system, called the **cryosphere**, plays an important role in the Earth's radiation balance.

Climate models predict that global warming over the next few decades will occur mainly in the polar regions. As polar ice begins to melt, less sunlight is reflected into space, which raises the overall temperature and fuels further melting. This process is called **ice albedo feedback**. Since satellite monitoring began in 1979, Arctic sea ice cover has decreased about 10% per decade, falling to a startling new low in 2007.



Numerous factors influence the freezing point of sea water, including its salinity, or mineral content. In this Lesson we'll develop a formula for the freezing temperature of water in terms of its salinity.

Slope-Intercept Form

In earlier sections we learned that:

Equation of a Line.

1 The y-intercept of a line gives the **initial value** of y.

2 The **slope** of the line gives the **rate of change** of y with respect to x.

Comparing these observations with the form for a linear model, we see that

$$\begin{aligned} y &= (\text{starting value}) + (\text{rate}) \cdot x \\ y &= \qquad b \qquad\quad + \quad mx \end{aligned}$$

Usually we write the terms in the opposite order, like this: $y = mx + b$. We call this last equation the **slope-intercept form** for a line.

Slope-Intercept Form.

We can write the equation of a line in the form

$$y = mx + b$$

where m is the **slope** of the line, and b is the y**-intercept**.

Checkpoint 1.74 QuickCheck 1. Choose the correct statement about the equation $y = 8 - 6x$

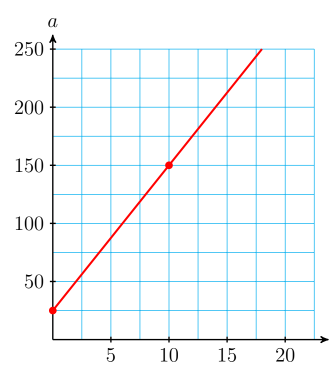
- ⊙ The slope is 8 and the y-intercept is -6.
- ⊙ The slope is -6 and the y-intercept is 8.
- ⊙ The slope is $-6x$ and the y-intercept is 8.
- ⊙ The slope is 8 and the y-intercept is 6.

Answer. Choice 2

Solution. The slope is -6 and the y-intercept is 8.

Example 1.75

The graph shows the amount of garbage G, in tons, that has been deposited at a dumpsite t years after new regulations go into effect.



a Choose two points on the graph and compute the slope, including units.

b Find an equation for the graph shown.

c State the meaning of the slope and the vertical intercept.

Solution.

a Two points on the line are $(0, 25)$ and $(10, 150)$ The slope is

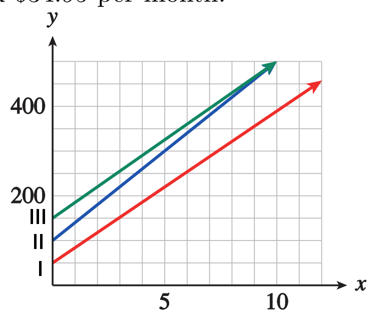
$$m = \frac{\Delta G}{\Delta t} = \frac{125 \text{ tons}}{10 \text{ years}} = 12.5 \text{ tons/year}$$

b The vertical intercept of the graph is $(0, 25)$, so $b = 25$, and

$$G = mt + b = 12.5t + 25$$

c The slope tells us that garbage is accumulating at a rate of 12.5 tons per year. The vertical intercept tells us that when the new regulations went into effect, the dump held 25 tons of garbage.

Checkpoint 1.76 Practice 1. Delbert decides to use DSL for his Internet service.Earthlink charges a $99 activation fee and $39.95 per month, DigitalRain charges $50 for activation and $34.95 per month, and FreeAmerica charges $149 for activation and $34.95 per month.



a. Write a formula for Delbert's Internet costs under each plan. Use x for the variable.

   Earthlink: $E =$_________

   DigitalRain: $D =$_________

   FreeAmerica: $F =$_________

b. Match Delbert's Internet cost under each company with its graph shown above.

   Line I: (☐ Earthlink ☐ DigitalRain ☐ FreeAmerica)

   Line II: (☐ Earthlink ☐ DigitalRain ☐ FreeAmerica)

   Line III: (☐ Earthlink ☐ DigitalRain ☐ FreeAmerica)

Answer 1. $99 + 39.95x$

Answer 2. $50 + 34.95x$

Answer 3. $149 + 34.95x$

Answer 4. DigitalRain

Answer 5. Earthlink

Answer 6. FreeAmerica

Solution.

a. Earthlink: $E = 99 + 39.95x$; DigitalRain: $D = 50 + 34.95x$; FreeAmerica: $F = 149 + 34.95x$

b. DigitalRain: I; Earthlink: II; FreeAmerica: III

Coordinate Formula for Slope

Slope measures the change in one variable with respect to unit changes in another.To calculate the **net change** between two points on a number line, we can subtract their coordinates.

$$\text{net change} = \text{final value} - \text{starting value}$$

For example, if you walk from 3rd street to 8th street, your distance, s, from the center of town has increased by 5 blocks, or

$$\Delta s = 8 - 3 = 5$$

If the temperature T drops from $28°$ to $22°$, it has decreased by $6°$, or

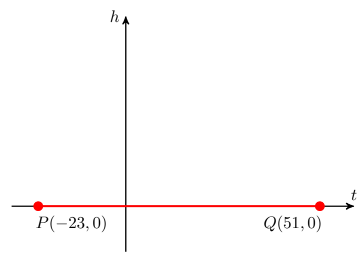
$$\Delta T = 22 - 28 = -6$$

The net change is positive if the variable increases, and negative if it decreases.

For the graph shown at right, the net change in t-coordinate from P to Q is

$$\Delta t = 51 - (-23) = 74$$

We can use the notion of net change to write a coordinate formula for computing slope.



Coordinate Formula for Slope.

If (x_1, y_1) and (x_2, y_2) are two points on a line, then the slope of the line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

as long as $x_1 \neq x_2$

Note 1.77 Notice that the numerator of the slope formula, $y_2 - y_1$, gives the net change in y, or Δy, and the denominator, $x_2 - x_1$, gives the net change in x, or Δx. The coordinate formula is equivalent to our definition of slope, $m = \dfrac{\Delta y}{\Delta x}$.

Checkpoint 1.78 QuickCheck 2. In the coordinate formula for slope, why do we require that $x_1 \neq x_2$?

- ⊙ A) The slope cannot be zero.
- ⊙ B) The two points cannot be the same.
- ⊙ C) The denominator of a fraction cannot be zero.
- ⊙ D) The denominator must equal 1.

Answer. C) The ... be zero.

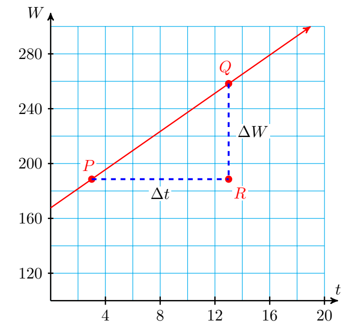
Solution. The denominator of a fraction cannot be zero.

Example 1.79

The graph shows wine consumption, W, in the US, in millions of cases, starting in 1990. In 1993, Americans drank 188.6 million cases of wine.

a Find the slope of the graph from 1993 to 2003.

b State the slope as a rate of change. What does the slope tell us about this problem?



Solution.

a If $t = 0$ in 1990, then in 1993, $t = 3$, and in 2003, $t = 13$. Thus, the points $P(3, 188.6)$ and $Q(13, 258.3)$ lie on the line. We want to compute the slope,

$$m = \frac{\Delta W}{\Delta t} = \frac{W_2 - W_1}{t_2 - t_1}$$

between these two points. Think of moving from P to Q in two steps, first moving horizontally to the right from P to the point R, and then vertically from R to Q. The coordinates of R are $(13, 188.6)$. (Do you see why?) Then

$$\Delta t = t_2 - t_1 = 13 - 3 = 10$$
$$\Delta W = W_2 - W_1 = 258.3 - 188.6 = 69.7$$

and thus

$$m = \frac{W_2 - W_1}{t_2 - t_1} = \frac{258.3 - 188.6}{13 - 3} = 6.97$$

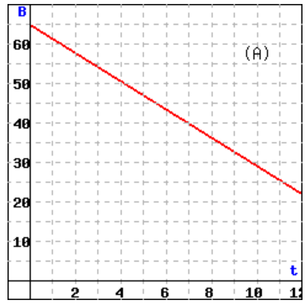
b The slope gives us a rate of change, and the units of the variables help us interpret the slope in context.

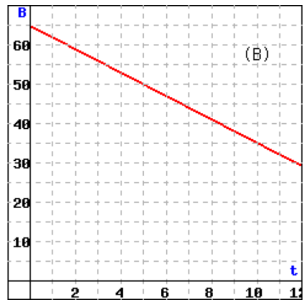
$$\frac{\Delta W}{\Delta t} = \frac{258.3 - 188.6 \text{ millions of cases}}{13 - 3 \text{ years}} = 6.97 \text{ millions of cases/year}$$

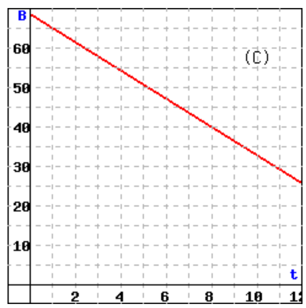
Over the ten years between 1993 and 2003, wine consumption in the US increased at a rate of 6.97 million cases per year.

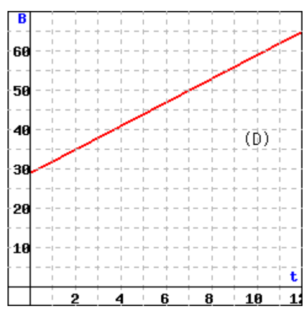
Checkpoint 1.80 **Practice 2.** In 1991, there were 64.6 burglaries per 1000 households in the United States. The number of burglaries reported annually declined at a roughly constant rate over the next decade, and in 2001 there were 28.7 burglaries per 1000 households. (Source: U.S. Department of Justice)

a. Sketch a line that goes through the two points (t, B) given in the problem, where $t = 0$ in 1990.









Choose the correct graph.

- ⊙ (A)
- ⊙ (B)
- ⊙ (C)
- ⊙ (D)

b. Find the slope of the line, including units.

m is ___ (☐ burglaries per decade ☐ burglaries per 1000 households per year ☐ households per burglary per year ☐ burglaries per year)

What does the slope tell us about this problem?

- ⊙ The burglary rate declined by 3.59 burglaries per decade

- ⊙ The burglary rate declined by 3.59 burglaries per 1000 households every year
- ⊙ The burglary rate declined by 3.59 burglaries per year

(in the years 1991 to 2001).

Answer 1. (C)

Answer 2. -3.59

Answer 3. burglaries per 1000 households per year

Answer 4. The burglary ... every year

Solution.

a. Because $t = 0$ corresponds to 1990, we have the two points $(1, 64.6)$ and $(11, 28.7)$. The line joining those two points corresponds to the line with equation $y = 68.19 - 3.59t$

b. We use the two points $(1, 64.6)$ and $(11, 28.7)$ to compute the slope: -3.59 burglaries per 1000 households per year. From 1991 to 2001, the burglary rate declined by 3.59 burglaries per 1000 households every year.

Point-Slope Formula

Now consider using the slope formula for a different problem. If we know the slope of a line and the coordinates of one point on the line, we can use the coordinate formula for slope to find the y-coordinate of any other point on the line.

Instead of *evaluating* the formula to find m, we *substitute* the values we know for m and (x_1, y_1). If we then plug in the x-coordinate of any unknown point, we can solve for y.

Checkpoint 1.81 QuickCheck 3. A line has slope $\dfrac{-3}{4}$ and passes through the point $(1, -4)$. Which equation can you use to find the y-coordinate of the point on the line with the x-coordinate of 6?

- ⊙ $\dfrac{-3}{4} = \dfrac{1-6}{y+4}$
- ⊙ $\dfrac{-3}{4} = \dfrac{y+4}{1-6}$
- ⊙ $\dfrac{-3}{4} = \dfrac{y-6}{-4-1}$
- ⊙ $\dfrac{-3}{4} = \dfrac{y+4}{6-1}$

Answer. Choice 4

Solution. $\dfrac{-3}{4} = \dfrac{y+4}{6-1}$

Remember that the equation for a line is really just a formula that gives the y-coordinate of any point on the line in terms of its x-coordinate. So, if we know the slope of a particular line and one point on the line, we can use the coordinate formula for slope to find its equation.

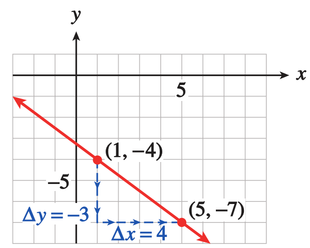
Example 1.82

a Graph the line that passes through the point $(1, -4)$ and has slope $= \frac{-3}{4}$.

b Find an equation for the line in part (a).

Solution.

a We first plot the given point, $(1, -4)$, and then use the slope to find another point on the line. The slope is $m = \frac{-3}{4} = \frac{\Delta y}{\Delta x}$, so starting from $(1, -4)$ we move down 3 units and then 4 units to the right. This brings us to the point $(5, -7)$. We draw the line through these two points.



b To find an equation for the line, we start with the slope formula,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

We substitute $\frac{-3}{4}$ for the slope, m, and $(1, -4)$ for (x_1, y_1). For the second point, (x_2, y_2), we substitute the variable point (x, y) to obtain

$$m = \frac{y + 4}{x - 1}$$

This is an equation for the line, but if we want to solve for y, we first multiply both sides by x—1.

$$\mathbf{(x-1)}\frac{-3}{4} = \frac{y+4}{x-1}\mathbf{(x-1)}$$

$$\frac{-3}{4}(x-1) = y + 4 \qquad \text{Apply the distributive law.}$$

$$\frac{-3}{4}x + \frac{3}{4} = y + 4 \qquad \text{Subtract 4 from both sides.}$$

$$\frac{-3}{4}x - \frac{13}{4} = y \qquad \frac{3}{4} - 4 = \frac{3}{4} - \frac{16}{4} = \frac{-13}{4}$$

When we use the slope formula to find the equation of a line, we substitute a variable point (x, y) for the second point. This version of the formula,

$$m = \frac{y - y_1}{x - x_1}$$

is called the **point-slope form** for a linear equation. It is sometimes stated in another version by clearing the fraction to get

$$\mathbf{(x - x_1)}m = \frac{y - y_1}{x - x_1}\mathbf{(x - x_1)}$$

$$(x - x_1)m = y - y_1$$

$$y = y_1 + m(x - x_1)$$

Point-Slope Form.

The equation of the line that passes through the point (x_1, y_1) and has slope m is

$$y = y_1 + m(x - x_1)$$

You may also see the formula written in an alternate version:

$$y - y_1 = m(x - x_1) \qquad \text{or} \qquad \frac{y - y_1}{x - x_1} = m$$

Checkpoint 1.83 Practice 3. Use the point-slope form to find the equation of the line that passes throughthe point $(-3, 5)$ and has slope -1.4.

Answer: __________

Answer. $1.4x + y = 0.8$

Solution. $y = 5 - 1.4(x + 3)$

Finding a Linear Model

Now we are ready to find the equation promised in the introduction to this section: a formula for the freezing temperature of water in terms of its salinity.

When we have two data points for a linear model, we can find its equation using two steps: first we compute the slope of the line, then we use the point-slope formula.

Example 1.84

Sea water does not freeze at exactly 32°F because of its salinity.The temperature at which water freezes depends on its dissolved mineral content. A common unit for measuring salinity is parts per thousand, or ppt. For example, salinity of 8 ppt means 8 grams of dissolved salts in each kilogram of water. Here are some data for the freezing temperature of water.

Salinity (ppt), S	8	12	20
Freezing temperature (°F), T	31.552	31.328	30.88

a Do these data points describe a linear model? Why or why not?

b Find a linear equation for freezing temperature, T, in terms of salinity, S.

Step 1: Find the slope.

Step 2: Use the point-slope formula.

c What is the salinity of water that freezes at 32°F?

d Sea water has an average salinity of 35 ppt. What is the freezing point of sea water?

Solution.

a We check to see if the slope between each pair of points is the same.

$$m = \frac{T_2 - T_1}{S_2 - S_1} = \frac{31.328 - 31.552}{12 - 8} = -0.056$$

$$m = \frac{T_3 - T_2}{S_3 - S_2} = \frac{30.88 - 31.328}{20 - 12} = -0.056$$

Because the slopes are the same, the data could describe a linear model.

b Step 1: In part (a) we found the slope: $m = -0.056$

Step 2: We substitute the slope and either point into the point-slope formula.

$$\begin{aligned} T - T_1 &= m(S - S_1) \\ T - 30.88 &= -0.056(S - 20) \\ T - 30.88 &= -0.056S + 1.12 \\ T &= -0.056S + 32 \end{aligned}$$

c If $T = 32$ in our equation, then S must be zero, so pure water (no salinity) freezes at 32°F.

d We substitute 35 for S in our equation.

$$T = -0.056(35) + 32 = 30.04$$

Sea water freezes at 30.04°F.

Checkpoint 1.85 Practice 4. The fee for registering a new car is given by a linear equation that depends on the car's value. The fee for a $15,000 car is $128.50, and the fee for a $25,000 car is $193.50.

a. Find a linear equation that gives the registration fee F for a new car that cost V dollars.

   Answer: ___________

b. Use your equation to estimate the registration fee for a car that costs $22,000.

   $___

c. What is the slope of your line? What does the slope mean in this situation?

   The slope is _____ and it tells

   - ⊙ the steepness of the car's value.
   - ⊙ the fraction of the car's value that is added into the registration fee.
   - ⊙ the intest rate of the loan.
   - ⊙ how much the car's value decreases.

Answer 1. $F - 0.0065V = 31$

Answer 2. 174

Answer 3. 0.0065

Answer 4. Choice 2

Solution.

a. $F = 31 + 0.0065V$

b. \$174

c. 0.0065: The registration fee increases by 0.65% of the value of the car.

Summary

In this section, we studied three different formulas associated with linear equations: the slope-intercept formula, the coordinate formula for slope, and the point-slope formula. How are these formulas related, and how are they different?

1 The **slope-intercept form**, $y = mx + b$, is just a special case of the **point-slope formula**. If the given point (x_1, y_1) happens to be the y-intercept $(0, b)$, then the point-slope formula reduces to the familiar form:

$$\begin{aligned} y &= y_1 + m(x - x_1) && \text{Substitute } b \text{ for } y_1 \text{ and } 0 \text{ for } x_1. \\ y &= b + m(x - 0) && \text{Simplify.} \\ y &= mx + b \end{aligned}$$

We can use the (shorter) slope-intercept form if we are lucky enough to know the y-intercept of the line.

2 What is the difference between the **slope formula**

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and the **point-slope formula**

$$m = \frac{y - y_1}{x - x_1}?$$

They are really the same formula, but they are used for different purposes:

a The slope formula is used to calculate the slope when we know two points. We know (x_1, y_1) and (x_2, y_2), and we are looking for m.

b The point-slope formula is used to find the equation of a line. We know (x_1, y_1) and m, and we are looking for $y = mx + b$.

Problem Set 1.5

Warm Up

For Problems 1 and 2, complete the table of values and graph the line, then answer the questions below.

1. $y = 8 - 2x$

x	-1	0	2	3	4
y					

2. $y = 2 + \frac{1}{2}x$

x	-2	0	1	3	5
y					

a What is the initial value for each line; that is, what is the y-value when $x = 0$?

Line 1: Line 2:

b Look at the table for each line. How much does y increase or decrease for each 1-unit increase in x? What is this value called?

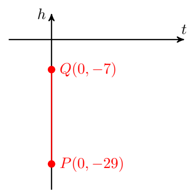
Line 1: Line 2:

c Compare your answers to parts (a) and (b) with the equation for each line. What do you observe?

For Problems 3 and 4, calculate the net change in coordinate from P to Q.

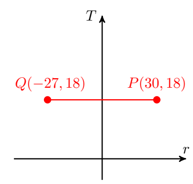
3.

a



$\Delta h =$

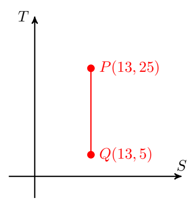
b



$\Delta r =$

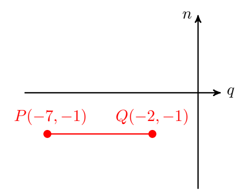
4.

a



$\Delta T =$

b



$\Delta q =$

Skills Practice

For Problems 5–8,

a Write the equation in slope-intercept form.

b State the slope and the y-intercept of its graph.

5. $3x + 2y = 1$

6. $2x - \dfrac{3}{2}y = 3$

7. $4.2 - 0.3y = 6.6$

8. $5x - 4y = 0$

For Problems 9 and 10,

a Sketch the graph of the line with the given slope and y-intercept.

b Write an equation for the line.

c Find the x-intercept of the line.

9. $m = \dfrac{-5}{3}$ and $b = -6$

10. $m = \dfrac{3}{4}$ and $b = -2$

In Problems 11 and 12, choose the correct graph for each equation. The scales on both axes are the same.

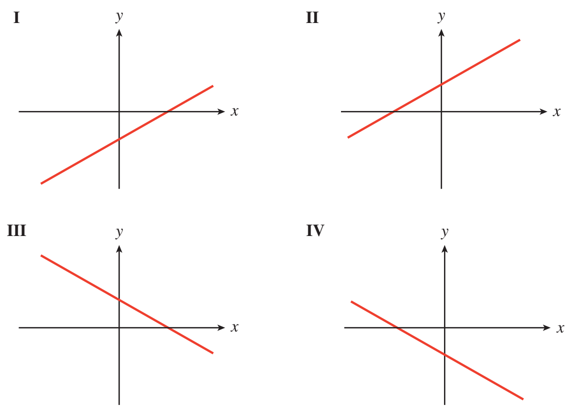
11.

a $y = \dfrac{3}{4}x + 2$

c $y = \dfrac{3}{4}x - 2$

b $y = \dfrac{-3}{4}x + 2$

d $y = \dfrac{-3}{4}x - 2$



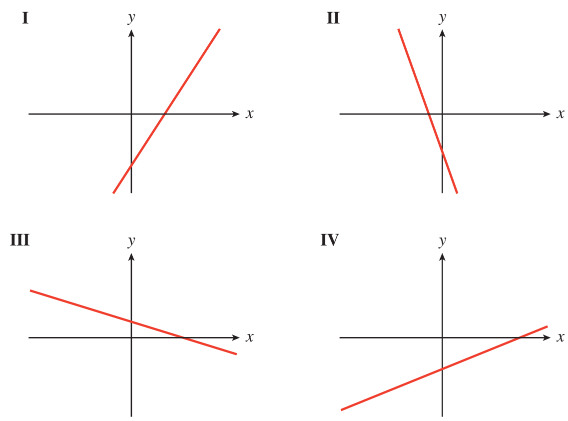
12.

a $m < 0$, $b > 0$

b $m > 1$, $b < 0$

c $0 < m < 1$, $b < 0$

d $m < -1$, $b < 0$



For Problems 13 and 14,

a Graph the line that passes through the given point and has the given slope.

b Write an equation for the line in point-slope form.

c Put your equation from part (b) into slope-intercept form.

13. $(2, -5)$; $m = -3$

14. $(2, -1)$; $m = \dfrac{5}{3}$

For Problems 15 and 16, find an equation for the line that goes through the given points. Put your equation into slope-intercept form.

15. $(-16, -24)$, $(8, 72)$

16. $(-5, 65)$, $(20, -145)$

In Problems 17 and 18, choose the correct graph for each equation. The scales on both axes are the same.

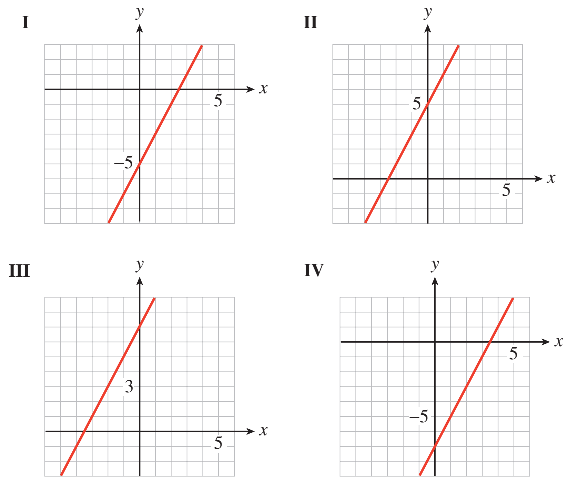
17.

a $y = 1 + 2(x + 3)$

b $y = -1 + 2(x - 3)$

c $y = -1 + 2(x + 3)$

d $y = 1 + 2(x - 3)$



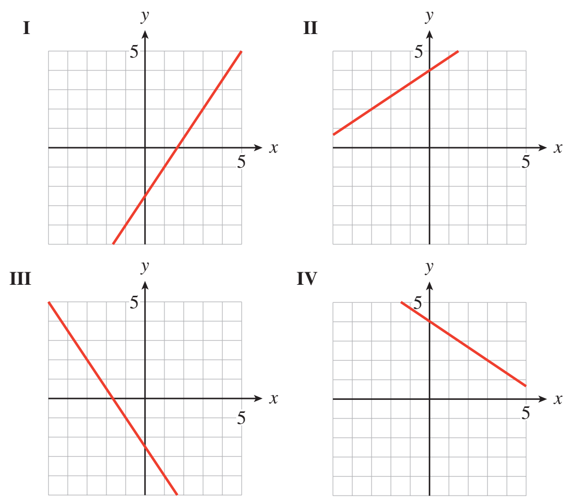
18.

a $y = 2 - \frac{2}{3}(x - 3)$

b $y = 2 - \frac{3}{2}(x + 3)$

c $y = 2 + \frac{3}{2}(x - 3)$

d $y = 2 + \frac{2}{3}(x + 3)$



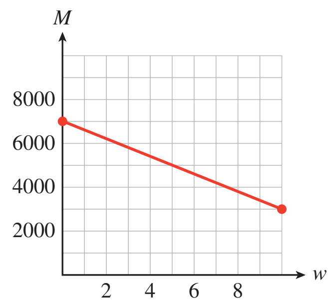
Applications

In Problems 19 and 20,

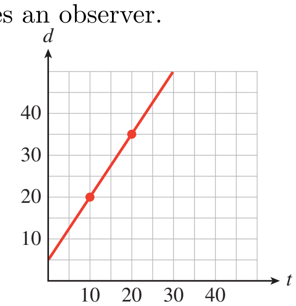
a Find a formula for the function whose graph is shown.

b Say what the slope and the vertical intercept tell us about the problem.

19. The graph shows the amount of money, M (in dollars), in Tammy's bank account w weeks after she loses all sources of income.



20. The graph shows the distance, d (in meters), traveled by a train t seconds after it passes an observer.



21. The boiling point of water changes with altitude and is approximated by the formula

$$B = 212 - 0.0018h$$

where B is in degrees and h is in feet. State the slope and vertical intercept of the graph, including units, and explain their meaning in this context.

22. The height of a woman in centimeters is related to the length of her femur (in centimeters) by the formula

$$H = 2.47x + 54.10$$

State the slope and the vertical intercept of the graph, including units, and explain their meaning in this context.

In Problems 23–26, we find a linear model from two data points.

a Make a table showing the coordinates of two data points for the model. (Which variable should be plotted on the horizontal axis?)

b Find a linear equation relating the variables.

c State the slope of the line, including units, and explain its meaning in the context of the problem.

23. Flying lessons cost \$645 for an 8-hour course and \$1425 for a 20-hour course. Both prices include a fixed insurance fee.

a Write an equation for the cost, C, of flying lessons in terms of the length, h, of the course in hours. *Hint*: Find two points (h, C) given in the problem. Find the equation of the line through those points.

b Explain the meaning of the slope and the vertical intercept.

24. On an international flight, a passenger may check two bags each weighing 70 kilograms, or 154 pounds, and one carry-on bag weighing 50 kilograms, or 110 pounds. Express the weight, p, of a bag in pounds in terms of its weight, k, in kilograms.

25. A radio station in Detroit, Michigan, reports the high and low temperatures in the Detroit/Windsor area as 59°F and 23°F, respectively. A station in Windsor, Ontario, reports the same temperatures as 15°C and −5°C. Express the Fahrenheit temperature, F, in terms of the Celsius temperature, C.

26. Ms. Randolph bought a used car in 2010. In 2012, the car was worth $9000, and in 2015 it was valued at $4500. Express the value, V, of Ms. Randolph's car in terms of the number of years, t, she has owned it.

27. If the temperature on the ground is 70° Fahrenheit, the formula

$$T = 70 - \frac{3}{820}h$$

gives the temperature at an altitude of h feet.

a What is the temperature at an altitude of 4100 feet?

b At what altitude is the temperature 34° Fahrenheit?

c Choose appropriate WINDOW settings and graph the equation $y = 70 - \frac{3}{820}x$.

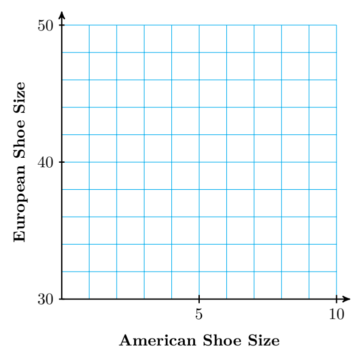
d Find the slope and explain its meaning for this problem.

e Find the intercepts and explain their meanings for this problem.

28. European shoe sizes are scaled differently than American shoe sizes. The table shows the European equivalents for various American shoe sizes.

American shoe size	5.5	6.5	7.5	8.5
Europoean shoe size	37	38	39	40

a Use the grid below to plot the data and draw a line through the data points.



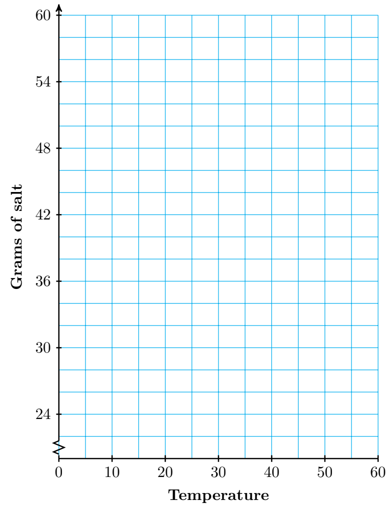
b Calculate the slope of your line. Estimate the y-intercept from the graph.

c Find an equation that gives the European shoe size, E, in terms of the American shoe size, A.

29. The table shows the amount of ammonium chloride salt, in grams, that can be dissolved in 100 grams of water at different temperatures.

Temperature (°C)	10	12	15	21	25	40	52
Salt (grams)	33	34	35.5	38.5	40.5	48	54

a Use the grid below to plot the data and draw a straight line through the points. Estimate the y-intercept of your graph.



b Calculate the slope of the line.

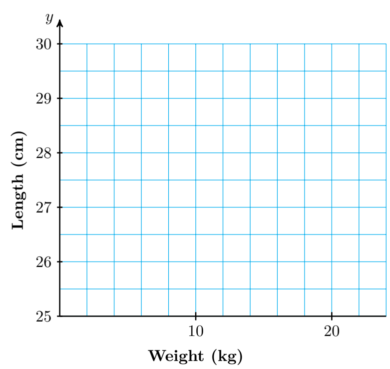
c Use the point-slope formula to find an equation for the line.

d At what temperature will 46 grams of salt dissolve?

30. A spring is suspended from the ceiling. The table shows the length of the spring in centimeters as it is stretched by hanging various weights from it.

Weight, kg	3	4	8	10	12	15	22
Length, cm	25.76	25.88	26.36	26.6	26.84	27.2	28.04

a Plot the data on graph paper and draw a straight line through the points. Estimate the y-intercept of your graph.



b Find an equation for the line.

c If the spring is stretched to 27.56 cm, how heavy is the attached weight?

Chapter Summary and Review

Glossary

- mathematical model
- linear model
- evaluate an expressions
- solve an equation
- increasing graph
- decreasing graph
- inequality
- intercepts
- linear equation
- solution of an equation
- graph of an equation
- ordered pair
- rate
- rate of change
- net change
- slope
- slope-intercept form
- point-slope form

Key Concepts

1 A **mathematical model** is a simplified description of reality that helps us understand a system or process.

2 We can describe a relationship between variables with a table of values, a graph, or an equation.

3 Linear models have equations of the form:

$$y = (\text{starting value}) + (\text{rate of change}) \cdot x$$

4 The **general form** for a linear equation is: $Ax + By = C$.

5 We can use the **intercepts** to graph a line. The intercepts are also useful for interpreting a model.

6 The **graph** of an equation in two variables is just a picture of all its solutions.

7 Lines have constant slope.

8 The slope of a line gives us the **rate of change** of one variable with respect to another.

9 Formulas for Linear Models

- **slope:** $m = \dfrac{\Delta y}{\Delta x} = \dfrac{Y_2 - y_1}{x_2 - x_1}$, $x_1 \neq x_2$
- **slope-intercept form:** $y = b + mx$
- **point-slopet form:** $y = y_1 + m(x - x_1)$

10 The **slope-intercept** form is useful when we know the initial value and the rate of change.

11 The **point-slope** form is useful when we know the rate of change and one point on the line.

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Chapter 1 Review Problems

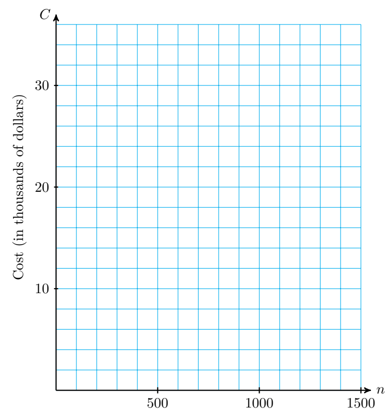
1. Last year, Pinwheel Industries introduced a new model calculator. It cost \$2000 to develop the calculator and \$20 to manufacture each one.

a Complete the table of values showing the total cost, C, of producing n calculators.

n	100	500	800	1200	1500
C					

b Write an equation that expresses C in terms of n.

c Graph the equation by hand.



d What is the cost of producing 1000 calculators? Illustrate this as a point on your graph.

e How many calculators can be produced for \$10,000? Illustrate this as a point on your graph.

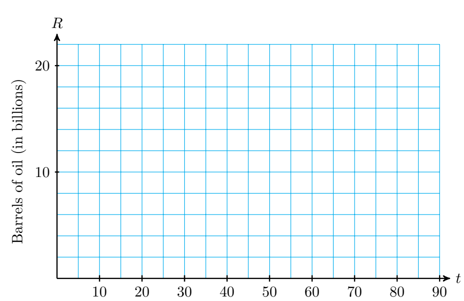
2. The world's oil reserves were 2100 billion barrels in 2005; total annual consumption is 28 billion barrels.

a Complete the table of values that shows the remaining oil reserves R in terms of time t (in years since 2005).

t	5	10	15	20	25
R					

b Write an equation that expresses R in terms of t.

c Find the intercepts and graph the equation by hand.

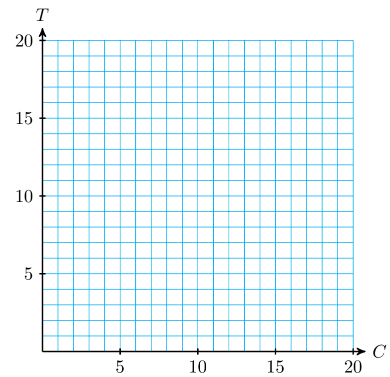


d What do the intercepts tell us about the world's oil supply?

3. Alida plans to spend part of her vacation in Atlantic City and part in Saint-Tropez. She estimates that after airfare her vacation will cost \$60 per day in Atlantic City and \$100 per day in Saint-Tropez. She has \$1200 to spend after airfare.

a Write an equation that relates the number of days, C, Alida can spend in Atlantic City and the number of days, T, in Saint-Tropez.

b Find the intercepts and graph the equation by hand.



c If Alida spends 10 days in Atlantic City, how long can she spend in Saint-Tropez?

d What do the intercepts tell us about Alida's vacation?

For Problems 4–9, graph the equation on graph paper. Use the most convenient method for each problem.

4. $\dfrac{x}{6} - \dfrac{y}{12} = 1$

5. $50x = 40y - 20,000$

6. $1.4x + 2.1y = 8.4$

7. $3x - 4y = 0$

8. $3x - 4y = 0$

9. $4x = -12$

10. The table shows the amount of oil, B (in thousands of barrels), left in a tanker t minutes after it hits an iceberg and springs a leak.

t	0	10	20	30
B	800	750	700	650

a Write a linear function for B in terms of t.

b Choose appropriate window settings on your calculator and graph your function.

c Give the slope of the graph, including units, and explain the meaning of the slope in terms of the oil leak.

11. An interior decorator bases her fee on the cost of a remodeling job. The accompanying table shows her fee, F, for jobs of various costs, C, both given in dollars.

C	5000	10,000	20,000	50,000
F	1000	1500	2500	5500

a Write a linear function for F in terms of C.

b Choose appropriate window settings on your calculator and graph your function.

c Give the slope of the graph, including units, and explain the meaning of the slope in terms of the the decorator's fee.

For Problems 12–15, find the slope of the line segment joining the points.

12. $(-1, 4),\ (3, -2)$

13. $(5, 0),\ (2, -6)$

14. $(6.2, 1.4),\ (-2.3, 4, 8)$

15. $(0, -6.4),\ (-5.6, 3.4)$

For Problems 16–17, which tables could describe variables related by a linear equation?

16.

a

r	E
1	5
2	$\frac{5}{2}$
3	$\frac{5}{3}$
4	$\frac{5}{4}$
5	1

b

s	t
10	6.2
20	9.7
30	12.6
40	15.8
50	19.0

17.

a

w	A
2	−13
4	−23
6	−33
8	−43
10	−53

b

x	G
0	0
2	5
4	10
8	20
16	40

For Problems 18–19, the table gives values for a linear equation in two variables. Fill in the missing values.

18.

d	V
−5	−4.8
−2	−3
	−1.2
6	1.8
10	

19.

q	S
−8	−8
−4	36
3	
	200
9	264

20. The planners at AquaWorld want the small water slide to have a slope of 25%. If the slide is 20 feet tall, how far should the end of the slide be from the base of the ladder?

For Problems 21–24, find the slope and y-intercept of the line.

21. $2x - 4y = 5$

22. $\frac{1}{2}x + \frac{2}{3}y = \frac{5}{6}$

23. $8.4x + 2.1y = 6.3$

24. $y - 3 = 0$

For Problems 25 and 26,

a Graph by hand the line that passes through the given point with the given slope.

b Find the equation of the line.

25. $(-4, 6);\ \ m = \frac{-2}{3}$

26. $(2, -5);\ \ m = \frac{3}{2}$

27. The rate at which air temperature decreases with altitude is called the lapse rate. In the troposphere, the layer of atmosphere that extends from the Earth's surface to a height of about 7 miles, the lapse rate is about 3.6°F for every 1000 feet. (Source: Ahrens, 1998)

a If the temperature on the ground is 62°F, write an equation for the temperature, T, at an altitude of h feet.

b What is the temperature outside an aircraft flying at an altitude of 30,000 feet? How much colder is that than the ground temperature?

c What is the temperature at the top of the troposphere?

For Problems 28 and 29, find an equation for the line passing through the two givenpoints.

28. $(3, -5),\ \ (-2, 4)$

29. $(0, 8),\ \ (4, -2)$

30. The population of Maple Rapids was 4800 in 2005 and had grown to 6780 by 2020. Assume that the population increases at a constant rate.

a Make a table of values showing two data points.

b Find a linear equation that expresses the population, P, of Maple Rapids in terms of the number of years, t, since 2005.

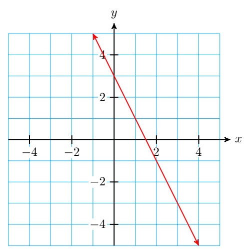
c State the slope of the line, including units, and explain its meaning in the context of the problem.

For Problems 31 and 32,

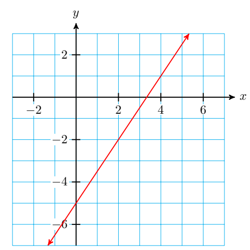
a Find the slope and y-intercept of the line.

b Write an equation for the line.

31.



32.



33. What is the slope of the line whose intercepts are $(-5, 0)$ and $(0, 3)$?

34.

a Find the x- and y-intercepts of the line $\dfrac{x}{4} - \dfrac{y}{6} = 1$.

b What is the slope of the line in part (a)?

35.

a Find the x- and y-intercepts of the line $y = 2 + \dfrac{3}{2}(x - 4)$.

b Find the point on the line whose x-coordinate is 4. Can there be more than one such point?

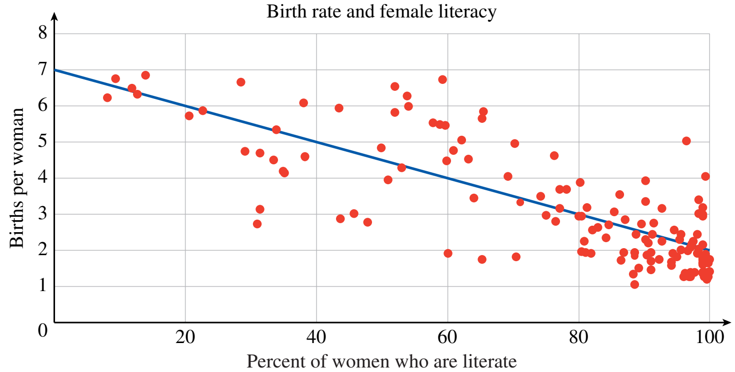
36. Find an equation in slope-intercept form for the line of slope $\dfrac{6}{5}$ that passes through $(-3, -4)$.

Chapter 2

Applications of Linear Models

You may have heard that mathematics is the language of science. In fact, professionals in nearly every discipline take advantage of mathematical methods to analyze data, identify trends, and predict the effects of change. This process is called **mathematical modeling**. A **model** is a simplified representation of reality that helps us understand a process or phenomenon. Because it is a simplification, a model can never be completely accurate. Instead, it should focus on those aspects of the real situation that will help us answer specific questions. Here is an example.

The world's population is growing at different rates in different nations. Many factors, including economic and social forces, influence the birth rate. Is there a connection between birth rates and education levels? The figure shows the birth rate plotted against the female literacy rate in 148 countries. Although the data points do not all lie precisely on a line, we see a generally decreasing trend: the higher the literacy rate, the lower the birth rate. The **regression line** provides a model for this trend, and a tool for analyzing the data. In this chapter we study the properties of linear models and some techniques for fitting a linear model to data.



Investigation 2.1 Water Level. When sailing upstream in a canal or a river that has rapids, ships must sometimes negotiate locks to raise them to a higher water level. Suppose your ship is in one of the lower locks, at an elevation of 20 feet. The next lock is at an elevation of 50 feet. Water begins to flow from the higher lock to the lower one, raising your level by 1 foot per minute, and simultaneously lowering the water level in the next lock by 1.5 feet per minute.

1 Fill in the table

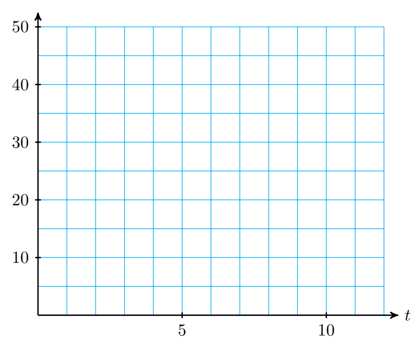
t (minutes)	Lower lock water level	Upper lock water level
0		
2		
4		
6		
8		
10		

2 Let t stand for the number of minutes the water has been flowing.

a Write an equation for L, the water level in the lower lock after t minues.

b Write an equation for U, the water level in the upper lock after t minues.

3 Graph both your equations on the grid.



4 When will the water level in the two locks be 10 feet apart?

5 When will the water level in the two locks be the same?

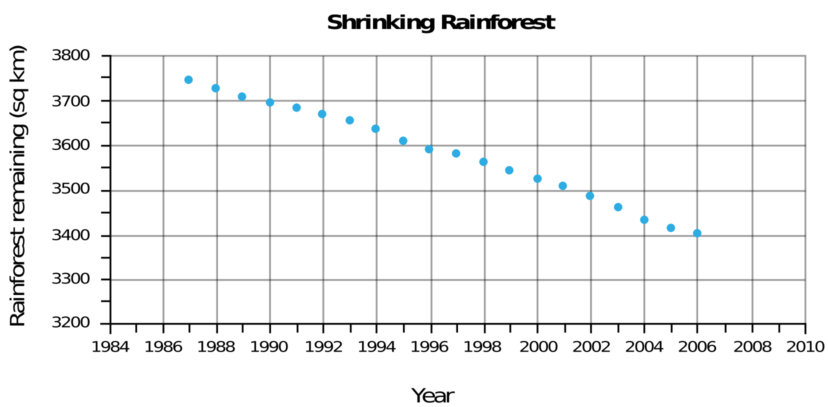
6 Write an equation you can use to verify your answer to part (5), and solve it.

Linear Regression

Shrinking Rain Forest

The Amazon Basin in South America contains over half of the planet's rain forest. The Amazon rain forest is home to the largest collection of plant and animal species in the world, including more than one-third of all living species. During the 1960s, colonists began cutting down the rain forest to clear land for agriculture. The construction of the Trans-Amazonian Highway in the early 1970s opened large forest areas to development by settlers and commercial interests, increasing the rate of deforestation.
Environmentalists are concerned about the loss of biodiversity which will result from destruction of the forest, and about the release of the carbon contained within the vegetation, which could accelerate global warming.
In Brazil, the Instituto Nacional de Pesquisas Espaciais (INPE, or National Institute of Space Research) uses Landsat satellite photos to monitor the pace of deforestation. According to their data, the original Amazon rain forest biome in Brazil of 4,100,000 square kilometers was reduced to 3,413,000 square kilometers by 2005, representing a loss of 16.8%. The figures for 1987 to 2006 are shown at right, and a plot of the data appears below

Year	Remaining forest (thousands sq km)
1987	3745
1988	3274
1989	3706
1990	3692
1991	3681
1992	3667
1993	3652
1994	3637
1995	3608
1996	3590
1997	3577
1998	3560
1999	3542
2000	3524
2001	3506
2002	3485
2003	3460
2004	3432
2005	3413
2006	3400



Although the data points do not all lie exactly on a straight line, they are very close. One question we might ask is: If deforestation continues at the same rate, when will the Amazon rain forest disappear completely? In this section we learn to find a linear model that approximates a data set.

Line of Best Fit

In most cases, a mathematical model is not a perfect description of reality. Many factors can affect empirical data, including measurement error, environmental conditions, and the influence of related variables. Nonetheless, we can often find an equation that approximates the data in a useful way.

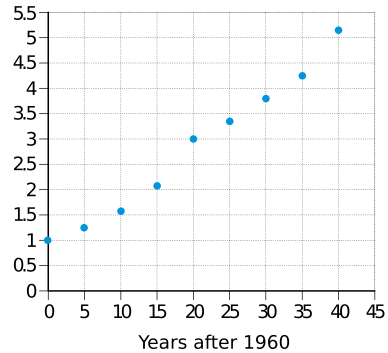
Example 2.1

The table shows the minimum wage in the US at five-year intervals. (Source: Economic Policy Institute)

Year	1960	1965	1970	1975	1980	1985	1990	1995	2000
Min. wage	1.00	1.25	1.60	2.10	3.10	3.35	3.80	4.25	5.15

a Let t represent the number of years after 1960, and plot the data. Are the data linear?

b Draw a line that "fits" the data.



Solution.

a The graph shown is called a **scatterplot**. The data are not strictly linear, because the slope is not constant: from 1960 to 1965, the minimum wage increased at an average rate of

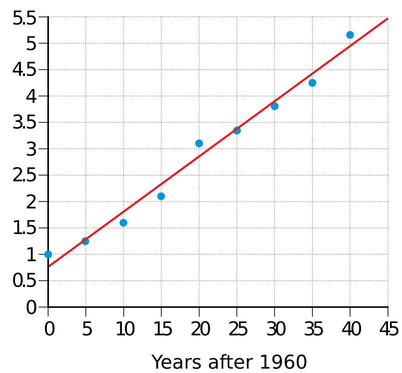
$$\frac{1.25 - 1.00}{5} = 0.05 \text{ dollars per year}$$

and from 1970 to 1975, the minimum wage increased at a rate of

$$\frac{2.10 - 1.60}{5} = 0.10 \text{ dollars per year}$$

However, the data points do appear to lie close to an imaginary line.

b We would like to draw a line that comes as close as possible to all the data points, even though it may not pass precisely through any of them. In particular, we try to adjust the line so that we have the same number of points above the line and below the line. One possible solution is shown in the figure at right.



A line that fits the data in a scatterplot is called a **regression line**. Drawing a regression line by eye is a subjective process. In the Activities Workbook we'll use a calculator to compute a particular regression line called the **least-squares regression line**, which is widely used in statistics and modeling.

We can still find an equation for a line of best fit using the point-slope formula. We choose two points on the line whose coordinates we can estimate fairly accurately. Note that these two points need not be any of the original data points.

Checkpoint 2.2 QuickCheck 1. Explain how to find the equation of a regression line.

- ⊙ A) Choose two data points and use the point-slope formula.
- ⊙ B) Use the point-slope formula with the first and last data points.
- ⊙ C) The regression line must pass through two data points, and we use the point-slope formula for those points.
- ⊙ D) Choose two points on the regression line and use the point-slope formula.

Answer. D) Choose ... slope formula.

Solution. Choose two points on the regression line and use the point-slope formula.

Checkpoint 2.3 Practice 1. The regression line in the Example above appears to pass through the points $(5, 1.25)$ and $(25, 3.35)$. Use those points to find an equation for the regression line.

Answer. $y - 0.105x = 0.725$

Solution. $y = 0.105x + 0.725$

Interpolation and Extrapolation

Example 2.4

An outdoor snack bar collected the following data showing the number of cups of cocoa, C, they sold when the high temperature for the day was $T°$ Celsius.

Temperature (°C), T	2	4	5	8	10	11	12	15	16	18
Cups of cocoa, C	45	42	42	35	25	25	17	16	15	6

a Make a scatterplot of the data, and draw a regression line

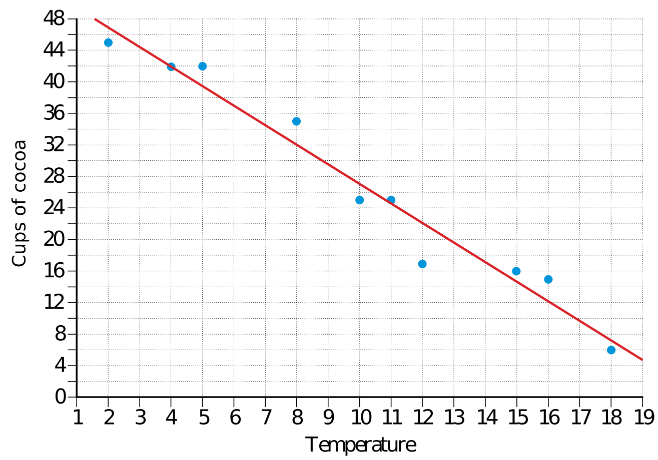
b Read values from your line for the number of cups of cocoa that will be sold when the temperature is 8°C and when the temperature is 16°C.

c Find an equation for the regression line.

d Use your equation to predict the number of cups of cocoa that will be sold when the temperature is 9°C, and when the temperature is 24°C.

Solution.

a The scatterplot and a regression line are shown in the figure.



The regression line need not pass through any of the data points, but it should be as close as possible. We try to draw the regression line so that there are an equal number of data points above and below the line.

b The points $(8, 32)$ and $(16, 12)$ appear to lie on the regression line. According to this model, the snack bar will sell 32 cups of cocoa when the temperature is 8°C, and 12 cups when it is 16°C. These values are close to the actual data, but not exact.

c To find an equation for the regression line, we use two points on the line—not data points! We will use $(8, 32)$ and $(16, 12)$. First we compute the slope

$$m = \frac{C_2 - C_1}{T_2 - T_1} = \frac{12 - 32}{16 - 8} = -2.5$$

Next, we apply the point-slope formula. We'll use the point $(16, 12)$.

$$C = C_1 + m(T - T_1)$$
$$C = 12 - 2.5(T - 16)$$

d When $T = \mathbf{9}$,

$$C = -2.5(\mathbf{9}) + 52 = 29.5$$

We predict that the snack bar will sell 29 or 30 cups of cocoa when the temperature is 9°C. When $T = \mathbf{24}$,

$$C = -2.5(\mathbf{24}) + 52 = -8$$

Because the snack bar cannot sell -8 cups of cocoa, this prediction is not useful. (What is the Fahrenheit equivalent of 24°C?)

Using a regression line to estimate values between known data points is called **interpolation**. If the data points lie fairly close to the regression line, then interpolation will usually give a fairly accurate estimate. In the Example above, the estimate of 29 or 30 cups of cocoa at 9°C seems reasonable in the context of the data.

Making predictions beyond the range of known data is called **extrapolation**. Extrapolation can often give useful information, but if we try to extrapolate too far beyond our data, we may get unreasonable results. The conditions that produced the data may no longer hold, as in the Example above, or other unexpected conditions may arise to alter the situation.

Checkpoint 2.5 QuickCheck 2. True or false.

a. A scatterplot is a type of linear model. (☐ True ☐ False)

b. A regression line should give the same y-values as the data points. (☐ True ☐ False)

c. We use interpolation to estimate the y-value at a data point. (☐ True ☐ False)

d. Extrapolation is usually more reliable than interpolation. (☐ True ☐ False)

Answer 1. False

Answer 2. False

Answer 3. False

Answer 4. False

Solution. The statements are all false.

Checkpoint 2.6 Practice 2.

a. Use your regression equation from the previous Example to predict the number of cups of cocoa sold when the temperature is $-10°$C.

__ cups

b. Predict the number of cups of cocoa sold when the temperature is 7°C.

__ cups

c. Which prediction is more likely to be accurate? Why?

- ⊙ (a), because it is extrapolation.
- ⊙ (a), because it is interpolation.
- ⊙ (b), because it is extrapolation.
- ⊙ (b), because it is interpolation.

Answer 1. 77

Answer 2. 34.5

Answer 3. (b ... interpolation.

Solution.

a. 77

b. 34 or 35

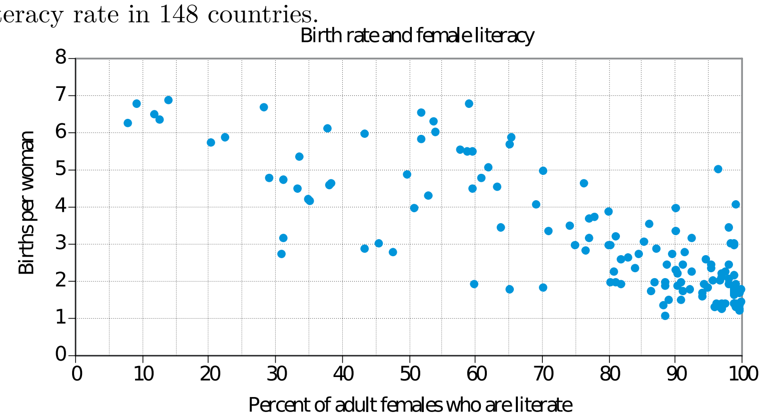
c. (b), because it is interpolation

Scatterplots

The data in a scatterplot may show a linear trend, even though the individual points are not clustered closely around a line. Scattering of data is common in the social sciences, where many variables may influence a particular situation. Nonetheless, by analyzing the data, we may be able to detect a connection between some of the variables.

Example 2.7

The world's population is growing at different rates in different nations. Many factors, including economic and social forces, influence the birthrate. Is there a connection between birth rates and education levels? The figure below shows the birth rate plotted against the female literacy rate in 148 countries.



a Draw a line of best fit for the data points.

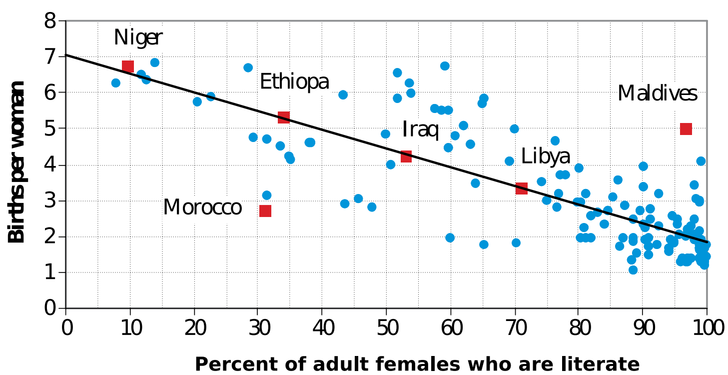
b Locate on the scatterplot the points representing the following nations

Country	Literacy rate	Birth rate
Ethiopa	33.8	5.33
Iraq	53.0	4.28
Libya	71.0	3.34
Maldives	96.4	5.02
Morocco	31.0	2.73
Niger	9.4	6.75

c Data points that lie far from the regression line are called **outliers**. Which of the nations listed in part (b) could be considered outliers?

Solution.

a



b The figure above shows the regression line and the data points for eachof the nations in part (b).

c Morocco and the Maldives could be considered outliers.

Checkpoint 2.8 Practice 3. The equation for the least-squares regression line in the previous Example is

$$y = 7.04 - 0.05x$$

a. What values for the input variable make sense for the model?

   From _ to __

   What is the largest value predicted by the model for the output variable

   What is the smallest value predicted by the model for the output variable

b. State the slope of the regression line, including units.

   - ⊙ A) biths per woman
   - ⊙ B) births per woman per percentage point
   - ⊙ C) literacy per percentage point
   - ⊙ D) women per birth

   Explain what the slope means in the context of the data.

   The birth rate decreases by :

   - ⊙ A) 0.05 births per woman for each percentage point increase in the female literacy rate.
   - ⊙ B) 0.05 births per year among literate women.
   - ⊙ C) 0.05 births per country for each percentage point increase in the female literacy rate.
   - ⊙ D) −0.05 births per women.

Answer 1. 0
Answer 2. 100
Answer 3. 7.04
Answer 4. 2.04
Answer 5. −0.05
Answer 6. B) ... percentage point
Answer 7. Choice 1
Solution.

a. From 0 to 100; largest: 7.04; smallest: 2.04

b. −0.05 births per woman per percentage point. The birth rate decreases by 0.05 births per woman for each percentage point increase in the female literacy rate.

Problem Set 2.1

Warm Up

1. Choose the correct graph for each equation. The scales on both axes are the same.

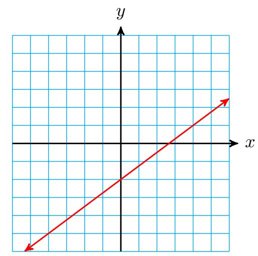
a $y = \dfrac{3}{4}x + 2$

b $y = \dfrac{-3}{4}x + 2$

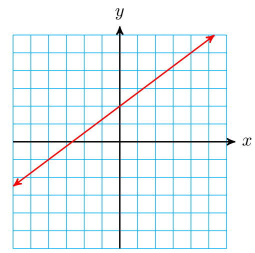
c $y = \dfrac{3}{4}x - 2$

d $y = \dfrac{-3}{4}x - 2$

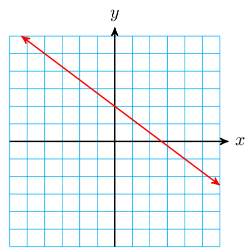
I



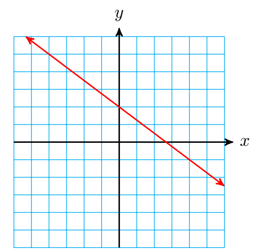
II



III



III



2. Choose the correct graph for each equation. The scales on both axes are the same.

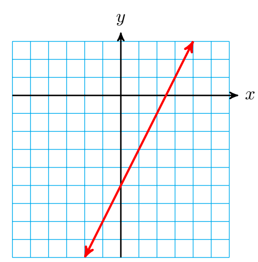
a $y = 1 + 2(x + 3)$

b $y = -1 + 2(x - 3)$

c $y = -1 + 2(x + 3)$

d $y = 1 + 2(x - 2)$

I



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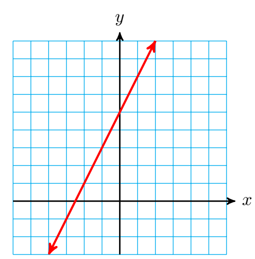

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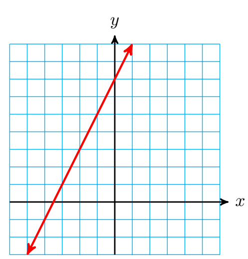
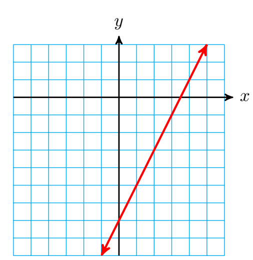


IV



In Problems 3 and 4, find a linear model from two data points as follows:

a Make a table showing the coordinates of two data points for the model. (Which variable should be plotted on the horizontal axis?)

b Make a table showing the coordinates of two data points for the model. (Which variable should be plotted on the horizontal axis?)

c State the slope of the line, including units, and explain its meaning in the context of the problem.

3. On an international flight a passenger may check two bags each weighing 70 kilograms, or 154 pounds, and one carry-on bag weighing 50 kilograms, or 110 pounds. Express the weight, p, of a bag in pounds in terms of its weight, i, in kilograms.

4. Ms. Randolph bought a used car in 2010. In 2012 the car was worth \$9000, and in 2015 it was valued at \$4500. Express the value, V, of Ms. Randolph's car in terms of the number of years, t, she has owned it.

Skills Practice

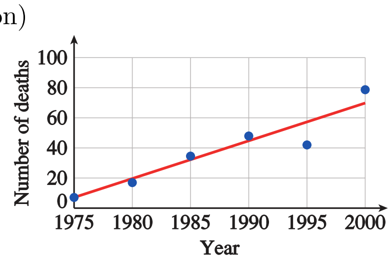
5.

a Find the slope, the C-intercept, and the T-intercept for the regression line in Example 2.4, p. 81.

b Explain the meaning of the slope and the intercepts for this situation.

6. The number of manatees killed by watercraft in Florida waters has been increasing since 1975. Data are given at 5-year intervals in the table, and a scatterplot with regression line is shown below. (Source: Florida Fish and Wildlife Conservation Commission)

Year	Manatee deaths
1975	6
1980	16
1985	33
1990	47
1995	42
2000	78



An equation for the regression line is $y = 4.7 + 2.6t$

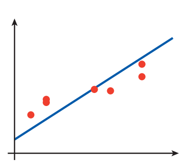
a Use the regression equation to estimate the number of manatees killed by watercraft in 1998.

b What does the slope of the regression line mean in this situation?

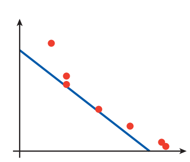
c Which data point might be considered an outlier?

In Problems 7 and 8, the regression lines can be improved by adjusting either m or b. Draw a line that fits the data points more closely.

7.



8.



For Problems 9 and 10,

a Use linear interpolation to give approximate answers.

b What is the meaning of the slope in the context of the problem?

9. Newborn blue whales are about 24 feet long and weigh 3 tons. The young whale nurses for 7 months, at which time it is 53 feet long. Estimate the length of a 1-year-old blue whale.

10. A truck on a slippery road is moving at 24 feet per second when the driver steps on the brakes. The truck needs 3 seconds to come to a stop. Estimate the truck's speed 2 seconds after the brakes were applied.

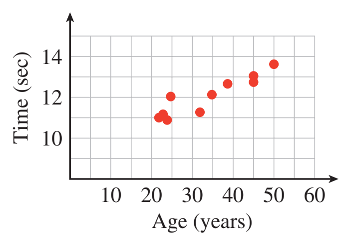
In Problems 11 and 12, use linear interpolation or extrapolation to answer the questions.

11. The temperature of an automobile engine is 9° Celsius when the engine is started and 51°C seven minutes later. Use a linear model to predict the engine temperature for both 2 minutes and 2 hours after it started. Are your predictions reasonable?

12. The elephant at the City Zoo becomes ill and loses weight. She weighed 10,012 pounds when healthy and only 9641 pounds a week later. Predict her weight after 10 days of illness.

Applications

13. The scatterplot shows the ages of 10 army drill sergeants and the time it took each to run 100 meters, in seconds.



a What was the hundred-meter time for the 25-year-old drill sergeant?

b How old was the drill sergeant whose hundred-meter time was 12.6 seconds?

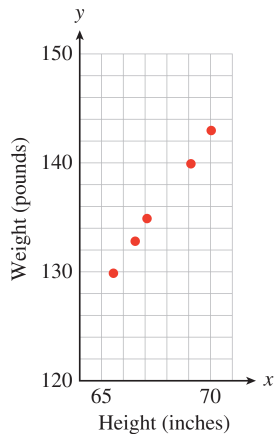
c Use a straightedge to draw a line of best fit through the data points.

d Use your line of best fit to predict the hundred-meter time of a 28-year-old drill sergeant.

e Choose two points on your regression line and find its equation.

f Use the equation to predict the hundred-meter time of a 40-year-old drill sergeant and a 12 year-old drill sergeant. Are these predictions reasonable?

14. The scatterplot shows the weights in pounds and the heights in inches of a team of distance runners.



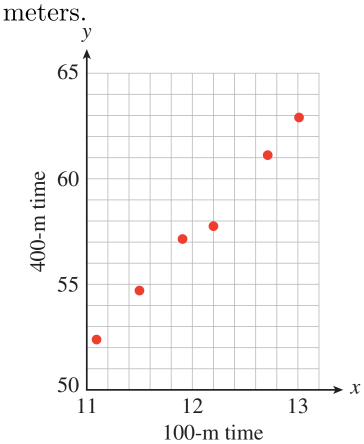
a Use a straightedge to draw a line that fits the data.

b Use your line to predict the weight of a 65-inch tall runner and the weight of a 71-inch tall runner.

c Use your answers from part (b) to approximate the equation of your regression line.

d Use your answer from part(c) to predict the weight of a runner who is 68 inches tall.

15. The scatterplot shows the best times for various women running the 400 meters and the 100 meters.



a Use a straightedge to draw a line that fits the data.

b Use your line to predict the 400-meter time of a woman who runs the 100-meter dash in 11.2 seconds, and the 400-meter time of a woman who runs the 100-meter dash in 13.2 seconds.

c Use your answers from part (b) to approximate the equation of your regression line.

d Use your answer from part(c) to predict the the 400-meter time of a womon who runs the 100-meter dash in 12.1 seconds.

16. The table shows the minimum wage in the United States at five-year intervals. (Source: Economic Policy Institute)

Year	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005	2010
Minimum wage	1.00	1.25	1.60	2.10	3.10	3.35	3.80	4.25	5.15	5.15	7.25

a Let t represent the number of years after 1960 and plot the data. Draw a line of best fit for the data points.

b Find an equation for your regression line.

c Estimate the minimum wage in 1972.

d Predict the minimum wage in 2025.

17. With Americans' increased use of faxes, pagers, and cell phones, new area codes are being created at a steady rate. The table shows the number of areacodes in the US each year. (Source: USA Today, NeuStar, Inc.)

Year	1997	1998	1999	2000	2001	2002	2003
Number of area codes	151	186	204	226	239	262	274

a Let t represent the number of years after 1995 and plot the data. Draw a line of best fit for the data points.

b Find an equation for your regression line.

c How many area codes do you predict for 2010?

18. The table shows the amount of carbon released into the atmosphere annually from burning fossil fuels, in billions of tons, at 5-year intervals from 1950 to 1995. (Source: www.worldwatch.org)

Year	1950	1955	1960	1965	1970	1975	1980	1985	1990	1995
Carbon emissions	1.6	2.0	2.5	3.1	4.0	4.5	5.2	5.3	5.9	6.2

a Let t represent the number of years after 1950 and plot the data. Scale the t-axis from 0 to 50 by 5's, and the C-axis from 0 to 7 by 0.5's.

b Draw a line of best fit for the data points.

c Find an equation for your regression line.

d Estimate the amount of carbon released in 1992.

19. Male birds with the largest repertoire of songs are the first to acquire mates in the spring. The table shows the number of different songs sung by several sedge warblers, and the days on which they acquired their mates, where day 1 is April 20. (Source: Krebs and Davies, 1993)

Number of songs, x	41	38	34	32	30	25	24	24	23	14
Pairing day, y	20	24	25	21	24	27	31	35	40	42

a Plot the data points on graph paper, scale the x-axis from 0 to 65 by 5's, and the y-axis from 0 to 60 by 5's.

b The least-squares regression line is

$$y = -0.85x + 53$$

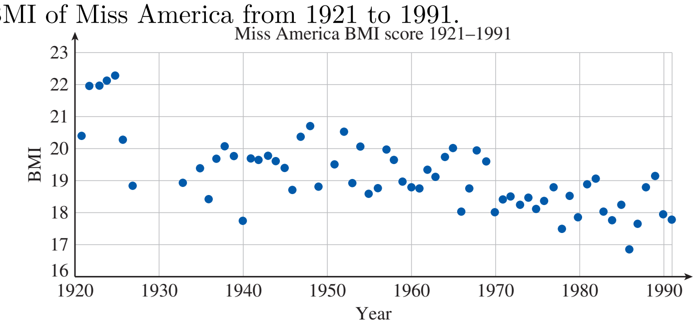
Graph this line on the same grid with the data. (Make a short table of values and plot the points.)

c What does the slope of the regression line tell us about sedge warblers?

d Use extrapolation to estimate when a sedge warbler that knows 10 songs can expect to find a mate.

e What do the intercepts of the regression line represent? Do these values make sense for this situation?

20. One measure of a person's physical fitness is the body mass index, or BMI. Your body mass index is the ratio of your weight in kilograms to the square of your height in meters. The points on the scatterplot show the BMI of Miss America from 1921 to 1991.



a Use a straightedge to draw a line of best fit on the scatterplot above.

b The equation of the least-squares regression line for the data is

$$y = 20.69 - 0.04t$$

where t is the number of years since 1920. On the figure above, relabel the horizontal axis with values of t. Then graph this line and compare to your estimated line of best fit.

c Do thinner people have higher or lower BMI scores than fatter people? Use the definition of BMI to explain your reasoning.

d The Center for Disease Control considers a BMI between 18.5 and 24.9 to be healthy. In 2002, Miss America was 5'3" tall and weighed 110 pounds. Calculate her BMI. (You will need to convert inches to meters and pounds to kilograms.)

e What BMI score does the regression line predict for Miss America 2002?

f There are no data points for the years 1928 to 1932. What happened during those years that might cause this gap?

Linear Systems

Systems of Equations

A biologist wants to know the average weights of two species of birds in a wildlife preserve. She sets up a feeder whose platform is actually a scale and mounts a camera to monitor the feeder. She waits until the feeder is occupied only by members of the two species she is studying, robins and thrushes. Then she takes a picture, which records the number of each species on the scale and the total weight registered.

From her two best pictures, she obtains the following information. The total weight of 3 thrushes and 6 robins is 48 ounces, and the total weight of 5 thrushes and 2 robins is 32 ounces. The biologist writes two equations about the photos. She begins by assigning variables to the two unknown quantities:

Average weight of a thrush: t
Average weight of a robin: r

In each of the photos,

$$(\text{weight of thrushes}) + (\text{weight of robins}) = \text{total weight}$$

Thus

$$3t + 6r = 48$$
$$5t + 2r = 32$$

This pair of equations is an example of a **linear system of two equations in two unknowns** (or a 2×2 linear system, for short). A **solution** to the system is an ordered pair of numbers, (t, r), that satisfies both equations in the system.

Checkpoint 2.9 QuickCheck 1. Choose the solution of the system:

$$3x - 2y = 19$$
$$4x + 5y = 10$$

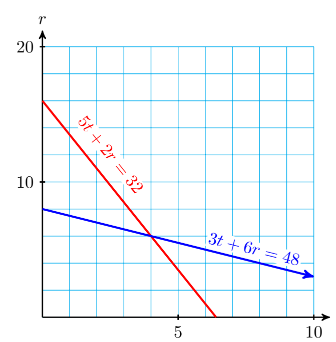
- $(7, 1)$
- $(0, 2)$
- $(5, -2)$
- $(2, 5)$

Answer. Choice 3

Solution. $(5, -2)$ satiesfies both equations.

Solving Systems by Graphing

Every point on the graph of an equation represents a solution to that equation. A solution to both equations must be a point that lies on both graphs. Therefore, a solution to the system is a point where the two graphs intersect. The figure at right shows a graph of the system about robins and thrushes. The two lines appear to intersect at the point $(4, 6)$, so we expect that the values $t = \mathbf{4}$ and $r = \mathbf{6}$ are the solution to the system. We can check by verifying that these values satisfy both equations in the system.



$$3(\mathbf{4}) + 6(\mathbf{6}) \stackrel{?}{=} 48 \qquad \text{True}$$
$$5(\mathbf{4}) + 2(\mathbf{6}) \stackrel{?}{=} 32 \qquad \text{True}$$

Both equations are true, so we have found the solution: a thrush weighs 4 ounces on average, and a robin weighs 6 ounces.

Checkpoint 2.10 QuickCheck 2. True or false.

a. The point where two lines cross is called the intercept (☐ True ☐ False)

b. The intercepts of a line are the points where it intersects the axes. (☐ True ☐ False)

c. The solution of a system may occur at an intercept. (☐ True ☐ False)

d. The words intercept and intersect mean the same thing (☐ True ☐ False)

Answer 1. False
Answer 2. True
Answer 3. True
Answer 4. False
Solution.

a. False

b. True

c. True

d. False

We can use a calculator to graph the equations in a system.

Example 2.11

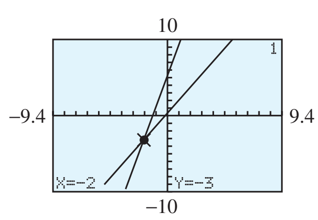
Use your calculator to solve the system by graphing.

$$y = 1.7x + 0.4$$
$$y = 4.1x + 5.2$$

Solution. We set the graphing window to

$$\text{Xmin} = -9.4 \qquad \text{Ymin} = -10$$
$$\text{Xmax} = -9.4 \qquad \text{Ymax} = 10$$

and enter the two equations. We can see in the figure that the two lines intersect in the third quadrant. We use the TRACE key to find the coordinates of the intersection point, $(-2, -3)$. The solution to the system is $x = -2$, $y = -3$.



The values we obtain from a calculator may be only approximations, so it is a good idea to check the solution algebraically. In the example above, we find that both equations are true when we substitute $x = \mathbf{-2}$ and $y = -3$.

$$-3 = 1.7(\mathbf{-2}) + 0.4 \qquad \text{True}$$
$$-3 = 4.2(\mathbf{-2}) + 5.2 \qquad \text{True}$$

Checkpoint 2.12 Practice 1.

a. Solve the system of equations

$$y = -0.7x + 6.9$$
$$y = 1.2x - 6.4$$

by graphing. Use the window

$$\text{Xmin} = -9.4 \quad \text{Xmax} = 9.4$$
$$\text{Ymin} = -10 \quad \text{Ymax} = 10$$

b. Verify algebraically that your solution satisfies both equations

Answer: ______

Answer. $(7, 2)$

Solution.

a. $(7, 2)$

b.

$$2 = -0.7(7) + 6.9 \quad \text{Yes!}$$
$$2 = 1.2(7) - 6.4 \quad \text{Yes!}$$

The Intersect Feature

Because the TRACE feature does not show every point on a graph, we may not find the exact solution to a system by tracing the graphs.

Example 2.13

Solve the system

$$3x - 2.8y = 21.06$$
$$2x + 1.2y = 5.3$$

Solution. We can graph this system in the standard window by solving each equation for y. We enter

$$Y_1 = (21.06 - 3X)/-2.8$$
$$Y_2 = (5.3 - 2X)/1.2$$

and then press ZOOM 6. If we trace along the graph to the intersection point, we will not find the same coordinates on both lines. The intersection point is not displayed in this window. Instead, we can use the **intersect** feature of the calculator.

Using the arrow keys, we position the Trace bug as close to the intersection point as possible. We then press 2nd CALC to see the **Calculate** menu. We press 5 for **intersect**, and respond to each of the calculator's questions, **First curve?**, **Second curve?**, and **Guess?**, by pressing ENTER. The calculator then displays the intersection point, $x = 4.36$, $y = -2.85$.

We can substitute these values into the original system to check that they satisfy both equations.

$$3(\mathbf{4.36}) - 2.8(\mathbf{-2.85}) = 21.06 \quad \text{True}$$
$$2(\mathbf{4.36}) + 1.2(\mathbf{-2.85}) = 5.3 \quad \text{True}$$

Checkpoint 2.14 Practice 2. Solve the system of equations

$$y = 47x - 1930$$
$$y + 19x = 710$$

by graphing. Estimate the intercepts of each graph to help you choose a suitable window, and use the intersect feature to locate the solution.

Answer: ______

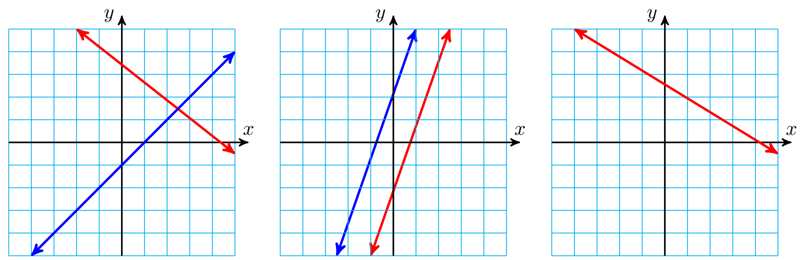
Answer. $(40, -50)$

Solution. $(40, -50)$ is the intersection of the two lines.

How does the calculator find the exact coordinates of the intersection point? In the next section we'll learn how to find the solution of a system using algebra.

Inconsistent and Dependent Systems

Because two straight lines do not always intersect at a single point, a 2×2 system of linear equations does not always have a unique solution. In fact, there are three possibilities, as illustrated below.



Consistent and independent: one solution

Inconsistent: no solution

Dependent: infinitely many solutions

Definition 2.15 Solutions of Linear Systems.

There are three types of 2×2 linear system :

1 **Consistent and independent system**. The graphs of the two lines intersect in exactly one point. The system has exactly one solution.

2 **Inconsistent system**. The graphs of the equations are parallel lines and hence do not intersect. An inconsistent system has no solutions.

3 **Dependent system**. All the solutions of one equation are also solutions to the second equation, and hence are solutions of the system. The graphs of the two equations are the same line. A dependent system has infinitely many solutions.

Example 2.16

Solve each system.

a

$$y = -x + 5$$
$$2x + 2y = 3$$

b

$$x = \frac{2}{3}y + 3$$
$$3x - 2y = 9$$

Solution.

a We use the calculator to graph both equations on the same axes. First, we rewrite the second equation in slope-intercept form by solving for y.

$$\begin{aligned} 2x + 2y &= 3 && \text{Substract } 2x \text{ from both sides.} \\ 2y &= -2x + 3 && \text{Divide both sides by 2.} \\ y &= -x + 1.5 \end{aligned}$$

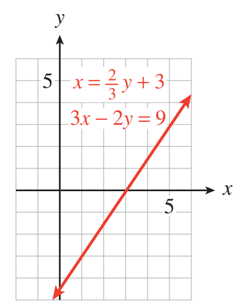
We enter the equations as

$$\begin{aligned} Y_1 &= -X + 5 \\ Y_2 &= -X + 1.5 \end{aligned}$$

You should see that the lines do not intersect within the viewing window; they appear to be parallel. If we look again at the equations of the lines, we recognize that both have slope -1 but different y-intercepts, so they are parallel. Because parallel lines never meet, there is no solution to the system.

b This time we will graph by hand. We begin by writing each equation in slope-intercept form.

$$\begin{aligned} x &= \frac{2}{3}y + 3 && \text{Subtract 3.} \\ x - 3 &= \frac{2}{3}y && \text{Multiply by } \frac{3}{2}. \\ \frac{3}{2}x - \frac{9}{2} &= y \end{aligned}$$



For the second equation,

$$\begin{aligned} 3x - 2y &= 9 && \text{Subtract } 3x. \\ -2y &= -3x + 9 && \text{Divide by } -2. \\ y &= \frac{3}{2}x - \frac{9}{2} \end{aligned}$$

The two equations are actually different forms of the same equation. They are equivalent, so they share the same line as a graph. Every point on the first line is also a point on the second line, and hence a solution of the system. The system has infinitely many solutions.

Checkpoint 2.17 QuickCheck 3. Complete the statements.

a. If two lines have the same slope but different y-intercepts, the system is (☐ dependent ☐ inconsistent ☐ consistent and independent)

b. If two lines have the same slope and the same y-intercept, the system is (☐ dependent ☐ inconsistent ☐ consistent and independent)

c. If two lines have the same y-intercept but different slopes, the system is (☐ dependent ☐ inconsistent ☐ consistent and independent)

d. If both lines are horizontal, the system is (☐ dependent ☐ inconsistent

☐ consistent and independent)

Answer 1. inconsistent

Answer 2. dependent

Answer 3. consistent and independent

Answer 4. dependent

Solution.

a. Inconsistent

b. Dependent

c. Consistent and independent

d. Dependent

Checkpoint 2.18 Practice 3.

a. Graph the system

$$y = -3x + 6$$
$$6x + 2y = 15$$

by hand, using either the intercept method or the slope-intercept method.

b. Identify the system as dependent, inconsistent, or consistent and independent.

(☐ dependent ☐ inconsistent ☐ consistent and independent)

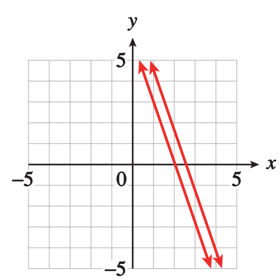
Answer. inconsistent

Solution.

a. A graph is below.

b. Inconsistent: The graph is two parallel lines.

Graph for part (a):



Applications

Systems of equations are useful in many applied problems. One example involves supply and demand. The owner of a retail business must try to balance the demand for his product from consumers with the supply he can obtain from manufacturers. Supply and demand both vary with the price of the product: consumers usually buy fewer items if the price increases, but manufacturers will be willing to supply more units of the product if its price increases.

The **demand equation** gives the number of units of the product that consumers will buy at a given price. The **supply equation** gives the number of units that theproducer will supply at that price. The price at which the supply and demand are equal is called the **equilibrium price**. This is the price at which the consumer and the producer agree to do business.

Example 2.19

A woolens mill can produce $400x$ yards of fine suit fabric if it can charge x dollars per yard. The mill's clients in the garment industry will buy $6000 - 100x$ yards of wool fabric at a price of x dollars per yard. Find the equilibrium price and the amount of fabric that will change hands at that price.

Solution.

- *Step 1.*

  We choose variables for the unknown quantities.

Price per yard:	x
Number of yards:	y

- *Step 2.*

  The supply equation tells us how many yards of fabric the mill will produce for a price of x dollars per yard.

  $$y = 400x$$

  The demand equation tells us how many yards of fabric the garment industry will buy at a price of x dollars per yard.

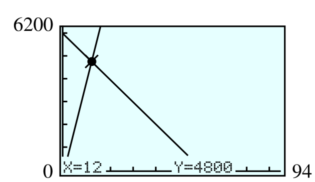
  $$y = 6000 - 100x$$

- *Step 3.*

  We graph the two equations on the same set of axes, as shown below. We set the window values to

  $$\text{Xmin} = 0 \qquad \text{Xmax} = 94$$
  $$\text{Ymin} = 0 \qquad \text{Ymax} = 6200$$

  and use the TRACE or the **intersect** command to locate the solution. The graphs intersect at the point $(12, 4800)$.

- *Step 4.*

  The equilibrium price is \$12 per yard, and the mill will sell 4800 yards of fabric at that price.

A business venture calculates its **profit** by subtracting its **costs** from its **revenue**, the amount of money it takes in from sales.

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

Cost is usually calculated as the sum of fixed costs, or overhead, and variable costs,the cost of labor and materials to produce its product. Revenue is the product of the selling price per item times the number of items sold. If the company's revenue exactly equals its costs (so that their profit is zero), we say that the business venture will **break even**.

Checkpoint 2.20 Practice 4. The Aquarius jewelry company determines that each production run to manufacture a pendant involves an initial setup cost of $200 and $4 for each pendant produced. The pendants sell for $12 each.

a. Express the cost C of production in terms of the number x of pendants produced.

   $C =$______

b. Express the revenue R in terms of the number x of pendants sold.

   $R =$______

c. Graph the revenue and cost on the same set of axes. (Find the intercepts of each equation to help you choose a window for the graph.) State the solution of the system.

   Solution _____ Give the solution as an ordered pair.

d. How many pendants must be sold for the Aquarius company to break even on a particular production run?

   The must sell __ pendants to break even.

Answer 1. $200 + 4x$

Answer 2. $12x$

Answer 3. $(25, 300)$

Answer 4. 25

Solution.

a. $C = 200 + 4x$

b. $R = 12x$

c. $(25, 300)$

d. They must sell 25 pendants in order to break even.

Problem Set 2.2

Warm Up

1. Graph by the intercept method.

$$4x = \frac{4}{3}y = 12$$

2. Graph by the slope-intercept method.

$$y = 8 - \frac{5}{2}x$$

3. Solve $0.4(30 - x) + 0.8x = 0.65(30)$

4. Write each equation in the form $ax+by=c$, where a, b, and c, are integers.

a $4y = 6x - 300$

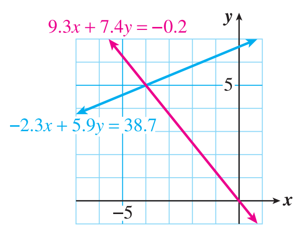
b $24 - \frac{2}{3}y = \frac{3}{4}x$

Skills Practice

For Problems 5 and 6, solve the system of equations using the graph. Then verify that your solution satisfies both equations.

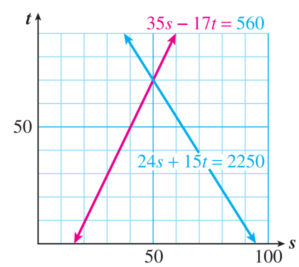
5.

$$-2.3x + 5.9y = 38.7$$
$$9.3x + 7.4y = -0.2$$



6.

$$35s - 17t = 560$$
$$24s + 15t = 2250$$



For Problems 7 and 8

a Solve the system of equations by graphing. Use the "friendly" window

Xmin = −9.4	Ymin = −10
Xmax = 9.4	Ymax = 10

b Verify algebraically that your solution satisfies both equations.

7.

$$y = 2.6x + 8.2$$
$$y = 1.8 - 0.6x$$

8.

$$y = 7.2 - 2.1x$$
$$-2.8x + 3.7y = 5.5$$

For Problems 9–12,

a Graph the system by hand.

b Identify the system as dependent, inconsistent, or consistent and independent.

9.

$$2x = y + 4$$
$$8x - 4y = 9$$

10.

$$2t + 12 = -6s$$
$$12s + 4t = 24$$

11.

$$-3x = 4y + 12$$
$$\frac{1}{2}x + 2 = \frac{-2}{3}y$$

12.

$$w - 3z = 6$$
$$2w + x = 8$$

Applications

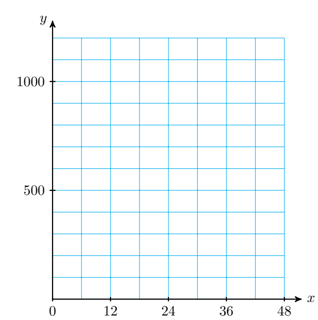
13. Francine wants to join a health club and has narrowed it down to two choices. The Sportshaus charges an initiation fee of \$500 and \$10 per month. Fitness First has an initiation fee of \$50 and charges \$25 per month.

a Let x stand for the number of months Francine uses the health club. Write equations for the total cost of each health club for x months.

b Complete the table for the total cost of each club.

x	Sporthaus total cost	Fitness First total cost
6		
12		
18		
24		
30		
36		
42		
48		

c Graph both equations on the grid.



d When will the total cost of the two health clubs be equal?

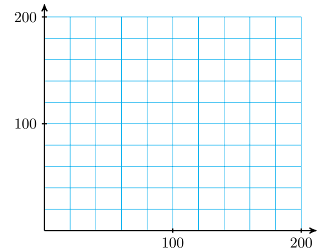
14. The Bread Alone Bakery has a daily overhead of \$90. It costs \$0.60 to bake each loaf of bread, and the bread sells for \$1.50 per loaf.

a Write an equation for the cost C in terms of the number of loaves, x.

b Write an equation for the revenue R in terms of the number of loaves, x.

c Graph the revenue, R, and cost, C, on the same set of axes. State the solution of the system.

d How many loaves must the bakery sell to break even on a given day?

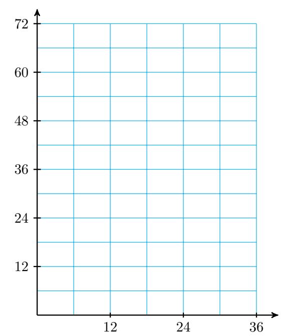


15. The manager for Books for Cooks plans to spend \$300 stocking a new diet cookbook. The paperback version costs her \$5, and the hardback costs \$10. She finds that she will sell three times as many paperbacks as hardbacks. How many of each should she buy?

a Let x represent the number of hardbacks and y the number of paperbacks she should buy. Write an equation about the cost of the books.

b Write a second equation about the number of each type of book.

c Graph both equations and solve the system using the grid.



d Answer the question in the problem.

16. There were 42 passengers on a local airplane flight. First-class fare was \$80, and coach fare was \$64. If the revenue for the flight totaled \$2880, how many first-class and how many coach passengers paid for the flight?

a Write algebraic expressions to fill in the table.

	Number of tickets	Cost per ticket	Revenue
First-class	x		
Coach	y		
Total			

b Write an equation about the number of tickets sold.

c Write a second equation about the revenue from the tickets.

d Graph both equations on graph paper and solve the system. (**Hint**: Find the intercepts of each equation to help you choose scales for the axes.)

e Answer the question in the problem.

17. Mel's Pool Service can clean $1.5x$ pools per week if it charges x dollars per pool, and the public will book $120 - 2.5x$ pool cleanings at x dollars per pool.

a What is the supply equation?

b What is the demand equation?

c Graph both equations in the window

Xmin = 0	Ymin = 0
Xmax = 50	Ymax = 125

d Find the equilibrium price and the number of pools Mel will clean at that price.

18.

a Explain how you can tell, without graphing, that the system has no solution.

$$3x = y + 3$$
$$6x - 2y = 12$$

b Explain how you can tell, without graphing, that the system has infinitely many solutions.

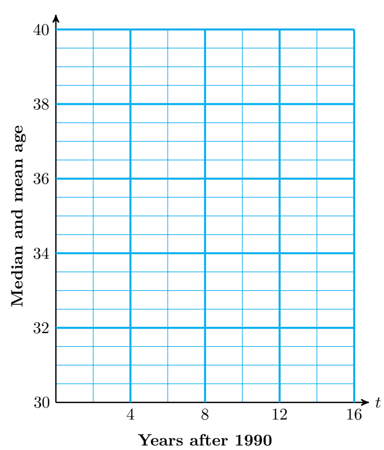
$$-x + 2y = 4$$
$$3x = 6y - 12$$

19. According to the Bureau of the Census, the average age of the U.S. population is steadily rising. The table gives data for two types of average, the mean and the median, for the ages of women. (The "mean" is the familiar average, and the "median" is the middle age: half of all women are older and half younger than the median.)

Date	1990	1992	1994	1996	1998
Median age	34.0	34.6	35.2	35.8	36.4
Mean age	36.6	36.8	37.0	37.3	37.5

a Which is growing more rapidly, the mean age or the median age?

b Plot the data for median age versus the date, using 1990 as $t = 0$. Draw a line through the data points.



c What is the meaning of the slope of the line in part (b)?

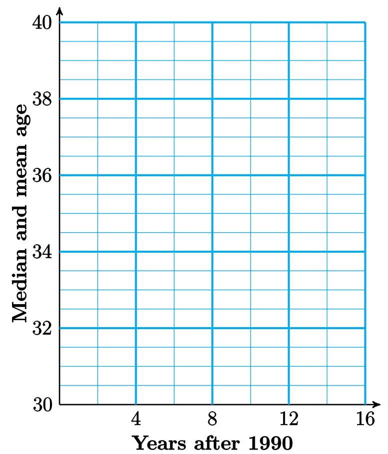
d Plot the data for mean age versus date on the same grid. Draw a line that fits the data.

e Use your graph to estimate when the mean age and the median age were the same.

f If the median age of women in the United States is greater than the mean age, what does this tell you about the population of women?

20. The table below gives data for the mean and median ages for men in the United States. Repeat Problem 19 for these data.

Date	1990	1992	1994	1996	1998
Median age	31.6	32.2	32.9	33.5	34.0
Mean age	33.8	34.0	34.3	34.5	34.9



Algebraic Solution of Systems

Substitution Method

You are probably familiar with the substitution method for solving a system of equations. Here is a summary of the steps for solving a system by substitution.

To Solve a System by Substitution.

1 Choose one of the variables in one of the equations. (It is best to choose a variable whose coefficient is 1 or -1.) Solve the equation for that variable.

2 Substitute the result of Step 1 into the other equation. This gives an equation in one variable.

3 Solve the equation obtained in Step 2. This gives the solution value for one of the variables.

4 Substitute this value into the result of Step 1 to find the solution value of the other variable.

Example 2.21

Staci stocks two kinds of sleeping bags in her sporting goods store, a standard model and a down-filled model for colder temperatures. From past experience, she estimates that she will sell twice as many of the standard variety as of the down-filled. She has room to stock 84 sleeping bags at a time. How many of each variety should Staci order?

Solution. We begin by choosing variables for the unknown quantities.

Number of standard sleeping bags: x
Number of down-filled sleeping bags: y

We write two equations about the variables.

Staci needs twice as many standard model as down-filled: $x = 2y$
The total number of sleeping bags is 84: $x + y = 84$

We will solve this system using substitution. Notice that the first equation is already solved for x in terms of y. We substitute $\mathbf{2y}$ for x in the second equation to obtain

$$\begin{aligned} \mathbf{2y} + y &= 84 \\ 3y &= 84 \end{aligned}$$

Solving for y, we find $y = \mathbf{28}$. Finally, substitute this value into the first equation to find

$$x = 2(\mathbf{28}) = 56$$

The solution to the system is $x = 56$, $y = 28$. Staci should order 56 standard sleeping bags and 28 down-filled bags.

Checkpoint 2.22 QuickCheck 1. What is the first step in the substitution method?

- ⊙ Solve both equations for y.
- ⊙ Substitute a value for x.
- ⊙ Solve one of the equations for one of the variables.
- ⊙ Add the equations together.

Answer. Choice 3

Solution. Solve one of the equations for one of the variables.

Checkpoint 2.23 Practice 1. Follow the suggested steps to solve the system by substitution:

$$\begin{aligned} 3y - 2x &= 3 \\ x - 2y &= -2 \end{aligned}$$

- Step 1

  Solve the second equation for x in terms of y.

  $x =$_____

- Step 2

  Substitute your expression for x into the first equation.

  $3y - 2($_____$) = 3$

- Step 3

  Solve the equation you got in Step 2.

$y =$__

- Step 4

  You now have the solution value for y. Substitute that value into your result from Step 1 to find the solution value for x.

  $x =$__

  As always, you should check that your solution values satisfy both equations in the system.

  The solution to the system is ____ (Give the solution as an ordered pair.)

Answer 1. $2y - 2$

Answer 2. $2y - 2$

Answer 3. 1

Answer 4. 0

Answer 5. $(0, 1)$

Solution. Follow the suggested steps to solve the system by substitution:

$$\begin{aligned} 3y - 2x &= 3 \\ x - 2y &= -2 \end{aligned}$$

- Step 1

  $x = 2y - 2$

- Step 2

  $3y - 2(2y - 2) = 3$

- Step 3

  $y = 1$

- Step 4

  $x = 0$

  The solution to the system is $(0, 1)$

Elimination Method

A second algebraic method for solving systems is called **elimination**. As with the substitution method, we obtain an equation in a single variable, but we do it by eliminating one of the variables in the system. We must first put both equations into the general linear form $Ax + By = C$.

Example 2.24

Solve the system

$$\begin{aligned} 5x &= 2y + 21 \\ 2y &= 19 - 3x \end{aligned}$$

Solution. First, we rewrite each equation in the form $Ax + By = C$.

$$\begin{aligned} 5x &= 2y + 21 \\ -2y &= -2y \\ \hline 5x - 2y &= 21 \end{aligned}$$

Subtract $2y$ from both sides.

$$\begin{aligned} 2y &= 19 - 3x \\ +3x &= +3x \\ \hline 3x + 2y &= 19 \end{aligned}$$

Add $3x$ to both sides.

We add the equations together by adding the left side of the first equation to the left side of the second equation, and then adding the two right sides together, as follows:

$$\begin{aligned} 5x - 2y &= 21 \\ \underline{3x + 2y} &= \underline{19} \\ 8x \qquad &= 40 \end{aligned}$$

Note that the y-terms canceled, or were **eliminated**. Solving the new equation, $8x = 40$, we find that $x = 5$. We are not finished yet, because we must still find the value of y. We can substitute our value for x into either of the original equations, and solve for y. We'll use the second equation, $3x + 2y = 19$:

$$\begin{aligned} 3(\mathbf{5}) + 2y &= 19 && \text{Subtract 15 from both sides.} \\ 2y &= 4 && \text{Divide by 2.} \\ y &= 2 \end{aligned}$$

Thus, the solution is the point $(5, 2)$.

Checkpoint 2.25 QuickCheck 2. In order to use the elimination method, in what form should you write the equations?

- ⊙ Point-slope form
- ⊙ Slope-intercept form
- ⊙ General linear form
- ⊙ Coordinate form

Answer. General linear form

Solution. The general linear form is usually the most convenient form when using the elimination method to solve a system.

In the previous Example, the elimination method worked because the coefficients of y in the two equations were opposites, 2 and -2. This caused the y-terms to cancel out when we added the two equations together. What if the coefficients of neither x nor y are opposites? Then we must multiply one or both of the equations in the system by a suitable constant. Consider the system

$$\begin{aligned} 4x + 3y &= 7 \\ 3x + y &= -1 \end{aligned}$$

We can eliminate the y-terms if we multiply each term of the second equation by $\mathbf{-3}$, so that the coefficients of y will be opposites:

$$\mathbf{-3}(3x + y = -1) \longrightarrow -9x - 3y = 3$$

Be careful to multiply each term by -3, not just the y-term. We can now replace the second equation by its new version to obtain this system:

$$\begin{aligned} 4x + 3y &= 7 \\ -9x - 3y &= 3 \end{aligned}$$

Checkpoint 2.26 Practice 2. Follow the suggested steps to solve the system

$$\begin{aligned} 4x + 3y &= 7 \\ -9x - 3y &= 3 \end{aligned}$$

- Step 1

  Add the equations together.

  ___ = ___

- Step 2

  Solve the resulting equation for x.

  $x =$__

- Step 3

  Substitute your value for x into either original equation and solve for y.

  $y =$__

- Step 4

  Check that your solution satisfies both original equations.

  The solution to the system is ____ (Give the solution as an ordered pair.)

Answer 1. $-5x$
Answer 2. 10
Answer 3. -2
Answer 4. 5
Answer 5. $(-2, 5)$
Solution.

- Step 1

  $-5x = 10$

- Step 2

  $x = -2$

- Step 3

  $y = 5$

- Step 4

$$\begin{aligned} 4(-2) + 3(5) &= 7 \\ -9(-2) - 3(5) &= 3 \end{aligned}$$

  The solution to the system is $(-2, 5)$

Checkpoint 2.27 QuickCheck 3. Before adding the two equations together, we must arrange it so that the coefficients of one of the variables are (☐ equal ☐ opposites ☐ reciprocals)

Answer. opposites

Solution. opposites

When we add a multiple of one equation to the other we are making a **linear combination**of the equations. The method of elimination is also called the method of linear combinations. Sometimes we need to multiply both equations by suitable constants in order to eliminate one of the variables.

Example 2.28

Use linear combinations to solve the system

$$\begin{aligned} 5x - 2y &= 22 \\ 2x - 5y &= 13 \end{aligned}$$

Solution. This time we choose to eliminate the x-terms. We must arrange things so that the coefficients of the x-terms are opposites, so we look for the smallest integer that both 2 and 5 divide into evenly. (This number is called the **lowest common multiple**, or **LCM**, of 2 and 5.) The LCM of 2 and 5 is 10. We want one of the coefficients of x to be 10, and the other to be -10.

To achieve this, we multiply the first equation by 2 and the second equation by -5.

$$\begin{aligned} \mathbf{2}(5x - 2y = 22) \quad &\rightarrow \quad 10x - 4y = 44 \\ \mathbf{-5}(2x - 5y = 13) \quad &\rightarrow \quad -10x + 25y = -65 \end{aligned}$$

Adding these new equations eliminates the x-term and yields an equation in y.

$$\begin{aligned} 10x - 4y &= 44 \\ \underline{-10x + 25y} &= \underline{-65} \\ 21y &= -21 \end{aligned}$$

We solve for y to find $y = -1$. Finally, we substitute $y = \mathbf{-1}$ into the first equation and solve for x.

$$\begin{aligned} 5x - 2(\mathbf{-1}) &= 22 \\ 5x &= 20 \\ x &= 4 \end{aligned}$$

The solution of the system is the point $(4, -1)$.

Checkpoint 2.29 QuickCheck 4. What is the name of the smallest integer that two integers, a and b, divide into evenly?

- ⊙ A) The divisor
- ⊙ B) The quotient
- ⊙ C) The lowest common multiple
- ⊙ D) The lowest common factor

Answer. C) ... common multiple

Solution. The lowest common multiple

Checkpoint 2.30 Practice 3. Follow the suggested steps to solve the system by elimination

$$\frac{3}{2}x = y + \frac{13}{2}$$
$$y - 5 = \frac{-7}{3}x$$

First, clear the fractions: multiply the first equation by 2:
__=_____,
and the second equation by 3.
_____=___

- Step 1

  Write each equation in the form $Ax + By = C$.

- Step 2

  Eliminate the y-terms: Multiply each equation by an appropriate constant.

- Step 3

  Add the new equations and solve the result for x.

  $x =$__

- Step 4

  Substitute your value for x into the second equation and solve for y.

  $y =$__

  The solution to the system is ____ (Give the solution as an ordered pair.)

Answer 1. $3x$
Answer 2. $2y + 13$
Answer 3. $3y - 15$
Answer 4. $-7x$
Answer 5. 3
Answer 6. -2
Answer 7. $(3, -2)$
Solution.

$$3x = 2y + 13$$
$$3y - 15 = -7x$$

- Step 1

$$3x - 2y = 13$$
$$7x + 3y = 15$$

- Step 2

$$9x - 6y = 39$$
$$14x + 6y = 30$$

- Step 3

  $x = 3$

- Step 4

  $y = -2$

  The solution to the system is $(3, -2)$.

Which method should you choose to solve a particular system? Both methods work on any linear system, but the substitution method is easier if one of the variables in one of the equations has coefficient 1 or −1.

Inconsistent and Dependent Systems

Recall that a system of two linear equations does not always have a unique solution; it may be inconsistent or dependent. The elimination method will reveal whether the system falls into one of these two cases.

Example 2.31

Solve each system by elimination.

a

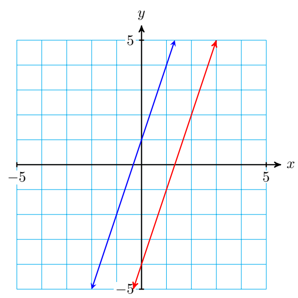
$$\begin{aligned} 3x - y &= 4 \\ -6x + 2y &= 2 \end{aligned}$$

b

$$\begin{aligned} x - 2y &= 3 \\ 2x - 4y &= 6 \end{aligned}$$

Solution.

a

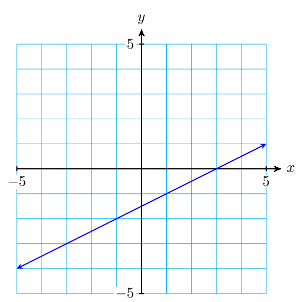


To eliminate the y-terms, we multiply the first equation by 2 and add:

$$\begin{aligned} 6x - 2y &= 8 \\ -6x + 2y &= 2 \\ \hline 0x + 0y &= 10 \end{aligned}$$

Both variables are eliminated, and we are left with the false statement $0 = 10$. There are no values of x or y that will make this equation true, so the system has no solutions. The graph shows that the system is inconsistent.

b



To eliminate the x-terms, we multiply the first equation by −2 and add:

$$\begin{aligned} -2x + 4y &= -6 \\ 2x - 4y &= 6 \\ \hline 0x + 0y &= 0 \end{aligned}$$

We are left with the true but unhelpful equation $0 = 0$. The two equations are in fact equivalent (one is a constant multiple of the other), so the system is dependent. The graph of both equations is shown in the figure.

Inconsistent and Dependent Systems.

When using elimination to solve a system:

1 If combining the two equations results in an equation of the form

$$0x + 0y = k \qquad (k \neq 0)$$

then the system is inconsistent.

2 If combining the two equations results in an equation of the form

$$0x + 0y = 0$$

then the system is dependent.

Checkpoint 2.32 Practice 4. Identify the system as dependent, inconsistent, or consistent and independent

$$x + 3y = 6$$
$$2x - 12 = -6y$$

- ⊙ dependent
- ⊙ inconsistent
- ⊙ consistent and independent

Answer. dependent
Solution. Dependent

Problem Set 2.3

Warm Up

1. Solve the system by **substitution**:

$$2x = y + 7$$
$$2y = 14 - 3x$$

- Step 1

  Choose one of the equations, and solve for one of the variables.

- Step 2

  Substitute this new expression for x or y into the other equation.

- Step 3

  Notice that the new equation has only one variable. Solve the equation.

- Step 4

  Substitute the answer to Step 3 into the result of Step 1 to find the other variable. Write your solution as an ordered pair.

2. Solve the system by **elimination**:

$$3x + 5y = 1$$
$$2x - 3y = 7$$

- Step 1

  We choose to eliminate x. Multiply the first equation by 2 and the second equation by -3. (Why?)

$$\mathbf{2}[3x + 5y = 1] \rightarrow$$

$$\mathbf{-3}[2x - 3y = 7] \rightarrow$$

- Step 2

  The coefficients of x are now opposites. Add the two equations together.

- Step 3

  Solve the new equation for the remaining variable.

- Step 4

  Use substitution to find the other variable. Write your solution as an ordered pair.

Skills Practice

For Problems 3-6, solve the system by substitution or by linear combinations.

3.

$$\begin{aligned} 3m + n &= 7 \\ 2m &= 5n - 1 \end{aligned}$$

4.

$$\begin{aligned} 2r &= s + 7 \\ 2s &= 14 - 3r \end{aligned}$$

5.

$$\begin{aligned} 2u - 3v &= -4 \\ 5u + 2v &= 9 \end{aligned}$$

6.

$$\begin{aligned} 3x + 5y &= 1 \\ 2x - 3y &= 7 \end{aligned}$$

For Problems 7 and 8, clear the fractions in each equation first, then solve the system by substitution or by linear combinations.

7.

$$\begin{aligned} \frac{x}{4} &= \frac{y}{3} - \frac{5}{12} \\ \frac{y}{5} &= \frac{1}{2} - \frac{x}{10} \end{aligned}$$

8.

$$\begin{aligned} \frac{t}{3} &= \frac{w}{3} + 2 \\ \frac{w}{3} &= \frac{t}{6} - 1 \end{aligned}$$

For Problems 9 and 10, use linear combinations to identify each system as dependent, inconsistent, or consistent and independent.

9.

$$\begin{aligned} 2n - 5p &= 6 \\ \frac{15p}{2} + 9 &= 3n \end{aligned}$$

10.

$$\begin{aligned} -3x &= 4y + 8 \\ \frac{1}{2}x + \frac{4}{3} &= \frac{-2}{3}y \end{aligned}$$

Applications

11. Benham will service $2.5x$ copy machines per week if he can charge x dollars per machine, and the public will pay for $350 - 4.5x$ jobs at x dollars per machine.

a Write the supply and demand equations for servicing copy machines.

b Find the equilibrium price and the number of copy machines Benham will service at that price.

12. Delbert answered 13 true-false and 9 fill-in questions correctly on his last test and got a score of 71. If he had answered 9 true-false and 13 fill-ins correctly, he would have made an 83. How many points was each type of problem worth?

13. Paul needs 40 pounds of 48% silver alloy to finish a collection of jewelry. How many pounds of 45% silver alloy should he melt with 60% silver alloy to obtain the alloy he needs?

a Choose variables for the unknown quantities, and fill in the table.

	Pounds	% silver	Amount of silver
First alloy			
Second alloy			
Mixture			

b Write one equation about the amount of alloy Paul needs.

c Write a second equation about the amount of silver in the alloys.

d Solve the system and answer the question in the problem.

14. Amal plans to make 10 liters of a 17% acid solution by mixing a 20% acid solution with a 15% acid solution. How much of each should she use?

a Choose variables for the unknown quantities, and fill in the table.

	Liters	% acid	Amount of acid
First solution			
Second solution			
Mixture			

b Write one equation about the amount of solution Amal needs.

c Write a second equation about the acid in the solution.

d Solve the system and answer the question in the problem.

15. Rani kayaks downstream for 45 minutes and travels a distance of 6000 meters. On the return journey upstream, she covers only 4800 meters in 45 minutes. How fast is the current in the river, and how fast would Rani kayak in still water? (Give your answers in meters per minute.)

a Choose variables for the unknown quantities, and fill in the table.

	Rate	Time	Distance
Downstream			
Upstream			

b Write one equation about the downstream trip.

c Write a second equation about the return trip.

d Solve the system and answer the question in the problem.

16. Because of prevailing winds, a flight from Detroit to Denver, a distance of 1120 miles, took 4 hours on Econoflite, and the return trip took 3.5 hours. What were the speed of the airplane and the speed of the wind?

a Choose variables for the unknown quantities, and fill in the table.

	Rate	Time	Distance
Detroit to Denver			
Denver to Detroit			

b Write one equation about the trip from Detroit to Ddenver.

c Write a second equation about the return trip.

d Solve the system and answer the question in the problem.

17. Geologists can calculate the distance from their seismograph to the epicenter of an earthquake by timing the arrival of the P and S waves. They know that, for this earthquake, P waves travel at about 5.4 miles per second and S waves travel at 3.0 miles per second. If the P waves arrived 3 minutes before the S waves, how far away is the epicenter of the quake?

18. A cup of rolled oats provides 310 calories. A cup of rolled wheat flakes provides 290 calories. A new breakfast cereal combines wheat and oats to provide 302 calories per cup. How much of each grain does 1 cup of the cereal include?

a Choose variables for the unknown quantities, and fill in the table.

	Cups	Calories per cup	Calories
Oat flakes			
Wheat flakes			
Mixture			

b Write one equation about the amounts of each grain.

c Write a second equation about the number of calories.

d Solve the system and answer the question in the problem.

19. Acme Motor Company is opening a new plant to produce chassis for two of its models, a sports coupe and a wagon. Each sports coupe requires a riveter for 3 hours and a welder for 4 hours; each wagon requires a riveter for 4 hours and a welder for 5 hours. The plant has available 120 hours of riveting and 155 hours of welding per day. How many of each model of chassis can it produce in a day?

a Choose variables for the unknown quantities, and fill in the table.

	Sports coupes	Wagons	Total
Hours of riveting			
Hours of welding			

b Write one equation about the hours of riveting.

c Write a second equation about the hours of weldings.

d Solve the system and answer the question in the problem.

20. Carmella has \$1200 invested in two stocks; one returns 8% per year and the other returns 12% per year. The income from the 8% stock is \$3 more than the income from the 12% stock. How much did she invest in each stock?

a Choose variables for the unknown quantities, and fill in the table.

	Principal	Interest Rate	Interest
First stock			
Second stock			
Total			

b Write one equation about the amounts Carmella invested.

c Write a second equation about Carmella's annual interest.

d Solve the system and answer the question in the problem.

Gaussian Reduction

3×3 Linear Systems

A **solution** to an equation in three variables, such as

$$x + 2y - 3z = -4$$

is an **ordered triple** of numbers that satisfies the equation. For example, $(0, -2, 0)$ and $(-1, 0, 1)$ are solutions to the equation above, but $(1, 1, 1)$ is not. You can verify this by substituting the coordinates into the equation to see if a true statement results.

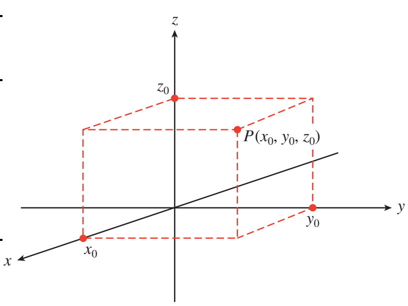
For $(0, -2, 0)$: $\mathbf{0} + 2(\mathbf{-2}) - 3(\mathbf{0}) = -4$ True

For $(-1, 0, 1)$: $\mathbf{-1} + 2(\mathbf{0}) - 3(\mathbf{1}) = -4$ True

For $(1, 1, 1)$: $\mathbf{1} + 2(\mathbf{1}) - 3(\mathbf{1}) = -4$ Not true

A linear equation in three variables has infinitely many solutions.

An ordered triple (x, y, z) can be represented geometrically as a point in space using a three-dimensional Cartesian coordinate system, as shown in the figure. The graph of a linear equation in three variables is a plane. A solution to a **system of three linear equations in three variables** is an ordered triple that satisfies each equation in the system. That triple represents a point where all three planes intersect.



- If the planes intersect in a single point, the system has a unique solution.
- If there is no point that lies on all three planes (for instance, if at least two of the planes are parallel), the system is inconsistent.
- If all three planes are the same, or if they intersect in a line, the system is dependent.

It is impractical to solve 3×3 systems by graphing. Even when technology for producing three-dimensional graphs is available, we cannot read coordinates on such graphs with any confidence. Thus, we will restrict our attention to algebraic methods of solving such systems.

Checkpoint 2.33 QuickCheck 1.

a. A solution to an equation in three variables is an

⊙ A) ordered pair that satisfies the equation.

⊙ B) ordered triple that satisfies the equation

⊙ C) arbitrary number that satisfies the equation

.

b. The graph of a linear equation in three variables is a (☐ point ☐ line ☐ plane ☐ parabola) .

c. A linear 3×3 system has a unique solution if the three planes intersect (☐ a single point ☐ a line ☐ a plane) .

d. Three planes may also intersect (☐ in exacty 2 points ☐ in exactly 3 points ☐ in a line) or (☐ be the same plane ☐ in a parabola ☐ in a circle) .

Answer 1. B) ... the equation

Answer 2. plane

Answer 3. a single point

Answer 4. in a line

Answer 5. be the same plane

Solution.

a. ordered triple that satisfies the equation

b. plane

c. a single point

d. in a line; be the same plane

Back-Substitution

The strategy for solving a 3×3 system is the same as the strategy for 2×2 systems: we would like to reduce the system to an equation in a single variable. Once we have found the value for that variable, we substitute its value into the other equations to find the remaining unknowns.

A special case of this technique is called **back-substitution**. It works when one of the equations involves exactly one variable, and a second equation involves that same variable and just one other variable. A 3×3 system with these properties is said to be in .

Example 2.34

Solve the system

$$\begin{aligned} x + 2y + 3z &= 2 \\ -2y - 4z &= -2 \\ 3z &= -3 \end{aligned}$$

Solution. We begin by solving the third equation to find $z = -1$. Then we substitute $\mathbf{-1}$ for z in the second equation and solve for y.

$$\begin{aligned} -2y - 4(\mathbf{-1}) &= -2 \\ -2y + 4 &= -2 \\ -2y &= -6 \end{aligned}$$

$$y = 3$$

Finally, we substitute $\mathbf{-1}$ for z and $\mathbf{3}$ for y into the first equation to find x.

$$\begin{aligned} x + 2(\mathbf{3}) + 3(\mathbf{-1}) &= 2 \\ x + 6 - 3 &= 2 \\ x &= -1 \end{aligned}$$

The solution is the ordered triple $(-1, 3, -1)$. You should verify that this triple satisfies all three equations of the system.

Checkpoint 2.35 Practice 1. Use back-substitution to solve the system

$$\begin{aligned} 2x + 2y + z &= 10 \\ y - 4z &= 9 \\ 3z &= -6 \end{aligned}$$

Answer. $(5, 1, -2)$

Solution. $(5, 1, -2)$ satisfies all three equations.

Gaussian Reduction

The method for solving a general 3×3 linear system is called **Gaussian reduction**, after the German mathematician Carl Gauss. We use linear combinations to reduce the system to triangular form, and then use back-substitution to find the solutions.

Checkpoint 2.36 QuickCheck 2.

a. The method for solving a 3×3 linear system is called (☐ completing the square. ☐ Gaussian reduction. ☐ induction method.) .

b. A special case of this method is called (☐ back-substitution. ☐ negative reciprocals. ☐ linear regression. ☐ point-slope.) .

c. The special case works on systems

- ⊙ A) when one equation involves exactly one variable, and a second equation involves that same variable and just one other variable.
- ⊙ B) with more equations than unknowns.
- ⊙ C) having only two unknowns.

.

d. To reduce a system to the special form, we use (☐ guess-and-check. ☐ linear combinations. ☐ the discriminant.) .

Answer 1. Gaussian reduction.

Answer 2. back-substitution.

Answer 3. A) when ... other variable.

Answer 4. linear combinations.

Solution.

a. Gaussian reduction

b. back-substitution

c. when one equation involves exactly one variable, and a second equation involves that same variable and just one other variable

d. linear combinations

To obtain the triangular form, we eliminate one of the variables from each of thethree equations by considering them in pairs. This results in a 2×2 system that we can solve using elimination.

Example 2.37

Solve the system:

$$\begin{aligned} x + 2y - 3z &= -4 \qquad (1)\\ 2x - y + z &= 3 \qquad (2)\\ 3x + 2y + z &= 10 \qquad (3) \end{aligned}$$

Solution. We can choose any one of the three variables to eliminate first. For this example, we will eliminate x. Next, we choose two of the equations, say (1) and (2), and use a linear combination: We multiply Equation (1) by -2 and add the result to Equation (2) to produce Equation (4).

$$\begin{aligned} -2x - 4y + 6z &= 8 \qquad (1a)\\ 2x - y + z &= 3 \qquad (2)\\ \hline -5y + 7z &= 11 \qquad (4) \end{aligned}$$

Now we have an equation involving only two variables. But we need *two* equations in two unknowns to find the solution. So we choose a different pair of equations, say (1) and (3), and eliminate x again. We multiply Equation (1) by -3 and add the result to Equation (3) to obtain Equation (5).

$$\begin{aligned} -3x - 6y + 9z &= 12 \qquad (1b)\\ 3x + 2y + z &= 10 \qquad (3)\\ \hline -4y + 10z &= 22 \qquad (5) \end{aligned}$$

We now form a 2×2 system with our new Equations (4) and (5).

$$\begin{aligned} -5y + 7z &= 11 \qquad (4)\\ -4y + 10z &= 22 \qquad (5) \end{aligned}$$

Finally, we eliminate one of the remaining variables to obtain an equation in a single variable. We choose to eliminate y, so we add 4 times Equation (4) to -5 times Equation (5) to obtain Equation (6).

$$\begin{aligned} -20y + 28z &= 44 \qquad (4a)\\ 20y - 50z &= -110 \qquad (5a)\\ \hline -22z &= -66 \qquad (6) \end{aligned}$$

Now we are ready to form a triangular system. We choose one of the original equations (in three variables), one of the equations from our 2×2 system, and our final equation in one variable. We choose Equations (1), (4), and (6).

$$\begin{aligned} x + \quad 2y - \quad 3z &= -4 \qquad (1) \\ -5y + \quad 7z &= 11 \qquad (4) \\ -22z &= -66 \qquad (6) \end{aligned}$$

This new system has the same solutions as the original system, and we can solveit by back-substitution. We first solve Equation (6) to find $z = 3$. Substituting 3 for z in Equation (4), we find

$$\begin{aligned} -5y + 7(\mathbf{3}) &= 11 \\ -5y + 21 &= 11 \\ -5y &= -10 \end{aligned}$$

So $y = 2$. Finally, we substitute **3** for z and **2** for y into Equation (1) to find

$$\begin{aligned} x + 2(\mathbf{2}) - 3(\mathbf{3}) &= -4 \\ x + 4 - 9 &= -4 \\ x &= 1 \end{aligned}$$

The solution to the system is the ordered triple $(1, 2, 3)$. You should verify that this triple satisfies all three of the original equations.

We summarize the method for solving a 3×3 linear system as follows.

Steps for Solving a 3×3 Linear System.

1 Clear each equation of fractions and put it in standard form.

2 Choose two of the equations and eliminate one of the variables by forming a linear combination.

3 Choose a different pair of equations and eliminate the *same* variable.

4 Form a 2×2 system with the equations found in steps (2) and (3). Eliminate one of the variables from this 2×2 system by using a linear combination.

5 Form a triangular system by choosing among the previous equations. Use back-substitution to solve the triangular system.

Checkpoint 2.38 QuickCheck 3. Suppose you eliminate y from two equations as Step 2 of Gaussian reduction. What should you do for Step 3?

- ⊙ Form a triangular system.
- ⊙ Eliminate either x or z.
- ⊙ Eliminate y again from a different pair of equations.
- ⊙ Substitute the value for y into the equations.

Answer. Choice 3

Solution. Eliminate y again from a different pair of equations.

Checkpoint 2.39 Practice 2. Use Gaussian reduction to solve the system

$$\begin{aligned} x - 2y + z &= -1 && (1) \\ \frac{2}{3}x + \frac{1}{3}y - z &= -1 && (2) \\ 3 + 3y - 2z &= 10 && (3) \end{aligned}$$

Follow the steps:

- Step 1: Clear the fractions from Equation (2).
- Step 2: Eliminate z from Equations (1) and (2).
- Step 3: Eliminate z from Equations (1) and (3).
- Step 4: Eliminate x from your new 2×2 system.
- Step 5: Form a triangular system and solve by back-substitution.

Answer. $(2, 2, 1)$
Solution. $(2, 2, 1)$

Inconsistent and Dependent Systems

If, at any step in forming linear combinations, we obtain an equation of the form

$$0x + 0y + 0z = k, \qquad (k \neq 0)$$

then the system is inconsistent and has no solution. If we don't obtain such an equation, but we do obtain one of the form

$$0x + 0y + 0z = 0$$

then the system is dependent and has infinitely many solutions.

Example 2.40

Solve the systems.

a
$$\begin{aligned} 3x + y - 2z &= 1 && (1) \\ 6x + 2y - 4z &= 5 && (2) \\ 2x - y + 3z &= -1 && (3) \end{aligned}$$

b
$$\begin{aligned} -x + 3y - z &= -2 && (1) \\ 2x + y - 4z &= -1 && (2) \\ 2x - 6y + 2z &= 4 && (3) \end{aligned}$$

Solution.

a To eliminate y from Equations (1) and (2), we multiply Equation (1) by -2 and add the result to Equation (2).

$$\begin{aligned} -6x - 2y + 4z &= -2 \\ 6x + 2y - 4z &= \underline{\ 5} \\ \hline 0x + 0y + 0z &= \ 3 \end{aligned}$$

Because the resulting equation has no solution, the system is *inconsistent*.

b To eliminate x from Equations (1) and (3), we multiply Equation (1) by 2 and add Equation (3).

$$-2x + 6y - 2z = -4$$

$$\begin{aligned} \underline{2x - 6y + 2z} &= \underline{\ 4} \\ 0x + 0y + 0z &= \ 0 \end{aligned}$$

Because the resulting equation vanishes, the system is dependent and has infinitely many solutions.

Checkpoint 2.41 Practice 3. Decide whether the system is inconsistent, dependent, or consistent andindependent.

$$\begin{aligned} x + 3y - z &= 4 \\ -2x - 6y + 2z &= 4 \\ x + 2y - z &= 3 \end{aligned}$$

- ⊙ Inconsistent
- ⊙ Dependent
- ⊙ Consistent and independent

Answer. Inconsistent
Solution. Inconsistent

Checkpoint 2.42 QuickCheck 4. True or false.

a. If a system is dependent, it has no solutions. (☐ True ☐ False)

b. The equation $0x + 0y + 0z = 0$ has no solution (☐ True ☐ False)

c. If a 3×3 system is dependent, all three equations are the same.. (☐ True ☐ False)

d. Gaussian reduction will reveal whether a system is dependent or inconsistent. (☐ True ☐ False)

Answer 1. False
Answer 2. False
Answer 3. False
Answer 4. True
Solution.

a. False

b. False

c. False

d. True

Applications

Here are some problems that can be modeled by a system of three linear equations.

Example 2.43

One angle of a triangle measures 4° less than twice the second angle, and the third angle is 20° greater than the sum of the first two. Find the measure of each angle.

Solution.

- *Step 1:*.

  We represent the measure of each angle by a separate variable.

First angle:	x
Second angle:	y
Third angle:	z

- *Step 2:*.

  We write the conditions stated in the problem as three equations.

x is 4° less than 2 times y :	$x = 2y - 4$
z is 20° more than $x + y$:	$z = x + y + 20$
the sum of the angles of a triangle is 180° :	$x + y + z = 180$

- *Step 3:*.

  We follow the steps for solving a 3×3 linear system.

  1. We write the three equations in standard form.

$$\begin{aligned} x - 2y \quad &= -4 \qquad (1) \\ x + \ y - z &= -20 \qquad (2) \\ x + \ y + z &= 180 \qquad (3) \end{aligned}$$

2–3. Because Equation (1) has no z-term, it will be most efficient to eliminate z from Equations (2) and (3). We add these two equations.

$$\begin{aligned} x + \ y - \ z &= -20 \qquad (2) \\ \underline{x + \ y + \ z} &= \underline{180} \qquad (3) \\ 2x + 2y \quad &= 160 \qquad (4) \end{aligned}$$

4. We form a 2×2 system from Equations (1) and (4). We add the two equations to eliminate the variable y, yielding

$$\begin{aligned} x - 2y &= -4 \qquad (1) \\ \underline{2x + 2y} &= \underline{160} \qquad (4) \\ 3x \quad &= 156 \qquad (5) \end{aligned}$$

5. We form a triangular system using Equations (2), (1), and (5). We use back-substitution to complete the solution.

$$\begin{aligned} x + \ y + z &= 180 \qquad (2) \\ x - 2y \quad &= -4 \qquad (1) \\ 3x \quad &= 156 \qquad (5) \end{aligned}$$

We divide both sides of Equation (5) by 3 to find $x = 52$. We substitute 52 for x in Equation (1) and solve for y to find

$$\begin{aligned} \mathbf{52} - 2y &= -4 \\ y &= 28 \end{aligned}$$

We substitute **52** for x and **28** for y in Equation (3) to find

$$\begin{aligned} \mathbf{52} + \mathbf{28} + z &= 180 \\ z &= 100 \end{aligned}$$

- *Step 4:.*

  The angles measure 52°, 28°, and 100°.

Problem Set 2.4

Warm Up

For Problems 1 and 2, solve the system by elimination.

1.

$$\begin{aligned} 4x - 3 &= 3y \\ 25 + 5x &= -2y \end{aligned}$$

2.

$$\begin{aligned} \frac{2x}{3} + \frac{8y}{9} &= \frac{4}{3} \\ \frac{x}{2} + 2 &= \frac{y}{3} \end{aligned}$$

3. Karen has \$2000, part of it invested in bonds paying 10%, and the rest in a certificate account at 8%. Her annual income from the two investments is \$184. How much did Karen invest at each rate?

a Choose variables for the unknown quantities, and fill in the table.

	Principal	Interest rate	Interest
Bonds			
Certificate			
Total		——	

b Write one equation about the amount Karen invested.

c Write a second equation about Kaaren's annual interest.

4. The pharmacist at Glenoaks Hospital was asked to supply 10 liters of a 20% solution of iodine. She has on hand iodine solutions in 15% strength and 40% strength. How much of each should she mix to make the required solution?

a Choose variables for the unknown quantities, and fill in the table.

	Liters	Strength	Amount of iodine
15% Solution			
40% solution			
20% solution			

b Write one equation about the number of liters of solution.

c Write a second equation about the amount of iodine.

Skills Practice

For Problems 5 and 6, use back-substitution to solve the system.

5. $$\begin{aligned} 2x + 3y - z &= -7 \\ y - 2z &= -6 \\ 5z &= 15 \end{aligned}$$

6. $$\begin{aligned} 2x + z &= 5 \\ 3y + 2z &= 6 \\ 5x &= 20 \end{aligned}$$

For Problems 7–12, use Gaussian reduction to solve the system.

7. $$\begin{aligned} x + y + z &= 0 \\ 2x - 2y + z &= 8 \\ 3x + 2y + z &= 2 \end{aligned}$$

8. $$\begin{aligned} x - 2y + 4z &= -3 \\ 3x + y - 2z &= 12 \\ 2x + y - z &= 11 \end{aligned}$$

9. $$\begin{aligned} 3x - 4y + 2z &= 20 \\ 4x + 3y - 3z &= -4 \\ 2x - 5y + 5z &= 24 \end{aligned}$$

10. $$\begin{aligned} 4x + z &= 3 \\ 2x - y &= 2 \\ 3y + 2z &= 0 \end{aligned}$$

11. $$\begin{aligned} 4x + 6y + 3z &= -3 \\ 2x - 3y - 2z &= 5 \\ -6x + 6y + 2z &= -5 \end{aligned}$$

12. $$\begin{aligned} x - \frac{1}{2}y - \frac{1}{2}z &= 4 \\ x - \frac{3}{2}y - 2z &= 3 \\ \frac{1}{4}x + \frac{1}{4}y - \frac{1}{4}z &= 0 \end{aligned}$$

For Problems 13 and 14, when you decide which variable to eliminate first, take advantage of the fact that one of the variables is already missing from each equation.

13. $$\begin{aligned} x &= -y \\ x + z &= \frac{5}{6} \\ y - 2z &= -\frac{7}{6} \end{aligned}$$

14. $$\begin{aligned} x &= y + \frac{1}{2} \\ y &= z + \frac{5}{4} \\ 2z &= x - \frac{7}{4} \end{aligned}$$

For Problems 15 and 16, decide whether the system is inconsistent or dependent.

15. $$\begin{aligned} 3x - 2y + z &= 6 \\ 2x + y - z &= 2 \\ 4x + 2y - 2z &= 3 \end{aligned}$$

16. $$\begin{aligned} x &= 2y - 7 \\ y &= 4z + 3 \\ x - 8z &= -1 \end{aligned}$$

Applications

Use a system of equations to solve Problems 17-22.

17. The perimeter of a triangle is 155 inches. Side x is 20 inches shorter than side y, and side y is 5 inches longer than side z. Find the lengths of the sides of the triangle.

18. One angle of a triangle measures 10° more than a second angle, and the third angle is 10° more than six times the measure of the smallest angle. Find the measure of each angle.

19. The Java Shoppe sells a house brand of coffee that is only 2.25% caffeine for \$6.60 per pound. The house brand is a mixture of Colombian coffee that sells for \$6 per pound and is 2% caffeine, French roast that sells for \$7.60 per pound and is 4% caffeine, and Sumatran that sells for \$6.80 per pound and is 1% caffeine. How much of each variety is in a pound of house brand?

20. Vegetable Medley is made of carrots, green beans, and cauliflower. The package says that 1 cup of Vegetable Medley provides 29.4 milligrams of vitamin C and 47.4 milligrams of calcium. One cup of carrots contains 9 milligrams of vitamin C and 48 milligrams of calcium. One cup of green beans contains 15 milligrams of vitamin C and 63 milligrams of calcium. One cup of cauliflower contains 69 milligrams of vitamin C and 26 milligrams of calcium. How much of each vegetable is in 1 cup of Vegetable Medley?

21. Reliable Auto Company wants to ship 1700 Status Sedans to three major dealers in Los Angeles, Chicago, and Miami. From past experience Reliable figures that it will sell twice as many sedans in Los Angeles as in Chicago. It costs \$230 to ship a sedan to Los Angeles, \$70 to Chicago, and \$160 to Miami. If Reliable Auto has \$292,000 to pay for shipping costs, how many sedans should it ship to each city?

22. A farmer has 1300 acres on which to plant wheat, corn, and soybeans. The seed costs \$6 for an acre of wheat, \$4 for an acre of corn, and \$5 for an acre of soybeans. An acre of wheat requires 5 acre-feet of water during the growing season, an acre of corn requires 2 acre-feet, and an acre of soybeans requires 3 acre-feet. If the farmer has \$6150 to spend on seed and can count on 3800 acre-feet of water, how many acres of each crop should she plant in order to use all her resources?

Linear Inequalities in Two Variables

Graphs of Inequalities in Two Variables

Definition 2.44

A **solution** to an inequality in two variables is an ordered pair of numbers that satisfies the inequality.

Example 2.45

Find a solution to the inequality $x + y \geq 10,000$ for $x = 2000$

Solution. For $x = 2000$, we have

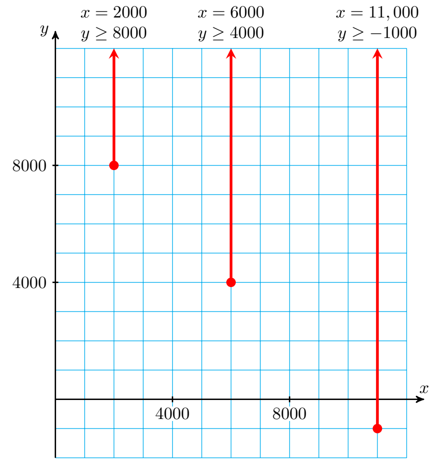
$$2000 + y \geq 10,000 \qquad \text{or} \qquad > y \geq 8000$$

You can check that any value of y greater than or equal to 8000 provides a solution when $x = 2000$. For example, $(2000, 9000)$ is a solution because $2000 + 9000 > 10,000$.

We can gain some insight into the nature of the solutions if we rewrite the inequality as

$$y \geq -x + 10,000$$

This inequality says that for each x-value, we must choose points with y-values greater than or equal to $-x + 10,000$. As we saw in the Example above, when $x = 2000$, the solutions have y-values greater than or equal to 8000. Solutions for $x = 2000$, $x = 6000$, and $x = 11,000$ are shown in the figure. Do you see a pattern emerging?



Checkpoint 2.46 Practice 1.

a. Find one y-value that satisfies the inequality $y - 3x < 6$ for each of the x-values in the table.

x	1	0	-2
y	—	—	—

b. Graph the line $y - 3x = 6$. Then plot your solutions from part (a) on the same grid.

Answer 1. 5

Answer 2. 0

Answer 3. -3

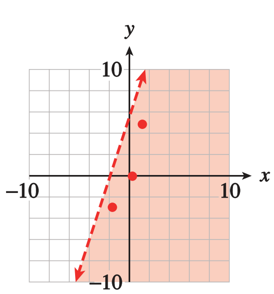
Solution.

a.

x	1	0	-2
y	**5**	**0**	**-3**

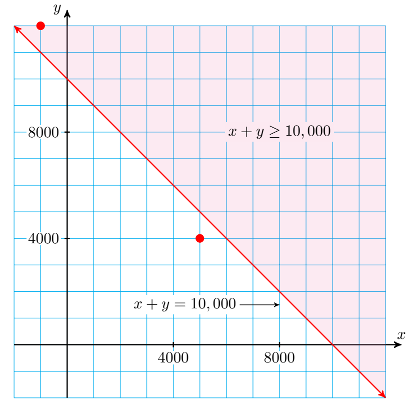
(Many answers are possible.)

b. A graph is below.



The graph of the inequality must show all the points whose coordinates are solutions. We found some solutions to the inequality $x+y \geq 10,000$ in the Example above by choosing a value for x and solving for y. A more efficient way to find all the solutions of the inequality is to start with the graph of the corresponding equation

$$y = -x + 10,000$$



The graph is a straight line with slope -1. Any point *above* this line has a y-coordinate greater than $-x + 10,000$ and hence satisfies the inequality. Thus, the graph of the inequality includes all the points on or above the line $y = -x + 10,000$, as shown by the shaded region in the figure. Of course, the shaded points are also solutions to the original inequality, $x + y \geq 10,000$.

For example, the point $(-1000, 12,000)$ is a solution, because

$$-1000 + 12,000 \geq 10,000$$

On the other hand, the point $(5000, 4000)$ does not lie in the shaded region because its coordinates do not satisfy the inequality.

Linear Inequalities

A **linear inequality** can be written in the form

$$ax + by + c \leq 0 \qquad \text{or} \qquad ax + by + c \geq 0$$

The solutions consist of the line $ax + by + c = 0$ and a **half-plane** on one side of that line. We shade the half-plane to show that all its points are included in the solution set. To decide which side of the line to shade, we can solve the inequality for y in terms of x. If we obtain

$$y \geq mx + b \qquad (\text{ or } y > mx + b)$$

then we shade the half-plane *above* the line. If the inequality is equivalent to

$$y \leq mx + b \qquad (\text{ or } y < mx + b)$$

then we shade the half-plane *below* the line.

Example 2.47

Graph $4x - 3y \geq 12$

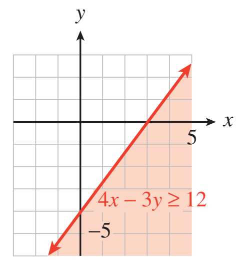
Solution. We solve the inequality for y.

$$4x - 3y \geq 12 \qquad \text{Subtract } 4x \text{ from both sides.}$$
$$-3y \geq -4x + 12 \qquad \text{Divide both sides by } -3.$$
$$y \leq \frac{4}{3}x - 4$$

We graph the corresponding line

$$y = \frac{4}{3}x - 4$$

with y-intercept is -4 and slope $m = \frac{4}{3}$. Finally, we shade the half-plane below the line. The completed graph is shown below.



Caution 2.48 Be careful when isolating y. We must remember to reverse the direction of the inequality whenever we multiply or divide by a negative number. For example, the inequality

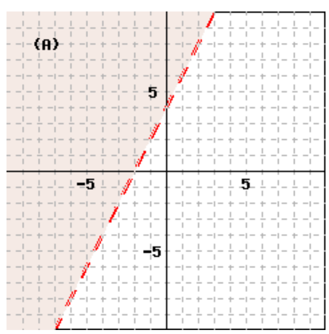
$$-2y > 6x - 8$$

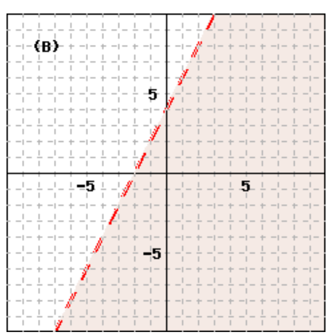
is equivalent to

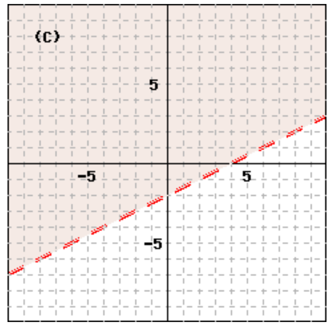
$$y < -3x + 4$$

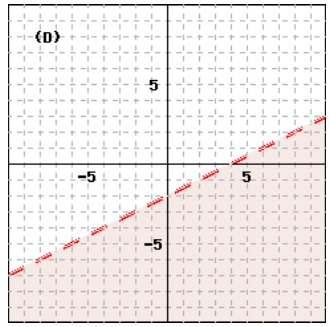
An inequality that uses $>$ or $<$ instead of $\geq$ or $\leq$ is called **strict**. The graph of a strict inequality includes only the half-plane and not the line. In that case we use a dashed line for the graph of the equation $ax + by + c = 0$ to show that it is not part of the solution.

Checkpoint 2.49 **Practice 2.** Graph the solutions of the inequality $4x - 2y < -8$.









Select the appropriate graph.

⊙ (A)

⊙ (B)

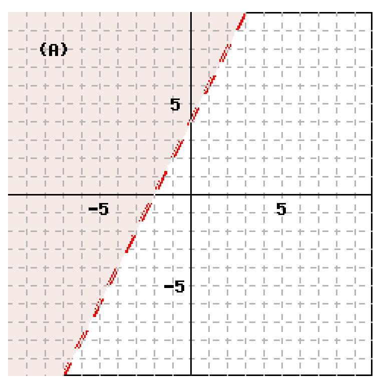
⊙ (C)

⊙ (D)

⊙ none of the above

Answer. (A)

Solution.



Checkpoint 2.50 QuickCheck 1.

a. The solutions of a linear inequality in two variables consist of a boundary line and a (☐ single point ☐ quadrant ☐ semicircle ☐ half-plane) .

b. An inequality that uses $>$ or $<$ instead of $\geq$ or $\leq$ is called (☐ strict ☐ non-strict ☐ compound ☐ conditional) .

c. If we multiply or divide an inequality by a negative number we must

(☐ reverse ☐ simplify ☐ copy ☐ obey) the direction of the inequality symbol.

d. The shaded region shows all the (☐ planes ☐ solutions ☐ constants ☐ slopes) of the inequality.

Answer 1. half-plane

Answer 2. strict

Answer 3. reverse

Answer 4. solutions

Solution.

a. half-plane

b. strict

c. reverse

d. solutions

Using a Test Point

A second method for graphing inequalities does not require us to solve for y. Once we have graphed the boundary line, we can decide which half-plane to shade by using a **test point**. The test point can be any point that is not on the boundary line itself.

Example 2.51

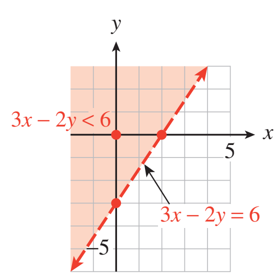
Graph the solutions of the inequality $3x - 2y < 6$

Solution. First, we graph the line $3x - 2y = 6$, as shown below. We will use the intercept method. The intercepts are $(2, 0)$ and $(0, -3)$, so we sketch the boundary line through those points.

Next, we choose a test point. Because $(0, 0)$ does not lie on the line, we choose it as our test point. We substitute the coordinates of the test point into the inequality to obtain

$$3(\mathbf{0}) - 2(\mathbf{0}) < 6$$

Because this is a true statement, $(0, 0)$ is a solution of the inequality. Since all the solutions lie on the same side of the boundary line, we shade the half-plane that contains the test point. In this example, the boundary line is a dashed line because the original inequality was strict.



Here is a summary of our test point method for graphing inequalities.

To Graph an Inequality Using a Test Point:.

1 Graph the corresponding equation to obtain the boundary line.

2 Choose a test point that does not lie on the boundary line.

3 Substitute the coordinates of the test point into the inequality.

a If the resulting statement is true, shade the half-plane that includes the test point.

b If the resulting statement is false, shade the half-plane that does not include the test point.

If the inequality is strict, make the boundary line a dashed line.

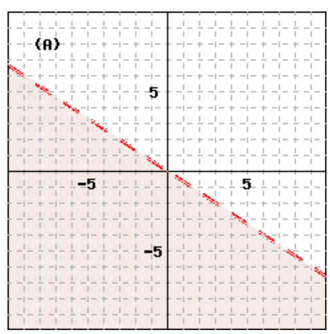
We can choose any point for the test point as long as it does not lie on the boundary line. We chose $(0, 0)$ in Example 2.51, p. 131 because the coordinates are easy to substitute into the inequality. If the test point is a solution, we shade the half-plane including that point. If the test point is not a solution, we shade the other half-plane. For example, suppose we had chosen $(5, 0)$ as the test point in Example 2.51, p. 131. When we substitute its coordinates into the inequality, we find

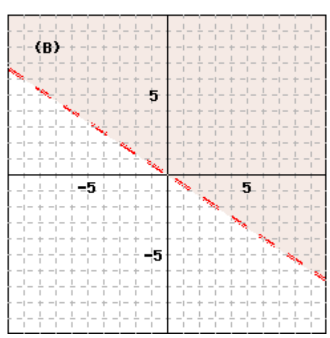
$$3(5) - 2(0) < 6$$

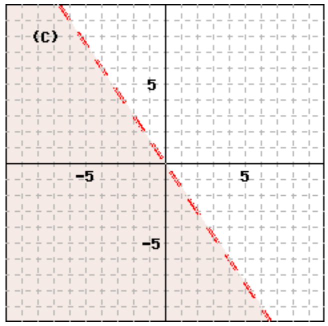
which is a false statement. Thus, $(5, 0)$ is not a solution to the inequality, so the solutions must lie on the other side of the boundary line. Note that using $(5, 0)$ as the test point gives us the same solutions we found in Example 2.51, p. 131.

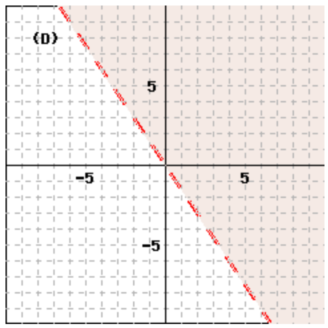
Checkpoint 2.52 Practice 3. Graph the solutions of the inequality $y > \frac{-3}{2}x$

1. Graph the line $y = \frac{-3}{2}x$. (Use the slope-intercept method.)
2. Choose a test point. (Do not choose $(0, 0)$!)
3. Decide which side of the line to shade. (☐ Above the line ☐ Below the line)
4. Should the boundary line be dashed or solid? (☐ Dashed ☐ Solid)









Select the appropriate graph.

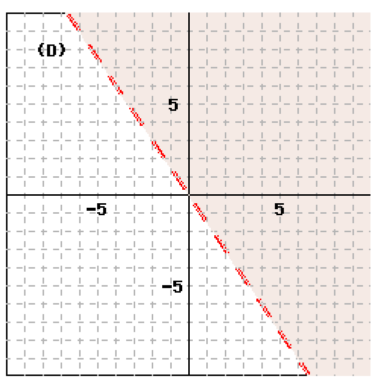
- ⊙ (A)
- ⊙ (B)
- ⊙ (C)
- ⊙ (D)
- ⊙ none of the above

Answer 1. Above the line

Answer 2. Dashed

Answer 3. (D)

Solution.



Caution 2.53 We cannot choose a test point that lies on the boundary line. In Practice 3 above, we cannot use $(0,0)$ as a test point, because it lies on the line $y = \frac{-3}{2}x$. In this case, we should choose some other point, such as $(0,1)$ or $(1,2)$.

Recall that the equation of a vertical line has the form

$$x = k$$

where k is a constant, and a horizontal line has an equation of the form

$$y = k$$

Similarly, the inequality $x \geq k$ may represent the inequality in two variables

$$x + 0y \geq k$$

Its graph is then a region in the plane.

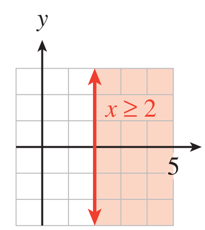
Example 2.54

Graph $x \geq 2$ in the plane.

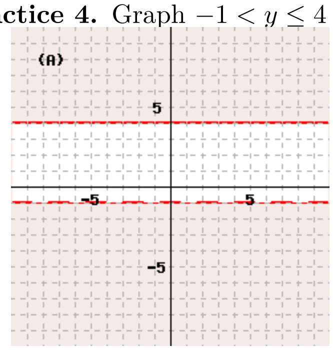
Solution. First, we graph the equation $x = 2$; its graph is a vertical line. Because the origin does not lie on this line, we can use it as a test point. Substitute 0 for x (there is no y) into the inequality to obtain

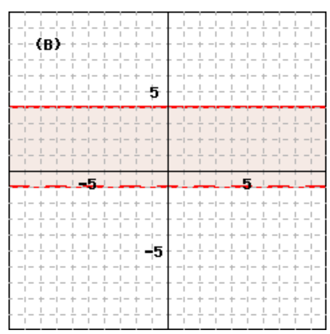
$$0 \geq 2$$

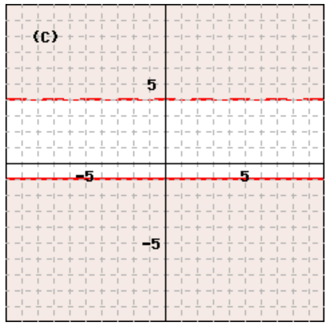
Since this is a false statement, we shade the half-plane that does not contain the origin. We see in the figure below that the graph of the inequality contains all points whose x-coordinates are greater than or equal to 2.

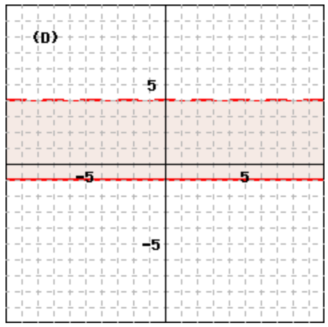


Checkpoint 2.55 **Practice 4.** Graph $-1 < y \leq 4$ in the plane.









Select the appropriate graph.

⊙ (A)

⊙ (B)

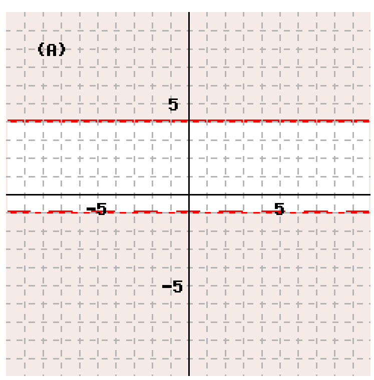
⊙ (C)

⊙ (D)

⊙ none of the above

Answer. (B)

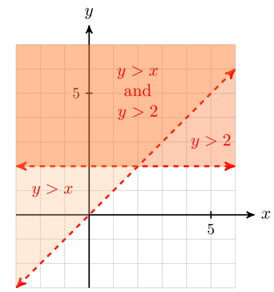
Solution.



Systems of Inequalities

Some applications are best described by a system of two or more inequalities. The solutions to a system of inequalities include all points that are solutions to each inequality in the system. The graph of the system is the intersection of the shaded regions for each inequality in the system. For example, the figure at right shows the solutions of the system

$$y > x \qquad \text{and} \qquad y > 2$$



Example 2.56

Laura is a finicky eater, and she dislikes most foods that are high in calcium. Her morning cereal satisfies some of her calcium requirements, but she needs an additional 500 milligrams of calcium, which she will get from a combination of broccoli, at 160 milligrams per serving, and zucchini, at 30 milligrams per serving. Draw a graph representing the possible combinations of broccoli and zucchini that fulfill Laura's calcium requirements.

Solution.

- *Step 1:*.

  Number of servings of broccoli: x

  Number of servings of zucchini: y

- *Step 2:*.

  To get enough calcium, Laura must choose x and y so that

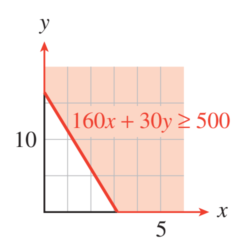
  $$160x + 30y \geq 500$$

  It makes no sense to consider negative values of x or of y, since Laura cannot eat a negative number of servings. Thus, we have two more inequalities to satisfy:

  $$x \geq 0 \qquad \text{and} \qquad y \geq 0$$

- *Step 3:*.

  We graph all three inequalities on the same axes. The inequalities $x \geq 0$ and $y \geq 0$ restrict the solutions to lie in the first quadrant. The solutions common to all three inequalities are shown at right.



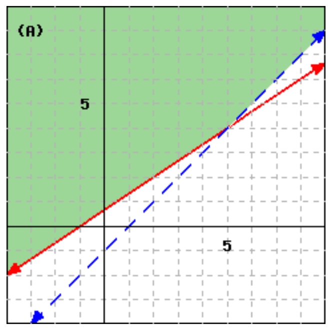
- *Step 4:*.

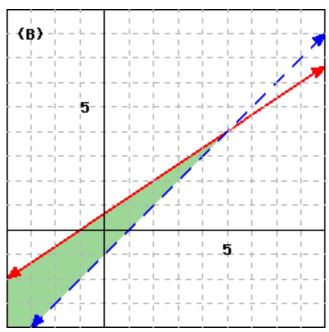
  Laura can choose any combination of broccoli and zucchini represented by points in the shaded region. For example, the point $(3, 1)$ is a solution to the system of inequalities, so Laura could choose to eat 3 servings of broccoli and 1 serving of zucchini.

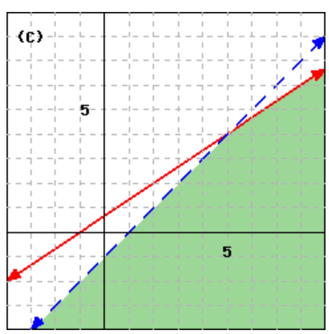
Checkpoint 2.57 Practice 5. Use the following steps to graph the solutions of the system

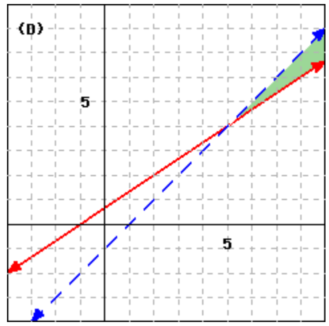
$$\begin{aligned} 3y - 2x &\leq 2 \\ y &> x - 1 \end{aligned}$$

1. Graph the boundary line $3y - 2x = 2$.
2. Lightly shade the solutions of the inequality $3y - 2x \leq 2$.
3. Graph the boundary line $y = x - 1$.
4. Lightly shade the solutions of $y > x - 1$.
5. Shade the intersection of the two solutions sets.







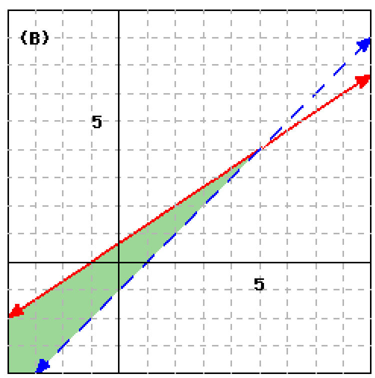


Which of the above is the best match for the solution to the system of inequalities?

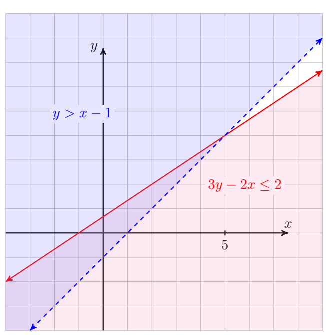
- ⊙ (A)
- ⊙ (B)
- ⊙ (C)
- ⊙ (D)

Answer. (B)

Solution.



A graph is also shown below.



To describe the solutions of a system of inequalities, it is useful to locate the **vertices**, or corner points, of the boundary.

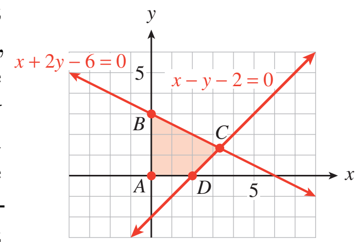
Example 2.58

Graph the solution set of the system below and find the coordinates of its vertices.

$$\begin{aligned} x - y - 2 &\le 0 \\ x + 2y - 6 &\le 0 \\ x \ge 0, \qquad y &\le 0 \end{aligned}$$

Solution.

The last two inequalities, $x \ge 0$ and $y \ge 0$, restrict the solutions to the first quadrant. We graph the line $x-y-2=0$, and use the test point $(0,0)$ to shade the half-plane including the origin. Finally we graph the line $x-2y-6=0$ and again use the test point $(0,0)$ to shade the half-plane below the line. The intersection of the shaded regions is shown at right.



To find the coordinates of the vertices A, B, C, and D, we solve simultaneously the equations of the two lines that intersect at each vertex. Thus,

For A, we solve the system $\begin{aligned} x &= 0 \\ y &= 0 \end{aligned}$ to find $(0,0)$

For B, we solve the system $\begin{aligned} x &= 0 \\ x + 2y &= 6 \end{aligned}$ to find $(0,3)$

For C, we solve the system $\begin{aligned} x + 2y &= 6 \\ x - y &= 2 \end{aligned}$ to find $\left(\dfrac{10}{3}, \dfrac{4}{3}\right)$

For D, we solve the system $\begin{aligned} y &= 0 \\ x - y &= 2 \end{aligned}$ to find $(2,0)$

The vertices are the points $(0,0)$, $(0,3)$, $\left(\dfrac{10}{3}, \dfrac{4}{3}\right)$, and $(2,0)$.

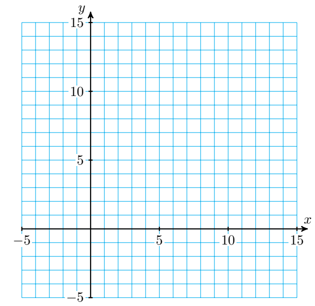
Problem Set 2.5

Warm Up

1. Ivana owns two hotels, that earn annual profits of x and y dollars, respectively. She would like her total annual profit from the two hotels to be (exactly) \$10,000.

a Fill in the table with some possible values for x and y, in thousands of dollars. (Note that it is possible for one of the hotels to sustain a loss, or negative profit.)

x				
y				



b Plot the ordered pairs from your table on the grid.

c Write an equation that describes this problem.

d Graph all solutions of the equation.

2. Now suppose that Ivana would like her profit to be greater than \$10,000.

a List four pairs of values for x and y that achieve this goal.

x				
y				

b Plot the four points on the grid in Problem 1.

c Write an inequality for x and y that describes this situation.

d Shade in the region that contains all solutions of the inequality. Note that all the solutions of the inequality lie on one side of the line.

3.

a Explain how the graph of the equation $x + y = 10,000$ in Problem 1 is different from the graph of the inequality $x + y > 10,000$ in Problem 2.

b Describe the graph of the inequality $x + y \geq 10,000$.

4. Graph the solutions of the inequality

$$3x - y \geq -2$$

a Write the equation $3x - y = -2$ in slope-intercept form

b Use the slope-intercept method to graph the line.

$b =$

$m = \dfrac{\Delta y}{\Delta x} =$

c Use a test point to locate the solutions of the inequality. We can use any point for a test point, as long as it does not lie on the line! We'll use $(1, -4)$. Plot this point on your graph. Use algebra to decide whether $(1, -4)$ is a solution of the inequality $3x - y \leq -2$.

d Which side of the line includes all the solutions of the inequality? Shade that side of the line.

e Can you suggest an easier test point instead of $(1, -4)$?

Skills Practice

For Problems 5–14, graph the inequality.

5. $y > 2x + 4$

6. $y > \frac{4}{3}x$

7. $y < 9 - 3x$

8. $2x + 5y \geq 10$

9. $x + 4y \geq -6$

10. $x > 2y - 5$

11. $y < \frac{1}{2}x$

12. $0 \geq x + 3y$

13. $x \geq -3$

14. $-1 < y \leq 4$

For Problems 15 and 16, graph the system of inequalities.

15. $y > 2, \quad x \geq -2$

16. $y \leq -x, \quad y < 2$

For Problems 17 and 18:

a Graph the system of inequalities

b Find the coordinates of the vertices

17. $2x + 3y - 6 < 0$
$x \geq 0,\ y \geq 0$

18. $3x + 2y < 6$
$x \geq 0,\ y \geq 0$

Graph each system of inequalities.

19. $x + y \leq 6$
$x + y \geq 4$

20. $2x - y \leq 4$
$x + 2y > 6$

Graph each system of inequalities and find the coordinates of the vertices.

21. $5y - 3x \leq 15$
$x + y \leq 11$
$x \geq 0,\ y \geq 0$

22. $2y \leq x$
$2x \leq y + 12$
$x \geq 0,\ y \geq 0$

23. $x + y \geq 3$
$2y \leq x + 8$
$2y + 3x \leq 24$
$x \geq 0,\ y \geq 0$

24. $3y - x \geq 3$
$y - 4x \geq -10$
$y - 2 \leq x$
$x \geq 0,\ y \geq 0$

Applications

For Problems 25–28, graph the set of solutions to the problem. For each system, $x \geq 0$ and $y \geq 0$.

25. Vassilis plans to invest at most $10,000 in two banks. One bank pays 6% annual interest and the other pays 5% annual interest. Vassilis wants at least $540 total annual interest from his two investments. Write a system of four inequalities for the amount Vassilis can invest in the two accounts, and graph the system.

26. Jeannette has 180 acres of farmland for growing wheat or soy. She can get a profit of $36 per acre for wheat and $24 per acre for soy. She wants to have a profit of at least $5400 from her crops. Write a system of four inequalities for the number of acres she can use for each crop, and graph the solutions.

27. Gary's pancake recipe includes corn meal and whole wheat flour. Corn meal has 2.4 grams of linoleic acid and 2.5 milligrams of niacin per cup. Whole wheat flour has 0.8 grams of linoleic acid and 5 milligrams of niacin per cup. These two ingredients should not exceed 3 cups total. The mixture should provide at least 3.2 grams of linoleic acid and at least 10 milligrams of niacin. Write a system of five inequalities for the amount of corn meal and the amount of whole wheat flour Gary can use, and graph the solutions.

28. Cho and his brother go into business making comic book costumes. They need 1 hour of cutting and 2 hours of sewing to make a Batman costume. They need 2 hours of cutting and 1 hour of sewing to make a Wonder Woman costume. They have available at most 10 hours per day for cutting and at most 8 hours per day for sewing. They must make at least one costume each day to stay in business. Write a system of five inequalities for the number of each type of costume Cho can make, and graph the solutions.

Chapter Summary and Review

Glossary

- scatterplot
- regression line
- interpolation
- extrapolation
- linear system
- solution of a system
- ordered pair
- inconsistent system
- dependent system
- equilibrium price
- substitution method
- elimination method
- linear combination
- ordered triple
- back-substitution
- triangular form
- Gaussian reduction

Key Concepts

1 We can approximate a linear pattern in a **scatterplot** using a **regression line**.

2 We can use **interpolation** or **extrapolation** to make estimates and predictions.

3 If we extrapolate too far beyond the known data, we may get unreasonable results.

4 A solution to a 2×2 linear system is an ordered pair that satisfies both equations.

5 A solution to a 2×2 linear system is a point where the two graphs intersect.

6 The graphs of the equations in an **inconsistent system** are parallel lines and hence to do not intersect.

7 The graphs of the two equations in a **dependent system** are the same line.

8 If a company's revenue exactly equals its costs (so that their profit is zero), we say that the business venture will **break even**.

9 For solving a 2×2 linear system, the **substitution method** is easier if one of the variables in one of the equations has a coefficient of 1 or -1.

10 If a **linear combination** of the equations in a system results in an equation of the form

$$0x + 0y = kk \qquad (k \neq 0)$$

then the system is **inconsistent**. If an equation of the form

$$0x + 0y = 0$$

results, then the system is **dependent**

11 The **point-slope** form is useful when we know the rate of change and one point on the line.

12 The solutions of the **linear inequality**

$$ax + by + c \leq 0 \qquad \text{or} \qquad ax + by + c \geq 0$$

consists of the line $ax + by + c = 0$ and a **half-plane** on one side of that line.

To Graph an Inequality Using a Test Point.

13

1 Graph the corresponding equation to obtain the boundary line.

2 Choose a test point that does not lie on the boundary line.

3 Substitute the coordinates of the test point into the inequality.

a If the resulting statement is true, shade the half-plane that includes the test point.

b If the resulting statement is false, shade the half-plane that does not include the test point.

4 If the inequality is strict, make the boundary line a dashed line.

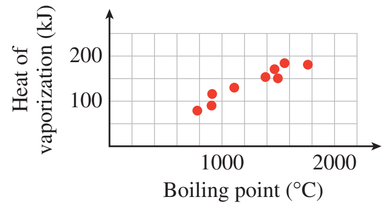
14 The solutions to a system of inequalities includes all points that are solutions to all the inequalities in the system. The graph of the system is the intersectionof the shaded regions for each inequality in the system.

15 A solution to an equation in three variables is an **ordered triple** of numbers that satisfies the equation.

16 To solve a 3×3 linear system, we use linear combinations to reduce the system to **triangular form**, and then use **back-substitution** to find the solutions.

Chapter 2 Review Problems

1. The scatterplot shows the boiling temperature of various substances on the horizontal axis and their heats of vaporization on the vertical axis. (The heat of vaporization is the energy needed to change the substance from liquid to gas at its boiling point.)



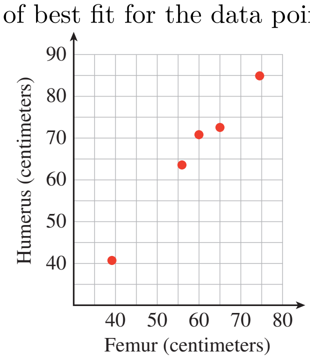
a Use a straightedge to estimate a line of best fit for the scatterplot.

b Use your line to predict the heat of vaporization of silver, whose boiling temperature is 2160°C.

c Find the equation of the regression line.

d Use the regression line to predict the heat of vaporization of potassium bromide, whose boiling temperature is 1435°C.

2. **Archaeopteryx** is an extinct creature with characteristics of both birds and reptiles. Only six fossil specimens are known, and only five of those include both a femur (leg bone) and a humerus (forearm bone) The scatterplot shows the lengths of femur and humerus for the five Archaeopteryx specimens. Drawa line of best fit for the data points.



a Predict the humerus length of an Archaeopteryx whose femur is 40 centimeters

b Predict the humerus length of an Archaeopteryx whose femur is 75 centimeters

c Use your answers from parts (b) and (c) to approximate the equation of a regression line.

d Use your answer to part (d) to predict the humerus length of an Archaeopteryx whose femur is 60 centimeters.

e Use your calculator and the given points on the scatterplot to find the least squares regression line. Compare the score this equation gives for part (d) with what you predicted earlier. The ordered pairs defining the data are $(38, 41)$, $(56, 63)$, $(59, 70)$, $(64, 72)$, $(74, 84)$.

3. In 1986 the space shuttle Challenger exploded because of O-ring failure on a morning when the temperature was about 30°F. Previously there had been 1 incident of O-ring failure when temperature was 70°F and 3 incidents whenthe temperature was 54°F. Use linear extrapolation to estimate the number of incidents of O-ring failure you would expect when the temperature is 30°F.

4. Thelma typed a 19-page technical report in 40 minutes. She required only 18 minutes for an 8-page technical report. Use linear interpolation to estimate how long Thelma would require to type a 12-page technical report.

For problems 5–6, solve the system by graphing. Use the **ZDecimal** window.

5. $y = -2.9x - 0.9$
$y = 1.4 - 0.6x$

6. $y = 0.6x - 1.94$
$y = -1.1x + 1.29$

For problems 7–10, solve the system by using substitution or elimination.

7. $x + 5y = 18$
$x - y = -3$

8. $x + 5y = 11$
$2x + 3y = 8$

9. $\frac{2}{3}x - 3y = 8$
$x - \frac{3}{4}y = 12$

10. $3x = 5y - 6$
$3y = 10 - 11x$

For problems 11–14, decide whether the system is inconsistent, dependent, or consistent and independent.

11. $2x - 3y = 4$
$x + 2y = 7$

12. $2x - 3y = 4$
$6x - 9y = 4$

13. $2x - 3y = 4$
$6x - 9y = 12$

14. $x - y = 6$
$x + y = 6$

For problems 15–20, solve the system using Gaussian reduction.

15. $x + 3y - z = 3$
$2x - y + 3z = 1$
$3x + 2y + z = 5$

16. $x + y + z = 2$
$3x - y + z = 4$
$2x + y + 2z = 3$

17. $x + z = 5$
$y - z = -8$
$2x + z = 7$

18. $x + 4y + 4z = 0$
$3x + 2y + z = -4$
$2x - 4y + z = -11$

19. $\frac{1}{2}x + y + z = 3$
$x - 2y - \frac{1}{3}z = -5$
$\frac{1}{2}x - 3y - \frac{2}{3}z = -6$

20. $\frac{3}{4}x - \frac{1}{2}y + 6z = 2$
$\frac{1}{2}x + y - \frac{3}{4}z = 0$
$\frac{1}{4}x + \frac{1}{2}y - \frac{1}{2}z = 0$

Solve Problems 21–26 by writing and solving a system of linear equations in twoor three variables.

21. A math contest exam has 40 questions. A contestant scores 5 points for each correct answer, but loses 2 points for each wrong answer. Lupe answered all the questions and her score was 102. How many questions did she answer correctly?

22. A game show contestant wins \$25 for each correct answer he gives but loses \$10 for each incorrect response. Roger answered 24 questions and won \$355. How many answers did he get right?

23. Barbara wants to earn \$500 a year by investing \$5000 in two accounts, a savings plan that pays 8% annual interest and a high-risk option that pays 13.5% interest. How much should she invest in each account?

24. An investment broker promises his client a 12% return on her funds. If the broker invests \$3000 in bonds paying 8% interest, how much must he invest in stocks, with an estimated rate of return equal to 15% interest, to keep his promise?

25. The perimeter of a triangle is 30 centimeters. The length of one side is 7 centimeters shorter than the second side, and the third side is 1 centimeter longer than the second side. Find the length of each side.

26. A company ships its product to three cities: Boston, Chicago, and Los Angeles. The cost of shipping is \$10 per crate to Boston, \$5 per crate to Chicago, and \$12 per crate to Los Angeles. The company's shipping budget for April is \$445. It has 55 crates to ship, and demand for their product is twice as high in Boston as in Los Angeles. How many crates should the company ship to each destination?

For Problems 27–30, graph the inequality.

27. $3x - 4y < 12$

28. $x > 3y - 6$

29. $y < \dfrac{-1}{2}$

30. $-4 \le x < 2$

For Problems 31–34, graph the solutions to the system of inequalities.

31. $y > 3, \quad x \le 2$

32. $y \ge x, \quad x > 2$

33. $3x - 6 < 6, \quad x + 2y > 6$

34. $x - 3y > 3, \quad y < x + 2$

For Problems 35–38

a Graph the solutions to the system of inequalities.

b Find the coordinates of the vertices.

35. $3x - 4y \le 12$
$x \ge 0, \quad y \le 0$

36. $x - 2y \le 6$
$y \le x$
$x \ge 0, \quad y \ge 0$

37. $x + y \le 5$
$y \ge x$
$y \ge 2, \quad x \ge 0$

38. $x - y \le -3$
$x + y \le 6$
$x \le 4$
$x \ge 0, \quad y \ge 0$

39. Ruth wants to provide cookies for the customers at her video rental store. It takes 20 minutes to mix the ingredients for each batch of peanut butter cookies and 10 minutes to bake them. Each batch of granola cookies takes 8 minutes to mix and 10 minutes to bake. Ruth does not want to use the oven more than 2 hours a day, or to spend more than 2 hours a day mixing ingredients. Write a system of inequalities for the number of batches of peanut butter cookies and of granola cookies that Ruth can make in one day, and graph the solutions.

40. A vegetarian recipe calls for 32 ounces of a combination of tofu and tempeh. Tofu provides 2 grams of protein per ounce and tempeh provides 1.6 grams of protein per ounce. Graham would like the dish to provide at least 56 grams of protein. Write a system of inequalities for the amount of tofu and the amountof tempeh for the recipe, and graph the solutions.

Chapter 3

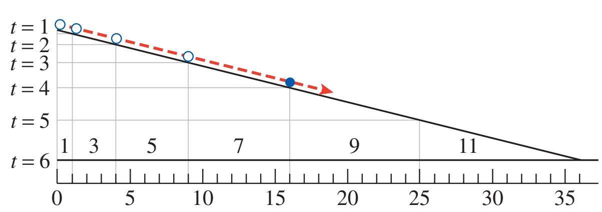
Quadratic Models

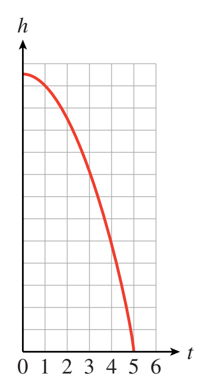
The models we have explored so far are linear models; their graphs are straight lines. In this chapter, we investigate problems where the graph may change from increeasing to decreasing, or vice versa. The simplest sort of function that models this behavior is a quadratic function, one that involves the square of the variable.

Around 1600, Galileo began to study the motion of falling objects. He used a ball rolling down an inclined plane or ramp to slow down the motion, but he had no accurate way to measure time; clocks had not been invented yet. So he used water running into a jar to mark equal time intervals.

After many trials, Galileo found that the ball traveled 1 unit of distance down the plane in the first time interval, 3 units in the second time interval, 5 units in the third time interval, and so on, as shown in the figure, with the distances increasing through odd units of distance as time went on.

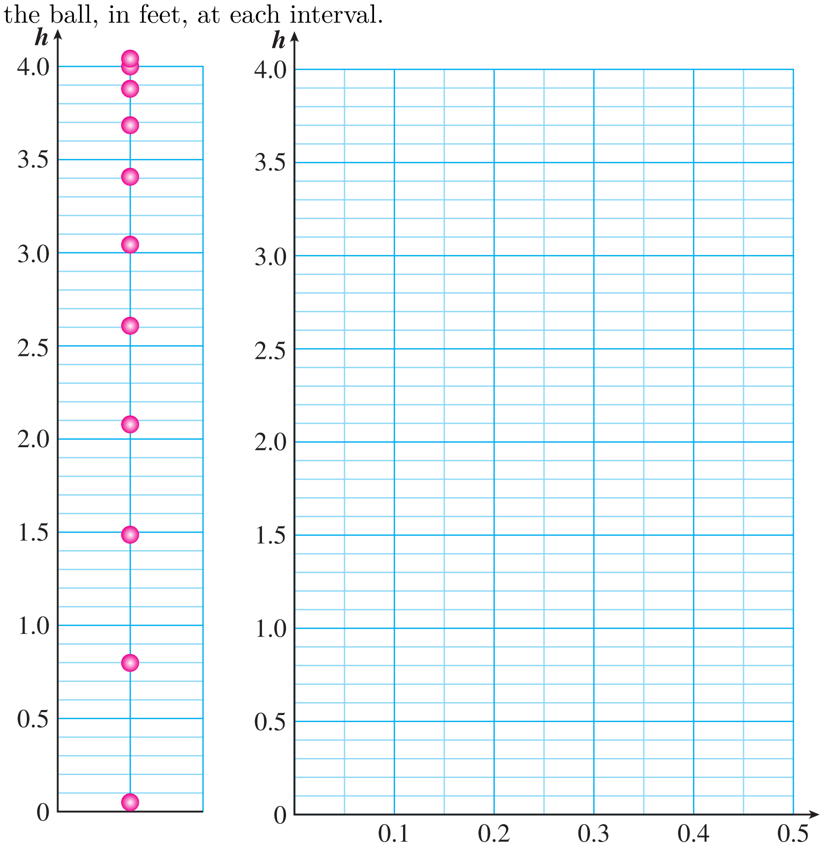
Time	Distance traveled	Total distance
1	1	1
2	3	5
3	5	9
4	7	16
5	9	25





The total distance traveled by the ball can be modeled by the equation, $d = kt^2$, where k is a constant. Galileo found that this relationship holds no matter how steep he made the ramp. If we plot the height of the ball (rather than the distance traveled) as a function of time, we obtain a portion of the graph of a quadratic function.

Investigation 3.1 Falling. Suppose you drop a small object from a height and let it fall under the influence of gravity. Does it fall at the same speed throughout its descent? The diagram shows a sequence of photographs of a steel ball falling onto a table. The photographs were taken using a stroboscopic flash at intervals of 0.05 second, and a scale on the left side shows the height of the ball, in feet, at each interval.



1 Complete the table showing the height of the ball at each 0.05-second interval. Measure the height at the bottom of the ball in each image. The first photo was taken at time $t = 0$.

t	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
h											

2 Plot the points in the table. Connect the points with a smooth curve to sketch a graph of height versus elapsed time. Is the graph linear?

3 Use your graph to estimate the time elapsed when the ball is 3.5 feet high, and when it is 1 foot high.

4 What was the change in the ball's height during the first quarter second, from $t = 0$ to $t = 0.25$? What was the change in the ball's height from $t = 0.25$ to $t = 0.50$?

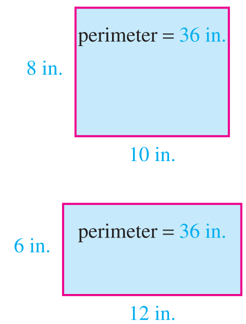
5 Add to your graph a line segment connecting the points at $t = 0$ and $t = 0.25$, and a second line segment connecting the points at $t = 0.25$ and $t = 0.50$. Compute the slope of each line segment.

6 What do the slopes in part (5) represent in terms of the problem?

7 Use your answers to part (4) to verify algebraically that the graph is not linear.

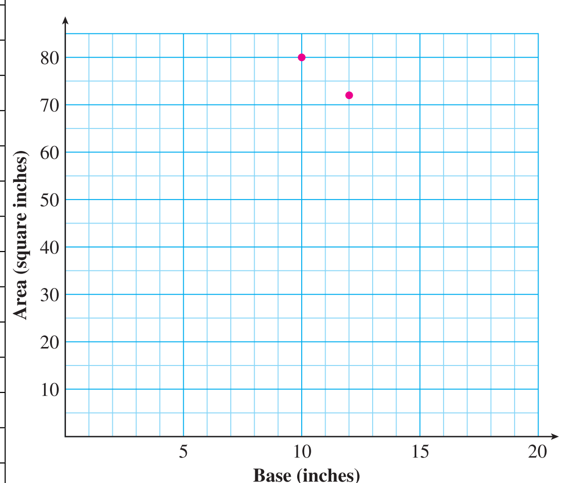
Investigation 3.2 Perimeter and Area.

Do all rectangles with the same perimeter, say 36 inches, have the same area? Two different rectangles with perimeter 36 inches are shown. The first rectangle has base 10 inches and height 8 inches, and its area is 80 square inches. The second rectangle has base 12 inches and height 6 inches. Its area is 72 square inches.



1 The table shows the bases of various rectangles, in inches. Each rectangle has a perimeter of 36 inches. Fill in the height and the area of each rectangle. (To find the height of the rectangle, reason as follows: The base plus the height makes up half of the rectangle's perimeter.)

Base	Height	Area
10	8	80
12	6	72
3		
14		
5		
17		
19		
2		
11		
4		
16		
15		
1		
6		
8		
13		
7		



2 What happens to the area of the rectangle when we change its base? On the grid above, plot the points with coordinates (Base, Area). (For this graph we will not use the heights of the rectangles.) The first two points, $(10, 80)$ and $(12, 72)$, are shown. Connect your data points with a smooth curve.

3 What are the coordinates of the highest point on your graph?

4 Each point on your graph represents a particular rectangle with perimeter 36 inches. The first coordinate of the point gives the base of the rectangle, and the second coordinate gives the area of the rectangle. What is the largest area you found among rectangles with perimeter 36 inches? What is the base for that rectangle? What is its height?

5 Describe the rectangle that corresponds to the point $(13, 65)$.

6 Find two points on your graph with vertical coordinate 80.

7 If the rectangle has area 80 square inches, what is its base? Why are there two different answers here? Describe the rectangle corresponding to each answer.

8 Now we'll write an algebraic expression for the area of the rectangle in terms of its base. Let x represent the base of the rectangle. First, express the height of the rectangle in terms of x. (Hint: If the perimeter of the rectangle is 36 inches, what is the sum of the base and the height?) Now write an expression for the area of the rectangle in terms of x.

9 Use your formula from part (8) to compute the area of the rectangle when the base is 5 inches. Does your answer agree with the values in your table and the point on your graph?

10 Use your formula to compute the area of the rectangle when $x = 0$ and when $x = 18$. Describe the "rectangles" that correspond to these data points.

11 Continue your graph to include the points corresponding to $x = 0$ and to $x = 18$.

Extraction of Roots

Introduction

So far you have learned how to solve linear equations. In linear equations, the variable cannot have any exponent other than 1, and for this reason such equations are often called **first-degree**. Now we'll consider second-degree equations, or **quadratic** equations. A quadratic equation includes the square of the variable.

Quadratic Equations.

A **quadratic equation** can be written in the standard form

$$ax^2 + bx + c = 0$$

where $a,\ b,$ and c are constants, and a is not zero.

Checkpoint 3.1 QuickCheck 1. Which of the following equations are quadratic?

☐ \(3x+2x^2=1\) ☐ \(4z^2 -2z^3+2=0\) ☐ \(36y-16=0\)
☐ \(v^2 =6v\)

Solution. (A) and (D) are quadratic.

We would like to be able to solve quadratic equations, use them in applications, and graph quadratic equations in two variables. Let's begin by considering some simple examples.

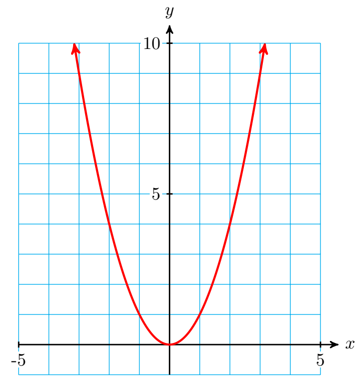
Graphs of Quadratic Equations

The simplest quadratic equation in two variables is

$$y = x^2$$

Its graph is not a straight line, but a curve called a **parabola**, shown in the figure. You can verify the table of values below for this parabola.

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9



$y = x^2$

Caution 3.2 Be careful when squaring negative numbers. To evaluate the square of a negative number on a calculator, we must enclose the number in parentheses. For example,

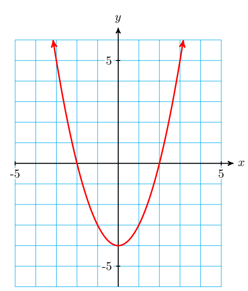
$$(-3)^2 = (-3)(-3) = 9, \quad \text{but} \quad -3^2 = -3 \cdot 3 = -9$$

Example 3.3

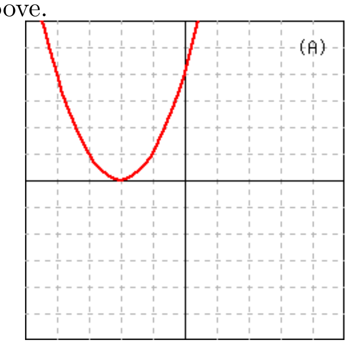
Graph the parabola $y = x^2 - 4$

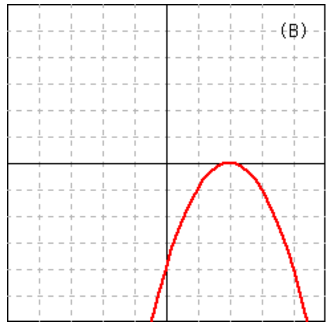
Solution. We make a table of values and plot the points. The graph is shown below.

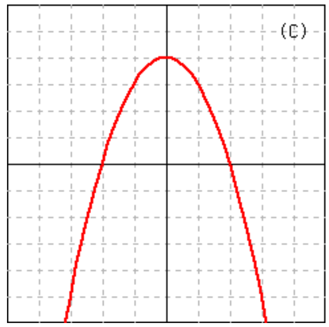
x	-3	-2	-1	0	1	2	3
y	5	0	-3	-4	-3	0	5

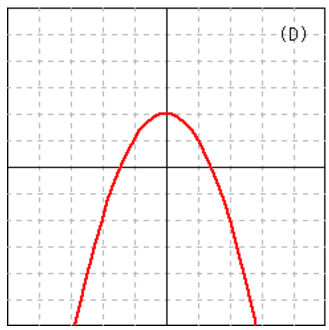


Checkpoint 3.4 Practice 1. Graph the parabola $y = 4 - x^2$ on the same grid in the Example above.







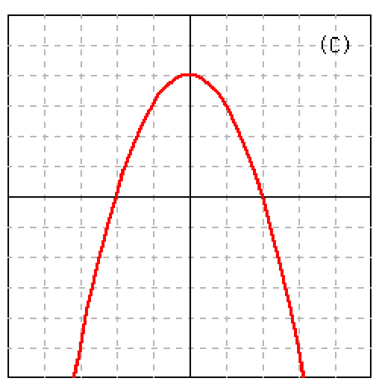


Which of the following is the best match for the graph of the parabola $y = 4 - x^2$ (using the same scale grid as in the Example)?

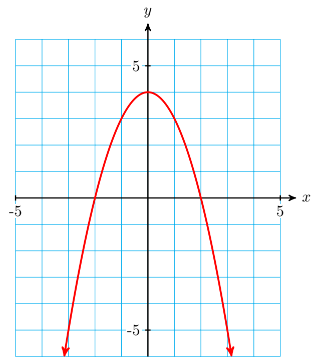
- ⊙ (A)
- ⊙ (B)
- ⊙ (C)
- ⊙ (D)

Answer. (C)

Solution.



A graph is also shown below.

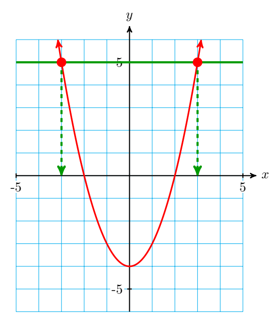


Solving Quadratic Equations

How can we solve a quadratic equation? Consider the equation

$$x^2 - 4 = 5$$

First, we can solve it graphically. Look again at the graph of $y = x^2 - 4$ from Example 1.



We would like to find the x-values that make $y = 5$. The horizontal line $y = 5$ intersects the graph at two points with y-coordinate 5, and their x-coordinates are the solutions of the equation. Thus, there are two solutions, namely 3 and -3.

Algebraically, we solve the equation as follows.

- First, we isolate the variable. We add 4 to both sides, yielding $x^2 = 9$.
- Because x is squared in this equation, we perform the opposite operation, or take square roots, in order to solve for x.

$x^2 = 9$	Take square roots of both sides.
$x = \pm\sqrt{9} = \pm 3$	Remember that every positive number has two square roots.

The solutions are 3 and -3, as we saw on the graph.

Note 3.5 Notice that we have found two solutions for this quadratic equation, whereas linear equations have at most one solution. (Sometimes they have no solution at all.) We shall see that every quadratic equation has two solutions, which may be equal. The solutions may also be complex numbers, which we'll study in Chapter 4.

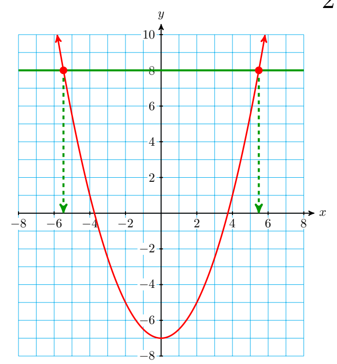
Example 3.6

Solve the equation

$$\frac{1}{2}x^2 - 7 = 8$$

graphically and algebraically.

Solution. The figure shows the graph of $y = \frac{1}{2}x^2 - 7$.



We would like to find the x-values that make $y = 8$. The horizontal line $y = 8$ intersects the graph at two points with x-coordinates approximately 5.5 and -5.5. These are the solutions of the equation.

Algebraically, we solve the equation as follows.

- First, we isolate the variable. We add 7 to both sides, then multiply by 2, yielding $x^2 = 30$.
- Because x is squared in this equation, we perform the opposite operation, or take square roots, in order to solve for x.

$x^2 = 30$	Take square roots of both sides.
$x = \pm\sqrt{30}$	Remember that every positive number has two square roots.

- We use a calculator to find that $\sqrt{30}$ is approximately 5.477, or about 5.5, as we saw on the graph.

Caution 3.7 It is important to make a distinction betweeen exact values and decimal approximations.

- For the example above, the exact solutions are $\pm\sqrt{30}$.
- The values from the calculator, ± 5.477, are decimal approximations to the solutions, rounded to thousandths.

Checkpoint 3.8 QuickCheck 2. Which solutions are exact values, and which are approximations?

a. $x^2 = 40$, $x = \pm 6.32455532$ (☐ exact solutions ☐ approximations)

b. $t^2 = \frac{81}{64}$, $t = \pm 1.125$ (☐ exact solutions ☐ approximations)

c. $w^2 = 50$, $w = \pm 5\sqrt{2}$ (☐ exact solutions ☐ approximations)

d. $b^2 = (0.632)^2$, $b = \pm 0.632$ (☐ exact solutions ☐ approximations)

Answer 1. approximations

Answer 2. exact solutions

Answer 3. exact solutions

Answer 4. exact solutions

Solution. Only the solutions to (a) are approximations.

We can now solve quadratic equations of the form $ax^2+c=0$, by isolating x on one side of the equation, and then taking the square root of each side. This method for solving quadratic equations is called **extraction of roots**.

Extraction of Roots.

To solve a quadratic equation of the form

$$ax^2 + c = 0$$

1 Isolate x on one side of the equation.

2 Take the square root of each side.

Checkpoint 3.9 **Practice 2.** Solve by extracting roots $\dfrac{3x^2 - 8}{4} = 10$

$x =$______ Enter solutions separated by a comma.

Hint. First isolate x^2. Then take the square root of both sides.

Answer. $4, -4$

Solution. ± 4

In the next Example, we compare the steps for *evaluating a quadratic expression* and for *solving a quadratic equation*.

Example 3.10

Tux the cat falls off a tree branch 20 feet above the ground. His height t seconds later is given by $h = 20 - 16t^2$.

a How high is Tux above the ground 0.5 second later?

b How long does Tux have to get in position to land on his feet before he reaches the ground?

Solution.

a We evaluate the formula for $t = 0.5$. We substitute **0.5** for t into the formula, and simplify.

$$\begin{aligned} h &= 20 - 16(\mathbf{0.5})^2 && \text{Compute the power.} \\ &= 20 - 16(0.25) && \text{Multiply, then subtract.} \\ &= 20 - 4 = 16 \end{aligned}$$

Tux is 16 feet above the ground after 0.5 second. You can also use your calculator to simplify the expression for h by entering

20 [−] 16 [×] 0.5 [x^2] [ENTER]

b We would like to find the value of t when Tux's height, h, is zero. We substitute $h = \mathbf{0}$ into the equation to obtain

$$\mathbf{0} = 20 - 16t^2$$

To solve this equation we use extraction of roots. We first isolate t^2 on one side of the equation.

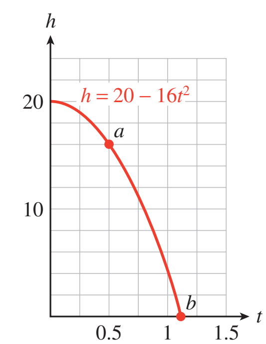
$$16t^2 = 20 \qquad \text{Divide by 16.}$$
$$t^2 = \frac{20}{16} = 1.25$$

Next, we take the square root of both sides of the equation to find

$$t = \pm\sqrt{1.25} \approx \pm 1.118$$

Only the positive solution makes sense here, so Tux has approximately 1.12 seconds to be in position for landing.

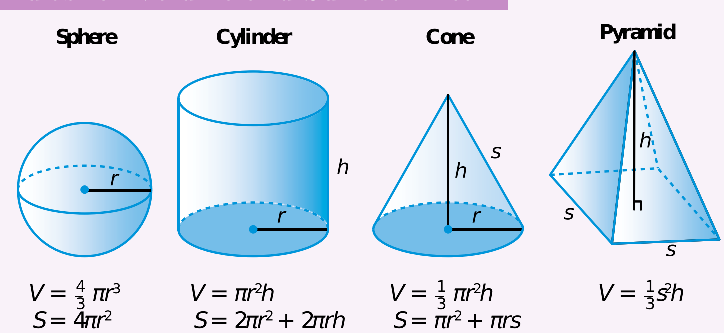
A graph of the Tux's height after t seconds is shown below. The points corresponding to parts (a) and (b) are labeled.



Geometric Formulas

The formulas for the volume and surface area of some everyday objects, such as cylinders and cones, involve quadratic expressions. We can use extraction of roots to solve problems involving these objects.

Formulas for Volume and Surface Area.



Example 3.11

The volume of a can of soup is 582 cubic centimeters, and its height is 10.5 centimeters. What is the radius of the can, to the nearest tenth of a centimeter?

Solution. The volume of a cylinder is given by the formula $V = \pi r^2 h$.

We substitute **582** for V and **10.5** for h, then solve for r.

$$\begin{aligned} \mathbf{582} &= \pi r^2(\mathbf{10.5}) && \text{Divide both sides by } 10.5\pi. \\ 17.643 &= r^2 && \text{Take square roots.} \\ 4.2 &= r \end{aligned}$$

The radius of the can is 4.2 centimeters.

Checkpoint 3.12 **Practice 3.** The glass pyramid at the Louvre in Paris has a square base, is 21.64 meters tall, and encloses a volume of 9049.68 cubic meters. Use the formula $V = \dfrac{1}{3}s^2h$ to find the length of the base. Round your answer to hundredths.

Answer: ____ meters

Answer. $\sqrt{\frac{3\cdot 9049.68}{21.64}}$

Solution. 35.42 m

Solving Formulas

Sometimes it is useful to solve a formula for one variable in terms of the others. You might want to know what radius you need to build cones of various fixed volumes. In that case, it is more efficient to solve the volume formula for r in terms of v.

Example 3.13

The formula $V = \dfrac{1}{3}\pi r^2 h$ gives the volume of a cone in terms of its height and radius. Solve the formula for r in terms of V and h.

Solution. Because the variable we want is squared, we use extraction of roots. First, we multiply both sides by 3 to clear the fraction.

$$\begin{aligned} 3V &= \pi r^2 h && \text{Divide both sides by } \pi h. \\ \frac{3V}{\pi h} &= r^2 && \text{Take square roots.} \\ \pm\sqrt{\frac{3V}{\pi h}} &= r \end{aligned}$$

Because the radius of a cone must be a positive number, we use only the positive square root: $r = \sqrt{\dfrac{3V}{\pi h}}$.

Checkpoint 3.14 **Practice 4.** Find a formula for the radius of a circle in terms of its area, A.

$r =$______________

Hint. Start with the formula for the area of a circle: $A =$
Solve for r in terms of A.

Answer. $\sqrt{\frac{A}{\pi}}$

Solution. $r = \sqrt{\dfrac{A}{\pi}}$

Checkpoint 3.15 QuickCheck 3. Match each quantity with the appropriate units.

a. Height of a cylinder (☐ I ☐ II ☐ III ☐ IV)

b. Volume of a cone (☐ I ☐ II ☐ III ☐ IV)

c. Surface area of a sphere (☐ I ☐ II ☐ III ☐ IV)

d. Area of a triangle (☐ I ☐ II ☐ III ☐ IV)

I. Square meters

II. Feet

III. Cubic centimeters

IV. Kilograms

Answer 1. II
Answer 2. III
Answer 3. I
Answer 4. I
Solution.

a. II

b. III

c. I

d. I

More Extraction of Roots

We can also use extraction of roots to solve quadratic equations of the form

$$a(x-p)^2 = q$$

We start by isolating the squared expression, $(x-p)^2$.

Example 3.16

Solve the equation $3(x-2)^2 = 48$.

Solution. First, we isolate the perfect square, $(x-2)^2$.

$$\begin{aligned} 3(x-2)^2 &= 48 && \text{Divide both sides by 3.} \\ (x-2)^2 &= 16 && \text{Take the square root of each side.} \\ x-2 &= \pm\sqrt{16} = \pm 4 \end{aligned}$$

This gives us two equations for x,

$$\begin{aligned} x-2 &= 4 \quad \text{or} \quad x-2 = -4 && \text{Solve each equation.} \\ x &= 6 \quad \text{or} \quad x = -2 \end{aligned}$$

The solutions are 6 and -2. You can check that both of these solutions satisfy the original equation.

Checkpoint 3.17 Practice 5. Solve $2(5x+3)^2 = 38$ by extracting roots.

a. Give your answers as exact values, separating the solutions with a comma.

   Note: Enter "sqrt(2)" to get $\sqrt{2}$, and take care to use parentheses appropriately.

   Use the "Preview My Answers" button to see if you have entered valid syntax.

b. Find approximations for the solutions to two decimal places, separating the solutions with a comma.

Answer 1. $\frac{-3+\sqrt{19}}{5}, \frac{-3-\sqrt{19}}{5}$

Answer 2. $0.27178, -1.47178$

Solution.

a. $x = \dfrac{-3 \pm \sqrt{19}}{5}$

b. $x \approx -1.47$ or $x \approx 0.27$

Checkpoint 3.18 QuickCheck 4. True or false.

a. The first step in extraction of roots is to take square roots. (☐ True ☐ False)

b. The solutions of a quadratic equation are always of the form $\pm k$. (☐ True ☐ False)

c. Your calculator gives exact decimal values for square roots of integers. (☐ True ☐ False)

d. The coefficients of a quadratic equation are called parabolas. (☐ True ☐ False)

Answer 1. False

Answer 2. False

Answer 3. False

Answer 4. False

Solution.

a. False

b. False

c. False

d. False

An Application: Compound Interest

Many savings accounts offer interest compounded annually: at the end of each year the interest earned is added to the principal, and the interest for the next year is computed on this larger sum of money. After n years, the amount of money in the account is given by the formula

$$A = P(1+r)^n$$

where P is the original principal and r is the interest rate, expressed as a decimal fraction.

Example 3.19

Carmella invests $3000 in an account that pays an interest rate r compounded annually.

a Write an expression for the amount of money in Carmella's account after two years.

b What interest rate would be necessary for Carmella's account to grow to $3500 in two years?

Solution.

a We use the formula $A = P(1+r)^n$ with $P = 3000$ and $n = 2$. Carmella's account balance will be

$$A = 3000(1+r)^2$$

b We substitute **3500** for A in the equation.

$$\mathbf{3500} = 3000(1+r)^2$$

This is a quadratic equation in the variable r, which we can solve by extraction of roots. First, we isolate the perfect square.

$$\begin{aligned} 3500 &= 3000(1+r)^2 && \text{Divide both sides by 3000.} \\ 1.1\overline{6} &= (1+r)^2 && \text{Take square roots.} \\ \pm 1.0801 &\approx 1+r && \text{Subtract 1 from both sides.} \\ r \approx 0.0801 \text{or} \quad & r \approx -2.0801 \end{aligned}$$

Because the interest rate must be a positive number, we discard the negative solution. Carmella needs an account with interest rate $r \approx 0.0801$, or over 8%, in order to have an account balance of $3500 in two years.

The formula for compound interest also applies to calculating the effects of inflation. For instance, if there is a steady inflation rate of 4% per year, then in two years the price of an item that costs $100 now will be

$$\begin{aligned} A &= P(1+r)^2 \\ 100 &= (1+0.04)^2 = 108.16 \end{aligned}$$

Checkpoint 3.20 Practice 6. The average cost of dinner and a movie two years ago was $36. This year the average cost is $38.16. What was the rate of inflation over the past two years? (Round to two decimal places.)

Answer: ___%

Answer. $100\sqrt{\frac{38.16}{36}} - 100$

Solution. 2.96%

Problem Set 3.1

Warm Up

1. Simplify.

a $4 - 2\sqrt{64}$ b $\dfrac{4-\sqrt{64}}{2}$ c $\sqrt{9-4(-18)}$

2. Give a decimal approximation rounded to thousandths.

a $5\sqrt{3}$ b $\dfrac{-2}{3}\sqrt{21}$ c $-3+2\sqrt{6}$

3. Use the definition of square toot to simplify the expression.

a $\sqrt{29}(\sqrt{29})$ b $(\sqrt{7})^2$ c $\dfrac{6}{\sqrt{6}}$

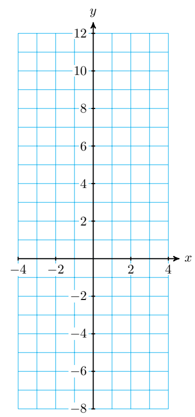
4. Solve. Remember that every positive number has two square roots.

a $3x^2 = 147$ b $4x^2 = 25$ c $3x^2 = 15$

5.

a Complete the table and graph $y = 2x^2 - 5$.

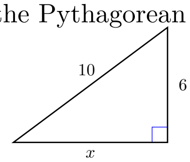
x	-3	-2	-1	0	1	2	3
y							

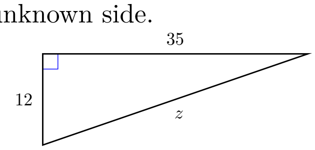


b Use the graph to solve the equation $2x^2 - 5 = 7$. Show your work on the graph. How many solutions did you find?

c Solve the equation $2x^2 - 5 = 7$ algebraically, by "undoing" each operation.

6. Use the Pythagorean theorem to find the unknown side.

a 

b 

Skills Practice

For problems 7–14, Solve by extracting roots. Give exact values for your answers.

7. $3x^2 - 9 = 0$

8. $\dfrac{3x^2}{5} = 6$

9. $(2x-1)^2 = 16$

10. $4(x-1)^2 = 12$

11. $(x - \dfrac{2}{3})^2 = \dfrac{5}{9}$

12. $81(x + \dfrac{1}{3})^2 = 1$

13. $3(8x-7)^2 = 24$

14. $2(5x-12)^2 = 48$

For problems 15 and 16, solve by extracting roots. Round your answers to two decimal places.

15. $5x^2 - 97 = 3.2x^2 - 38$

16. $17 - \dfrac{x^2}{4} = 43 - x^2$

For problems 17 and 18,

a Use technology to graph the quadratic equation in the suggested window.

b Use your graph to find two solutions for the equation in part (b).

c Check your solutions algebraically, using mental arithmetic.

17.

a $y = 3(x - 4)^2$

Xmin = −5 Ymin = −20
Xmax = 15 Ymax = 130

b $3(x - 4)^2 = 108$

18.

a $y = \frac{1}{2}(x + 3)^2$

Xmin = −15 Ymin = −5
Xmax = 5 Ymax = 15

b $\frac{1}{2}(x + 3)^2 = 8$

For problems 19–22, solve the formula for the specified variable.

19. $F = \frac{mv^2}{r}$, for v

20. $S = 4\pi r^2$, for r

21. $L = \frac{8}{\pi^2}T^2$, for T

22. $s = \frac{1}{2}gt^2$, for t

Applications

For problems 23 and 24,

a Make a sketch of the situation described, and label a right triangle.

b Use the Pythagorean theorem to solve each problem.

23. The size of a TV screen is the length of its diagonal. If the width of a 35-inch TV screen is 28 inches, what is its height?

24. 24 a 30-meter pine tree casts a shadow of 30 meters, how far is the tip of the shadow from the top of the tree?

25. You plan to deposit your savings of \$1600 in an account that compounds interest annually.

a Write a formula for the amount, A, in your savings account after two years in terms of the interest rate, r.

b Complete the table showing your account balance after two years for various interest rates.

r	0.02	0.04	0.06	0.08
A				

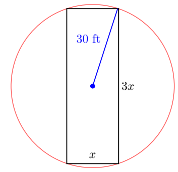
c To the nearest tenth of a percent, what interest rate will you require if you want your \$1600 to grow to \$2000 in two years?

d Use your calculator to graph the formula for the account balance. Locate the point on the graph that corresponds to the amount in part (b).

26. Two years ago Carol's living expenses were \$1200 per month. This year the same items cost Carol \$1400 per month. What was the annual inflation rate for the past two years?

27.

What size rectangle will fit inside a circle of radius 30 feet if the length of the rectangle must be three times its width?



28. A storage box for sweaters is consturcted from a square sheet of cardboard measuring x inches on a side. The volume of the box, in cubic inches, is

$$V = 10(x - 20)^2$$

If the box should have a volume of 1960 cubic inches, what size cardboard square is needed?

29. A large bottle of shampoo is 20 centimeters tall and cylindrical in shape.

a Write a formula for the volume of the bottle in terms of its radius.

b Complete the table of values for the volume equation. If you cut the radius of the bottle in half, by what factor does the volume decrease?

r	1	2	3	4	5	6	7	8
V								

c What radius should the bottle have if it must hold 240 milliliters of shampoo? (A milliliter is equal to one cubic centimeter.)

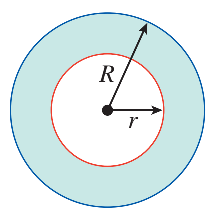
d Use your calculator to graph the volume equation. (Use the table to help you choose a suitable window.) Locate the point on the graph that corresponds to the bottle in part (c). Make a sketch of your graph, and lable the scales on the axes.

30.

The area of a ring is given by the formula

$$A = \pi R^2 - \pi r^2$$

where R is the radius of the outer circle, and r is the radius of the inner circle.



a Suppose the inner radius of the ring is kept fixed at $r = 4$ centimeters, but the radius of the outer circle, R, is allowed to vary. Find the area of the ring when the outer radius is 6 centimeters, 8 centimeters, and 12 centimeters.

b Graph the area equation, with $r = 4$, in the window

$$\text{Xmin} = 0 \qquad \text{Xmax} = 14.1$$
$$\text{Ymin} = 0 \qquad \text{Ymax} = 400$$

Use the Trace to verify your answers to part (a).

c Trace along the curve to the point $(9.75, 248.38217)$. What do the coordinates of this point represent?

d Use your graph to estimate the outer radius of the ring when its area is 100 square centimeters.

e Write and solve an equation to answer part(d).

For Problems 31 and 32, solve for x in terms of a, b, and c.

31.

a $\dfrac{ax^2}{b} = c$

b $\dfrac{bx^2}{c} - a = 0$

32.

a $(x-a)^2 = 16$

b $(ax+b)^2 = 9$

Intercepts, Solutions, and Factors

In the last section, we used extraction of roots to solve quadratic equations of the form

$$a(x-p)^2 = q$$

But this technique will not work on quadratic equations that also include a linear term, bx. Recall that the most general type of quadratic equation looks like

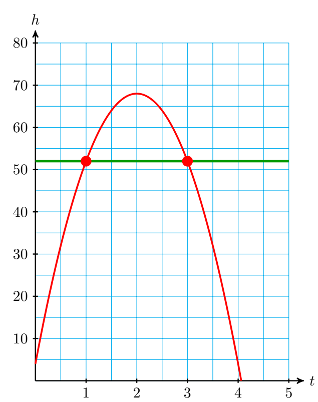
$$ax^2 + bx + c = 0$$

Here is an example.

Suppose a baseball player pops up, that is, she hits the baseball straight up into the air. The height, h, of the baseball after t seconds is given by a formula from physics. This formula takes into account the initial speed of the ball (64 feet per second) and its height when it was hit (4 feet).

$$h = -16t^2 + 64t + 4$$

The graph of this equation is shown below.



We would like to know when the baseball was exactly 52 feet high. To find out, we must solve the equation

$$-16t^2 + 64t + 4 = 52$$

where we have substituted 52 for the height, h. We can use the graph to solve this equation, by finding points with h-coordinate 52. You can see that there are two such points, with t-coordinates 1 and 3, so the baseball is 52 feet high at 1 second, and again on the way down at 3 seconds.

Can we solve the equation algebraically? Not with the techniques we know, because there are two terms containing the variable t, and they cannot be combined. We will need a new method. To find this method, we are going to study the connection between:

1 the **factors** of $ax^2 + bx + c$,

2 the **solutions** of the quadratic equation $ax^2 + bx + c = 0$, and

3 the x-intercepts of the graph of $y = ax^2 + bx + c = 0$.

Note 3.21 If you would like to review multiplying binomials (the "FOIL" method) or factoring quadratic trinomials, please see the Algebra Toolkit for this section.

Zero-Factor Principle

The method we will learn now is not like extraction of roots, or solving linear equations, where we "undid" in reverse order each operation performed on the variable, like peeling an onion. This new method will seem less direct. It relies on applying a property of our number sustem.

Can you multiply two numbers together and obtain a product of zero? Only if one of the two numbers happens to be zero. (Try it yourself.)

Zero-Factor Principle.

The product of two factors equals zero if and only if one or both of the factors equals zero. In symbols,

$$ab = 0 \quad \text{if and only if} \quad a = 0 \quad \text{or} \quad b = 0 \quad \text{(or both)}$$

Checkpoint 3.22 **QuickCheck 1.** Fill in the blanks.

a. If the sum of two numbers is zero, the numbers must be (☐ undefined ☐ opposites ☐ reciprocals ☐ numerator ☐ zero) .

b. If a fraction equals zero, the (☐ undefined ☐ opposites ☐ reciprocals ☐ numerator ☐ zero) must be zero.

c. If the product of two numbers is zero, one of the numbers must be (☐ undefined ☐ opposites ☐ reciprocals ☐ numerator ☐ zero) .

d. If the divisor in a quotient is zero, the quotient is (☐ undefined ☐ opposites ☐ reciprocals ☐ numerator ☐ zero) .

Answer 1. opposites

Answer 2. numerator

Answer 3. zero

Answer 4. undefined

Solution.

a. opposites

b. numerator

c. zero

d. undefined

Here is the simplest possible application of the Zero-Factor Principle (ZFP): For what value(s) of x is the equation $3x = 0$ true? You could divide both sides by 3, but you can also see that the product $3x$ can equal zero only if one of its factors is zero, so x must be zero!

The ZFP is true even if the numbers a and b are represented by algebraic expressions, such as $x - 6$ or $x + 2$. For example, if

$$(x-6)(x+2) = 0$$

then it must be true that either $x - 6 = 0$ or $x + 2 = 0$. This is how we can use the ZFP to solve quadratic equations.

Example 3.23

Solve the equation $x^2 - 4x - 12 = 0$

Solution. We can factor the expression $x^2 - 4x - 12$, and write the equation as

$$(x-6)(x+2) = 0$$

Now it is in the form $ab = 0$, with $a = x - 6$ and $b = x + 2$, so the ZFP tells us that either $x - 6 = 0$ or $x + 2 = 0$. We solve each of these equations.

$$\begin{aligned} x - 6 &= 0 \quad \text{or} \quad x + 2 = 0 \qquad \text{Solve each equation.} \\ x &= 6 \quad \text{or} \quad x = -2 \end{aligned}$$

Once again we see that a quadratic equation has two solutions. You can check that both of these values satisfy the original equation.

Checkpoint 3.24 Practice 1. Solve the equation $x^2 - 11x + 24 = 0$
$x =$_____ Use a comma to separate different solutions.

Answer. $3, 8$

Solution. $x = 3$ or $x = 8$

X-Intercepts of a Parabola

Recall that the x-intercept of a line is the point where $y = 0$, or where the line crosses the x-axis. We find the x-intercept by setting $y = 0$ in the equation of the line, and solving for x. We can find the x-intercepts of a parabola the same way.

Example 3.25

Find the x-intercepts of the graph of $y = x^2 - 4x - 12$

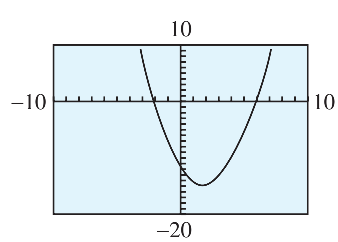
Solution. To find the x-intercepts of the graph, we set $y = \mathbf{0}$ and solve the equation

$$\mathbf{0} = x^2 - 4x - 12$$

But this is the same equation we solved in the last Example, because

$$x^2 - 4x - 12 = (x-6)(x+2)$$

The solutions of that equation were 6 and -2, so the x-intercepts of the graph are $(6, 0)$ and $(-2, 0)$. You can see this by graphing the equation on your calculator, as shown in the figure.



We can state a general result: The x-intercepts of the graph of

$$y = ax^2 + bx + c$$

are the solutions of the equation

$$0 = ax^2 + bx + c$$

So we can always solve a quadratic equation to find the x-intercepts of a parabola (if there are any).

And we can use this relationship the other way round, too: If we know the x-intercepts of the graph of $y = ax^2 + bx + c$, we also know the solutions of the equation $ax^2 + bx + c = 0$.

Checkpoint 3.26 Practice 2. Use technology to graph the equation

$$y = 2x^2 - 3x - 9$$

and find the x-intercepts of the graph. Use your answers to solve the equation

$$2x^2 - 3x - 9 = 0$$

Check your solutions by factoring and applying the ZFP.

$x =$______ Use a comma to separate different solutions.

Answer. 3, $\frac{-3}{2}$

Solution. $x = 3$ or $x = \dfrac{-3}{2}$

Solving Quadratic Equations by Factoring

Now we'll consider some other quadratic equations. Before we apply the ZFP, we must write the equation so that one side is zero.

Example 3.27

Solve $3x(x+1) = 2x + 2$

Solution. First, we write the equation in standard form.

$$\begin{aligned} 3x(x+1) &= 2x + 2 && \text{Apply the distributive law to the left side.} \\ 3x^2 + 3x &= 2x + 2 && \text{Subtract } 2x + 2 \text{ from both sides.} \\ 3x^2 + x - 2 &= 0 \end{aligned}$$

Now we factor the left side to obtain

$$\begin{aligned} (3x-2)(x+1) &= 0 && \text{Apply the zero-factor principle.} \\ 3x - 2 = 0 \quad \text{or} \quad x + 1 &= 0 && \text{Solve each equation.} \\ x = \frac{2}{3} \quad \text{or} \quad x &= -1 \end{aligned}$$

The solutions are $\dfrac{2}{3}$ and -1.

Caution 3.28 When we apply the zero-factor principle, one side of the equation must be zero. For example, to solve the equation

$$(x-2)(x-4) = 15$$

it is *incorrect* to set each factor equal to 15! (There are many ways that the product of two numbers can equal 15; it is not necessary that one of the numbers be 15.)

We must first simplify the left side and write the equation in standard form. (The correct solutions are 7 and -1; check that you can find these solutions.)

We summarize the factoring method for solving quadratic equations as follows.

To Solve a Quadratic Equation by Factoring.

1 Write the equation in standard form.

2 Factor the left side of the equation.

3 Apply the zero-factor principle: Set each factor equal to zero.

4 Solve each equation. There are two solutions (which may be equal).

Checkpoint 3.29 **Practice 3.** Solve by factoring: $(t-3)^2 = 3(9-t)$
$t =$______ Use a comma to separate different solutions.

Answer. $-3, 6$

Solution. $t = -3$ or $t = 6$

Checkpoint 3.30 **QuickCheck 2.** Which technique, extracting roots or factoring, is better-suited to each equation?

a. $4x^2 - 12x = 0$ (☐ extracting roots ☐ factoring) .

b. $6(4x-1)^2 = 18$ (☐ extracting roots ☐ factoring)

c. $(x+4)^2 = 16x$ (☐ extracting roots ☐ factoring) .

d. $9x^2 - 42 = 0$ (☐ extracting roots ☐ factoring) .

Answer 1. factoring

Answer 2. extracting roots

Answer 3. factoring

Answer 4. extracting roots

Solution. Extracting roots applies to (b) and (d).

Example 3.31

The height, h, of a baseball t seconds after being hit is given by

$$h = -16t^2 + 64t + 4$$

When will the baseball reach a height of 64 feet?

Solution. We substitute **64** for h in the formula, and solve for t.

$-16t^2 + 64t + 4 = 64$	Write the equation in standard form.
$16t^2 - 64t + 60 = 0$	Factor 4 from the left side.
$4(4t^2 - 16t + 15) = 0$	Factor the quadratic expression.
$4(2t-3)(2t-50) = 0$	Set each variable factor equal to zero.
$2t - 3 = 0$ or $2t - 5 = 0$	Solve each equation.
$t = \frac{3}{2}$ or $t = \frac{5}{2}$	

There are two solutions. At $t = \frac{3}{2}$ seconds, the ball reaches a height of 64 feet on the way up, and at $t = \frac{5}{2}$ seconds, the ball is 64 feet high on its way down.

Caution 3.32 In the Example above, the factor of 4 does not affect the solutions of the equation at all. You can understand why this is true by looking at some graphs. Use technology to graph the equation

$$y_1 = x^2 - 4x + 3$$

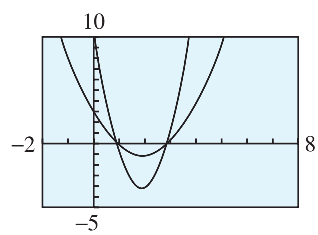
in the window

$$\text{Xmin} = -2 \quad \text{Ymin} = -5$$
$$\text{Xmax} = 8 \quad \text{Ymax} = 10$$

Notice that when $y = 0$, $x = 1$ or $x = 3$. These two points are the x-intercepts of the graph. Now on the same window graph

$$y_2 = 4(x^2 - 4x + 3)$$

as shown below.



This graph has the same x-values when $y = 0$. The factor of 4 makes the graph "skinnier," but does not change the location of the x-intercepts.

Checkpoint 3.33 Practice 4.

a. Solve by factoring $4t - t^2 = 0$.

   $t =$_____ Use a comma to separate solutions.

b. Solve by factoring $20t - 5t^2 = 0$.

   $t =$_____ Use a comma to separate solutions.

c. Graph $y = 4t - t^2$ and $y = 20t - 5t^2$ together in the window

$$\text{Xmin} = -2 \quad \text{Ymin} = -20$$
$$\text{Xmax} = 6 \quad \text{Ymax} = 25$$

   and locate the horizontal intercepts on each graph.

   $t =$_____ Use a comma to separate values.

Answer 1. $0, 4$

Answer 2. $0, 4$

Answer 3. $0, 4$

Solution. $t = 0$ and $t = 4$

Checkpoint 3.34 QuickCheck 3. Match each equation with its solutions.

a. $3t(t-2)=0$ (☐ i ☐ ii ☐ iii ☐ iv)

b. $t^2-t=2$ (☐ i ☐ ii ☐ iii ☐ iv)

c. $3t^2=12$ (☐ i ☐ ii ☐ iii ☐ iv)

d. $t(t-3)=2(t-3)$ (☐ i ☐ ii ☐ iii ☐ iv)

i. $-1, 2$

ii. $0, 2$

iii. $2, 3$

iv. $-2, 2$

Answer 1. ii
Answer 2. i
Answer 3. iv
Answer 4. iii
Solution.

a. ii

b. i

c. iv

d. iii

An Application

Here is another example of how quadratic equations arise in applications.

Example 3.35

The size of a rectangular computer monitor screen is taken to be the length of its diagonal. If the length of the screen should be 3 inches greater than its width, what are the dimensions of a 15-inch monitor?

Solution. We express the two dimensions of the screen in terms of a single variable:

$$\text{Width of screen: } w$$
$$\text{Length of screen: } w+3$$

We apply the Pythagorean theorem to write an equation:

$$w^2+(w+3)^2=15^2$$

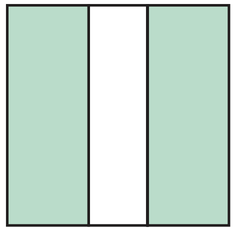
To solve this equation, we begin by simplifying the left side

$$\begin{aligned} w^2+w^2+6w+9&=225 && \text{Write the equation in standard form.}\\ 2w^2+6w-216&=0 && \text{Factor 2 from the left side.}\\ 2(w^2+3w-108)&=0 && \text{Factor the quadratic expression.}\\ 2(w-9)(w+12)&=0 && \text{Set each factor equal to zero.} \end{aligned}$$

$w - 9 = 0$ or $w + 12 = 0$ Solve each equation.
$w = 9$ or $w = -12$

Because the width of the screen cannot be a negative number, the width is 9 inches, and the length is $w + 3 = 12$ inches.

Checkpoint 3.36 Practice 5.



Francine is designing the layout for a botanical garden. The plan includes a square herb garden, with a path 5 feet wide through the center of the garden, as shown above. To include all the species of herbs, the planted area must be 300 square feet. Find the dimensions of the herb garden.

Answer: __ feet by __ feet

Answer 1. 20

Answer 2. 20

Solution. 20 feet by 20 feet

More About Solutions of Quadratic Equations

As we have seen in ehe examples above, the solutions of the quadratic equation

$$a(x - r_1)(x - r_2) = 0$$

are r_1 and r_2. This is called the **factored form** of the quadratic equation. Thus, we know the two solutions of a quadratic equation, we can work backwards to reconstruct the equation.

Example 3.37

Find a quadratic equation whose solutions are $\frac{1}{2}$ and -3.

Solution. Each solution corresponds to a factor of the equation, so the equation must look like this:

$$\left(x - \frac{1}{2}\right)(x - (-3)) = 0$$

or, simplifying:

$$\left(x - \frac{1}{2}\right)(x + 3) = 0$$

We multiply the factors together to obtain

$$x^2 + \frac{5}{2}x - \frac{3}{2} = 0$$

This is an equation that works, but we can make a "nicer" one if we clear the fractions. We can multiply both sides of the equation by 2.

We know that multiplying by a constant does not change the solutions of the equation.

$$\mathbf{2}\left(x^2 + \frac{5}{2}x - \frac{3}{2}\right) = \mathbf{2}(0)$$
$$2x^2 + 5x - 3 = 0$$

By factoring, we can check that this equation really does have the given solutions.

$$0 = 2x^2 + 5x - 3 = (2x - 1)(x + 3)$$

From here, you can see that the solutions are indeed $\frac{1}{2}$ and -3.

Checkpoint 3.38 **Practice 6.** Find a quadratic equation with integer coefficients whose solutions are $\frac{2}{3}$ and -5.

____________ $= 0$

Answer. $3x^2 + 13x - 10$

Solution. $3x^2 + 13x - 10 = 0$

A quadratic equation in one variable always has two solutions. In some cases, the solutions may be equal. For example, the equation

$$x^2 - 2x + 1 = 0$$

can be solved by factoring as follows:

$$(x - 1)(x - 1) = 0 \qquad \text{Apply the zero-factor principle.}$$
$$x - 1 = 0 \quad \text{or} \quad x - 1 = 0$$

Both of these equations have solution $x = 1$. We say that 1 is a solution of **multiplicity two**, meaning that it occurs twice as a solution of the quadratic equation.

Checkpoint 3.39 **QuickCheck 4.** True or false.

a. We find the x-intercepts of a graph by setting $x = 0$. (☐ True ☐ False)

b. If z is a solution of a quadratic equation, then $(x - z)$ is a factor of the left side in standard form. (☐ True ☐ False)

c. We can factor a constant from both sides of a quadratic equation without changing its solutions. (☐ True ☐ False)

d. To solve a quadratic equation by factoring, we should factor each side of the equation. (☐ True ☐ False)

Answer 1. False

Answer 2. True

Answer 3. True

Answer 4. False

Solution.

a. False

b. True

c. True

d. False

Problem Set 3.2

Warm Up

For Problems 1-4, write each product as a polynomial in simplest form.

1.

a $(b+6)(2b-3)$

b $(3z-8)(4z-1)$

2.

a $(4z-3)^2$

b $(2d+8)^2$

3.

a $3p(2p-5)(p-3)$

b $2v(v+4)(3v-4)$

4.

a $-50(1+r)^2$

b $12(1-t))^2$

5. Factor if possible.

a x^2-16

b x^2-16x

c $x^2-8x+16$

d x^2+16

6. Solve if possible.

a $x^2-16=0$

b $x^2-16x=0$

c $x^2-8x+16=0$

d $x^2+16=0$

For Problems 7-12, factor completely.

7. $x^2-7x+10$

8. x^2-225

9. $w^2-4w-32$

10. $2z^2+11z-40$

11. $9n^2+24n+16$

12. $4n^2-28n+49$

Skills Practice

For Problems 13–20, solve by factoring.

13. $2a^2+5a-3=0$

14. $3b^2-4b-4=0$

15. $2x^2-6x=0$

16. $3y^2-6y=-3$

17. $x(2x-3)=-1$

18. $t(t-3)=2(t-3)$

19. $z(3z+2)=(z+2)^2$

20. $(v+2)(v-5)=8$

For problems 21-24, solve by extracting roots.

21. $3(8x-7)^2=24$

22. $81\left(x+\frac{1}{3}\right)^2=1$

23. $(ax-b)^2=25$

24. $100=\pi x^2-16\pi$

For problems 25 and 26, graph the function in the ZInteger window, and locate the x-intercepts of the graph. Use the x-intercepts to write the quadratic expression in factored form.

25. $y=0.1(x^2-3x-270)$

26. $y=-0.06(x^2-22x-504)$

27. Use technology to graph all three equations in the same window. What do you notice about the x-intercepts?

a $y = x^2 + 2x - 15$

b $y = 3(x^2 + 2x - 15)$

c $y = 0.2(x^2 + 2x - 15)$

28. Write a quadratic equation with the given solutions. Give your answers in standard form with integer coefficients.

a -2 and 1

b -3 and $\frac{1}{2}$

c $\frac{-1}{4}$ and $\frac{3}{2}$

Applications

29. Delbert stands at the top of a 300-foot cliff and throws his algebra book directly upward with a velocity of 20 feet per second. The height of his book above the ground t seconds later is given by the equation

$$h = -16t^2 + 20t + 300$$

where h is in feet.

a Use your calculator to make a table of values for the height equation, with increments of 0.5 second.

b Graph the height equation on your calculator. Use your table of values to help you choose appropriate WINDOW settings.

c What is the highest altitude Delbert's book reaches? When does it reach that height? Use the TRACE feature to find approximate answers first. Then use the Table feature to improve your estimate.

d When does Delbert's book pass him on its way down? (Delbert is standing at a height of 300 feet.) Use the intersect command.

e Write and solve an equation to answer the question: How long will it take Delbert's book to hit the ground at the bottom of the cliff?

30. The annual increase I in the deer population in a national park is given by the formula

$$I = 1.2x - 0.0002x^2$$

where x is the size of the population that year.

a Make a table of values for I for $0 \leq x \leq 7000$. Use increments of 500 in x.

b How much will a population of 2000 deer increase? A population of 5000 deer? A population of 7000 deer?

c Use your calculator to graph the annual increase versus the size of the population, x, for $0 \leq x \leq 7000$. Use your table from part (b) to help you choose appropriate values for Ymin and Ymax.

d What do the x-intercepts tell us about the deer population?

e Estimate the population size that results in the largest annual increase. What is that increase?

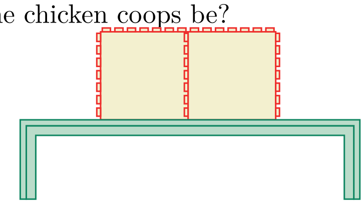
31. One end of a ladder is 10 feet from the base of a wall, and the other end reaches a window in the wall. The ladder is 2 feet longer than the height of the window.

a Choose a variable for the height of the window. Make a sketch of the situation described, and label the sides of a right triangle.

b Write a quadratic equation about the height of the window.

c Solve your equation to find the height of the window.

32. Irene would like to enclose two adjacent chicken coops of equal size against the henhouse wall. She has 66 feet of chicken wire fencing, and she would like the total area of the two coops to be 360 square feet. What should the dimensions of the chicken coops be?



We'll use three methods to solve this problem: a table of values, a graph, and an algebraic equation.

a Make a table by hand that shows the areas of coops of various widths, as shown below.

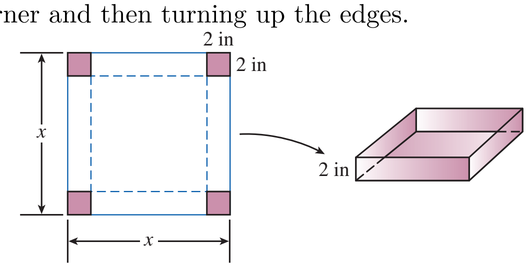
Width	Length	Area
10	340	3400
20	320	6400

Continue the table until you find a pair of chicken coops whose area is 360 square feet. (Be careful computing the length of the coops: look at the diagram above.)

b Write an expression for the length of the two coops if their width is x. Then write an expression for the area of the coops if their width is x. Graph the equation for A, and use the graph to find the pair of coops whose area is 360 square feet. (Is there more than one solution?)

c Write an equation for the area A of the two coops in terms of their width, x. Solve your equation algebraically for $A = 360$.

33. A box is made from a square piece of cardboard by cutting 2-inch squares from each corner and then turning up the edges.



a If the piece of cardboard is x inches square, write expressions for the length, width, and height of the box. Then write an expression for the volume, V, of the box in terms of x.

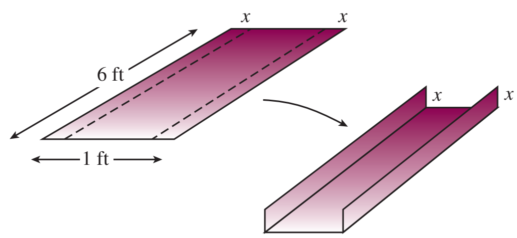
b Use your calculator to make a table of values showing the volumes of boxes made from cardboard squares of side 4 inches, 5 inches, and so on.

c Graph your expression for the volume on your calculator. What is the smallest value of x that makes sense for this problems

d Use your table or your graph to find what size cardboard you need to make a box with volume 50 cubic inches.

e Write and solve a quadratic equation to answer part (d).

34. A length of rain gutter is made from a piece of aluminum 6 feet long and 1 foot wide.



a If a strip of width x is turned up along each long edge, write expressions for the length, width and height of the gutter. Then write an expression for the volume V of the gutter in terms of x.

b Use your calculator to make a table of values showing the volumes of various rain gutters formed by turning up edges of 0.1 foot, 0.2 foot, and so on.

c Graph your expression for the volume. What happens to V as x increases?

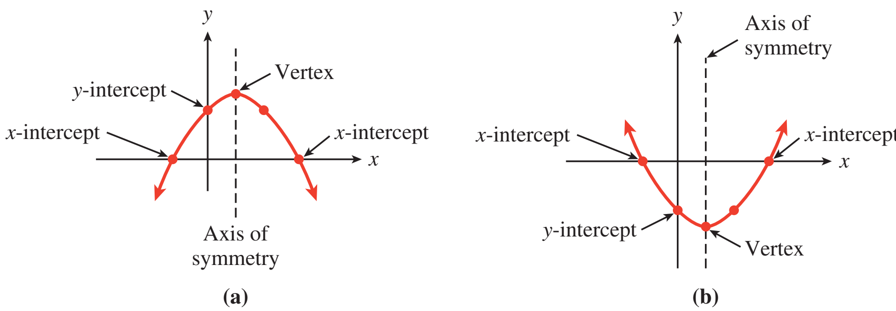
d Use your table or your graph to discover how much metal should be turned up along each long edge so that the gutter has a capacity of $\frac{3}{4}$ cubic foot of rainwater.

e Write and solve a quadratic equation to answer part (d).

Graphing Parabolas

Introduction

The graph of the quadratic equation $y = ax^2 + bx + c$ is called a **parabola**. Some parabolas are shown below.



All these parabolas share certain features.

- The graph has either a highest point (if the parabola opens downward, as in figure (a) or a lowest point (if the parabola opens upward, as in figure (b). This high or low point is called the **vertex** of the graph.
- The parabola is symmetric about a vertical line, called the **axis of symmetry**, that runs through the vertex.
- A parabola has a y-intercept, and it may have zero, one, or two x-intercepts.
- If there are two x-intercepts, they are equidistant from the axis of symmetry.

The values of the constants a, b, and c determine the location and orientation of the parabola. We'll consider each of these constants separately.

Checkpoint 3.40 QuickCheck 1. Which point on a parabola always lies on the axis of symmetry?

- ⊙ The x-intercept
- ⊙ The y-intercept
- ⊙ The vertex
- ⊙ The origin

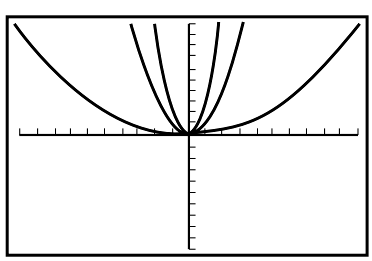
Answer. Choice 3

Solution. The vertex

The Graph of $y = ax^2$

Use your calculator to graph the following three equations in the standard window, as shown below:

$$y = x^2$$
$$y = 3x^2$$
$$y = 0.1x^2$$



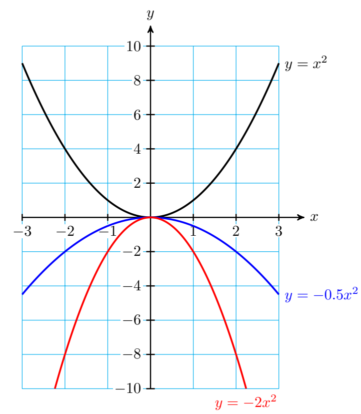
You can see that the graph of $y = 3x^2$ is narrower than the basic parabola, and the graph of $y = 0.1x^2$ is wider. As x increases, the graph of $y = 3x^2$ increases faster than the basic parabola, and the graph of $y = 0.1x^2$ increases more slowly. Compare the corresponding x-values for the three graphs shown in the table.

x	$y = x^2$	$y = 3x^2$	$y = 0.1x^2$
-2	4	12	0.4
1	1	3	0.1
3	9	27	0.9

For each x-value, the points on the graph of $y = 3x^2$ are higher than the points on the basic parabola, while the points on the graph of $y = 0.1x^2$ are lower. Multiplying by a positive constant greater than 1 stretches the graph vertically, and multiplying by a positive constant less than 1 squashes the graph vertically.

What about negative values for a? Consider the graphs of

$$y = x^2$$
$$y = -2x^2$$
$$y = -0.5x^2$$



We see that multiplying x^2 by a negative constant reflects the graph about the x-axis. These parabolas open downward.

The Graph of $y = ax^2$.

- The parabola opens upward if $a > 0$
- The parabola opens downward if $a < 0$
- The magnitude of a determines how wide or narrow the parabola is.
- The vertex, the x-intercepts, and the y-intercept all coincide at the origin.

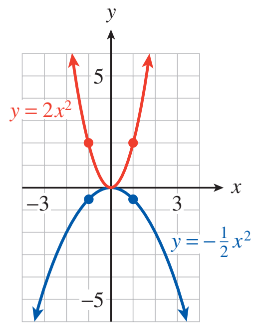
Example 3.41

Sketch by hand a graph of each quadratic equation.

a $y = 2x^2$ b $y = -\dfrac{1}{2}x^2$

Solution. Both equations have the form $y = ax^2$. The graph of $y = 2x^2$ opens upward because $a = 2 > 0$, and the graph of $y = -\dfrac{1}{2}x^2$ opens downward because $a = -\dfrac{1}{2} < 0$.

To make a reasonable sketch by hand, it is enough to plot a few *guidepoints*; the points with x-coordinates 1 and -1 are easy to compute.



x	$y = 2x^2$	$y = -\frac{1}{2}x^2$
-1	2	$-\dfrac{1}{2}$
0	0	0
1	2	$-\dfrac{1}{2}$

We sketch parabolas through each set of guidepoints, as shown at left.

Checkpoint 3.42 QuickCheck 2. True or false.

a. If $a > 1$, the graph of $y = ax^2$ is wider than the basic parabola. (☐ True ☐ False)

b. The vertex of a parabola lies on its axis of symmetry. (☐ True ☐ False)

c. If the y-intercept is negative, the parabola opens downward. (☐ True ☐ False)

d. A parabola may have one, two, or three x-intercepts. (☐ True ☐ False)

Answer 1. False
Answer 2. True
Answer 3. False
Answer 4. False
Solution.

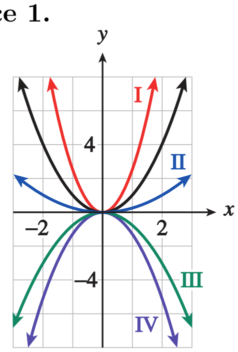
a. False

b. True

c. False

d. False

Checkpoint 3.43 Practice 1.



Match each parabola in the figure above with its equation. The basic parabola is shown in black.

a. $y = -\dfrac{3}{4}x^2$ (☐ I ☐ II ☐ III ☐ IV)

b. $y = \frac{1}{4}x^2$ (☐ I ☐ II ☐ III ☐ IV)

c. $y = \frac{5}{2}x^2$ (☐ I ☐ II ☐ III ☐ IV)

d. $y = -\frac{5}{4}x^2$ (☐ I ☐ II ☐ III ☐ IV)

Answer 1. III

Answer 2. II

Answer 3. I

Answer 4. IV

Solution.

a. III

b. II

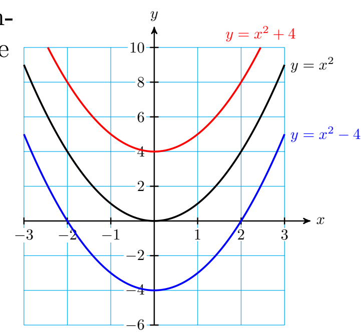
c. I

d. IV

The Graph of $y = x^2 + c$

Next, we consider the effect of the constant term, c, on the graph. Compare the graphs of

$$y = x^2$$
$$y = x^2 + 4$$
$$y = x^2 - 4$$



The graph of $y = x^2 + 4$ is shifted upward four units compared to the basic parabola, and the graph of $y = x^2 - 4$ is shifted downward four units. Look at the table, which shows the y-values for the three graphs.

x	$y = x^2$	$y = x^2 + 4$	$y = x^2 - 4$
-1	1	5	-3
0	0	4	-4
2	4	8	0

Each point on the graph of $y = x^2 + 4$ is four units higher than the corresponding point on the basic parabola, and each point on the graph of $y = x^2 - 4$ is four units lower. In particular, the vertex of the graph of $y = x^2 + 4$ is the point $(0, 4)$, and the vertex of the graph of $y = x^2 - 4$ is the point $(0, -4)$.

The Graph of $y = x^2 + c$.

Compared th the graph of $y = x^2$, the graph of $y = x^2 + c$

- is shifted upward by c units if $c > 0$
- is shifted downward by c units if $c < 0$

Example 3.44

Sketch graphs for the following quadratic equations.

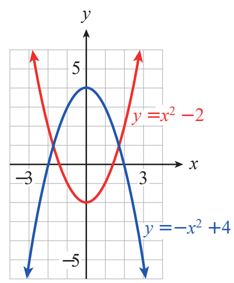
a $y = x^2 - 2$ b $y = -x^2 + 4$

Solution.

a The graph of $y = x^2 - 2$ is shifted downward by two units, compared to the basic parabola. The vertex is the point $(0, -2)$ and the x-intercepts are the solutions of the equation

$$0 = x^2 - 2$$

or $\sqrt{2}$ and $-\sqrt{2}$. The graph is shown below.



b The graph of $y = -x^2 + 4$ opens downward and is shifted 4 units up, compared to the basic parabola. Its vertex is the point $(0, 4)$. Its x-intercepts are the solutions of the equation

$$0 = -x^2 + 4$$

or 2 and -2. You can verify both graphs with your graphing calculator.

Checkpoint 3.45 QuickCheck 3. Match each equation with the description of its graph.

a. $y = x^2 - 3$ (☐ i ☐ ii ☐ iii ☐ iv)

b. $y = -3x^2$ (☐ i ☐ ii ☐ iii ☐ iv)

c. $y = 3 - x^2$ (☐ i ☐ ii ☐ iii ☐ iv)

d. $y = \frac{1}{3}x^2$ (☐ i ☐ ii ☐ iii ☐ iv)

i. Wider than the basic parabola.

ii. Shifted down 3 units from the basic parabola.

iii. Opens downward, narrower than the basic parabola.

iv. Opens downward, shifted up 3 units.

Answer 1. ii

Answer 2. iii

Answer 3. iv

Answer 4. i

Solution.

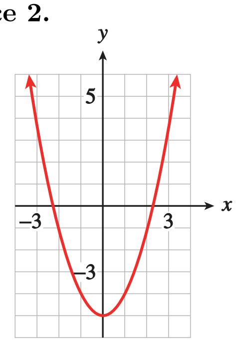
a. ii

b. iii

c. iv

d. i

Checkpoint 3.46 Practice 2.



a. Find an equation for the parabola shown above.

$y =$_______

b. Give the x- and y-intercepts of the graph.

x-intercepts: ____________ Note: Use "sqrt(2)" to get $\sqrt{2}$, and separate different ordered pairs with a comma.

y-intercept: ________________

Answer 1. $x^2 - 5$

Answer 2. $\left(\sqrt{5}, 0\right), \left(-\left(\sqrt{5}\right), 0\right)$

Answer 3. $(0, -5)$

Solution.

a. $y = x^2 - 5$

b. x-intercepts: $(-\sqrt{5}, 0), (\sqrt{5}, 0)$; y-intercept: $(0, -5)$

The Graph of $y = ax^2 + bx$

How does the linear term, bx, affect the graph?

Example 3.47

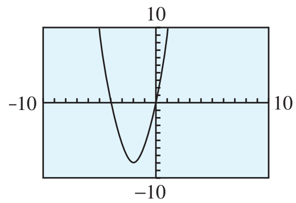
Describe the graph of the equation

$$y = 2x^2 + 8x$$

Solution. The graph in the standard window is shown below. We see that the axis of symmetry for this parabola is not the y-axis: the graph is shifted to the left, compared to the basic parabola. We find the y-intercepts of the graph by setting y equal to zero:

$$\begin{aligned} 0 &= 2x^2 + 8x \\ &= 2x(x + 4) \end{aligned}$$

The solutions of this equation are 0 and -4, so the x-intercepts are the points $(0, 0)$ and $(-4, 0)$.



We can find the vertex of the graph by using the symmetry of the parabola. The x-coordinate of the vertex lies exactly half-way between the x-intercepts, so we average their x-coordinates to find:

$$x = \frac{1}{2}[0 + (-4)] = -2$$

To find the y-coordinate of the vertex, substitute $x = -2$ into the equation for the parabola:

$$\begin{aligned} y &= 2(\mathbf{-2})^2 + 8(\mathbf{-2}) \\ &= 8 - 16 = -8 \end{aligned}$$

Thus, the vertex is the point $(-2, -8)$.

Checkpoint 3.48 Practice 3.

a. Find the x-intercepts and the vertex of the parabola $y = 6x - x^2$.

x-intercepts: __________ Note: Use a comma to separate different points.

Vertex: _____

b. Verify your answers by graphing the function in the window

$$\begin{aligned} \text{Xmin} &= -9.4 \quad \text{Xmax} = 9.4 \\ \text{Ymin} &= -10 \quad \text{Ymax} = 10 \end{aligned}$$

Answer 1. $(0, 0), (6, 0)$

Answer 2. $(3, 9)$

Solution. x-intercepts: $(0, 0)$ and $(6, 0)$; vertex: $(3, 9)$

Checkpoint 3.49 QuickCheck 4. True or false.

a. The axis of symmetry of $y = ax^2 + bx$ is shifted horizontally, compared to the basic parabola. (☐ True ☐ False)

b. The x-coordinate of the vertex is the average of the x-intercepts. (☐ True ☐ False)

c. The y-coordinate of the vertex is the same as the y-intercept. (☐ True ☐ False)

d. We can use extraction of roots to solve $ax^2 + bx = 0$. (☐ True ☐ False)

Answer 1. True

Answer 2. True

Answer 3. False

Answer 4. False

Solution.

a. True

b. True

c. False

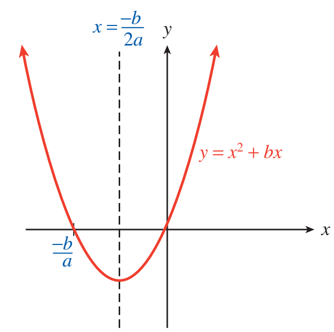
d. False

A Formula for the Vertex

We can find a formula for the vertex of any parabola of the form

$$y = ax^2 + bx$$

First, find the x-intercepts of the graph by setting y equal to zero and solving for x.



$$\begin{aligned} 0 &= ax^2 + bx && \text{Factor.} \\ &= x(ax+b) && \text{Set each factor equal to zero.} \\ x = 0 \quad &\text{or} \quad ax + b = 0 && \text{Solve for } x. \\ x = 0 \quad &\text{or} \quad x = \frac{-b}{a} \end{aligned}$$

The x-intercepts are the points $(0, 0)$ and $(\frac{-b}{a}, 0)$.

Next, we find the x-coordinate of the vertex by taking the average of the two x-intercepts.

$$x = \frac{1}{2}\left[0 + \left(\frac{-b}{a}\right)\right] = \frac{-b}{2a}$$

Now we have a formula for the x-coordinate of the vertex.

Vertex of a Parabola.

For the graph of $y = ax^2 + bx$, the x-coordinate of the vertex is

$$x_v = \frac{-b}{2a}$$

We find the y-coordinate of the vertex by substituting its x-coordinate into the equation for the parabola.

Example 3.50

a Find the vertex of the graph of $f(x) = -1.8x^2 - 16.2x$.

b Find the x-intercepts of the graph.

Solution.

a The x-coordinate of the vertex is

$$x_v = \frac{-b}{2a} = \frac{-(-16.2)}{2(-1.8)} = -4.5$$

To find the y-coordinate of the vertex, evaluate $f(x)$ at $x = \mathbf{-4.5}$.

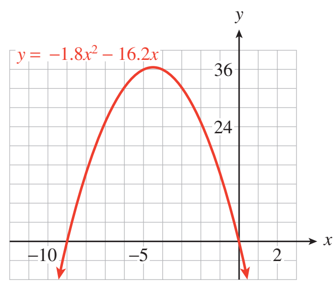
$$y_v = -1.8(\mathbf{-4.5})^2 - 16.2(\mathbf{-4.5}) = 36.45$$

The vertex is $(-4.5, 36.45)$.

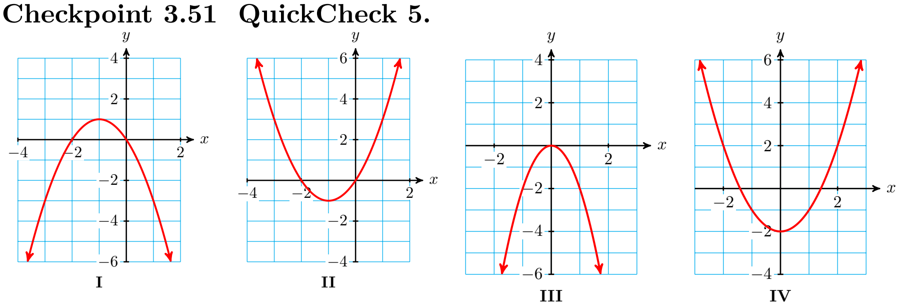
b To find the x-intercepts of the graph, set $y = 0$ and solve.

$-1.8x^2 - 16.2x = 0$		Factor.
$-x(1.8x + 16.2) = 0$		Set each factor equal to zero.
$-x = 0$	$1.8x + 16.2 = 0$	Solve each equation.
$x = 0$	$x = -9$	

The x-intercepts of the graph are $(0, 0)$ and $(-9, 0)$. The graph is shown below.



Checkpoint 3.51 QuickCheck 5.



Match each equation with its graph.

a. $y = x^2 - 2$ (☐ i ☐ II ☐ III ☐ IV)

b. $y = -x^2 - 2x$ (☐ I ☐ II ☐ III ☐ IV)

c. $y = x^2 + 2x$ (☐ I ☐ II ☐ III ☐ IV)

d. $y = -2x^2$ (☐ I ☐ II ☐ III ☐ IV)

Answer 1. IV

Answer 2. I

Answer 3. II

Answer 4. III

Solution.

a. IV

b. I

c. II

d. III

Checkpoint 3.52 **Practice 4.** Find the intercepts and the vertex of the graph of $y = 6.4x - 3.6x^2$.

Intercepts: ______________ Note: Use a comma to separate different points.

Vertex: ________

Answer 1. $(0,0), (1.8,0)$

Answer 2. $(0.9, 2.916)$

Solution. $(0,0)$,$(1.8,0)$, $(0.9, 2.916)$

Problem Set 3.3

Warm Up

For problems 1 and 2, evaluate.

1. $2x^2 - 3x - 1$, for $x = -2$

2. $3 + 4x - 3x^2$, for $x = -3$

For problems 3 and 4, which technique would you use to solve the equation, extracting roots or factoring? Then solve the equation.

3.

a $3x^2 - 15 = 0$

b $3x^2 - 15x = 0$

c $(2x - 3)^2 = 9$

d $2x^2 - 3x = 9$

4.

a $20x - 2x^2$

b $20 = 2x^2$

c $4x^2 = 2 + 2x$

d $4(x + 2)^2 - 1 = 0$

For Problems 5 and 6, factor the right side of the formula.

5. $A = \frac{1}{2}bh + \frac{1}{2}h^2$

6. $S = 2\pi R^2 + 2\pi RH$

For Problems 7 and 8, solve the equation.

7. $-9x = 81x^2$

8. $0 = -140x - 4x^2$

Skills Practice

9. Match each equation with its graph. In each equation, $k > 0$.

a $y = x^2 + k$

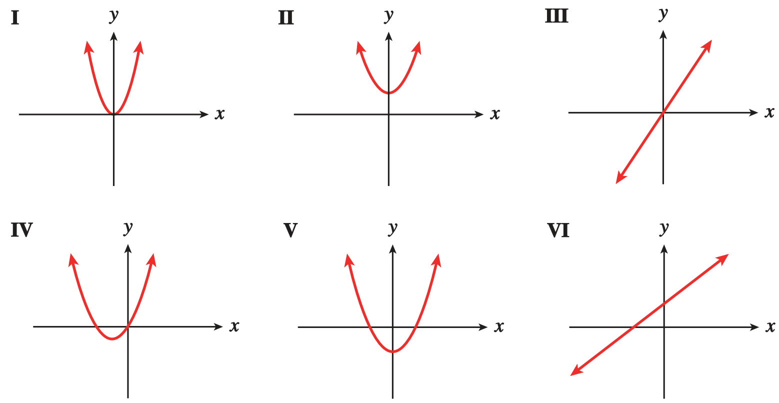
b $y = x^2 + kx$

c $y = kx^2$

d $y = kx$

e $y = x + k$

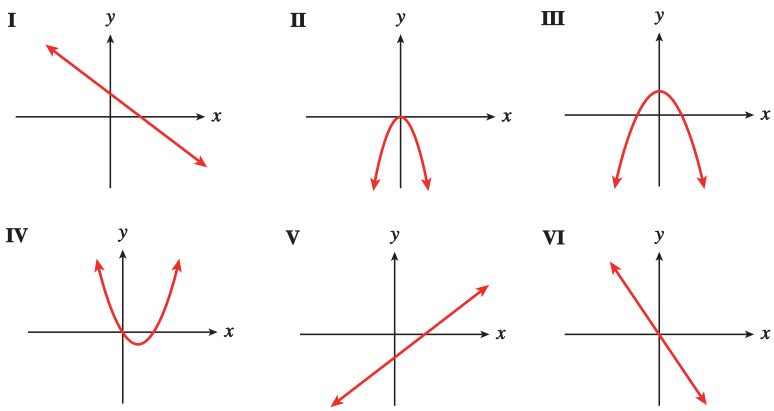
f $y = x^2 - k$



10. Match each equation with its graph. In each equation, $k > 0$.

a $y = -kx$ c $y = k - x^2$ e $y = k - x$

b $y = -kx^2$ d $y = x - k$ f $y = x^2 - kx$



For Problems 11 and 12:

a Describe what each graph will look like compared to the basic parabola.

b Sketch the graph by hand and label the coordinates of three points on the graph.

11.

i $y = 2x^2$ ii $y = \frac{1}{2}x^2$ iii $y = -x^2$

12.

i $y = x^2 + 2$ ii $y = x^2 - 9$ iii $y = 100 - x^2$

For Problems 13–16, find the x-intercepts and the vertex of the graph. Then sketch the graph by hand.

13. $y = x^2 - 4x$ **14.** $y = x^2 + 2x$

15. $y = 3x^2 + 6x$ **16.** $y = -2x^2 + 5x$

17. $y = 40x - 2x^2$ **18.** $y = 144 - 12x^2$

Applications

19. Commercial fishermen rely on a steady supply of fish in their area. To avoid overfishing, they adjust their harvest to the size of the population. The formula

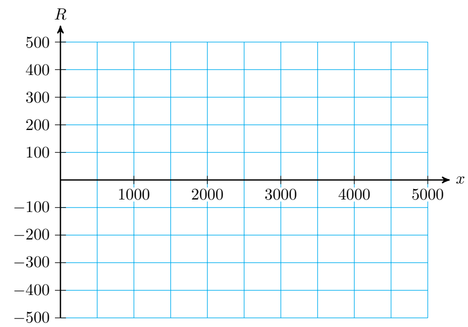
$$R = 0.4x - 0.001x^2$$

gives the annual rate of growth, in tons per year, of a fish population of biomass x tons.

a Find the vertex of the graph. What does it tell us about the fish population?

b Find the x-intercepts of the graph. What do they tell us about the fish population?

c Sketch the graph for $0 \le x \le 5000$.



d For what values of x does the fish population decrease rather than increase? Suggest a reason why the population might decrease.

20. After it lands on Earth, the distance the space shuttle travels is given by

$$d = vT + \frac{v^2}{2a}$$

where v is the shuttle's velocity in ft/sec at touchdown, T is the pilot's reaction time before the brakes are applied, and a is the shuttle's deceleration.

a Suppose that for a particular landing, $T = 0.5$ seconds and $a = 12$ ft/sec. Write the equation for d in terms of v, using these values.

b Find the coordinates of the vertex and the horizontal intercepts of the graph of your equation. Use these points to help you graph the equation.

c Explain the meaning of the vertex and the intercepts, if any, in this context. As v increases, what happens to d?

d The runway at Edwards Air Force base is 15,000 feet long. Graph your equation again in an appropriate (larger) window, and use it to estimate the answer to the question: What is the maximum velocity the shuttle can have at touchdown and still stop on the runway?

For problems 21–24, graph the set of equations in the standard window on your calculator. Describe how the **parameters** (constants) a,b, and c in each equation transform the graph, compared to the basic parabola.

21.

a $y = x^2$

b $y = 3x^2$

c $y = \frac{1}{4}x^2$

d $y = -2x^2$

22.

a $y = x^2$

b $y = x^2 + 1$

c $y = x^2 + 3$

d $y = x^2 - 6$

23.

a $y = x^2 - 4x$

b $y = x^2 + 4x$

c $y = 4x - x^2$

d $y = -x^2 - 4x$

24.

a $y = x^2 - 4x$

b $y = 2x^2 - 8x$

c $y = \frac{1}{2}x^2 - 2x$

d $y = -x^2 + 4x$

Completing the Square

Not every quadratic equation can be solved easily by factoring or by extraction of roots. If the solutions are not integers, the expression $x^2 + bx + c$ may be very difficult to factor. For example, the equation $x^2 + 3x - 1 = 0$ cannot be solved easily by either of these methods. In this section we learn a method that will solve *any* quadratic equation.

Squares of Binomials

We can use extraction of roots to solve equations of the form

$$a(x + p)^2 + q = 0$$

where the left side of the equation includes the square of a binomial, or a **perfect square**. It turns out that we can write any quadratic equation in this form.

Study the following squares of binomials.

Square of binomial $(x+p)^2$	p	$2p$	p^2
1. $(x + \mathbf{5})^2 = x^2 + 10x + 25$	$\mathbf{5}$	$2(\mathbf{5}) = 10$	$\mathbf{5}^2 = 25$
2. $(x - \mathbf{3})^2 = x^2 - 6x + 9$	$\mathbf{-3}$	$2(\mathbf{-3}) = -6$	$(\mathbf{-3})^2 = 9$
3. $(x - \mathbf{12})^2 = x^2 - 24x + 144$	$\mathbf{-12}$	$2(\mathbf{-12}) = -24$	$(\mathbf{-12})^2 = 144$

In each case, the square of the binomial is a **quadratic trinomial**,

$$(x + p)^2 = x^2 + 2px + p^2$$

The coefficient of the linear term, $2p$, is twice the constant in the binomial, and the constant term of the trinomial, p^2, is its square.

Checkpoint 3.53 QuickCheck 1. What is the linear term of $(x + 6)^2$?

- ⊙ x^2
- ⊙ $6x$
- ⊙ $12x$
- ⊙ 36

Answer. Choice 3

Solution. $12x$ is the linear term of $x^2 + 12x + 36$.

We would like to reverse the process and write a quadratic expression as the square of a binomial. For example, what constant term can we add to

$$x^2 - 16x$$

to produce a perfect square trinomial? Compare the expression to the formula above:

$$x^2 + \mathbf{2p}x + p^2 = (x+p)^2$$
$$x^2 - \mathbf{16}x + \ ? \ = (x + \ ? \ )^2$$

We see that $2p = -16$, so

$$p = \frac{1}{2}(-16) = \mathbf{-8} \quad \text{and} \quad p^2 = (-8)^2 = \mathbf{64}$$

We substitute these values for p^2 and p into the equation to find

$$x^2 - 16x + \mathbf{64} = (x \mathbf{-8})^2$$

You can check that in the resulting trinomial, the constant term is equal to *the square of one-half the coefficient of* x. In other words, we can find the constant term by taking one-half the coefficient of x and then squaring the result. Adding a constant term obtained in this way is called **completing the square**.

Example 3.54

Complete the square by adding an appropriate constant; write the result as the square of a binomial.

a $x^2 - 12x +$ ______ b $x^2 + 5x +$ ______

Solution.

a One-half of -12 is -6, so the constant term is $(-6)^2$, or 36. We add 36 to obtain

$$x^2 - 12x + \mathbf{36} = (x-6)^2 \qquad p = \tfrac{1}{2}(-12) = -6, \quad p^2 = (-6)^2 = 36$$

b One-half of 5 is $\frac{5}{2}$, so the constant term is $\left(\frac{5}{2}\right)^2$, or $\frac{25}{4}$. We add $\frac{25}{4}$ to obtain

$$x^2 + 5x + \mathbf{\frac{25}{4}} = \left(x + \frac{5}{2}\right)^2 \qquad p = \tfrac{1}{2}(5) = \tfrac{5}{2}, \quad p^2 = (\tfrac{5}{2})^2 = \tfrac{25}{4}$$

Checkpoint 3.55 QuickCheck 2. True or False.

a. Every quadratic equation can be solved by factoring. (☐ True ☐ False)

b. Every expression of the form $x^2 + bx$ can be turned into a perfect square by adding an appropriate constant. (☐ True ☐ False)

c. The coefficient of the linear term in the expansion of $(x+p)^2$ is twice the constant term. (☐ True ☐ False)

d. To complete the square means to square the expression. (☐ True ☐ False)

Answer 1. False

Answer 2. True

Answer 3. False

Answer 4. False

Solution.

a. False

b. True

c. False

d. False

Checkpoint 3.56 Practice 1. Complete the square by adding an appropriate constant; write the result as the square of a binomial.

a. $x^2 - 18x + \_\_ = (x + \_\_)^2$

b. $x^2 + 9x + \_\_ = (x + \_\_)^2$

Hint. For part (a): $p = \frac{1}{2}(-18) =$ ________ , $p^2 =$ _________
For part (b): $p = \frac{1}{2}(9) =$ ________ , $p^2 =$ _________

Answer 1. 81

Answer 2. -9

Answer 3. $\frac{81}{4}$

Answer 4. $\frac{9}{2}$

Solution.

a. $x^2 - 18x + 81 = (x - 9)^2$

b. $x^2 + 9x + \dfrac{81}{4} = \left(x + \dfrac{9}{2}\right)^2$

Solving Quadratic Equations by Completing the Square

Now we will use completing the square to solve quadratic equations. First, we will solve equations in which the coefficient of the squared term is 1. Consider the equation

$$x^2 - 6x - 7 = 0$$

Step 1 To begin, we move the constant term to the other side of the equation, to get

$$x^2 - 6x\_\_\_\_\_ = 7$$

Step 2 Next, we complete the square on the left. Because

$$p = \frac{1}{2}(-6) = -3 \quad \text{and} \quad p^2 = (-3)^2 = 9$$

we add 9 to *both* sides of our equation to get

$$x^2 - 6x + \mathbf{9} = 7 + \mathbf{9}$$

Step 3 The left side of the equation is now the square of a binomial, namely $(x-3)^2$. We write the left side in its square form and simplify the right side, which gives us

$$(x - 3)^2 = 16$$

Step 4 We can now use extraction of roots to find the solutions. Taking square roots of both sides, we get

$$x - 3 = 4 \qquad \text{or} \qquad x - 3 = -4 \qquad \text{Solve each equation.}$$

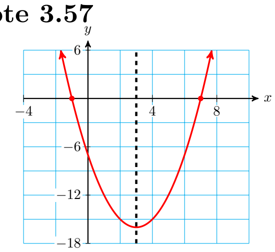
$$x = 7 \qquad \text{or} \qquad x = -1$$

The solutions are 7 and -1.

In Step 3, you can check that this equation is equivalent to the original one; if you expand the left side and collect like terms, you will return to the original equation:

$$(x-3)^2 = 16 \qquad \text{Expand the square.}$$
$$x^2 - 6x + 9 = 16 \qquad \text{Subtract 16 fron both sides.}$$
$$x^2 - 6c - 7 = 0$$

Note 3.57



The graph of $y = x^2 - 6x - 7$ is shown at left. Notice that the x-intercepts of the graph are $x = 7$ and $x = -1$, and the parabola is symmetric about the vertical line halfway between the intercepts, at $x = 3$.

We can also solve $x^2 - 6x - 7 = 0$ by factoring instead of completing the square. Of course, we get the same solutions by either method. In the next Example, we solve an equation that cannot be solved by factoring.

Example 3.58

Solve $x^2 - 4x - 3 = 0$ by completing the square.

Solution.

1 We write the equation with the constant term on the right side.

$$x^2 - 4x \, \_\_\_\_\_ = 3$$

2 We complete the square on the left side. The coefficient of x is -4, so

$$p = \frac{1}{2}(-4) = -2 \qquad \text{and} \qquad p^2 = (-2)^2 = 4$$

We add 4 to both sides of our equation:

$$x^2 - 4x + 4 = 3 + 4$$

3 We write the left side as the square of a binomial, and combine terms on the right side:

$$(x-2)^2 = 7$$

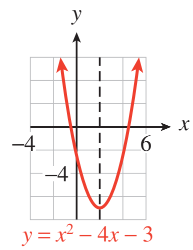
4 Finally, we use extraction of roots to obtain

$$x - 2 = \sqrt{7} \qquad \text{or} \qquad x - 2 = -\sqrt{7} \qquad \text{Solve each equation.}$$
$$x = 2 + \sqrt{7} \qquad \text{or} \qquad x = 2 - \sqrt{7}$$

These are the exact values of the solutions. We use a calculator to find decimal approximations for each solution:

$$2 + \sqrt{7} \approx 4.646 \qquad \text{and} \qquad 2 - \sqrt{7} \approx -0.646$$

These values are the x-intercepts of the graph of $y = x^2 - 4x - 3$, as shown below.



Checkpoint 3.59 QuickCheck 3. Put the steps for completing the square in the correct order:

a. Add to both sides of the equation.

b. Use extraction of roots.

c. Write the left side as a perfect square.

d. Write the equation with the constant term on the right side.

- Step 1: (☐ a ☐ b ☐ c ☐ d)
- Step 2: (☐ a ☐ b ☐ c ☐ d)
- Step 3: (☐ a ☐ b ☐ c ☐ d)
- Step 4: (☐ a ☐ b ☐ c ☐ d)

Answer 1. d
Answer 2. a
Answer 3. c
Answer 4. b
Solution. d, a, c, b

Checkpoint 3.60 Practice 2.

a. Follow the steps to solve by completing the square: $x^2 - 1 = 3x$.

1. Write the equation with the constant on the right.
   _____=__
2. Complete the square on the left:
   $p = \frac{1}{2}(-3) =$ ____, $p^2 =$ ____
   Add p^2 to both sides.
   _________=_____
3. Write the left side as a perfect square; simplify the right side.
   (___)^2 =___
4. Solve by extracting roots.
   Solutions: $x =$________________________
   List all the values that are solutions. Use a comma to separate different solutions.

b. Find approximations to two decimal places for the solutions.

Solutions: $x =$________

List all the values that are solutions. Use a comma to separate different solutions.

c. Graph the parabola $y = x^2 - 3x - 1$ in the window

$$\begin{array}{ll} \text{Xmin} = -4.7 & \text{Xmax} = 4.7 \\ \text{Ymin} = -5 & \text{Ymax} = 5 \end{array}$$

Answer 1. $x^2 - 3x$

Answer 2. 1

Answer 3. $-\frac{3}{2}$

Answer 4. $\frac{9}{4}$

Answer 5. $x^2 - 3x + \frac{9}{4}$

Answer 6. $1 + \frac{9}{4}$

Answer 7. $x + -\frac{3}{2}$

Answer 8. $\frac{13}{4}$

Answer 9. $\frac{3}{2} + \sqrt{\frac{13}{4}}, \frac{3}{2} - \sqrt{\frac{13}{4}}$

Answer 10. $3.30278, -0.302776$

Solution.

a. $x = \dfrac{3}{2} \pm \sqrt{\dfrac{13}{4}}$

b. $x \approx -0.30$ or $x \approx 3.30$

The General Case

Our method for completing the square works only if the coefficient of x^2 is 1. If we want to solve a quadratic equation whose lead coefficient is not 1, we first divide each term of the equation by the lead coefficient.

Example 3.61

Solve $2x^2 - 6x - 5 = 0$.

Solution.

1 Because the coefficient of x^2 is 2, we must divide each term of the equation by 2.

$$x^2 - 3x - \frac{5}{2} = 0$$

Now we proceed as before. Rewrite the equation with the constant on the right side.

$$x^2 - 3x \;\_\_\_\_\_\_ = \frac{5}{2}$$

2 Complete the square:

$$p = \frac{1}{2}(-3) = \frac{-3}{2} \quad \text{and} \quad p^2 = \left(\frac{-3}{2}\right)^2 = \frac{9}{4}$$

Add $\frac{9}{4}$ to both sides of our equation:

$$x^2 - 3x + \frac{9}{4} = \frac{5}{2} + \frac{9}{4}$$

3 Rewrite the left side as the square of a binomial and simplify the right side to get

$$\left(x-\frac{3}{2}\right)^2=\frac{19}{4}$$

4 Finally, extract roots and solve each equation for x.

$$x-\frac{3}{2}=\sqrt{\frac{19}{4}} \quad \text{or} \quad x-\frac{3}{2}=-\sqrt{\frac{19}{4}}$$

The solutions are $\dfrac{3}{2}+\sqrt{\dfrac{19}{4}}$ and $\dfrac{3}{2}-\sqrt{\dfrac{19}{4}}$.

Using a calculator, we can find decimal approximations for the solutions: 3.679 and −0.679.

Caution 3.62 In Example 3.58, p. 194, it is essential that we first divide each term of the equation by 2, the coefficient of x^2. The following attempt at a solution is *incorrect*.

$$\begin{aligned} 2x^2-6x&=5\\ 2x^2-6x+9&=5+9\\ (2x-3)^2&=14 \quad \rightarrow \quad \text{Incorrect!}\end{aligned}$$

You can check that $(2x-3)^2$ is not equal to $2x^2-6x+9$. We have not written the left side of the equation as a perfect square, so the solutions we obtain by extracting roots will not be correct.

Checkpoint 3.63 Practice 3.

a. Follow the steps to solve by completing the square:

$$-4x^2-36x-65=0.$$

1. Divide each term by −4. Write the equation with the constant on the right.

   _____=___

2. Complete the square on the left:

   $p=\frac{1}{2}(9)=$ ____, $p^2=$ ____

   Add p^2 to both sides.

   ________=______

3. Write the left side as a perfect square; simplify the right side.

   (____)^2 =___

4. Solve by extracting roots.

   Solutions: $x=$______________________

   List all the values that are solutions. Use a comma to separate different solutions.

b. Graph $y=-4x^2-36x-65$ in the window

$$\begin{aligned}&\text{Xmin}=-9.4 \quad \text{Xmax}=0\\ &\text{Ymin}=-10 \quad \text{Ymax}=20\end{aligned}$$

Answer 1. $x^2 + 9x$

Answer 2. $-\frac{65}{4}$

Answer 3. $\frac{9}{2}$

Answer 4. $\frac{81}{4}$

Answer 5. $x^2 + 9x + \frac{81}{4}$

Answer 6. $-\frac{65}{4} + \frac{81}{4}$

Answer 7. $x + \frac{9}{2}$

Answer 8. 4

Answer 9. $-\frac{5}{2}, -\frac{13}{2}$

Solution. $x = \dfrac{-13}{2}$, $x = \dfrac{-5}{2}$

Here is a summary of the steps for solving quadratic equations by completing the square.

To Solve a Quadratic Equation by Completing the Square.

1. a Write the equation in standard form.

   b Divide both sides of the equation by the coefficient of the quadratic term, and subtract the constant term from both sides.

2. Complete the square on the left side:

   a Multiply the coefficient of the first-degree term by one-half, then square the result.

   b Add the value obtained in (a) to both sides of the equation.

3. Write the left side of the equation as the square of a binomial. Simplify the right side.

4. Use extraction of roots to finish the solution.

Problem Set 3.4

Warm Up

1. Solve by factoring

$$2x^2 + 3x - 20 = 0$$

2. Solve by extraction of roots

$$5(2x + 3)^2 = 80$$

3. Expand each square.

a $(x + 5)^2 =$

b $(x - 6)^2 =$

c $(x - 12)^2 =$

d $(x + 15)^2 =$

Skills Practice

4. Add a term to make a square of a binomial.

a $x^2 - 14x +$______

b $x^2 + 16x +$______

c $x^2 +$______ $+ 36$

d $x^2 -$______ $+ 9$

For Problems 5 and 6, which of the expressions are squares of binomials?

5.

a $x^2 + 4x + 16$

b $x^2 - 5x + \frac{25}{4}$

c $x^2 + 16x + 169$

6.

a $x^2 - 6x + 12$

b $x^2 + 9x + 81$

c $x^2 - \frac{3}{2}x + \frac{9}{4}$

For Problems 7-10, solve by completing the square.

7. $x^2 + 9x + 20 = 0$

8. $x^2 = 3 - 3x$

9. $x^2 + 5x = 5$

10. $3x^2 + 12x + 2 = 0$

11. $2x^2 - 4x - 3 = 0$

12. $3x^2 + x = 4$

For Problems 13-16, choose the best method, then solve the equation

13. $x^2 - 3x = 40$

14. $(x - 3)^2 = 40$

15. $3x^2 + 5x = 12$

16. $3x^2 + 6x = 15$

Applications

For Problems 17–20,

a Use completing the square to find the x-intercepts of the graph.

b Find the vertex of the graph

17. $y = 4x^2 - 2x - 3$

18. $y = 2x^2 - 3x - 5$

19. $y = 5x^2 + 8x - 4$

20. $y = 3x^2 - x - 4$

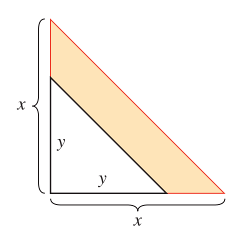
21. The diagonal of a rectangle is 20 inches. The width of the rectangle is 4 inches shorter than its length.

a Write a quadratic equation about the length of the rectangle.

b Solve your equation to find the dimensions of the rectangle.

22. The city park used 136 meters of fence to enclose its rectangular rock garden. The diagonal path across the middle of the garden is 52 meters long. What are the dimensions of the garden?

23. The sail pictured is a right triangle of base and height x. It has a colored stripe along the hypotenuse and a white triangle of base and height y in the lower corner.



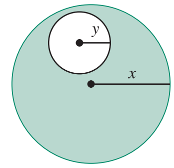
a Write an expression for the area of the colored stripe.

b Express the area of the stripe in factored form.

c The sail is $7\frac{1}{2}$ feet high and the white triangle is $4\frac{1}{2}$ feet high. Use your answer to part (b) to calculate mentally the area of the stripe.

24.

An hors d'oeuvres tray has radius x, and the dip container has radius y.



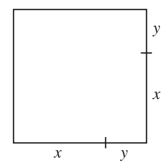
a Write an expression for the area for the chips (the shaded region).

b Express the area in factored form.

c The tray has radius $8\frac{1}{2}$ inches and the space for the dip has radius $2\frac{1}{2}$ inches. Use your answer to part (b) to calculate mentally the area for the chips. Express your answer as a multiple of π.

25.

a Write an expression for the area of the square.

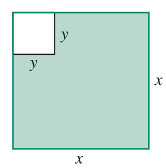


b Express the area as a polynomial.

c Divide the square into four pieces whose areas are given by the terms of your answer to part (b).

26.

a Write an expression for the area of the shaded region.



b Express the area in factored form.

c By making one cut in the shaded region, rearrange the pieces into a rectangle whose area is given by your answer to part (b).

For Problems 27-30, solve by completing the square. Your answers will involve a, b, or c (or a combination of these).

27. $x^2 + 2x + c = 0$

28. $x^2 + bx - 4 = 0$

29. $x^2 + bx + c = 0$

30. $ax^2 - 4x + 9 = 0$

For Problems 31–36, solve the formula for the indicated variable.

31. $V = \pi(r - 3)^2 h$, for r

32. $E = \frac{1}{2}mv^2 + mgh$, for v

33. $V = 2(s^2 + t^2)w$, for t

34. $x^2 y - y^2 = 0$, for y

35. $(2y + 3x)^2 = 9$, for y

36. $4x^2 - 9y^2 = 36$, for y

Chapter 3 Summary and Review

Glossary

- quadratic equation
- parabola
- extraction of roots
- compound interest
- factor
- multiplicity
- vertex
- axis of symmetry
- quadratic trinomial
- complete the square

Key Concepts

1 A **quadratic** equation has the standard form $ax^2 + bx = c = 0$, where $a,\ b,$ and c are constants and a is not equal to zero.

2 The graph of a quadratic equation $y = ax^2 + bx = c$ is called a **parabola**. The **basic parabola** is the graph of $y = x^2$.

Extraction of Roots.

3 To solve a quadratic equation of the form

$$ax^2 + c = 0$$

1 Isolate x on one side of the equation.

2 Take the square root of each side.

4 Every quadratic equation has two solutions, which may be the same.

5 Solutions of a quadratic equation can be given as exact values or as decimal approximations.

Formulas for Volume and Surface Area.

6

1 Sphere $V = \frac{4}{3}\pi r^3$ $S = 4\pi r^2$

2 Cylinder $V = \pi r^2 h$ $S = 2\pi r^2 + 2\pi rh$

3 Cone $V = \frac{1}{3}\pi r^2 h$ $S = \pi r^2 + \pi rs$

4 Square Pyramid $V = \frac{1}{3}s^2 h$

7 The formula for interest compounded annually is

$$A = P(1+r)^n$$

8 **Zero-Factor Principle**: The product of two factors equals zero if and only if one or both of the factors equals zero. In symbols,

$$ab = 0 \ \text{ if and only if } \ a = 0 \ \ |textor \ \ b = 0 \ \text{ or both}$$

To Solve a Quadratic Equation by Factoring.

9

1 Write the equation in standard form.

2 Factor the left side of the equation.

3 Apply the zero-factor principle: Set each factor equal to zero.

4 Solve each equation. There are two solutions (which may be equal).

10 Each solution of a quadratic equation corresponds to a non-constant factor in the factored form.

11 The value of the constant a in the factored form of a quadratic equation does not affect the solutions.

12 The x-intercepts of the graph of $y = ax^2 + bx + c$ are the solutions of the equation $ax^2 + bx + c = 0$.

The Graph of $y = ax^2$.

13

- The parabola opens upward if $a > 0$
- The parabola opens downward if $a < 0$
- The magnitude of a determines how wide or narrow the parabola is.
- The vertex, the x-intercepts, and the y-intercept all coincide at the origin.

The Graph of $y = x^2 + c$.

14 Compared to the graph of $y = x^2$, the graph of $y = x^2 + c$

- is shifted upward by c units if $c > 0$
- is shifted downward by c units if $c < 0$

15 For the graph of $y = ax^2 + bx + c$, the x-coordinate of the vertex is

$$x_v = \frac{-b}{2a}$$

16 The square of a binomial is a quadratic trinomial,

$$(x + p)^2 = x^2 + 2px + p^2$$

To Solve a Quadratic Equation by Completing the Square.

17

1. a Write the equation in standard form.
   b Divide both sides of the equation by the coefficient of the quadratic term, and subtract the constant term from both sides.
2. Complete the square on the left side:
   a Multiply the coefficient of the first-degree term by one-half, then square the result.
   b Add the value obtained in (a) to both sides of the equation.
3. Write the left side of the equation as the square of a binomial. Simplify the right side.
4. Use extraction of roots to finish the solution.

Chapter 3 Review Problems

For Problems 1–6, solve by extraction of roots.

1. $x^2 + 7 = 13 - 2x^2$

2. $\dfrac{2x^2}{5} - 7 = 9$

3. $3(x + 4)^2 = 60$

4. $(7x - 1)^2 = 15$

5. $(2x - 5)^2 = 9$

6. $(3.5 - 0.2x)^2 = 1.44$

For Problems 7–10, solve by factoring.

7. $(w + 1)(2w - 3) = 3$

8. $6y = (y + 1)^2 + 3$

9. $4x - (x + 1)(x + 2) = -8$

10. $3(x + 2)^2 = 15 + 12x$

For Problems 11 and 12, write a quadratic equation with integer coefficients and with the given solutions.

11. $\dfrac{-3}{4}$ and 8

12. $\dfrac{5}{3}$ and $\dfrac{5}{3}$

For Problems 13 and 14, graph the equation using the ZDecimal setting. Locate the x-intercepts, and use them to write the quadratic expression in factored form.

13. $y = x^2 - 0.6x - 7.2$

14. $y = -x^2 + 0.7x + 2.6$

For Problems 15–18,

a Find the coordinates of the vertex and the intercepts.

b Sketch the graph.

15. $y = \dfrac{1}{2}x^2$

16. $y = x^2 - 4$

17. $y = x^2 - 9x$

18. $y = -2x^2 - 4x$

For Problems 19–22, solve by completing the square.

19. $x^2 - 4x - 6 = 0$

20. $x^2 + 3x = 3$

21. $2x^2 + 3 = 6x$

22. $3x^2 = 2x + 3$

For Problems 23–26, solve the formula for the indicated variable.

23. $K = \frac{1}{2}mv^2$, for v

24. $a^2 + b^2 = c^2$, for b

25. $V = \frac{1}{3}s^2h$, for h

26. $A = P(1 + r)^2$, for r

27. In a tennis tournament among n competitors, $\frac{n(n-1)}{2}$ matches must be played. If the organizers can schedule 36 matches, how many players should they invite?

28. The formula $S = \frac{n(n-1)}{2}$ gives the sum of the first positive integers. How many consecutive integers must be added to make a sum of 91?

29. Lewis invested \$2000 in an account that compounds interest annually. He made no deposits or withdrawals after that. Two years later he closed the account, withdrawing \$2464.20. What interest rate did Lewis earn?

30. Earl borrowed \$5500 from his uncle for 2 years with interest compounded annually. At the end of two years he owed his uncle \$6474.74. What was the interest rate on the loan?

31. The perimeter of an equilateral triangle is 36 inches. Find its altitude. (Hint: The altitude is the perpendicular bisector of the base.)

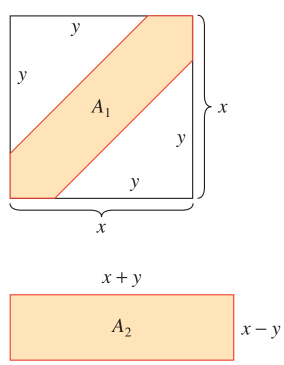
32. The base of an isosceles triangle is one inch shorter than the equal sides, and the altitude of the triangle is two inches shorter than the equal sides. What is the length of the equal sides?

33. A car traveling at 50 feet per second (about 34 miles per hour) can stop in 2.5 seconds after applying the brakes hard. The distance the car travels, in feet, t seconds after applying the brakes is $d = 50t - 10t^2$. How long does it take the car to travel 40 feet?

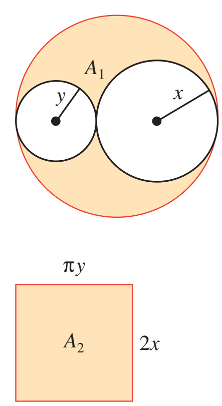
34. You have 300 feet of wire fence to mark off a rectangluar Christmas tree lot with a center-divider, using a brick wall as one side of the lot. If you would like to enclose a total area of 7500 square feet, what should be the dimensions of the lot?

For Problems 35 and 36, show that the shaded areas are equal.

35.

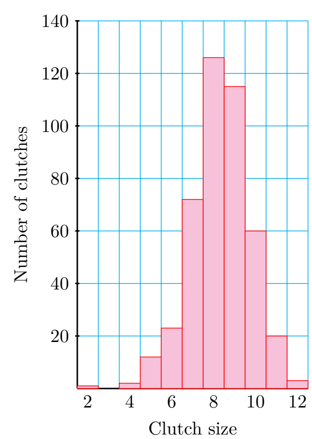


36.



Chapter 4

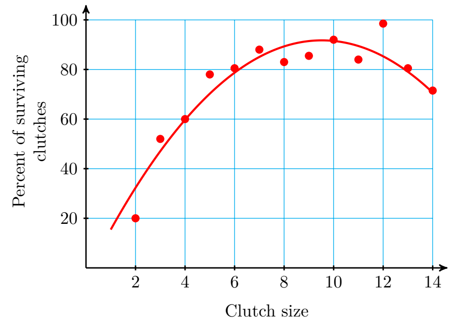
Applications of Quadratic Models



An important part of modeling is optimization, finding the best possible solution to a particular problem. For qudaratic models, this often involves finding the vertex of the graph where the maximum or minumum value of the output variable occurs. For example, biologists conducted a four-year study of the nesting habits of a species of wrens, The bar graph shows the clutch size (the number of eggs) in 433 nests. Why is 8 or 9 eggs the most common clutch size? Does that number increase the birds' chances of survival?

The average weight of the young birds decrreases as the size of the brood increases, and the survival of individual nestlings is linked to their weight. Which clutch size produces the largest average number of survivors?

The graph shows the number of survivors for each clutch size in the study, along with the curve of best fit. The equation for the curve is $y = -0.0105x^2 + 0.2x - 0.035$. Looking at the graph, the optimum clutch size for maximizing the survival of the nestlings is about 9 eggs. How does this optimum clutch size compare with the most frequently observed clutch size in the study?

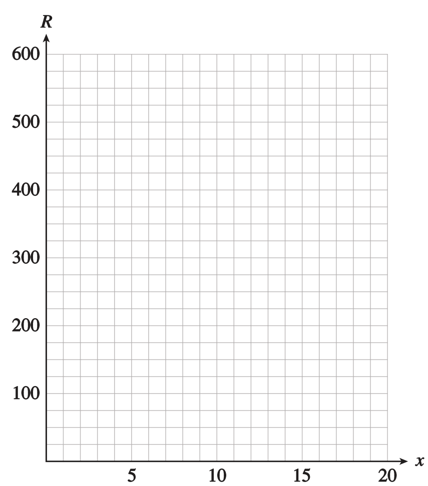


Investigation 4.1 Revenue from Theater Tickets. The local theater group sold tickets to its opening-night performance for \$5 and drew an audience of 100 people. The next night they reduced the ticket price by \$0.25 and 10 more people attended; that is, 110 people bought tickets at \$4.75 apiece. In fact, for each \$0.25 reduction in ticket price, 10 additional tickets can be sold.

1 Fill in the table

No. of price reductions	Price of ticket	No. of tickets sold	Total revenue
0	5.00	100	500.00
1	4.75	110	522.50
2			
3			
4			
5			
6			
8			
10			

2 On the grid below, plot *Total revenue* on the vertical axis versus *Number of price reductions* on the horizontal axis. Use the data from your table.



3 Let x represent the *number of price reductions*, as in the first column of your table. Write algebraic expressions in terms of x for

The *price of a ticket* after x price reductions:

Price =

The *number of tickets* sold at that price:

Number =

The *total revenue* from ticket sales:

$$\text{Revenue} =$$

4 Enter your expressions for the price of a ticket, the number of tickets sold, and the total revenue into the calculator as Y_1, Y_2, and Y_3. Use the Table feature to verify that your algebraic expressions agree with your table from part (1).

5 Use your calculator to graph your expression for total revenue in terms of x. Use your table to choose appropriate window settings that show the high point of the graph and both x-intercepts.

6 What is the maximum revenue possible from ticket sales? What price should the theater group charge for a ticket to generate that revenue? How many tickets will the group sell at that price?

Quadratic Formula

A New Formula

Instead of completing the square every time we solve a new quadratic equation, we can complete the square on the general quadratic equation,

$$ax^2 + bx + c = 0, \qquad (a \neq 0)$$

and obtain a formula for the solutions of any quadratic equation. Here is the resulting formula.

The Quadratic Formula.

The solutions of the equation $\quad ax^2 + bx + c = 0, \quad (a \neq 0) \quad$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note 4.1 The formula gives us the solutions of a particular quadratic equation in terms of its coefficients, a, b, and c. We know that there should be two solutions, and the symbol $\pm$ is used to represent the two expressions

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

in a single formula.

Checkpoint 4.2 QuickCheck 1. Which of the following is a correct statement of the quadratic formula?

- ⊙ $x = -b \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$
- ⊙ $x = \dfrac{-b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$
- ⊙ $x = -b \pm \dfrac{b - \sqrt{4ac}}{2a}$

⊙ $x = -b \pm \dfrac{\sqrt{b^2 - ac}}{a}$

Answer. Choice 2

Solution. $x = \dfrac{-b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$

To solve a quadratic equation using the quadratic formula, all we have to do is substitute the coefficients a, b, and c into the formula.

Example 4.3

Solve $2x^2 + 1 = 4x$.

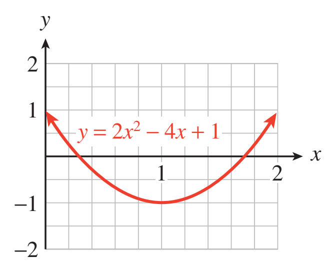
Solution. Write the equation in standard form as

$$2x^2 - 4x + 1 = 0$$

We substitute **2** for a, **−4** for b, and **1** for c into the quadratic formula, then simplify.

$$\begin{aligned} x &= \frac{-(\mathbf{-4}) \pm \sqrt{(\mathbf{-4})^2 - 4(\mathbf{2})(\mathbf{1})}}{2(\mathbf{2})} \\ &= \frac{4 \pm \sqrt{8}}{4} \end{aligned}$$

Using a calculator, we find that the solutions are approximately 1.7 and 0.3. These values are the x-intercepts of the graph of $y = 2x^2 - 4x + 1$, as shown in the figure.



Checkpoint 4.4 **Practice 1.** Use the quadratic formula to solve $x^2 - 3x = 1$.

Solutions: $x =$ ____________________

Hint. Write the equation in standard form.

Substitute $a = 1$, $b = -3$, $c = -1$ into the quadratic formula.

Simplify.

Answer. $\frac{3+\sqrt{13}}{2}, \frac{3-\sqrt{13}}{2}$

Solution. $x = \dfrac{3 \pm \sqrt{13}}{2}$

Applications

We have now seen four different algebraic methods for solving quadratic equations:

1. Factoring
2. Extraction of roots

3. Completing the square
4. Quadratic formula

Factoring and extraction of roots are relatively fast and simple, but they do not work on all quadratic equations. The quadratic formula will work on any quadratic equation.

Checkpoint 4.5 QuickCheck 2. Match each equation with the most efficient method of solution.

a. $6(x-4)^2 = 120$ (☐ I ☐ II ☐ III ☐ IV)

b. $x^2 - 9x + 20 = 0$ (☐ I ☐ II ☐ III ☐ IV)

c. $1.4x^2 - 6.2x + 2.5 = 0$ (☐ I ☐ II ☐ III ☐ IV)

d. $x^2 - 20x = 44$ (☐ I ☐ II ☐ III ☐ IV)

I. Factoring

II. Extraction of roots

III. Completing the square

IV. Quadratic formula

Answer 1. II
Answer 2. I
Answer 3. IV
Answer 4. III
Solution.

a. II

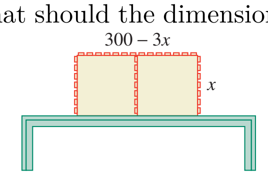
b. I

c. IV

d. III

Example 4.6

The owners of a day-care center plan to enclose a divided play area against the back wall of their building, as shown below. They have 300 feet of picket fence and would like the total area of the playground to be 6000 square feet. Can they enclose the playground with the fence they have, and if so, what should the dimensions of the playground be?



Solution. Suppose the width of the play area is x feet. Because there are three sections of fence along the width of the play area, that leaves $300 - 3x$ feet of fence for its length.

The area of the play area should be 6000 square feet, so we have the equation

$$x(300 - 3x) = 6000$$

This is a quadratic equation. In standard form,

$$\begin{aligned} 3x^2 - 300x + 6000 &= 0 \qquad \text{Divide each term by 3.} \\ x^2 - 100x + 2000 &= 0 \end{aligned}$$

The left side cannot be factored, so we use the quadratic formula with $a = \mathbf{1}$, $b = \mathbf{-100}$, and $c = \mathbf{2000}$.

$$\begin{aligned} x &= \frac{-(\mathbf{-100}) \pm \sqrt{(\mathbf{-100})^2 - 4(\mathbf{1})(\mathbf{2000})}}{2(\mathbf{1})} \\ &= \frac{100 \pm \sqrt{2000}}{2} \approx \frac{100 \pm 44.7}{2} \end{aligned}$$

When we evalute this last expression, we get two different positive values, $x = 72.4$ or $x = 27.6$. Both values give solutions to the problem. To find the length of the playground in each case, we substitute x into $300 - 3x$.

- If the width of the play area is 72.4 feet, the length is $300-3(72.4)$, or 82.8 feet.
- If the width is 27.6 feet, the length is $300 - 3(27.6)$, or 217.2 feet.

The dimensions of the play area can be 72.4 feet by 82.8 feet, or it can be 27.6 feet by 217.2 feet.

Checkpoint 4.7 **Practice 2.** The height of a baseball is given by the equation

$$h = -16t^2 + 64t + 4$$

where t is the time in seconds. Find two times when the ball is at a height of 20 feet. Round your answers to two decimal places, and separate the values with a comma.

$t =$______________ sec.

Hint. Set $h = 20$, then write the equation in standard form.
Divide each term by -16.
Use the quadratic formula to solve.

Answer. $3.73205, 0.267949$

Solution. 0.27 sec, 3.73 sec

Complex Numbers

Not all quadratic equations have solutions that are real numbers. For example, when we try to solve the equation $x^2 + 4 = 0$, , we find

$$\begin{aligned} x^2 &= -4 \\ x &= \pm\sqrt{-4} \end{aligned}$$

Although square roots of negative numbers such as $\sqrt{-4}$ are not real numbers, they occur frequently in mathematics and its applications. Mathematicians in the sixteenth century gave them the name **imaginary numbers**, which reflected the mistrust with which they were viewed at the time. Today, however, such numbers are well understood and are used routinely by scientists and engineers.

To help us work with imaginary numbers, we define a new number, i, the **imaginary unit**, whose square is -1.

Definition 4.8 Imaginary Unit.

$$i = \sqrt{-1} \quad \text{or} \quad i^2 = -1$$

With this new number we define the principal square root of any negative real number as follows.

Imaginary Numbers.

If $a \geq 0$,

$$\sqrt{-a} = \sqrt{-1}\sqrt{a} = i\sqrt{a}$$

Thus, the square root of any negative real number can be written as the product of a real number and i. Every negative real number has two imaginary square roots. For example, the square roots of -9 are $3i$ and $-3i$. You can verify that

$$(3i)^2 = 9i^2 = 9(-1) = -9 \quad \text{and} \quad (-3i)^2 = (-3)^2 i^2 = 9(-1) = -9$$

Example 4.9

a

$$\begin{aligned}\sqrt{-4} &= \sqrt{-1}\sqrt{4}\\ &= i\sqrt{4} = 2i\end{aligned}$$

b

$$\begin{aligned}\sqrt{-3} &= \sqrt{-1}\sqrt{3}\\ &= i\sqrt{3}\end{aligned}$$

Checkpoint 4.10 Practice 3. Simplify.

a. $-\sqrt{-36} =$___

b. $(5i)^2 =$___

Answer 1. $-6i$

Answer 2. -25

Solution.

a. $-6i$

b. -25

The solutions of many quadratic equations involve imaginary numbers.

Example 4.11

Solve $2x^2 - x + 2 = 0$.

Solution. For this equation, $a = \mathbf{2}$, $b = \mathbf{-1}$, and $c = \mathbf{2}$. We substitute these values into the quadratic formula to obtain

$$x = \frac{-(\mathbf{-1}) \pm \sqrt{(\mathbf{-1})^2 - 4(\mathbf{2})(\mathbf{2})}}{2(\mathbf{2})} = \frac{1 \pm \sqrt{-15}}{4}$$

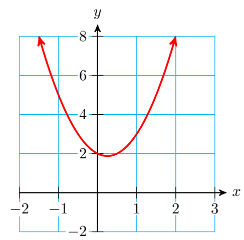
We write the solutions as

$$\frac{1 \pm i\sqrt{15}}{4} \qquad \text{or} \qquad \frac{1}{4} \pm \frac{\sqrt{15}}{4}i$$

Because the solutions are not real numbers, the graph of

$$y = 2x^2 - x + 2$$

has no x-intercepts, as shown below.



The sum of a real number and an imaginary number is called a **complex number**.

Checkpoint 4.12 Practice 4. Use extraction of roots to solve $(2x+1)^2+9 = 0$. Write your answers as complex numbers, separated with a comma.

$x =$ ______________

Answer. $\frac{-1}{2} + \frac{3}{2}i, \frac{-1}{2} - \frac{3}{2}i$

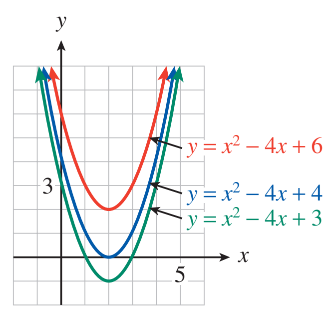
Solution. $\dfrac{-1}{2} + \dfrac{3}{2}i, \dfrac{-1}{2} - \dfrac{3}{2}i$

Number of x-Intercepts

The graph of the quadratic equation

$$y = ax^2 + bx + c$$

may have two, one, or no x-intercepts, according to the number of distinct real-valued solutions of the equation $ax^2 + bx + c = 0$. For example, consider the three graphs shown at right.



- The graph of
$$y = x^2 - 4x + 3$$
has two x-intercepts, because the equation
$$x^2 - 4x + 3 = 0$$
has two real-valued solutions, $x = 1$ and $x = 3$.

- The graph of
$$y = x^2 - 4x + 4$$
has only one x-intercept, because the equation
$$x^2 - 4x + 4 = 0$$
has only one (repeated) real-valued solution, $x = 2$.

- The graph of

  $$y = x^2 - 4x + 6$$

  has no x-intercepts, because the equation

  $$x^2 - 4x + 6 = 0$$

  has no real-valued solutions.

A closer look at the quadratic formula reveals useful information about the solutions of quadratic equations. The sign of the number under the radical determines how many solutions the equation has. For the three equations above, we calculate as follows:

$y = x^2 - 4x + 3$	$y = x^2 - 4x + 4$	$y = x^2 - 4 + 6$
two x-intercepts	one x-intercept	no x-intercepts
$x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(3)}}{2}$	$x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(4)}}{2}$	$x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(6)}}{2}$
$= \dfrac{4 \pm \sqrt{\mathbf{4}}}{2}$	$= \dfrac{4 \pm \sqrt{\mathbf{0}}}{2}$	$= \dfrac{4 \pm \sqrt{\mathbf{-12}}}{2}$
two real solutions	one repeated solution	no real solutions

From these examples, we see that the solutions of a quadratic equation always occur in **conjugate pairs**,

$$\frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

For example, if we know that one solution of a particular quadratic equation is $3 + \sqrt{2}$, the other solution must be $3 - \sqrt{2}$. If one solution is $5 - 3i$, the other solution must be $5 + 3i$.

The expression $b^2 - 4ac$, which appears under the radical in the quadratic formula, is called the **discriminant**, D, of the equation. The value of the discriminant determines the nature of the solutions of the equation. In particular, if the discriminant is negative, the equation has no real-valued solutions; the solutions are complex numbers.

The Discriminant.

The **discriminant** of a quadratic equation is

$$D = b^2 - 4ac$$

1 If $D > 0$, there are two unequal real solutions.

2 If $D = 0$, there is one solution of multiplicity two.

3 If $D < 0$, there are two complex conjugate solutions.

Example 4.13

Use the discriminant to discover how many x-intercepts the graph has.

a $y = x^2 - x - 3$ b $y = 2x^2 + x + 1$ c $y = x^2 - 6x + 9$

Solution. We set $y = 0$ and compute the discriminant for the resulting equation.

a $D = b^2 - 4ac = (-1)^2 - 4(1)(-3) = 13 > 0$. The equation has two real, unequal solutions, and the graph has two x-intercepts.

b $D = b^2 - 4ac = 1^2 - 4(2)(1) = -7 < 0$.

The equation has no real solutions, so the graph has no x-intercepts.

c $D = b^2 - 4ac = (-6)^2 - 4(1)(9) = 0$.

The equation has one real solution of multiplicity two, and the graph has a single x-intercept.

Checkpoint 4.14 QuickCheck 3. True or False.

a. The discriminant is part of the quadratic formula. (☐ True ☐ False)

b. We use the discriminant to calculate the solutions of a quadratic equation. (☐ True ☐ False)

c. If the discriminant is negative, both x-intercepts of the graph are negative. (☐ True ☐ False)

d. If a quadratic equation won't factor, its graph has no x-intercepts. (☐ True ☐ False)

Answer 1. True
Answer 2. False
Answer 3. False
Answer 4. False
Solution.

a. True

b. False

c. False

d. False

Checkpoint 4.15 Practice 5. Use the discriminant to discover how many x-intercepts the graph of each equation has.

a. $y = x^2 + 5x + 7$ (☐ None ☐ 1 ☐ 2 ☐ Other)

b. $y = -\dfrac{1}{2}x^2 + 4x - 8$ (☐ None ☐ 1 ☐ 2 ☐ Other)

Answer 1. None
Answer 2. 1
Solution.

a. None: the discriminant is negative.

b. One: the discriminant is 0.

Solving Formulas

Sometimes it is useful to solve a quadratic equation for one variable in terms of the others.

Example 4.16

Solve $x^2 - xy + y = 2$ for x in terms of y.

Solution. We first write the equation in standard form as a quadratic equation in the variable x.

$$x^2 - yx + (y-2) = 0$$

Expressions in y are treated as constants with respect to x, so that $a = \mathbf{1}$, $b = \mathbf{-y}$, and $c = \mathbf{y-2}$. We substitute these expressions into the quadratic formula.

$$\begin{aligned} x &= \frac{-(\mathbf{-y}) \pm \sqrt{(\mathbf{-y})^2 - 4(\mathbf{1})(\mathbf{y-2})}}{2(\mathbf{1})} \\ &= \frac{y \pm \sqrt{y^2 - 4y + 8}}{2} \end{aligned}$$

Checkpoint 4.17 **Practice 6.** Solve $2x^2 + kx + k^2 = 1$ for x in terms of k.

Solutions: $x =$ ____________________________

Answer. $\frac{-k+\sqrt{8-7k^2}}{4}$, $\frac{-k-\sqrt{8-7k^2}}{4}$

Solution. $x = \dfrac{-k \pm \sqrt{8-7k^2}}{4}$

Problem Set 4.1

Warm Up

For Problems 1 and 2, simplify according to the order of operations.

1.

a $8 - 2\sqrt{36-16}$

b $\dfrac{4 + 4\sqrt{4(16)}}{4}$

2.

a $\dfrac{6 + 3\sqrt{36 - 4(3)}}{2(3)}$

b $\dfrac{\sqrt{10^2 + 4(11)} - \sqrt{9 + 3(6)}}{12 - \sqrt{1 + 5(16)}}$

3.

a Write the formulas for the area and perimeter of a rectangle.

b Find the area and perimeter of each rectangle:

i 4 ft by 6 ft

ii 3 ft by 8 ft

4.

a Do all rectangles with the same area have the same perimeter?

b Write expressions for the area and perimeter of each rectangle:

i w in by $w - 3$ in

ii w cm by $160 - 2w$ cm

Skills Practice

For Problems 5-10, use the quadratic formula to solve. Round your answers to three decimal places.

5. $0 = x^2 + x - 1$

6. $3z^2 = 4z + 1$

7. $0 = x^2 - \frac{5}{3}x + \frac{1}{3}$

8. $-5.2z^2 + 176z + 1218 = 0$

9. $2x^2 = 7.5x - 6.3$

10. $x = \frac{1}{2}x^2 - \frac{3}{4}$

11.

a Graph $y = 6x^2 - 7x - 3$ in the window

$$\text{Xmin} = -2 \quad \text{Ymin} = -6$$
$$\text{Xmax} = 3 \quad \text{Ymax} = 4$$

b Estimate the x-intercepts of the graph.

c Use the quadratic formula to solve the equation

$$6x^2 - 7x - 3 = 0$$

How do the solutions compare to your estimates in part (b)?

12.

a Solve by the quadratic formula

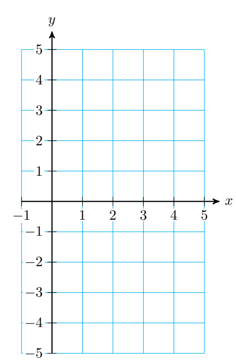
$$2x^2 - 1 = 5x$$

Find decimal approximations for your answers. Round to hundredths.

b Graph $y = 2x^2 - 5x - 1$ in the window

$$\text{Xmin} = -1 \quad \text{Ymin} = -5$$
$$\text{Xmax} = 5 \quad \text{Ymax} = 5$$

Sketch your graph on the grid.



c How are your solutions from part (a) related to the graph?

13.

a Graph the three equations

$$y = x^2 - 6x + 5$$

$$y = x^2 - 6x + 9$$
$$y = x^2 - 6x + 12$$

in the window

$$\text{Xmin} = -2 \quad \text{Ymin} = -5$$
$$\text{Xmax} = 7.4 \quad \text{Ymax} = 15$$

Use the Trace to locat the x-intercepts of each graph.

b Use the quadratic formula to find the solutions of each equation.

$$x^2 - 6x + 5 = 0$$
$$x^2 - 6x + 9 = 0$$
$$x^2 - 6x + 12 = 0$$

How are your answers related to the graphs in part (a)?

14. Use the discriminant to determine the nature of the solutions of each equation.

a $3x^2 + 26 = 17x$

c $16x^2 - 712x + 7921 = 0$

b $4x^2 + 23x = 19$

d $0.03x^2 = 0.05x - 0.12$

15. Here is one solution of a quadratic equation. Find the other solution, then write a quadratic equation in standard form that has those solutions.

a $2 + \sqrt{5}$

b $4 - 3i$

16.

a What is the sum of the two solutions of the quadratic equation $ax^2 + bx + c = 0$? (Hint: The two solutions are given by the quadratic formula.)

b What is the product of the two solutions of the quadratic equation $ax^2 + bx + c = 0$? (Hint: Do not try to multiply the two solutions given by the quadratic formula! Think about the factored form of the equation.)

For Problems 17-22, use the quadratic formula to solve the equation for the indicated variable.

17. $h = 4t - 16t^2$, for t

18. $A = 2w^2 + 4lw$, for w

19. $s = vt - \dfrac{1}{2}at^2$, for t

20. $3x^2 + xy + y^2 = 2$, for y

21. $S = \dfrac{n^2 + n}{2}$, for n

22. $A = \pi r^2 + \pi rs$, for r

Applications

23. A car traveling at s miles per hour on a wet road surface requires approximately d feet to stop, where d is given by the equation

$$d = \frac{s^2}{12} + \frac{s}{2}$$

a Make a table showing the stopping distance, d, for speeds of 10, 10, . . ., 100 miles per hour. (Use the **Table** feature of your calculator.)

b Graph the equation for d in terms of s. Use your table values to help you choose appropriate window settings.

c Insurance investigators at the scene of an accident find skid marks 100 feet long leading up to the point of impact. Write and solve an equation to discover how fast the car was traveling when it put on the brakes. Verify your answer on your graph.

24. A high diver jumps from the 10-meter springboard. His height in meters above the water t seconds after leaving the board is given by

$$h = -4.9t^2 + 8t + 10$$

a Make a table of values showing the diver's altitude at 0.25-second intervals after he jumps from the springboard. (Use the Table feature of your calculator.)

b Graph the equation. Use your table of values to choose appropriate window settings.

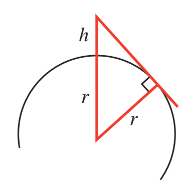
c How long is it before the diver passes the board on the way down?

d How long is it before the diver hits the water?

e Find points on your graph that correspond to your answers to parts (c) and (d).

25.

When you look down from a height, say a tall building or a mountain peak, your line of sight is tangent to the Earth at the horizon, as shown in the figure.



a The radius of the earth is 6370 kilometers. How far can you see from an airplane at an altitude of 10,000 meters? (You will need to use the Pythagorean theorem.)

b How high would the airplane have to be in order for you to see a distance of 300 kilometers?

26. Refer to the figure in Problem 25.

a Suppose you are standing on top of the Petronas Tower in Kuala Lumpur, 1483 feet high. How far can you see on a clear day? (The radius of the Earth is 3960 miles. Don't forget to convert the height of the Petronas Tower to miles.)

b How tall a building should you stand on in order to see 100 miles?

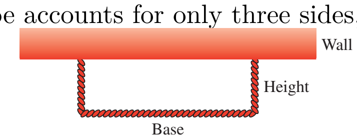
27. The volume of a large aquarium at the zoo is 2160 cubic feet. The tank is 10 feet wide, and its length is 6 feet less than twice its height.

a Sketch the aquarium, and label its dimensions in terms of a variable.

b Write an equation for the the variable.

c Find the dimensions of the aquarium.

28. You have 72 feet of rope to enclose a rectangular display area against one wall of an exhibit hall. The area enclosed depends upon the dimensions of the rectangle you make. Because the wall makes one side of the rectangle, the length of the rope accounts for only three sides, as shown below.



a Let h stand for the height of a rectangle, and write an algebraic expression for the base of the rectangle.

b Write an expression for the area of the rectangle.

c If you would like to enclose 640 square feet of display space, what should the dimensions of the rectangle be? (There are two possible solutions.)

d What should the dimensions be if you would like to enclose exactly 600 square feet of space?

The Vertex

Finding the Vertex

In Section 3.3 we saw that the vertex of the graph of $y = ax^2 + bx$ has x-coordinate given by

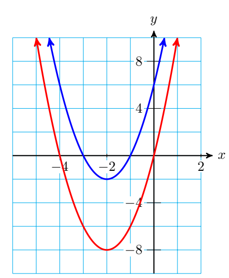
$$x_v = \frac{-b}{2a}$$

Now we'll see that the same formula holds for any parabola. Use your calculator to graph the two parabolas

$$y = 2x^2 + 8x$$
$$y = 2x^2 + 8x + 6$$

in the window

$$\text{Xmin} = -6 \quad \text{Ymin} = -6$$
$$\text{Xmax} = 4 \quad \text{Ymax} = 10$$



The graphs are shown at right. Now compare the two graphs. You should notice that:

- The second graph is identical to the first, except shifted upward by 6 units.
- For both graphs, the x-coordinate of the vertex is $x_v = -2$.

We see that the x-coordinate of the vertex is not affected by an upward shift. The formula for the x-coordinate of the vertex still holds.

Vertex of a Parabola.

For the graph of $y = ax^2 + bx + c = 0$, the x-coordinate of the vertex is

$$x_v = \frac{-b}{2a}$$

Example 4.18

Find the vertex of the graph of $f(x) = -2x^2 + x + 1$.

Solution. For this equation, $a = -2$, $b = 1$, and $c = 1$. The x-coordinate of the vertex is given by

$$x_v = \frac{-b}{2a} = \frac{-1}{2(-2)} = \frac{1}{4}$$

To find the y-coordinate of the vertex, we substitute $x = \frac{1}{4}$ into the equation. We can do this by hand to find

$$\begin{aligned} y_v &= -2\left(\frac{1}{4}\right)^2 + \frac{1}{4} + 1 \\ &= -2\left(\frac{1}{16}\right) + \frac{4}{16} + \frac{16}{16} = \frac{18}{16} = \frac{9}{8} \end{aligned}$$

So the coordinates of the vertex are $\left(\frac{1}{4}, \frac{9}{8}\right)$. Alternatively, we can use the calculator to evaluate $-2x^2 + x + 1$ for $x = 0.25$. We enter

[(-)] 2 [×] 0.25 [x^2] [+] 0.25 [+] 1

and press [ENTER]. The calculator returns the y-value 1.125. Thus, the vertex is the point $(0.25, 1.125)$, which is the decimal equivalent of $\left(\frac{1}{4}, \frac{9}{8}\right)$.

Checkpoint 4.19 **Practice 1.** Find the vertex of the graph of $y = 3x^2 - 6x + 4$.

Decide whether the vertex is a maximum point or a minimum point of the graph. (☐ maximum ☐ minimum)

Answer 1. $(1, 1)$

Answer 2. minimum

Solution. $(1, 1)$, minimum

Once we know the vertex of a parabola, we can make a quick sketch of the graph.

Example 4.20

Sketch a graph of $y = x^2 + 3x + 1$

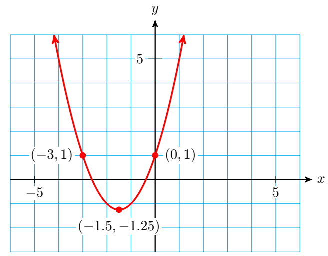
Solution. The vertex of the graph is given by

$$\begin{aligned} x_v &= \frac{-b}{2a} = \frac{-3}{2(1)} = -1.5 \\ y_v &= (-1.5)^2 + 3(-1.5) + 1 = -1.25 \end{aligned}$$

The vertex is the point $(-1.5, -1.25)$.

We set $x = 0$ to find the y-intercept, $(0, 1)$. Now, the axis of symmetry of the parabola is the vertical line that passes through its vertex. That line is $x = -1.5$, so the y-intercept lies 1.5 units to the right of the axis. There must be another point on the parabola with the same y-coordinate as the intercept but 1.5 units to the left of the axis. This point is $(-3, 1)$.

By plotting the vertex, the y-intercept, and its symmetric point, we can make a quick sketch of the parabola, as shown below.



If we need more accuracy in our graph, we can find and plot more points, including the x-intercepts.

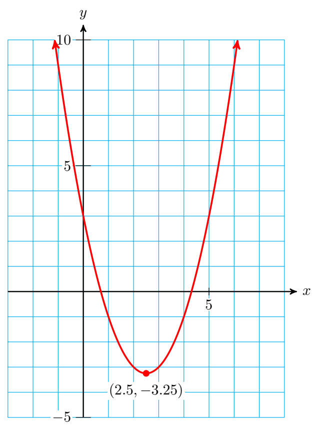
Checkpoint 4.21 Practice 2. Find the vertex and sketch a graph of $y = x^2 - 5x + 3$

Vertex: ________

Answer. $(2.5, -3.25)$

Solution. $(2.5, -3.25)$; a graph is below.

$y = x^2 - 5x + 3$:



Maximum or Minimum Values

Many quadratic models arise as the product of two variables, one of which increases while the other decreases. For example, in the Investigation for Chapter 3 we looked at the areas of different rectangles with the same perimeter. The area of a rectangle is the product of its length and its width, or $A = lw$. If we require that the rectangle have a certain perimeter, then as we increase its length, we must also decrease its width.

Another example is the formula for the revenue from sales of an item:

$$\text{Revenue} = (\text{price of one item}) \times (\text{number of items sold})$$

Usually, when the price of an item increases, the number of items sold decreases.

Finding the maximum or minimum value for a variable expression is a common problem in applications. For example, if you own a company that manufactures blue jeans, you might like to know how much to charge for your jeans in order to maximize your revenue. As you increase the price of the jeans, your revenue may increase for a while. But if you charge too much for the jeans, consumers will not buy as many pairs, and your revenue may actually start to decrease. Is there some optimum price you should charge for a pair of jeans in order to achieve the greatest revenue?

Example 4.22

Late Nite Blues finds that it can sell $600 - 15x$ pairs of jeans per week if it charges x dollars per pair. (Notice that as the price increases, the number of pairs of jeans sold decreases.)

a Write an equation for the revenue as a function of the price of a pair of jeans.

b Graph the function.

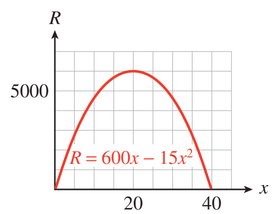
c How much should Late Nite Blues charge for a pair of jeans in order to maximize its revenue?

Solution.

a Using the formula for revenue stated above, we find

$$\begin{aligned}\text{Revenue} &= (\text{price of one item})(\text{number of items sold})\\ R &= x(600 - 15x)\\ R &= 600x - 15x^2\end{aligned}$$

b We recognize the equation as quadratic, so the graph is a parabola. You can use your calculator to verify the graph below.



c As you can see from the graph, the maximum value of R occurs at the vertex of the parabola. Thus,

$$\begin{aligned}x_v &= \frac{-b}{2a} = \frac{-600}{2(-15)} = 20\\ y_v &= 600(\mathbf{20}) - 15(\mathbf{20})^2 = 6000\end{aligned}$$

The revenue takes on its maximum value when $x = 20$, and the maximum value is $R = 6000$. This means that Late Nite Blues should charge \$20 for a pair of jeans in order to maximize revenue at \$6000 a week.

Note 4.23 If the equation relating two variables is quadratic, then the maximum or minimum value is easy to find: It is the value at the vertex. If the parabola opens downward, as in Example 4.22, p. 222, there is a maximum value at the vertex. If the parabola opens upward, there is a minimum value at the vertex.

Checkpoint 4.24 QuickCheck 1. To find the maximum or minimum value of a quadratic expression, we should:

⊙ Set $y = 0$ and solve for x.

⊙ Factor the expression.

⊙ Use the quadratic formula.

⊙ Find the vertex.

Answer. Choice 4

Solution. Find the vertex.

Checkpoint 4.25 Practice 3. The Metro Rail service sells $1200-80x$ tickets each day when it charges x dollars per ticket.

a. Write an equation for the revenue, R, as a function of the price of a ticket.

$R =$__________

b. What ticket price will return the maximum revenue?

$___

What is the maximum revenue?

$_____

Answer 1. $1200x - 80x^2$

Answer 2. 7.5

Answer 3. 4500

Solution.

a. $R = 1200x - 80x^2$

b. $7.50, $4500

The Vertex Form for a Parabola

Consider the quadratic equation

$$y = 2(x-3)^2 - 8$$

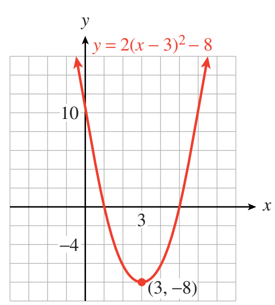
By expanding the squared expression and collecting like terms, we can rewrite the equation in standard form as

$$y = 2(x^2 - 6x + 9) - 8$$
$$y = 2x^2 - 12x + 10$$

The vertex of this parabola is

$$x_v = \frac{-(-12)}{2(2)} = 3$$
$$y_v = 2(3)^2 - 12(3) + 10 = -8$$

and its graph is shown at right.



Notice that the coordinates of the vertex, $(3, -8)$, are apparent in the original equation; we don't need to do any computation to find the vertex.

$$y = 2(x - \mathbf{3})^2 \quad \mathbf{-8}$$
$$\qquad\qquad \boldsymbol{x_v} \qquad \boldsymbol{y_v}$$

This equation is an example of the **vertex form** for a quadratic function.

Vertex Form for a Quadratic Equation.

A quadratic equation $y = ax^2 + bx + c$, $a \neq 0$, can be written in the vertex form

$$y = a(x - x_v)^2 + y_v$$

where the vertex of the graph is (x_v, y_v).

Example 4.26

Find the vertex of the graph of $y = -3(x-4)^2 + 6$. Is the vertex a maximum or a minimum point of the graph?

Solution. We compare the equation to the vertex form to see that the coordinates of the vertex are $(4, 6)$. For this equation, $a = -3 < 0$, so the parabola opens downward. The vertex is the maximum point of the graph.

To understand why the vertex form works, substitute $x_v = \mathbf{4}$ into $y = -3(x-4)^2 + 6$ from Example 4.26, p. 224 to find

$$y = -3(\mathbf{4} - 4)^2 + 6 = 6$$

which confirms that when $x = 4$, $y = 6$. Next, notice that if x is any number except 4, the expression $-3(x-4)^2$ is negative, so $y < 6$. Therefore, 6 is the maximum value for y on the graph, so $(4, 6)$ is the high point or vertex.

You can also rewrite $y = -3(x-4)^2 + 6$ in standard form and use the formula $x_v = \dfrac{-b}{2a}$ to confirm that the vertex is the point $(4, 6)$.

Checkpoint 4.27 QuickCheck 2. Which of the following is the vertex form for a parabola with vertex $(-2, 3)$?

- ⊙ $y = 4(x-2) + 3$
- ⊙ $y = 4(x-2)^2 + 3$
- ⊙ $y = 4(x+2)^2 + 3$
- ⊙ $y = 4(x-3)^2 + 2$

Answer. Choice 3

Solution. $y = 4(x+2)^2 + 3$

Checkpoint 4.28 Practice 4.

a. Find the vertex of the graph of $y = 5 - \dfrac{1}{2}(x+2)^2$.

Vertex: _____

b. Write the equation of the parabola in standard form.

$y =$______________

Answer 1. $(-2, 5)$

Answer 2. $\frac{-1}{2}x^2 - 2x + 3$

Solution.

a. $(-2, 5)$

b. $y = -\dfrac{1}{2}x^2 - 2x + 3$

Any quadratic equation in vertex form can be written in standard form by expanding, and any quadratic equation in standard form can be put into vertex form by completing the square.

Example 4.29

Write the equation $y = 3x^2 - 6x - 1$ in vertex form and find the vertex of its graph.

Solution. We factor the lead coefficient, 3, from the variable terms, leaving a space to complete the square.

$$y = 3(x^2 - 2x\_\_\_\_\_) - 1$$

Next, we complete the square inside parentheses. Take half the coefficient of x and square the result:

$$p = \frac{1}{2}(-2) = -1, \quad \text{and} \quad p^2 = (-1)^2 = 1$$

We must add 1 to complete the square. However, we are really adding 3(1) to the right side of the equation, so we must also subtract 3 to compensate:

$$y = 3(x^2 - 2x \mathbf{+ 1}) - 1 \mathbf{- 3}$$

The expression inside parentheses is now a perfect square, and the vertex form is

$$y = 3(x - 1)^2 - 4$$

The vertex of the parabola is $(1, -4)$.

Checkpoint 4.30 **Practice 5.** Write the equation $y = 2x^2 + 12x + 13$ in vertex form, and find the vertex of its graph.

The vertex form is $y = a(x - x_v)^2 + y_v$, where

$a =$__

$x_v =$__

$y_v =$__

The vertex of the graph is _____

Hint.

1. Factor 2 from the variable terms.
2. Complete the square inside parentheses.
3. Subtract $2p^2$ outside parentheses.
4. Write the vertex form.

Answer 1. 2

Answer 2. -3

Answer 3. -5

Answer 4. $(-3, -5)$

Solution. $y = 2(x + 3)^2 - 5$; $(-3, -5)$

Using the Vertex Form

If we know the vertex of a parabola and one other point, we can use the vertex form to find its equation.

Example 4.31

When Andre practices free-throws at the park, the ball leaves his hands at a height of 7 feet, and reaches the vertex of its trajectory 10 feet away at a height of 11 feet.

a Find a quadratic equation for the ball's trajectory.

b Do you think Andre's free-throw would score on a regulation basketball court, where the hoop is 15 feet from the shooter and 10 feet high?

Solution.

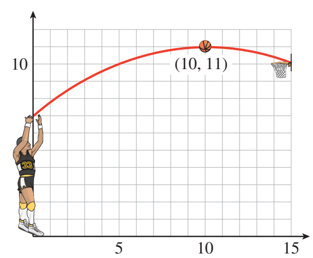
a If Andre's feet are at the origin, then the vertex of the ball's trajectory is the point $(\mathbf{10}, \mathbf{11})$, and its y-intercept is $(0, 7)$. We start with the vertex form for a parabola.

$$y = a(x - x_v)^2 + y_v$$
$$y = a(x - \mathbf{10})^2 + \mathbf{11}$$

We use the point $(0, 7)$ to find the value of a.

$$\mathbf{7} = a(\mathbf{0} - 10)^2 + 11$$
$$7 = 100a + 11$$
$$a = -0.04$$

The equation of the trajectory is $y = -0.04(x - 10)^2 + 11$.



b We'd like to know if the point $(15, 10)$ is on the trajectory of Andre's free-throw. We substitute $x = \mathbf{15}$ into the equation.

$$y = -0.04(\mathbf{15} - 10)^2 + 11$$
$$y = -0.04(25) + 11 = 10$$

From our computations, we see that the point $(15, 10)$ is indeed on the trajectory. However, because Andre's shot will probably hit the backboard just where the hoop attaches and bounce off, so it is unlikely that his shot will score.

Checkpoint 4.32 QuickCheck 3. Why do we need to know a second point besides the vertex to find the equation of a parabola?

- ⊙ We need two points to find the equation of a line.
- ⊙ To find the value of a.
- ⊙ We must know one of the x-intercepts.

⊙ We must know the y-intercept.

Answer. Choice 2

Solution. To find the value of a for the vertex form $y = a(x - x_v)^2 + y_v$.

Checkpoint 4.33 Practice 6. A parabola has its vertex at $(16, 80)$ and one of its x-intercepts at $(40, 0)$. Find an equation for the parabola.

$y =$______________

Answer. $\frac{-5}{36}(x-16)^2 + 80$

Solution. $y = \dfrac{-5}{36}(x-16)^2 + 80$

Problem Set 4.2

Warm Up

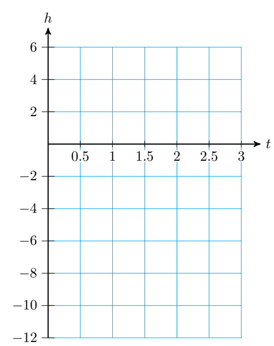
1. Francine throws a wrench into the air from the bottom of a trench 12 feet deep. Its height t seconds later is given in feet by

$$h = -12 + 32t - 16t^2$$

a Complete the table of values.

t	0	0.25	0.5	0.75	1	0.25	1.5
h							

b Graph the equation.



c Find the vertex of the graph.

d What does the h-coordinate of the vertex tell us about the wrench?

e What does the t-coordinate of the vertex tell us about the wrench?

2.

a Write an equation for a parabola that has x-intercepts at $(2, 0)$ and $(-3, 0)$.

b Write an equation for another parabola that has the same x-intercepts.

3.

a Write an equation for a parabola that opens upward with x-intercepts $(-1, 0)$ and $(4, 0)$.

b Write an equation for a parabola that opens downward with x-intercepts $(-1, 0)$ and $(4, 0)$.

4.

a The math club has \$68 in the treasury. Annual dues are \$4. If x more students join, write an expression for the amount of money in the treasury.

b The monthly dues for Rafael's condo association are \$120. However, for each new tenant who moves in, the dues will be reduced by \$5. Write an expression for the dues if x new tenants move in.

Skills Practice

For Problems 5 and 6:

a Find the vertex of the parabola.

b Sketch the graph.

5. $y = x^2 + 4x + 7$

6. $y = x^2 - 6x + 10$

For Problems 7 and 8, sketch a graph of the parabola. What is the vertex of each graph?

7.

a $y = (x - 3)^2$

b $y = -(x - 3)^2$

c $y = -(x - 3)^2 + 4$

8.

a $y = (x + 4)^2$

b $y = \frac{1}{2}(x + 4)^2$

c $y = 3 + \frac{1}{2}(x + 4)^2$

For Problems 9 and 10:

a Find the vertex of the parabola.

b Sketch the graph.

c Write the equation in standard form.

9. $y = 2(x - 3)^2 + 4$

10. $y = -\frac{1}{2}(x + 4)^2 - 3$

11.

a Write an equation for a parabola whose vertex is the point $(-2, 6)$. (Many answers are possible.)

b Find the value of a if the y-intercept of the parabola in part (a) is 18.

12.

a Write an equation for a parabola with vertex at $(0, -3)$ and one x-intercept at $(2, 0)$.

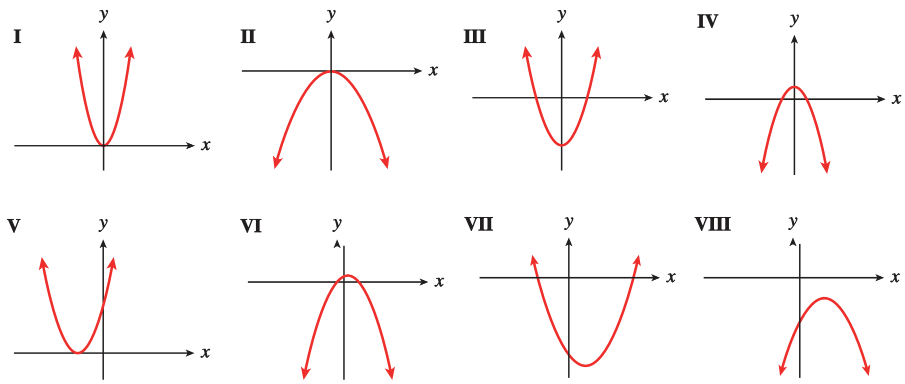
b Write an equation for a parabola with vertex at $(0, -3)$ and no x-intercepts.

For Problems 13 and 14, write the equation in the form $a(x - p)^2 + q$ by completing the square.

13. $y = 3x^2 + 6x - 2$

14. $y = -2x^2 - 8x + 3$

For Problems 15 and 16, match each equation with one of the eight graphs shown.



15.

a $y = 1 - x^2$

b $y = (x + 2)^2$

c $y = 2x^2$

d $y = (x - 4)(x + 2)$

16.

a $y = -2 - (x - 2)^2$

b $y = x - x^2$

c $y = x^2 - 4$

d $y = -0.5x^2$

Applications

17. Gavin has rented space for a booth at the county fair. As part of his display, he wants to rope off a rectangular area with 80 yards of rope.

a Let w represent the width of the roped-off rectangle, and write an expression for its length. Then write an expression in terms of w for the area A of the roped-off space.

b What is the largest area that Gavin can rope off? What will the dimensions of the rectangle be?

18. A breeder of horses wants to fence two adjacent rectangular grazing areas along a river with 600 meters of fence.

a Write an expression for the total area, A, of the grazing land in terms of the width, w, of the rectangles.

b What is the largest area she can enclose?

19. The owner of a motel has 60 rooms to rent. She finds that if she charges \$20 per room per night, all the rooms will be rented. For every \$2 that she increases the price of a room, three rooms will stand vacant.

a Complete the table. The first two rows are filled in for you.

No. of price increases	Price of a room	No. of rooms rented	Total revenue
0	20	60	1200
1	22	57	1254
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			

b Let stand for the number of \$2 price increases the owner makes. Write algebraic expressions for the price of a room, the number of rooms that will be rented, and the total revenue earned at that price.

c Use your calculator to make a table of values for your algebraic expressions. Let Y_1 stand for the price of a room, Y_2 for the number of rooms rented, and Y_3 for the total revenue. Verify the values you calculated in part (a).

d Use your table to find a value of x that causes the total revenue to be zero.

e Use your graphing calculator to graph your formula for total revenue.

f What is the lowest price that the owner can charge for a room if she wants her revenue to exceed \$1296 per night? What is the highest price she can charge to obtain this revenue?

g What is the maximum revenue the owner can earn in one night? How much should she charge for a room to maximize her revenue? How many rooms will she rent at that price?

20. A travel agent offers a group rate of \$2400 per person for a week in London if 16 people sign up for the tour. For each additional person who signs up, the price per person is reduced by \$100.

a Let x represent the number of additional people who sign up. Write expressions for the number of people signed up, the price per person, and the total revenue.

b How many people must sign up for the tour in order for the travel agent to maximize her revenue?

21. In skeet shooting, the clay pigeon is launched from a height of 4 feet and reaches a maximum height of 164 feet at a distance of 80 feet from the launch site.

a Write an equation for the height of the clay pigeon in terms of the horizontal distance it has traveled.

b If the shooter misses the clay pigeon, how far from the launch site will it hit the ground?

22. The batter in a softball game hits the ball when it is 4 feet above the ground. The ball reaches the greatest height on its trajectory, 35 feet, directly above the head of the left-fielder, who is 200 feet from home plate.

a Write an equation for the height of the softball in terms of its horizontal distance from home plate.

b Will the ball clear the left-field wall, which is 10 feet tall and 375 feet from home plate?

23. The rate at which an antigen precipitates during an antigen–antibody reaction depends on the amount of antigen present. For a fixed quantity of antibody, the time required for a particular antigen to precipitate is given in minutes by

$$t = 2w^2 - 20w + 54$$

where w is the quantity of antigen present, in grams. For what quantity of antigen will the reaction proceed most rapidly, and how long will the precipitation take?

24.

a Graph in the standard window two lines and a parabola:

$$\begin{aligned} Y_1 &= x + 2 \\ Y_2 &= 4 - x \\ Y_3 &= (x + 2)(4 - x) \end{aligned}$$

b What do you notice about the x-intercepts of the graphs?

c What do you notice about the vertex of the parabola and the intersection point of the two lines?

25.

a Graph in the standard window two lines and a parabola:

$$\begin{aligned} Y_1 &= x + 4 \\ Y_2 &= x - 2 \\ Y_3 &= (x + 4)(x - 2) \end{aligned}$$

b What are the x-intercepts of the parabola?

c By referring to your graph, complete the table showing whether the y-values are positive or negative in each region.

	$x < -4$	$-4 < x < 2$	$x > 2$
Y_1			
Y_2			
Y_3			

Curve Fitting

Earlier, we used linear regression to fit a line to a collection of data points. In this section we'll see how to fit a quadratic equation to a collection of data points.

Finding a Quadratic Equation through Three Points

Every linear equation can be written in the form

$$y = mx + b$$

To find a specific line we must find values for the two **parameters** (constants) m and b. We need two data points in order to find those two parameters. A quadratic equation, however, has three parameters, a, b, and c:

$$y = ax^2 + bx + c$$

To find these parameters we need three data points.

Example 4.34

Find values for a, b, and c so that the points $(1, 3)$, $(3, 5)$, and $(4, 9)$ lie on the graph of $y = ax^2 + bx + c$.

Solution. We substitute the coordinates of each of the three points into the equation of the parabola to obtain three equations:

$$\begin{aligned} \mathbf{3} &= a(\mathbf{1})^2 + b(\mathbf{1}) + c \\ \mathbf{5} &= a(\mathbf{3})^2 + b(\mathbf{3}) + c \\ \mathbf{9} &= a(\mathbf{4})^2 + b(\mathbf{4}) + c \end{aligned}$$

or, equivalently,

$$\begin{aligned} a + b + c &= 3 && (1) \\ 9a + 3b + c &= 5 && (2) \\ 16a + 4b + c &= 9 && (3) \end{aligned}$$

This is a system of three equations in the three unknowns a, b, and c. To solve the system, we use Gaussian reduction. We first eliminate c. We subtract Equation (1) from Equation (2) to obtain

$$8a + 2b = 2 \qquad (4)$$

and then subtract Equation (1) from Equation (3) to get

$$15a + 3b = 6 \qquad (5)$$

We now have a system of two linear equations in two variables:

$$\begin{aligned} 8a + 2b &= 2 && (4) \\ 15a + 3b &= 6 && (5) \end{aligned}$$

Next, we eliminate b from Equations (4) and (5): we add -3 times Equation (4) to 2 times Equation (5) to get

$$\begin{array}{rcrcr} -24a & - & 6b & = & -6 \\ 30a & + & 6b & = & 12 \\ \hline 6a & & & = & 6 \end{array} \qquad \begin{array}{r} -3 \times (4) \\ 2 \times (5) \end{array}$$

or $a = 1$. We substitute 1 for a in Equation (4) to find

$$\begin{aligned} 8(\mathbf{1}) + 2b &= 2 & \text{Solve for } b. \\ b &= -3 \end{aligned}$$

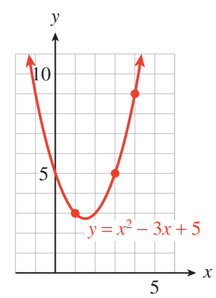
Finally, we substitute -3 for b and 1 for a in Equation (1) to find

$$\begin{aligned} \mathbf{1} + (\mathbf{-3}) + c &= 3 & \text{Solve for } c. \\ c &= 5 \end{aligned}$$

Thus, the equation of the parabola is

$$y = x^2 - 3x + 5$$

The parabola and the three points are shown below.



Checkpoint 4.35 QuickCheck 1. Fill in the blanks. To find the equation for a parabola:

a. We can use the formula $y = a(x-x_v)^2+y_v$ if we know the (☐ Gaussian reduction ☐ x-intercepts ☐ vertex ☐ three ☐ one) and one other point.

b. We can use the formula $y = a(x-r_1)(x-r_2)$ if we know the (☐ Gaussian reduction ☐ x-intercepts ☐ vertex ☐ three ☐ one) and one other point.

c. Otherwise, we must know at least (☐ Gaussian reduction ☐ x-intercepts ☐ vertex ☐ three ☐ one) points on the graph.

d. In that case, we use (☐ Gaussian reduction ☐ x-intercepts ☐ vertex ☐ three ☐ one) to solve for the parameters of the equation.

Answer 1. vertex

Answer 2. x-intercepts

Answer 3. three

Answer 4. Gaussian reduction

Solution.

a. vertex

b. x-intercepts

c. three

d. Gaussian reduction

Checkpoint 4.36 Practice 1.

a. Find the equation of a parabola

$$y = ax^2 + bx + c$$

that passes through the points $(0, 80)$, $(15, 95)$, and $(25, 55)$.

$y =$______________

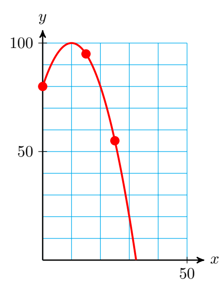
b. Plot the data points and sketch the parabola.

Answer. $\frac{-1}{5}x^2 + 4x + 80$

Solution.

a. $y = \dfrac{-1}{5}x^2 + 4x + 80$

b. A graph is below.



Applications

The simplest way to fit a parabola to a set of data points is to pick three of the points and find the equation of the parabola that passes through those three points.

Example 4.37

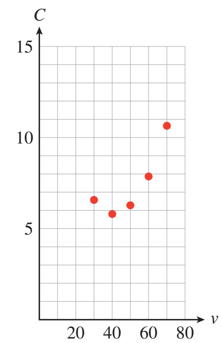
Major Motors Corporation is testing a new car designed for in-town driving. The data below show the cost of driving the car at different speeds. The speeds, v, are given in miles per hour, and the cost, C, includes fuel and maintenance for driving the car 100 miles at that speed.

v	30	40	50	60	70
C	6.50	6.00	6.20	7.80	10.60

Find a possible quadratic model, $C = av^2 + bv + c$, that expresses C in terms of v.

Solution.

When we plot the data, it is clear that the relationship between v and C is not linear, but it may be quadratic, as shown at right. We will use the last three data points, $(50, 6.20)$, $(60, 7.80)$, and $(70, 10.60)$, to fit a parabola to the data. We would like to find the coefficients a, b, and c of a parabola $C = av^2+bv+c$ that includes the three data points. This gives us a system of equations:



$$2500a + 50b + c = 6.20 \quad (1)$$
$$3600a + 60b + c = 7.8 \quad (2)$$
$$4900a + 70b + c = 10.6 \quad (3)$$

Eliminating c from Equations (1) and (2) yields Equation (4), and eliminating c from Equations (2) and (3) yields Equation (5).

$$1100a + 10b = 1.60 \quad (4)$$
$$1300a + 10b = 2.8 \quad (5)$$

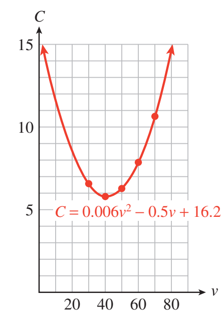
Eliminating b from Equations (4) and (5) gives us

$$200a = 1.20$$
$$a = 0.006$$

We substitute this value into Equation (4) to find $b = -0.5$, then substitute both values into Equation (1) to find $c = 16.2$.
Thus, our quadratic model is

$$C = 0.006v^2 - 0.5v + 16.2$$

The graph of this equation, along with the data points, is shown at right.



As was the case with linear regression, the graph of the regression equation may not pass through all of the data points, but it should be close to most of them.

Checkpoint 4.38 Practice 2. The data below show Americans' annual per capita consumption of chicken for several years since 1985.

Year	1986	1987	1988	1989	1990
Pounds of chicken	51.3	55.5	57.4	60.8	63.6

a. Use the values for 1987 through 1989 to fit a quadratic equation to the data, $C = at^2 + bt + c$, where t is measured in years since 1985.

$C =$________________

b. What does your equation predict for per capita chicken consumption in 1990?

Answer: ___ lbs

c. Sketch the graph of your equation and the given data. Does your model provide a good fit for the data?

Answer 1. $0.75t^2 + (-1.85)\,t + 56.2$

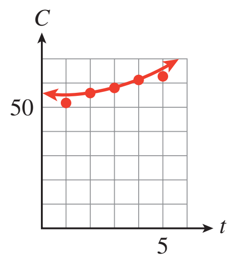
Answer 2. 65.7

Solution.

a. $C = 0.75t^2 - 1.85t + 56.2$

b. 65.7 lbs

c. A graph is below.



Checkpoint 4.39 **QuickCheck 2.** True or False

a. We can plot the data points to see what type of curve is appropriate as a model. (☐ True ☐ False)

b. We write a system of equations in which the y-coordinates of the data points are the unknown values. (☐ True ☐ False)

c. A good regression equation will pass through all of the data points. (☐ True ☐ False)

d. According to the model in the previous Example, higher speeds always result in higher driving costs. (☐ True ☐ False)

Answer 1. True

Answer 2. False

Answer 3. False

Answer 4. False

Solution.

a. True

b. False

c. False

d. False

Using a Calculator for Quadratic Regression

We can use a graphing calculator to find an approximate quadratic fit for a set of data. The procedure is similar to the steps for linear regression.

Example 4.40

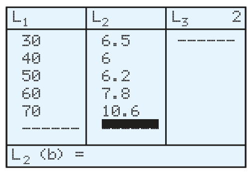
a Use your calculator to find a quadratic fit for the data in Example 37, p. 234.

b How many of the given data points actually lie on the graph of the quadratic approximation?

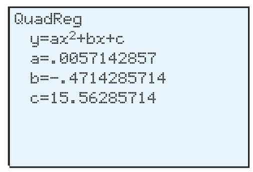
Solution.

a We press STAT ENTER and enter the data under columns L_1 and L_2, as shown below. Next, we calculate the quadratic regression equation and store it in Y_1 by pressing STAT → 5 VARS → 1 1 ENTER.

The regression equation has the form $y = ax^2 + bx + c$, where $a = 0.0057$, $b = -0.47$, and $c = 15.56$. Notice that a, b, and c are all close to the values we computed in Example 4.37, p. 234.



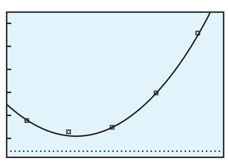
(a)



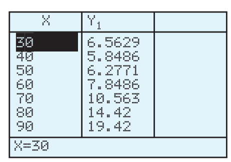
(b)

b Next, we will graph the data and the regression equation. We press Y= and select *Plot1*, then press ZOOM 9 to see the graph shown below. The parabola seems to pass close to all the data points.

However, try using either the *value* feature or a table to find the y-coordinates of points on the regression curve. By comparing these y-coordinates with our original data points, we find that none of the given data points lies precisely on the parabola.



(a)



(b)

Checkpoint 4.41 Practice 3. To test the effects of radiation, a researcher irradiated male mice with various dosages and bred them with unexposed female mice. The table below shows the fraction of fertilized eggs that survived, as a function of the radiation dosage. (Source: Strickberger, Monroe W., 1976)

Radiation (rems)	100	300	500	700	900	1100	1500
Relative survival of eggs	0.94	0.700	0.544	0.424	0.366	0.277	0.195

a. Enter the data into your calculator and create a scatterplot. Does the graph appear to be linear? Does it appear to be quadratic? (☐ linear ☐ quadratic)

b. Fit a quadratic regression equation to the data and graph the equation on the scatterplot.

$y =$ ______________________

Answer 1. quadratic

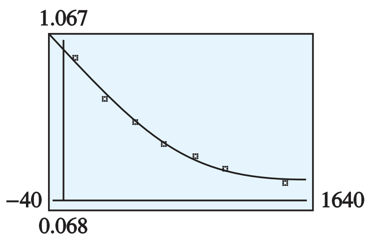
Answer 2. $(3.65014 \times 10^{-7})x^2 - 0.001x + 1.02$

Solution.

a. The graph appears to be quadratic.

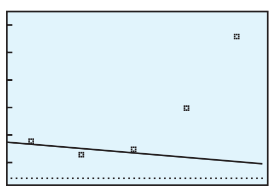
b. $y = 3.65 \times 10^{-7}x^2 - 0.001x + 1.02$

Graph for part (a):



Choosing an Appropriate Model

We must be careful that our data set gives a complete picture of the situation we want to model. A regression equation may fit a particular collection of data and still be a poor model if the rest of the data diverge from the regression graph.



In Example 4.37, p. 234, suppose Major Motors had collected only the first three data points and fit a line through them, as shown at left. This regression line gives poor predictions for the cost of driving at 60 or 70 miles per hour.

Example 4.42

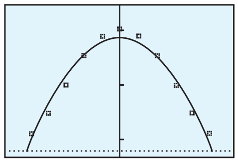
Delbert records the height of the tip of the minute hand on the classroom's clock at different times. The data are shown in the table, where time is measured in minutes since noon. (A negative time indicates a number of minutes before noon.) Find a quadratic regression equation for the data and use it to predict the height of the minute hand's tip at 40 minutes past noon. Do you believe this prediction is valid?

Time (minutes)	−25	−20	−15	−10	−5	0	5	10	15	20	25
Height (feet)	7.13	7.50	8.00	8.50	8.87	9.00	8.87	8.50	8.80	7.50	7.13

Solution. We enter the time data under L_1 and the height data under L_2. Then we calculate and store the quadratic regression equation in Y_1, as we did in Example 4.40, p. 237. The regression equation is

$$y = -0.00297x^2 + 0x + 8.834$$

From either the graph of the regression equation or from the table (see figure below), we can see that the fit is not perfect, although the curve certainly fits the data better than any straight line could.



(a)

X	Y_1
-25	6.9762
-20	7.645
-15	8.1652
-10	8.5368
-5	8.7597
0	8.834
5	8.7597

X=-25

(b)

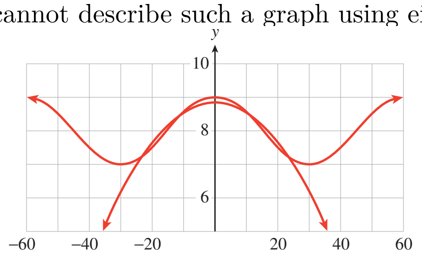
X	Y_1
10	8.5368
15	8.1652
20	7.645
25	6.9762
30	6.1588
35	5.1927
40	4.078

X=40

(c)

If we scroll down the table, we find that this equation predicts a height of approximately 4.08 feet at time 40 minutes. (See figure (c).) This is a preposterous estimate! The position of the minute hand at 40 minutes after noon should be the same as it was exactly one hour earlier (at 20 minutes before noon), when it was 7.50 feet.

Caution 4.43 Using the wrong type of function to fit the data is a common error in making predictions. In the Example above, we know that the minute hand of a clock repeats its position every 60 minutes. The graph of the height of its tip oscillates up and down, repeating the same pattern over and over. We cannot describe such a graph using either a linear or a quadratic function.



The graph of the height is shown at left, along with the graph of our quadratic regression equation. You can see that the regression equation fits the actual curve only on a small interval.

Even though your calculator can always compute a regression equation,that equation is not necessarily appropriate for your data. Choosing a reasonable type of regression equation for a particular data set requires knowledge of different kinds of models and the physical or natural laws that govern the situation at hand.

Checkpoint 4.44 QuickCheck 3. True or False

a. Your calculator can choose the correct type of regression equation for a data set. (☐ True ☐ False)

b. It is only necessary to use the first and last data points to compute a regression equation. (☐ True ☐ False)

c. A regression equation may fit some of the data but still be a poor model. (☐ True ☐ False)

d. A good regression equation should fit all the data points exactly. (☐ True ☐ False)

Answer 1. False
Answer 2. False
Answer 3. True
Answer 4. False
Solution.

a. False

b. False

c. True

d. False

Checkpoint 4.45 **Practice 4.** A speeding motorist slams on the brakes when she sees an accident directly ahead of her. The distance she has traveled t seconds after braking is shown in the table.

Time (seconds)	0	0.5	1.0	1.5	2.0	2.5
Distance (feet)	0	51	95	131	160	181

a. Enter the data into your calculator and create a scatterplot. Fit a quadratic regression equation to the data and graph the equation on the scatterplot.

   Regression equation: $y =$_____________ Use x for the time in seconds.

b. Use your regression equation to find the vertex of the parabola: ________

   What do the coordinates represent in terms of the problem?

Answer 1. $-15x^2 + 109.957x - 0.0714286$

Answer 2. $(3.66524, 201.438)$

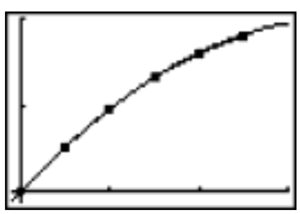
Solution.

a. A graph is below.

   $y = -15x^2 + 110x - 0.07$

b. $(3.67, 201)$: The car came to a stop in 3.67 seconds, after sliding 201 feet.

Graph for part (a):



Problem Set 4.3

Warm Up

For Problems 1 and 2, state the vertex, the y-intercept, and the x-intercepts of the parabola.

1. $y = -2(x-4)^2 + 10$

2. $y = \dfrac{1}{3}(x+1)^2 - 4$

3. You are in charge of selling tickets to a one-woman show at a local art gallery. Tickets to the opening night were priced at \$25, and you sold 30 tickets.

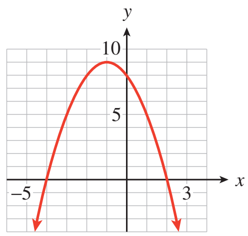
   a Every night after the opening, you reduce the ticket price by \$2. What is the ticket price after x nights?

   b Every night after the opening, you sell 4 more tickets than the previous night. How many tickets did you sell x nights after the opening?

4. The point $(-3, 8)$ lies on the graph of $y = ax^2 + bx + c$. Write an equation involving a, b, and c.

For Problems 5–8, find an equation for the parabola. Use the vertex form or the factored form of the equation, whichever is more appropriate.

5.



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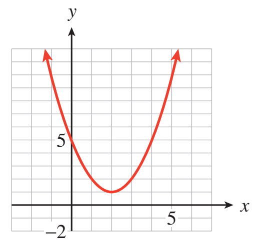

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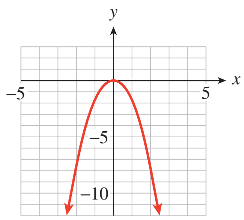

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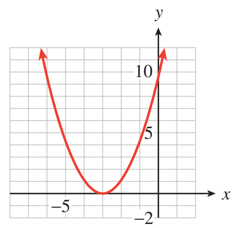


Skills Practice

For Problems 9–12, solve the system by elimination. Begin by eliminating c.

9.

$$\begin{aligned} a+b+c &= -3 \\ a-b+c &= -9 \\ 4a+2b+c &= -6 \end{aligned}$$

10.

$$\begin{aligned} a+b+c &= 10 \\ 4a+2b+c &= 19 \\ 9a+3b+c &= 38 \end{aligned}$$

11.

$$\begin{aligned} a-b+c &= 12 \\ 4a-2b+c &= 19 \\ 9a+3b+c &= 4 \end{aligned}$$

12.

$$\begin{aligned} 4a+2b+c &= 14 \\ 9a-3b+c &= -41 \\ 16a-4b+c &= -70 \end{aligned}$$

13. Find values for a, b, and c so that the graph of $y = ax^2 + bx + c$ includes the points $(-1, 0)$, $(2, 12)$, and $(-2, 8)$.

14. Find values for a, b, and c so that the graph of $y = ax^2 + bx + c$ includes the points $(-1, 2)$, $(1, 6)$, and $(2, 11)$.

Applications

15. The data show the number of people of certain ages who were the victims of homicide in a large city last year.

Age	10	20	30	40
Number of victims	12	62	72	40

a Use the first three data points to fit a quadratic equation to the data, $N = ax^2 + bx + c$, where x represents age.

b What does your equation predict for the number of 40-year-olds who were the victims of homicide?

c Sketch the graph of your quadratic equation and the given data on the same axes.

16. Sara plans to start a side business selling eggs. She finds that the total number of eggs produced each day depends on the number of hens confined in the henhouse, as shown in the table.

Number of hens, n	15	20	25	30	36	39
Number of eggs, E	15	18	20	21	21	20

a Use the first three data points to find a quadratic model, $E = an^2 + bn + c$.

b Plot the data and sketch the model on the same axes.

c What does the model predict for the number of eggs produced when 39 hens are confined in the henhouse?

17. Find a quadratic model for the number of diagonals that can be drawn in a polygon of n sides. Some data are provided.

Sides	4	5	6	7
Diagonals	2	5	9	14

18. You are driving at a speed of 60 miles per hour when you step on the brakes. Find a quadratic model for the distance in feet that your car travels in t seconds after braking. Some data are provided.

Seconds	1	2	3	4
Feet	81	148	210	267

19. In the 1990's, an outbreak of mad cow disease (Creutzfeldt-Jakob disease) alarmed health officials in England. The table shows the number of deaths each year from the disease.

Year	94	95	96	97	98	99	00	01	02	03	04
Deaths	0	3	10	10	18	15	28	20	17	19	9

(Source: www.cjd.ed.ac.uk/vcjdqsep05)

a The Health Protection Agency determined that a quadratic model was the best-fitting model for the data. Find a quadratic regression equation for the data.

b Use your model to estimate when the peak of the epidemic occurred, and how many deaths from mad cow disease were expected in 2005.

20. The table shows the height in kilometers of a star-flare at various times after it exploded from the surface of a star.

Time (seconds)	0.2	0.4	0.6	0.8	1.0	1.2
Height (kilometers)	6.8	12.5	17.1	20.5	22.8	23.9

a Find the equation of the least-squares regression line for the height of the flare in terms of time.

b Use the regression line to predict the height of the flare 1.4 seconds after it exploded.

c Make a scatterplot of the data and draw the regression line on the same axes.

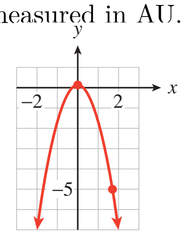
d Find the quadratic regression equation for the height in terms of time.

e Use the quadratic regression equation to predict the height of the flare 1.4 seconds after it exploded.

f Draw the quadratic regression curve on the graph from part (c).

g Which model is more appropriate for the height of the star-flare, linear or quadratic? Why?

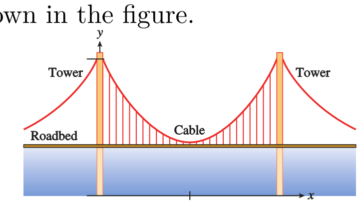
21. Some comets move about the sun in parabolic orbits. In 1973 the comet Kohoutek passed within 0.14 AU (**astronomical units**), or 21 million kilometers of the sun. Imagine a coordinate system superimposed on a diagram of the comet's orbit, with the sun at the origin, as shown below. The units on each axis are measured in AU.



a The comet's closest approach to the sun (called perihelion) occurred at the vertex of the parabola. What were the comet's coordinates at perihelion?

b When the comet was first discovered, its coordinates were $(1.68, -4.9)$. Find an equation for comet Kohoutek's orbit in vertex form.

22. The Akashi Kaikyo bridge in Japan is the longest suspension bridge in the world, with a main span of 1991 meters. Its main towers are 297 meters tall. The roadbed of the bridge is 14 meters thick and clears the water below by 65 meters. The cables on a suspension bridge hang in the shape of parabolas. Imagine a coordinate system superimposed on the diagram of the bridge, as shown in the figure.



a Find the coordinates of the vertex and one other point on the cable.

b Use the points from part (a) to find an equation for the shape of the cable in vertex form.

Quadratic Inequalities

Solving Inequalities Graphically

The easiest way to solve a quadratic inequality is with a graph.

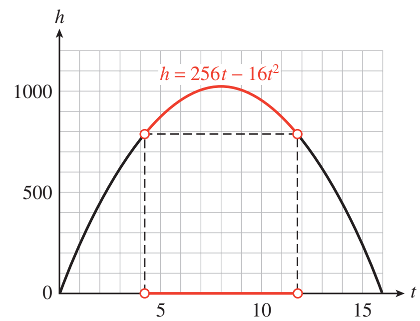
Example 4.46

The Chamber of Commerce in River City plans to put on a Fourth of July fireworks display. City regulations require that fireworks at public gatherings explode higher than 800 feet above the ground. The mayor particularly wants to include the Freedom Starburst model, which is launched from the ground. Its height after t seconds is given by

$$h = 256t - 16t^2$$

When should the Starburst explode in order to satisfy the safety regulation?

Solution. We can get an approximate answer to this question by looking at the graph of the rocket's height, shown below.



We would like to know when the rocket's height is greater than 800 feet, or, in mathematical terms, for what values of t is $h > 800$? The answer to this question is the solution of the inequality

$$256t - 16t^2 > 800$$

Points on the graph with $h > 800$ are shown in color, and the t-coordinates of those points are marked on the horizontal axis. If the Freedom Starburst explodes at any of these times, it will satisfy the safety regulation.

From the graph, the safe time interval runs from approximately 4.25 seconds to 11.75 seconds after launch. The solution of the inequality is the set of all t-values greater than 4.25 but less than 11.75.

Note 4.47 The solution set in Example 4.46, p. 243 is called a **compound inequality**, because it involves more than one inequality symbol. We write this inequality as

$$4.25 < t < 11.75$$

and read "t greater than 4.25 but less than 11.75."

Checkpoint 4.48 **QuickCheck 1.** Which is the correct way to write "x is greater than 3 and less than 8"?

- ⊙ $8 > x < 3$
- ⊙ $3 < x < 8$
- ⊙ $8 < x < 3$
- ⊙ $3 < x > 8$

Answer. Choice 2

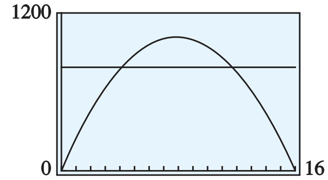
Solution. $3 < x < 8$

Technology 4.49 **Solving an Inequality With a Graphing Calculator.** You can use your graphing calculator to solve the problem in Example 4.46,

p. 243. Graph the two functions

$$Y_1 = 256X - 16X^2$$
$$Y_2 = 800$$

on the same screen. Use WINDOW settings to match the graph in Example 4.46, p. 243.



Then use the intersect feature to find the points where the two graphs intersect, at about $x = 4.26$ and $x = 11.74$. Both these points y-coordinate 800, and between the points the parabola is above the line, so $h > 800$ when $4.26 < t < 11.74$.

Checkpoint 4.50 Practice 1.

a. Graph the function $y = x^2 - 2x - 9$ in the window

$$\text{Xmin} = -9.4 \quad \text{Xmax} = 9.4$$
$$\text{Ymin} = -10 \quad \text{Ymax} = 10$$

b. Use the graph to solve the inequality $x^2 - 2x - 9 \leq 6$.

Solution: ____________

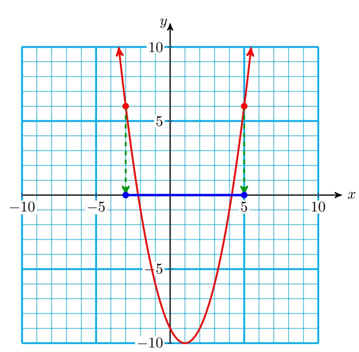
Answer. $-3 \leq x \leq 5$

Solution.

a. A graph is below.

b. $-3 \leq x \leq 5$

$y = x^2 - 2x - 9$:



Using the x-Intercepts

Because it is easy to decide whether the y-coordinate of a point on a graph is positive or negative (the point lies above the x-axis or below the x-axis), we often rewrite a given inequality so that one side is zero.

Example 4.51

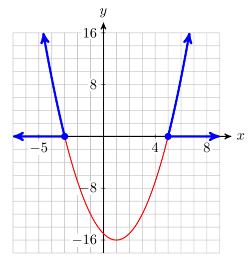
Use a graph to solve $x^2 - 2x - 3 \geq 12$

Solution. We first write the inequality with zero on one side:

$$x^2 - 2x - 15 \geq 0.$$

We would like to find points on the graph of $y = x^2 - 2x - 15$ that have y-coordinates greater than or equal to zero. A graph of the equation is shown below.

You can check that the x-intercepts of the graph are -3 and 5. The points shown in blue on the graph lie above the x-axis and have $y \geq 0$, so the x-coordinates of these points are the solutions of the inequality.



Note that the solutions lie in two intervals, less than -3 or greater than 5. Thus, the solution is $x \leq -3$ or $x \geq 5$.

Caution 4.52 In Example 4.51, p. 246 above, the solution of the inequality $x^2 - 2x - 15 \geq 0$ is the set of values

$$x \leq -3 \quad \text{or} \quad x \geq 5$$

This set is another type of compound inequality, and its graph consists of two pieces. Therefore, we cannot write the solution as a single inequality. For instance, it would be *incorrect* to describe the solution set as $-3 \geq x \geq 5$, because this notation implies that $-3 \geq 5$. We must write the solution as two parts: $x \leq -3$ or $x \geq 5$.

Checkpoint 4.53 QuickCheck 2. Which is the correct way to write "x is less than 6 or greater than 10"?

- ⊙ $6 < x > 10$
- ⊙ $10 < x < 6$
- ⊙ $x > 6$ or $x > 10$
- ⊙ $10 > x < 6$

Answer. Choice 3

Solution. $x < 6$ or $x > 10$

Checkpoint 4.54 Practice 2. Follow the steps below to solve the inequality $36 + 6x - x^2 \leq 20$.

a. Rewrite the inequality so that the right side is zero.

   ___________ ≤ 0

b. Graph the equation $y = 16 + 6x - x^2$.

c. Locate the points on the graph with y-coordinate less than zero, and mark the x-coordinates of the points on the x-axis.

Answer. $16 + 6x - x^2$

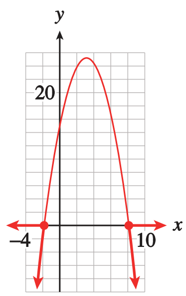
Solution.

a. $16 + 6x - x^2 \leq 0$

b. A graph is below.

c. See graph

$y = 16 + 6x - x^2$.



Interval Notation

The solution set in Example 4.46, p. 243, namely $4.26 < t < 11.74$, is called an interval. An **interval** is a set that consists of all the real numbers between two numbers a and b. An interval may include one or both of its endpoints.

Interval Notation.

1 The **closed interval** $[a, b]$ is the set $a \leq x \leq b$.

2 The **open interval** (a, b) is the set $a < x < b$.

3 Intervals may also be **half-open** or **half-closed**.

4 The **infinite interval** $[a, \infty)$ is the set $x \geq a$.

5 The **infinite interval** $(-\infty, a]$ is the set $x \leq a$.

The symbol ∞, infinity, does not represent a specific real number; it indicates that the interval continues forever along the real line. A set consisting of two or more intervals is called the **union** of the intervals. For example, the solution to Example 4.51, p. 246 is denoted in interval notation by $(-\infty, -3) \cup (5, \infty)$.

Many solutions of inequalities are intervals or unions of intervals.

Example 4.55

Write the solution sets with interval notation, and graph the solution set on a number line.

a $3 \leq x < 6$

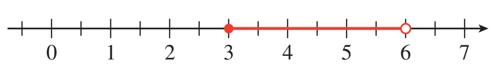
b $x \geq -9$

c $x \leq 1$ or $x > 4$

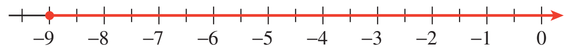
d $-8 < x \leq -5$ or $-1 \leq x < 3$

Solution.

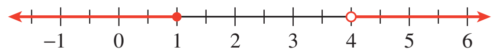
a $[3, 6)$. This is called a half-open or half-closed interval.



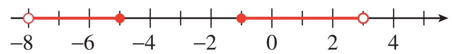
b $[-9, \infty)$. We always use round brackets next to the symbol ∞ because ∞ is not a specific number and is not included in the set.



c $(-\infty, 1] \cup (4, \infty)$. The word "or" describes the union of two sets.



d $(-8, -5] \cup [-1, 3)$.



Checkpoint 4.56 Practice 3. Write the solutions to Practice 1 and Practice 2 in interval notation.

Practice 1 answer: _____

Practice 1 answer: ______________

Answer 1. $[-3, 5]$

Answer 2. $(-\infty, -2] \cup [8, \infty)$

Solution. Practice 1: $[-3, 5]$

Practice 2: $(-\infty, -2] \cup [8, \infty)$

Checkpoint 4.57 QuickCheck 3. True or False

a. An open interval includes only the numbers between its endpoints. (☐ True ☐ False)

b. Infinite intervals include the number ∞. (☐ True ☐ False)

c. The union of two intervals includes all numbers that lie in both intervals. (☐ True ☐ False)

d. An interval must be either closed or open. (☐ True ☐ False)

Answer 1. True

Answer 2. False

Answer 3. True

Answer 4. False

Solution.

a. True

b. False

c. True

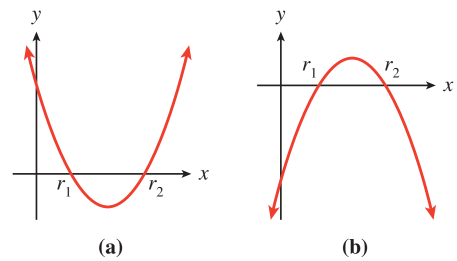
d. False

Solving Quadratic Inequalities Algebraically

Although a graph is very helpful in solving inequalities, it is not completely necessary. Every quadratic inequality can be put into one of the forms

$$ax^2 + bx + c < 0, \qquad ax^2 + bx + c > 0$$
$$ax^2 + bx + c \leq 0, \qquad ax^2 + bx + c \geq 0$$

All we really need to know is whether the corresponding parabola $y = ax^2 + bx + c$ opens upward or downward. Consider the parabolas shown below.



The parabola in figure (a) opens upward. It crosses the x-axis at two points, $x = r_1$ and $x = r_2$. At these points, $y = 0$.

- The graph lies below the x-axis between r_1 and r_2, so the solutions to the inequality $y < 0$ lie between r_1 and r_2.
- The graph lies above the x-axis for x-values less than r_1 or greater than r_2, so the solutions to the inequality $y > 0$ are $x < r_1$ or $x > r_2$.

If the parabola opens downward, as in figure (b), the situation is reversed. The solutions to the inequality $y > 0$ lie between the x-intercepts, and the solutions to $y < 0$ lie outside the x-intercepts.

From the graphs, we see that the x-intercepts are the boundary points between the portions of the graph with positive y-coordinates and the portions with negative y-coordinates. To solve a quadratic inequality, we need only locate the x-intercepts of the corresponding graph and then decide which intervals of the x-axis produce the correct sign for y.

To solve a quadratic inequality algebraically.

1 Write the inequality in standard form: One side is zero, and the other has the form $ax^2 + bx + c$.

2 Find the x-intercepts of the graph of $y = ax^2 + bx + c$ by setting $y = 0$ and solving for x.

3 Make a rough sketch of the graph, using the sign of a to determine whether the parabola opens upward or downward.

4 Decide which intervals on the x-axis give the correct sign for y.

Example 4.58

Solve the inequality $36 + 6x - x^2 \le 20$ algebraically.

Solution.

1 We subtract 20 from both sides of the inequality so that we have 0 on the right side.

$$16 + 6x - x^2 \le 0$$

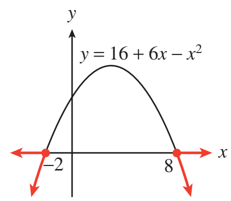
2 Consider the equation $y = 16+6x-x^2$. To locate the x-intercepts, we set $y = 0$ and solve for x.

$16 + 6x - x^2 = 0$	Multiply each term by -1.
$x^2 - 6x - 16 = 0$	Factor the left side.
$(x - 8)(x + 2) = 0$	Apply the zero-factor principle.

$$x - 8 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = 8 \quad \text{or} \quad x = -2$$

The x-intercepts are $x = -2$ and $x = 8$.

3 Make a rough sketch of the graph of $y = 16 + 6x - x^2$, as shown below. Because $a = -1 < 0$, the graph is a parabola that opens downward.



4 We are interested in points on the graph for which $y \leq 0$. The points with negative y-coordinates (that is, points below the x-axis) lie outside the x-intercepts of the graph, so the solution of the inequality is $x \leq -2$ or $x \geq 8$. Or, using interval notation, the solution is $(-\infty, -2] \cup [8, \infty)$.

Caution 4.59 Many people think that the inequality signs in the solution should point in the same direction as the sign in the original problem, and hence would incorrectly write the solution to Example 4.58, p. 249 as $x \leq -2$ or $x \leq 8$. However, you can see from the graph that this is incorrect. Remember that the graph of a quadratic equation is a parabola, not a straight line!

Checkpoint 4.60 **Practice 4.** Solve $x^2 < 20$.

Answer: ____________________

Hint.

1. Write the inequality in standard form.
2. Find the x-intercepts of the corresponding graph. Use extraction of roots.
3. Make a rough sketch of the graph.
4. Decide which intervals on the x-axis give the correct sign for y.

Answer. $-\left(\sqrt{20}\right) < x < \sqrt{20}$

Solution. $-\sqrt{20} < x < \sqrt{20}$

Checkpoint 4.61 **QuickCheck 4.** Which is the correct solution for $x^2 > 16$?

- ⊙ $x > 4$
- ⊙ $x > 4$ or $x > -4$
- ⊙ $4 < x < -4$
- ⊙ $x < -4$ or $x > 4$

Answer. Choice 4

Solution. $x < -4$ or $x > 4$

Applications

If we cannot find the x-intercepts of the graph by factoring or extraction of roots, we can use the quadratic formula.

Example 4.62

TrailGear, Inc. manufactures camping equipment. The company finds that the profit from producing and selling x alpine parkas per month is given, in dollars, by

$$P = -0.8x^2 + 320x - 25,200$$

How many parkas should the company produce and sell each month if it must keep the profits above \$2000?

Solution.

1 We would like to solve the inequality

$$-0.8x^2 + 320x - 25,200 > 2000$$

or, subtracting 2000 from both sides,

$$-0.8x^2 + 320x - 27,200 > 0$$

2 Consider the equation

$$y = -0.8x^2 + 320x - 27,200$$

We locate the x-intercepts of the graph by setting $y = 0$ and solving for x. We will use the quadratic formula to solve the equation

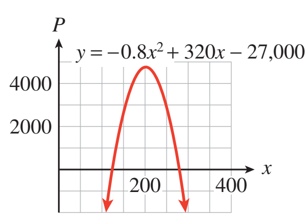
$$-0.8x^2 + 320x - 27,200 = 0$$

so $a = \mathbf{-0.8}$, $b = \mathbf{320}$, and $c = \mathbf{-27,200}$. We substitute these values into the quadratic formula to obtain

$$\begin{aligned} x &= \frac{-(\mathbf{320}) \pm \sqrt{(\mathbf{320})^2 - 4(-0.8)(\mathbf{-27,200})}}{2(\mathbf{-0.8})} \\ &= \frac{-320 \pm \sqrt{102,400 - 87,040}}{-1.6} \\ &= \frac{-320 \pm \sqrt{15,360}}{-1.6} \end{aligned}$$

To two decimal places, the solutions to the equation are 122.54 and 277.46.

3 The graph of the equation is a parabola that opens downward, because the coefficient of x^2 is negative.



4 The graph lies above the x-axis, and hence $y > 0$, for x-values between the two x-intercepts, that is, for $122.54 < x < 277.46$. Because we cannot produce a fraction of a parka, we restrict the interval to the closest whole number x-values included, namely 123 and 277.

Thus, TrailGear can produce as few as 123 parkas or as many as 277 parkas per month to keep its profit above \$2000.

Checkpoint 4.63 Practice 5. Solve the inequality $10 - 8x + x^2 > 4$.
Answer: ______________________

Hint.

1. Write the inequality in standard form.
2. Find the x-intercepts of the corresponding graph. Use extraction of roots.
3. Make a rough sketch of the graph.
4. Decide which intervals on the x-axis give the correct sign for y.

Answer. $\left(-\infty, 4-\sqrt{10}\right] \cup \left[4+\sqrt{10}, \infty\right)$

Solution. $(-\infty, 4-\sqrt{10}] \cup [4+\sqrt{10}, \infty)$

Checkpoint 4.64 QuickCheck 5. If you cannot find the x-intercepts of a parabola by factoring, what should you do?

- ⊙ A) Panic
- ⊙ B) Use the$y$-intercept instead
- ⊙ C) Use the quadratic formula
- ⊙ D) Go on to the next problem

Answer. Choice 3

Solution. Use the quadratic formula

Problem Set 4.4

Warm Up

For Problems 1–4,

a Find the x-intercepts of the parabola.

b Decide whether the parabola opens up or down.

1. $y = x^2 - 2x - 24$

2. $y = 40 - x^2$

3. $y = 12 - (x-3)^2$

4. $y = x^2 + 3x + 1$

For Problems 5 and 6, write the set with interval notation, and graph the set on a number line.

5.

a $0 \le x < 4$

b $8 > x > 5$

6.

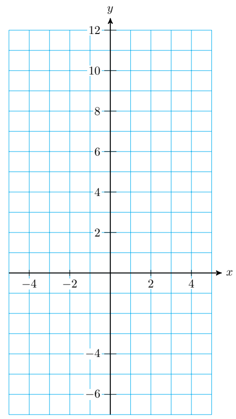
a $x \le 1$

b $x \ge 3$ or $x \le -3$

Skills Practice

7.

a Graph the equation $y = x^2 - 2x - 3$ on the grid.

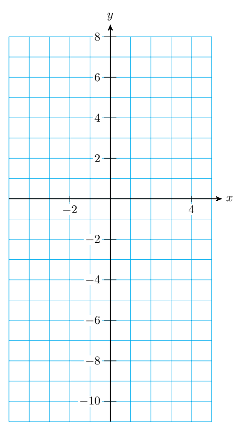


b Darken the portion of the x-axis for which $y > 0$.

c Solve the inequality $x^2 - 2x - 3 > 0$

8.

a Graph the equation $y = x^2 + 2x - 8$ on the grid.

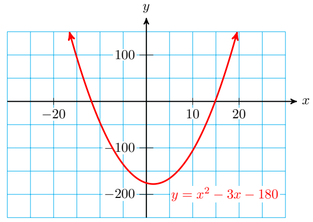


b Darken the portion of the x-axis for which $y < 0$.

c Solve the inequality $x^2 + 2x - 8 < 0$

For Problems 9 and 10, use the graph to solve the equation and the inequality.

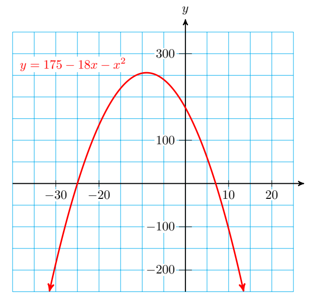
9.



a $x^2 - 3x - 180 = 0$

b $x^2 - 3x - 180 > 0$

10.



a $175 - 18x - x^2 = 0$

b $175 - 18x - x^2 < 0$

For Problems 11 and 12, graph the parabola in the window

Xmin = −9.4	Xmax = 9.4
Ymin = −25	Ymax = 25

Then use the graph to solve the inequalities. Write your answers in interval notation.

11. $y = x^2 - 3x - 18$
For parts (c) and (d), it may be helpful to graph $Y_2 = -8$ as well.

a $x^2 - 3x - 18 > 0$

b $x^2 - 3x - 18 < 0$

c $x^2 - 3x - 18 \leq -8$

d $x^2 - 3x - 18 \geq -8$

12. $y = 16 - x^2$
For parts (c) and (d), it may be helpful to graph $Y_2 = 7$ as well.

a $16 - x^2 > 0$

b $16 - x^2 < 0$

c $16 - x^2 \leq 7$

d $16 - x^2 \geq 7$

For Problems 13–16, solve the inequality. It may be helpful to sketch a rough graph.

13. $(x - 3)(x + 2) > 0$

14. $(x + 3)(x - 4) \leq 0$

15. $k(k - 4) \geq 0$

16. $t^2 - 36 < 0$

For Problems 17–24, solve the inequality algebraically. Write your answers in interval notation, and round to two decimal places if necessary.

17. $q^2 + 9q + 18 < 0$

18. $28 - 3x - x^2 \geq 0$

19. $2z^2 - 7z > 4$

20. $4x^2 + x \geq -2x^2 + 2$

21. $5 - v^2 < 0$

22. $x^2 - 4x + 1 \geq 0$

23. $-3 - m^2 < 0$

24. $w^2 - w + 4 \leq 0$

For Problems 25 and 26, solve the inequality by graphing. Use the window setting

Xmin = −9.4	Xmax = 9.4
Ymin = −64	Ymax = 62

25. $x^2 - 1.4x - 20 < 9.76$

26. $-6x^2 - 36x - 20 \leq 25.36$

Applications

27. The x-intercepts of the graph of $y = 2x^2 + bx + c$ are -4.2 and 2.6. What are the solutions of the inequality $2x^2 + bx + c > 0$?

28. The x-intercepts of the graph of $y = -x^2 + bx + c$ are $-2 \pm \sqrt{17}$. Which of the following are solutions of the inequality $-x^2 + bx + c \geq 0$?

a 2

b -6

c $\sqrt{17}$

d $2\sqrt{17}$

For Problems 29 and 30,

a Solve the problem by writing and solving an inequality.

b Graph the equation and verify your solution on the graph.

29. A fireworks rocket is fired from ground level. Its height in feet t seconds after launch is given by

$$h = 320t - 16t^2$$

During what time interval is the rocket higher than 1024 feet?

30. The volume of a cylindrical can should be between 21.2 and 21.6 cubic inches. If the height of the can is 5 inches, what values for the radius (to the nearest hundredth of an inch) will produce an acceptable can?

For Problems 31 and 32, recall that

$$\text{Revenue} = (\text{number of items sold}) \cdot (\text{ price per item})$$

31. Green Valley Nursery sells $120 - 10p$ boxes of rose food per month at a price of p dollars per box. It would like to keep its monthly revenue from rose food over \$350. In what range should it price a box of rose food?

32. The Locker Room finds that it sells $1200 - 30p$ sweatshirts each month when it charges$p$dollars per sweatshirt. It would like its revenue from sweatshirts to be over \$9000 per month. In what range should it keep the price of a sweatshirt?

33. A farmer inherits an apple orchard on which 60 trees are planted per acre. Each tree yields 12 bushels of apples. Experimentation has shown that for each tree removed per acre, the yield per tree increases by $\frac{1}{2}$ bushel.

a Write algebraic expressions for the number of trees per acre and for the yield per tree if x trees per acre are removed.

b Write a quadratic equation for the total yield per acre if x trees are removed per acre.

c What is the maximum yield per acre that can be achieved by removing trees? How many trees per acre should be removed to achieve this yield?

d How many trees should the farmer remove per acre in order to harvest at least 850 bushels per acre?

e Graph your equation for total yield in the window

Xmin = 0	Xmax = 94
Ymin = 0	Ymax = 1000

and use your graph to verify your answers to parts (e) and (f).

Chapter 4 Summary and Review

Glossary

- quadratic formula
- imaginary unit
- imaginary number
- conjugate pair
- complex number
- maximum value
- minimum value
- vertex form
- compound inequality
- interval notation
- open interval
- closed interval

Key Concepts

The Quadratic Formula.

1 The solutions of the equation $ax^2 + bx + c = 0, \quad (a \neq 0)$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2 We have four methods for solving quadratic equations: extraction of roots, factoring, completing the square, and the quadratic formula. The first two methods are faster, but they do not work on all equations. The last two methods work on any quadratic equation.

3 The **imaginary unit**, i, is defined by $i = \sqrt{-1}$.

4 The square root of any negative number can be written as the product of a real number (the square root of its absolute value) and i.

5 The sum of a real number and an **imaginary number** is called a **complex number**.

6 The graph of the quadratic equation $y = ax^2 + bx + c$ may have two, one, or no x-intercepts, according to the number of distinct real-valued solutions of the equation $ax^2 + bx + c = 0$.

The Discriminant.

7 The **discriminant** of a quadratic equation is

$$D = b^2 - 4ac$$

1 If $D > 0$, there are two unequal real solutions.

2 If $D = 0$, there is one solution of multiplicity two.

3 If$D < 0$, there are two complex conjugate solutions.

8 Every quadratic equation has two solutions, which may be the same.

9 For the graph of $y = ax^2 + bx + c$, the x-coordinate of the vertex is

$$x_v = \frac{-b}{2a}$$

To find the -coordinate of the vertex, we substitute x_v into the formula for the parabola.

10 Quadratic models may arise as the product of two variables.

11 The maximum or minimum of a quadratic equation occurs at the vertex of its graph.

Vertex Form for a Quadratic Equation.

12 A quadratic equation $y = ax^2 + bx + c,\ a \neq 0,$ can be written in the vertex form

$$y = a(x - x_v)^2 + y_v$$

where the vertex of the graph is (x_v, y_v).

13 We can convert a quadratic equation to vertex form by completing the square.

14 We need three points to determine a parabola.

15 We can use the method of elimination to find the equation of a parabola through three points.

16 If we know the vertex of a parabola, we need only one other point to find its equation.

17 We can use quadratic regression to fit a parabola to a collection of data points from a quadratic model.

18 We can use a graphical technique to solve quadratic inequalities.

To solve a quadratic inequality algebraically.

19

1 Write the inequality in standard form: One side is zero, and the other has the form $ax^2 + bx + c$.

2 Find the x-intercepts of the graph of $y = ax^2 + bx + c$ by setting $y = 0$ and solving for x.

3 Make a rough sketch of the graph, using the sign of a to determine whether the parabola opens upward or downward.

4 Decide which intervals on the x-axis give the correct sign for y.

20 We can write the solution set to a quadratic inequality in interval notation.

Interval Notation.

21

1 The **closed interval** $[a, b]$ is the set $a \leq x \leq b$.

2 The **open interval** (a, b) is the set $a < x < b$.

3 Intervals may also be **half-open** or **half-closed**.

4 The **infinite interval** $[a, \infty)$ is the set $x \geq a$.

5 The **infinite interval** $(\infty, a]$ is the set $x \leq a$.

To graph a quadratic equation $y = ax^2 + bx + c$.

22

1 Determine whether the parabola opens upward (if $a > 0$) or downward (if $a < 0$).

2 Locate the vertex of the parabola.

a The x-coordinate of the vertex is $x_v = \dfrac{-b}{2a}$.

b Find the y-coordinate of the vertex by substituting x_v into the equation of the parabola.

3 Locate the x-intercepts (if any) by setting $y = 0$ and solving for x.

4 Locate the y-intercept by evaluating y for $x = 0$.

5 Locate the point symmetric to the y-intercept across the axis of symmetry.

Chapter 4 Review Problems

For Problems 1–4, solve by using the quadratic formula.

1. $\dfrac{1}{2}x^2 + 1 = \dfrac{3}{2}x$

2. $x^2 - 3x + 1 = 0$

3. $x^2 - 4x + 2 = 0$

4. $2x^2 + 2x = 3$

For Problems 5 and 6, solve the formula for the indicated variable.

5. $h = 6t + 3t^2, \quad$ for t

6. $D = \dfrac{n^2 - 3n}{2}, \quad$ for n

For Problems 7–10, use the discriminant to determine the nature of the solutions of the equation.

7. $4x^2 - 12x + 9 = 0$

8. $2t^2 + 6t + 5 = 0$

9. $2y^2 = 3y - 4$

10. $\dfrac{x^2}{4} = x + \dfrac{5}{4}$

11. The height, h, of an object t seconds after being thrown from ground level is given by

$$h = v_0 t - \frac{1}{2} g t^2$$

where v_0 is its starting velocity and g is a constant that depends on gravity. On the moon, the value of g is approximately 5.6. Suppose you hit a golf ball on the moon with an upwards velocity 100 feet per second.

a Write an equation for the height of the golf ball t seconds after you hit it.

b Graph your equation in the window

Xmin = 0	Xmax = 47
Ymin = 0	Ymax = 1000

c Use the TRACE key to estimate the maximum height the golf ball reaches.

d Use your equation to calculate when the golf ball will reach a height of 880 feet.

12. An acrobat is catapulted into the air from a springboard at ground level. His height h in meters is given by the formula

$$h = -4.9t^2 + 14.7t$$

where t is the time in seconds from launch. Use your calculator to graph the acrobat's height versus time. Set the WINDOW values on your calculator to

Xmin = 0	Xmax = 4.7
Ymin = 0	Ymax = 12

a Use the TRACE key to find the coordinates of the highest point on the graph. When does the acrobat reach his maximum height, and what is that height?

b Use the formula to find the height of the acrobat after 2.4 seconds.

c Use the TRACE key to verify your answer to part (b). Find another time when the acrobat is at the same height.

d Use the formula to find two times when the acrobat is at a height of 6.125 meters. Verify your answers on the graph.

e What are the coordinates of the horizontal intercepts of your graph? What do these points have to do with the acrobat?

For Problems 13-16,

a Find the coordinates of the vertex and the intercepts.

b Sketch the graph.

13. $y = x^2 - x - 12$

14. $y = -2x^2 + x - 4$

15. $y = -x^2 + 2x + 4$

16. $y = x^2 - 3x + 4$

17. Find the equation for a parabola whose vertex is $(15, -6)$ and that passes through the point $(3, 22.8)$.

18.

a Find the vertex of the graph of $y = -2(x - 1)^2 + 5$

b Write the equation of the parabola in standard form.

19. The total profit Kiyoshi makes from producing and selling x floral arrangements is

$$P = -0.4x^2 + 36x$$

a How many floral arrangements should Kiyoshi produce and sell to maximize his profit?

b What is his maximum profit?

c Verify your answers on a graph.

20. The Metro Rail service sells $1200 - 80x$ fares each day when it charges x dollars per fare.

a Write an equation for the revenue in terms of the price of a fare.

b What fare will return the maximum revenue?

c What is the maximum revenue?

d Verify your answers on a graph.

21. A beekeeper has beehives distributed over 60 square miles of pastureland. When she places four hives per square mile, each hive produces about 32 pints of honey per year. For each additional hive per square mile, honey production drops by 4 pints per hive.

a Write an equation for the total production of honey, in pints, in terms of the number of additional hives per square mile.

b How many additional hives per square mile should the beekeeper install in order to maximize honey production?

22. A small company manufactures radios. When it charges \$20 for a radio, it sells 500 radios per month. For each dollar the price is increased, 10 fewer radios are sold per month.

a Write an equation for the monthly revenue in terms of the price increase over \$20.

b What should the company charge for a radio in order to maximize its monthly revenue?

23. Find values of a , b, and c so that the graph of the parabola $y = ax^2 + bx + c$ contains the points $(-1, -4)$, $(0, -6)$ and $(4, 6)$.

24. Find a parabola that fits the following data points.

x	-8	-4	2	4
y	10	18	0	-14

25. The height of a cannonball was observed at 0.2-second intervals after the cannon was fired, and the data recorded in the table below.

Time (seconds)	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
Height(metes)	10.2	19.2	27.8	35.9	43.7	51.1	58.1	64.7	71.0	76.8

a Find the equation of the least-squares regression line for height in terms of time.

b Use the linear regression equation to predict the height of the cannonball at 3 seconds and at 4 seconds after it was fired.

c Make a scatterplot of the data and draw the regression line on the same axes.

d Find the quadratic regression equation for height in terms of time.

e Use quadratic regression equation to predict the height of the cannonball at 3 seconds and at 4 seconds after it was fired.

f Make a scatterplot of the data and draw the regression curve on the same axes.

g Which model is more appropriate for the height of the cannonball, linear or quadratic? Why?

26. Max took a sequence of photographs of an explosion spaced at equal time intervals. From the photographs he was able to estimate the height and vertical velocity of some debris from the explosion, as shown in the table. (Negative velocities indicate that the debris is falling back to earth.)

Velocity (meters/second)	67	47	27	8	-12	-31
Height (meters)	8	122	196	232	228	185

a Enter the data into your calculator and create a scatterplot. Fit a quadratic regression equation to the data, and graph the equation on the scatterplot.

b Use your regression equation to find the vertex of the parabola. What do the coordinates represent, in terms of the problem? What should the velocity of the debris be at its maximum height?

For Problems 27–32, solve the inequality algebraically, and give your answers in interval notation. Verify your solutions by graphing.

27. $(x-3)(x+2) > 0$

28. $y^2 - y - 12 \leq 0$

29. $2y^2 - y \leq 3$

30. $3z^2 - 5z > 2$

31. $s^2 \leq 4$

32. $4t^2 > 12$

33. The Sub Station sells $220 - \frac{1}{4}p$ submarine sandwiches at lunchtime if it sells them at p cents each.

a Write a formula for the Sub Station's daily revenue in terms of p.

b What range of prices can the Sub Station charge if it wants to keep its daily revenue from subs over \$480?

34. When it charges p dollars for an electric screwdriver, Handy Hardware will sell $30 - \frac{1}{2}p$ screwdrivers per month.

a Write a formula in terms of p for Handy Hardware's monthly revenue from screwdrivers.

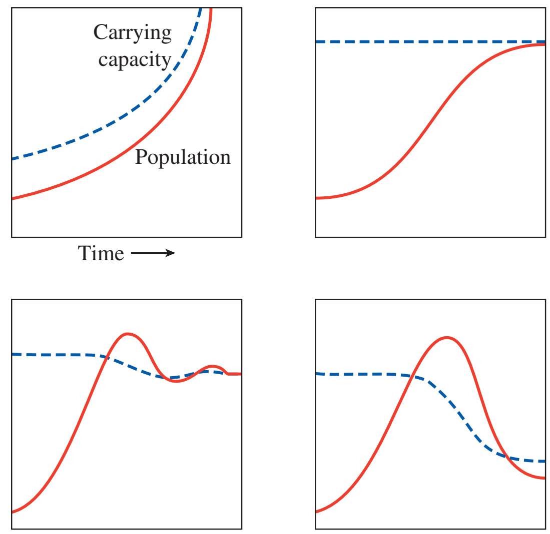
b How much should Handy charge per screwdriver if it wants the monthly revenue from the screwdrivers to be over $400?

Chapter 5

Functions and Their Graphs

World3 is a computer model developed by a team of researchers at MIT. The model tracks population growth, use of resources, land development, industrial investment, pollution, and many other variables that describe human impact on the planet.

The figure below is taken from *Limits to Growth: The 30-Year Update*. The graphs represent four possible answers to World3's core question: How may the global population and economy interact with and adapt to Earth's limited carrying capacity (the maximum it can sustain) over the coming decades?



Source: Meadows, Randers, and Meadows, 2004

Each of the graphs represents a nonlinear function. A **function** is a special relationship between two variables, and we have already encountered linear and quadratic funtions. In this chapter we examine the properties and features of some basic nonlinear functions, and how they may be used as mathematial models.

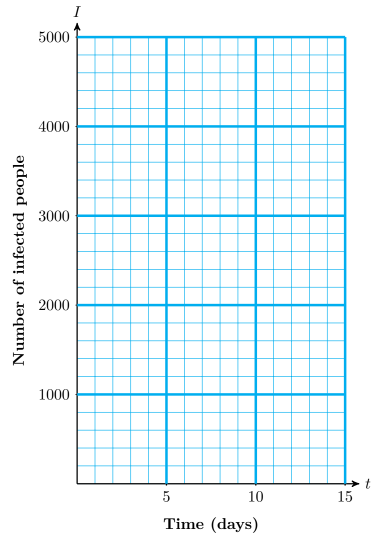
Investigation 5.1 Epidemics. A contagious disease whose spread is unchecked can devastate a confined population. For example, in the early 16th-century, Spanish troops introduced smallpox into the Aztec population in Central America, and the resulting epidemic contributed significantly to the fall of Montezuma's empire.

Suppose that an outbreak of cholera follows severe flooding in an isolated town of 5000 people. Initially (Day 0), 40 people are infected. Every day after that, 25% of those still healthy fall ill.

1 At the beginning of the first day (Day 1), how many people are still healthy? How many will fall ill during the first day? What is the total number of people infected after the first day?

2 Check your results against the table below. Subtract the total number of infected residents from 5000 to find the number of healthy residents at the beginning of the second day. Then fill in the rest of the table for 10 days. (Round off decimal results to the nearest whole number.)

Day	Number healthy	New patients	Total infected
0	5000	40	40
1	4960	1240	1280
2			
3			
4			
5			
6			
7			
8			
9			
10			



3 Use the last column of the table to plot the total number of infected residents, I, against time, t, on the grid. Connect your data points with a smooth curve.

4 Do the values of I approach some largest value? Draw a dotted horizontal line at that value of I. Will the values of I ever exceed that value?

5 What is the first day on which at least 95% of the population is infected?

6 Look back at the table. What is happening to the number of new patients each day as time goes on? How is this phenomenon reflected in the graph? How would your graph look if the number of new patients every day were a constant?

7 Summarize your work: In your own words, describe how the number of residents infected with cholera changes with time. Include a description of your graph.

Functions

Definitions and Notation

We often want to predict values of one variable from the values of a related variable. For example, when a physician prescribes a drug in a certain dosage, she needs to know how long the dose will remain in the bloodstream. A sales manager needs to know how the price of his product will affect its sales. A **function** is a special type of relationship between variables that allows us to make such predictions.

We have already seen some examples of functions. For instance, suppose it costs \$800 for flying lessons, plus \$30 per hour to rent a plane. If we let C represent the total cost for t hours of flying lessons, then

$$C = 800 + 30t \qquad (t \geq 0)$$

The variable t is called the **input** variable, and C is the **output** variable. Given a value of the input, we can calculate the corresponding output value using the formula for the function. Thus, for example

$$\begin{aligned} &\text{when} \quad t = \mathbf{0}, \quad C = 800 + 30(\mathbf{0}) = 800 \\ &\text{when} \quad t = \mathbf{4}, \quad C = 800 + 30(\mathbf{4}) = 920 \\ &\text{when} \quad t = \mathbf{10}, \quad C = 800 + 30(\mathbf{10}) = 1100 \end{aligned}$$

We can display the relationship between two variables by a table or by ordered pairs. The input variable is the first component of the ordered pair, and the output variable is the second component. For the example above we have:

t	C	(t, C)
0	800	$(0, 800)$
4	920	$(4, 920)$
10	1100	$(10, 1100)$

Note that there can be only one value of C for each value of t. We say that "C is a **function** of t."

Definition 5.1 Definition of Function.

A **function** is a relationship between two variables for which a exactly one value of the **output** variable is determined by each value of the **input** variable.

Checkpoint 5.2 QuickCheck 1. What distinguishes a function from other variable relationships?

- ⊙ A) There cannot be two output values for a single input value.
- ⊙ B) We can display the variables as ordered pairs.
- ⊙ C) The variables are related by a formula.
- ⊙ D) The values of the input and output variables must be different.

Answer. A) There ... input value.

Solution. There cannot be two output values for a single input value.

Example 5.3

a The distance, d, traveled by a car in 2 hours is a function of its speed, r. If we know the speed of the car, we can determine the distance it travels by the formula $d = r \cdot 2$.

b The cost of a fill-up with unleaded gasoline is a function of the number of gallons purchased. The gas pump represents the function by displaying the corresponding values of the input variable (number of gallons) and the output variable (cost).

c Score on the Scholastic Aptitude Test (SAT) is not a function of score on an IQ test, because two people with the same score on an IQ test may score differently on the SAT; that is, a person's score on the SAT is not uniquely determined by his or her score on an IQ test.

Checkpoint 5.4 Practice 1.

a. As part of a project to improve the success rate of freshmen, the counseling department studied the grades earned by a group of students in English and algebra. Do you think that a student's grade in algebra is a function of his or her grade in English? (☐ Yes ☐ No)

Explain why or why not.

- ⊙ A) Each value of x has exactly one value of y associated with it.
- ⊙ B) Two students with the same grade English can have different grades in algebra.
- ⊙ C) Two students with the same grade math will also have the same grade in English.
- ⊙ D) Two students with the same grade math can have different grades in English.

b. Phatburger features a soda bar, where you can serve your own soft drinks in any size. Do you think that the number of calories in a serving of Zap Kola is a function of the number of fluid ounces? (☐ Yes ☐ No)

Explain why or why not.

- ⊙ A) The number of calories is proportional to the number of fluid ounces.
- ⊙ B) Two servings with the same calories will have different fluid ounces.
- ⊙ C) Two servings with the same flid ounces will have different calories.

Answer 1. No

Answer 2. Choice 2

Answer 3. Yes

Answer 4. A) The ... fluid ounces.

Solution.

a. No, students with the same grade in English can have different grades in algebra.

b. Yes, the number of calories is proportional to the number of fluid ounces.

A function can be described in several different ways. In the following examples, we consider functions defined by tables, by graphs, and by equations.

Functions Defined by Tables

When we use a table to describe a function, the first variable in the table (the left column of a vertical table or the top row of a horizontal table) is the input variable, and the second variable is the output. We say that the output variable *is a function of* the input.

Example 5.5

a The table below shows data on sales compiled over several years by the accounting office for Eau Claire Auto Parts, a division of Major Motors. In this example, the year is the input variable, and total sales is the output. We say that total sales, S, *is a function of* t.

Year (t)	Total sales (S)
2000	\$612,000
2001	\$663,000
2002	\$692,000
2003	\$749,000
2004	\$904,000

b The table below gives the cost of sending a letter by first-class mail in 2020.

Weight in ounces (w)	Postage (P)
$0 < w \leq 1$	\$0.50
$1 < w \leq 2$	\$0.65
$2 < w \leq 3$	\$0.80
$3 < w \leq 4$	\$0.95
$4 < w \leq 5$	\$1.10
$5 < w \leq 6$	\$1.25
$6 < w \leq 7$	\$1.40

If we know the weight of the article being mailed, we can find the postage from the table. For instance, a catalog weighing 4.5 ounces would require \$1.10 in postage. In this example, w is the input variable and p is the output variable. We say that *p is a function of w*.

c The table below records the age and cholesterol count for 20 patients tested in a hospital survey.

Age	Cholesterol count		Age	Cholesterol count
53	217		**51**	**209**
48	232		53	241
55	198		49	186
56	238		**51**	**216**
51	**227**		57	208
52	264		52	248
53	195		50	214
47	203		56	271
48	212		53	193
50	234		48	172

According to these data, cholesterol count is *not* a function of age, because several patients who are the same age have different cholesterol levels. For example, three different patients are 51 years old but have cholesterol counts of 227, 209, and 216, respectively. Thus, we cannot determine a *unique* value of the output variable (cholesterol count) from the value of the input variable (age). Other factors besides age must influence a person's cholesterol count.

Note 5.6 Note that several different inputs for a function can have the same output. For example, the inputs 4.5 and 4.25 in part (b) of the Example above have output \$1.10. However, a single input cannot have more than one output, as illustrated in part (c) of the Example.

Checkpoint 5.7 Practice 2. Decide whether each table describes y as a function of x. Explain your choice.

a.

x	3.5	2.0	2.5	3.5	2.5	4.0	2.5	3.0
y	2.5	3.0	2.5	4.0	3.5	4.0	2.0	2.5

Is y a function of x? (☐ Yes ☐ No)

- ⊙ Each value of x has exactly one value of y associated with it.
- ⊙ For example, $x = 3.5$ corresponds both to $y = 2.5$ and also to $y = 4.0$

b.

x	-3	-2	-1	0	1	2	3
y	17	3	0	-1	0	3	17

Is y a function of x? (☐ Yes ☐ No)

- ⊙ Each value of x has exactly one value of y associated with it.
- ⊙ For example, $y = 3$ corresponds both to $x = -2$ and also to $x = 2$

Answer 1. No

Answer 2. Choice 2

Answer 3. Yes

Answer 4. Choice 1

Solution.

a. No, for example, $x = 3.5$ corresponds both to $y = 2.5$ and also to $y = 4$.

b. Yes, each value of x has exactly one value of y associated with it.

Checkpoint 5.8 QuickCheck 2. How would you know if a table of values does not come from a function?

- ⊙ A) The output values are all the same.
- ⊙ B) The input values are not evely spaced.
- ⊙ C) Two different input values have the same output value.
- ⊙ D) Two different output values have the same input value.

Answer. D) Two ... input value.

Solution. Two different output values have the same input value.

Functions Defined by Graphs

We can also use a graph to define a function. The input variable is displayed on the horizontal axis, and the output variable on the vertical axis.

Example 5.9

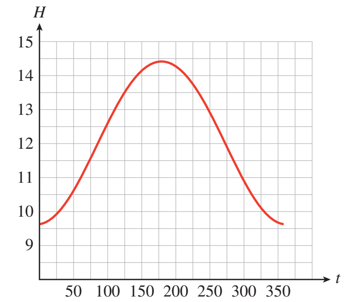
The graph shows the number of hours, H, that the sun is above the horizon in Peoria, Illinois, on day t, where $t = 0$ on January 1.

a Which variable is the input, and which is the output?

b How many hours of sunlight are there in Peoria on day 150?

c On which days are there 12 hours of sunlight?

d What are the maximum and minimum values of H, and when do these values occur?



Solution.

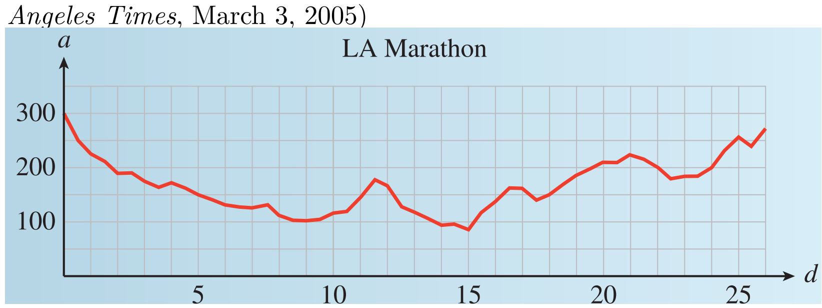
a The input variable, t, appears on the horizontal axis. The number of daylight hours, H, is a function of the date. The output variable appears on the vertical axis.

b The point on the curve where $t = 150$ has $H \approx 14.1$, so Peoria gets about 14.1 hours of daylight when $t = 150$, which is at the end of May.

c $H = 12$ at the two points where $t \approx 85$ (in late March) and $t \approx 270$ (late September).

d The maximum value of 14.4 hours occurs on the longest day of the year, when $t \approx 170$, about three weeks into June. The minimum of 9.6 hours occurs on the shortest day, when $t \approx 355$, about three weeks into December.

Checkpoint 5.10 Practice 3. The graph shows the elevation in feet, a, of the Los Angeles Marathon course at a distance d miles into the race. (Source: *Los Angeles Times*, March 3, 2005)



a. Which variable is the input, and which is the output?

- ⊙ The input variable is d, and the output variable is a.
- ⊙ The input variable is a, and the output variable is d.

b. What is the elevation at mile 20?

Answer: ___ feet

c. At what distances is the elevation 150 feet?

The relevant distances (to the nearest half-mile) separated by commas: miles

d. What are the maximum and minimum values of a, and when do these values occur?

   The maximum elevation is $a =$___ feet which occurs at $d =$__.

e. The runners pass by the Los Angeles Coliseum at about 4.2 miles into the race. What is the elevation there?

   Approximately (within 5) ___ feet

Answer 1. Choice 1

Answer 2. 210

Answer 3. $5, 11, 12, 16, 17.5, 18$

Answer 4. 300

Answer 5. 0

Answer 6. 165

Solution.

a. The input variable is d, and the output variable is a.

b. Approximately 210 feet

c. Approximately where $d \approx 5$, $d \approx 11$, $d \approx 12$, $d \approx 16$, $d \approx 17.5$, and $d \approx 18$

d. The maximum value of 300 feet occurs at the start, when $d = 0$. The minimum of 85 feet occurs when $d \approx 15$.

e. Approximately 165 feet

Functions Defined by Equations

Example 5.11, p. 271 illustrates a function defined by an equation.

Example 5.11

As of 2020, One World Trade Center in New York City is the nation's tallest building, at 1776 feet. If an algebra book is dropped from the top of One World Trade Center, its height above the ground after t seconds is given by the equation

$$h = 1776 - 16t^2$$

Thus, after **1** second the book's height is

$$h = 1776 - 16(\mathbf{1})^2 = 1760 \text{ feet}$$

After **2** seconds its height is

$$h = 1776 - 16(\mathbf{2})^2 = 1712 \text{ feet}$$

For this function, t is the input variable and h is the output variable. For any value of t, a unique value of h can be determined from the equation for h. We say that *h is a function of t*.

Checkpoint 5.12 Practice 4. Write an equation that gives the volume, V, of a sphere as a function of its radius, r.

$V =$_______

Answer. $\frac{4}{3}\pi r^3$

Solution. $V = \dfrac{4}{3}\pi r^3$

Checkpoint 5.13 QuickCheck 3. Name three ways to describe a function.

- ⊙ A) By inputs, outputs, or evaluation
- ⊙ B) By tables, equations, or graphs
- ⊙ C) By the intercepts, the slope, or the vertex
- ⊙ D) By numbers, letters, or diagrams

Answer. B) By ... , or graphs

Solution. By tables, equations, or graphs

Function Notation

There is a convenient notation for discussing functions. First, we choose a letter, such as f, g, or h (or F, G, or H), to name a particular function. (We can use any letter, but these are the most common choices.)

For instance, in Example 5.11, p. 271, the height, h, of a falling algebra book is a function of the elapsed time, t. We might call this function f. In other words, f is the name of the relationship between the variables h and t. We write

$$h = f(t)$$

which means "h is a function of t, and f is the name of the function."

Caution 5.14 The new symbol $f(t)$, read "f of t," is another name for the height, h. The parentheses in the symbol $f(t)$ do not indicate multiplication. (It would not make sense to multiply the name of a function by a variable.) Think of the symbol $f(t)$ as a single variable that represents the output value of the function.

With this new notation we may write

$$h = f(t) = 1776 - 16t^2$$

or just

$$f(t) = 1776 - 16t^2$$

instead of

$$h = 1776 - 16t^2$$

to describe the function.

Note 5.15 Perhaps it seems complicated to introduce a new symbol for h, but the notation $f(t)$ is very useful for showing the correspondence between specific values of the variables h and t.

Example 5.16

In Example 5.11, p. 271, the height of an algebra book dropped from the top of One World Trade Center is given by the equation

$$h = 1776 - 16t^2$$

We see that

when $t = 1$	$h = 1760$
when $t = 2$	$h = 1712$

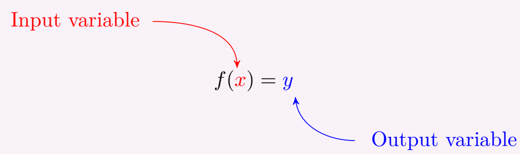
Using function notation, these relationships can be expressed more concisely as

$$f(1) = 1760 \quad \text{and} \quad f(2) = 1712$$

which we read as "f of 1 equals 1760" and "f of 2 equals 1712." The values for the input variable, t, appear *inside* the parentheses, and the values for the output variable, h, appear on the other side of the equation.

Remember that when we write $y = f(x)$, the symbol $f(x)$ is just another name for the output variable.

Function Notation.



Checkpoint 5.17 QuickCheck 4. True or False.

a. The notation $f(t)$ indicates the product of f and t. (☐ True ☐ False)

b. If $y = f(x)$, then $f(x)$ gives the value of the input variable. (☐ True ☐ False)

c. If Q is a function of M, we may write $M = f(Q)$. (☐ True ☐ False)

d. In the equation $d = g(n)$, the letters d, g, and n are variables. (☐ True ☐ False)

Answer 1. False

Answer 2. False

Answer 3. False

Answer 4. False

Solution.

a. False

b. False

c. False

d. False

Checkpoint 5.18 Practice 5. Let F be the name of the function defined by the graph in Example 5.9, p. 269, the number of hours of daylight in Peoria t days after January 1.

a. Use function notation to state that H is a function of t.

- ⊙ $F = H(t)$
- ⊙ $H = F(t)$
- ⊙ $t = F(H)$
- ⊙ $H = t(F)$

b. What does the statement $F(15) = 9.7$ mean in the context of the problem?

- ⊙ A) The sun is 9.7 degrees above the horizon in Peoria on January 15.
- ⊙ B) The sun is above the horizon in Peoria for 15 hours on January 10.
- ⊙ C) The sun is above the horizon in Peoria for 9.7 hours on January 16.

Answer 1. Choice 2

Answer 2. C) The ... January 16.

Solution.

a. $H = F(t)$

b. The sun is above the horizon in Peoria for 9.7 hours on January 16.

Checkpoint 5.19 QuickCheck 5. Use function notation to write the statement "L defines w as a function of p."

- ⊙ $L = w(p)$
- ⊙ $w = L(p)$
- ⊙ $p = L(w)$
- ⊙ $L = p(w)$

Answer. Choice 2

Solution. $w = L(p)$

Using Function Notation

Finding the value of the output variable that corresponds to a particular value of the input variable is called **evaluating the function**.

Example 5.20

Let g be the name of the postage function defined by the table in Example 5.5, p. 267 b. Find $g(1)$, $g(3)$, and $g(6.75)$.

Solution. According to the table,

$$\begin{array}{llll} \text{when } w = 1, & p = 0.50 & \text{so} & g(1) = 0.50 \\ \text{when } w = 3, & p = 0.80 & \text{so} & g(3) = 0.80 \\ \text{when } w = 6.75, & p = 1.40 & \text{so} & g(6.75) = 1.40 \end{array}$$

Thus, a letter weighing 1 ounce costs \$0.50 to mail, a letter weighing 3 ounces costs \$0.80, and a letter weighing 6.75 ounces costs \$1.40.

We can also find the input (or inputs) corresponding to a given output. For example, if $p = g(w)$ is the postage function, we solve the equation $g(w) = 0.65$ by finding all input values, w, that correspond to the output \$0.65. According to the table in Example 2b, any value of w greater than 1 but less than or equal to 2 is a solution.

Checkpoint 5.21 Practice 6. When you exercise, your heart rate should increase until it reaches your target heart rate. The table shows target heart rate, $r = f(a)$, as a function of age.

a	20	25	30	35	40	45	50	55	60	65	70
r	150	146	142	139	135	131	127	124	120	116	112

a. Find $f(25)$ and $f(50)$.

$f(25) =$___

$f(50) =$___

b. Find a value of a for which $f(a) = 135$.

$a =$___

Answer 1. 146

Answer 2. 127

Answer 3. 40

Solution.

a. $f(25) = 146$, $f(50) = 127$

b. $a = 40$

Checkpoint 5.22 QuickCheck 6. If $n = f(a)$, what are the input and output variables?

- ⊙ f is the output and n is the input
- ⊙ a is the input and f is the output
- ⊙ a is the input and n is the output
- ⊙ $f(a)$ is the input and n is the output

Answer. Choice 3

Solution. a is the input and n is the output

To evaluate a function described by an equation, we simply substitute the given input value into the equation to find the corresponding output, or function value.

Example 5.23

The function H is defined by $H = f(s) = \dfrac{\sqrt{s+3}}{s}$. Evaluate the function at the following values.

a $s = 6$ b $s = -1$

Solution.

a $f(\mathbf{6}) = \dfrac{\sqrt{\mathbf{6}+3}}{\mathbf{6}} = \dfrac{\sqrt{9}}{6} = \dfrac{3}{6} = \dfrac{1}{2}$. Thus, $f(6) = \dfrac{1}{2}$.

b $f(\mathbf{-1}) = \dfrac{\sqrt{\mathbf{-1}+3}}{\mathbf{-1}} = \dfrac{\sqrt{2}}{-1} = -\sqrt{2}$. Thus, $f(-1) = -\sqrt{2}$.

Checkpoint 5.24 Practice 7. Complete the table displaying ordered pairs for the function $f(x) = 5 - x^3$. Evaluate the function to find the corresponding $f(x)$-value for each value of x.

x	$f(x)$	
-2	___	$f(\mathbf{-2}) = 5 - (\mathbf{-2})^3 =$
0	___	$f(\mathbf{0}) = 5 - \mathbf{0}^3 =$
1	___	$f(\mathbf{1}) = 5 - \mathbf{1}^3 =$
3	___	$f(\mathbf{3}) = 5 - \mathbf{3}^3 =$

Answer 1. 13

Answer 2. 5

Answer 3. 4

Answer 4. -22

Solution.

x	$f(x)$
-2	13
0	5
1	4
3	-22

Problem Set 5.1

Warm Up

For Problems 1-4, evaluate.

1. $2x - x^2$ for $x = -4$

2. $\dfrac{2z-3}{z+2}$ for $z = \dfrac{1}{2}$

3. $\sqrt{36 - (r+1)^2}$ for $r = 3$

4. $-t^3 + 3t^2$ for $t = -2$

For Problems 5-8, solve.

5. $4 - 5x - 2x^2 = 1$

6. $6(2x-8)^2 = 20$

7. $\dfrac{1}{2x-9} = 3$

8. $5\sqrt{8+x} = 20$

Skills Practice

9. $x = h(v) = 2v^2 - 3v + 1$

a Which variable is the input, and which is the output?

b Evaluate $h(-2)$.

c Solve $h(v) = 6$.

10. $A = g(r) = 750(1+r)^2$

a Which variable is the input, and which is the output?

b Evaluate $g(0.04)$.

c Solve $g(r) = 874.80$.

For Problems 11 and 12, evaluate the function.

11. $F(x) = \dfrac{1-x}{2x-3}$

a $F(0)$

b $F(-3)$

c $F(\frac{5}{2})$

d $F(9.8)$

12. $E(t) = \sqrt{t-4}$

a $E(16)$

b $E(4)$

c $E(7)$

d $E(4.2)$

Applications

13. Which of the following tables define the second variable as a function of the first variable? Explain why or why not.

a

x	t
-1	2
0	9
1	-2
0	-3
-1	5

b

y	w
0	8
1	12
3	7
5	-3
7	4

c

x	y
-3	8
-2	3
-1	0
0	-1
1	0
2	3
3	8

d

s	t
2	5
4	10
6	15
8	20
6	25
4	30
2	35

e

r	-4	-2	0	2	4
v	6	6	3	6	8

f

p	-5	-4	-3	-2	-1
d	-5	-4	-3	-2	-1

14. Which of the following tables define the second variable as a function of the first variable? Explain why or why not.

a

Pressure (p)	Volume (v)
15	100.0
20	75.0
25	60.0
30	50.0
35	42.8
40	37.5
45	33.3
50	30.0

c

Temperature (T)		Humidity (h)
Jan. 1	34°F	42%
Jan. 2	36°F	44%
Jan. 3	35°F	47%
Jan. 4	29°F	50%
Jan. 5	31°F	52%
Jan. 6	35°F	51%
Jan. 7	34°F	49%

b

Frequency (f)	Wavelength (w)
5	60.0
10	30.0
20	15.0
30	10.0
40	7.5
50	6.0
60	5.0
70	4.3

d

Inflation rate (I)		Unemployment rate (U)
1972	5.6%	5.1%
1973	6.2%	4.5%
1974	10.1%	4.9%
1975	9.2%	7.4%
1976	5.8%	6.7%
1977	5.6%	6.8%
1978	6.7%	7.4%

15. The function described in Problem 14(a) is called g, so that $v = g(p)$. Find the following:

a $g(25)$

b $g(40)$

c x so that $g(x) = 50$

16. The function described in Problem 14(b) is called h, so that $w = h(f)$. Find the following:

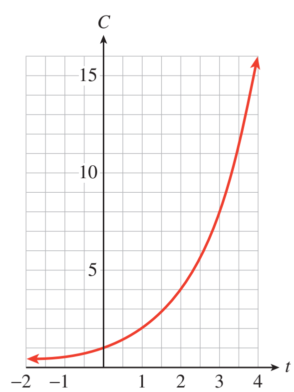
a $h(20)$

b $h(60)$

c x so that $h(x) = 10$

For Problems 17—24, use the graph of the function to answer the questions.

17. The graph shows $C = h(t)$, where C stands for the number of customers (in thousands) signed up for a new movie streaming service, measured in months after their advertising campaign at $t = 0$ in January.



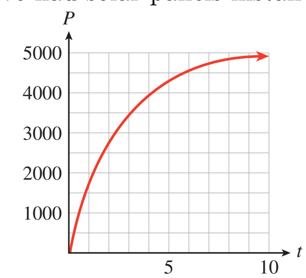
a When did the service have 2000 customers? Write your answer with function notation.

b How long did it take that number to double?

c How long did it take for the number to double again?

d How many customers signed up between March and April (months 2 and 3)?

18. The graph shows P as a function of t. P is the number of houses in Cedar Grove who have had solar panels installed t years after 2000.



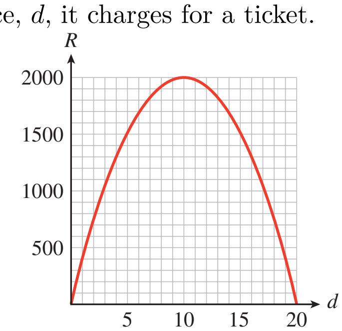
a When did 3500 houses have solar panels? Write your answer using function notation.

b How many houses had solar panels in 2005? Write your answer using function notation.

c The number of houses with solar panels in Cedar Grove seems to be leveling off at what number?

d How many houses had solar panels installed between 2001 and 2004?

19. The graph shows the revenue, $R = f(d)$, a movie theater collects as a function of the price, d, it charges for a ticket.



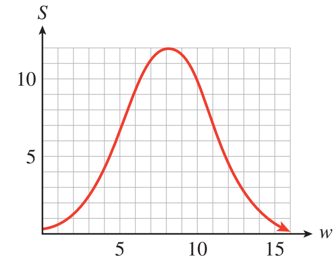
a Estimate the revenue if the theater charges \$12.00 for a ticket.

b What should the theater charge for a ticket in order to collect \$1500 in revenue?

c Write your answers to parts (a) and (b) using function notation.

d For what values of d is $R > 1800$?

20. The graph shows $S = g(w)$. S represents the weekly sales of a best-selling book, in thousands of dollars, w weeks after it is released.



a In which weeks were sales over \$7000?

b In which week did sales fall below \$5000 on their way down?

c For what values of w is $S > 3.4$?

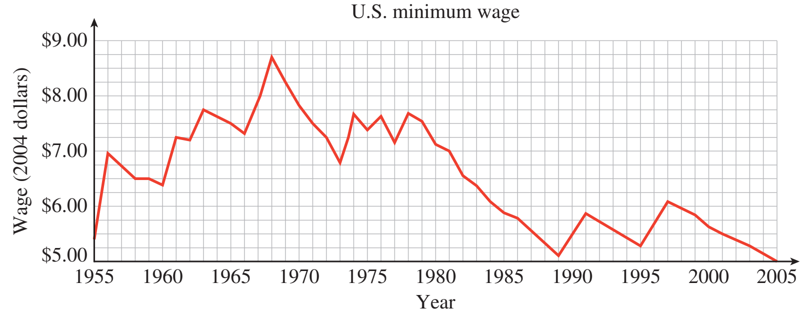
21. The graph shows the U.S. unemployment rate, $U = F(t)$, where t, represents years. Give your answers to the questions below in function notation. (Source: U.S. Bureau of Labor Statistics)

a When did the unemployment rate reach its highest value, and what was its highest value?

b When did the unemployment rate fall to its lowest value, and what was its lowest value?

c Give two years in which the unemployment rate was 4.5%.

22. The graph shows the federal minimum wage, M, over the pasr five decades, adjusted for inflation to reflect its buying power in 2004 dollars. (Source: www.infoplease.com)



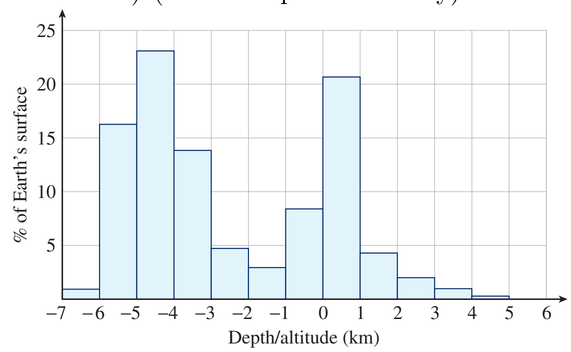
a Is M a function of t? Support your answer.

b What is the largest function value on the graph? Write your answer with function notation, and explain what it means in this problem.

c Give two years in which the minimum wage was worth \$8 in 2004 dollars. Does this fact mean that M is not a function of t?

Why or why not?

23. The bar graph shows the percent of Earth's surface that lies at various altitudes or depths below the surface of the oceans. (Depths are given as negative altitudes.) (Source: Open University)



a Read the graph and complete the table.

Altitude (km)	Percent of Earth's surface
-7 to -6	
-6 to -5	
-5 to -4	
-4 to -3	
-3 to -2	
-2 to -1	
-1 to 0	
0 to 1	
1 to 2	
2 to 3	
3 to 4	
4 to 5	

b What is the most common altitude? What is the second most common altitude?

c Approximately what percent of the Earth's surface is below sea level?

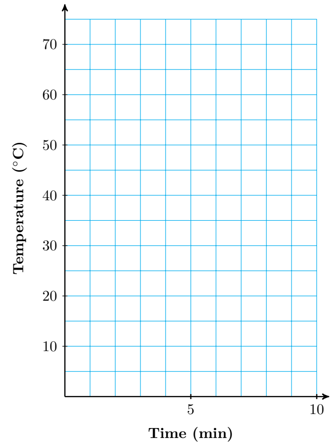
d The height of Mt. Everest is 8.85 kilometers. Can you think of a reason why it is not included in the graph?

24. Energy is necessary to raise the temperature of a substance, and it is also needed to melt a solid substance to a liquid. The table shows data from heating a solid sample of stearic acid. Heat was applied at a constant rate throughout the experiment.

Time (minutes)	0	0.5	1.5	2	2.5	3	4	5	6	7	8	8.5	9	9.5	10
Temperature (deg C)	19	29	40	48	53	55	55	55	55	55	55	64	70	73	74

a Did the temperature rise at a constant rate? Describe the temperature as a function of time.

b Graph the temperature as a function of time.



c What is the melting point of stearic acid? How long did it take the sample to melt?

25. The number of compact cars that a large dealership can sell at price p is given by

$$N(p) = \frac{12,000,000}{p}$$

a Evaluate $N(6000)$ and explain what it means.

b As p increases, does $N(p)$ increase or decrease? Support your answer with calculations.

c Solve the equation $F(p) = 400$, and explain what it means.

26. The distance, d, in miles that a person can see on a clear day from a height, h, in feet is given by

$$d = G(h) = 1.22\sqrt{h}$$

a Evaluate $G(20,320)$ and explain what it means.

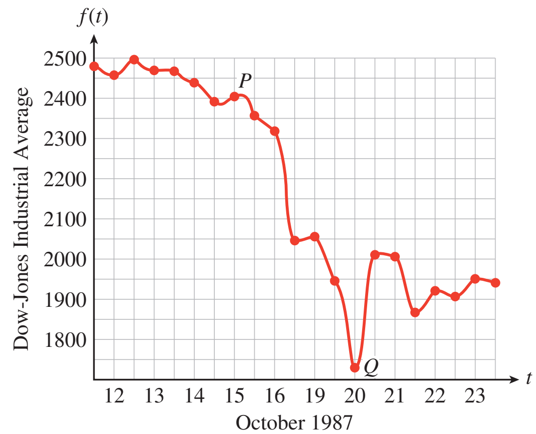
b As h increases, does d increase or decrease? Support your answer with calculations.

c Estimate the height you need in order to see 100 miles. Write your answer with function notation.

Graphs of Functions

Reading Function Values from a Graph

The graph below shows the Dow-Jones Industrial Average (the average value of the stock prices of 500 major companies) recorded at noon each day during the stock market correction around October 10, 1987 ("Black Monday").



The graph describes a function because there is only one value of the output, DJIA, for each value of the input, t. There is no formula that gives the DJIA for a particular day; but it is still a function, defined by its graph. The value of $f(t)$ is specified by the vertical coordinate of the point with the given t-coordinate.

Example 5.25

a The coordinates of point P on the DJIA graph are $(15, 2412)$. What do the coordinates tell you about the function f?

b If the DJIA was 1726 at noon on October 20, what can you say about the graph of f?

Solution.

a The coordinates of point P tell us that $f(15) = 2412$, so the DJIA was 2412 at noon on October 15.

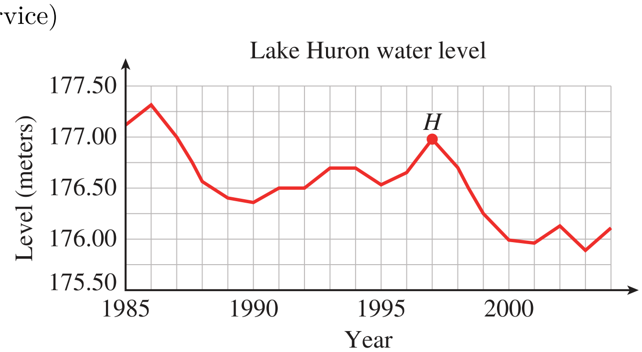
b We can say that $f(20) = 1726$, so the point $(20, 1726)$ lies on the graph of f. This point is labeled Q in the figure above.

The coordinates of each point on the graph of the function give the output for a specific input.

Graph of a Function.

The point (a, b) lies on the graph of the function f if and only if $f(a) = b$.

Checkpoint 5.26 Practice 1. The water level in Lake Huron alters unpredictably over time. The graph below gives the average water level, $L = f(t)$, in meters in the year t over a 20-year period. (Source: The Canadian Hydrographic Service)



a. The coordinates of point H on the graph are $(1997, 176.98)$. What do the coordinates tell you about the function f?

- ⊙ $f(1997) = 176.98$; the average water level was 176.98 meters in 1997.
- ⊙ $f(176.98) = 176.98$; the average water level was 176.98 meters in 1997.

b. The average water level in 2004 was 176.11 meters. Write this fact in function notation. What can you say about the graph of f?

- ⊙ $f(176.11) = 2004$. The point $(176.11, 2004)$ lies on the graph of f.
- ⊙ $f(2004) = 176.11$. The point $(2004, 176.11)$ lies on the graph of f.

Answer 1. Choice 1

Answer 2. Choice 2

Solution.

a. $f(1997) = 176.98$; the average water level was 176.98 meters in 1997.

b. $f(2004) = 176.11$. The point $(2004, 176.11)$ lies on the graph of f.

The second coordinate of a point on the graph is the function value for the first coordinate.

Functions and Coordinates.

Each point on the graph of the function f has coordinates $(x, f(x))$ for some value of x.

Checkpoint 5.27 QuickCheck 1. True or False.

a. If $(12, 5)$ lies on the graph of f, then $f(5) = 12$. (☐ True ☐ False)

b. If $g(p) = w$, then (p, w) lies on the graph of g. (☐ True ☐ False)

c. We can find the function value at x by finding the x-coordinate of the corresponding point on the graph of f. (☐ True ☐ False)

d. Every y-coordinate on the graph of F represents a function value for F. (☐ True ☐ False)

Answer 1. False

Answer 2. True

Answer 3. False

Answer 4. True

Solution.

a. False

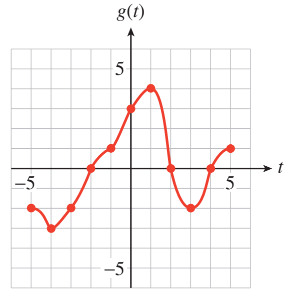
b. True

c. False

d. True

Example 5.28

The figure shows the graph of a function g.



a Find $g(-2)$ and $g(5)$.

b For what value(s) of t is $g(t) = -2$?

c What is the largest, or maximum, value of $g(t)$? For what value of t does the function take on its maximum value?

d On what intervals is g increasing?

Solution.

a To find $g(-2)$, we look for the point with t-coordinate -2. The point $(-2, 0)$ lies on the graph of g, so $g(-2) = 0$. Similarly, the point $(5, 1)$ lies on the graph, so $g(5) = 1$.

b We look for points on the graph with y-coordinate -2. Because the points $(-5, -2)$, $(-3, -2)$, and $(3, -2)$ lie on the graph, we know that $g(-5) = -2$, $g(-3) = -2$, and $g(3) = -2$. Thus, the t-values we want are -5, -3, and 3.

c The highest point on the graph is $(1, 4)$, so the largest y-value is 4. Thus, the maximum value of $g(t)$ is 4, and it occurs when $t = 1$.

d A graph is increasing if the y-values get larger as we read from left to right. The graph of g is increasing for t-values between -4 and 1, and between 3 and 5. Thus, g is increasing on the intervals $(-4, 1)$ and $(3, 5)$.

Checkpoint 5.29 **QuickCheck 2.** True or False.

a. A graph is called increasing if its x-values increase. (☐ True ☐ False)

b. The maximum function value is the y-coordinate of the highest point on the graph. (☐ True ☐ False)

c. If we say that f is increasing on the interval $(2, 7)$, we mean that the function values increased from 2 to 7. (☐ True ☐ False)

d. It is not possible for a function of x to take on the same value at two different x-values. (☐ True ☐ False)

Answer 1. False

Answer 2. True

Answer 3. False

Answer 4. False

Solution.

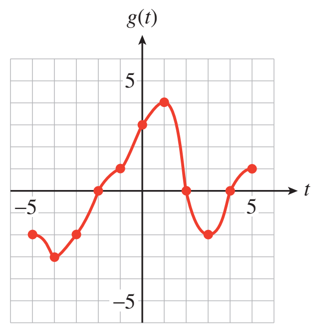
a. False

b. True

c. False

d. False

Checkpoint 5.30 Practice 2. Refer to the graph of the function g shown in Example 5.28, p. 285.



a. $g(0) =$___

b. List the value(s) of t for which $g(t) = 0$. Separate different values with commas.

c. What is the smallest, or minimum, value of $g(t)$?

Minimum: ___

For what value of t does the function take on its minimum value?

$t =$___

d. Select all the intervals listed below where g is decreasing.

☐ from \(-5 \) to \(-4 \) ☐ from \(-5 \) to \(-2 \) ☐ from \(2 \) to \(4 \) ☐ from \(1 \) to \(3 \) ☐ None of the above

Answer 1. 3

Answer 2. $-2, 2, 4$

Answer 3. -3

Answer 4. -4

Solution.

a. 3

b. $-2, 2, 4$

c. -3; $t = -4$

d. $(-5, -4)$ and $(1, 3)$

Constructing the Graph of a Function

We can construct a graph for a function described by a table or an equation. We make these graphs the same way we graph equations in two variables: by plotting points whose coordinates satisfy the equation.

Example 5.31

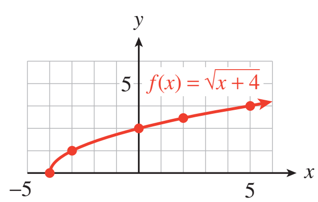
Graph the function $f(x) = \sqrt{x+4}$

Solution. We choose several convenient values for x and evaluate the function to find the corresponding $f(x)$-values. For this function we cannot choose x-values less than -4, because the square root of a negative number is not a real number.

$$f(\mathbf{-4}) = \sqrt{\mathbf{-4}+4} = \sqrt{0} = 0$$
$$f(\mathbf{-3}) = \sqrt{\mathbf{-3}+4} = \sqrt{1} = 1$$
$$f(\mathbf{0}) = \sqrt{\mathbf{0}+4} = \sqrt{4} = 2$$
$$f(\mathbf{2}) = \sqrt{\mathbf{2}+4} = \sqrt{6} \approx 2.45$$
$$f(\mathbf{5}) = \sqrt{\mathbf{5}+4} = \sqrt{9} = 3$$

The results are shown in the table.

x	$f(x)$
-4	0
-3	1
0	2
2	$\sqrt{6}$
5	3



Points on the graph have coordinates $(x, f(x))$, so the vertical coordinate of each point is given by the value of $f(x)$. We plot the points and connect them with a smooth curve, as shown in the figure. Notice that no points on the graph have x-coordinates less than -4.

Checkpoint 5.32 QuickCheck 3. How do we find the value of $f(3)$ from a graph of f?

- ⊙ Find 3 on the x-axis, move vertically to the point, then horizontally to the y-axis.
- ⊙ Find 3 on the y-axis, move horizontally to the point, then vertically to the x-axis.
- ⊙ Substitue 3 for x into the formula for the function.
- ⊙ Substitue 3 for y into the formula for the function.

Answer. Choice 1

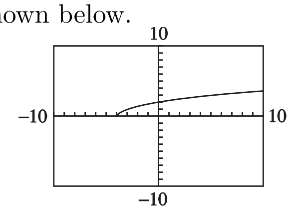
Solution. Find 3 on the x-axis, move vertically to the point, then horizontally to the y-axis.

Technology 5.33 Using Technology to Graph a Function. We can also use a graphing utility to obtain a table and graph for the function in Example 5.31, p. 287. We graph a function just as we graphed an equation.

For this function, we enter

$$Y_1 = \sqrt{}(X + 4)$$

and press ZOOM 6 for the standard window. Your calculator does not use the $f(x)$ notation for graphs, so we will continue to use Y_1, Y_2, etc. for the output variable. Don't forget to enclose $x+4$ in parentheses, because it appears under a radical. The graph is shown below.



Checkpoint 5.34 Practice 3. Let $f(x) = x^3 - 2$

Complete the table of values and sketch a graph of the function.

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
$f(x)$	__	_	__	_	__	_	_

Answer 1. -10

Answer 2. -3

Answer 3. $-\frac{17}{8}$

Answer 4. -2

Answer 5. $-\frac{15}{8}$

Answer 6. -1

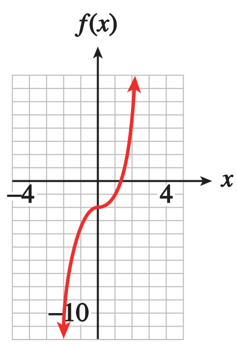
Answer 7. 6

Solution.

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
$f(x)$	-10	-3	$\frac{-17}{8}$	-2	$\frac{-15}{8}$	-1	6

The graph is shown below.

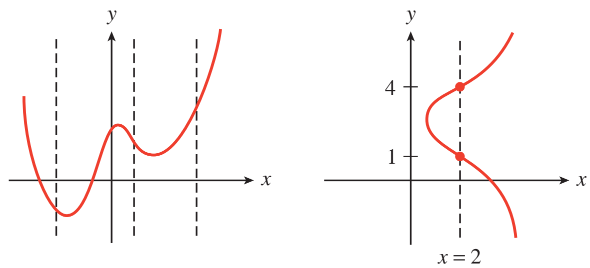
$f(x) = x^3 - 2$



The Vertical Line Test

In a function, two different outputs cannot be related to the same input. This restriction means that two different ordered pairs cannot have the same first coordinate. What does it mean for the graph of the function?

Consider the graph shown in Figure (a). Every vertical line intersects the graph in at most one point, so there is only one point on the graph for each x-value. This graph represents a function.



(a) (b)

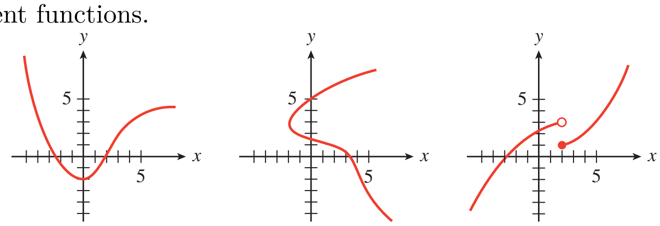
In Figure (b), however, the line $x = 2$ intersects the graph at two points, $(2, 1)$ and $(2, 4)$. Two different y-values, 1 and 4, are related to the same x-value, 2. This graph cannot be the graph of a function.

The Vertical Line Test.

A graph represents a function if and only if every vertical line intersects the graph in at most one point.

Example 5.35

Use the vertical line test to decide which of the graphs in the figure represent functions.



(a) (b) (c)

Solution.

- Graph (a) represents a function, because it passes the vertical line test.
- Graph (b) is not the graph of a function, because the vertical line at (for example) $x = 1$ intersects the graph at two points.
- For graph (c), notice the break in the curve at $x = 2$: The solid dot at $(2, 1)$ is the only point on the graph with $x = 2$; the open circle at $(2, 3)$ indicates that $(2, 3)$ is not a point on the graph. Thus, graph (c) is a function, with $f(2) = 1$.

Checkpoint 5.36 QuickCheck 4. What does the vertical line test tell us?

⊙ A) If the graph is a vertical line.

⊙ B) If the graph is increasing.

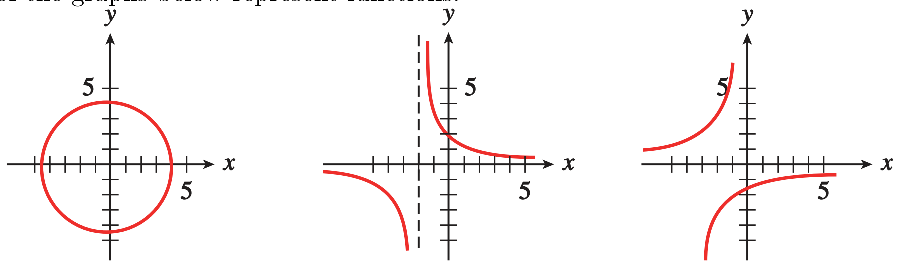
⊙ C) If the graph is decreasing.

⊙ D) If the graph is a function.

Answer. D) If ... a function.

Solution. The vertical line test tells us if the graph is a function.

Checkpoint 5.37 Practice 4. Use the vertical line test to determine which of the graphs below represent functions.



(a) (b) (c)

☐ (a) ☐ (b) ☐ (c) ☐ None of the above

Solution. Only (b) is a function.

Graphical Solution of Equations and Inequalities

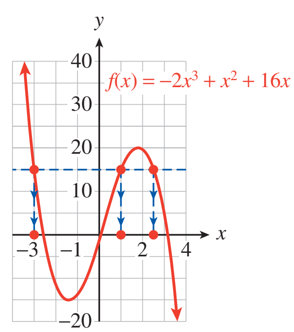
We have used graphs to solve linear and quadratic equations and inequalities. We can also use a graphical technique to solve equations and inequalities involving other functions.

Example 5.38

Use a graph of $f(x) = -2x^3 + x^2 + 16x$ to solve the equation

$$-2x^3 + x^2 + 16x = 15$$

Solution. If we sketch in the horizontal line $y = 15$, we can see that there are three points on the graph of f that have y-coordinate 15, as shown below. The x-coordinates of these points are the solutions of the equation.



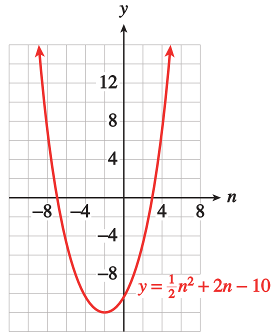
From the graph, we see that the solutions are $x = -3$, $x = 1$, and approximately $x = 2.5$. We can verify each solution algebraically.

For example, if $x = \mathbf{-3}$, we have

$$\begin{aligned} f(\mathbf{-3}) &= -2(\mathbf{-3})3 + (\mathbf{-3})^2 + 16(\mathbf{-3}) \\ &= -2(-27) + 9 - 48 \\ &= 54 + 9 - 48 = 15 \end{aligned}$$

so -3 is a solution. Similarly, you can check that $x = 1$ and $x = 2.5$ are solutions.

Checkpoint 5.39 Practice 5.



Use the graph of $y = \frac{1}{2}n^2 + 2n - 10$ shown above to solve

$$\frac{1}{2}n^2 + 2n - 10 = 6$$

and verify your solutions algebraically.
$n =$__________

Answer. $-8, 4$

Solution. $-8, 4$

Checkpoint 5.40 QuickCheck 5. To solve an equation graphically, it is helpful to sketch in a (☐ vertical ☐ horizontal ☐ slanted) line. We are looking for _ values that give the required _ value.

Answer 1. horizontal

Answer 2. x

Answer 3. y

Solution. horizontal line; x; y

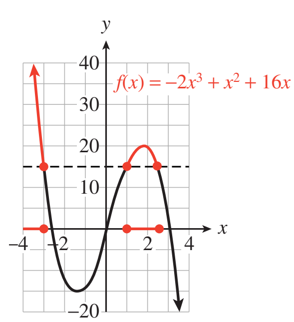
Example 5.41

Use the graph in Example 5.38, p. 290 to solve the inequality

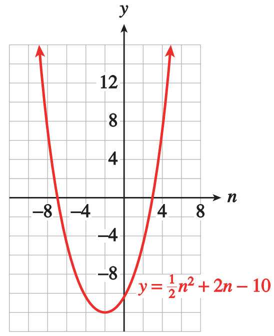
$$-2x^3 + x^2 + 16x \geq 15$$

Solution. We first locate all points on the graph that have y-coordinates greater than or equal to 15. The x-coordinates of these points are the solutions of the inequality.

The figure below shows the points in red, and their x-coordinates as intervals on the x-axis. The solutions are $x \leq -3$ and $1 \leq x \leq 2.5$, or in interval notation, $(-\infty, -3] \cup [1, 2.5]$.



Checkpoint 5.42 Practice 6.



Use the graph above from Checkpoint 5.2.15 to solve the inequality

$$\frac{1}{2}n^2 + 2n - 10 < 6$$

Answer: ____________

Answer. $-8 < n < 4$

Solution. $-8 < n < 4$

More about Notation

To simplify the notation, we sometimes use the same letter for the output variable and for the name of the function. In the next example, C is used in this way.

Example 5.43

TrailGear decides to market a line of backpacks. The cost, C, of manufacturing backpacks is a function of the number of backpacks produced, x, given by the equation

$$C = C(x) = 3000 + 20x$$

where $C(x)$ is in dollars. Find the cost of producing 500 backpacks.

Solution. To find the value of C that corresponds to $x = \mathbf{500}$, we evaluate $C(\mathbf{500})$:

$$C(\mathbf{500}) = 3000 + 20(\mathbf{500}) = 13,000$$

The cost of producing 500 backpacks is \$13,000.

Checkpoint 5.44 Practice 7. The volume of a sphere of radius r centimeters is given by

$$V = V(r) = \frac{4}{3}\pi r^3$$

Evaluate $V(10)$ and explain what it means.

$V(10) =$____________, which represents

- ⊙ A) the volume (in cm) of a sphere whose radius is 10 cu. cm
- ⊙ B) the radius (in cm) of a sphere whose volume is 10 cm
- ⊙ C) the volume (in sq. cm) of a sphere whose radius is 10 cm
- ⊙ D) the volume (in cu. cm) of a sphere whose radius is 10 cm

Answer 1. 4188.79

Answer 2. D) the ... is 10 cm

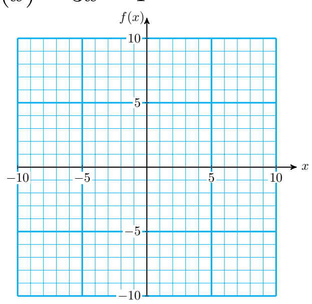
Solution. $V(10) = 4000\pi/3 \approx 4188.79$ cm^3 is the volume of a sphere whose radius is 10 cm.

Problem Set 5.2

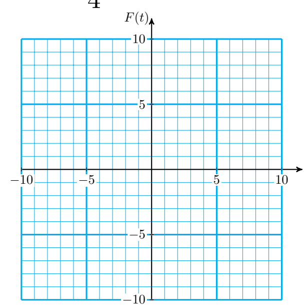
Warm Up

For problems 1–4, sketch a graph of the linear or quadratic function by hand, and label the significant points.

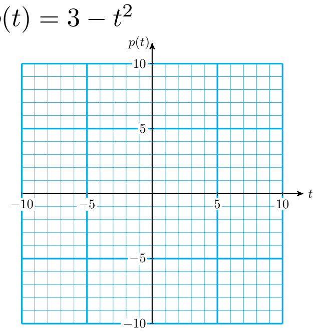
1. $f(x) = 3x - 4$



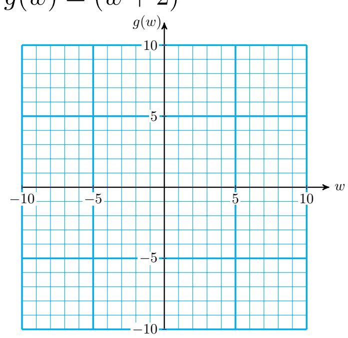
2. $F(t) = \dfrac{-3}{4}t + 60$



3. $p(t) = 3 - t^2$

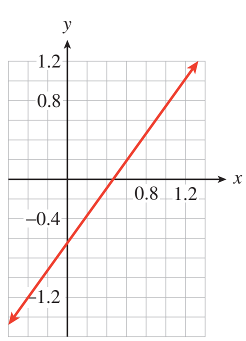


4. $g(w) = (w + 2)^2$

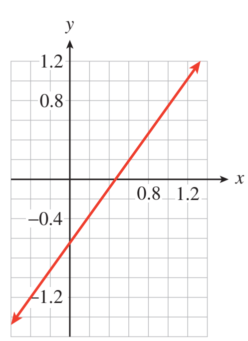


5. Use the graph of $y = 1.4x - 0.64$ to solve the inequalities. Show your method on the graph.

a $1.4x - 0.64 > 0.2$

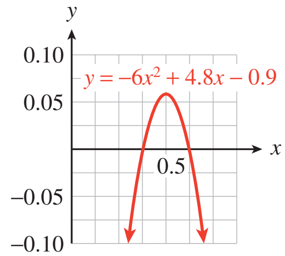


b $-1.2 > 1.4x - 0.64$

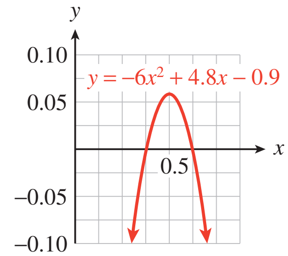


6. Use the graphs to solve the inequalities. Show your method on the graph.

a $-6x^2 + 4.8x - 0.9 \geq 0$



b $5x^2 + 7.5x + 1.8 \leq 0$



Skills Practice

For Problems 7–10,

a Make a table of values and sketch a graph of the function by plotting points. (Use the suggested x-values.)

b Use your calculator to make a table of values and to graph the function. Compare the calculator's graph with your sketch.

7. $g(x) = x^3 + 4$
$x = -2, -1, \ldots, 2$

8. $h(x) = 2 + \sqrt{x}$
$x = 0, 1, \ldots, 9$

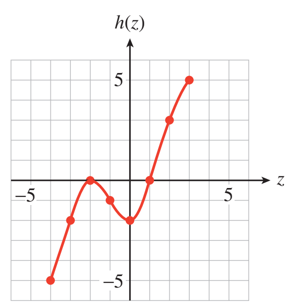
9. $G(x) = \sqrt{4 - x}$
$x = -5, -4, \ldots, 4$

10. $w(x) = x^3 - 8x$
$x = -4, -3, \ldots, 4$

Applications

For Problems 11–14, use the graph.

11.



a Find $h(-3)$, $h(1)$, and $h(3)$.

b For what value(s) of z is $h(z) = 3$?

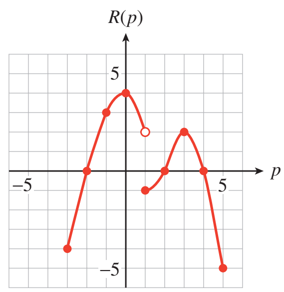
c Find the intercepts of the graph. List the function values given by the intercepts.

d What is the maximum value of $h(z)$?

e For what value(s) of z does h take on its maximum value?

f On what intervals is the function increasing? Decreasing?

12.



a Find $R(1)$ and $R(3)$.

b For what value(s) of p is $R(p) = 2$?

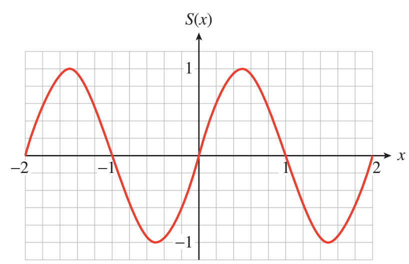
c Find the intercepts of the graph. List the function values given by the intercepts.

d Find the maximum and minimum values of $R(p)$.

e For what value(s) of p does R take on its maximum and minimum values?

f On what intervals is the function increasing? Decreasing?

13.



a Find $S(0)$, $S\left(\dfrac{1}{6}\right)$, and $S(-1)$.

b Estimate the value of $S\left(\dfrac{1}{3}\right)$ from the graph.

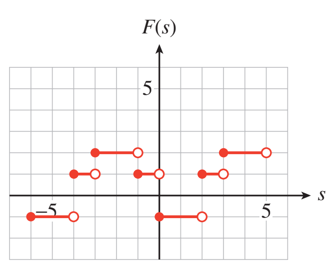
c For what value(s) of x is $S(x) = \dfrac{-1}{2}$?

d Find the maximum and minimum values of $S(x)$.

e For what value(s) of x does S take on its maximum and minimum values?

f On what intervals is the function increasing? Decreasing?

14.



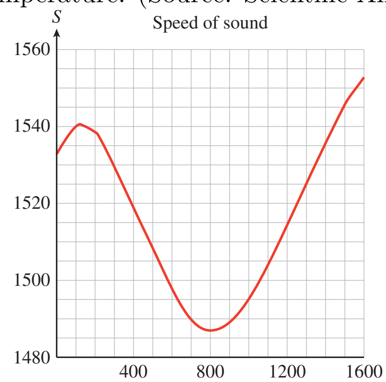
a Find $F(-3)$, $F(-2)$ and $F(2)$.

b For what value(s) of s is $F(s) = -1$?

c Find the maximum and minimum values of $F(s)$.

d For what value(s) of s does F take on its maximum and minimum values?

15. The graph shows the speed of sound in the ocean as a function of depth, $S = f(d)$. The speed of sound is affected both by increasing water pressure and by dropping temperature. (Source: Scientific American)



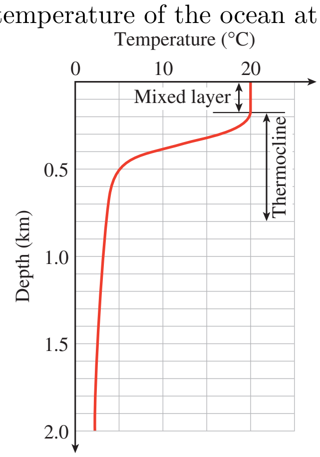
a Evaluate $f(1000)$ and explain its meaning.

b Solve $f(d) = 1500$ and explain its meaning.

c At what depth is the speed of sound the slowest, and what is the speed? Write your answer with function notation.

d Describe the behavior of $f(d)$ as d increases.

16. The figure shows the temperature of the ocean at various depths.



a Is depth a function of temperature?

b Is temperature a function of depth?

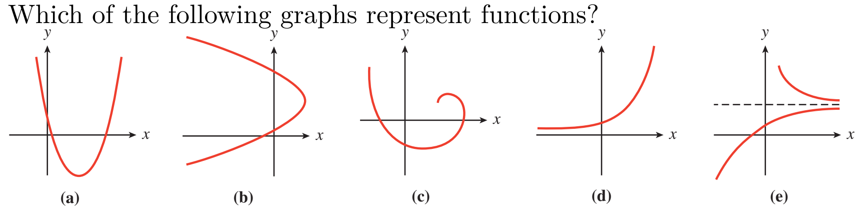
c The axes in this figure are scaled in an unusual way. Why is it useful to present the graph in this way?

d What is the difference in temperature between the surface of the ocean and the deepest level shown?

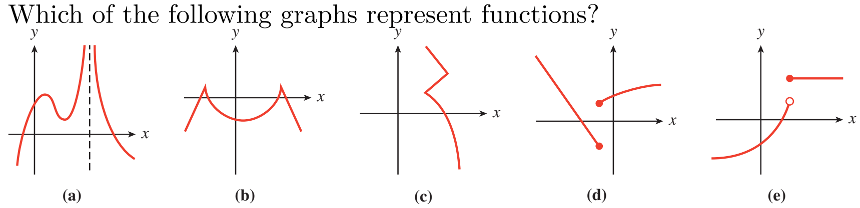
e Over what depths does the temperature change most rapidly?

f What is the average rate of change of temperature with respect to depth in the region called the thermocline?

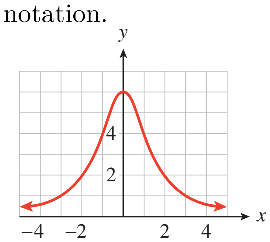
17. Which of the following graphs represent functions?



18. Which of the following graphs represent functions?



19. The figure shows the graph of $g(x) = \dfrac{12}{2+x^2}$. Use the graph to solve the following equations and inequalities. Show your work on the graph. Write your answers in interval notation.



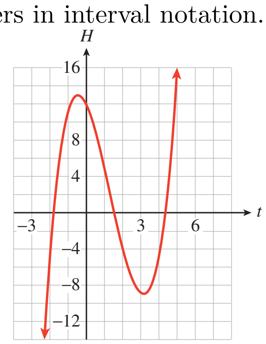
a $\dfrac{12}{2+x^2} = 4$

b $\dfrac{12}{2+x^2} > 4$

c $1 \le \dfrac{12}{2+x^2} \le 2$

d $\dfrac{12}{2+x^2} \le 6$

20. The figure shows the graph of $H(t) = t^3 - 4t^2 - 4t - 12$. Use the graph to solve the following equations and inequalities. Show your work on the graph. Write your answers in interval notation.



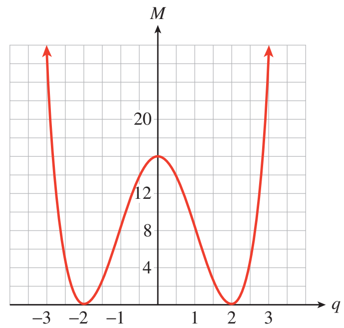
a $t^3 - 4t^2 - 4t - 12 = -4$

b $t^3 - 4t^2 - 4t - 12 = 16$

c $t^3 - 4t^2 - 4t - 12 < -4$

d $t^3 - 4t^2 - 4t - 12 > 6$

21. The figure shows a graph of $M = g(q)$.



a Find all values of q for which

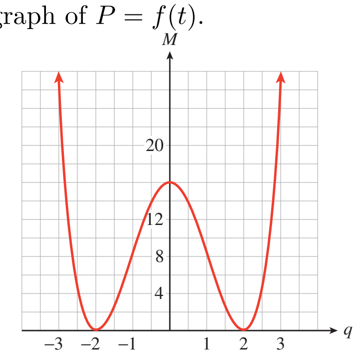
I $g(q) = 0$

II $g(q) = 16$

III $g(q) < 6$

b For what values of q is $g(q)$ increasing?

22. The figure shows a graph of $P = f(t)$.



a Find all values of t for which

I $f(t) = 3$

II $f(t) > 4.5$

III $2 \leq f(t) \leq 4$

b For what values of q is $g(q)$ increasing?

For Problems 23 and 24, graph each function in the "friendly" window

Xmin $= -9.4$	Xmax $= 9.4$
Ymin $= -10$	Ymax $= 10$

Then answer the questions about the graph.

23. $g(x) = \sqrt{36 - x^2}$

a Find the intercepts of the graph. Write your answers in function notation.

b Find all points on the graph for which $g(x) = 6.4$.

c Explain why there are no points on the graph with $x > 6$ or $x < -6$.

24. $F(x) = 0.5x^3 - 4x$

a Estimate the coordinates of the turning points of the graph, where the graph changes from increasing to decreasing or vice versa.

b Estimate the coordinates of the x-intercepts.

c Write an equation of the form $F(a) = b$ for each turning point and each x-intercept.

For Problems 25 and 26,

a Compute $f(0)$ and $g(0)$.

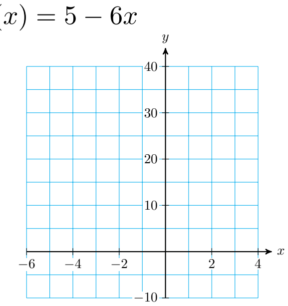
b Find all values of x for which $f(x) = 0$.

c Find all values of x for which $g(x) = 0$.

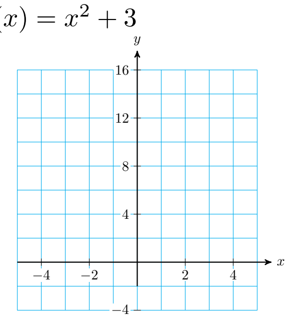
d Find all values of x for which $f(x) = g(x)$.

e Graph f and g on the same axes, and illustrate your answers to (a)–(d) as points on the graph.

25. $f(x) = 2x^2 + 3x, \quad g(x) = 5 - 6x$



26. $f(x) = 2x^2 - 2x, \quad g(x) = x^2 + 3$



Some Basic Graphs

In this section we study the graphs of some important basic functions. Many functions fall into families or classes of similar functions, and recognizing the appropriate family for a given situation is an important part of modeling.

We'll need two new algebraic operations.

Cube Roots

You are familiar with square roots. Every non-negative number has two square roots, defined as follows.

$$s \quad \text{is a square root of} \quad n \quad \text{if} \quad s^2 = n$$

There are several other kinds of roots, one of which is called the **cube root**. We define the cube root as follows.

Definition 5.45 Cube Root.

b is the **cube root** of a if b cubed equals a. In symbols, we write

$$b = \sqrt[3]{a} \quad \text{if} \quad b^3 = a$$

Square roots of negative numbers are not real (they are complex), but every real number has a real cube root. For example,

$$\begin{aligned} 4 &= \sqrt[3]{64} && \text{because} && 4^3 = 64 \\ -3 &= \sqrt[3]{-27} && \text{because} && (-3)^3 = -27 \end{aligned}$$

Simplifying radicals occupies the same position in the order of operations as computing powers: after parentheses, and before products and quotients.

Example 5.46

Simplify each expression.

a $3\sqrt[3]{-8}$

b $2 - \sqrt[3]{-125}$

Solution.

a $3\sqrt[3]{-8} = 3(-2) = -6$

b $2 - \sqrt[3]{-125} = 2 - (-5) = 7$

Checkpoint 5.47 Practice 1. Simplify each expression.

a. $5 - 3\sqrt[3]{64} =$____

b. $\dfrac{6 - \sqrt[3]{-27}}{2} =$____

Answer 1. -7

Answer 2. $\frac{9}{2}$

Solution.

a. -7

b. $\dfrac{9}{2}$

Checkpoint 5.48 QuickCheck 1. True or False.

a. A negative number has a negative cube root. (☐ True ☐ False)

b. A negative number has a negative square root. (☐ True ☐ False)

c. A positive number has a negative square root. (☐ True ☐ False)

d. A positive number has a negative cube root. (☐ True ☐ False)

Answer 1. True

Answer 2. False

Answer 3. True

Answer 4. False

Solution.

a. True

b. False

c. True

d. False

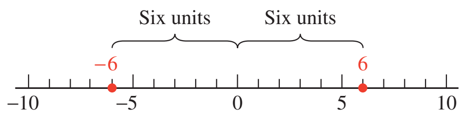
Note 5.49 We can use the calculator to find cube roots as follows. Press the MATH key to get a menu of options. Option 4 is labeled $\sqrt[3]{\;}$; this is the cube root key. To find the cube root of, say, 15.625, we key in

MATH 4 15.625) ENTER

and the calculator returns the result, 2.5. Thus, $\sqrt[3]{15.625} = 2.5$. You can check this result by verifying that $2.5^3 = 15.625$.

Absolute Value

We use the absolute value to discuss problems involving distance. For example, consider the number line below. Starting at the origin, we travel in opposite *directions* to reach the two numbers 6 and -6, but the *distance* we travel in each case is the same.



The distance from a number c to the origin is called the **absolute value** of c, denoted by $|c|$. Because distance is never negative, the absolute value of a number is always positive (or zero). Thus, $|6| = 6$ and $|-6| = 6$. In general, we define the absolute value of a number x as follows.

Definition 5.50 Absolute Value.

The **absolute value** of x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This definition is called **piecewise**, because the formula has two pieces. It says that the absolute value of a positive number (or zero) is the same as the number. To find the absolute value of a negative number, we take the opposite of the number, which is then positive. For instance,

$$|-6| = -(-6) = 6$$

Checkpoint 5.51 QuickCheck 2. If $|x| = -x$, what can you say about x?

- ⊙ x must be zero.
- ⊙ x must be negative.
- ⊙ x must be zero or negative.
- ⊙ This cannot happen for any value of x.

Answer. Choice 3

Solution. x must be zero or negative if $|x| = -x$.

Absolute value bars act like grouping devices in the order of operations: you should complete any operations that appear inside absolute value bars before you compute the absolute value.

Example 5.52

Simplify each expression.

a $|3-8|$

b $|3|-|8|$

Solution.

a We simplify the expression inside the absolute value bars first.

$$|3-8| = |-5| = 5$$

b We simplify each absolute value; then subtract.

$$|3|-|8| = 3-8 = -5$$

Checkpoint 5.53 Practice 2. Simplify each expression.

a. $12-3\,|-6| =$__

b. $-7-3\,|2-9| =$__

Answer 1. -6

Answer 2. -28

Solution.

a. -6

b. -28

Checkpoint 5.54 QuickCheck 3. True or False: $|a+b| = |a|+|b|$ (☐ True ☐ False)

Answer. False

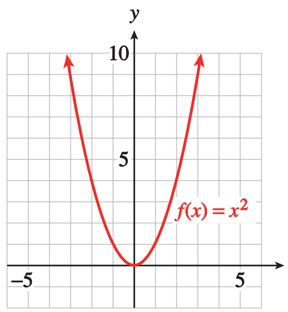
Solution. False

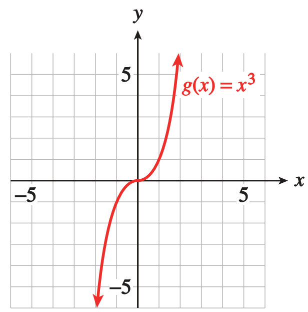
Eight Basic Graphs

Most of the graphs in this section will be new to you, but many useful graphs are variations of the eight basic functions shown below.

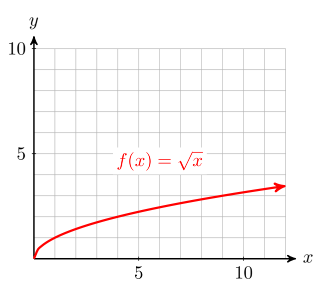
Consider the first pair of graphs. You have already studied the graph of $f(x) = x^2$, the basic parabola. Compare that graph with the graph of $g(x) = x^3$. Notice several differences in the shape of the two graphs. Once you have a good idea of the shape of a graph, up can make a quick sketch with just a few "guide points." For these two graphs, complete a short table of values to find useful guide points:

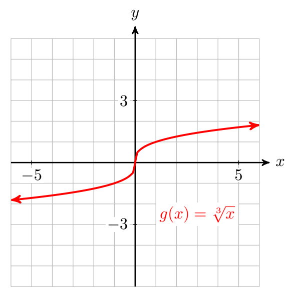
x	-2	-1	0	1	2
$f(x)$					
$g(x)$					





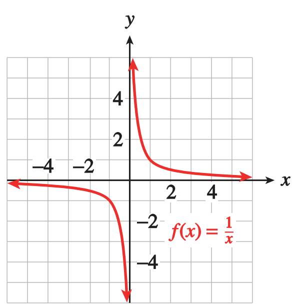
The next pair of graphs are $f(x) = \sqrt{x}$ and $g(x) = \sqrt[3]{x}$. Once again, notice the differences in the two graphs. For example, we cannot take the square root of a negative number, but we can take its cube root. How is this reflected in the graphs?

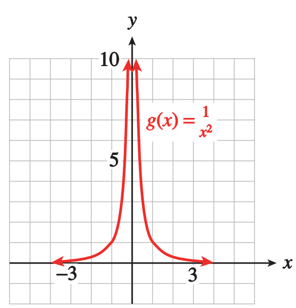




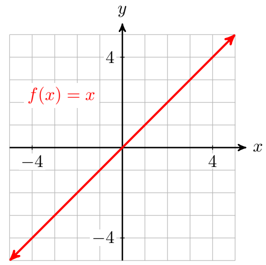
The next pair of functions, $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2}$, are both undefined at $x = 0$, so thier graphs do not include any points with x-coordinate zero. For very small positive values of x, both $f(x)$ and $g(x)$ get very large. As x gets closer to zero, the graphs approach the vertical line $x = 0$ (the y-axis). This line is called a **vertical asymptote** for the graph.

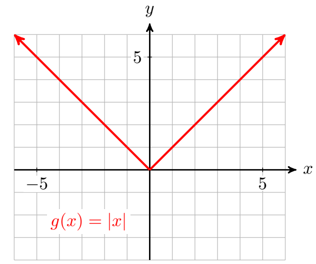
Also, notice that for very large values of x, both $f(x)$ and $g(x)$ get very close to zero. Their graphs approach the horizontal line $y = 0$ (the x-axis). This line is called the **horizontal asymptote** for the graph.





Finally, compare the familiar graph of $f(x) = x$ with the graph of $g(x) = |x|$. The piecewise definition of $|x|$ means that we graph $y = x$ in the first quadrant (where $x \geq 0$), and $y = -x$ in the first quadrant $x < 0$). The result is the V-shaped graph shown below.





Because they are fundamental to further study of mathematics and its applications, you should become familiar with the properties of these eight graphs, and be able to sketch them easily from memory, using their basic shapes and a few guidepoints.

Problem Set 5.3

Warm Up

1. Evaluate each function.

a $f(x) = -2x^3 - 3x^2;\quad f(-2)$

b $g(x) = \dfrac{x-1}{x^2+2x};\quad g(-1)$

For Problems 2 and 3, compute each cube root. Round your answers to three decimal places if necessary. Verify your answers by cubing them.

2.

a $\sqrt[3]{512}$

b $\sqrt[3]{-125}$

c $\sqrt[3]{-0.064}$

d $\sqrt[3]{1.728}$

3.

a $\sqrt[3]{9}$

b $\sqrt[3]{258}$

c $\sqrt[3]{-0.02}$

d $\sqrt[3]{-3.1}$

For Problems 4–6, simplify each by following the order of operations.

4.

a $\dfrac{6 - 2\sqrt[3]{64}}{2}$

b $2\sqrt[3]{-125} - \sqrt[3]{6^2 - 3^2}$

5.

a $\sqrt[3]{\dfrac{8-1}{64-8}}$

b $\dfrac{4 + \sqrt[3]{-216}}{8 - \sqrt[3]{8}}$

6.

a $\sqrt[3]{3^3 + 4^3 + 5^3}$

b $\sqrt[3]{9^3 + 10^3 - 1^3}$

For problems 7–10, simplify the expression according to the order of operations.

7.

a $-|-9|$

b $-(-9)$

c $2 - (-6)$

d $2 - |-6|$

8.

a $|-8| - |12|$

b $|-8 - 12|$

c $|-3| + |-5|$

d $\big|-3 + (-5)\big|$

9.

a $4 - 9\,|2-8|$

b $2 - 5\,|-6-3|$

c $|-4-5|\,\big|1-3(-5)\big|$

10.

a $|-3+7|\,\big|-2(6-10)\big|$

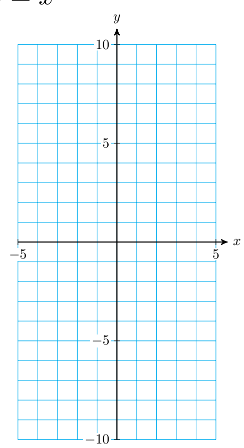
b $\big||-5|-|-6|\big|$

c $\big||4|-|-6|\big|$

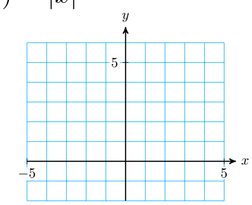
Skills Practice

For Problems 11–16, sketch the graph of the function by hand, paying attention to the shape of the graph. Carefully plot at least three "guide points" to ensure accuracy. If possible, plot the points with x-coordinates -1, 0, and 1.

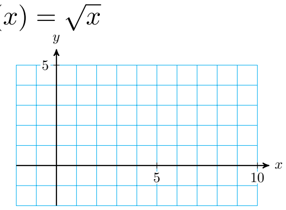
11. $f(x) = x^3$



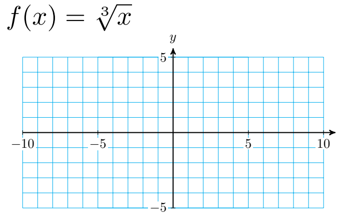
12. $f(x) = |x|$



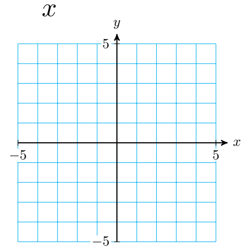
13. $f(x) = \sqrt{x}$



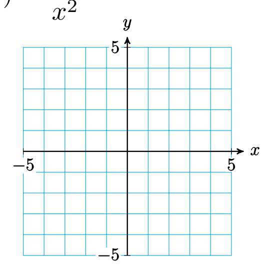
14. $f(x) = \sqrt[3]{x}$



15. $f(x) = \dfrac{1}{x}$



16. $f(x) = \dfrac{1}{x^2}$



17.

a Use your calculator to graph $f(x) = x^2$ and $g(x) = x^3$ on the same axes for $0 \le x \le 1$. Which function is greater on that interval?

b Use your calculator to graph $f(x) = \sqrt{x}$ and $g(x) = \sqrt[3]{x}$ on the same axes for $0 \le x \le 1$. Which function is greater on that interval?

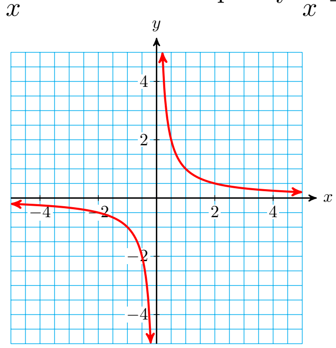
18.

a Use your calculator to graph $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2}$ on the same axes for $0 \le x \le 1$. Which function is greater on that interval?

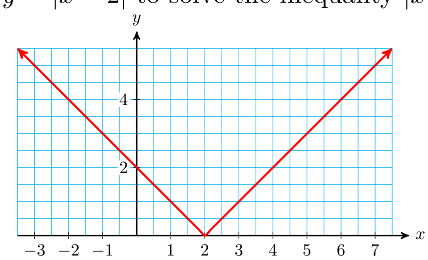
b Use your calculator to graph $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2}$ on the same axes for $1 \le x \le 4$. Which function is greater on that interval?

Applications

19. Use the graph of $y = \dfrac{1}{x}$ to solve the inequality $\dfrac{1}{x} \le 2$.



20. Use the graph of $y = |x - 2|$ to solve the inequality $|x - 2| > 1$.



For Problems 21–26, graph the functions in the same window on your calculator. Describe how the graphs in parts (b) and (c) are different from the basic graph.

21.

a $f(x) = x^3$

b $g(x) = x^3 - 2$

c $h(x) = x^3 + 1$

22.

a $f(x) = |x|$

b $g(x) = |x - 2|$

c $h(x) = |x + 1|$

23.

a $f(x) = \dfrac{1}{x}$

b $g(x) = \dfrac{1}{x + 1.5}$

c $h(x) = \dfrac{1}{x - 1}$

24.

a $f(x) = \dfrac{1}{x^2}$

b $g(x) = \dfrac{1}{x^2} + 2$

c $h(x) = \dfrac{1}{x^2} - 1$

25.

a $f(x) = \sqrt{x}$

b $g(x) = -\sqrt{x}$

c $h(x) = \sqrt{-x}$

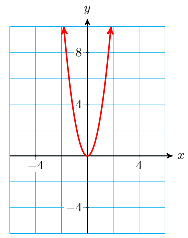
26.

a $f(x) = \sqrt[3]{x}$

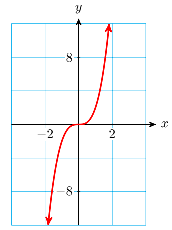
b $g(x) = -\sqrt[3]{x}$

c $h(x) = \sqrt[3]{-x}$

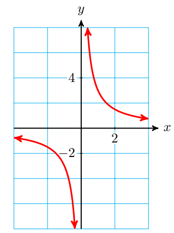
27. Match each graph with its equation.



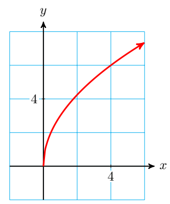
(a)



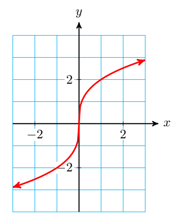
(b)



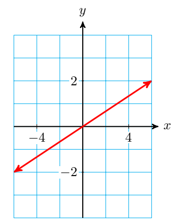
(c)



(d)



(e)

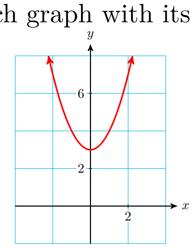


(f)

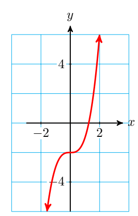
i $f(x) = 3\sqrt{x}$ iii $f(x) = \dfrac{x}{3}$ v $f(x) = 2\sqrt[3]{x}$

ii $f(x) = 2x^3$ iv $f(x) = \dfrac{3}{x}$ vi $f(x) = 3x^2$

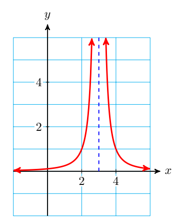
28. Match each graph with its equation.



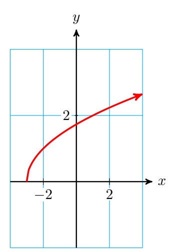
(a)



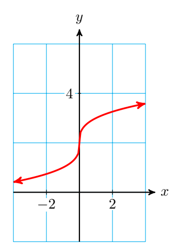
(b)



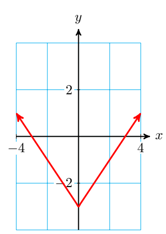
(c)



(d)



(e)



(f)

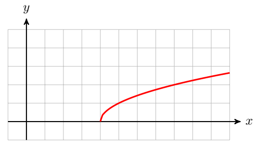
i $f(x) = x^3 - 2$ iii $f(x) = \dfrac{1}{(x-3)^2}$ v $f(x) = x^2 + 3$

ii $f(x) = \sqrt[3]{x} + 2$ iv $f(x) = |x| - 3$ vi $f(x) = \sqrt{x+3}$

For Problems 29 and 30, each graph is a variation of one of the eight basic graphs. Identify the basic graph for each.

29.

a



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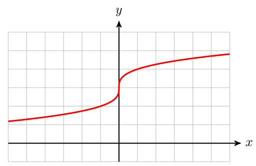
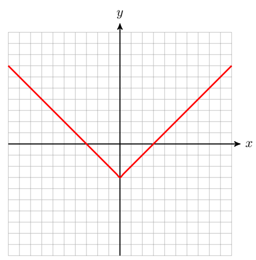


c



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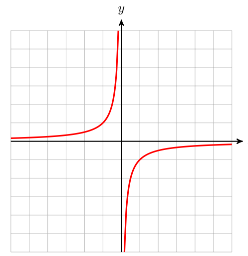
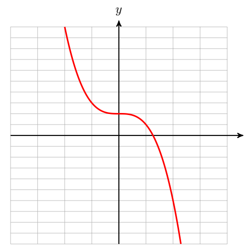


e



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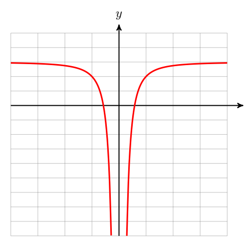
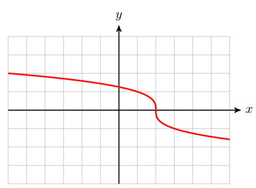
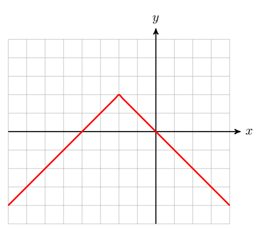


30.

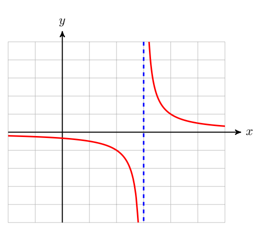
a



b



c



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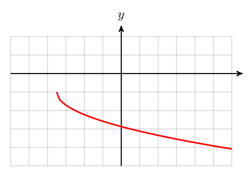

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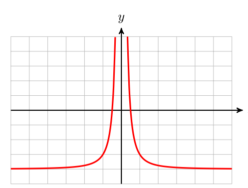

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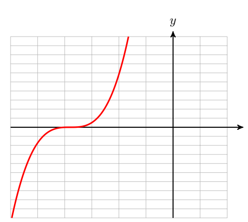
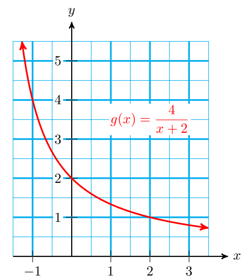


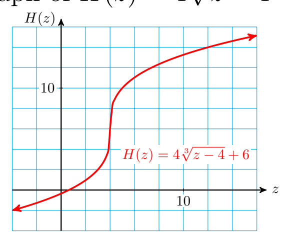
For Problems 31 and 32, use the graph to estimate the solutions to the inequalities.

31. The figure shows a graph of $g(x) = \dfrac{4}{x+2}$.



a $g(x) > 1$ b $g(x) < 3$

32. The figure shows a graph of $H(z) = 4\sqrt[3]{z-4} + 6$.



a $H(z) > 14$ b $H(z) < 6$

For Problems 33 and 34, graph the function in the window

$$\text{Xmin} = -47 \qquad \text{Xmax} = 47$$
$$\text{Ymin} = -31 \qquad \text{Ymax} = 31$$

Use the graph to solve each equation or inequality. Check your solutions algebraically.

33. Graph $F(x) = 4\sqrt{x-25}$.

a Solve $4\sqrt{x-25} = 16$

b Solve $4\sqrt{x-25} = -16$

c Solve $8 < 4\sqrt{x-25} \leq 24$

34. Graph $G(x) = 20 - 0.001(x-8)^4$.

a Solve $20 - 0.001(x-8)^4 = 26.2$

b Solve $20 - 0.001(x-8)^4 = -8.561$

c Solve $20 - 0.001(x-8)^4 \geq 10$

Direct Variation

Two types of functions are widely used in modeling and are known by special names: **direct variation** and **inverse variation**.

Direct Variation

Two variables are **directly proportional** (or just **proportional**) if the ratios of their corresponding values are always equal. Consider the functions described in the tables below. The first table shows the price of gasoline as a function of the number of gallons purchased.

Gallons of gasoline	Total price	Price/Gallons
4	\$9.60	$\dfrac{9.60}{4} = 2.40$
6	\$14.40	$\dfrac{14.40}{6} = 2.40$
8	\$19.20	$\dfrac{19.20}{8} = 2.40$
12	\$28.80	$\dfrac{28.80}{12} = 2.40$
15	\$36.00	$\dfrac{36.00}{15} = 2.40$

Years	Population	People/Years
10	432	$\dfrac{432}{10} \approx 43$
20	932	$\dfrac{932}{20} \approx 47$
30	2013	$\dfrac{2013}{30} \approx 67$
40	4345	$\dfrac{4345}{40} \approx 109$
50	9380	$\dfrac{9380}{50} \approx 188$
60	$20,251$	$\dfrac{20,251}{60} \approx 338$

The ratio $\dfrac{\text{total price}}{\text{number of gallons}}$, or price per gallon, is the same for each pair of values in the first table. This agrees with everyday experience: The price per gallon of gasoline is the same no matter how many gallons you buy. Thus, the total price of a gasoline purchase is directly proportional to the number of gallons purchased.

The second table shows the population of a small town as a function of the town's age. The ratio $\dfrac{\text{number of people}}{\text{number of years}}$ gives the average rate of growth of the population in people per year. You can see that this ratio is *not* constant; in fact, it increases as time goes on. Thus, the population of the town is *not* proportional to its age.

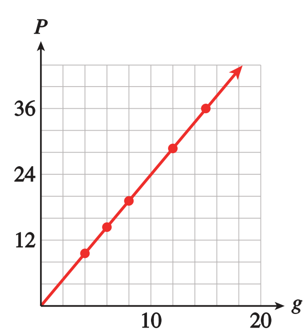
Checkpoint 5.55 **QuickCheck 1.** When does a table represent direct variation?

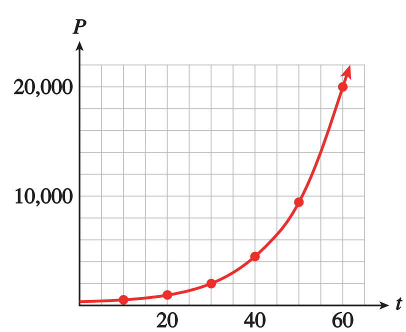
- ⊙ If it has a constant slope
- ⊙ If it includes the point $(0, 0)$
- ⊙ If the ratio output/input is constant
- ⊙ If each output is double the previous one

Answer. Choice 3

Solution. If the ratio output/input is constant

The graphs of these two functions are shown below.





We see that the price, P, of a fill-up is a linear function of the number of gallons, g, purchased. This should not be surprising if we write an equation relating the variables g and P. Because the ratio of their values is constant, we can write

$$\frac{P}{g} = k$$

where k is a constant. In this example, the constant k is 2.40, the price of gasoline per gallon. Solving for P in terms of g, we have

$$P = kg = 2.40g$$

which we recognize as the equation of a line through the origin.

In general, we make the following definition.

Direct Variation.

y **varies directly** with x if

$$y = kx$$

where k is a positive constant called the **constant of variation**.

If y varies directly with x, we may also say that y is directly proportional to x. The two phrases mean the same thing.

Example 5.56

a The circumference, C, of a circle varies directly with its radius, r, because

$$C = 2\pi r$$

The constant of variation is 2π, or about 6.28.

b The amount of interest, I, earned in one year on an account paying 7% simple interest, varies directly with the principal, P, invested, because

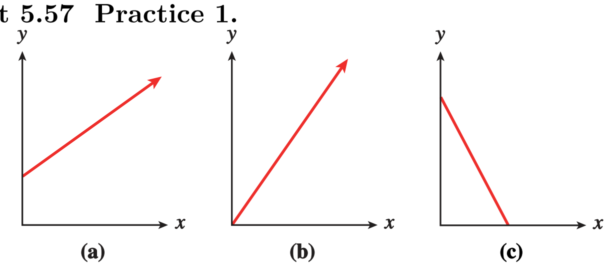
$$I = 0.07P$$

Direct variation defines a linear function of the form

$$y = f(x) = kx$$

The positive constant k in the equation $y = kx$ is just the slope of the graph, so it tells us how rapidly the graph increases. Compared to the standard form for a linear function, $y = b + mx$, the constant term, b, is zero, so the graph of a direct variation passes through the origin.

Checkpoint 5.57 Practice 1.



Which of the graphs above could represent direct variation? Explain why.

- ⊙ (a) The graph is a straight line that increases.
- ⊙ (b) The graph is a straight line through the origin.
- ⊙ (c) The graph is a straight line that decreases.

⊙ None of the above

Answer. (b) ... the origin.

Solution. (b): The graph is a straight line through the origin.

Checkpoint 5.58 QuickCheck 2.

a. A table describes direct variation if the ratio of corresponding entries is (☐ linear ☐ constant ☐ increasing ☐ positive) .

b. A graph describes direct variation if it is a (☐ straight line through the origin ☐ line with positive slope ☐ basic graph ☐ constant of variation) .

c. An equation describes direct variation if it has the form $y =$______.

d. If two variables vary directly, we may also say that they are (☐ growing exponentially ☐ constant ☐ independent ☐ directly proportional) .

Answer 1. constant

Answer 2. straight line through the origin

Answer 3. kx

Answer 4. directly proportional

Solution.

a. constant

b. straight line through the origin

c. $y = kx$

d. directly proportional

The Scaling Property of Direct Variation

Because the graph of $y = kx$ passes through the origin, direct variation has the following **scaling** property: if we double the value of x, then the value of y will double also. In fact, increasing x by any factor causes y to increase by the same factor. Look again at the table for the price of buying gasoline. Doubling the number of gallons of gas purchased, say, from 4 gallons to 8 gallons or from 6 gallons to 12 gallons, causes the total price to double also.

Checkpoint 5.59 QuickCheck 3. You invest $800 for one year at 7% simple interest. The interest earned is

$I = 0.07(800) =$_____ dollars

If you increase the investment by a factor of, say, 1.6 to 1.6(800) or $1280, the interest will be

$I = 0.07(1280) =$_____ dollars

The original interest is increased by a factor of _____.

Answer 1. 56

Answer 2. 89.6

Answer 3. 1.6

Solution. $56; $89.60; 1.6 (the factor by which you increased the investment)

Example 5.60

a Tuition at Woodrow University is \$400 plus \$30 per unit. Is the tuition proportional to the number of units you take?

b Imogen makes a 15% commission on her sales of environmentally friendly products marketed by her co-op. Do her earnings vary directly with her sales?

Solution.

a Let u represent the number of units you take, and let $T(u)$ represent your tuition. Then

$$T(u) = 400 + 30u$$

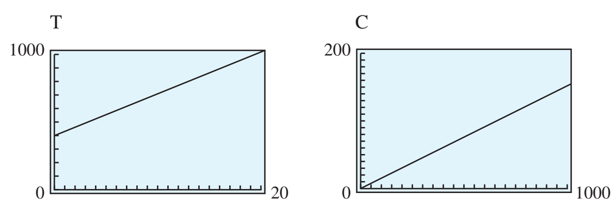
Thus, $T(u)$ is a linear function of u, but the T-intercept is 400, not 0. Your tuition is *not* proportional to the number of units you take, so this is not an example of direct variation. You can check that doubling the number of units does not double the tuition. For example,

$$T(6) = 400 + 30(6) = 580$$

and

$$T(12) = 400 + 30(12) = 760$$

Tuition for 12 units is not double the tuition for 6 units. The graph of $T(u)$ in figure (a) does not pass through the origin.



(a) (b)

b Let S represent Imogen's sales, and let $C(S)$ represent her commission. Then

$$C(S) = 0.15S$$

Thus, $C(S)$ is a linear function of S with a C-intercept of zero, so Imogen's earnings do vary directly with her sales.

Checkpoint 5.61 Practice 2. Which tables could represent direct variation? Explain why. (Hint: What happens to y if you multiply x by a constant?)

a.

x	1	2	3	6	8	9
y	2.5	5	7.5	15	20	22.5

b.

x	2	3	4	6	8	9
y	2	3.5	5	7	8.5	10

- ⊙ (a): If we multiply x by c, y is also multiplied by c.
- ⊙ (b): If we increase x, we also increase y.
- ⊙ None of the above

Answer. Choice 1

Solution. (a): If we multiply x by c, y is also multiplied by c.

Finding a Formula for Direct Variation

If we know any one pair of values for the variables in a direct variation, we can find the constant of variation. Then we can use the constant to write a formula for one of the variables as a function of the other.

Example 5.62

If you kick a rock off the rim of the Grand Canyon, its speed, v, varies directly with the time, t, it has been falling. The rock is falling at a speed of 39.2 meters per second when it passes a lizard on a ledge 4 seconds later.

a Express v as a function of t.

b What is the speed of the rock after it has fallen for 6 seconds?

c Sketch a graph of $v(t)$ versus t.

Solution.

a Because v varies directly with t, there is a positive constant k for which

$$v = kt.$$

We substitute $v = \mathbf{39.2}$ and $t = \mathbf{4}$ and solve for k to find

$$\begin{aligned} 39.2 &= k(4) && \text{Divide both sides by 4.} \\ k &= 9.8 \end{aligned}$$

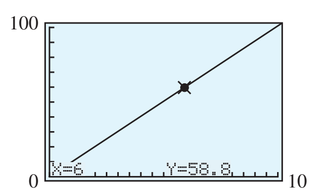
Replacing k by 9.8, we find the formula $v(t) = 9.8t$.

b We evaluate the function found in part (a) for $t = 6$.

$$v = 9.8(\mathbf{6}) = 58.8$$

At $t = 6$ seconds, the rock is falling at a speed of 58.8 meters per second.

c You can use your calculator to graph the function $v = 9.8t$. The graph is shown below.



Checkpoint 5.63 **Practice 3.** The volume of a bag of rice, in cups, is directly proportional to the weight of the bag. A 2-pound bag contains 3.5 cups of rice.

a. Express the volume, V, of a bag of rice as a function of its weight, w.

__=_____

b. How many cups of rice are in a 15-pound bag?

Answer 1. V; $1.75w$

Answer 2. 26.25

Solution.

a. $V = 1.75w$

b. 26.25

Checkpoint 5.64 QuickCheck 3. How can you find the value of the constant of variation?

- ⊙ Double the value of x.
- ⊙ Find the y-intercept of the graph.
- ⊙ Substitute a pair of related values into $y = x$.
- ⊙ All of the above.

Answer. Choice 3

Solution. Substitute a pair of related values into $y = x$.

Direct Variation with a Power of x

In many situations, y is proportional to a power of x, instead of x itself.

Direct Variation with a Power.

y **varies directly** with a power of x if

$$y = kx^n$$

where k and n are positive constants.

Example 5.65

The surface area of a sphere varies directly with the *square* of its radius. A balloon of radius 5 centimeters has surface area 100π square centimeters, or about 314 square centimeters. Find a formula for the surface area of a sphere as a function of its radius.

Solution. If S stands for the surface area of a sphere of radius r, then

$$S = f(r) = kr^2$$

To find the constant of variation, k, we substitute the values of S and r.

$$\mathbf{100\pi} = k(\mathbf{5})^2$$
$$4\pi = k$$

Thus, $S = f(r) = 4\pi r^2$.

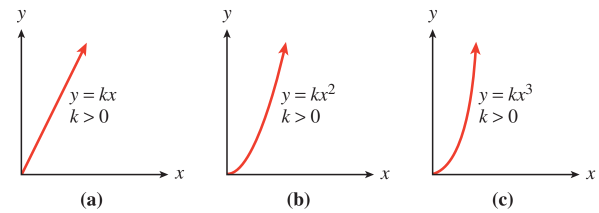
Checkpoint 5.66 Practice 4. The volume of a sphere varies directly with the *cube* of its radius. A balloon of radius 5 centimeters has volume $\dfrac{500\pi}{3}$ cubic centimeters, or about 524 cubic centimeters. Find a formula for the volume, V, of a sphere as a function of its radius, r.

__=_________

Answer. V; $\frac{4}{3}\pi r^3$

Solution. $V = \frac{4}{3}\pi r^3$

In any example of direct variation, as the input variable increases through positive values, the output variable increases also. Thus, a direct variation is an increasing function, as we can see when we consider the graphs of some typical direct variations shown below.



Caution 5.67 The graph of a direct variation always passes through the origin, so when the input is zero, the output is also zero. Thus, the functions $y = 3x+2$ and $y = 0.4x^2 - 2.3$, for example, are not direct variation, even though they are increasing functions for positive x.

Checkpoint 5.68 QuickCheck 4. True or False.

a. Every increasing function is a direct variation. (☐ True ☐ False)

b. Every direct variation is an increasing function. (☐ True ☐ False)

c. The graph of every direct variation is a straight line. (☐ True ☐ False)

d. The graph of every direct variation passes through the origin. (☐ True ☐ False)

Answer 1. False

Answer 2. True

Answer 3. False

Answer 4. True

Solution.

a. False

b. True

c. False

d. True

Even without an equation, we can check whether a table of data describes direct variation or merely an increasing function. If y varies directly with x^n, then $y = kx^n$, or, equivalently, $\frac{y}{x^n} = k$.

Test for Direct Variation.

If the ratio $\frac{y}{x^n}$ is constant, then y varies directly with x^n.

Example 5.69

Delbert collects the following data and would like to know if y varies directly with the square of x. What should he calculate?

x	2	5	8	10	12
y	6	16.5	36	54	76

Solution. If y varies directly with x^2, then $y = kx^2$, or $\frac{y}{x^2} = k$. Delbert should calculate the ratio $\frac{y}{x^2}$ for each data point.

x	2	5	8	10	12
y	6	16.5	36	54	76
$\frac{y}{x^2}$	1.5	0.66	0.56	0.54	0.53

Because the ratio $\frac{y}{x^2}$ is not constant, y does not vary directly with x^2.

Checkpoint 5.70 **Practice 5.** Does B vary directly with the cube of r? Explain your decision.

r	0.1	0.3	0.5	0.8	1.2
B	0.072	1.944	9.0	16.864	124.416

- ⊙ Yes, $\frac{B}{r^3}$ is constant.
- ⊙ No, B is not r^3.

Answer. Choice 1

Solution. Yes, $\frac{B}{r^3}$ is constant.

Scaling

If y varies directly with x, then doubling x causes y to double also. But what if y varies directly with a *power* of x :

- Is the area of a 16-inch circular pizza double the area of an 8-inch pizza?
- If you double the dimensions of a model of a skyscraper, will its weight double also?

You probably know that the answer to both of these questions is *No.* The area of a circle is proportional to the *square* of its radius, and the volume (and hence the weight) of an object is proportional to the *cube* of its linear dimension. Variation with a power of x produces a different scaling effect.

Example 5.71

The Taipei 101 building is 1671 feet tall, and in 2006 it was the tallest skyscraper in the world. Show that doubling the dimensions of a model of the Taipei 101 building produces a model that weighs 8 times as much.

Solution. The Taipei 101 skyscraper is approximately box shaped, so its volume is given by the product of its linear dimensions, $V = lwh$. The weight of an object is proportional to its volume, so the weight, W, of the model is

$$W = klwh$$

where the constant k depends on the material of the model. If we double the length, width, and height of the model, then

$$\begin{aligned}W_{\text{new}} &= k(2l)(2w)(2h)\\ &= 2^3(klwh) = 8W_{\text{old}}\end{aligned}$$

The weight of the new model is $2^3 = 8$ times the weight of the original model.

Checkpoint 5.72 Practice 6. Use the formula for the area of a circle to show that doubling the diameter of a pizza quadruples its area.

The formula for the area of a circle of radius r is $A =$_____

If we double the diameter, the new radius is _____

Substitute this expression into the area formula to get the area of the new circle $A_{\text{new}} =$______

So A_{new} is __ times the original area.

Answer 1. πr^2

Answer 2. $2r$

Answer 3. $\pi(2r)^2$

Answer 4. 4

Solution. $A = \pi r^2$

Doubling the diameter means doubling the radius.

$$\begin{aligned}A_{\text{new}} &= \pi(2r)^2\\ &= 4\pi r^2\\ &= 4A_{\text{old}}\end{aligned}$$

In general, if y varies directly with a power of x, that is, if $y = kx^n$, then doubling the value of x causes y to increase by a factor of 2^n. In fact, if we multiply x by any positive number c, then

$$\begin{aligned}y_{\text{new}} &= k(cx)^n\\ &= c^n(kx^n) = c^n(y_{\text{old}})\end{aligned}$$

so the value of y is multiplied by c^n.

We will call n the **scaling exponent**, and you will often see variation described in terms of scaling. For example, we might say that "the area of a circle scales as the square of its radius." (In many applications, the power n is called the *scale factor*, even though it is not a factor but an exponent.)

Problem Set 5.4

Warm Up

1. Solve

a $\dfrac{a}{a-2} = \dfrac{25}{30}$

b $\dfrac{7}{90} = \dfrac{126}{5(t-2)^2}$

2. Sketch a graph in the first quadrant.

a $f(x) = 2x$

x	0	$\frac{1}{2}$	1	2
$f(x)$				

b $g(x) = 2x^2$

x	0	$\frac{1}{2}$	1	2
$g(x)$				

c $h(x) = \dfrac{2}{x}$

x	0	$\frac{1}{2}$	1	2
$h(x)$				

Skills Practice

For Problems 3 and 4,

a Use the values in the table to find the constant of variation, k, and write y as a function of x.

b Fill in the rest of the table with the correct values.

c What happens to y when you double the value of x?

3. y varies directly with x.

x	2	5		12	
y		1.5	2.4		4.5

4. y varies directly with the square of x.

x	3	6		12	
y		24	54		150

For Problems 5–8, decide whether

a y varies directly with x

b y varies directly with x^2

c y does not vary directly with a power of x

Explain why your choice is correct. If your choice is (a) or (b), find the constant of variation, k.

5.

x	2	3	5	8
y	2.0	4.5	12.5	32.0

6.

x	2	4	6	9
y	12	28	44	68

7.

x	1.5	2.4	5.5	8.2
y	3.0	7.2	33	73.8

8.

x	1.2	2.5	6.4	12
y	7.20	31.25	204.80	720.00

Applications

9. Delbert's credit card statement lists three purchases he made while on a business trip in the Midwest. His company's accountant would like to know the sales tax rate on the purchases.

Price of item	18	28	12
Tax	1.17	1.82	0.78
Tax/Price			

a Compute the ratio of the tax to the price of each item. Is the tax proportional to the price? What is the tax rate?

b Express the tax, T, as a function of the price, p, of the item.

c Sketch a graph of the function by hand, and label the scales on the axes.

10. At constant acceleration from rest, the distance traveled by a race car is proportional to the square of the time elapsed. The highest recorded road-tested acceleration is 0 to 60 miles per hour in 3.07 seconds, which produces the following data.

Time (seconds)	2	2.5	3
Distance (feet)	57.32	89.563	128.97
Distance/Time^2			

a Compute the ratios of the distance traveled to the square of the time elapsed. What was the acceleration, in feet per second squared?

b Express the distance traveled, d, as a function of time in seconds, t.

c Sketch a graph of the function by hand, and label the scales on the axes.

11. The weight of an object on the Moon varies directly with its weight on Earth. A person who weighs 150 pounds on Earth would weigh only 24.75 pounds on the Moon.

a Find a function that gives the weight m of an object on the Moon in terms of its weight w on Earth. Complete the table and graph your function in a suitable window.

w	50	100	200	400
m				

b How much would a person weigh on the Moon if she weighs 120 pounds on Earth?

c A piece of rock weighs 50 pounds on the Moon. How much will it weigh back on Earth?

d If you double the weight of an object on Earth, what will happen to its weight on the Moon?

12. The length of a rectangle is 10 inches, and its width is 8 inches. Suppose we increase the length of the rectangle while holding the width constant.

a Fill in the table.

Length	Width	Perimeter	Area
10	8		
12	8		
15	8		
20	8		

b Does the perimeter vary directly with the length?

c Write a formula for the perimeter of the rectangle in terms of its length.

d Does the area vary directly with the length?

e Write a formula for the area of the rectangle in terms of its length.

13. Hubble's law says that distant galaxies are receding from us at a rate that varies directly with their distance. (The speeds of the galaxies are measured using a phenomenon called redshifting.) A galaxy in the constellation Ursa Major is 980 million light-years away and is receding at a speed of 15,000 kilometers per second.

a Find a function that gives the speed, v, of a galaxy in terms of its distance, d, from Earth. Complete the table and graph your function in a suitable window. (Distances are given in millions of light-years.)

d	500	1000	2000	4000
m				

b How far away is a galaxy in the constellation Hydra that is receding at 61,000 kilometers per second?

c A galaxy in Leo is 1240 million light-years away. How fast is it receding from us?

14. The length, L, of a pendulum varies directly with the square of its period, T, the time required for the pendulum to make one complete swing back and forth. The pendulum on a grandfather clock is 3.25 feet long and has a period of 2 seconds.

a Express L as a function of T. Complete the table and graph your function in a suitable window.

T	1	5	10	20
L				

b How long is the Foucault pendulum in the Pantheon in Paris, which has a period of 17 seconds?

c A hypnotist uses a gold pendant as a pendulum to mesmerize his clients. If the chain on the pendant is 9 inches long, what is the period of its swing?

d In order to double the period of a pendulum, how must you vary its length?

15. The amount of power, P, generated by a windmill varies directly with the cube of the wind speed, w. A windmill on Oahu, Hawaii, produces 7300 kilowatts of power when the wind speed is 32 miles per hour.

a Express the power as a function of wind speed. Complete the table and graph your function in a suitable window.

w	10	20	40	80
P				

b How much power would the windmill produce in a light breeze of 15 miles per hour?

c What wind speed is needed to produce 10,000 kilowatts of power?

d If the wind speed doubles, what happens to the amount of power generated?

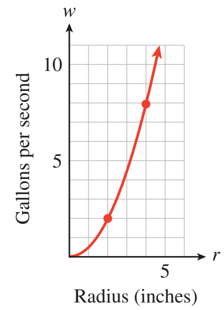
The functions described by a table or a graph in Problems 16-19 are examples of direct variation.

a Find an algebraic formula for the function, including the constant of variation, k.

b Answer the question in the problem.

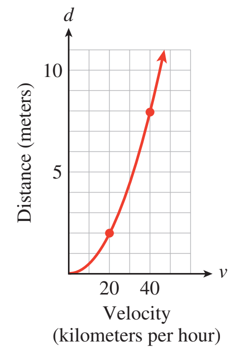
16.

A wide pipe can handle a greater water flow than a narrow pipe. The graph shows the water flow through a pipe, w, as a function of its radius, r. How great is the water flow through a pipe of radius of 10 inches?



17.

The faster a car moves, the more difficult it is to stop. The graph shows the distance, d, required to stop a car as a function of its velocity, v, before the brakes were applied. What distance is needed to stop a car moving at 100 kilometers per hour?



18. The maximum height attained by a cannonball depends on the speed at which it was shot. The table shows maximum height as a function of initial speed. What height is attained by a cannonball whose initial upward speed was 100 feet per second?

Speed (ft/sec)	Height (ft)
40	200
50	31.25
60	450
70	612.5

19. The strength of a cylindrical rod depends on its diameter. The greater the diameter of the rod, the more weight it can support before collapsing. The table shows the maximum weight supported by a rod as a function of its diameter. How much weight can a 1.2-centimeter rod support before collapsing?

Diameter (cm)	Weight (newtons)
0.5	150
1.0	600
1.5	1350
2.0	2400

20. The wind resistance, W, experienced by a vehicle on the freeway varies directly with the square of its speed, v.

a If you double your speed, what happens to the wind resistance?

b If you drive one-third as fast, what happens to the wind resistance?

c If you decrease your speed by 10%, what happens to the wind resistance?

Inverse Variation

Inverse Variation

How long does it take to travel a distance of 600 miles? The answer depends on your average speed. If you are on a bicycle trip, your average speed might be 15 miles per hour. In that case, your traveling time will be

$$T = \frac{D}{R} = \frac{600}{15} = 40 \text{ hours}$$

(Of course, you will have to add time for rest stops; the 40 hours are just your travel time.)

If you are driving your car, you might average 50 miles per hour. Your travel time is then

$$T = \frac{D}{R} = \frac{600}{50} = 12 \text{ hours}$$

If you take a commercial air flight, the plane's speed might be 400 miles per hour, and the flight time would be

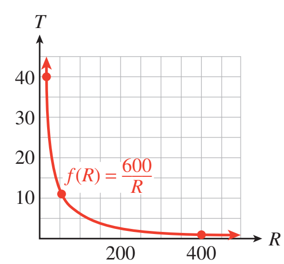
$$T = \frac{D}{R} = \frac{600}{400} = 1.5 \text{ hours}$$

You can see that for higher average speeds, the travel time is shorter. In other words, the time needed for a 600-mile journey is a decreasing function of average speed. In fact, a formula for the function is

$$T = f(R) = \frac{600}{R}$$

This function is an example of **inverse variation**. A table of values and a graph of the function are shown below.

R	T
10	60
15	40
20	30
50	12
200	3
400	1.5



Inverse Variation.

y **varies inversely** with x if

$$y = \frac{k}{x},\ x \neq 0$$

where k is a positive constant.

Caution 5.73 Inverse variation describes a decreasing function, but not every decreasing function represents inverse variation. People sometimes mistakenly use the phrase *varies inversely* to describe any decreasing function, but if y varies inversely with x, the variables must satisfy an equation of the form $y = \dfrac{k}{x}$, or $xy = k$.

To decide whether two variables truly vary inversely, we can check whether their product is constant. For instance, in the preceding travel-time example, we see from the table that $RT = 600$.

R	10	15	20	50	200	400
T	60	40	30	12	3	1.5
RT	600	600	600	600	600	600

Checkpoint 5.74 QuickCheck 1. How can you test whether a table for $y = f(x)$ represents inverse variation?

- ⊙ Check whether xy is a constant.
- ⊙ Check whether the function is decreasing.
- ⊙ Check whether y is the reciprocal of x.
- ⊙ Check whether y/x is a constant.

Answer. Choice 1

Solution. Check whether xy is a constant.

Finding a Formula for Inverse Variation

If we know that two variables vary inversely and we can find one pair of corresponding values for the variables, we can determine k, the constant of variation.

Example 5.75

The amount of current, I, that flows through a circuit varies inversely with the resistance, R, on the circuit. An iron with a resistance of 12 ohms draws 10 amps of current.

a Write a formula that gives current as a function of the resistance.

b Complete the table and graph your function in a suitable window.

R	1	2	10	20
I				

c How much current is drawn by a light bulb with a resistance of 533.3 ohms?

d What is the resistance of a toaster that draws 12.5 amps of current?

Solution.

a Because I varies inversely with R, we know that $I = \dfrac{k}{R}$. To find the constant k, we substitute **10** for I and **12** for R.

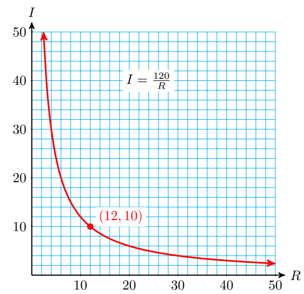
$$10 = \frac{k}{12} \qquad \text{Solve for } k.$$
$$k = 10 \cdot 12 = 120$$

So the formula is $I = \dfrac{120}{R}$.

b We use the formula to complete the table.

R	1	2	10	20
I	120	60	12	6

A graph of I as a function of R is shown below.



c We substitute **533.3** for R in the formula.

$$I = \frac{120}{\mathbf{533.3}} = 0.225$$

The light bulb draws 0.225 amps of current.

d We substitute **12.5** for I in the formula and solve for R.

$$\mathbf{12.5} = \frac{120}{R} \qquad \text{Solve for } R.$$

$$R = \frac{120}{12.5} = 9.6$$

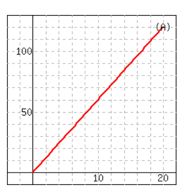
The toaster has a resistance of 9.6 ohms.

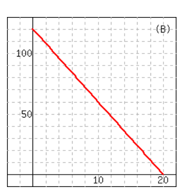
Checkpoint 5.76 **Practice 1.** Delbert's officemates want to buy a $120 gold watch for a colleague who is retiring. The cost per person is inversely proportional to the number of people who contribute.

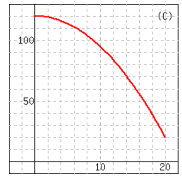
a. Express the cost per person, C, as a function of the number of people, p, who contribute.

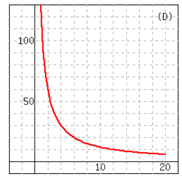
__=_________

b. Sketch the function on the domain $0 \le p \le 20$.









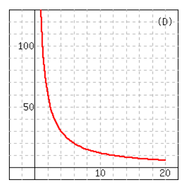
The graph of cost per person is

- ⊙ (A)
- ⊙ (B)
- ⊙ (C)
- ⊙ (D)
- ⊙ none of the above

Answer 1. C; $\frac{120}{p}$

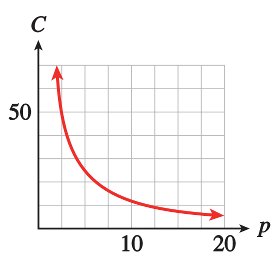
Answer 2. (D)

Solution.

a. 

A graph is also shown below.

b. $x = -9$ or $x = 2$



Inverse Variation with a Power

We can also define inverse variation with a power of the variable.

Inverse Variation with a Power.

y **varies inversely** with x^n if

$$y = \frac{k}{x^n},\ x \neq 0$$

where k and n are positive constants.

We may also say that y is **inversely proportional** to x^n.

Example 5.77

The intensity of electromagnetic radiation, such as light or radio waves, varies inversely with the square of the distance from its source. Radio station KPCC broadcasts a signal that is measured at 0.016 watt per square meter by a receiver 1 kilometer away.

a Write a formula that gives signal strength as a function of distance.

b If you live 5 kilometers from the station, what is the strength of the signal you will receive?

c Graph your function in the window

$$\text{Xmin} = 0 \qquad \text{Xmax} = 6$$
$$\text{Ymin} = 0 \qquad \text{Ymax} = 0.02$$

Solution.

a Let I stand for the intensity of the signal in watts per square meter, and d for the distance from the station in kilometers. Then $I = \dfrac{k}{d^2}$. To find the constant k, we substitute **0.016** for I and 1 for d. Solving for k gives us

$$\mathbf{0.016} = \frac{k}{1^2}$$
$$k = 0.016(1^2) = 0.016$$

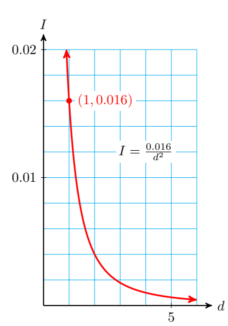
Thus, $I = \dfrac{0.016}{d^2}$.

b Now we can substitute **5** for d and solve for I.

$$I = \frac{0.016}{\mathbf{5}^2} = 0.00064$$

At a distance of 5 kilometers from the station, the signal strength is 0.00064 watt per square meter.

c The graph is shown below.

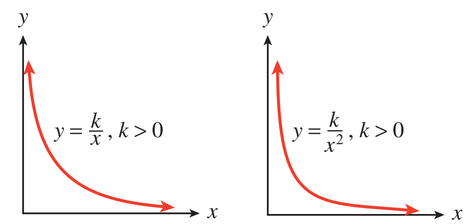


To summarize, an inverse variation is an example of a decreasing function, but not every decreasing function describes an inverse variation.

Test for Inverse Variation.

If the product yx^n is constant and n is positive, then y varies inversely with x^n.

The graphs of some typical inverse variations are shown below. They are versions of the basic graphs you studied in Section 5.3, but restricted to positive x-values only.



(a) **(b)**

Checkpoint 5.78 QuickCheck 2. Use the table to decide whether H could vary inversely with m^2.

m	0.05	0.20	0.25	0.4
H	240	15	9.6	3.75

(☐ Yes ☐ No)

Answer. Yes

Solution. Yes, the product Hm^2 is the constant 0.6 for all pairs (m, H) in the table.

In Section 5.4, we considered the scaling property of direct variation. If $y = kx$ and you double the value of x, then the value of y doubles also. If $y = kx^2$ and you double the value of x, then the value of y is multiplied by a factor of $2^2 = 4$.

What happens when you double the input of an inverse variation?

Example 5.79

The weight, w, of an object varies inversely with the square of its distance, d, from the center of the Earth. Thus,

$$w = \frac{k}{d^2}$$

If you double your distance from the center of the Earth, what happens to your weight? What if you triple the distance?

Solution. Suppose you weigh W pounds at distance D from the center of the Earth. Then $W = \frac{k}{D^2}$. At distance $2D$, your weight will be

$$w = \frac{k}{(2D)^2} = \frac{k}{4D^2} = \frac{1}{4} \cdot \frac{k}{D^2} = \frac{1}{4}W$$

Your new weight will be $\frac{1}{4}$ of your old weight. By a similar calculation, you can check that by tripling the distance, your weight will be reduced to $\frac{1}{9}$ of its original value.

Checkpoint 5.80 Practice 2. The amount of force, F, (in pounds) needed to loosen a rusty bolt with a wrench is inversely proportional to the length, l, of the wrench. Thus,

$$F = \frac{k}{l}$$

If you increase the length of the wrench by 50% so that the new length is $1.5l$, what happens to the amount of force required to loosen the bolt?

- ⊙ $F_{\text{new}} = \frac{1}{2}F_{\text{old}}$
- ⊙ $F_{\text{new}} = 2F_{\text{old}}$
- ⊙ $F_{\text{new}} = \frac{3}{2}F_{\text{old}}$
- ⊙ $F_{\text{new}} = \frac{2}{3}F_{\text{old}}$

Answer. Choice 4

Solution. $F_{\text{new}} = \frac{2}{3}F_{\text{old}}$

Checkpoint 5.81 QuickCheck 3. Match each formula with its description below.

a. Direct variation (☐ i ☐ ii ☐ iii ☐ iv)

b. Direct variation with a power (☐ i ☐ ii ☐ iii ☐ iv)

c. Inverse variation (☐ i ☐ ii ☐ iii ☐ iv)

d. Inverse variation with a power (☐ i ☐ ii ☐ iii ☐ iv)

i. $I = \frac{50}{d^2}$

ii. $S = 15.3h^2$

iii. $N = \frac{2600}{P}$

iv. $C = 85A$

Answer 1. iv
Answer 2. ii
Answer 3. iii
Answer 4. i
Solution.

a. iv

b. ii

c. iii

d. i

Problem Set 5.5

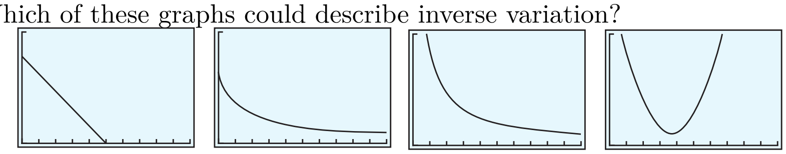
Warm Up

For Problems 1-4, choose variables and write an equation relating them. Which equations describe inverse variation?

1. Rachel spends one-third of her income on rent.
2. There were two candidates in the election for mayor of Centerville, Smith and Jones. A total of 4800 votes were cast (no write-in votes).
3. Water is leaking from a 2000-gallon tank at a rate of one cup per day. (There are 16 cups in a gallon.) The amount of water left in the tank is a function of the amount leaked out.
4. Craig is planning to tile the floor of a 250-square foot room. He is deciding what size tile to use, and how many tiles he will need.

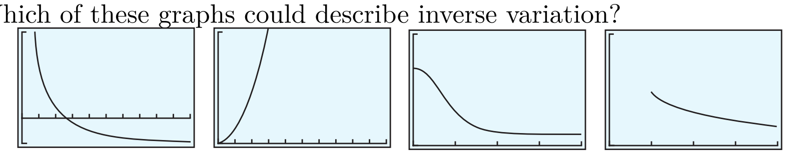
Skills Practice

5. Which of these graphs could describe inverse variation?



(a) (b) (c) (d)

6. Which of these graphs could describe inverse variation?



(a) (b) (c) (d)

For Problems 7 and 8,

a Use the values in the table to find the constant of variation, k, and write y as a function of x.

b Fill in the rest of the table with the correct values.

c What happens to y when you double the value of x?

7. y varies inversely with x.

x	4		20	30	
y		15	6		3

8. y varies inversely with the square of x.

x	0.2		2	4	
y		80		1.25	0.8

For Problems 9–12, decide whether

a y varies inversely with x

b y varies inversely with x^2

c y does not vary inversely with a power of x

Explain why your choice is correct. If your choice is (a) or (b), find the constant of variation, k.

9.

x	0.5	2	3	6
y	288	18	8	2

10.

x	0.5	2	4	5
y	100.0	25.0	12.5	10.0

11.

x	1	1.3	3	4
y	4.0	3.7	2.0	1.0

12.

x	0.5	2	3	5
y	180.00	11.25	5.00	1.80

Applications

13. The marketing department for a paper company is testing wrapping paper rolls in various dimensions to see which shape consumers prefer. All the rolls contain the same amount of wrapping paper.

Width (feet)	2	2.5	3
Length (feet)	12	9.6	8
Length × Width			

a Compute the product of the length and width for each roll of wrapping paper. What is the constant of inverse proportionality?

b Express the length, L, of the paper as a function of the width, w, of the roll.

c Sketch a graph of the function by hand, and label the scales on the axes.

14. The force of gravity on a 1-kilogram mass is inversely proportional to the square of the object's distance from the center of the Earth. The table shows the force on the object, in newtons, at distances that are multiples of the Earth's radius.

Distance (Earth radii)	1	2	4
Force (newtons)	9.8	2.45	0.6125
Force × distance^2			

a Compute the products of the force and the square of the distance. What is the constant of inverse proportionality

b Express the gravitational force, F, on a 1-kilogram mass as a function of its distance, r, from the Earth's center, measured in Earth radii

c Sketch a graph of the function by hand, and label the scales on the axes.

15. Computer monitors produce a magnetic field. The effect of the field, B, on the user varies inversely with his or her distance, d, from the screen. The field from a certain color monitor was measured at 22 milligauss 4 inches from the screen.

a Express the field strength as a function of distance from the screen. Complete the table and graph your function in a suitable window.

d	1	2	12	24
B				

b What is the field strength 10 inches from the screen?

c An elevated risk of cancer can result from exposure to field strengths of 4.3 milligauss. How far from the screen should the computer user sit to keep the field level below 4.3 milligauss?

d If you double your distance from the screen, how does the field strength change?

16. Boyle's law says that the pressure on a gas is inversely proportional to the volume it occupies. For example, deep-sea divers who return to the surface too rapidly get "the bends" when nitrogen bubbles in the blood expand. Suppose a submarine at a depth of 100 meters, where the pressure is is 10.7 atmospheres, releases a bubble of volume 1.5 cubic centimeters.

a Find a formula for the volume of the bubble as a function of the pressure.

b What will the volume of the bubble be when it reaches the surface, where the pressure is 1 atmosphere?

c Graph your function.

17. After the 2017 wildfires, California needs to replant 129,000,000 trees. The amount of time this will take is inversely proportional to the number of workers planting trees. On average, one worker can plant 2000 tree seedlings each day.

a How many days would it take 100 workers to plant the trees?

b Write a formula for the number of working days, D, it will take n workers to plant the trees.

c How many workers would be needed to plant the trees in 300 working days?

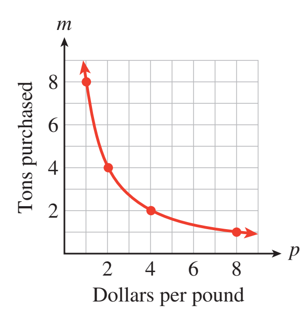
The functions described by a table or a graph in Problems 18–21 are examples of inverse variation.

a Find a formula for the function, including the constant of variation, k.

b Answer the question in the problem.

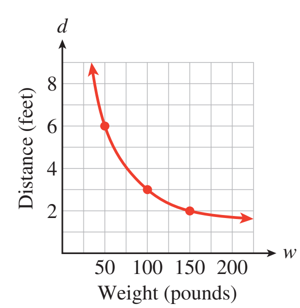
18.

If the price of mushrooms goes up, the amount consumers are willing to buy goes down. The graph shows the number of tons of shiitake mushrooms, m, sold in California each week as a function of their price, p. If the price of shiitake mushrooms rises to \$10 per pound, how many tons will be sold?



19.

When an adult plays with a small child on a seesaw, the adult must sit closer to the pivot point to balance the seesaw. The graph shows this distance, d, as a function of the adult's weight, w. How far from the pivot must Kareem sit if he weighs 280 pounds?



20. The thermocline is a layer of ocean water where the temperature changes rapidly. The table shows the temperature of the water as a function of depth in the thermocline. What is the ocean temperature at a depth of 500 meters?

Depth (m)	Temperature (°C)
200	20
400	10
1000	4

21. The shorter the length of a vibrating guitar string, the higher the frequency of the vibrations. The fifth string is 65 centimeters long and is tuned to A (with a frequency of 220 vibrations per second). The placement of the fret relative to the bridge changes the effective length of the guitar string. The table shows frequency as a function of effective length. How far from the bridge should the fret be placed for the note C (256 vibrations per second)?

Length (cm)	Frequency
55	260
57.2	250
65	220
71.5	200

22. The intensity of illumination, I, from a lamp varies inversely with the square of your distance, d, from the lamp.

a If you double your distance from a reading lamp, what happens to the illumination?

b If you triple the distance, what happens to the illumination?

c If you increase the distance by 25%, what happens to the illumination?

Functions as Models

The Shape of the Graph

To create a good model we first decide what kind of function to use. What sort of function has the right shape to describe the process we want to model? Should it be increasing or decreasing, or some combination of both? Is the slope constant or is it changing?

In Examples 5.82, p. 333 and 5.84, p. 335, we investigate how the shape of a graph illustrates the nature of the process it models.

Example 5.82

Forrest leaves his house to go to school. For each of the following situations, sketch a possible graph of Forrest's distance from home as a function of time.

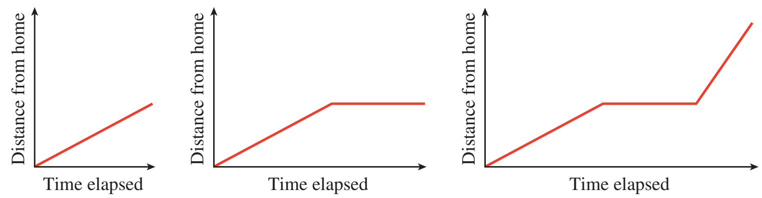
a Forrest walks at a constant speed until he reaches the bus stop.

b Forrest walks at a constant speed until he reaches the bus stop; then he waits there until the bus arrives.

c Forrest walks at a constant speed until he reaches the bus stop, waits there until the bus arrives, and then the bus drives him to school at a constant speed.

Solution.

a The graph is a straight-line segment, as shown in figure (a). It begins at the origin because at the instant Forrest leaves the house, his distance from home is 0. (In other words, when $t = 0, y = 0$.) The graph is a straight line because Forrest has a constant speed. The slope of the line is equal to Forrest's walking speed.

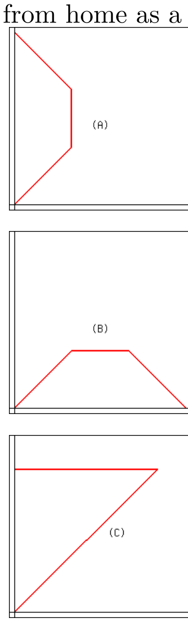


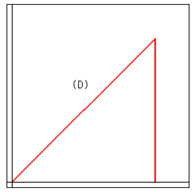
(a) **(b)** **(c)**

b The first part of the graph is the same as part (a). But while Forrest waits for the bus, his distance from home remains constant, so the graph at that time is a horizontal line, as shown in figure (b). The line has slope 0 because while Forrest is waiting for the bus, his speed is 0.

c The graph begins like the graph in part (b). The last section of the graph represents the bus ride. It has a constant slope because the bus is moving at a constant speed. Because the bus (probably) moves faster than Forrest walks, the slope of this segment is greater than the slope for the walking section. The graph is shown in figure (c).

Checkpoint 5.83 **Practice 1.** Erin walks from her home to a convenience store, where she buys some cat food, and then walks back home. Sketch a possible graph of her distance from home as a function of time.

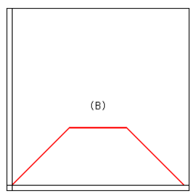




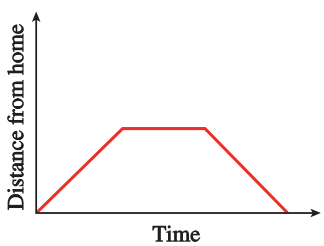
The graph of Erin's distance from home is

- ⊙ (A)
- ⊙ (B)
- ⊙ (C)
- ⊙ (D)
- ⊙ none of the above

Answer. (B)
Solution.



A graph is also shown below.



The graphs in Example 5.82, p. 333 are portions of straight lines. We can also consider graphs that bend upward or downward. The bend is called the **concavity** of the graph.

Example 5.84

The two functions described in this example are both increasing functions, but they increase in different ways. Match each function to its graph and to the appropriate table of values.

a The number of flu cases reported at an urban medical center during an epidemic is an increasing function of time, and it is growing at a faster and faster rate.

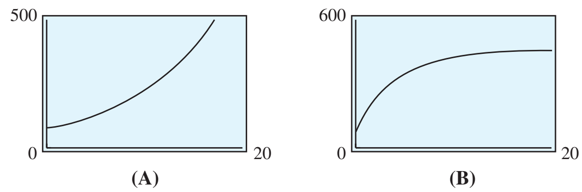
b The temperature of a potato placed in a hot oven increases rapidly at first, then more slowly as it approaches the temperature of the oven.

(1)

x	0	2	5	10	15
y	70	89	123	217	383

(2)

x	0	2	5	10	15
y	70	219	341	419	441



Solution.

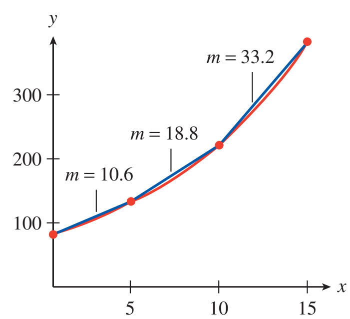
a The number of flu cases is described by graph(A) and table (1). The function values in table (1) increase at an increasing rate. We can see this by computing the rate of change over successive time intervals.

$$x = 0 \text{ to } x = 5: \qquad m = \frac{\Delta y}{\Delta x} = \frac{123 - 70}{5 - 0} = 10.6$$

$$x = 5 \text{ to } x = 10: \qquad m = \frac{\Delta y}{\Delta x} = \frac{217 - 123}{10 - 5} = 18.8$$

$$x = 10 \text{ to } x = 15: \qquad m = \frac{\Delta y}{\Delta x} = \frac{383 - 217}{15 - 10} = 33.2$$

The increasing rates can be seen inthe figure below; the graph bends upward as the slopes increase.



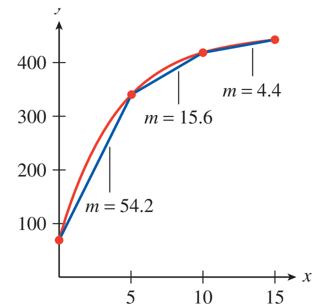
b The temperature of the potato is described by graph(B) and table (2). The function values in table (2) increase, but at a decreasing rate.

$$x = 0 \text{ to } x = 5: \qquad m = \frac{\Delta y}{\Delta x} = \frac{341 - 70}{5 - 0} = 54.2$$

$$x = 5 \text{ to } x = 10: \qquad m = \frac{\Delta y}{\Delta x} = \frac{419 - 341}{10 - 5} = 15.6$$

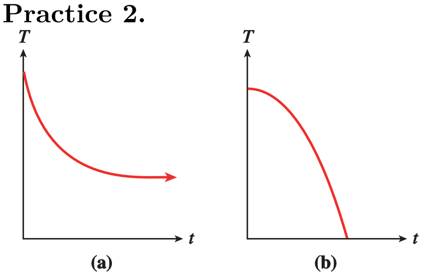
$$x = 10 \text{ to } x = 15: \qquad m = \frac{\Delta y}{\Delta x} = \frac{441 - 419}{15 - 10} = 4.4$$

The decreasing slopes can be seen in the figure below. The graph is increasing but bends downward.



A graph that bends upward is called **concave up**, and one that bends down is **concave down**.

Checkpoint 5.85 Practice 2.



Francine bought a cup of cocoa at the cafeteria. The cocoa cooled off rapidly at first, and then gradually approached room temperature. Which graph more accurately reflects the temperature of the cocoa as a function of time? (☐ (a) ☐ (b))

Explain why. (Select all that apply.)

☐ The graph has a steep negative slope at first, corresponding to an initial rapid drop in the temperature of the cocoa. ☐ The graph becomes closer to a horizontal line, corresponding to the cocoa approaching room temperature. ☐ The graph has a slight negative slope at first, corresponding to an initial rapid drop in the temperature of the cocoa. ☐ The graph becomes grows steeper and steeper, corresponding to the cocoa approaching room temperature. ☐ None of the above

Is the graph you chose concave up or concave down? (☐ Concave up ☐ Concave down)

Answer 1. (a)

Answer 2. AB

Solution. Graph (a): The graph has a steep negative slope at first, corresponding to an initial rapid drop in the temperature of the cocoa. The graph becomes closer to a horizontal line, corresponding to the cocoa approaching room temperature. The graph is concave up.

Checkpoint 5.86 QuickCheck 1. Match each type of graph with its properties.

a. Increasing, concave up (☐ i ☐ ii ☐ iii ☐ iv)

b. Increasing, concave down (☐ i ☐ ii ☐ iii ☐ iv)

c. Decreasing, concave up (☐ i ☐ ii ☐ iii ☐ iv)

d. Decreasing, concave down (☐ i ☐ ii ☐ iii ☐ iv)

i. y-values decrease, slopes decrease

ii. y-values decrease, slopes increase

iii. y-values increase, slopes decrease

iv. y-values increase, slopes increase

Answer 1. iv
Answer 2. iii
Answer 3. ii
Answer 4. i
Solution.

a. iv

b. iii

c. ii

d. i

Using the Basic Functions as Models

We have considered situations that can be modeled by linear or quadratic functions. In this section we'll look at a few of the other basic functions.

Example 5.87

Choose one of the eight basic functions to model each situation, and sketch a possible graph.

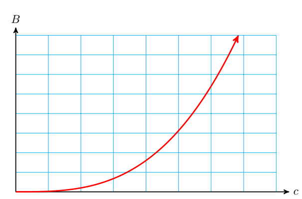
a The number of board-feet, B, that can be cut from a Ponderosa pine is a function of the *cube* of the circumference, c, of the tree at a standard height.

b The manager of an appliance store must decide how many coffee-makers to order every quarter. The optimal order size, Q, is a function of the *square root* of the annual demand for coffee-makers, D.

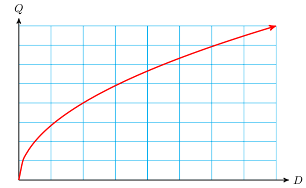
c The loudness, or intensity, I, of the music at a concert is a function of the *reciprocal of the square* of your distance, d, from the speakers.

Solution.

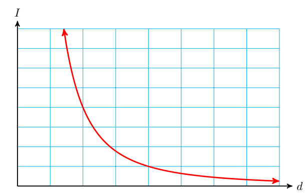
a $B = kc^3$



b $Q = k\sqrt{D}$



c $I = \dfrac{k}{d^2}$



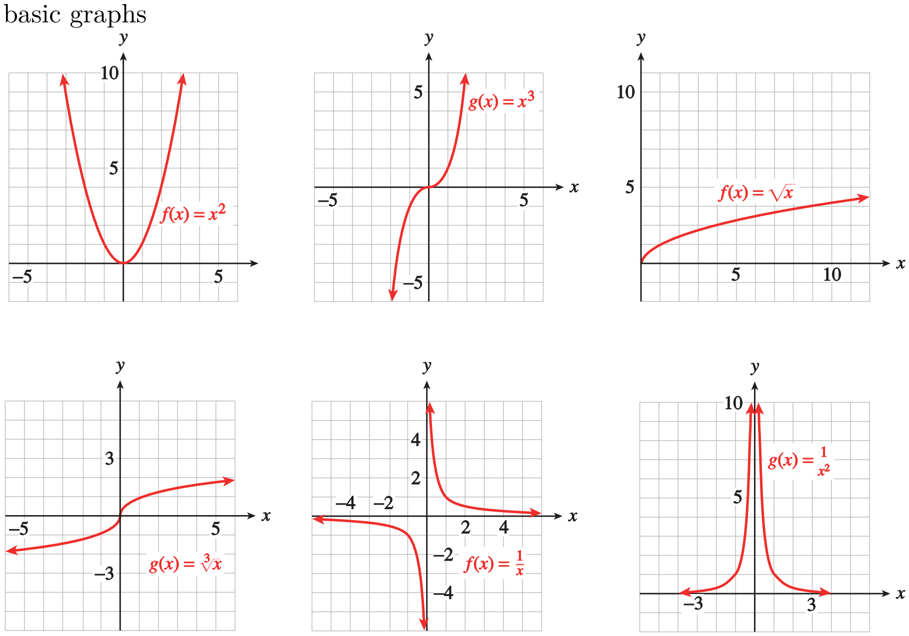
Checkpoint 5.88 QuickCheck 2. Which graph is concave down wherever it is defined?

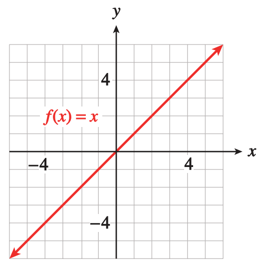
- ⊙ $y = x^3$
- ⊙ $y = \dfrac{1}{x}$
- ⊙ $y = \sqrt{3}x$
- ⊙ $y = \sqrt{x}$

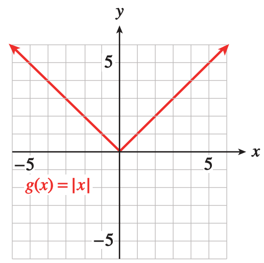
Answer. Choice 4

Solution. $y = \sqrt{x}$

Checkpoint 5.89 Practice 3. Graphs of the eight basic functions: Eight basic graphs







Choose one of the eight basic functions to model each situation, and sketch a possible graph.

a. The contractor for a new hotel is estimating the cost of the marble tile for a circular lobby. The cost, C, is a function of the square of the diameter, D, of the lobby.

Basic function: (☐ quadratic ☐ cubic ☐ square root ☐ cube root ☐ reciprocal ☐ inverse square ☐ linear ☐ absolute value)

b. Investors are deciding whether to support a windmill farm. The wind speed, v, needed to generate a given amount of power is a function of the cube root of the power, P.

Basic function: (☐ quadratic ☐ cubic ☐ square root ☐ cube root ☐ reciprocal ☐ inverse square ☐ linear ☐ absolute value)

c. The frequency, F, of the note produced by a violin string is a function of the reciprocal of the length, L, of the string.

Basic function: (☐ quadratic ☐ cubic ☐ square root ☐ cube root ☐ reciprocal ☐ inverse square ☐ linear ☐ absolute value)

Answer 1. quadratic

Answer 2. cube root

Answer 3. reciprocal

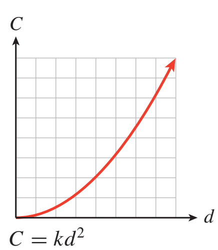
Solution.

a. $C = kD^2$; a graph is below.

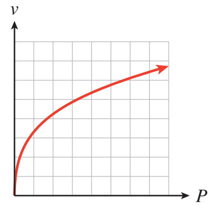
b. $v = k\sqrt[3]{P}$; a graph is below.

c. $F = \dfrac{k}{L}$; a graph is below.

a

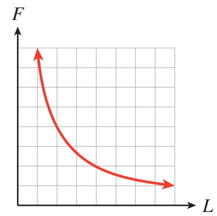


b



$v = k\sqrt[3]{P}$

c



$F = \frac{k}{L}$

The next Example illustrates an application of the function $f(x) = \sqrt{x}$.

Example 5.90

The speed of sound is a function of the temperature of the air in kelvins. (The temperature, T, in kelvins is given by $T = C + 273$, where C is the temperature in degrees Celsius.) The table shows the speed of sound, s, in meters per second, at various temperatures, T.

T (°K)	0	20	50	100	200	400
T (m/sec)	0	89.7	141.8	200.6	283.7	401.2

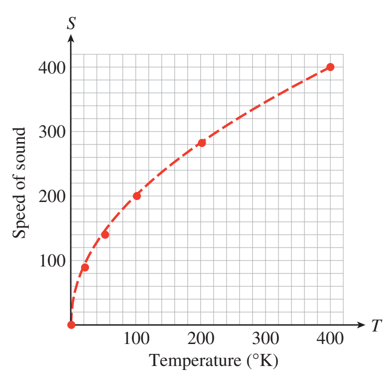
a Plot the data to obtain a graph. Which of the basic functions does your graph most resemble?

b Find a value of k so that $s = kf(T)$ fits the data.

c On a summer night when the temperature is 20° Celsius, you see a flash of lightning, and 6 seconds later you hear the thunderclap. Use your function to estimate your distance from the thunderstorm.

Solution.

a The graph of the data is shown below. The shape of the graph reminds us of the square root function, $y = \sqrt{x}$.



b We are looking for a value of k so that the function $f(T) = k\sqrt{T}$ fits the data. We substitute one of the data points into the formula and solve for k. If we choose the point $(100, 200.6)$, we obtain

$$200.6 = k\sqrt{100}$$

and solving for k yields $k = 20.06$. We can check that the formula $s = 20.06\sqrt{T}$ is a good fit for the rest of the data points as well. Thus, we suggest the function

$$f(T) = 20.06\sqrt{T}$$

as a model for the speed of sound.

c First, we use the model to calculate the speed of sound at a temperature of 20° Celsius. The Kelvin temperature is

$$T = 20 + 273 = 293$$

so we evaluate $s = f(T)$ for $T = 293$.

$$f(293) = 20.06\sqrt{293} \approx 343.4$$

Thus, s is approximately 343.4 meters per second.

The lightning and the thunderclap occur simultaneously, and the speed of light is so fast (about 30,000,000 meters per second) that we see the lightning flash as it occurs. So if the sound of the thunderclap takes 6 seconds after the flash to reach us, we can use our calculated speed of sound to find our distance from the storm.

$$\begin{aligned}\text{distance} &= \text{speed} \times \text{time}\\ &= (343.4 \text{ m/sec})(6 \text{ sec}) = 2060.4 \text{ meters}\end{aligned}$$

The thunderstorm is 2060 meters, or about 1.3 miles, away.

Checkpoint 5.91 Practice 4. The ultraviolet index (UVI) is issued by the National Weather Service as a forecast of the amount of ultraviolet radiation expected to reach Earth around noon on a given day. The data show how much exposure to the sun people can take before risking sunburn.

UVI	2	3	4	5	6	8	10	12
Minutes to burn (more sensitive)	30	20	15	12	10	7.5	6	5
Minutes to burn (more sensitive)	150	100	75	60	50	37.5	30	25

a. Plot m, the minutes to burn, against u, the UVI, to obtain two graphs, one for people who are more sensitive to sunburn, and another for people less sensitive to sunburn. Which of the basic functions do your graphs most resemble?

- ⊙ $y = x^2$
- ⊙ $y = x^3$
- ⊙ $y = \sqrt{x}$
- ⊙ $y = \sqrt[3]{x}$
- ⊙ $y = \dfrac{1}{x}$

- ⊙ $y = \frac{1}{x^2}$
- ⊙ $y = x$
- ⊙ $y = |x|$

b. For each graph, find a value of k so that $m = kf(u)$ fits the data.

More sensitive: $k =$____

Less sensitive: $k =$____

Answer 1. Choice 5

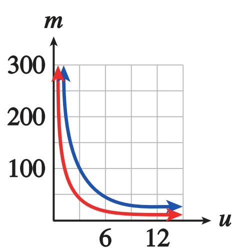
Answer 2. 60

Answer 3. 300

Solution.

a. The graphs are shown below. They resemble $f(x) = \frac{1}{x}$.

b. More sensitive: $k = 60$, Less sensitive: $k = 300$



Checkpoint 5.92 QuickCheck 3. To decide which basic function might model a set of data, we can

- ⊙ plot the data.
- ⊙ look at the variables.
- ⊙ choose scales for the axes.
- ⊙ calculate the slope.

Answer. plot the data.

Solution. plot the data.

At this point, a word of caution is in order. There is more to choosing a model than finding a curve that fits the data. A model based purely on the data is called an **empirical model**. However, many functions have similar shapes over small intervals of their input variables, and there may be several candidates that model the data. Such a model simply describes the general shape of the data set; the parameters of the model do not necessarily correspond to any actual process.

In contrast, **mechanistic models** provide insight into the biological, chemical, or physical process that is thought to govern the phenomenon under study. Parameters derived from mechanistic models are quantitative estimates of real system properties. Here is what GraphPad Software has to say about modeling:

"Choosing a model is a scientific decision. You should base your choice on your understanding of chemistry or physiology (or genetics, etc.). The choice should not be based solely on the shape of the graph.

"Some programs . . . automatically fit data to hundreds or thousands of equations and then present you with the equation(s) that fit the data best. Using such a program is appealing because it frees you from the need to choose an equation. The problem is that the program has no understanding of the scientific context of your experiment. The equations that fit the data best are unlikely to correspond to scientifically meaningful models. You will not be able to interpret the best-fit values of the variables, and the results are unlikely to be useful for data analysis."

(Source: *Fitting Models to Biological Data Using Linear and Nonlinear Regression*, Motulsky & Christopoulos, GraphPad Software, 2003)

The Absolute Value and Distance

The absolute value function is used to model problems involving distance. Recall that the absolute value of a number gives the distance from the origin to that number on the number line.

Checkpoint 5.93 QuickCheck 4. To find the absolute value of a number, we can

- ⊙ change its sign.
- ⊙ subtract it from 0.
- ⊙ find its distance from 0.
- ⊙ square it.

Answer. square it.

Solution. find its distance from 0.

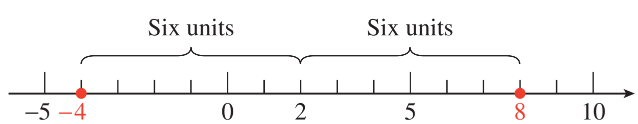
We can find the distance from a number x to some point other than the origin, say a, by computing $|x - a|$. For instance, the distance on the number line from $x = -2$ to $a = 5$ is

$$|-2 - 5| = |-7| = 7 \text{ units}$$

Distance and Absolute Value.

The distance between two points x and a is given by $|x - a|$.

For example, the equation $|x - 2| = 6$ means "the distance between x and 2 is 6 units." The number x could be to the left or the right of 2 on the number line. Thus, the equation has two solutions, 8 and -4, as shown below.



Example 5.94

Write each statement using absolute value notation. Illustrate the solutions on a number line.

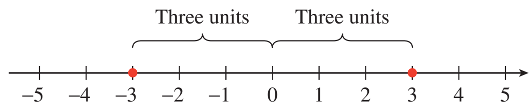
a x is three units from the origin.

b p is two units from 5.

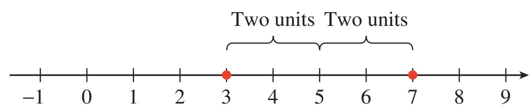
c a is within four units of -2.

Solution. First, we restate each statement in terms of distance.

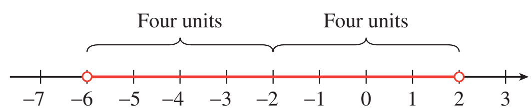
a The distance between x and the origin is three units, or $|x| = 3$. Thus, x can be 3 or -3.



b The distance between p and 5 is two units, or $|p - 5| = 2$. If we count two units on either side of 5, we see that p can be 3 or 7.



c The distance between a and -2 is less than four units, or $\left|a - (-2)\right| < 4$, or $|a + 2| < 4$. Count four units on either side of -2, to find -6 and 2. Then a is between -6 and 2, or $-6 < a < 2$.



Checkpoint 5.95 **QuickCheck 5.** The notation $|x - 3| = 5$ means

- ⊙ x is 3 units bigger than 5.
- ⊙ the distance between x and 5 is 3 units.
- ⊙ 5 and 3 are x units apart.
- ⊙ the distance between x and 3 is 5 units.

Answer. Choice 4

Solution. the distance between x and 3 is 5 units.

Checkpoint 5.96 **Practice 5.** Write each statement using absolute value notation; then illustrate the solutions on a number line.

a. x is five units away from -3.

______________=__

b. x is at least six units away from 4.

______________≥__

Answer 1. $|x + 3|$

Answer 2. 5

Answer 3. $|x - 4|$

Answer 4. 6

Solution.

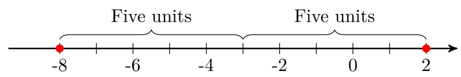
a. $|x + 3| = 5$

A number line is shown below.

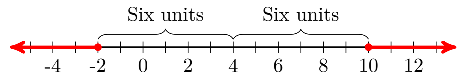
b. $|x - 4| \geq 6$

A numberline is shown below.

a

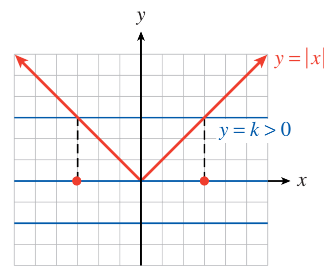


b



Absolute Value Equations

A graph can help us analyze an absolute value equation. For example, we know that the simple equation $|x| = 5$ has two solutions, $x = 5$ and $x = -5$.
In fact, we can see from the graph at right that the equation $|x| = k$ has two solutions if $k > 0$, one solution if $k = 0$, and no solution if $k < 0$.



Checkpoint 5.97 **QuickCheck 6.** Which statement is true?

- ⊙ The graph of $y = |2x - 8|$ has no negative inputs.
- ⊙ The equation $|2x - 8| = -4$ has two solutions.
- ⊙ Depending on x, $|2x - 8|$ can equal $2x - 8$ or $8 - 2x$.
- ⊙ The graph of $y = |2x - 8|$ is a straight line.

Answer. Choice 3

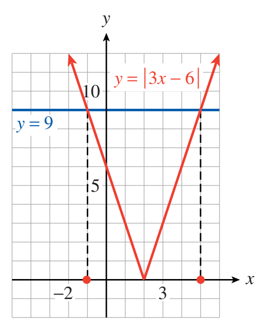
Solution. Depending on x, $|2x - 8|$ can equal $2x - 8$ or $8 - 2x$.

Example 5.98

a Use a graph of $y = |3x - 6|$ to solve the equation $|3x - 6| = 9$.

b Use a graph of $y = |3x - 6|$ to solve the equation $|3x - 6| = -2$.

Solution.



a The graph shows the graphs of $y = |3x - 6|$ and $y = 9$. We see that there are two points on the graph of $y = |3x - 6|$ that have $y = 9$, and those points have x-coordinates $x = -1$ and $x = 5$. We can verify algebraically that the solutions are -1 and 5.

$$x = -1:$$

$$|3(-1) - 6| = |-9| = 9$$

$$x = 5:$$

$$|3(\mathbf{5}) - 6| = |9| = 9$$

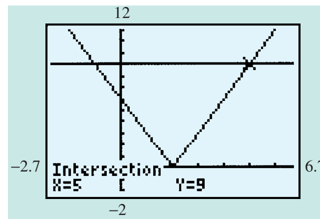
b There are no points on the graph of $y = |3x - 6|$ with $y = -2$, so the equation $|3x - 6| = -2$ has no solutions.

Technology 5.99 Solving Absolute Value Equations. We can use a graphing calculator to solve the equations in Example 5.98, p. 346.

The graph shows the graphs of $Y_1 = \text{abs}(3X - 6)$ and $Y_2 = 9$ in the window

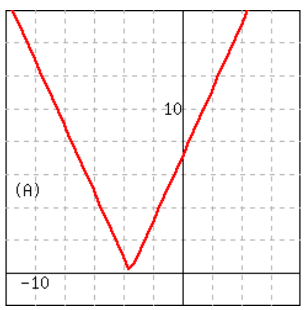
$$\text{Xmin} = -2.7 \qquad \text{Xmax} = 6.7$$
$$\text{Ymin} = -2 \qquad \text{Ymax} = 12$$

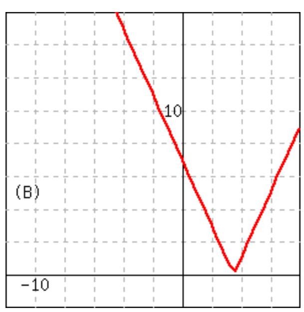
We use the Trace or the *intersect* feature to locate the intersection points at $(-1, 9)$ and $(5, 9)$.

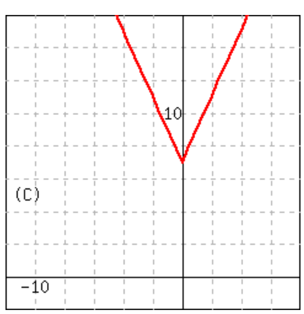


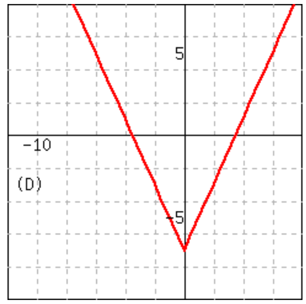
Checkpoint 5.100 Practice 6.

a. Graph $y = |2x + 7|$ for $-12 \le x \le 8$.









The graph of $y = |2x + 7|$ is

- ⊙ (A)
- ⊙ (B)
- ⊙ (C)
- ⊙ (D)
- ⊙ none of the above

b. Use your graph to solve the equation $|2x + 7| = 11$.

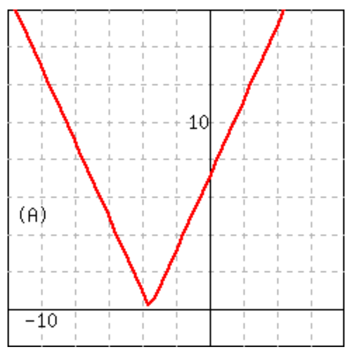
$x =$________

Answer 1. (A)

Answer 2. $-9, 2$

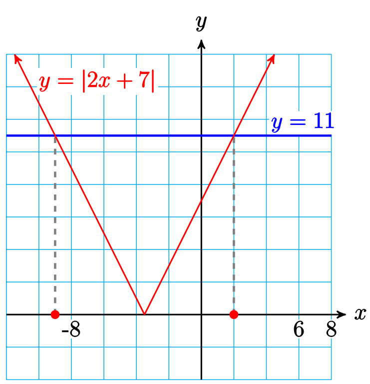
Solution.

a.



A graph is also shown below for part (b).

b. $x = -9$ or $x = 2$



To solve an absolute value equation algebraically, we use the definition of absolute value.

Example 5.101

Solve the equation $|3x-6|=9$ algebraically.

Solution. We write the piecewise definition of $|3x-6|$.

$$|3x-6| = \begin{cases} 3x-6 & \text{if} \quad 3x-6 \geq 0, \text{ or } \; x \geq 2 \\ -(3x-6) & \text{if} \quad 3x-6 < 0, \text{ or } \; x < 2 \end{cases}$$

Thus, the absolute value equation $|3x-6|=9$ is equivalent to two regular equations:

$$3x-6=9 \quad \text{or} \quad -(3x-6)=9$$

or, by simplifying the second equation,

$$3x-6=9 \quad \text{or} \quad 3x-6=-9$$

Solving these two equations gives us the same solutions we found in Example 5.98, p. 346, namely $x=5$ and -1.

In general, we have the following strategy for solving absolute value equations.

Absolute Value Equations.

The equation

$$|ax+b|=c \qquad (c>0)$$

is equivalent to

$$ax+b=c \quad \text{or} \quad ax+b=-c$$

Checkpoint 5.102 **Practice 7.** Solve $|2x+7|=11$ algebraically.

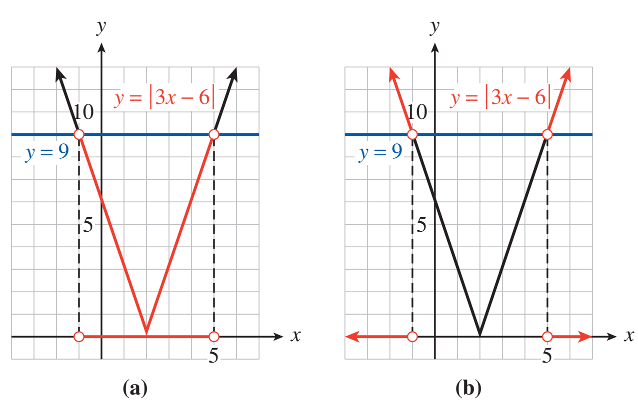
$x=$________

Answer. $-9, 2$

Solution. $x=-9$ or $x=2$

Absolute Value Inequalities

We can also use graphs to solve absolute value inequalities. Look again at the graph of $y=|3x-6|$ in figure (a) below.



Because of the V-shape of the graph, all points with y-values less than 9 lie between the two solutions of $|3x-6|=9$, that is, between -1 and 5. Thus, the solutions of the inequality $|3x-6|<9$ are $-1<x<5$.

On the other hand, to solve the inequality $|3x-6|>9$, we look for points on the graph with y-values greater than 9. In figure (b), we see that these points have x-values outside the interval between -1 and 5. In other words, the solutions of the inequality $|3x-6|>9$ are $x<-1$ or $x>5$.

Thus, we can solve an absolute value inequality by first solving the related equation.

Absolute Value Inequalities.

Suppose the solutions of the equation $|ax+b|=c$ are r and s, with $r<s$. Then

1. The solutions of $|ax+b|<c$ are
$$r<x<s$$
2. The solutions of $|ax+b|>c$ are
$$x<r \quad \text{or} \quad x>s$$

Checkpoint 5.103 QuickCheck 8. If $|x-3|>5$, then

- ⊙ either $x-3>5$ or $x-3>-5$.
- ⊙ $5>x-3>-5$.
- ⊙ either $x-3>5$ or $x-3<-5$.
- ⊙ $-5>x-3<5$.

Answer. Choice 3

Solution. either $x-3>5$ or $x-3<-5$.

Example 5.104

Solve $|4x-15|<0.01$

Solution. First, we solve the equation $|4x-15|=0.01$. There are two cases:

$$\begin{aligned} 4x-15 &= 0.01 \\ 4x &= 15.01 \\ x &= 3.7525 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 4x-15 &= -0.01 \\ 4x &= 14.99 \\ x &= 3.7475 \end{aligned}$$

Because the inequality symbol is $<$, the solutions of the inequality are between these two values: $3.7475<x<3.7525$. In interval notation, the solutions are $(3.7475, 3.7525)$.

Checkpoint 5.105 Practice 8.

a. Solve the inequality $|2x+7|<11$

b. Solve the inequality $|2x+7|>11$

Answer 1. $-9<x<2$

Answer 2. $x<-9$ or $x>2$

Solution.

a. $(-9, 2)$

b. $(-\infty, -9) \cup (2, \infty)$

Problem Set 5.6

Warm Up

In Problems 1–6, state the intervals (if any) on which the function is

a increasing and concave up

b increasing and concave down

c decreasing and concave up

d decreasing and concave down

Give your answers in interval notation.

1. $f(x) = x^2$

2. $f(x) = x^3$

3. $f(x) = \sqrt{x}$

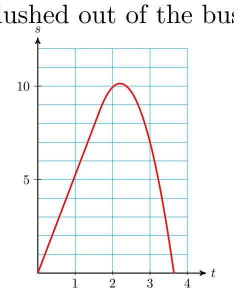
4. $f(x) = \sqrt[3]{x}$

5. $f(x) = \dfrac{1}{x}$

6. $f(x) = \dfrac{1}{x^2}$

Skills Practice

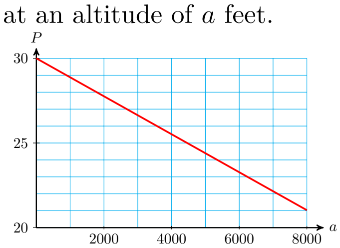
7. The graph defines a function, h, that shows the height, s, in meters, of a duck t seconds after it is flushed out of the bushes.



a Use function notation to state that s is a function of t.

b What does the statement $h(3) = 7$ mean in this context?

8. The graph defines a function, f, that shows the atmospheric pressure, P, in inches of mercury, at an altitude of a feet.



a Use function notation to state that P is a function of a.

b What does the statement $f(1500) = 28.3$ mean in this context?

In Problems 9 and 10, use the table of values to answer the questions.

a Based on the given values, is the function increasing or decreasing?

b Could the function be concave up, concave down, or linear?

9.

x	0	1	2	3	4
$f(x)$	1	1.5	2.25	3.375	5.0625

10.

x	0	1	2	3	4
$g(x)$	1	0.8	0.64	0.512	0.4096

For Problems 11 and 12, plot the data; then decide which of the basic functions could describe the data.

11.

x	0.5	1	2	3	4
y	12	6	3	2	1.5

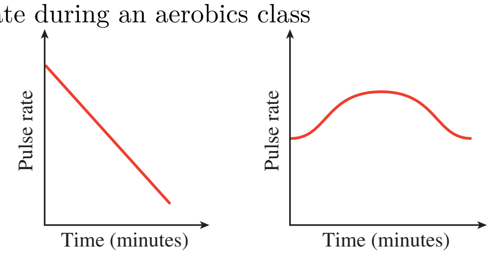
12.

x	0	0.5	1	2	3
y	0	0.0125	0.1	0.8	2.7

Applications

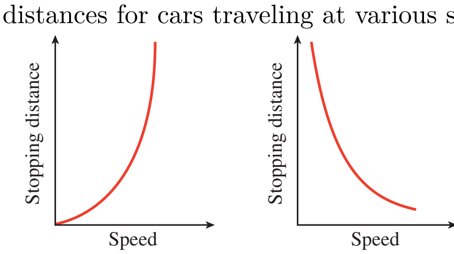
In Problems 13–16, which graph best illustrates each of the following situations?

13. Your pulse rate during an aerobics class



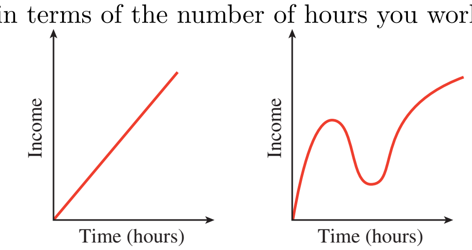
(a) (b)

14. The stopping distances for cars traveling at various speeds



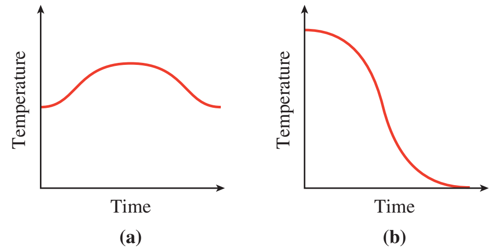
(a) (b)

15. Your income in terms of the number of hours you worked



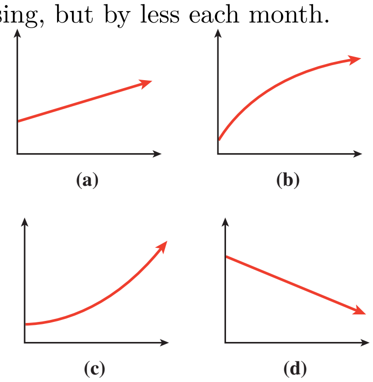
(a) (b)

16. Your temperature during an illness

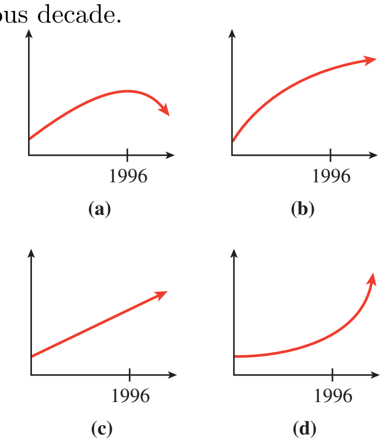


Choose the graph that depicts the function described in Problems 17 and 18.

17. Inflation is still rising, but by less each month.



18. The price of wheat was rising more rapidly in 1996 than at any time during the previous decade.



In Problems 19 and 20, match each graph with the function it illustrates.

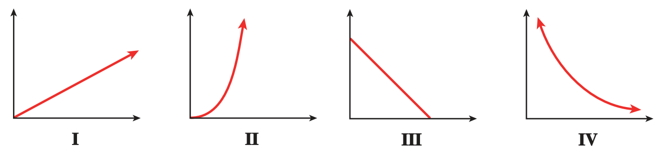
19.

a The volume of a cylindrical container of constant height as a function of its radius

b The time it takes to travel a fixed distance as a function of average speed

c The simple interest earned at a given interest rate as a function of the investment

d The number of Senators present versus the number absent in the U.S. Senate



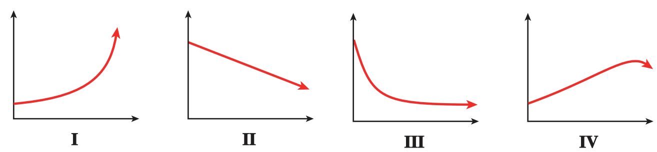
20.

a Unemployment was falling but is now steady.

b Inflation, which rose slowly until last month, is now rising rapidly.

c The birthrate rose steadily until 1990 but is now beginning to fall.

d The price of gasoline has fallen steadily over the past few months.



Each situation in Problems 21–26 can be modeled by a transformation of a basic function. Name the basic function and sketch a possible graph.

21. The volume of a hot air balloon, as a function of its radius

22. The length of a rectangle as a function of its width, if its area is 24 square feet

23. The time it takes you to travel 600 miles, as a function of your average speed

24. The sales tax on a purchase, as a function of its price

25. The width of a square skylight, as a function of its area

26. The sales tax on a purchase, as a function of its price

27. Four different functions are described below. Match each description with the appropriate table of values and with its graph.

a As a chemical pollutant pours into a lake, its concentration is a function of time. The concentration of the pollutant initially increases quite rapidly, but due to the natural mixing and self-cleansing action of the lake, the concentration levels off and stabilizes at some saturation level.

b An overnight express train travels at a constant speed across the Great Plains. The train's distance from its point of origin is a function of time.

c The population of a small suburb of a Florida city is a function of time. The population began increasing rather slowly, but it has continued to grow at a faster and faster rate.

d The level of production at a manufacturing plant is a function of capital outlay, that is, the amount of money invested in the plant. At first, small increases in capital outlay result in large increases in production, but eventually the investors begin to experience diminishing returns on their money, so that although production continues to increase, it is at a disappointingly slow rate.

1

x	1	2	3	4	5	6	7	8
y	60	72	86	104	124	149	179	215

2

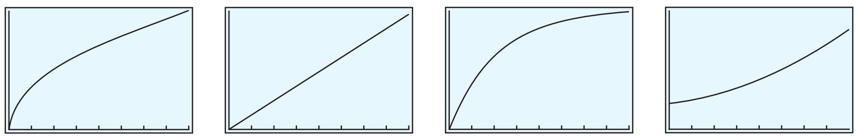
x	1	2	3	4	5	6	7	8
y	60	85	103	120	134	147	159	169

3

x	1	2	3	4	5	6	7	8
y	60	120	180	240	300	360	420	480

4

x	1	2	3	4	5	6	7	8
y	60	96	118	131	138	143	146	147



(A) (B) (C) (D)

28. Four different functions are described below. Match each description with the appropriate table of values and with its graph.

a Fresh water flowing through Crystal Lake has gradually reduced the phosphate concentration to its natural level, and it is now stable.

b The number of bacteria in a person during the course of an illness is a function of time. It increases rapidly at first, then decreases slowly as the patient recovers.

c A squirrel drops a pine cone from the top of a California redwood. The height of the pine cone is a function of time, decreasing ever more rapidly as gravity accelerates its descent.

d Enrollment in Ginny's Weight Reduction program is a function of time. It began declining last fall. After the holidays, enrollment stabilized for a while but soon began to fall off again.

1

x	0	1	2	3	4
y	160	144	96	16	0

2

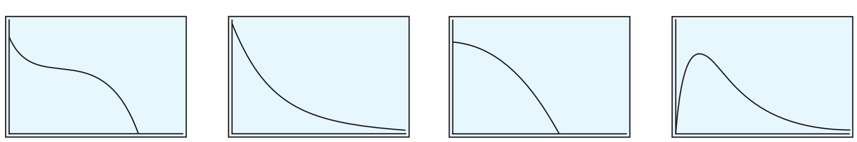
x	0	1	2	3	4
y	20	560	230	90	30

3

x	0	1	2	3	4
y	480	340	240	160	120

4

x	0	1	2	3	4
y	250	180	170	150	80

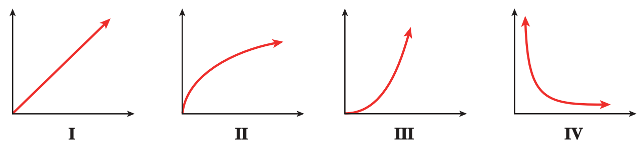


(A) (B) (C) (D)

29. The table shows the radii, r, of several gold coins, in centimeters, and their value, v, in dollars.

Radius	0.5	1	1.5	2	2.5
Value	200	800	1800	3200	5000

a Which graph represents the data?



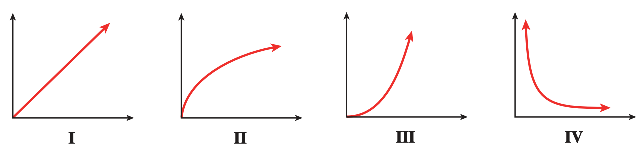
b Which equation describes the function?

i $v = k\sqrt{r}$ ii $v = kr$ iii $v = kr^2$ iv $v = \frac{k}{r}$

30. The table shows how the amount of water, A, flowing past a point on a river is related to the width, W, of the river at that point.

Width (feet)	11	23	34	46
Amount of water (ft^3/sec)	23	34	41	47

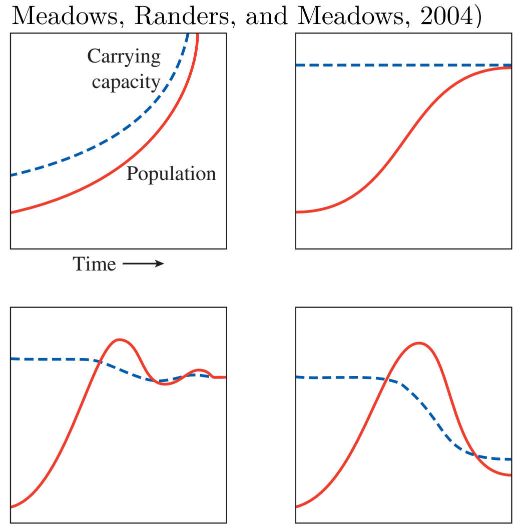
a Which graph represents the data?



b Which equation describes the function?

i $A = k\sqrt{W}$ ii $A = kW$ iii $A = kW^2$ iv $A = \frac{k}{W}$

31. As the global population increases, many scientists believe it is approaching, or has already exceeded, the maximum number the Earth can sustain. This maximum number, or carrying capacity, depends on the finite natural resources of the planet -- water, land, air, and materials -- but also on how people use and preserve the resources. The graphs show four different ways that a growing population can approach its carrying capacity over time. (Source: Meadows, Randers, and Meadows, 2004)



Source: Meadows, Randers, and Meadows, 2004

Match each graph to one of the scenarios described in (a)-(d), and explain your choice.

a Sigmoid growth: the population levels off smoothly below the carrying capacity.

b Overshoot and collapse: the population exceeds the carrying capacity with severe damage to the resource base, and is forced to decline rapidly to achieve a new balance with a reduced carrying capacity.

c Continued growth: the carrying capacity is far away, or growing faster than the population.

d Overshoot and oscillation: the population exceeds the carrying capacity without inflicting permanent damage, then oscillates around the limit before leveling off.

Absolute Value

In Problems 1–8,

a Use absolute value notation to write each expression as an equation or an inequality. (It may be helpful to restate each sentence using the word *distance*.)

b Illustrate the solutions on a number line.

1. x is six units from the origin.

2. a is seven units from the origin.

3. The distance from p to -3 is five units.

4. The distance from q to -7 is two units.

5. t is within three units of 6.

6. w is no more than one unit from -5.

7. b is at least 0.5 unit from -1.

8. m is more than 0.1 unit from 8.

9. Graph $y = |x + 3|$. Use your graph to solve the following equations and inequalities.

a $|x + 3| = 2$ b $|x + 3| \leq 4$ c $|x + 3| > 5$

10. Graph $y = |x - 2|$. Use your graph to solve the following equations and inequalities.

a $|x - 2| = 5$ b $|x - 2| < 8$ c $|x - 2| \geq 4$

11. Graph $y = |2x - 8|$. Use your graph to solve the following equations and inequalities.

a $|2x - 8| = 0$ b $|2x - 8| = -2$ c $|2x - 8| < -6$

12. Graph $y = |4x + 8|$. Use your graph to solve the following equations and inequalities.

a $|4x + 8| = 0$ b $|4x + 8| < 0$ c $|4x + 8| > -3$

For Problems 13-24, solve.

13. $|2x - 1| = 4$ **14.** $|3x - 1| = 5$ **15.** $0 = |7 + 3q|$

16. $|-11 - 5t| = 0$ **17.** $4 = \dfrac{|b + 2|}{3}$ **18.** $6\,|n + 2| = 9$

19. $\big|2(w - 7)\big| = 1$ **20.** $2 = \left|\dfrac{a - 4}{5}\right|$ **21.** $|c - 2| + 3 = 1$

22. $5 = 4 - |h + 3|$ **23.** $-7 = |2m + 3|$ **24.** $|5r - 3| = -2$

For Problems 25-36, solve.

25. $|2x + 6| < 3$ **26.** $|5 - 3x| \leq 1$ **27.** $7 \leq |3 - 2d|$

28. $10 < |3r + 2|$ **29.** $|6s + 15| > -3$ **30.** $|8b - 12| < -4$

31. $|t - 1.5| < 0.1$

32. $|z - 2.6| \leq 0.1$

33. $|T - 3.25| \geq 0.05$

34. $|P - 0.6| > 0.01$

35. $-1 \geq \left|\dfrac{n-3}{2}\right|$

36. $-0.1 \leq |9(p+2)|$

Chapter 5 Summary and Review

Glossary

- function
- input variable
- output variable
- cube root
- absolute value
- proportional
- direct variation
- inverse variation
- constant of variation
- concavity
- scaling
- horizontal asymptote
- vertical asymptote

Key Concepts

1 A function can be described in words, by a table, by a graph, or by an equation.

Function Notation.

2

Input variable

$$f(x) = y$$

Output variable

3 Finding the value of the output variable that corresponds to a particular value of the input variable is called **evaluating the function**.

4 The point (a, b) lies on the graph of the function f if and only if $f(a) = b$.

5 Each point on the graph of the function f has coordinates $(x, f(x))$ for some value of x.

The Vertical Line Test.

6 A graph represents a function if and only if every vertical line intersects the graph in at most one point.

7 We can use a graphical technique to solve equations and inequalities.

8 b is the **cube root** of a if b cubed equals a. In symbols, we write

$$b = \sqrt[3]{a} \quad \text{if} \quad b^3 = a$$

Absolute Value.

9 The **absolute value** of x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

10 Absolute value bars act like grouping devices in the order of operations: you should complete any operations that appear inside absolute value bars before you compute the absolute value.

11 The maximum or minimum of a quadratic function occurs at the vertex.

12 Eight basic functions and their graphs are important in applications:

$$f(x) = x \qquad f(x) = |x| \qquad f(x) = x^2 \qquad f(x) = x^3$$
$$f(x) = \sqrt{x} \qquad f(x) = \sqrt[3]{x} \qquad f(x) = \frac{1}{x} \qquad f(x) = \frac{1}{x^2}$$

13 Two variables are **directly proportional** if the ratios of their corresponding values are always equal.

Direct Variation.

14 y **varies directly** with x if

$$y = kx$$

where k is a positive constant called the **constant of variation**.

15 Direct variation defines a linear function of the form

$$y = f(x) = kx$$

The positive constant k in the equation $y = kx$ is just the slope of the graph.

16 Direct variation has the following **scaling** property: increasing x by any factor causes y to increase by the same factor.

Direct Variation with a Power.

17

y **varies directly** with a power of x if

$$y = kx^n$$

where k and n are positive constants.

18 If the ratio $\dfrac{y}{x^n}$ is constant, then y varies directly with x^n.

Inverse Variation.

19 y **varies inversely** with x if

$$y = \frac{k}{x},\ x \neq 0$$

where k is a positive constant.

Inverse Variation with a Power.

20 y **varies inversely** with x^n if

$$y = \frac{k}{x^n},\ x \neq 0$$

where k and n are positive constants.

21 If the product yx^n is constant and n is positive, then y varies inversely with x^n.

22 A graph that bends upward is called **concave up**, and one that bends downward is **concave down**.

Chapter 5 Review Problems

Which of the tables in Problems 1–4 describe functions? Why or why not?

1.

x	-2	-1	0	1	2	3
y	6	0	1	2	6	8

2.

p	3	-3	2	-2	-2	0
q	2	-1	4	-4	3	0

3.

Student	Score on IQ test	Score on SAT test
(A)	118	649
(B)	98	450
(C)	110	590
(D)	105	520
(E)	98	490
(F)	122	680

4.

Student	Correct answers on math quiz	Quiz grade
(A)	13	85
(B)	15	89
(C)	10	79
(D)	12	82
(E)	16	91
(F)	18	95

5. The total number of barrels of oil pumped by the AQ oil company is given by the formula

$$N(t) = 2000 + 500t$$

where N is the number of barrels of oil t days after a new well is opened. Evaluate $N(10)$ and explain what it means.

6. The number of hours required for a boat to travel upstream between two cities is given by the formula

$$H(v) = \frac{24}{v-8}$$

where v represents the boat's top speed in miles per hour. Evaluate $H(16)$ and explain what it means.

For Problems 7-10, evaluate the function for the given values.

7. $F(t) = \sqrt{1+4t^2}$, $F(0)$ and $F(-3)$

8. $G(x) = \sqrt[3]{x-8}$, $G(0)$ and $G(20)$

9. $h(v) = 6 - |4-2v|$, $h(8)$ and $h(-8)$

10. $m(p) = \dfrac{120}{p+15}$, $m(5)$ and $m(-40)$

11. $P(x) = x^2 - 6x + 5$

a Compute $P(0)$.

b Find all values of x for which $P(x) = 0$.

12. $R(x) = \sqrt{4-x^2}$

a Compute $R(0)$.

b Find all values of x for which $R(x) = 0$.

For Problems 13 and 14, refer to the graphs to answer the questions.

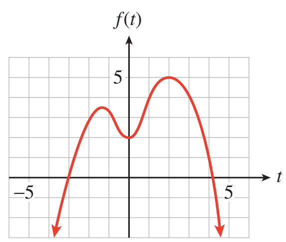
13.

a Find $f(-2)$ and $f(2)$.

b For what value(s) of t is $f(t) = 4$?

c Find the t- and $f(t)$-intercepts of the graph.

d What is the maximum value of f? For what value(s) of t does f take on its maximum value?



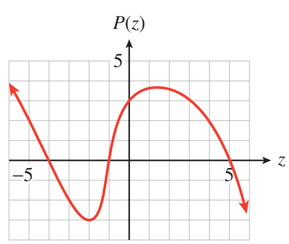
14.

a Find $P(-3)$ and $P(3)$.

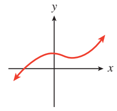
b For what value(s) of z is $P(z) = 2$?

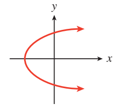
c Find the z- and $P(z)$-intercepts of the graph.

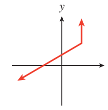
d What is the minimum value of P? For what value(s) of z does P take on its minimum value?

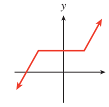


Which of the graphs in Problems 15–18 represent functions?

15. 

16. 

17. 

18. 

For Problems 19–22, graph the function by hand.

19. $f(t) = -2t + 4$

20. $g(s) = \dfrac{-2}{3}s - 2$

21. $p(x) = 9 - x^2$

22. $q(x) = x^2 - 16$

For Problems 23–26, graph the given function on a graphing calculator. Then use the graph to solve the equations and inequalities. Round your answers to one decimal place if necessary.

23. $y = \sqrt[3]{x}$

a Solve $\sqrt[3]{x} = 0.8$

b Solve $\sqrt[3]{x} = 1.5$

c Solve $\sqrt[3]{x} > 1.7$

d Solve $\sqrt[3]{x} \leq 1.26$

24. $y = \dfrac{1}{x}$

a Solve $\dfrac{1}{x} = 2.5$

b Solve $\dfrac{1}{x} = 0.3125$

c Solve $\dfrac{1}{x} \geq 0.\overline{2}$

d Solve $\dfrac{1}{x} < 5$

25. $y = \dfrac{1}{x^2}$

a Solve $\dfrac{1}{x^2} = 0.03$

b Solve $\dfrac{1}{x^2} = 6.25$

c Solve $\dfrac{1}{x^2} > 0.16$

d Solve $\dfrac{1}{x^2} \leq 4$

26. $y = \sqrt{x}$

a Solve $\sqrt{x} = 0.707$

b Solve $\sqrt{x} = 1.7$

c Solve $\sqrt{x} < 1.5$

d Solve $\sqrt{x} \geq 1.3$

In Problems 27–30, y varies directly or inversely with a power of x. Find the power of x and the constant of variation, k. Write a formula for each function of the form $y = kx^n$ or $y = \dfrac{k}{x^n}$.

27.

x	y
2	4.8
5	30.0
8	76.8
11	145.2

28.

x	y
1.4	75.6
2.3	124.2
5.9	318.6
8.3	448.2

29.

x	y
0.5	40.0
2.0	10.0
4.0	5.0
8.0	2.5

30.

x	y
1.5	320.0
2.5	115.2
4.0	45.0
6.0	20.0

31. The distance s a pebble falls through a thick liquid varies directly with the square of the length of time t it falls.

a If the pebble falls 28 centimeters in 4 seconds, express the distance it will fall as a function of time.

b Find the distance the pebble will fall in 6 seconds.

32. The volume, V, of a gas varies directly with the temperature, T, and inversely with the pressure, P, of the gas.

a If $V = 40$ when $T = 300$ and $P = 30$, express the volume of the gas as a function of the temperature and pressure of the gas.

b Find the volume when $T = 320$ and $P = 40$.

33. The demand for bottled water is inversely proportional to the price per bottle. If Droplets can sell 600 bottles at \$8 each, how many bottles can the company sell at \$10 each?

34. The intensity of illumination from a light source varies inversely with the square of the distance from the source. If a reading lamp has an intensity of 100 lumens at a distance of 3 feet, what is its intensity 8 feet away?

35. A person's weight, w, varies inversely with the square of his or her distance, r, from the center of the Earth.

a Express w as a function of r. Let k stand for the constant of variation.

b Make a rough graph of your function.

c How far from the center of the Earth must Neil be in order to weigh one-third of his weight on the surface? The radius of the Earth is about 3960 miles.

36. The period, T, of a pendulum varies directly with the square root of its length, L.

a Express T as a function of L. Let k stand for the constant of variation.

b Make a rough graph of your function.

c If a certain pendulum is replaced by a new one four-fifths as long as the old one, what happens to the period?

For Problems 37 and 38, sketch a graph to illustrate the situations.

37. Inga runs hot water into the bathtub until it is about half full. Because the water is too hot, she lets it sit for a while before getting into the tub. After several minutes of bathing, she gets out and drains the tub. Graph the water level in the bathtub as a function of time, from the moment Inga starts filling the tub until it is drained.

38. David turns on the oven and it heats up steadily until the proper baking temperature is reached. The oven maintains that temperature during the time David bakes a pot roast. When he turns the oven off, David leaves the oven door open for a few minutes, and the temperature drops fairly rapidly during that time. After David closes the door, the temperature continues to drop, but at a much slower rate. Graph the temperature of the oven as a function of time, from the moment David first turns on the oven until shortly after David closes the door when the oven is cooling.

For Problems 39–42, sketch a graph by hand for the function.

39. y varies directly with x^2. The constant of variation is $k = 0.25$.

40. y varies directly with x. The constant of variation is $k = 1.5$.

41. y varies inversely with x. The constant of variation is $k = 2$.

42. y varies inversely with x^2. The constant of variation is $k = 4$.

In Problems 43 and 44,

a Plot the points and sketch a smooth curve through them.

b Use your graph to discover the equation that describes the function.

43.

x	$g(x)$
2	12
3	8
4	6
6	4
8	3
12	2

44.

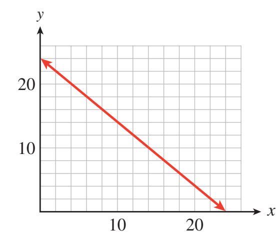
x	$F(x)$
-2	8
-1	1
0	0
1	-1
2	-8
3	-27

In Problems 45–50,

a Use the graph to complete the table of values.

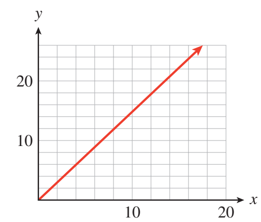
b By finding a pattern in the table of values, write an equation for the graph.

45.



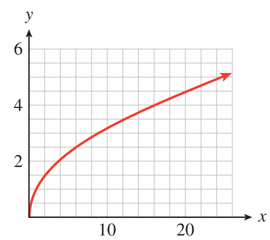
x	0	4	8		16	
y				10		2

46.



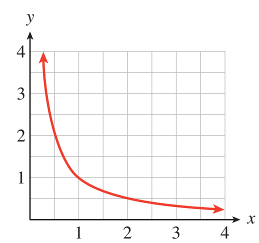
x	0	4	10		14	
y				18		24

47.



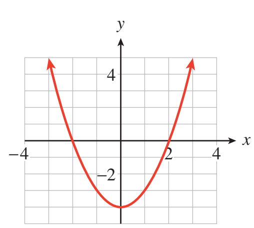
x	0		4		16	25
y		1		3		

48.



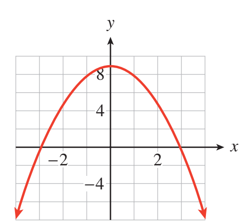
x		0.5	1	1.5		4
y	4				0.5	

49.



x	-3	-2		0	1	2
y			-3			

50.



x	-3	-2		0	1	
y			8			-7

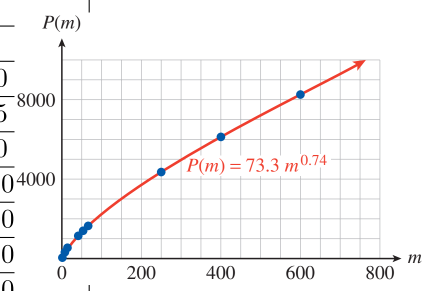
Chapter 6

Powers and Roots

We next turn our attention to a large and useful family of functions, called power functions. Here is an example of a **power function** with fractional exponents.

In 1932, Max Kleiber published a remarkable equation for the metabolic rate of an animal as a function of its mass. The table at right shows the mass of various animals in kilograms and their metabolic rates, in kilocalories per day. A plot of the data, resulting in the famous "mouse-to-elephant" curve, is shown in the figure.

Animal	Mass (kg)	Metabolic rate (kcal/day)
Mouse	0.02	3.4
Rat	0.2	28
Guinea pig	0.8	48
Cat	3.0	150
Rabbit	3.5	165
Dog	15.5	520
Chimpanzee	38	1110
Sheep	50	1300
Human	65	1660
Pig	250	4350
Cow	400	6080
Polar bear	600	8340
Elephant	3670	48, 800



Kleiber modeled his data by the power function

$$P(m) = 73.3m^{0.74}$$

where P is the metabolic rate and m is the mass of the animal. Kleiber's rule initiated the use of **allometric equations**, or power functions of mass, in physiology.

Investigation 6.1 Inflating a Balloon. If you blow air into a balloon, what do you think will happen to the air pressure inside the balloon as it expands? Here is what two physics books have to say:

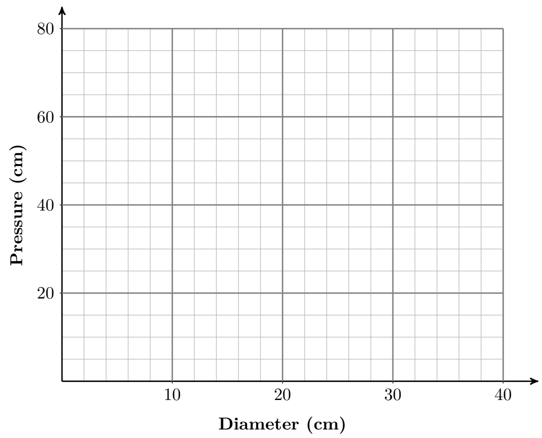
"The greater the pressure inside, the greater the balloon's volume." Jay Boleman, Physics, a Window on Our World

"Contrary to the process of blowing up a toy balloon, the pressure required to force air into a bubble decreases with bubble size." Francis Sears, Mechanics, Heat, and Sound

1 On the basis of these two quotations and your own intuition, sketch a graph of pressure as a function of the diameter of the balloon. Describe your graph: Is it increasing or decreasing? Is it concave up (bending upward) or concave down (bending downward)?

2 Two high school students, April Leonardo and Tolu Noah, decided to see for themselves how the pressure inside a balloon changes as the balloon expands. Using a column of water to measure pressure, they collected the following data while blowing up a balloon. Graph their data on the grid below.

Diameter (cm)	Pressure (cm H_2O)
5.7	60.6
7.3	57.2
8.2	47.9
10.7	38.1
12	37.1
14.6	31.9
17.5	28.1
20.5	26.4
23.5	28
25.2	31.4
26.1	34
27.5	37.2
28.4	37.9
29	40.7
30	43.3
30.6	46.6
31.3	50
32.2	61.9



3 Describe the graph of April and Tolu's data. Does the graph confirm the predictions of the physics books?

4 As the diameter of the balloon increases from 5 cm to 20 cm, the pressure inside decreases. Can we find a function that describes this portion of the graph? Here is some information:

- Pressure is the force per unit area exerted by the balloon on the air inside, or $P = \dfrac{F}{A}$.
- The balloon is spherical, so its surface area, A, is given by $A = \pi d^2$.

Because the force increases as the balloon expands, we will try a power function of the form $F = kd^p$, where k and p are constants, to see if it fits the data. Combine the three equations, $P = \dfrac{F}{A}$, $A = \pi d^2$, and $F = kd^p$, to express P as a power function of d.

5 Graph the function $P = 211d^{-0.7}$ on the same grid with the data. Do the data support the hypothesis that P is a power function of d?

6 What is the value of the exponent p in $F = kd^p$?

Integer Exponents

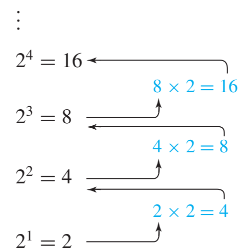
You know that that a positive integer exponent tells us how many times its base occurs as a factor in an expression. For example,

$$4a^3b^2 \quad \text{means} \quad 4aaabb$$

What is the meaning of a negative exponent?

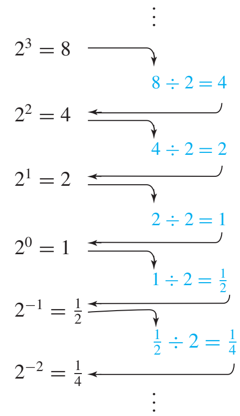
Negative Exponents

Study the list of powers of 2 shown in Table (a) and observe the pattern as we move up the list from bottom to top. Each time the exponent increases by 1 we multiply by another factor of 2. We can continue up the list as far as we like.



a.

If we move back down the list, we divide by 2 at each step, until we get to the bottom of the list, $2^1 = 2$.
What if we continue the list in the same way, dividing by 2 each time we decrease the exponent? The results are shown in Table (b).
As we continue to divide by 2, we generate fractions whose denominators are powers of 2. In particular,



b.

$$2^{-1} = \frac{1}{2} = \frac{1}{2^1} \qquad \text{and} \qquad 2^{-2} = \frac{1}{4} = \frac{1}{2^2}$$

Based on these observations, we make the following definitions.

Definition of Negative and Zero Exponents.

$$a^{-n} = \frac{1}{a^n} \qquad (a \neq 0)$$
$$a^0 = 1 \qquad (a \neq 0)$$

These definitions tell us that if the base a is not zero, then any number raised to the zero power is 1, and that a negative exponent denotes a reciprocal.

Example 6.1

a $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

b $9x^{-2} = 9 \cdot \frac{1}{x^2} = \frac{9}{x^2}$

Caution 6.2

1. A negative exponent does *not* mean that the power is negative! For example,
$$2^{-3} \neq -2^3$$

2. In Example 6.1, p. 368b, note that
$$9x^{-2} \neq \frac{1}{9x^2}$$
The exponent, -2, applies only to the base x, not to 9.

Checkpoint 6.3 QuickCheck 1. True or false.

a. Negative exponents are used to denote reciprocals. (☐ True ☐ False)

b. We can write $\dfrac{1}{4x^3}$ as $4x^{-3}$. (☐ True ☐ False)

c. Any number raised to the zero power is zero. (☐ True ☐ False)

d. $(-4)^{-3} = 4^3$ (☐ True ☐ False)

Answer 1. True
Answer 2. False
Answer 3. False
Answer 4. False
Solution.

a. True

b. False

c. False

d. False

Checkpoint 6.4 Practice 1. Write each expression without using negative exponents.

a. $5^{-4} =$__________

b. $5x^{-4} =$__________

Answer 1. $\frac{1}{5^4}$
Answer 2. $\frac{5}{x^4}$
Solution.

a. $\dfrac{1}{5^4}$

b. $\dfrac{5}{x^4}$

In the next Example, we see how to evaluate expressions that contain negative exponents and how to solve equations involving negative exponents.

Example 6.5

The body mass index, or BMI, is one measure of a person's physical fitness. Your body mass index is defined by

$$\text{BMI} = wh^{-2}$$

where w is your weight in kilograms and h is your height in meters. The World Health Organization classifies a person as obese if his or her BMI is 25 or higher.

a Calculate the BMI for a woman who is 1.625 meters (64 inches) tall and weighs 54 kilograms (120 pounds).

b For a fixed weight, how does BMI vary with height?

c The world's heaviest athlete is the amateur sumo wrestler Emanuel

Yarbrough, who weighs 319 kg (704 pounds). What height would Yarbrough have to be to have a BMI under 25?

Solution.

a We evaluate the formula to find

$$\text{BMI} = 54(1.625^{-2}) = 54\left(\frac{1}{1.625^2}\right) = 20.45$$

b $\text{BMI} = wh^{-2} = \dfrac{w}{h^2}$, so BMI varies inversely with the square of height. That is, for a fixed weight, BMI decreases as height increases.

c To find the height that gives a BMI of 25, we solve the equation $25 = 319h^{-2}$. Note that the variable h appears in the denominator of a fraction, so we begin by clearing the denominator—in this case we multiply both sides of the equation by h^2.

$$\begin{aligned} 25 &= \frac{319}{h^2} && \text{Multiply both sides by } h^2. \\ 25h^2 &= 319 && \text{Divide both sides by 25.} \\ h^2 &= 12.76 && \text{Extract square roots.} \\ h &\approx 3.57 \end{aligned}$$

To have a BMI under 25, Yarbrough would have to be over 3.75 meters, or 11 feet 8 inches tall. (In fact, he is 6 feet 8 inches tall.)

Checkpoint 6.6 Practice 2. Solve the equation $0.2x^{-3} = 1.5$

$x =$__________

Hint. Rewrite without a negative exponent.
Clear the fraction.
Isolate the variable.

Answer. $\sqrt[3]{\frac{2}{15}}$

Solution. $x = \sqrt[3]{\dfrac{2}{15}} \approx 0.51$

Power Functions

The functions that describe direct and inverse variation are part of a larger family of functions called **power functions**.

Definition 6.7 Power Function.

A function of the form

$$f(x) = kx^p$$

where k and p are nonzero constants, is called a **power function**.

Examples of power functions are

$$V(r) = \frac{4}{3}\pi r^3 \quad \text{and} \quad L(T) = 0.8125T^2$$

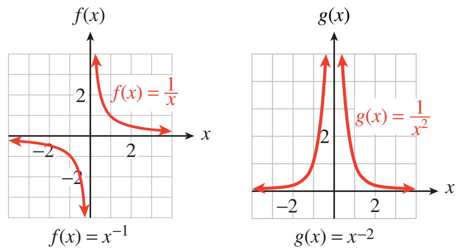
In addition, the basic functions

$$f(x) = \frac{1}{x} \quad \text{and} \quad g(x) = \frac{1}{x^2}$$

which we studied in Chapter 5, p. 263 can be written as

$$f(x) = x^{-1} \quad \text{and} \quad g(x) = x^{-2}$$

Their graphs are shown below. Note that power functions with negative exponents are undefined at zero.



Example 6.8

Which of the following are power functions?

a $f(x) = \frac{1}{3}x^4 + 2$ b $g(x) = \frac{1}{3x^4}$ c $h(x) = \frac{x+6}{x^3}$

Solution.

a This is not a power function, because of the addition of the constant term.

b We can write $g(x) = \frac{1}{3}x^{-4}$, so g is a power function.

c This is not a power function, but it can be treated as the sum of two power functions, because $h(x) = x^{-2} + 6x^{-3}$.

Checkpoint 6.9 QuickCheck 2. True or false.

a. The exponent in a power function must be positive. (☐ True ☐ False)

b. Direct and inverse variation are examples of power functions. (☐ True ☐ False)

c. If f is a power function, then $f(0) = 0$. (☐ True ☐ False)

d. The sum of two power functions is not always a power function. (☐ True ☐ False)

Answer 1. False
Answer 2. True
Answer 3. False
Answer 4. True

Solution.

a. False

b. True

c. False

d. True

Checkpoint 6.10 Practice 3. Write each function as a power function in the form $y = kx^p$.

For this exercise, enter rational numbers in decimal form. For example, enter “0.5” rather than “1/2”.

a. $f(x) = \dfrac{12}{x^2} =$__________

b. $g(x) = \dfrac{1}{4x} =$__________

c. $h(x) = \dfrac{2}{5x^6} =$__________

Answer 1. $12x^{-2}$

Answer 2. $\frac{1}{4}x^{-1}$

Answer 3. $\frac{2}{5}x^{-6}$

Solution.

a. $f(x) = 12x^{-2}$

b. $g(x) = \dfrac{1}{4}x^{-1} = 0.25x^{-1}$

c. $h(x) = \dfrac{2}{5}x^{-6} = 0.4x^{-6}$

Most applications are concerned with positive variables only, so many models use only the portion of the graph in the first quadrant.

Example 6.11

The intensity of the radiation from our Sun varies inversely with distance.

a Use a negative exponent to write a formula for the intensity I as a function of d, $I = f(d)$.

b The planet Mercury is 0.387 AU (Astronomical Units) distant from the Sun, and the intensity of radiation at its surface is 9147 watts per square meter. Find the constant of proportionality in the formula for I.

c Graph $I = f(d)$.

d Earth is 1 AU from the Sun. What is the intensity of the Sun's radiation at the surface of the Earth?

e The surface of Jupiter receives 50.63 watts per square meter of the Sun's radiation. How far is Jupiter from the Sun?

Solution.

a If we use k for the constant of proportionality, then $I = \dfrac{k}{d^2}$. Rewriting this equation with a negative exponent gives $I = kd^{-2}$.

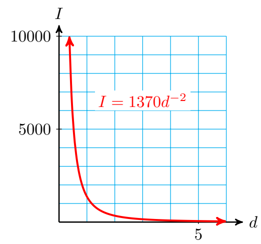
b We substitute $T = 9147$ and $d = 0.387$ to obtain

$$9147 = k(0.387)^{-2}$$

$$9147 = \frac{k}{(0.387)^2} \qquad \text{Multiply both sides by } 0.387^2.$$
$$1369.9 = k$$

Thus, $I = f(d) = 1370d^{-2}$.

c We evaluate the function for several values of d, and use a calculator to obtain the graph shown.



d We substitute $d = 1$ into the formula to obtain

$$I = 1370(1)^{-2} = 1370$$

Earth receives 1370 watts/m^2 of solar radiation.

e We substitute $I = \mathbf{50.63}$ into the formula, and solve for d.

$$\mathbf{50.63} = 1370d^{-2} = \frac{1370}{d^2} \qquad \text{Multiply both sides by } d^2.$$
$$50.63d^2 = 1370 \qquad \text{Divide both sides by 50.63.}$$
$$d^2 = \frac{1370}{50.63} = 27.059 \qquad \text{Extract roots.}$$
$$d = 5.202$$

Jupiter is 5.202 AU from the Sun, or about 484 million miles.

The function $I = \dfrac{k}{d^2}$ is an example of an **inverse square law**, because I varies inversely with the square of d. Such laws are fairly common in physics and its applications, because gravitational and other forces behave in this way. Here is another example of an inverse square law.

Checkpoint 6.12 Practice 4. Cell phone towers typically transmit signals at 10 watts of power. The signal strength varies inversely with the square of distance from the tower, and 1 kilometer away the signal strength is 0.8 picowatt. (A picowatt is 10^{-12} watt.) Cell phones can receive a signal as small as 0.01 picowatt. How far can you be from the nearest tower and still hope to have cell phone reception?

About __ km

Answer. $\sqrt{\frac{0.8}{0.01}}$

Solution. About 9 km

Working with Negative Exponents

The laws of exponents apply to all integer exponents, positive negative, and zero. When we allow negative exponents, we only need one version of the rule for computing quotients of powers, namely

$$\frac{a^m}{a^n} = a^{m-n} \qquad (a \neq 0)$$

For example, by applying this new version of the law we find

$$\frac{x^2}{x^5} = x^{2-5} = x^{-3}$$

which is consistent with our previous version of the rule, $\dfrac{x^2}{x^5} = \dfrac{1}{x^{5-2}} = \dfrac{1}{x^3}$

For reference, we restate the laws of exponents below. The laws are valid for all integer exponents m and n, and for $a,\ b \neq 0$.

Laws of Exponents.

I $a^m \cdot a^n = a^{m+n}$

II $\dfrac{a^m}{a^n} = a^{m-n}$

III $(a^m)^n = a^{mn}$

IV $(ab)^n = a^n b^n$

V $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Example 6.13

a. $x^3 \cdot x^{-5} = x^{3-5} = x^{-2}$ Apply the first law: Add exponents.

b. $\dfrac{8x^{-2}}{4x^{-6}} = \dfrac{8}{4}x^{-2-(-6)} = 2x^4$ Apply the second law: Subtract exponents.

c. $\left(5x^{-3}\right)^{-2} = 5^{-2}(x^{-3})^{-2} = \dfrac{x^6}{25}$ Apply laws IV and III.

You can check that each of the calculations in Example 6.13, p. 374 is shorter when we use negative exponents instead of converting the expressions into algebraic fractions.

Checkpoint 6.14 Practice 5. Simplify by applying the laws of exponents. Write without negative exponents.

a. $\left(2a^{-4}\right)\left(-4a^2\right) =$________

b. $\dfrac{(r^2)^{-3}}{3r^{-4}} =$________

Answer 1. $\frac{-8}{a^2}$

Answer 2. $\frac{1}{3r^2}$

Solution.

a. $\dfrac{-8}{a^2}$

b. $\dfrac{1}{3r^2}$

Caution 6.15 The laws of exponents do not apply to sums or differences of powers. We can add or subtract like terms, that is, powers with the same exponent. For example,

$$6x^{-2} + 3x^{-2} = 9x^{-2}$$

but *we cannot combine terms with different exponents* into a single term. Thus, for example,

$4x^2 - 3x^{-2}$	cannot be simplified
$x^{-1} + x^{-3}$	cannot be simplified

In the opening table we saw that $2^0 = 1$, and in fact $a^0 = 1$ as long as $a \neq 0$. Now we can see that this definition is consistent with the laws of exponents. The quotient of any (nonzero) number divided by itself is 1. But by applying the second law of exponents, we also have

$$1 = \frac{a^m}{a^m} = a^{m-m} = a^0$$

Zero as Exponent.

$$a^0 = 1, \quad \text{if } a \neq 0$$

For example,

$$3^0 = 1, \quad (-528)^0 = 1, \quad \text{and} \quad (0.024)^0 = 1$$

Checkpoint 6.16 QuickCheck 3. True or false.

a. We cannot add or subtract terms with different exponents. (☐ True ☐ False)

b. $(a + b)^5 = a^5 + b^5$ is an application of the fourth law of exponents. (☐ True ☐ False)

c. If $x \neq 0$, then $\dfrac{k}{x^{-n}} = kx^n$. (☐ True ☐ False)

d. $(x^2)^{-3} = (x^{-2})^3$ because $\dfrac{1}{x^2 \cdot x^2 \cdot x^2} = \left(\dfrac{1}{x^2}\right)\left(\dfrac{1}{x^2}\right)\left(\dfrac{1}{x^2}\right)$. (☐ True ☐ False)

Answer 1. True

Answer 2. False

Answer 3. True

Answer 4. True

Solution.

a. True

b. False

c. True

d. True

Review of Scientific Notation

Scientists and engineers regularly encounter very large numbers such as,

$$5,980,000,000,000,000,000,000,000,$$

(the mass of the earth in kilograms) and very small numbers such as

$$0.000\ 000\ 000\ 000\ 000\ 000\ 000\ 001\ 67$$

(the mass of a hydrogen atom in grams) in their work. These numbers are easier to use when expressed in **scientific notation**.

To Write a Number in Scientific Notation.

1 Locate the decimal point so that there is exactly one nonzero digit to its left.

2 Count the number of places you moved the decimal point: this determines the power of 10.

  a If the original number is greater than 10, the exponent is positive.

  b If the original number is less than 1, the exponent is negative.

Example 6.17

Write each number in scientific notation.

a $478,000$ b 0.00032

Solution.

a

$$\begin{aligned} 478,000 &= 4.78000 \times 10^5 \quad \text{Move the decimal 5 places to the left.} \\ &= 4.78 \times 10^5 \end{aligned}$$

b

$$0.00032 = 00003.2 \times 10^{-4} \quad \text{Move the decimal 4 places to the right.}$$

Checkpoint 6.18 Practice 6. Write each number in scientific notation.

a. $0.063 =$_______

b. $1480 =$_______

Answer 1. 6.3×10^{-2}

Answer 2. 1.48×10^3

Solution.

a. 6.3×10^{-2}

b. 1.480×10^3

Your calculator displays numbers in scientific notation if they are too large or too small to fit in the display screen. Try squaring the number 123, 456, 789 on your calculator. Enter

123456789 [x^2]

and the calculator will display the result as

1.524157875 E 16

This is how the calculator displays the number $1.524157875 \times 10^{16}$. Notice that the power 10^{16} is displayed as E 16.

To enter a number in scientific form, we use the key labeled EE, possibly accessed by pressing 2nd ,. For example, to enter 3.26×10^{-18} we enter the keying sequence

3.26 EE (-) 18

Example 6.19

In 2019, the average American ate 98.6 kilograms of meat. It takes about 16 kilograms of grain to produce one kilogram of meat, and advanced farming techniques can produce about 6000 kilograms of grain on each hectare of arable land. (A hectare is 10,000 square meters, or just under two and a half acres.)

Now, the total land area of the Earth is about 13 billion hectares, but only about 11% of that land is arable. Is it possible for each of the 7.8 billion people on Earth to eat as much meat as Americans do?

Solution. First we'll compute the amount of meat necessary to feed every person on earth 110 kilograms per year. In scientific notation, the population of Earth is 7.8×10^9 people.

$$(7.8 \times 10^9 \text{ people}) \times (98.6 \text{ kg/person}) = 7.69 \times 10^{11} \text{ kg meat}$$

Next we'll compute the amount of grain needed to produce that much meat.

$$(16 \text{ kg grain/kg meat}) \times (7.69 \times 10^{11} \text{ kg meat}) = 1.23 \times 10^{13} \text{ kg grain}$$

Next we'll see how many hectares of land are needed to produce that much grain.

$$(1.23 \times 10^{13} \text{ kg grain}) \div (6000 \text{ kg grain/hectare}) = 2.05 \times 10^9 \text{ hectares}$$

Finally, we'll compute the amount of land on Earth suitable for grain production.

$$0.11 \times (13 \times 10^9 \text{ hectares}) = 1.43 \times 10^9 \text{ hectares}$$

The amount of arable land on Earth is less than the amount needed to produce that much grain. Thus, even if we use every hectare of arable land to produce grain for livestock, we won't have enough to provide every person on Earth with 98.6 kilograms of meat per year.

Problem Set 6.1

Warm Up

For Problems 1 and 2, simplify if possible.

1.

a $4z^2 - 6z^2$

b $4z^2(-6z^2)$

2.

a $4p^2 + 3p^3$

b $4p^2(3p^3)$

For Problems 3-6, use the laws of exponents to simplilfy.

3.

a $-4x(3xy)(xy^3)$

b $\dfrac{2a^3b}{8a^4b^5}$

4.

a $-5x^2(2xy)(5x^2)$

b $\dfrac{8a^2b}{12a^5b^3}$

5.

a $(2x^3y)^2(xy^3)4$

b $\dfrac{(xy)^2(-x^2y)^3}{(x^2y^2)^2}$

6.

a $(3xy^2)^3(2x^2y^2)^2$

b $\dfrac{(-x)^2(x^2)^4}{(x^2)^3}$

Skills Practice

For Problems 7–11, simplify. Write your answers as integers or common fractions; do not use a calculator!

7.

a 3^2

b 3^{-2}

c $(-3)^2$

d $(-3)^{-2}$

e -3^2

f -3^{-2}

8.

a $\dfrac{1}{2^3}$

b $\dfrac{1}{2^{-3}}$

c $\dfrac{-1}{2^3}$

d $\dfrac{-1}{2^{-3}}$

e $\left(\dfrac{1}{2}\right)^3$

f $\left(\dfrac{1}{2}\right)^{-3}$

9.

a $5 \cdot 2^3$

b $5 \cdot 2^{-3}$

c $\dfrac{5}{2^3}$

d $\dfrac{1}{5 \cdot 2^3}$

e $\dfrac{5}{2^{-3}}$

f $\dfrac{5^{-2}}{2^{-3}}$

10.

a $\dfrac{2^3}{5^{-2}}$

b $\dfrac{2^{-3}}{5^2}$

c $\dfrac{5^{-2}}{2^{-3}}$

d $(\frac{5}{2})^{-3}$

e $5^2 \cdot 2^{-3}$

f $5^{-2} \cdot 2^{-3}$

11.

a $2^{-1} + 4^{-1}$

b $2^{-1} \cdot 4^{-1}$

c $2 \cdot 4^{-1}$

d $2 + 4^{-1}$

e $(2 + 4)^{-1}$

f $(2 \cdot 4)^{-1}$

g $\left(2^{-1} + 4^{-1}\right)^{-1}$

h $\left(2^{-1} \cdot 4^{-1}\right)^{-1}$

For Problems 12 and 13, write without negative exponents and simplify.

12.

a $\dfrac{5}{4^{-3}}$

b $(2q)^{-5}$

c $-4x^{-2}$

d $\dfrac{8}{b^{-3}}$

13.
a $(m-n)^{-2}$
b $y^{-2}+y^{-3}$
c $2pq^{-4}$
d $\dfrac{-5y^{-2}}{x^{-5}}$

14. Write each expression as a power function using negative exponents.
a $F(r)=\dfrac{3}{r^4}$ b $G(w)=\dfrac{2}{5w^3}$ c $H(z)=\dfrac{1}{(3z)^2}$

For Problems 15–17, solve.

15. $6x^{-2}=3.84$

16. $12+0.04t^{-3}=175.84$

17. $100-0.15v^{-4}=6.25$

For Problems 18–20, use the laws of exponents to simplify and write without negative exponents.

18.
a $a^{-3}\cdot a^{8}$
b $5^{-4}\cdot 5^{-3}$
c $\dfrac{p^{-7}}{p^{-4}}$
d $\left(7^{-2}\right)^5$

19.
a $(4x^{-5})(5x^2)$ b $\dfrac{3u^{-3}}{9u^9}$ c $\dfrac{5^6t^0}{5^{-2}t^{-1}}$

20.
a $\left(3x^{-2}y^3\right)^{-2}$ b $\left(\dfrac{6a^{-3}}{b^2}\right)^{-2}$ c $\dfrac{5h^{-3}\left(h^4\right)^{-2}}{6h^{-5}}$

For Problems 21 and 22, write each expression as a sum of terms of the form kx^p.

21.
a $\dfrac{x}{3}+\dfrac{3}{x}$ b $\dfrac{x-6x^2}{4x^3}$

22.
a $\dfrac{2}{x^4}\left(\dfrac{x^2}{2}+\dfrac{x}{2}-\dfrac{1}{4}\right)$ b $\dfrac{x^2}{3}\left(\dfrac{2}{x^4}-\dfrac{1}{3x^2}+\dfrac{1}{2}\right)$

For Problems 23–26, use the distributive law to write each product as a sum of power functions.

23. $x^{-1}(x^2-3x+2)$

24. $-3t^{-2}(t^2-2-4t^{-2})$

25. $2u^{-3}(-2u^3-u^2+3u)$

26. $\dfrac{-1}{2}z^{-3}(-2z^2+3z-4)$

For Problems 27 and 28, factor as indicated, writing the second factor with positive exponents only.

27. $4x^2+16x^{-2}=4x^{-2}(\ \ ?\ \ )$

28. $3a^{-3}-3a+a^3=a^{-3}(\ \ ?\ \ )$

29. Write each number in scientific notation.
a 285
b $8,372,000$
c 0.024
d 0.000523

30. Write each number in standard notation.
a 4.8×10^3
b 8.31×10^{12}
c 8.0×10^{-1}
d 4.31×10^{-5}

Applications

31. Let $g(x) = x^{-3}$. Complete the tables.

a

x	1	2	4.5	6.2	9.3
$g(x)$					

b What happens to the values of $g(x)$ as the values of x increase? Explain why.

c

x	1.5	0.6	0.1	0.03	0.002
$g(x)$					

d What happens to the values of $g(x)$ as the values of x decrease toward 0? Explain why.

32.

(a) Use your calculator to graph each of the following functions on the window

$$\text{Xmin} = -5 \qquad \text{Xmax} = 5$$
$$\text{Ymin} = -2 \qquad \text{Ymax} = 10$$

i. $f(x) = x^2$

ii. $f(x) = x^{-2}$

iii. $f(x) = \dfrac{1}{x^2}$

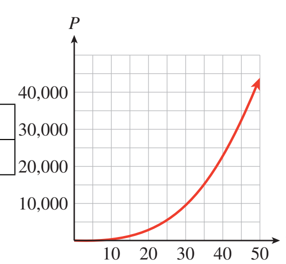
iv. $f(x) = \left(\dfrac{1}{x}\right)^2$

(b) Which functions have the same graph? Explain your results.

33. When an automobile accelerates, the power, P, needed to overcome air resistance varies directly with a power of the speed, v.

(a) Use the data and the graph to find the scaling exponent and the constant of variation. Then write a formula for P as a power function of v.

v (mph)	10	20	30	40
P (watts)	355	2840	9585	22,720



(b) Find the speed that requires 50,000 watts of power.

(c) If you increase your speed by 50%, by what factor does the power requirement increase?

34. Poiseuille's law for the flow of liquid through a tube can be used to describe blood flow through an artery. The rate of flow, F, in liters per minute is proportional to the fourth power of the radius, r, divided by the length, L, of the artery.

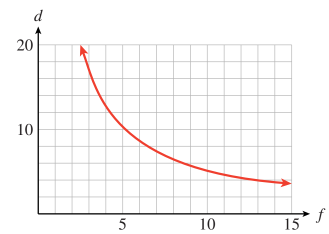
a Write a formula for the rate of flow as a power function of radius.

b If the radius and length of the artery are measured in centimeters, then the constant of variation, $k = 7.8 \times 10^5$, is determined by blood pressure and viscosity. If a certain artery is 20 centimeters long, what should its radius be in order to allow a blood flow of 5 liters per minute?

35. The f-stop setting on a camera regulates the size of the aperture and thus the amount of light entering the camera. The f-stop f is inversely proportional to the diameter, d, of the aperture.

(a) Use the data and the graph to find the constant of proportionality and write d as a power function of f. Values of d have been rounded to one decimal place.

f	2.8	4	5.6	8	11
d	17.9	12.5	8.9	6.3	4.5



(b) Why are the f-stop settings labeled with the values given in the table?

Hint. As you stop down the aperture from one f-value to the next, by what factor does d increase?

36. The lifetime of a star is roughly inversely proportional to the cube of its mass. Our Sun, which has a mass of one solar mass, will last for approximately 10 billion years.

(a) Write a power function for the lifetime, L, of a star in terms of its mass, m.

(b) Sketch a graph of the function using units of solar mass on the horizontal axis.

(c) How long will a star that is 10 times as massive as the Sun last?

(d) One solar mass is about 2×10^{30} kilograms. Rewrite your formula for L with the units of mass in kilograms.

(e) How long will a star that is half as massive as the Sun last?

36. In 2020, the public debt of the United States was 19.05 trillion dollars.

a Express this number in scientific notation.

b The population of the United States in 2020 was 331 million. What was the per capita debt (the debt per person) for that year?

37. The diameter of the galactic disk is about 1.2×10^{18} kilometers, and our Sun lies about halfway from the center of the galaxy to the edge of the disk. The Sun orbits the galactic center once every 240 million years.

a What is the speed of the Sun in its orbit, in kilometers per year?

b What is its speed in meters per second?

Roots and Radicals

In the previous section we saw that inverse variation can be expressed as a power function by using negative exponents. We can also use exponents to denote square roots and other radicals.

n^{th} Roots

Recall that s is a square root of b if $s^2 = b$, and s is a cube root of b if $s^3 = b$. In a similar way, we can define the fourth, fifth, or sixth root of a number. For instance, the fourth root of b is a number s whose fourth power is b.

nth Roots.

s is called an **nth root of** b if $s^n = b$.

We use the symbol $\sqrt[n]{b}$ to denote the nth root of b. An expression of the form $\sqrt[n]{b}$ is called a **radical**, b is called the **radicand**, and n is called the **index of the radical**.

Example 6.20

a $\sqrt[4]{81} = 3$ because $3^4 = 81$

b $\sqrt[5]{32} = 2$ because $2^5 = 32$

c $\sqrt[6]{64} = 2$ because $2^6 = 64$

d $\sqrt[4]{1} = 1$ because $1^4 = 1$

e $\sqrt[5]{100,000} = 10$ because $10^5 = 100,000$

Checkpoint 6.21 **Practice 1.** Evaluate each radical.

a. $\sqrt[4]{16} =$__

b. $\sqrt[5]{243} =$__

Answer 1. 2

Answer 2. 3

Solution.

a. 2

b. 3

Exponential Notation for Radicals

A convenient notation for radicals uses fractional exponents. Consider the expression $9^{1/2}$. What meaning can we attach to an exponent that is a fraction? The third law of exponents says that when we raise a power to a power, we multiply the exponents together:

$$(x^a)^b = x^{ab}$$

Therefore, if we square the number $9^{1/2}$, we get

$$\left(9^{1/2}\right)^2 = 9^{(1/2)(2)} = 9^1 = 9$$

Thus, $9^{1/2}$ is a number whose square is 9. But this means that $9^{1/2}$ is a square root of 9, or

$$9^{1/2} = \sqrt{9} = 3$$

Checkpoint 6.22 QuickCheck 1. Fill in the blanks.

a. Inverse variation can be expressed as a power function by using a (☐ power ☐ negative ☐ reciprocal ☐ inverse) exponent.

b. The number n in the expression $\sqrt[n]{b}$ is called the (☐ root ☐ exponent ☐ radicand ☐ index) of the radical.

c. We use (☐ fractional ☐ negative ☐ reciprocal ☐ inverse) exponents to denote radicals.

d. State the law of exponents that shows that $a^{1/2}$ is a square root of a.

- ⊙ $a^m \cdot a^n = a^{m+n}$
- ⊙ $\dfrac{a^m}{a^n} = a^{m-n}$
- ⊙ $(a^m)^n = a^{mn}$
- ⊙ $(ab)^n = a^n b^n$
- ⊙ $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Answer 1. negative

Answer 2. index

Answer 3. fractional

Answer 4. Choice 3

Solution.

a. negative

b. index

c. fractional

d. $(a^m)^n = a^{mn}$

In general, any nonnegative number raised to the 1/2 power is equal to the positive square root of the number, or

$$a^{1/2} = \sqrt{a}$$

Example 6.23

a $25^{1/2} = 5$

b $-25^{1/2} = -5$

c $(-25)^{1/2}$ is not a real number.

d $0^{1/2} = 0$

Checkpoint 6.24 Practice 2. Evaluate each power.

a. $4^{1/2} =$___

b. $4^{-2} =$___

c. $4^{-1/2} =$___

d. $\left(\dfrac{1}{4}\right)^{1/2} =$___

Answer 1. 2

Answer 2. $\frac{1}{16}$
Answer 3. $\frac{1}{2}$
Answer 4. $\frac{1}{2}$
Solution.

a. 2

b. $\dfrac{1}{16}$

c. $\dfrac{1}{2}$

d. $\dfrac{1}{2}$

The same reasoning works for roots with any index. For instance, $8^{1/3}$ is the cube root of 8, because

$$\left(8^{1/3}\right)^3 = 8^{(1/3)(3)} = 8^1 = 8$$

Thus, we make the following definition for fractional exponents.

Exponential Notation for Radicals.

For any integer $n \geq 2$ and for $a \geq 0$,

$$a^{1/n} = \sqrt[n]{a}$$

Example 6.25

a $81^{1/4} = \sqrt[4]{81} = 3$

b $125^{1/3} = \sqrt[3]{125} = 5$

Caution 6.26 Note that

$$25^{1/2} \neq \frac{1}{2}(25) \quad \text{and} \quad 125^{1/3} \neq \frac{1}{3}(125)$$

An exponent of $\frac{1}{2}$ denotes the square root of its base, and an exponent of $\frac{1}{3}$ denotes the cube root of its base.

Checkpoint 6.27 Practice 3. Write each power with radical notation, and then evaluate.

a. $32^{1/5}$ is

- ⊙ $\sqrt{32/5}$
- ⊙ $\sqrt[5]{32}$
- ⊙ $\sqrt{32^5}$

$32^{1/5} =$__

b. $625^{1/4}$ is

- ⊙ $\sqrt{625/4}$
- ⊙ $\sqrt{625^4}$
- ⊙ $\sqrt[4]{625}$

$625^{1/4} =$__

Answer 1. Choice 2
Answer 2. 2
Answer 3. Choice 3
Answer 4. 5
Solution.

a. $\sqrt[5]{32} = 2$

b. $\sqrt[4]{625} = 5$

Of course, we can use decimal fractions for exponents as well. For example,

$$\sqrt{a} = a^{1/2} = a^{0.5} \qquad \text{and} \qquad \sqrt[4]{a} = a^{1/4} = a^{0.25}$$

Example 6.28

a $100^{0.5} = \sqrt{100} = 10$ b $16^{0.25} = \sqrt[4]{16} = 2$

Checkpoint 6.29 Practice 4. Write each power with radical notation, and then evaluate.

a. $100,000^{0.2}$ is

- ⊙ $\sqrt[2]{100,000}$
- ⊙ $\sqrt[5]{100,000}$
- ⊙ $\sqrt{100,000 \cdot 5}$

$100,000^{0.2} =$__

b. $81^{0.25}$ is

- ⊙ $\sqrt{81/4}$
- ⊙ $\sqrt[4]{81}$
- ⊙ $\sqrt[25]{81}$

$81^{0.25} =$__

Answer 1. Choice 2
Answer 2. 10
Answer 3. Choice 2
Answer 4. 3
Solution.

a. $\sqrt[5]{100,000} = 10$

b. $\sqrt[4]{81} = 3$

Checkpoint 6.30 QuickCheck 2. True or false.

a. The exponent $\dfrac{1}{n}$ tells us to take a reciprocal. (☐ True ☐ False)

b. The exponent 0.2 denotes a fifth root. (☐ True ☐ False)

c. A power with a negative exponent is not a real number. (☐ True ☐ False)

d. The expressions x^{-3} and $x^{1/3}$ are the same. (☐ True ☐ False)

Answer 1. False
Answer 2. True
Answer 3. False
Answer 4. False
Solution.

a. False

b. True

c. False

d. False

Irrational Numbers

What about nth roots such as $\sqrt{23}$ and $5^{1/3}$ that cannot be evaluated easily? These are examples of **irrational numbers**. An irrational number is one that cannot be expressed as a quotient of two integers.

It is not possible to write down an exact decimal equivalent for an irrational number, but we can find an approximation to as many decimal places as we like. We can use a calculator to obtain decimal approximations for irrational numbers. For example, you can verify that

$$\sqrt{23} \approx 4.796 \qquad \text{and} \qquad 5^{1/3} \approx 1.710$$

Caution 6.31 The following keying sequence for evaluating the irrational number $7^{1/5}$ is incorrect:

7 [^] 1 [÷] 5 [ENTER]

You can check that this sequence calculates $\frac{7^1}{5}$, instead of $7^{1/5}$. Recall that according to the order of operations, powers are computed before multiplications or divisions. We must enclose the exponent 1/5 in parentheses and enter

7 [^] (1 [÷] 5) [ENTER]

Or, because $\frac{1}{5} = 0.2$, we can enter

7 [^] 0.2 [ENTER]

Working with Fractional Exponents

Fractional exponents simplify many calculations involving radicals. You should learn to convert easily between exponential and radical notation.

Example 6.32

Convert each radical to exponential notation.

a $\sqrt[3]{12} = 12^{1/3}$ b $\sqrt[4]{2y} = (2y)^{1/4}$ or $(2y)^{0.25}$

Caution 6.33 In part (b) of the Example above, the parentheses around $(2y)$ must not be omitted. The expression $2y^{1/4}$ means $2\sqrt[4]{y}$.

Remember that a negative exponent denotes a reciprocal.

Checkpoint 6.34 **Practice 5.** Convert each radical to exponential notation.

a. $\dfrac{1}{\sqrt[5]{ab}} =$__________

b. $3\sqrt[6]{w} =$__________

Answer 1. $(ab)^{-\frac{1}{5}}$

Answer 2. $3w^{\frac{1}{6}}$

Solution.

a. $(ab)^{-1/5}$

b. $3w^{1/6}$

Example 6.35

Convert each power to radical notation.

a $5^{1/2} = \sqrt{5}$

b $x^{0.2} = \sqrt[5]{x}$

c $2x^{1/3} = 2\sqrt[3]{x}$

d $8a^{-1/4} = \dfrac{8}{\sqrt[4]{a}}$

Note 6.36 In Example 6.35, p. 387d, note that the exponent $-1/4$ applies only to a, not to $8a$. Compare with $(8a)^{-1/4} = \dfrac{1}{\sqrt[4]{8a}}$

Checkpoint 6.37 Practice 6.

a. Convert $\dfrac{3}{\sqrt[4]{2x}}$ to exponential notation.

- ⊙ $\dfrac{3}{2x^{1/4}}$
- ⊙ $3(2x)^{-1/4}$
- ⊙ $\dfrac{3}{2}x^{-1/4}$

b. Convert $-5b^{0.125}$ to radical notation.

- ⊙ $\sqrt[8]{-5b}$
- ⊙ $-\sqrt[5]{b^8}$
- ⊙ $-5\sqrt[8]{b}$

Answer 1. Choice 2

Answer 2. Choice 3

Solution.

a. $3(2x)^{-1/4}$

b. $-5\sqrt[8]{b}$

Checkpoint 6.38 QuickCheck 3. True or false.

a. All expressions of the form $x^{1/n}$ represent irrational numbers. (☐ True ☐ False)

b. When we enter an exponent $1/n$ into a calculator, we must enclose it in parentheses. (☐ True ☐ False)

c. In the expression $16x^{1/4}$, the exponent does not apply to 16. (☐ True

☐ False)

d. You can use your calculator to find a decimal equivalent for an irrational number. (☐ True ☐ False)

Answer 1. False
Answer 2. True
Answer 3. True
Answer 4. False
Solution.

a. False

b. True

c. True

d. False

Using Fractional Exponents to Solve Equations

We know that raising to powers and taking roots are inverse operations, that is, each operation undoes the effects of the other. This relationship is especially easy to see when the root is denoted by a fractional exponent. For example, to solve the equation

$$x^5 = 250$$

we would take the fifth root of each side. But instead of using radical notation, we can raise both sides of the equation to the power $\dfrac{1}{5}$:

$$\left(x^5\right)^{1/5} = 250^{1/5}$$
$$x \approx 3.017$$

The third law of exponents tells us that $(x^a)^b = x^{ab}$, so

$$\left(x^5\right)^{1/5} = x^{(1/5)(5)} = x^1$$

Thus, to solve an equation involving a power function x^n, we first isolate the power, then raise both sides to the exponent $\dfrac{1}{n}$.

Example 6.39

For astronomers, the mass of a star is its most important property, but it is also the most difficult to measure directly. For many stars, their luminosity, or brightness, varies roughly as the fourth power of the mass.

a Our Sun has luminosity 4×10^{26} watts and mass 2×10^{30} kilograms. Because the numbers involved are so large, astronomers often use these solar constants as units of measure: The luminosity of the Sun is 1 solar luminosity, and its mass is 1 solar mass. Write a power function for the luminosity, L, of a star in terms of its mass, M, using units of solar mass and solar luminosity.

b The star Sirius is 23 times brighter than the Sun, so its luminosity is 23 solar luminosities. Estimate the mass of Sirius in units of solar mass.

Solution.

a Because L varies as the fourth power of M, we have

$$L = kM^4$$

Substituting the values of L and M for the Sun (namely, $L = 1$ and $M = 1$), we find

$$1 = k(1)^4$$

so $k = 1$ and $L = M^4$.

b We substitute the luminosity of Sirius, $L = 23$, to get

$$23 = M^4$$

To solve the equation for M, we raise both sides to the $\dfrac{1}{4}$ power.

$$\begin{aligned}(23)^{1/4} &= \left(M^4\right)^{1/4}\\ 2.1899 &= M\end{aligned}$$

The mass of Sirius is about 2.2 solar masses, or about 2.2 times the mass of the Sun.

Checkpoint 6.40 QuickCheck 4.

a. How do we solve the equation $x^8 = 32$?

- ⊙ Divide both sides of the equation by 8.
- ⊙ Add -8 to each side of the equation.
- ⊙ Multiply both sides by $\frac{1}{8}$.
- ⊙ Raise each side of the equation to the power $\frac{1}{8}$.

b. Which law of exponents justifies this strategy?

- ⊙ $a^m \cdot a^n = a^{m+n}$
- ⊙ $\dfrac{a^m}{a^n} = a^{m-n}$
- ⊙ $(a^m)^n = a^{mn}$
- ⊙ $(ab)^n = a^n b^n$
- ⊙ $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Answer 1. Choice 4

Answer 2. Choice 3

Solution.

a. Raise each side to the power 1/8.

b. $(x^a)^b = x^{ab}$

Power Functions

The size of a diamond is usually given by its weight w in carats. The diameter, D, of a diamond cut in a traditional round style is then a power function of its weight, given by

$$D = f(w) = kw^{1/3}$$

Example 6.41

a A diamond weighing one-quarter carat has diameter about 4.05 millimeters. Find the constant of proportionality, k, and write a formula for D as a function of w.

b Complete the table with the diameters of diamonds of various weights.

w (carats)	0	0.5	0.75	1.0	1.25	1.5	2.0	2.5	3.0
D (mm)									

c Sketch a graph of the function $D = f(w)$.

d What is the diameter of the Hope Diamond, which weighs 45.52 carats and is worth $300 million?

Solution.

a We substitute $w = \mathbf{0.25}$ and $D = \mathbf{4.05}$ into the equation and solve for k.

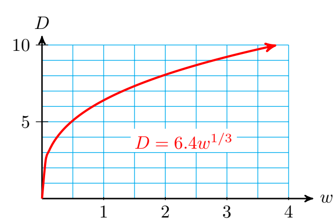
$$\mathbf{4.05} = k(\mathbf{0.25})^{1/3}$$
$$k = \frac{4.05}{0.25^{1/3}} = 6.429$$

Rounded to tenths, $k = 6.4$, and $D = 6.4w^{1/3}$.

b We evaluate the function for each of the weights given in the table. The diameters are rounded to tenths of a millimeter.

w (carats)	0	0.5	0.75	1.0	1.25	1.5	2.0	2.5	3.0
D (mm)	0	5.1	5.8	6.4	6.9	7.3	8.1	8.7	9.2

c We plot the points from the table and draw a smooth curve to obtain the graph shown.



d We evaluate the function for $w = 45.52$ to obtain

$$D = 6.4(45.52)^{1/3} = 22.85$$

According to our formula, the Hope Diamond is about 22.85 millimeters in diameter.

Of course, power functions can be expressed using any of the notations we have discussed. For example, the function in the Example above can be written as

$$f(w) = 6.4w^{1/3} \quad \text{or} \quad f(w) = 6.4w^{0.3\overline{3}} \quad \text{or} \quad f(w) = 6.4\sqrt[3]{w}$$

Checkpoint 6.42 Practice 8.

a. Complete the table of values for the power function $f(x) = x^{-1/2}$.

x	0.1	0.25	0.5	1	2
$f(x)$	—	—	—	—	—

x	4	8	10	20	200
$f(x)$	—	—	—	—	—

b. Sketch the graph of $y = f(x)$.

c. Write the formula for $f(x)$ with a decimal exponent, and with radical notation.

$f(x)$ is

- ⊙ x^{-2}
- ⊙ $x^{0.5}$
- ⊙ $x^{-0.5}$

and also, $f(x)$ is

- ⊙ $\dfrac{1}{\sqrt{x}}$
- ⊙ $-\sqrt{x/2}$
- ⊙ $\dfrac{-\sqrt{x}}{2}$

Answer 1. 3.16228

Answer 2. 2

Answer 3. 1.41421

Answer 4. 1

Answer 5. 0.707107

Answer 6. 0.5

Answer 7. 0.353553

Answer 8. 0.316228

Answer 9. 0.223607

Answer 10. 0.0707107

Answer 11. Choice 3

Answer 12. Choice 1

Solution.

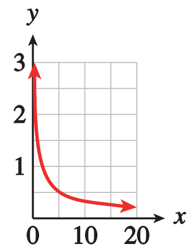
a.

x	0.1	0.25	0.5	1	2
$f(x)$	3.2	2	1.4	1	0.71

x	4	8	10	20	200
$f(x)$	0.5	0.35	0.32	0.22	0.1

b. A graph is below.

c. $f(x) = x^{-0.5}$, $f(x) = \dfrac{1}{\sqrt{x}}$



Solving Radical Equations

A **radical equation** is one in which the variable appears under a square root or other radical. The radical may be denoted by a fractional exponent. For example, the equation

$$5x^{1/3} = 32$$

is a radical equation because $x^{1/3} = \sqrt[3]{x}$. To solve the equation, we first isolate the power to get

$$x^{1/3} = 6.4$$

Then we raise both sides of the equation to the reciprocal of $\dfrac{1}{3}$, or 3.

$$\left(x^{1/3}\right)^3 = 6.4^3$$
$$x = 262.144$$

Example 6.43

When a car brakes suddenly, its speed can be estimated from the length of the skid marks it leaves on the pavement. A formula for the car's speed, in miles per hour, is

$$v = f(d) = (24d)^{1/2}$$

where the length of the skid marks, d, is given in feet.

a If a car leaves skid marks 80 feet long, how fast was the car traveling when the driver applied the brakes?

b How far will a car skid if its driver applies the brakes while traveling 80 miles per hour?

Solution.

a To find the velocity of the car, we evaluate the function for $d = 80$.

$$v = (24 \cdot 80)^{1/2}$$
$$= (1920)^{1/2} \approx 43.8178046$$

The car was traveling at approximately 44 miles per hour.

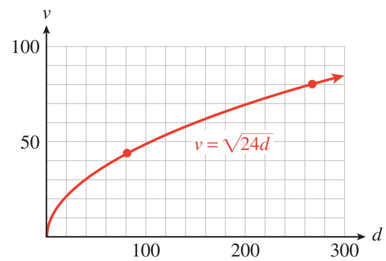
b We would like to find the value of d when the value of v is known. We substitute $v = 80$ into the formula and solve the equation

$$80 = (24d)^{1/2} \quad \text{Solve for } d.$$

Because d appears to the power $\frac{1}{2}$, we first square both sides of the equation to get

$$\begin{aligned} 80^2 &= \left((24d)^{1/2}\right)^2 && \text{Square both sides.} \\ 6400 &= 24d && \text{Divide by 24.} \\ 266.\overline{6} &= d \end{aligned}$$

You can check that this value for d works in the original equation. Thus, the car will skid approximately 267 feet. A graph of the function $v = (24d)^{1/2}$ is shown below, along with the points corresponding to the values in parts (a) and (b).



Note 6.44 Thus, we can solve an equation where one side is an nth root of x by raising both sides of the equation to the nth power. We must be careful when raising both sides of an equation to an even power, since extraneous solutions may be introduced. However, because most applications of power functions deal with positive numbers only, they do not usually involve extraneous solutions.

Checkpoint 6.45 QuickCheck 5.

a. How do we solve the equation $x^{1/6} = 12$?

- ⊙ Raise each side to the power 6.
- ⊙ Raise each side to the power $\frac{1}{6}$.
- ⊙ Multiply both sides by 6.
- ⊙ Multiply both sides by $\frac{1}{6}$.

b. How do we solve the equation $\sqrt[6]{x} = 12$?

- ⊙ Raise each side to the power 6.
- ⊙ Raise each side to the power $\frac{1}{6}$.
- ⊙ Multiply both sides by 6.
- ⊙ Multiply both sides by $\frac{1}{6}$.

c. How do we solve the equation $x^6 = 12$?

- ⊙ Raise each side to the power 6.
- ⊙ Raise each side to the power $\frac{1}{6}$.
- ⊙ Multiply both sides by 6.
- ⊙ Multiply both sides by $\frac{1}{6}$.

d. How do we evaluate the function $f(x) = x^{1/6}$ for $x = 12$?

- ⊙ Raise 12 to the power 6.
- ⊙ Raise 12 the power $\frac{1}{6}$.
- ⊙ Multiply 12 by 6.
- ⊙ Multiply 12 by $\frac{1}{6}$.

Answer 1. Choice 1

Answer 2. Choice 1

Answer 3. Choice 2

Answer 4. Choice 2

Solution.

a. Raise each side to the power 6.

b. Raise each side to the power 6.

c. Raise each side to the power $\dfrac{1}{6}$.

d. Raise 12 to the power $\dfrac{1}{6}$.

Checkpoint 6.46 Practice 9. Solve $5(x-1)^{1/4} = 10$

$x =$___

Answer. 17

Solution. 17

Roots of Negative Numbers

You already know that $\sqrt{-9}$ is not a real number, because there is no real number whose square is -9. Similarly, $\sqrt[4]{-16}$ is not a real number, because there is no real number r for which $r^4 = -16$. (Both of these radicals represent **complex numbers**.) In general, we cannot find an even root (square root, fourth root, and so on) of a negative number.

On the other hand, every positive number has two even roots that are real numbers. For example, both 3 and -3 are square roots of 9. The symbol $\sqrt{9}$ refers only to the positive, or **principal root**, of 9. If we want to refer to the negative square root of 9, we must write $-\sqrt{9} = -3$. Similarly, both 2 and -2 are fourth roots of 16, because $2^4 = 16$ and $(-2)^4 = 16$. However, the symbol $\sqrt[4]{16}$ refers to the principal, or positive, fourth root only. Thus,

$$\sqrt[4]{16} = 2 \quad \text{and} \quad -\sqrt[4]{16} = -2$$

Things are simpler for odd roots (cube roots, fifth roots, and so on). Every real number, whether positive, negative, or zero, has exactly one real-valued odd root. For example,

$$\sqrt[5]{32} = 2 \quad \text{and} \quad \sqrt[5]{-32} = -2$$

Here is a summary of our discussion.

Roots of Real Numbers.

1. Every positive number has two real-valued roots, one positive and one negative, if the index is even.
2. A negative number has no real-valued root if the index is even.

3. Every real number, positive, negative, or zero, has exactly one real-valued root if the index is odd.

Example 6.47

a $\sqrt[4]{-625}$ is not a real number.

b $-\sqrt[4]{625} = -5$

c $\sqrt[5]{-1} = -1$

d $\sqrt[4]{-1}$ is not a real number.

The same principles apply to powers with fractional exponents. Thus

$$(-32)^{1/5} = -2$$

but $(-64)^{1/6}$ is not a real number. On the other hand,

$$-64^{1/6} = -2$$

because the exponent $1/6$ applies only to 64, and the negative sign is applied after the root is computed.

Checkpoint 6.48 QuickCheck 6. Fill in the blanks.

a. We cannot take an (☐ even ☐ odd ☐ cube) root of a (☐ positive ☐ negative) number.

b. The principal root is the (☐ positive ☐ negative) root.

c. An odd root of a negative number is (☐ positive ☐ negative) .

d. An n^{th} root is the same as the (☐ nth ☐ 1/n) power.

Answer 1. even
Answer 2. negative
Answer 3. positive
Answer 4. negative
Answer 5. 1/n
Solution.

a. even, negative

b. positive

c. negative

d. $\dfrac{1}{n}$

Checkpoint 6.49 Practice 10. Evaluate each power, if possible. Enter "DNE" if it is not possible to evaluate.

a. $-81^{1/4} =$___

b. $(-81)^{1/4} =$___

c. $-64^{1/3} =$___

d. $(-64)^{1/3} =$___

Answer 1. -3

Answer 2. DNE or undefined

Answer 3. -4

Answer 4. -4

Solution.

a. -3

b. undefined

c. -4

d. -4

Problem Set 6.2

Skills Warm Up

1. Each radical below is equal to an integer. Use trial and error to find the root without a calculator.

a $\sqrt{169}$ c $\sqrt[4]{81}$ e $\sqrt[4]{1296}$

b $\sqrt[3]{64}$ d $\sqrt[5]{100,000}$ f $\sqrt[3]{343}$

2. Evaluate each power without using a calculator.

a $9^{1/2}$ c $64^{1/6}$ e $8^{-1/3}$

b $81^{1/4}$ d $32^{0.2}$ f $64^{-0.5}$

3. Use a calculator to approximate each irrational number to the nearest thousandth:

a $2^{1/2}$ c $\sqrt[4]{1.6}$ e $0.006^{-0.2}$

b $\sqrt[3]{75}$ d $365{-}1/3$ f $100^{0.25}$

4. Use the definition of a root to simplify without using a calculator.

a $(\sqrt[3]{125})^3 =$ d $\left(2\sqrt[3]{12}\right)^3$

b $(\sqrt[4]{2})^4 =$ e $\left(-a^3\sqrt[4]{a^2}\right)^4$

c $(\sqrt[3]{7})^3 =$ f $(-x^2\sqrt[3]{2x})^3 =$

Skills Practice

5. Evaluate each power, if possble.

a $-81^{1/4}$ c $-27^{1/3}$

b $(-81)^{1/4}$ d $(-27)^{1/3}$

6. Complete the table converting radicals to powers.

Radical	$\sqrt{x}$	$\sqrt[3]{x}$	$\sqrt[4]{x}$	$\sqrt[5]{x}$	$\frac{1}{\sqrt{x}}$	$\frac{1}{\sqrt[3]{x}}$	$\frac{1}{\sqrt[4]{x}}$	$\frac{1}{\sqrt[5]{x}}$
Exponent (Fraction)	$x^{1/2}$							
Exponent (Decimal)	$x^{0.5}$							

For Problems 7 and 8, write each expression in radical form.

7.

(a) $7^{1/2}$ (b) $3x^{1/4}$ (c) $(3x)^{0.25}$

8.
(a) $8^{-1/4}$ (b) $y(5x)^{-0.5}$ (c) $(y+2)^{1/3}$

For Problems 9 and 10, write each expression in exponential form.

9.
(a) $\sqrt{5}$ (b) $\sqrt[3]{4y}$ (c) $5\sqrt[3]{x}$

10.
(a) $\dfrac{2}{\sqrt[5]{3}}$ (b) $\sqrt[3]{y+2x}$ (c) $\dfrac{-1}{\sqrt[4]{3a-2b}}$

For Problems 11-14, write the expression as a sum of terms of the form kx^p.

11. $\dfrac{\sqrt{x}}{4} - \dfrac{2}{\sqrt{x}} + \dfrac{x}{\sqrt{2}}$

12. $\dfrac{6-\sqrt[3]{x}}{2\sqrt[3]{x}}$

13. $x^{-0.5}\left(x + x^{0.25} - x^{0.5}\right)$

14. $\dfrac{\frac{1}{2}x^{-1} + 2x^{-0.5} + x^{-0.25}}{2x^{0.5}}$

For Problems 15-20, solve.

15. $6.5x^{1/3} + 3.8 = 33.05$

16. $4(x+2)^{1/5} = 12$

17. $(2x-3)^{-1/4} = \dfrac{1}{2}$

18. $(5x-2)^{-1/3} = \dfrac{1}{4}$

19. $\sqrt[3]{x^2-3} = 3$

20. $\sqrt[3]{2x^2-15} = 5$

For Problems 21-26, solve for the indicated variable.

21. $T = 2\pi\sqrt{\dfrac{L}{g}}$ for L

22. $r = \sqrt{t^2 - s^2}$ for s

23. $d = \sqrt[3]{\dfrac{16Mr^2}{m}}$ for M

24. $R = \sqrt[4]{\dfrac{8Lvf}{\pi p}}$ for p

25. $T = \sqrt[4]{\dfrac{E}{SA}}$ for A

26. $r = \sqrt[3]{\dfrac{3V}{4\pi}}$ for V

27. Match each function with the description of its graph in the first quadrant.

I $f(x) = x^2$

III $h(x) = x^{1/2}$

II $g(x) = x^{-2}$

IV $f(x) = x^{-1/2}$

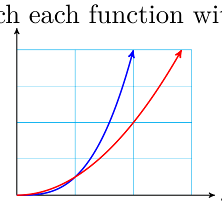
a Increasing and concave up

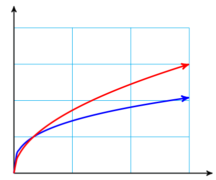
b Increasing and concave down

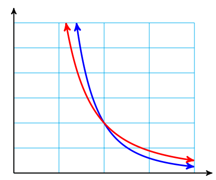
c Decreasing and concave up

d Decreasing and concave down

28. Match each function with its graph.

a  $f(x) = x^2,\ g(x) = x^3$

b  $f(x) = x^{1/2},\ g(x) = x^{1/3}$

c  $f(x) = x^{-2},\ g(x) = x^{-3}$

29. Write each expression as a power function.

a $G(x) = 3.7\sqrt[3]{x}$ b $H(x) = \sqrt[4]{85x}$ c $F(t) = \dfrac{25}{\sqrt[5]{t}}$

30. Graph each set of functions in the given window. What do you observe?

a $y_1 = \sqrt{x},\ y_2 = x^2,\ y_3 = x$

$$\text{Xmin} = 0 \qquad \text{Xmax} = 4$$
$$\text{Ymin} = 0 \qquad \text{Ymax} = 4$$

b $y_1 = \sqrt[3]{x},\ y_2 = x^3,\ y_3 = x$

$$\text{Xmin} = -4 \qquad \text{Xmax} = 4$$
$$\text{Ymin} = -4 \qquad \text{Ymax} = 4$$

Applications

31. When a ship moves through the water, it creates waves that impede its own progress. Because of this resistance, there is an upper limit to the speed at which a ship can travel, given in knots by

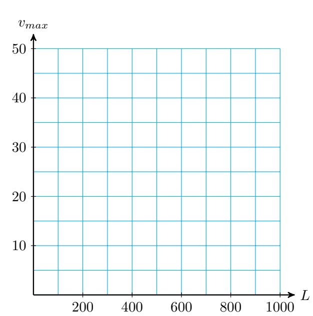
$$v_{\max} = 1.3\sqrt{L}$$

where L is the length of the vessel in feet. (Source: Gilner, 1972)

a Complete the table of values for $v_{\max}$ as a function of L.

L (feet)	200	400	600	800	1000
$v_{\max}$ (knots)					

b Graph maximum speed against vessel length.



c The world's largest ship, the oil tanker Jahre Viking, is 1054 feet long. What is its top speed?

d As a ship approaches its maximum speed, the power required increases sharply. Therefore, most merchant ships are designed to cruise at speeds no higher than

$$v_c = 0.8\sqrt{L}$$

Graph v_c on the same axes with v_{max}.

e What is the cruising speed of the Jahre Viking? What percent of its maximum speed is that?

32. If you walk in the normal way, your speed, v, in meters per second, is limited by the length of your legs, r, according to the formula

$$v \leq \sqrt{gr}$$

where g is the acceleration due to gravity. (Source: Alexander, 1992)

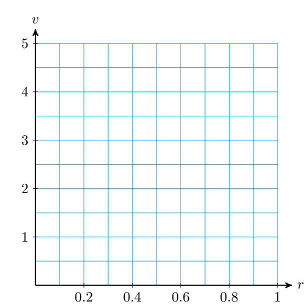
a Complete the table of values for v as a function of r. The constant, g, is approximately 10.

r (meters)	0.2	0.4	0.6	0.8	1.0
v (meters/sec)					

b A typical adult man has legs about 0.9 meter long. How fast can he walk?

c A typical four-year-old has legs 0.5 meter long. How fast can she walk?

d Graph maximum walking speed against leg length.



e Race-walkers can walk as fast as 4.4 meters per second by rotating their hips so that the effective length of their legs is increased. What is that effective length?

f On the moon the value of g is 1.6. How fast can a typical adult man walk on the moon?

33. The period of a pendulum is the time it takes for the pendulum to complete one entire swing, from left to right and back again. The greater the length, L, of the pendulum, the longer its period, T. In fact, if L is measured in feet, then the period is given in seconds by

$$T = 2\pi\sqrt{\frac{L}{32}}$$

(a) Write the formula for T as a power function in the form $f(x) = kx^p$.

(b) The Foucault pendulum in the Convention Center in Portland, Oregon is 90 feet long. What is its period?

(c) Choose a suitable window and graph the function $T = f(L)$. Label the point corresponding to point (b) on the graph.

34. The rate, r, in feet per second, at which water flows from a fire hose varies directly with the square root of the water pressure, P, in psi (pounds per square inch). What is the rate of water flow at a typical water pressure of 60 psi?

P (psi)	10	20	30	40
r (ft/sec)	38.3	54.1	66.3	76.5

a Find the value of k and write a power function relating the variables.

b What is the rate of water flow at a typical water pressure of 60 psi?

c Graph your function and verify your answer to part (b) on the graph.

35. Thanks to improvements in technology, the annual electricity cost of running most major appliances has decreased steadily since 1970. The average annual cost of running a refrigerator is given, in dollars, by the function

$$C(t) = 148 - 28t^{1/3}$$

where t is the number of years since 1970.

(a) How much did it cost to run a refrigerator in 1980? In 1990?

(b) When was the cost of running a refrigerator half of the cost in 1970? If the cost continues to decline according to the given function, when will it cost \$50 per year to run a refrigerator?

(c) Graph the function $C(t)$. Do you think that the cost will continue to decline indefinitely according to the given function? Why or why not?

36. A rough estimate for the radius of the nucleus of an atom is provided by the formula

$$r = kA^{1/3}$$

where A is the mass number of the nucleus and $k \approx 1.3 \times 10^{-13}$ centimeter.

(a) Estimate the radius of the nucleus of an atom of iodine-127, which has mass number 127. If the nucleus is roughly spherical, what is its volume?

(b) The nuclear mass of iodine-127 is 2.1×10^{-22} gram. What is the density of the nucleus? (Density is mass per unit volume.)

(c) Complete the table of values for the radii of various radioisotopes.

Element	Carbon	Potassium	Cobalt	Technetium	Radium
Mass number, A	14	40	60	99	226
Radius, r					

(d) Sketch a graph of r as a function of A. (Use units of 10^{-13} centimeter on the vertical axis.)

37. When the Concorde lands at Heathrow airport in London, the width w of the sonic boom felt on the ground is given in kilometers by the following formula:

$$w = 4\left(\frac{Th}{m}\right)^{1/2}$$

where T stands for the temperature on the ground in kelvins, h is the altitude of the Concorde when it breaks the sound barrier, and m is the drop in temperature for each gain in altitude of 1 kilometer. Find the width of the sonic boom if the ground temperature is 293 kelvins, the altitude of the Concorde is 15 kilometers, and the temperature drop is 4 kelvin per kilometer of altitude.

38. The Stefan-Boltzmann law says that the temperature, T, of the Sun, in kelvins, can be computed from the formula

$$sT^4 = \frac{L}{4\pi R^2}$$

where $L = 3.9 \times 10^{33}$ is the total luminosity of the Sun, $R = 9.96 \times 10^{10}$ centimeters is the radius of the Sun, and $s = 5.7 \times 10^{-5}$ is a constant governing radiation. Calculate the temperature of the Sun.

39.

(a) Graph the functions $f(x) = 4\sqrt[3]{x-9}$ and $g(x) = 12$ in the window

$$\text{Xmin} = 0 \qquad \text{Xmax} = 47$$
$$\text{Ymin} = -8 \qquad \text{Ymax} = 16$$

(b) Use the graph to solve the equation $4\sqrt[3]{x-9} = 12$.

(c) Solve the equation algebraically.

40.

(a) Graph the functions $f(x) = 6 + 2\sqrt[4]{12-x}$ and $g(x) = 10$ in the window

$$\text{Xmin} = -27 \qquad \text{Xmax} = 20$$
$$\text{Ymin} = 4 \qquad \text{Ymax} = 12$$

(b) Use the graph to solve the equation $6 + 2\sqrt[4]{12-x} = 10$.

(c) Solve the equation algebraically.

Rational Exponents

Powers of the Form $a^{m/n}$

In the last section, we considered powers of the form $a^{1/n}$, such as $x^{1/3}$ and $x^{-1/4}$, and saw that $a^{1/n}$ is equivalent to the root $\sqrt[n]{a}$. What about other fractional exponents? What meaning can we attach to a power of the form $a^{m/n}$?

Consider the power $x^{3/2}$. Notice that the exponent $\frac{3}{2} = 3(\frac{1}{2})$, and thus by the third law of exponents, we can write

$$(x^{1/2})^3 = x^{(1/2)^3} = x^{3/2}$$

In other words, we can compute $x^{3/2}$ by first taking the square root of x and then cubing the result. For example,

$$\begin{aligned} 100^{3/2} &= (\mathbf{100^{1/2}})^3 && \text{Take the square root of 100.} \\ &= \mathbf{10^3} = 1000 && \text{Cube the result.} \end{aligned}$$

Checkpoint 6.50 QuickCheck 1. True or false.

a. When we raise a power to a power, we multiply the exponents; in symbols

$$(a^m)^n = a^{mn}$$

(☐ True ☐ False)

b. $\dfrac{4}{3} = 4\left(\dfrac{1}{3}\right)$, and in general $\dfrac{p}{q} = p\left(\dfrac{1}{q}\right)$. (☐ True ☐ False)

c. To compute $(25^{1/2})^3$, we first take the square root of 25, then cube the result. (☐ True ☐ False)

d. The notation $16^{3/4}$ means to multipy 16 by $\dfrac{3}{4}$. (☐ True ☐ False)

Answer 1. True
Answer 2. True
Answer 3. True
Answer 4. False
Solution.

a. True

b. True

c. True

d. False

A fractional exponent represents a power and a root. The denominator of the exponent is the root, and the numerator of the exponent is the power. We will define fractional powers only when the base is a positive number.

Rational Exponents.

$$a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}, \quad a > 0, \;\; n \neq 0$$

To compute $a^{m/n}$, we can compute the nth root first, or the mth power, whichever is easier. For example,

$$8^{2/3} = \left(8^{1/3}\right)^2 = 2^2 = 4$$

or

$$8^{2/3} = \left(8^2\right)^{1/3} = 64^{1/3} = 4$$

Example 6.51

a $81^{3/4} = \left(81^{1/4}\right)^3 = 3^3 = 27$

b $-27^{5/3} = -\left(27^{1/3}\right)^5 = -3^5 = -243$

c $27^{-2/3} = \dfrac{1}{\left(27^{1/3}\right)^2} = \dfrac{1}{3^2} = \dfrac{1}{9}$

d $5^{3/2} = \left(5^{1/2}\right)^3 \approx (2.236)^3 \approx 11.180$

Note 6.52 You can verify all the calculations in Example 6.51, p. 403 on your calculator. For example, to evaluate $81^{3/4}$, key in

81 [^] (3 [÷] 4) [ENTER]

or simply

81 [^] 0.75 [ENTER]

Caution 6.53 When computing $81^{3/4}$ on a calculator, do not forget the parentheses around the exponent, $\dfrac{3}{4}$. The keying sequence

81 [^] 3 [÷] [ENTER] gives us the value of $\dfrac{81^3}{4}$, or $132,860.25$.

The parentheses tell the calculator to include all of the quotient $3 \div 4$ in the exponent, not just the 3.

Checkpoint 6.54 **Practice 1.** Evaluate each power.

a. $32^{-3/5} =$___

b. $-81^{1.25} =$___

Answer 1. $\frac{1}{8}$

Answer 2. -243

Solution.

a. $\dfrac{1}{8}$

b. -243

Power Functions

Perhaps the single most useful piece of information a scientist can have about an animal is its metabolic rate. The metabolic rate is the amount of energy the animal uses per unit of time for its usual activities, including locomotion, growth, and reproduction.

The basal metabolic rate, or BMR, sometimes called the resting metabolic rate, is the minimum amount of energy the animal can expend in order to survive.

Example 6.55

A revised form of Kleiber's rule states that the basal metabolic rate for many groups of animals is given by

$$B(m) = 70m^{0.75}$$

where m is the mass of the animal in kilograms and the BMR is measured in kilocalories per day.

a Calculate the BMR for various animals whose masses are given in the table.

Animal	Bat	Squirrel	Raccoon	Lynx	Human	Moose	Rhinoceros
Weight (kg)	0.1	0.6	8	30	70	360	3500
BMR (kcal/day)							

b Sketch a graph of Kleiber's rule for $0 < m \leq 400$.

c Do larger species eat more or less, relative to their body mass, than smaller ones?

Solution.

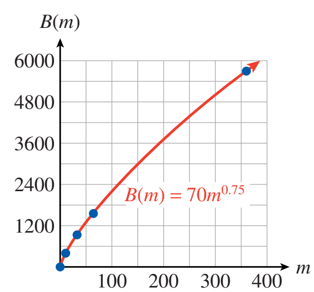
a We evaluate the function for the values of m given. For example, to calculate the BMR of a bat, we compute

$$B(0.1) = 70(0.1)^{0.75} = 12.4$$

A bat expends, and hence must consume, at least 12 kilocalories per day. We evaluate the function to complete the rest of the table. The values of BMR are rounded to the nearest whole number.

Animal	Bat	Squirrel	Raccoon	Lynx	Human	Moose	Rhinoceros
Weight (kg)	0.1	0.6	8	30	70	360	3500
BMR (kcal/day)	12	48	333	897	1694	5785	31,853

b We plot the data from the table to obtain the graph below.



c If energy consumption were proportional to body weight, the graph would be a straight line. But this graph is concave down, or bends downward. Larger species eat less than smaller ones, relative to their body weight. For example, a moose weighs 600 times as much as a squirrel, but its energy requirement is only 121 times the squirrel's.

Checkpoint 6.56 QuickCheck 2. True or false.

a. The graph of $f(x) = 70x^{3/4}$ is concave down. (☐ True ☐ False)

b. Energy consumption is not directly proportional to body weight. (☐ True ☐ False)

c. An animal twice as heavy as a squirrel uses twice as much energy as a squirrel. (☐ True ☐ False)

d. The graph of $f(x) = 70x^{3/4}$ passes through the origin. (☐ True ☐ False)

Answer 1. True
Answer 2. True
Answer 3. False
Answer 4. True
Solution.

a. True

b. True

c. False

d. True

Checkpoint 6.57 Practice 2.

a. Complete the table of values for the function $f(x) = x^{-3/4}$.

x	0.1	0.2	0.5	1
$f(x)$	____	____	____	____

x	2	5	8	10
$f(x)$	____	____	____	____

b. Sketch the graph of the function.

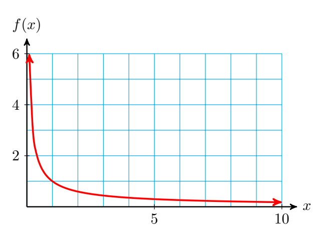
Answer 1. 5.62341
Answer 2. 3.3437
Answer 3. 1.68179
Answer 4. 1
Answer 5. 0.594604
Answer 6. 0.29907
Answer 7. 0.210224
Answer 8. 0.177828
Solution.

a.

x	0.1	0.2	0.5	1
$f(x)$	5.623	3.344	1.682	1

x	2	5	8	10
$f(x)$	0.595	0.299	0.210	0.178

b. A graph is below.



Radical Notation

Because $a^{1/n} = \sqrt[n]{a}$, we can write any power with a fractional exponent in radical form as follows.

Rational Exponents and Radicals.

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

Example 6.58

a $125^{4/3} = \sqrt[3]{125^4}$ or $\left(\sqrt[3]{125}\right)^4$

b $x^{0.4} = x^{2/5} = \sqrt[5]{x^2}$

c $6w^{-3/4} = \dfrac{6}{\sqrt[4]{w^3}}$

Checkpoint 6.59 Practice 3. Write each expression in radical notation.

a. $5t^{1.25}$

- ⊙ $\sqrt[4]{5t^5}$
- ⊙ $\sqrt[4]{(5t)^5}$
- ⊙ $5\sqrt[4]{t^5}$
- ⊙ $5\sqrt{(t/4)^5}$

b. $3m^{-5/3}$

- ⊙ $\dfrac{3}{\sqrt[3]{m^5}}$
- ⊙ $\dfrac{1}{\sqrt[3]{3m^5}}$
- ⊙ $\dfrac{1}{3\sqrt[3]{m^5}}$
- ⊙ $-3\sqrt[3]{m^5}$

Answer 1. Choice 3

Answer 2. Choice 1

Solution.

a. $5\sqrt[4]{t^5}$

b. $\dfrac{3}{\sqrt[3]{m^5}}$

Usually, we will want to convert from radical notation to fractional exponents, since exponential notation is easier to use.

Example 6.60

a $\sqrt{x^5} = x^{5/2}$

b $5\sqrt[4]{p^3} = 5p^{3/4}$

c $\dfrac{3}{\sqrt[5]{t^2}} = 3t^{-2/5}$

d $\sqrt[3]{2y^2} = \left(2y^2\right)^{1/3} = 2^{1/3}y^{2/3}$

Checkpoint 6.61 Practice 4. Convert to exponential notation.

a. $\sqrt[3]{6w^2} =$____________

b. $\sqrt[4]{\dfrac{v^3}{s^5}} =$____________

Answer 1. $1.81712w^{\frac{2}{3}}$

Answer 2. $v^{\frac{3}{4}}s^{-\frac{5}{4}}$

Solution.

a. $6^{1/3}w^{2/3}$

b. $v^{3/4}s^{-5/4}$

Checkpoint 6.62 QuickCheck 3. True or false.

a. $x^{3/5}$ represents the cube root of x to the fifth power. (☐ True ☐ False)

b. We can simplify $2x^{4/3}$ as $\sqrt[3]{2x^4}$. (☐ True ☐ False)

c. The notation $a^{-1/4}$ means $-a^4$. (☐ True ☐ False)

d. We can simplify $5b^{-3/2}$ as $\dfrac{1}{5b^{3/2}}$. (☐ True ☐ False)

Answer 1. False

Answer 2. False

Answer 3. False

Answer 4. False

Solution.

a. False

b. False

c. False

d. False

Operations with Rational Exponents

Powers with rational exponents—positive, negative, or zero—obey the laws of exponents, which we discussed in Section 6.1, p. 367. You may want to review those laws before studying the following examples.

Example 6.63

a

$$\frac{7^{0.75}}{7^{0.5}} = 7^{0.75-0.5} = 7^{0.25} \quad \text{Apply the second law of exponents.}$$

b

$$\begin{aligned} v \cdot v^{-2/3} &= v^{1+(-2/3)} \quad \text{Apply the first law of exponents.} \\ &= v^{1/3} \end{aligned}$$

c

$$\left(x^8\right)^{0.5} = x^{8(0.5))} = x^4 \quad \text{Apply the third law of exponents.}$$

d

$$\begin{aligned} \frac{\left(5^{1/2}y^2\right)^2}{\left(5^{2/3}y\right)^3} &= \frac{5y^4}{5^2y^3} \quad \text{Apply the fourth law of exponents.} \\ &= \frac{y^{4-3}}{5^{2-1}} = \frac{y}{5} \quad \text{Apply the second law of exponents.} \end{aligned}$$

Checkpoint 6.64 Practice 5. Simplify by applying the laws of exponents.

a. $x^{1/3}\left(x + x^{2/3}\right) =$__________

b. $\dfrac{n^{9/4}}{4n^{3/4}} =$__________

Answer 1. $x^{\frac{4}{3}} + x$

Answer 2. $\frac{n^{\frac{3}{2}}}{4}$

Solution.

a. $x^{4/3} + x$

b. $\dfrac{n^{3/2}}{4}$

Solving Equations

According to the third law of exponents, when we raise a power to another power, we multiply the exponents together. In particular, if the two exponents are reciprocals, then their product is 1. For example,

$$\left(x^{2/3}\right)^{3/2} = x^{(2/3)(3/2)} = x^1 = x$$

This observation can help us to solve equations involving fractional exponents. For instance, to solve the equation

$$x^{2/3} = 4$$

we raise both sides of the equation to the reciprocal power, 3/2. This gives us

$$\left(x^{2/3}\right)^{3/2} = 4^{3/2}$$

$$x = 8$$

The solution is 8.

Example 6.65

Solve $(2x+1)^{3/4} = 27$

Solution. We raise both sides of the equation to the reciprocal power, $\frac{4}{3}$.

$$\left[(2x+1)^{3/4}\right]^{4/3} = 27^{4/3} \quad \text{Apply the third law of exponents.}$$

$$2x+1 = 81 \quad \text{Solve as usual.}$$

$$x = 40$$

Checkpoint 6.66 Practice 6. Solve the equation $3.2z^{0.6} - 9.7 = 8.7$. Round your answer to two decimal places.

Answer: ____

Hint. Isolate the power.
Raise both sides to the reciprocal power.

Answer. $\left(\frac{18.4}{3.2}\right)^{\frac{1}{0.6}}$

Solution. 18.45

Checkpoint 6.67 QuickCheck 3. True or false.

a. The first step in solving an equation with a fractional exponent is to isolate the power. (☐ True ☐ False)

b. Then we raise both sides to the reciprocal of the exponent. (☐ True ☐ False)

c. We can use the laws of exponents to simplify expressions involving fractional exponents. (☐ True ☐ False)

d. The reciprocal of 0.6 is $\frac{1}{0.6}$. (☐ True ☐ False)

Answer 1. True

Answer 2. True

Answer 3. True

Answer 4. True

Solution.

a. True

b. True

c. True

d. True

Problem Set 6.3

Warm Up

1. Combining powers and roots

a Start with 8, square it, and then take the cube root of the result. That is:

$$\sqrt[3]{8^2} = \sqrt[3]{?} =$$

Now start with 8, take the cube root, then square the result. That is:

$$(\sqrt[3]{8})^2 = (?)^2 =$$

b Do you get the same answers for parts (a) and (b)? You should! Now try the same thing with some irrational numbers. Use your calculator, and round your answers to two decimal places.

$$\sqrt{3^5} = \qquad (\sqrt{3})^5 =$$

$$(\sqrt[3]{58})^4 = \qquad \sqrt[3]{58^4} =$$

c Choose the most convenient order of operations to evaluate each root without using a calculator.

i $\sqrt[5]{32^3}$ ii $-\sqrt[3]{27^4}$

2. Using exponents

a Simplify $(\sqrt[3]{8})^2$ using exponents as follows. Fill in the blanks with the correct exponents.

$$(\sqrt[3]{8})^2 = (8^{—})^2 = 8^{—}$$

b Use your calculator to compute $8^{2/3}$. Don't forget to put parentheses around the exponent.

c Approximate each power to the nearest thousandth.

1 $12^{5/6}$
2 $\sqrt[3]{6^4}$
3 $37^{-2/3}$
4 $4.7^{2.3}$

3. Evaluate each root without using a calculator.

(a) $\sqrt[4]{16y^{12}}$
(b) $\sqrt[5]{243x^{10}}$
(c) $\sqrt[3]{8x^9y^{27}}$
(d) $-\sqrt[4]{81a^8b^{12}}$
(e) $\sqrt[3]{64x^6y^{18}}$
(f) $-\sqrt[5]{32x^{25}y^5}$

4. Evaluate each power without using a calculator.

a $81^{3/4}$
b $125^{2/3}$
c $625^{0.75}$
d $16^{-3/2}$
e $8^{-4/3}$
f $32^{-1.6}$

Skills Practice

For Problems 5 and 6, write each power in radical form.

5.

(a) $y^{3/4}$ (b) $a^{-2/7}$ (c) $(st)^{-3/5}$

6.

(a) $5y^{2/3}$ (b) $6w^{-1.5}$ (c) $-3x^{0.4}y^{0.6}$

For Problems 7 and 8, write each expression with fractional exponents.

7.

(a) $\sqrt{y^3}$ (b) $6\sqrt[5]{(ab)^3}$ (c) $\dfrac{-2n}{\sqrt[8]{q^{11}}}$

8.

(a) $\sqrt[3]{ab^2}$ (b) $\dfrac{5}{\sqrt[3]{y^2}}$ (c) $\dfrac{S}{4\sqrt{VH^3}}$

For Problems 9–14, simplify by applying the laws of exponents. Write your answers with positive exponents only.

9. $4a^{6/5}a^{4/5}$ **10.** $(-2m^{2/3})^4$ **11.** $\dfrac{8w^{9/4}}{2w^{3/4}}$

12. $(-3u^{5/3})(5u^{-2/3})$ **13.** $\dfrac{k^{3/4}}{2k}$ **14.** $c^{-2/3}\left(\dfrac{2}{3}c^2\right)$

For Problems 15–20, solve. Round your answers to the nearest thousandth if necessary.

15. $x^{2/3} - 1 = 15$ **16.** $x^{-2/5} = 9$

17. $6 - 2.4x^{-5/4} = 8$ **18.** $2(5.2 - x^{5/3}) = 1.4$

19. $\dfrac{2}{3}(2y+1)^{0.2} = 6$ **20.** $1.3w^{0.3} + 4.7 = 5.2$

21. If $f(x) = (3x-4)^{3/2}$, find x so that $f(x) = 27$.

22. If $S(x) = 12x^{-5/4}$, find x so that $S(x) = 20$.

For Problems 23–26, use the distributive law to find the product.

23. $x^{1/3}(2x^{2/3} - x^{1/3})$ **24.** $3y^{-3/8}\left(\dfrac{1}{4}y^{-1/4} + y^{3/4}\right)$

25. $(2x^{1/4}+1)(x^{1/4}-1)$ **26.** $(a^{3/4}-2)^2$

For Problems 27–30, factor out the smallest power from each expression. Write your answers with positive exponents only.

27. $x^{3/2} + x = x(\ \ ?\ \ )$ **28.** $y^{3/4} - y^{-1/4} = y^{-1/4}(\ \ ?\ \ )$

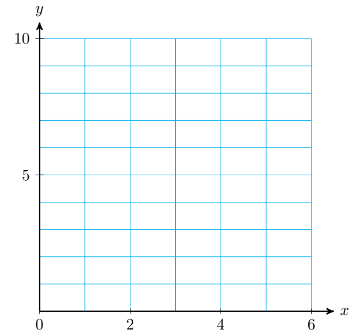
29. $a^{1/3} + 3 - a^{-1/3} = a^{-1/3}(\ \ ?\ \ )$ **30.** $4b - 6 + 4b^{-2/3} = 2b^{-2/3}(\ \ ?\ \ )$

31.

a Complete the table of values, then graph both functions on the grid, along with the graph of $y = x$.

$$f(x) = x^{4/3}, \quad g(x) = x^{3/2}$$

x	0	1	2	3	4	5	6
$f(x)$							
$g(x)$							



b Check your work by graphing the functions in the suggested window, and compare the graphs.

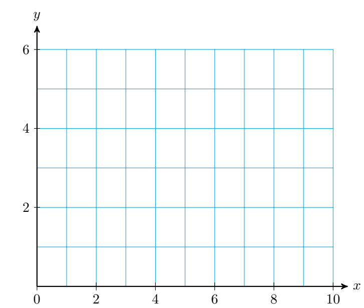
Xmin = 0	Xmax = 6
Ymin = 0	Ymax = 10

32.

a Complete the table of values, then graph both functions on the grid, along with the graph of $y = x$.

$$f(x) = x^{3/4}, \quad g(x) = x^{2/3}$$

x	0	2	4	6	8	10
$f(x)$						
$g(x)$						



b Check your work by graphing the functions in the suggested window, and compare the graphs.

Xmin = 0	Xmax = 10
Ymin = 0	Ymax = 4

For Problems 33 and 34, solve the equations, and locate the corresponding points on the graphs in Problems 31 and 32.

33.

a $x^{4/3} = 4$

b $x^{3/2} = 6$

34.

a $x^{3/4} = 4$

b $x^{2/3} = 6$

Applications

35. During a flu epidemic in a small town, health officials estimate that the number of people infected t days after the first case was discovered is given by

$$I(t) = 50t^{3/5}$$

a Complete the table of values.

t	5	10	15	20
$I(t)$				

b How long will it be before 300 people are ill?

c Graph the function $I(t)$ on your calculator, and verify your answer to part (b) on your graph.

36. The research division of an advertising firm estimates that the number of people who have seen their ads t days after the campaign begins is given by the function

$$N(t) = 2000t^{5/4}$$

a Complete the table of values.

t	6	10	14	20
$N(t)$				

b How long will it be before 75,000 people have seen the ads?

c Graph the function $n(t)$ on your calculator, and verify your answer to part (b) on your graph.

37. In the 1970s, Jared Diamond studied the number of bird species on small islands near New Guinea. He found that larger islands support a larger number of different species, according to the formula

$$S = 15.1A^{0.22}$$

where S is the number of species on an island of area A square kilometers. (Source: Chapman and Reiss, 1992)

(a) Fill in the table.

A	10	100	1000	5000	$10,000$
S					

(b) Graph the function on your calculator for $0 < A \leq 10,000$.

(c) How many species of birds would you expect to find on Manus Island, with an area of 2100 square kilometers? On Lavongai, which bird's area is 1140 square kilometers?

(d) How large must an island be in order to support 200 different species of bird?

38. The climate of a region has a great influence on the types of animals that can survive there. Extreme temperatures create difficult living conditions, so the diversity of wildlife decreases as the annual temperature range increases. Along the west coast of North America, the number of species of mammals, M, is approximately related to the temperature range, R, (in degrees Celsius) by the function

$$M = f(R) = 433.8R^{-0.742}$$

(Source: Chapman and Reiss, 1992)

(a) Graph the function for temperature ranges up to 30°C.

(b) How many species would you expect to find in a region where the temperature range is 10°C? Label the corresponding point on your graph.

(c) If 50 different species are found in a certain region, what temperature range would you expect the region to experience? Label the corresponding point on your graph.

(d) Evaluate the function to find $f(9)$, $f(10)$, $f(19)$, and $f(20)$. What do these values represent? Calculate the change in the number of species as the temperature range increases from 9°C to 10°C and from 19°C to 20°C. Which 1° increase results in a greater decrease in diversity? Explain your answer in terms of slopes on your graph.

39. The table at right shows the exponent, p, in the allometric equation

$$\text{variable} = k(\text{body mass})^p$$

for some variables related to mammals. (Source: Chapman and Reiss, 1992)

Variable	Exponent, p
Home range size	1.26
Lung volume	1.02
Brain mass	0.70
Heart rate	−0.25

a The average body mass of a dolphin is about 140 kilograms, twice the body mass of an average human male. Using the allometric equation above, calculate the ratio of the brain mass of a dolphin to that of a human.

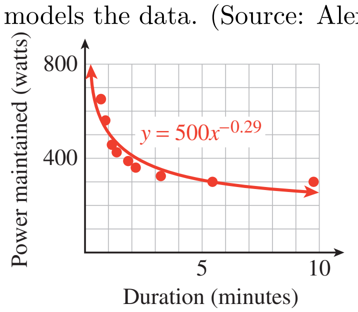
b A good-sized brown bear weighs about 280 kilograms, twice the weight of a dolphin. Calculate the ratio of the brain mass of a brown bear to that of a dolphin.

c Use a ratio to compare the heartbeat frequencies of a dolphin and a human, and those of a brown bear and a dolphin.

40. A bicycle ergometer is used to measure the amount of power generated by a cyclist. The scatterplot shows how long an athlete was able to sustain variouslevels of power output. The curve is the graph of

$$y = 500x^{0.29}$$

which approximately models the data. (Source: Alexander, 1992)



a As represented by the graph, which variable is the input, and which is the output?

b The athlete maintained 650 watts of power for 40 seconds. What power output does the equation predict for 40 seconds?

c The athlete maintained 300 watts of power for 10 minutes. How long does the equation predict that power output can be maintained?

d In 1979 a remarkable pedal-powered aircraft called the Gossamer Albatross was successfully flown across the English Channel. The flight took 3 hours. According to the equation, what level of power can be maintained for 3 hours?

e The Gossamer Albatross needed 250 watts of power to keep it airborne. For how long can 250 watts be maintained, according to the equation?

41. Birds' eggs typically lose 10%–20% of their mass during incubation. The embryo metabolizes lipid during growth, and this process releases water vapor through the porous shell. The incubation time for birds' eggs is a function of the mass of the egg and has been experimentally determined as

$$I(m) = 12.0m^{0.217}$$

where m is measured in grams and I in days. (Source: Burton, 1998)

a Calculate the incubation time (to the nearest day) for the wren, whose eggs weigh about 2.5 grams, and the greylag goose, whose eggs weigh 46 grams.

b The rate of water loss from the egg is also a function of its mass and appears to follow the rule

$$W(m) = 0.015m^{0.742}$$

Combine the functions $I(m)$ and $W(m)$ to calculate the fraction of the initial egg mass that is lost during the entire incubation period.

c Explain why your result shows that most eggs lose about 18% of their mass during incubation.

42. The incubation time for birds' eggs is given by

$$I(m) = 12.0m^{0.217}$$

where m is measured in grams and I in days. (See Problem 41.) Before hatching, the eggs take in oxygen at the rate of

$$O(m) = 22.2m^{0.77}$$

in milliliters per day. (Source: Burton, 1998)

a Combine the functions $I(m)$ and $O(m)$ to calculate the total amount of oxygen taken in by the egg during its incubation.

b Use your result from part (a) to explain why total oxygen consumption per unit mass is approximately inversely proportional to incubation time. (Oxygen consumption is a reliable indicator of metabolic rate, and it is reasonable that incubation time should be inversely proportional to metabolic rate.)

c Predict the oxygen consumption per gram of a herring gull's eggs, given that their incubation time is 26 days. (The actual value is 11 milliliters per day.)

43. Kepler's law gives a relation between the period, p, of a planet's revolution and its average distance, a, from the Sun:

$$p^2 = Ka^3$$

where $K = 1.243 \times 10^{24}$, a is measured in miles, and p is measured in years.

a Solve Kepler's law for p in terms of a.

b Find the period of Mars if its average distance from the Sun is 1.417×10^8 miles.

44. Refer to Kepler's law, $p^2 = Ka^3$, in Problem 43.

a Solve Kepler's law for a in terms of p.

b The period of Venus is 0.615 years. Find the distance from Venus to the Sun.

Working with Radicals

Sometimes radical notation is more convenient to use than exponents. In these cases, we usually simplify radical expressions algebraically before using a calculator to obtain decimal approximations.

Properties of Radicals

Because $\sqrt[n]{a} = a^{1/n}$, we can use the laws of exponents to derive two important properties that are useful in working with radicals.

Product Rule for Radicals.

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}, \qquad \text{for } a, b \geq 0$$

Quotient Rule for Radicals.

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, \qquad \text{for } a \geq 0, \quad b > 0$$

As examples, you can verify that

$$\sqrt{36} = \sqrt{4}\sqrt{9} \quad \text{and} \quad \sqrt[3]{\frac{1}{8}} = \frac{\sqrt[3]{1}}{\sqrt[3]{8}}$$

Caution 6.68 In general, it is *not* true that $\sqrt[n]{a+b}$ is equivalent to $\sqrt[n]{a} + \sqrt[n]{b}$, or that $\sqrt[n]{a-b}$ is equivalent to $\sqrt[n]{a} - \sqrt[n]{b}$.

For example, you can check that

$$\sqrt{9+16} \neq \sqrt{9} + \sqrt{16}$$
$$\text{and} \quad \sqrt[3]{27-8} \neq \sqrt[3]{27} - \sqrt[3]{8}$$

Example 6.69

Which of the following are true?

a Is $\sqrt{36+64} = \sqrt{36} + \sqrt{64}$?

b Is $\sqrt[3]{8(64)} = \sqrt[3]{8}\sqrt[3]{64}$?

c Is $\sqrt{x^2+4} = x + 2$?

d Is $\sqrt[3]{8x^3} = 2x$?

Solution. The statements in (b) and (d) are true, and both are examples of the first property of radicals.

Statements (a) and (c) are false.

Checkpoint 6.70 **QuickCheck 1.** Which of the following are true?

a. $\sqrt[3]{8a^3 - b^3} = 2a - b$? (☐ True ☐ False)

b. $\sqrt[5]{x^3y^4} = \sqrt[5]{x^3}\,\sqrt[5]{y^4}$? (☐ True ☐ False)

c. $\sqrt[4]{\dfrac{1}{w^4}} = \dfrac{1}{w}$, for $w > 0$? (☐ True ☐ False)

d. $\sqrt{1 + x^2} = 1 + x$? (☐ True ☐ False)

Answer 1. False

Answer 2. True

Answer 3. True

Answer 4. False

Solution. (b) and (c) are true.

Simplifying Radicals

Each time we use a calculator to approximate a radical, we lose accuracy, and in the course of a long calculation, the error accumulates. To avoid this problem, we simplify radicals by factoring out any perfect powers from the radicand.

Example 6.71

Simplify $\sqrt[3]{108}$

Solution. We look for perfect cubes that divide evenly into 108. The easiest way to do this is to try the perfect cubes in order:

$$1,\ 8,\ 27,\ 64,\ 125, \ldots$$

and so on, until we find one that is a factor. For this example, we find that $108 = 27 \cdot 4$. Applying the Product Rule, we write

$$\begin{aligned} \sqrt[3]{108} &= \sqrt[3]{27}\,\sqrt[3]{4} && \text{Simplify: } \sqrt[3]{27} = 3. \\ &= 3\sqrt[3]{4} \end{aligned}$$

This expression is considered simpler than the original radical because the new radicand, 4, is smaller than the original, 108.

Caution 6.72 Finding a decimal approximation for a radical is not the same as simplifying the radical. In the Example, p. 417 above, we can use a calculator to find

$$\sqrt[3]{108} \approx 4.762$$

but 4.762 is not the *exact* value for $\sqrt[3]{108}$.

In long calculations, too much error may be introduced by approximating each radical. However, $3\sqrt[3]{4}$ is equal to $\sqrt[3]{108}$, so their values are exactly the same. We can replace one expression by the other without losing accuracy.

Checkpoint 6.73 **Practice 1.** Simplify $\sqrt[4]{80}$

- ⊙ 20
- ⊙ $\sqrt{20}$
- ⊙ $4\sqrt[4]{5}$
- ⊙ $2\sqrt[4]{5}$

Answer. Choice 4

Solution. $\sqrt[4]{80} = \sqrt[4]{16 \cdot 5} = \sqrt[4]{16}\sqrt[4]{5} = 2\sqrt[4]{5}$

We can also simplify radicals containing variables. If the exponent on the variable is a multiple of the index, we can extract the variable from the radical. For instance,

$$\sqrt[3]{12} = x^{12/3} = x^4$$

(You can verify this by noting that $(x^4)^3 = x^{12}$.)

Caution 6.74 Do not confuse $\sqrt{a^{16}}$ with $\sqrt{16}$. Compare the two radicals:

$$\sqrt{16} = 4 \qquad \text{but} \qquad \sqrt{a^{16}} = a^8$$

Example 6.75

Simplify $\sqrt[3]{x^{11}}$

Solution. Because the exponent on the variable, 11, is not a multiple of the index, 3, we factor out the highest power that is a multiple.

$$\begin{aligned} \sqrt[3]{x^{11}} &= \sqrt[3]{x^9 \cdot x^2} && \text{Apply the Product Rule.} \\ &= \sqrt[3]{x^9} \cdot \sqrt[3]{x^2} && \text{Simplify } \sqrt[3]{x^9} = x^{9/3}. \\ &= x^3\sqrt[3]{x^2} \end{aligned}$$

Checkpoint 6.76 **Practice 2.** Simplify $\sqrt[4]{a^{15}}$

- ⊙ $3\sqrt[4]{a^3}$
- ⊙ $a^3\sqrt[4]{a^3}$
- ⊙ $a\sqrt[4]{11}$
- ⊙ $a^{12}\sqrt[4]{a^3}$

Answer. Choice 2

Solution. $\sqrt[4]{a^{15}} = \sqrt[4]{a^{12} \cdot a^3} = \sqrt[4]{a^{12}}\sqrt[4]{a^3} = a^3\sqrt[4]{a^3}$

To simplify a root of a monomial, we factor the coefficient and each power of a variable separately.

Example 6.77

Simplify each radical.

a $\sqrt{18x^5}$

b $\sqrt[3]{24x^6y^8}$

Solution.

a The index of the radical is 2, so we look for perfect square factors of $18x^5$. The factor 9 is a perfect square, and x^4 has an exponent divisible by 2. Thus,

$$\begin{aligned} \sqrt{18x^5} &= \sqrt{9x^4 \cdot 2x} && \text{Apply the Product Rule.} \\ &= \sqrt{9x^4}\sqrt{2x} && \text{Take square roots.} \\ &= 3x^2\sqrt{2x} \end{aligned}$$

b The index of the radical is 3, so we look for perfect cube factors of $24x^6y^8$. The factor 8 is a perfect cube, and x^6 and y^6 have exponents divisible by 3. Thus,

$$\begin{aligned} \sqrt[3]{24x^6y^8} &= \sqrt[3]{8x^6y^6 \cdot 3y^2} && \text{Apply the Product Rule.} \\ &= \sqrt[3]{8x^6y^6}\,\sqrt[3]{3y^2} && \text{Take cube roots.} \\ &= 2x^2y^2\,\sqrt[3]{3y^2} \end{aligned}$$

Checkpoint 6.78 Practice 3. Simplify $\sqrt[3]{250b^7}$

- ⊙ $5b^2\sqrt[3]{2b}$
- ⊙ $25b^4\sqrt[3]{2b}$
- ⊙ $5b^3\sqrt[3]{10b}$
- ⊙ $5b^6\sqrt[3]{10b}$

Answer. Choice 1

Solution. $\sqrt[3]{250b^7} = \sqrt[3]{125b^6 \cdot 2b} = \sqrt[3]{125b^6}\,\sqrt[3]{2b} = 5b^2\,\sqrt[3]{2b}$

Caution 6.79 It is worth stating again that the Product Rule applies only to *products* under the radical, not to sums or differences. For example,

$$\sqrt{4 \cdot 9} = \sqrt{4}\sqrt{9} = 2 \cdot 3, \quad \text{but} \quad \sqrt{4+9} \neq \sqrt{4} + \sqrt{9}$$

and

$$\sqrt[3]{x^3y^6} = \sqrt[3]{x^3}\,\sqrt[3]{y^6} = xy^2, \quad \text{but} \quad \sqrt[3]{x^3 - y^6} \neq x - y^2$$

Checkpoint 6.80 QuickCheck 2. True or false.

a. To simplify a radical means to find a decimal approximation. (☐ True ☐ False)

b. $\sqrt{x^{16}} = x^4$ (☐ True ☐ False)

c. We cannot simplify $\sqrt[3]{x^{10}}$. (☐ True ☐ False)

d. $\sqrt{83^5} = 83^2\sqrt{83}$ (☐ True ☐ False)

Answer 1. False
Answer 2. False
Answer 3. False
Answer 4. True
Solution.

a. False

b. False

c. False

d. True

Sums and Differences of Radicals

How can we add or subtract radicals? Keep in mind that, in general,

$$\sqrt[n]{a} + \sqrt[n]{b} \neq \sqrt[n]{a+b}$$

We cannot add or subtract expressions that occur under a radical. For example, you can check that

$$\sqrt{16} - \sqrt{4} \neq \sqrt{12} \qquad \text{and} \qquad \sqrt{7} + \sqrt{7} \neq \sqrt{14}$$

However, if two roots have the same index and identical radicands, they are said to be **like radicals**. We can add or subtract like radicals in the same way that we add or subtract like terms, namely by adding or subtracting their coefficients. For example, we know that

$$2r + 3r = (2+3)r = 5r$$

where r is a variable that can stand for any real number. In particular, if $r = \sqrt{x}$, we have

$$2\sqrt{x} + 3\sqrt{x} = (2+3)\sqrt{x} = 5\sqrt{x}$$

So we may add like radicals by adding their coefficients. The same idea applies to subtraction.

Like Radicals.

To add or subtract like radicals, we add or subtract their coefficients. We do not change the index or the radicand.

Example 6.81

a $3\sqrt{3} + 4\sqrt{3} = (3+4)\sqrt{3} = 7\sqrt{3}$

b $4\sqrt[3]{y} - 6\sqrt[3]{y} = (4-6)\sqrt[3]{y} = -2\sqrt[3]{y}$

Caution 6.82

1 In Example 6.81, p. 420a, $3\sqrt{3} + 4\sqrt{3} \neq 7\sqrt{6}$. Only the coefficients are added; the radicand does not change.

2 Sums of radicals with different radicands or different indices cannot be combined. Thus,

$\sqrt{11} + \sqrt{5} \neq \sqrt{16}$	Radicands are not the same.
$\sqrt[3]{10x} - \sqrt[3]{2x} \neq \sqrt[3]{8x}$	Radicands are not the same.
$\sqrt[3]{7} + \sqrt{7} \neq \sqrt[5]{7}$	Indices are not the same.

None of the expressions above can be simplified.

Checkpoint 6.83 **Practice 4.** Simplify $8\sqrt[3]{5b} + 4\sqrt[3]{5b} =$ ___________

Answer. $12\sqrt[3]{5b}$

Solution. $12\sqrt[3]{5b}$

Checkpoint 6.84 QuickCheck 3. True or false.

a. We combine like radicals the same way we combine like terms: by adding or subtracting their coefficients. (☐ True ☐ False)

b. To add radicals with different indices, we multiply the indices. (☐ True ☐ False)

c. $3\sqrt{5}+6\sqrt{5}=9\sqrt{10}$ (☐ True ☐ False)

d. Like radicals must have identical coefficients. (☐ True ☐ False)

Answer 1. True
Answer 2. False
Answer 3. False
Answer 4. False
Solution.

a. True

b. False

c. False

d. False

Sometimes we must simplify the roots in a sum or difference before we can recognize like radicals.

Example 6.85

Simplify $\sqrt[3]{40x^2}-3\sqrt[3]{16x^2}+\sqrt[3]{54x^2}$

Solution. We simplify each cube root by factoring perfect cubes from the radicals.

$$\begin{aligned}
&\sqrt[3]{40x^2}-3\sqrt[3]{16x^2}+\sqrt[3]{54x^2}=\\
&\quad=\sqrt[3]{\mathbf{8}\cdot 5x^2}-3\sqrt[3]{\mathbf{8}\cdot 2x^2}+\sqrt[3]{\mathbf{27}\cdot 2x^2} && \text{Extract roots.}\\
&\quad=\mathbf{2}\sqrt[3]{5x^2}-3\cdot\mathbf{2}\sqrt[3]{2x^2}+\mathbf{3}\sqrt[3]{2x^2}\\
&\quad=2\sqrt[3]{5x^2}-6\sqrt[3]{2x^2}+3\sqrt[3]{2x^2} && \text{Combine like radicals.}\\
&\quad=2\sqrt[3]{5x^2}-3\sqrt[3]{2x^2}
\end{aligned}$$

Checkpoint 6.86 Practice 5. Simplify $\sqrt{12x}+\sqrt{8x}-4\sqrt{18x}=$ ________
Answer. $2\sqrt{3x}-10\sqrt{2x}$
Solution. $2\sqrt{3x}-10\sqrt{2x}$

Products and Quotients of Radicals

Because of the Product and Quotient Rules, we can multiply or divide radicals of the same index.

$$\sqrt[n]{a}\sqrt[n]{b}=\sqrt[n]{ab} \qquad (a,b\geq 0)$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}} \qquad (a\geq 0, b>0)$$

These rules tell us that for products and quotients, the radicands do not have to be the same; only the indices must match.

Example 6.87

Simplify.

a $\sqrt[4]{6x^2}\sqrt[4]{8x^3}$

b $\dfrac{\sqrt[3]{16y^5}}{\sqrt[3]{y^2}}$

Solution.

a We apply the Product Rule to write the product as a single radical, then simplify.

$$\begin{aligned}\sqrt[4]{6x^2}\sqrt[4]{8x^3} &= \sqrt[4]{48x^5} && \text{Factor out perfect fourth powers.}\\ &= \sqrt[4]{16x^4}\sqrt[4]{3x} && \text{Simplify.}\\ &= 2x\sqrt[4]{3x}\end{aligned}$$

b We apply the Quotient Rule to write the quotient as a single radical, then reduce the fraction under the radical.

$$\begin{aligned}\frac{\sqrt[3]{16y^5}}{\sqrt[3]{y^2}} &= \sqrt[3]{\frac{16y^5}{y^2}} && \text{Reduce.}\\ &= \sqrt[3]{16y^3} && \text{Simplify: factor out perfect cubes.}\\ &= \sqrt[3]{8y^3}\sqrt[3]{2} = 2y\sqrt[3]{2}\end{aligned}$$

Checkpoint 6.88 **Practice 6.** Simplify.

a. $\sqrt{\dfrac{18x^5}{25y^4}} =$ ____________

b. $\sqrt[3]{2a^2}\sqrt[3]{6a^2} =$ ____________

Answer 1. $\frac{3x^2\sqrt{2x}}{5y^2}$

Answer 2. $a\sqrt[3]{12a}$

Solution.

a. $\dfrac{3x^2\sqrt{2x}}{5y^2}$

b. $a\sqrt[3]{12a}$

For products involving binomials, we apply the distributive law.

Example 6.89

Expand each product.

a $\sqrt{3}\left(\sqrt{2x}+\sqrt{6}\right)$

b $\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)$

Solution.

a We multiply each term within the parentheses by $2\sqrt{3}$.

$$\begin{aligned}&\mathbf{2\sqrt{3}}\left(\sqrt{2x}+\sqrt{6}\right) =\\ &= \mathbf{2\sqrt{3}}\cdot\sqrt{2x} + \mathbf{2\sqrt{3}}\cdot 5\sqrt{6} && \text{Apply the Product Rule.}\end{aligned}$$

$$= \mathbf{2}\sqrt{\mathbf{3} \cdot 2x} + \mathbf{2} \cdot 5\sqrt{\mathbf{3} \cdot 6} \qquad \text{Simplify the radicals.}$$
$$= 2\sqrt{6x} + 10\sqrt{18}$$
$$= 2\sqrt{6x} + 10 \cdot \sqrt{9}\sqrt{2}$$
$$= 2\sqrt{6x} + 30\sqrt{2}$$

b We use the "FOIL" method to expand the product.

$$\left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right) = \sqrt{x}\sqrt{x} + \sqrt{x}\sqrt{y} - \sqrt{x}\sqrt{y} - \sqrt{y}\sqrt{y}$$
$$= \sqrt{x^2} + \sqrt{xy} - \sqrt{xy} - \sqrt{y^2}$$
$$= x - y$$

Note 6.90 In part (a) of the Example, p. 422 above, observe that

$$2\sqrt{3} \cdot 5\sqrt{6} = 2 \cdot 5 \cdot \sqrt{3} \cdot \sqrt{6} = 10\sqrt{18}$$

We multiply together any expressions outside the radical, and apply the product rule to expressions under the radical.

Checkpoint 6.91 **Practice 7.** Expand $(\sqrt{5} - 2\sqrt{3})^2 =$ ____________

Answer. $17 - 4\sqrt{15}$

Solution. $17 - 4\sqrt{15}$

Checkpoint 6.92 **QuickCheck 3.** True or false.

a. We can only simplify products or quotients of like radicals. (☐ True ☐ False)

b. $4\left(3\sqrt{5}\right) = 3 \cdot 4 + 3\sqrt{5}$ (☐ True ☐ False)

c. $(\sqrt{3} + \sqrt{5})^2 = 3 + 5 = 8$ (☐ True ☐ False)

d. $(3\sqrt{x})^2 = 9x$ (☐ True ☐ False)

Answer 1. False

Answer 2. False

Answer 3. False

Answer 4. True

Solution.

a. False

b. False

c. False

d. True

Rationalizing the Denominator

It is easier to work with radicals if there are no roots in the denominators of fractions. We can use the fundamental principle of fractions to remove radicals from the denominator. This process is called **rationalizing the denominator**. For square roots, we multiply the numerator and denominator of the fraction by the radical in the denominator.

Example 6.93

Rationalize the denominator of each fraction.

a $\sqrt{\dfrac{1}{3}}$

b $\dfrac{\sqrt{2}}{\sqrt{50x}}$

Solution.

a First, we apply the Quotient Rule to write the radical as a quotient.

$$\sqrt{\frac{1}{3}} = \frac{\sqrt{1}}{\sqrt{3}} = \frac{1}{\sqrt{3}} \quad \text{Multiply numerator and denominator by } \sqrt{3}.$$
$$= \frac{1 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$$
$$= \frac{\sqrt{3}}{3}$$

b It is always best to simplify the denominator before rationalizing.

$$\frac{\sqrt{2}}{\sqrt{50x}} = \frac{\sqrt{2}}{5\sqrt{2x}}$$

Now we can see that we should multiply numerator and denominator by $\sqrt{2x}$ — not $\sqrt{50x}$!

$$\frac{\sqrt{2} \cdot \sqrt{2x}}{5\sqrt{2x} \cdot \sqrt{2x}} = \frac{\sqrt{4x}}{5(2x)} \quad \text{Apply the Product Rule.}$$
$$= \frac{2\sqrt{x}}{10x} = \frac{\sqrt{x}}{5x} \quad \text{Simplify.}$$

Checkpoint 6.94 Practice 8. Rationalize the denominator of $\dfrac{-\sqrt{3}}{\sqrt{7}} =$ _____

Answer. $\frac{-(\sqrt{21})}{7}$

Solution. $\dfrac{-\sqrt{21}}{7}$

If the denominator of a fraction is a *binomial* in which one or both terms is a radical, we can use a special building factor to rationalize it. First, recall that

$$(p-q)(p+q) = p^2 - q^2$$

In particular, then,

$$(\sqrt{b} - \sqrt{c})(\sqrt{b} + \sqrt{c}) = (\sqrt{b})^2 - (\sqrt{c})^2 = b - c$$

The product contains no radicals. Each of the two factors $\sqrt{b}-\sqrt{c}$ and $\sqrt{b}+\sqrt{c}$ is said to be the **conjugate** of the other.

Now consider a fraction of the form

$$\frac{a}{b + \sqrt{c}}$$

If we multiply the numerator and denominator of this fraction by the conjugate of the denominator, we get

$$\frac{a\mathbf{(b-\sqrt{c})}}{(b+\sqrt{c})\mathbf{(b-\sqrt{c})}}=\frac{ab-a\sqrt{c}}{b^2-(\sqrt{c})^2}=\frac{ab-a\sqrt{c}}{b^2-c}$$

The denominator of the fraction no longer contains any radicals—it has been rationalized.

Example 6.95

Rationalize the denominator: $\dfrac{x}{\sqrt{2}+\sqrt{x}}$.

Solution. We multiply numerator and denominator by the conjugate of the denominator, $\sqrt{2}-\sqrt{x}$.

$$\frac{x(\mathbf{\sqrt{2}-\sqrt{x}})}{(\sqrt{2}+\sqrt{x})(\mathbf{\sqrt{2}-\sqrt{x}})}=\frac{x(\sqrt{2}-\sqrt{x})}{2-x}$$

Checkpoint 6.96 Practice 9. Rationalize the denominator of $\dfrac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}=$

Answer. $3+\sqrt{6}$

Solution. $3+\sqrt{6}$

Checkpoint 6.97 QuickCheck 3. Fill in the blanks.

a. We rationalize the denominator to remove (☐ fractions ☐ decimals ☐ radicals ☐ variables) from the denominator.

b. To rationalize a binomial denominator, we multiply by its (☐ opposite ☐ reciprocal ☐ conjugate ☐ inverse) .

c. Before rationalizing, it is always best to (☐ simplify ☐ cross-multiply ☐ invert ☐ FOIL) .

d. $\dfrac{5}{\sqrt{5}}=$ _____ power.

Answer 1. radicals

Answer 2. conjugate

Answer 3. simplify

Answer 4. $\sqrt{5}$

Solution.

a. radicals

b. conjugate

c. simplify

d. $\sqrt{5}$

Problem Set 6.4

Warm Up

Each question in Problems 1–4 is followed by three examples. Use your calculator to decide if the examples are true or false, then circle the correct answer.

1. Is $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$? Yes or No

a $\sqrt{12} = \sqrt{8} + \sqrt{4}$

b $\sqrt{5} = \sqrt{2} + \sqrt{3}$

c $\sqrt{25} = \sqrt{9} + \sqrt{16}$

2. Is $\sqrt{a-b} = \sqrt{a} - \sqrt{b}$? Yes or No

a $\sqrt{36} = \sqrt{100} - \sqrt{64}$

b $\sqrt{20} = \sqrt{16} - \sqrt{4}$

c $\sqrt{9} = \sqrt{10} - \sqrt{1}$

3. Is $\sqrt{ab} = \sqrt{a}\sqrt{b}$? Yes or No

a $\sqrt{36} = \sqrt{9}\sqrt{4}$

b $\sqrt{12} = \sqrt{3}\sqrt{4}$

c $\sqrt{15} = \sqrt{3}\sqrt{5}$

4. Is $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$? Yes or No

a $\sqrt{\dfrac{144}{9}} = \dfrac{\sqrt{144}}{\sqrt{9}}$

b $\sqrt{\dfrac{64}{5}} = \dfrac{\sqrt{64}}{\sqrt{5}}$

c $\sqrt{\dfrac{5}{8}} = \dfrac{\sqrt{5}}{\sqrt{8}}$

5. Which of the four possible properties listed above are true, and which are false? Write the true properties below:

6. True or False. State which property (or non-property) above supports your answer.

(a) $\sqrt{9b^4} = \sqrt{9}\sqrt{b^4}$

(b) $\sqrt{4-x^2} = \sqrt{4} - \sqrt{x^2}$

(c) $\sqrt{\dfrac{3a}{16}} = \dfrac{\sqrt{3a}}{\sqrt{16}}$

(d) $\sqrt{2+w} = \sqrt{2} + \sqrt{w}$

(e) $\sqrt{\dfrac{x}{9}} = \dfrac{\sqrt{x}}{3}$

(f) $\sqrt{x^2+y^2} = x + y$

(g) $\sqrt{1-25b^2} = 1 - 5b$

(h) $\sqrt{7m^2} = m\sqrt{7}$

Skills Practice

For Problems 7–9, simplify. Assume that all variables represent positive numbers.

7.

a $\sqrt{18}$ b $\sqrt[3]{24}$ c $-\sqrt[4]{64}$

8.

a $\sqrt{60,000}$ b $\sqrt[3]{900,000}$ c $\sqrt[3]{\dfrac{-40}{27}}$

9.

a $\sqrt[3]{x^{10}}$ b $\sqrt{27z^3}$ c $\sqrt[4]{48a^9}$

For Problems 10–13, simplify.

10.

a $-\sqrt{18s}\sqrt{2s^3}$

b $\sqrt[3]{7h^2}\,\sqrt[3]{-49h}$

11.

a $\sqrt{16-4x^2}$

b $\sqrt[3]{8A^3+A^6}$

12.

a $\dfrac{\sqrt{45x^3y^3}}{\sqrt{5y}}$

b $\dfrac{\sqrt[3]{8b^7}}{\sqrt[3]{a^6b^2}}$

13.

a $\dfrac{ab}{\sqrt[3]{a^2b}}$

b $\dfrac{\sqrt{x^2y}}{\sqrt{xy}}$

For Problems 14-16, simplify and combine like terms.

14.

a $4\sqrt{3}+\sqrt{27}$

b $\sqrt{50x}+\sqrt{32x}$

15.

a $\sqrt[3]{81}+2\sqrt[3]{24}-3\sqrt[3]{3}$

b $3\sqrt[3]{16}-\sqrt[3]{2}-2\sqrt[3]{54}$

16.

a $2v\sqrt{v}+3\sqrt{v^3}-\dfrac{v^2}{\sqrt{v}}$

b $12\sqrt[3]{w^2}-\sqrt[3]{w}\left(-2\sqrt[3]{w}\right)+\dfrac{8w}{\sqrt[3]{w}}$

For Problems 17–20, multiply.

17.

a $3\sqrt{2}\left(\sqrt{6}+\sqrt{10}\right)$

b $\sqrt{3k}\left(3\sqrt{k}-k\sqrt{6k}\right)$

18.

a $\sqrt[3]{2}\left(\sqrt[3]{20}-2\sqrt[3]{12}\right)$

b $\sqrt[3]{3}\left(2\sqrt[3]{18}+\sqrt[3]{36}\right)$

19.

a $(\sqrt{x}-3)(\sqrt{x}+3)$

b $\left(\sqrt{2}-\sqrt{3}\right)\left(\sqrt{2}+2\sqrt{3}\right)$

20.

a $\left(\sqrt{5}-\sqrt{2}\right)^2$

b $\left(\sqrt{a}-2\sqrt{b}\right)^2$

21. Reduce if possible.

a $\dfrac{9-3\sqrt{5}}{3}$ b $\dfrac{-8+\sqrt{8}}{4}$ c $\dfrac{6a-\sqrt{18}}{6a}$

22. Write each expression as a single fraction in simplest form.

a $\dfrac{5}{4}+\dfrac{3\sqrt{2}}{2}$ b $\dfrac{3}{2a}+\dfrac{\sqrt{3}}{6a}$ c $\dfrac{3\sqrt{3}}{2}+3$

For Problems 23–26, rationalize the denominator.

23.

a $\sqrt{\dfrac{7x}{18}}$

b $\sqrt{\dfrac{2a}{b}}$

24.

a $\dfrac{6\sqrt{2}}{\sqrt{12v}}$

b $\dfrac{2\sqrt{3}}{\sqrt{2k}}$

25.

a $\dfrac{4}{1+\sqrt{3}}$

b $\dfrac{x}{x-\sqrt{3}}$

26.

a $\dfrac{\sqrt{6}-3}{2-\sqrt{6}}$

b $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

Applications

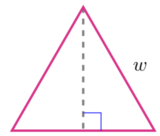
For Problems 27 and 28, verify by substitution that the number is a solution of the quadratic equation.

27. $x^2 + 4x - 1 = 0, \quad -2 + \sqrt{5}$

28. $4x^2 - 20x + 22 = 0, \quad \dfrac{5 - \sqrt{3}}{2}$

29.

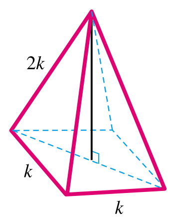
a Write an expression for the height of an equilateral triangle of side w.



b Write an expression for the area of the triangle.

30.

a Write an expression in terms of k for the height of the pyramid shown below.



b Write an expression in terms of k for the volume of the pyramid.

Radical Equations

Solving a Radical Equation

A **radical equation** is one in which the variable appears under a square root or other radical. We solve simple radical equations by raising both sides to the appropriate power. For example, to solve the equation

$$\sqrt{x+3} = 4$$

we square both sides to find

$$\begin{aligned}(\sqrt{x+3})^2 &= 4^2 \\ x + 3 &= 16\end{aligned}$$

You can check that $x = 13$ is the solution for this equation.

Example 6.98

Solve $4\sqrt[3]{x-9} = 12$

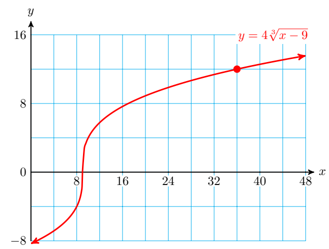
Solution. We first divide both sides of the equation by 4 to isolate the radical.

$$\sqrt[3]{x-9} = 3$$

Next, we cube both sides of the equation.

$$\left(\sqrt[3]{x-9}\right)^3 = 3^3$$
$$x - 9 = 27$$
$$x = 36$$

The solution is 36. We can also solve the equation graphically by graphing $y = 4\sqrt[3]{x-9}$, as shown in the figure. The point $(36, 12)$ lies on the graph, so $x = 36$ is the solution of the equation $4\sqrt[3]{x-9} = 12$



Checkpoint 6.99 Practice 1. Solve $6 + 2\sqrt[4]{12-v} = 10$

$v =$__

Hint:

- Isolate the radical.
- Raise each side to the fourth power.
- Complete the solution.

Answer. -4

Solution. $v = -4$

Extraneous Solutions

Whenever we raise both sides of an equation to an even power, it is possible to introduce false or **extraneous solutions**. For example, the equation

$$\sqrt{x} = -5$$

has no solution, because $\sqrt{x}$ is never a negative number. However, if we try to solve the equation by squaring both sides, we find

$$\left(\sqrt{x}\right)^{\mathbf{2}} = (-5)^{\mathbf{2}}$$
$$x = 25$$

You can check that 25 is *not* a solution to the original equation, $\sqrt{x} = -5$, because $\sqrt{25}$ does not equal -5.

Raising both sides of an equation to an odd power does not introduce extraneous solutions. However, if we raise both sides to an even power, we should check each solution in the original equation.

Checkpoint 6.100 QuickCheck 1. When we raise both sides of an equation to an (☐ even ☐ odd) power, it is possible to introduce (☐ essential ☐ extraneous) solutions.

Answer 1. even

Answer 2. extraneous

Solution. even; extraneous

Example 6.101

Solve the equation $\sqrt{x+2}+4=x$

Solution. First, we isolate the radical expression on one side of the equation. (This will make it easier to square both sides.)

$$\sqrt{x+2} = x-4 \qquad \text{Square both sides of the equation.}$$
$$\left(\sqrt{x+2}\right)^{\mathbf{2}} = (x-4)^{\mathbf{2}}$$
$$x+2 = x^2-8x+16 \qquad \text{Subtract } x+2 \text{ from both sides.}$$
$$x^2-9x+14=0 \qquad \text{Factor the left side.}$$
$$x=2 \quad \text{or} \quad x=7$$

Check

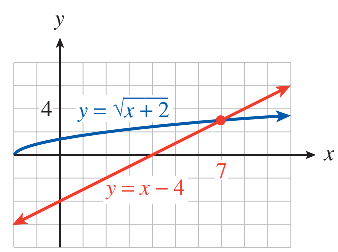
Does $\sqrt{\mathbf{2}+2}+4=\mathbf{2}$? No; 2 is not a solution.

Does $\sqrt{\mathbf{7}+2}+4=\mathbf{7}$? Yes; 7 is a solution.

The apparent solution 2 is extraneous. The only solution to the original equation is 7. We can verify the solution by graphing the equations

$$y_1=\sqrt{x+2} \quad \text{and} \quad y_2=x-4$$

as shown at right. The graphs intersect in only one point, $(7,3)$, so there is only one solution, $x=7$.



Caution 6.102 When we square both sides of an equation, it is *not* correct to square each term of the equation separately. Thus, in Example 6.101, p. 430, the original equation is *not* equivalent to

$$(\sqrt{x+2})^2+4^2=x^2 \qquad \text{Incorrect!}$$

This is because $(a+b)^2 \neq a^2+b^2$. Instead, we must square the *entire* left side of the equation as a binomial, like this,

$$(\sqrt{x+2}+4)^2=x^2$$

or we may proceed as shown in Example 6.101, p. 430.

Checkpoint 6.103 Practice 2. Solve $2x-5=\sqrt{40-3x}$

$x =$__

Hint:

- Square both sides.
- Solve the quadratic equation.
- Check for extraneous roots.

Answer. 5

Solution. $x=5$

Checkpoint 6.104 QuickCheck 2. True or false.

a. At most one of the solutions of a radical equation can be extraneous. (☐ True ☐ False)

b. We don't have to check for extraneous solutions if we cube both sides of an equation. (☐ True ☐ False)

c. We can square each term of an equation separately without changing its solutions. (☐ True ☐ False)

d. It is helpful to isolate the radical before squaring both sides. (☐ True ☐ False)

Answer 1. False

Answer 2. True

Answer 3. False

Answer 4. True

Solution.

a. False

b. True

c. False

d. True

Solving Formulas

We can also solve formulas involving radicals for one variable in terms of the others.

Example 6.105

Solve the formula $t = \sqrt{1+s^2}$ for s.

Solution. Because the variable we want is under a square root, we square both sides of the equation, to get

$$\begin{aligned} t^2 &= 1 + s^2 && \text{Subtract 1 from both sides.} \\ t^2 - 1 &= s^2 && \text{Take square roots.} \\ s &= \pm\sqrt{t^2 - 1} \end{aligned}$$

Checkpoint 6.106 Practice 3. Solve the formula $r - 2 = \sqrt[3]{V - Bh}$ for h.

$h =$ ______________

Answer. $\frac{V-(r-2)^3}{B}$

Solution. $h = \dfrac{V - (r-2)^3}{B}$

Equations with More than One Radical

Sometimes we need to square both sides of an equation more than once in order to eliminate all the radicals.

Example 6.107

Solve $\sqrt{x-7}+\sqrt{x}=7$.

Solution. First, we isolate the more complicated radical on one side of the equation. (This will make it easier to square both sides.) We subtract $\sqrt{x}$ from both sides to get

$$\sqrt{x-7}=7-\sqrt{x}$$

Now we square each side to remove one radical. Be careful when squaring the binomial $7-\sqrt{x}$.

$$\begin{aligned}(\sqrt{x-7})^{\mathbf{2}} &= (7-\sqrt{x})^{\mathbf{2}}\\ x-7 &= 49-14\sqrt{x}+x\end{aligned}$$

We collect like terms, and isolate the radical on one side of the equation.

$$\begin{aligned}-56 &= -14\sqrt{x} && \text{Divide both sides by } -14.\\ 4 &= \sqrt{x}\end{aligned}$$

Finally, we square both sides again to obtain

$$\begin{aligned}(4)^{\mathbf{2}} &= (\sqrt{x})^{\mathbf{2}}\\ 16 &= x\end{aligned}$$

Check

Does $\sqrt{\mathbf{16}-7}+\sqrt{\mathbf{16}}=7$? Yes. The solution is 16.

Caution 6.108 Recall that we cannot solve a radical equation by squaring each term separately. So it is *incorrect* to begin Example 107, p. 432 by writing

$$(\sqrt{x-7})^2+(\sqrt{x})^2=7^2 \qquad \text{Incorrect!}$$

We must square the *entire expression* on each side of the equal sign as one piece.

Checkpoint 6.109 **Practice 4.** Solve $\sqrt{3x+1}=6-\sqrt{9-x}$

$x =$_____

Hint:

- Square both sides.
- Isolate the radical term.
- Divide both sides by 4.
- Square both sides again.
- Solve the quadratic equation.
- Check for extraneous roots.

Answer. $5, 8$

Solution. $x=5$; $x=8$

Simplifying $\sqrt[n]{a^n}$

We have seen that raising to a power is the inverse operation for extracting roots, so that

$$(\sqrt[n]{a})^n = a$$

For example,

$$(\sqrt[4]{16})^4 = 16 \qquad \text{and} \qquad (\sqrt[5]{x})^5 = x$$

What if the power and root operations occur in the opposite order? Is it always true that $\sqrt[n]{a^n} = a$?

First, consider the case where the index is an odd number. For example,

$$\sqrt[3]{2^3} = \sqrt[3]{8} = 2 \qquad \text{and} \qquad \sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$$

Because every real number has exactly one nth root if n is odd, we see that,

$$\sqrt[n]{a^n} = a, \qquad \text{for } n \text{ odd}$$

However, if n is even, then a^n is positive, regardless of whether a itself is positive or negative, and hence $\sqrt[n]{a^n}$ is positive also. For example, if $a = -3$, then

$$\sqrt{(-3)^2} = \sqrt{9} = 3$$

In this case, $\sqrt{a^2}$ does not equal a, because a is negative but $\sqrt{a^2}$ is positive. We must be careful when taking even roots of powers. We have the following special relationship for even roots.

$$\sqrt[n]{a^n} = |a|, \qquad \text{for } n \text{ even}$$

We summarize our results in the box below.

Roots of Powers.

$\sqrt[n]{a^n} = a$	If n is odd.
$\sqrt[n]{a^n} = \lvert a\rvert$	If n is even.

In particular, note that it is not always true that $\sqrt{a^2} = a$, unless we know that $a \geq 0$. Otherwise, we can only assume that $\sqrt{a^2} = |a|$.

Example 6.110

a $\sqrt{16x^2} = 4|x|$ b $\sqrt{(x-1)^2} = |x-1|$

Checkpoint 6.111 Practice 5. Simplify.

a. $\sqrt[3]{-125x^6} =$________

b. $\sqrt[4]{16x^{12}} =$________

Answer 1. $-5x^2$

Answer 2. $2|x^3|$

Solution.

a. $-5x^2$

b. $2\left|x^3\right|$

Checkpoint 6.112 QuickCheck 3. True or false.

a. If an equation contains more than one radical, we square each term of the equation. (☐ True ☐ False)

b. $(7 - \sqrt{x})^2 = 49 - x$ (☐ True ☐ False)

c. If $\sqrt{x^2} \neq x$, then $\sqrt{x^2} = -x$. (☐ True ☐ False)

d. If n is even and $a \neq 0$, then a^n is always positive. (☐ True ☐ False)

Answer 1. False
Answer 2. False
Answer 3. True
Answer 4. True
Solution.

a. False

b. False

c. True

d. True

Problem Set 6.5

Warm Up

1. Solve the equation. Isolate the radical first, then square both sides of the equation.

a $-3\sqrt{z} + 14 = 8$

b $8 - 3\sqrt{9 + 2w} = -7$

2. Square the binomial.

a $(3x + 4)^2$

b $(2\sqrt{x} - 3)^2$

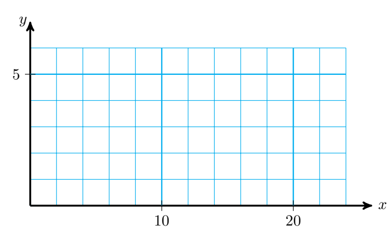
c $(4 - \sqrt{x + 3})^2$

d $(5 - 2\sqrt{2x - 5})^2$

3.

a Complete the table of values and graph $y = \sqrt{x - 4}$.

x	y
4	
5	
6	
10	
16	
19	
24	

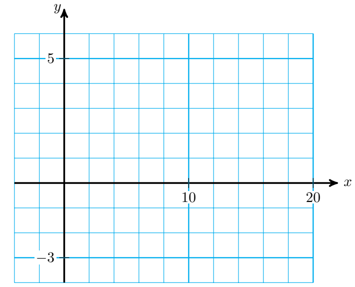


b Solve $\sqrt{x - 4} = 3$ graphically and algebraically. Do your answers agree?

4.

a Complete the table of values and graph $y = 4 - \sqrt{x + 3}$.

x	y
-3	
-2	
0	
1	
4	
8	
16	

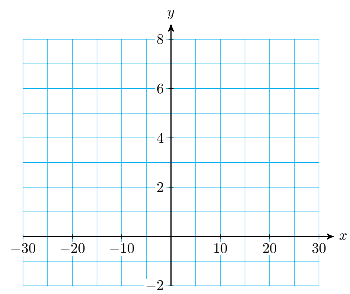


b Solve $4 - \sqrt{x+3} = 1$ graphically and algebraically. Do your answers agree?

5.

a Complete the table of values and graph $y = 4 - \sqrt[3]{x}$.

x	y
-25	
-20	
-15	
-10	
-5	
0	
5	
10	
15	
20	

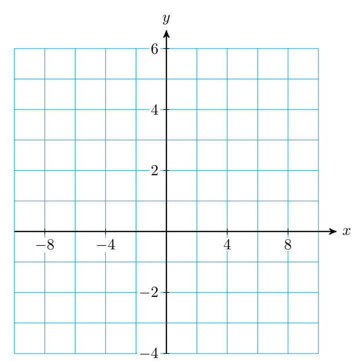


b Solve $4 - \sqrt[3]{x} = 6$ graphically and algebraically. Do your answers agree?

6.

a Complete the table of values and graph $y = 3 + \sqrt[3]{x-3}$.

x	y
-8	
-6	
-4	
-2	
0	
2	
4	
6	
8	



b Solve $3 + \sqrt[3]{x-3} = 1$ graphically and algebraically. Do your answers agree?

Skills Practice

For Problems 7–16, solve.

7. $3z + 4 = \sqrt{3z+10}$

8. $2x + 1 = \sqrt{10x+5}$

9. $\sqrt{2y-1} = \sqrt{3y-6}$

10. $\sqrt{x - 3\sqrt{x}} = 2$

11. $\sqrt{y+4} = \sqrt{y+20} - 2$

12. $\sqrt{x} + \sqrt{2} = \sqrt{x+2}$

13. $\sqrt[3]{2x-5} - 1 = 2$

14. $16 = 12 - \sqrt[3]{x+24}$

15. $15 - 2\sqrt[3]{x-4} = 9$

16. $2 = 8 - 3\sqrt[3]{x^3+1}$

For Problems 17 and 18, use absolute value bars as necessary to simplify the radicals.

17.

a $\sqrt{4x^2}$ b $\sqrt{(x-5)^2}$ c $\sqrt{x^2-6x+9}$

18.

a $\sqrt{9x^2y^4}$ b $\sqrt{(2x-1)^2}$ c $\sqrt{9x^2-6x+1}$

Applications

19. If you are flying in an airplane at an altitude of h miles, on a clear day you can see a distance of h miles to the horizon, where

$$d = 89.4\sqrt{h}$$

a At what altitude will you be able to see for a distance of 100 miles? How high is that in feet?

b Solve the formula for h in terms of d.

20. If a gun is fired vertically into the air, the time it takes the bullet to return to the ground is given approximately by

$$t = \frac{1}{2}\sqrt{\frac{2h}{g}}$$

where h is the greatest height the bullet reaches in meters, and g is the force of gravity, 9.8 meters per second squared.

a If the bullet returns to earth in 12 seconds, what was the greatest height it reached?

b Solve the formula for h in terms of g and t.

21. The height of a cylindrical storage tank is four times its radius. If the tank holds V cubic inches of liquid, its radius in inches is

$$r = \sqrt[3]{\frac{V}{12.57}}$$

a What is the volume of a tank whose radius is 3 inches?

b Solve the formula for V in terms of r.

22. In order for a windmill to generate P watts of power, the velocity of the wind, in miles per hour, must be

$$v = \sqrt[3]{\frac{P}{0.015}}$$

a How much power will a wind speed of 30 mph generate?

b Solve the formula for P in terms of v.

For Problems 23–26, solve the formula for the indicated variable.

23. $c = \sqrt{a^2 - b^2}$, for b

24. $x = a - \sqrt{h(2r - h)}$, for r

25. $D = S\sqrt[3]{1 - \dfrac{v}{W}}$, for W

26. $R = \dfrac{T}{1 - \sqrt[3]{1 - K}}$, for K

Chapter 6 Summary and Review

Glossary

- exponent
- power
- power function
- inverse square law
- scientific notation
- root
- radical
- radicand
- index
- irrational number
- principal root
- radical equation
- rationalize
- conjugate

Key Concepts

1 A positive integer exponent tells us how many copies of its base to multiply together.

Definition of Negative and Zero Exponents.

2

$$a^{-n} = \frac{1}{a^n} \qquad (a \neq 0)$$
$$a^0 = 1 \qquad (a \neq 0)$$

3 A negative exponent denotes a reciprocal, as long as the base is not zero. A negative exponent does not mean that the power is negative.

Power Function.

4 A function of the form

$$f(x) = kx^p$$

where k and p are nonzero constants, is called a **power function**.

Laws of Exponents.

5

I $a^m \cdot a^n = a^{m+n}$

II $\dfrac{a^m}{a^n} = a^{m-n}$

III $(a^m)^n = a^{mn}$

IV $(ab)^n = a^n b^n$

V $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

6 The laws of exponents are used to simplify products and quotients of powers, not for sums or differences of powers. We can combine (add or subtract) like terms, but we cannot combine terms with different exponents into a single term.

To Write a Number in Scientific Notation.

7

1 Locate the decimal point so that there is exactly one nonzero digit to its left.

2 Count the number of places you moved the decimal point: this determines the power of 10.

  a If the original number is greater than 10, the exponent is positive.

  b If the original number is less than 1, the exponent is negative.

nth Roots.

8 s is called an **nth root of** b if $s^n = b$.

Exponential Notation for Radicals.

9 For any integer $n \geq 2$ and for $a \geq 0$,

$$a^{1/n} = \sqrt[n]{a}$$

10 It is not possible to write down an exact decimal equivalent for an irrational number, but we can find an approximation to as many decimal places as we like.

11 To solve an equation involving x^n, we first isolate the power, then raise both sides to the exponent $\dfrac{1}{n}$.

12 We can solve an equation where one side is an nth root of x by raising both sides of the equation to the nth power. We must be careful when raising both sides of an equation to an even power, as extraneous solutions may be introduced.

Roots of Real Numbers.

13

(a) Every positive number has two real-valued roots, one positive and one negative, if the index is even.

(b) A negative number has no real-valued root if the index is even.

(c) Every real number, positive, negative, or zero, has exactly one real-valued root if the index is odd.

14 A fractional exponent represents a power and a root. The denominator of the exponent is the root, and the numerator of the exponent is the power. We will define fractional powers only when the base is a positive number.

15 Direct variation has the following **scaling** property: increasing x by any factor causes y to increase by the same factor.

Rational Exponents.

16

$$a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}, \qquad a > 0, \;\; n \neq 0$$

Rational Exponents and Radicals.

17

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

18 To solve an equation with a fractional exponent we first isolate the power. Then we raise both sides to the reciprocal of the exponent.

Product Rule for Radicals.

19

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}, \qquad \text{for } a, b \geq 0$$

Quotient Rule for Radicals.

20

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, \qquad \text{for } a \geq 0, \;\; b > 0$$

21 In general, it is *not* true that $\sqrt[n]{a+b}$ is equivalent to $\sqrt[n]{a} + \sqrt[n]{b}$, or that $\sqrt[n]{a-b}$ is equivalent to $\sqrt[n]{a} - \sqrt[n]{b}$.

22 We simplify radicals by factoring out any perfect powers from the radicand.

23 We can add or subtract like radicals in the same way that we add or subtract like terms, by adding or subtracting their coefficients.

24 We can use the fundamental principle of fractions to remove radicals from the denominator. This process is called **rationalizing the denominator.**

25 Whenever we raise both sides of an equation to an even power, it is possible to introduce false or extraneous solutions.

Roots of Powers.

26

$\sqrt[n]{a^n} = a$	If n is odd.
$\sqrt[n]{a^n} = \lvert a \rvert$	If n is even.

Chapter 6 Review Problems

For Problems 1–6, write without negative exponents and simplify.

1.
a $(-3)^{-4}$
b 4^{-3}

2.
a $(\frac{1}{2})^{-2}$
b $\dfrac{3}{5^{-2}}$

3.
a $(3m)^{-5}$
b $-7y{-}8$

4.
a $a^{-1} + a^{-2}$
b $\dfrac{3q^{-9}}{r^{-2}}$

5.
a $6c^{-7} \cdot 3^{-1}c^4$
b $\dfrac{11z^{-7}}{3^{-2}z^{-5}}$

6.
a $(2d^{-2}k^3)^{-4}$
b $\dfrac{2w^3(w^{-2})^{-3}}{5w^{-5}}$

7. The speed of light is approximately 186,000 miles per second.

a How long will it take light to travel a distance of 1 foot? (1 mile = 5280 feet) Express your answer in both scientific and standard notation.

b How long does it take sunlight to reach the Earth, a distance of 92,956,000 miles?

8. In April 2020, the national debt was over 24×10^{12} dollars. How many hours would it take you to earn an amount equal to the national debt if you were paid $20 per hour? Express your answer in standard notation, both in terms of hours and in terms of years.

9. In the twenty-first century, spacecraft may be able to travel at speeds of 3×10^7 meters per second, 1000 times their current speed. (At that speed you could circumnavigate the Earth in 1.3 seconds.)

a How long would it take to reach the Sun at this speed? The Sun is approximately 1.496×10^{11} meters from Earth.

b What fraction of the speed of light (3×10^8 meters per second) is this speed?

c How long would it take to reach Proxima Centauri, 4.2 light years from Earth? A light year is the distance that light can travel in one year.

10.

a Use the data in the table to calculate the density of each of the planets as follows: first find the volume of the planet, assuming planets are spherical. Then divide the mass of the planet by its volume.

Planet	Radius (km)	Mass (10^{20} kg)	Density (kg/m^3)
Mercury	2440	3302	
Venus	6052	48,690	
Earth	6378	59,740	
Mars	3397	6419	
Jupiter	71,490	18,990,000	
Saturn	60,270	5,685,000	
Uranus	25,560	866,200	
Neptune	24,765	1,028,000	
Pluto	1150	150	

b The planets are composed of three broad categories of materials: rocky materials, "icy" materials (including water), and the materials that dominate the Sun, namely hydrogen and helium. The density of rock varies from 3000 to 8000 kg/m^3. Which of the planets could be composed mainly of rock?

For Problems 11–14, write each power in radical form.

11.

a $25m^{1/2}$

b $8n^{-1/3}$

12.

a $(13d)^{2/3}$

b $6x^{2/5}y^{3/5}$

13.

a $(3q)^{-3/4}$

b $7(uv)^{3/2}$

14.

a $(a^2+b^2)^{0.5}$

b $(16-x^2)^{0.25}$

For Problems 15–18, write each radical as a power with a fractional exponent.

15.

a $2\sqrt[3]{x^2}$

b $\dfrac{1}{4}\sqrt[4]{x}$

16.

a $z^2\sqrt{z}$

b $z\sqrt[3]{z}$

17.

a $\dfrac{6}{\sqrt[4]{b^3}}$

b $\dfrac{-1}{3\sqrt[3]{b}}$

18.

a $\dfrac{-4}{(\sqrt[4]{a})^2}$

b $\dfrac{2}{(\sqrt{a})^3}$

19. According to the theory of relativity, the mass of an object traveling at velocity v is given by the function

$$m = \frac{M}{\sqrt{1-\dfrac{v^2}{c^2}}}$$

where M is the mass of the object at rest and c is the speed of light. Find the mass of a man traveling at a velocity of $0.7c$ if his rest mass is 80 kilograms.

20. The cylinder of smallest surface area for a given volume has a radius and height both equal to

$$\sqrt[3]{\frac{V}{\pi}}$$

Find the dimensions of the tin can of smallest surface area with volume 60 cubic inches.

21. Two businesswomen start a small company to produce saddle bags for bicycles. The number of saddle bags, q, they can produce depends on the amount of money, m, they invest and the number of hours of labor, w, they employ, according to the Cobb-Douglas formula

$$q = 0.6m^{1/4}w^{3/4}$$

where m is measured in thousands of dollars.

a If the businesswomen invest \$100,000 and employ 1600 hours of labor in their first month of production, how many saddle bags can they expect to produce?

b With the same initial investment, how many hours of labor would they need in order to produce 200 saddle bags?

22. A child who weighs w pounds and is h inches tall has a surface area (in square inches) given approximately by

$$S = 8.5h^{0.35}w^{0.55}$$

a What is the surface area of a child who weighs 60 pounds and is 40 inches tall?

b What is the weight of a child who is 50 inches tall and whose surface area is 397 square inches?

23. Membership in the Wildlife Society has grown according to the function

$$M(t) = 30t^{3/4}$$

where t is the number of years since its founding in 1970.

a Sketch a graph of the function $M(t)$.

b What was the society's membership in 1990?

c In what year will the membership be 810 people?

24. The heron population in Saltmarsh Refuge is estimated by conservationists at

$$P(t) = 360t^{-2/3}$$

where t is the number of years since the refuge was established in 1990.

a Sketch a graph of the function $P(t)$.

b How many heron were there in 1995?

c In what year will there be only 40 heron left?

25. A brewery wants to replace its old vats with larger ones. To estimate the cost of the new equipment, the accountant uses the 0.6 rule for industrial costs. This rule states that the cost of a new container is approximately $N = Cr^{0.6}$, where C is the cost of the old container and r is the ratio of the capacity of the new container to the old one.

a If an old vat cost \$5000, sketch a graph of N as a function of r for $0 \le r \le 5$.

b How much should the accountant budget for a new vat that holds 1.8 times as much as the old one?

26. If a quantity of air expands without changing temperature, its pressure in pounds per square inch is given by $P = kV^{-1.4}$, where V is the volume of the air in cubic inches and $k = 2.79 \times 10^4$.

a Sketch a graph of P as a function of V for $0 \le V \le 100$.

b Find the air pressure of an air sample when its volume is 50 cubic inches.

27. Shipbuilders find that the average cost of producing a ship decreases as more of those ships are produced. This relationship is called the experience curve, and is given by the equation

$$C = ax^{-b}$$

where C is the average cost per ship in millions of dollars and x is the number of ships produced. The value of the constant b depends on the complexity of the ship. (Source: Storch, Hammon, and Bunch, 1988)

a What is the significance of the constant of proportionality a? (Hint: What is the value of C if only one ship is built?)

b For one kind of ship, $b = \dfrac{1}{8}$, and the cost of producing the first ship is \$12 million. Write the equation for C as a function of x using radical notation.

c Compute the cost per ship when 2 ships have been built. By what percent does the cost per ship decrease? By what percent does the cost per ship decrease from building 2 ships to building 4 ships?

d By what percent does the average cost decrease from building n ships to building $2n$ ships? (In the shipbuilding industry, the average cost per ship usually decreases by 5 to 10% each time the number of ships doubles.)

28. A population is in a period of supergrowth if its rate of growth, R, at any time is proportional to P^k, where P is the population at that time and k is a constant greater than 1. Suppose R is given by

$$R = 0.015P^{1.2}$$

where P is measured in thousands and R is measured in thousands per year.

a Find R when $P = 20$, when $P = 40$, and when $P = 60$.

b What will the population be when its rate of growth is 5000 per year?

c Graph R and use your graph to verify your answers to parts (a) and (b).

In Problems 29 and 30, evaluate the function for the given values.

29. $Q(x) = 4x^{5/2}$

a $Q(16)$

b $Q(\frac{1}{4})$

c $Q(3)$

d $Q(100)$

30. $T(w) = -3w^{2/3}$

a $T(27)$

b $T(\frac{1}{8})$

c $T(20)$

d $T(1000)$

In Problems 31–42, solve.

31. $2\sqrt{w} - 5 = 21$

32. $16 - 3\sqrt{w} = -5$

33. $12 - \sqrt{5v+1} = 3$

34. $3\sqrt{17-4v} - 8 = 19$

35. $x - 3\sqrt{x} + 2 = 0$

36. $\sqrt{x+1} + \sqrt{x+8} = 7$

37. $(x+7)^{1/2} + x^{1/2} = 7$

38. $(y-3)^{1/2} + (y+4)^{1/2} = 7$

39. $\sqrt[3]{x+1} = 2$

40. $x^{2/3} + 2 = 6$

41. $(x-1)^{-3/2} = \dfrac{1}{8}$

42. $(2x+1)^{-1/2} = \dfrac{1}{3}$

For Problems 43–46, solve the formula for the indicated variable.

43. $t = \sqrt{\dfrac{2v}{g}}$, for g

44. $q - 1 = 2\sqrt{\dfrac{r^2-1}{3}}$, for r

45. $R = \dfrac{1+\sqrt{p^2+1}}{2}$, for p

46. $q = \sqrt[3]{\dfrac{1+r^2}{2}}$, for r

For Problems 47–52, write the radical or radical expression in simplest form.

47.

a $\sqrt{\dfrac{125p^9}{a^4}}$

b $\sqrt[3]{\dfrac{24v^2}{w^6}}$

48.

a $\dfrac{\sqrt{a^5b^3}}{\sqrt{ab}}$

b $\dfrac{\sqrt{x}\sqrt{xy^3}}{\sqrt{y}}$

49.

a $\sqrt[3]{8a^3 - 16b^6}$

b $\sqrt[3]{8a^3}\,\sqrt[3]{-16b^6}$

50.

a $\sqrt{4t^2 + 24t^6}$

b $\sqrt{4t^2}\sqrt{24t^6}$

51.

a $(x - 2\sqrt{x})^2$

b $(x - 2\sqrt{x})(x + 2\sqrt{x})$

52.

a $(\sqrt{2} - 2\sqrt{3})^2$

b $(\sqrt{2a} - 2\sqrt{b})(\sqrt{2a} + 2\sqrt{b})$

For Problems 53–56, rationalize the denominator.

53.

a $\dfrac{7}{\sqrt{5y}}$

b $\dfrac{6d}{\sqrt{2d}}$

54.

a $\sqrt{\dfrac{3r}{11s}}$

b $\sqrt{\dfrac{26}{2m}}$

55.

a $\dfrac{-3}{\sqrt{a}+2}$

b $\dfrac{-3}{\sqrt{z}-4}$

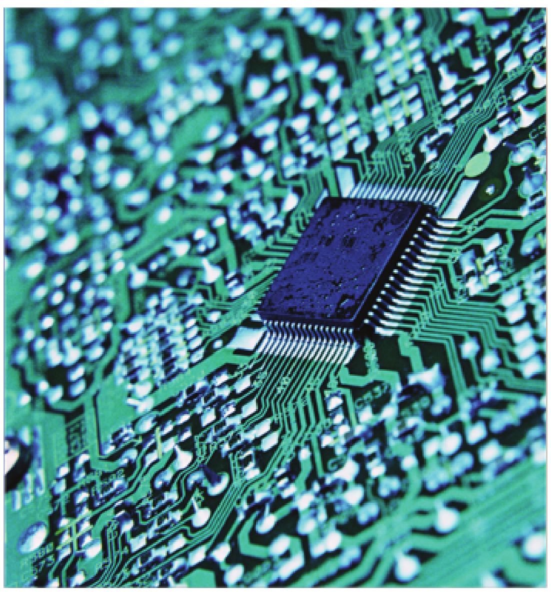
56.

a $\dfrac{2x-\sqrt{3}}{x-\sqrt{3}}$

b $\dfrac{m-\sqrt{3}}{5m+2\sqrt{3}}$

Chapter 7

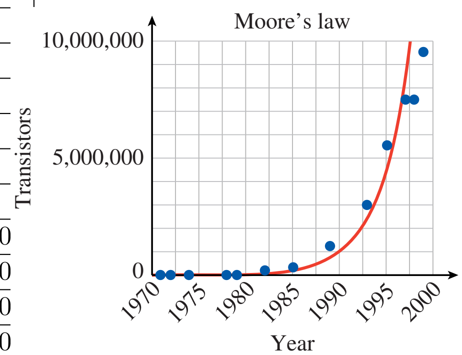
Exponential Functions



We next consider another important family of functions, called **exponential functions**. These functions describe growth by a constant factor in equal time periods. Exponential functions model many familiar processes, including the growth of populations, compound interest, and radioactive decay. Here is an example.

In 1965, Gordon Moore, the cofounder of Intel, observed that the number of transistors on a computer chip had doubled every year since the integrated circuit was invented. Moore predicted that the pace would slow down a bit, but the number of transistors would continue to double every 2 years. More recently, data density has doubled approximately every 18 months, and this is the current definition of **Moore's law**. Most experts, including Moore himself, expected Moore's law to hold for at least another two decades.

Year	Name of circuit	Transistors
1971	4004	2300
1972	8008	3300
1974	8080	6000
1978	8086	29,000
1979	8088	30,000
1982	80286	134,000
1985	80386	275,000
1989	90486	1,200,000
1993	Pentium	3,000,000
1995	Pentium Pro	5,500,000
1997	Pentium II	7,500,000
1998	Pentium II Xeon	7,500,000
1999	Pentium III	9,500,000



The data shown are modeled by the exponential function $N(t) = 2200(1.356)^t$, where t is the number of years since 1970.

Investigation 7.1 Population Growth. In a laboratory experiment, researchers establish a colony of 100 bacteria and monitor its growth. The colony triples in population every day.

t	$P(t)$
0	100
1	
2	
3	
4	
5	

$P(0) = 100$
$P(1) = 100 \cdot 3 =$
$P(2) = [100 \cdot 3] \cdot 3 =$
$P(3) =$
$P(4) =$
$P(5) =$

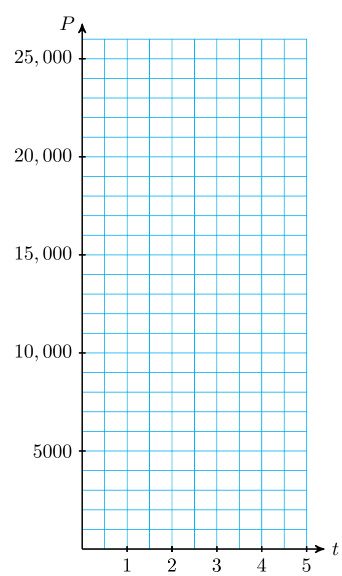
1 Fill in the table showing the population $P(t)$ of bacteria t days later.

2 Plot the data points from the table and connect them with a smooth curve.

3 Write a function that gives the population of the colony at any time t, in days. *Hint:* Express the values you calculated in part (1) using powers of 3. Do you see a connection between the value of t and the exponent on 3?

4 Graph your function from part (3) using a calculator. (Use the table to choose an appropriate window.) The graph should resemble your hand-drawn graph from part (2).

5 Evaluate your function to find the number of bacteria present after 8 days. How many bacteria are present after 36 hours?



Investigation 7.2 Exponential Decay. A small coal-mining town has been losing population since 1940, when 5000 people lived there. At each census thereafter (taken at 10-year intervals), the population declined to approximately 0.90 of its earlier figure.

t	$P(t)$
0	5000
10	
20	
30	
40	
50	

$P(0) = 5000$

$P(10) = 5000 \cdot 0.90 =$

$P(20) = [5000 \cdot 0.90] \cdot 0.90 =$

$P(3) =$

$P(4) =$

$P(5) =$

1 Fill in the table showing the population $P(t)$ of the town t years after 1940.

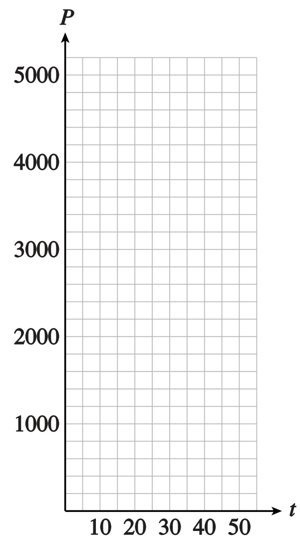
2 Plot the data points and connect them with a smooth curve.

3 Write a function that gives the population of the town at any time t in years after 1940.

Hint: Express the values you calculated in part (1) using powers of 0.90. Do you see a connection between the value of t and the exponent on 0.90?

4 Graph your function from part (3) using a calculator. (Use the table to choose an appropriate window.) The graph should resemble your hand-drawn graph from part (2).

5 Evaluate your function to find the population of the town in 1995. What was the population in 2000?



Exponential Growth and Decay

Introduction

The function in Investigation 7.1, p. 448 describes **exponential growth**. During each time interval of a fixed length, the population is multiplied by a certain constant amount. In this case, the bacteria population grows by a factor of 3 every day.

t	0	1	2	3	4
$P(t)$	100	300	900	2700	8100

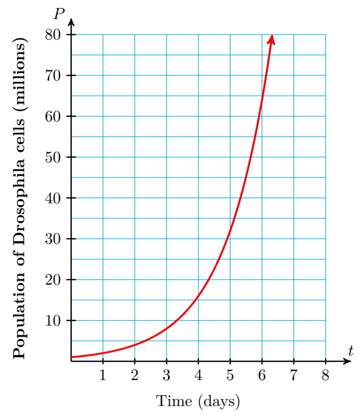
For this reason, we say that 3 is the **growth factor** for the function.

Growth Factors

Researchers often use cell lines from the fruit fly *Drosophila melanogaster* to study protein interactions related to cancer and other diseases. From 60% to 70% of human disease genes are found in *Drosophila* cells, and gene discoveries in the flies have led to parallel studies in vertebrates.

One milliliter of culture contains about 1 million *Drosophila* cells, and the population doubles every 24 hours. The table shows the population, $P(t)$, of *Drosophila* cells, in millions, as a function of time in days.

t	$P(t)$
0	1
1	2
2	4
3	8
4	16
5	32
6	64



Because the fruit fly population grows by a factor of 2 every day, the function $P(t)$ describes exponential growth. Functions that describe exponential growth can be expressed in a standard form.

Exponential Growth.

The function

$$P(t) = P_0 b^t$$

describes **exponential growth**, where $P_0 = P(0)$ is the **initial value** of the function and b is the **growth factor**.

For the *Drosophila* cell population, we have

$$P(t) = 1 \cdot 2^t$$

so $P_0 = 1$ and $b = 2$. You can see that the graph of the function is not linear. In fact, the population grows slowly at first, but eventually grows faster and faster.

Example 7.1

In 1985, there were about 1.2 million cell phone users world-wide. Since that time, the number has grown by a factor of 1.5 each year.

a Make a table of values and graph the function.

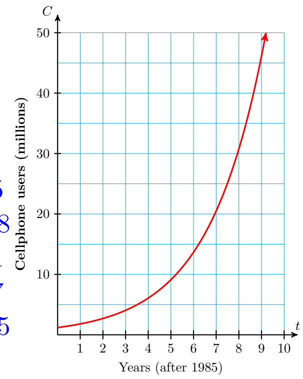
b Write a formula for the number, $C(t)$, of cell phone users t years after 1985.

c How many cell phone users does the formula predict for the year 2000?

Solution.

a We let $t = 0$ in 1985, so $C(0) = 1.2$, in millions. Each value of $C(t)$ can be obtained by multiplying the previous value by the growth factor, 1.5.

t	$C(t)$	
0	1.2	
1	1.8	$1.2 \times 1.5 = 1.8$
2	2.7	$1.8 \times 1.5 = 2.7$
3	4.05	$2.7 \times 1.5 = 4.05$
4	6.08	$4.05 \times 1.5 = 6.08$
5	9.1	$6.08 \times 1.5 = 9.1$
6	13.7	$9.1 \times 1.5 = 13.7$
7	20.5	$13.7 \times 1.5 = 20.5$



b The initial value of the function is $C_0 = C(0) = 1.2$ million. The annual growth factor is $b = 1.5$, so the formula is

$$C(t) = 1.2(1.5)^t$$

c The year 2000 is 15 years after 1985, so we evaluate the function for $t = \mathbf{15}$.

$$C(\mathbf{15}) = 1.2(1.5)^{\mathbf{15}} = 525.47$$

The formula predicts that over 525 million people used cell phones in 2000.

In the examples above, you can see that the graphs of the functions are not linear. In each case, the function grows slowly at first, but eventually grows faster and faster.

Caution 7.2 Be careful when evaluating exponential growth functions. In part (c) of the previous Example, p. 450, note that

$$1.2(1.5)^{15} = 1.2 \times 437.89 = 525.47 \qquad \text{Compute the power first.}$$

According to the order of operations, we compute the power 1.5^{15} first, and then multiply the result by 1.2.

Checkpoint 7.3 QuickCheck 1. A population grows according to the formula $P(t) = 800(1.06)^t$, where t is in years.

a. What was the starting value of the population? ____

b. What was the population one year later? ____

c. What does 1.06 tell you about the population?

- ⊙ A) The population grew by 1.06 each year.
- ⊙ B) The population grew by a factor of 1.06 each year.
- ⊙ C) The population began at 1.06.

d. Choose the correct first step to evaluate $800(1.06)^5$:

- ⊙ A) Multiply 800 by 1.06
- ⊙ B) Raise 1.06 to the 5th power

Answer 1. 800

Answer 2. 848

Answer 3. B) The ... each year.

Answer 4. B) Raise ... the 5th power

Solution.

a. 800

b. 848

c. The population grew by a factor of 1.06 each year.

d. Raise 1.06 to the 5th power

Checkpoint 7.4 Practice 1. A colony of rabbits started with 20 rabbits and doubles every 3 months.

a. Complete the table for the number of rabbits $P(t)$ after t months, and graph the function.

t	$P(t)$
0	___
3	___
6	___
9	___
12	___
15	___

b. Write a formula for the function $P(t)$. Note that the population is multiplied by 2 every 3 months. If you know the value of t, how do you find the corresponding exponent in $P(t)$?

$P(t) =$_______

c. How many rabbits will there be after 1 year? ____

Answer 1. 20

Answer 2. 40

Answer 3. 80

Answer 4. 160

Answer 5. 320

Answer 6. 640

Answer 7. $20 \cdot 2^{\frac{t}{3}}$

Answer 8. 320

Solution.

a.

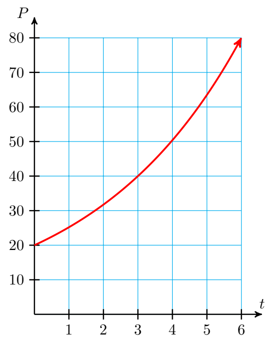
t	$P(t)$
0	20
3	40
6	80
9	160
12	320
15	640

A graph is below.

b. $P(t) = 20(2)^{t/3}$

c. 320 rabbits (after 12 months)

Graph for part (a):



Note 7.5 In Practice 1, the rabbit population doubled every 3 months, leading to the growth law

$$P(t) = 20(2)^{t/3}$$

To see the growth factor for the rabbit population, we use the third law of exponents to rewrite $2^{t/3}$.

$$(2^{1/3})^t = 2^{t(1/3)} = 2^{t/3}$$

So the growth factor for the rabbit population is $2^{1/3}$, or about 1.26. The rabbit population grows by a factor of 1.26 every month.

Comparing Linear Growth and Exponential Growth

It may be helpful to compare linear growth and exponential growth. Consider the two functions

$$L(t) = 5 + 2t \qquad \text{and} \qquad E(t) = 5 \cdot 2^t \quad (t \geq 0)$$

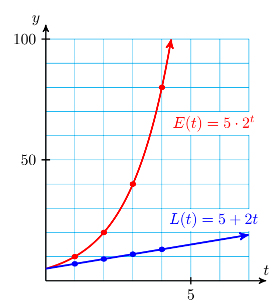
whose graphs are shown below.

t	$L(t)$
0	5
1	7
2	9
3	11
4	13

Slope $m = 2$

t	$E(t)$
0	5
1	10
2	20
3	40
4	80

Growth factor $b = 2$



L is a linear function with y-intercept 5 and slope 2; E is an exponential function with initial value 5 and growth factor 2. In a way, the growth factor of an exponential function is analogous to the slope of a linear function: Each measures how quickly the function is increasing.

However, for each unit increase in t, 2 units are *added* to the value of $L(t)$, whereas the value of $E(t)$ is *multiplied* by 2. An exponential function with growth factor 2 eventually grows much more rapidly than a linear function with slope 2, as you can see by comparing the graphs or the function values in the tables.

Example 7.6

A solar energy company sold \$80,000 worth of solar collectors last year, its first year of operation. This year its sales rose to \$88,000. The marketing department must estimate its projected sales for the next 3 years.

a If the marketing department predicts that sales will grow linearly, what sales total should it expect next year? Graph the projected sales figures over the next 3 years, assuming that sales will grow linearly.

b If the marketing department predicts that sales will grow exponentially, what sales total should it expect next year? Graph the projected sales figures over the next 3 years, assuming that sales will grow exponentially.

Solution.

a Let $L(t)$ represent the company's total sales t years after starting business, where $t = 0$ is the first year of operation. If sales grow linearly, then $L(t)$ has the form $L(t) = mt + b$. Because $L(0) = 80,000$, the intercept b is 80,000. The slope m of the graph is

$$\frac{\Delta S}{\Delta t} = \frac{8000 \text{ dollars}}{1 \text{ year}} = 8000 \text{ dollars/year}$$

where $\Delta S = 8000$ is the increase in sales during the first year. Thus, $L(t) = 8000t + 80,000$, and sales grow by adding \$8000 each year. The expected sales total for the next year is

$$L(2) = 8000(2) + 80,000 = 96,000$$

b Let $E(t)$ represent the company's sales assuming that sales will grow exponentially. Then $E(t)$ has the form $E(t) = E_0 b^t$, and the initial value is $E_0 = 80,000$. We find the growth factor in sales over the first year by dividing $E(1)$ by E_0:

$$E(1) = E_0 b^1$$

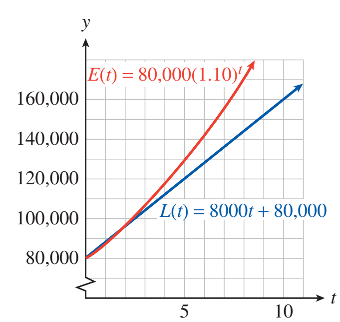
$$\text{so} \quad b = \frac{E(1)}{E_0} = \frac{88,000}{80,000} = 1.1$$

Thus, $E(t) = 80,000(1.1)^t$, and the expected sales total for the next year is

$$E(2) = 80,000(1.10)^2 = 96,800$$

We evaluate each function at several points to obtain the graphs shown in the figure.

t	$L(t)$	$E(t)$
0	80,000	80,000
1	88,000	88,000
2	96,000	96,800
3	104,000	106,480
4	112,000	117,128



Exponential Decay

In the examples above, exponential growth was modeled by increasing functions of the form

$$P(t) = P_0 b^t$$

where the growth factor, b, is a number greater than 1. If we multiply the function value by a number smaller than 1, the function values will decrease. Thus, if $0 < b < 1$, then $P(t) = P_0 b^t$ is a *decreasing* function. In this case, we say that the function describes **exponential decay**, and the constant b is called the **decay factor**.

Example 7.7

Before the introduction of disposable containers, soft drinks and draught beer were sold in refillable glass botles. During the second half of the last century, the percent of beer volume sold in refillable glass bottles declined by a factor of 0.942 each year.

a In 1944, 98% of beer was sold in refillable bottles. Write a formula for the percent of beer sold in refillable bottles as a function of t, the number of years after 1944.

b Graph the function from 1944 to 2000.

c In 1998, only 3.3% of beer was sold in refillable bottles. How well does the model predict this number?

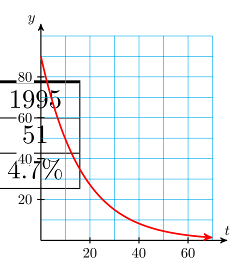
Solution.

a We let $t = 0$ in 1944, so that $P_0 = 98$. The formula is

$$P(t) = P_0 b^t = 90(0.942)^t$$

b We evaluate the formula for several values of t, and plot the data points.

Year	1950	1965	1980	1995
t	6	21	36	51
$P(t)$	68.5%	27.9%	11.4%	4.7%



c In 1998, $t = 54$, and

$$P(54) = 98(0.942)^{54} = 3.89$$

The model predicts that 3.89% of beer was sold in refillable bottles in 1998, just slightly above the actual figure.

Checkpoint 7.8 Practice 2. The number of perch in Hidden Lake has declined to 0.88 of its previous value every year since 2000, when the perch population was estimated at 8000.

a. Let $P(t)$ represent the perch population t years after 2000. Complete the table.

t	2	8	10	15	20
$P(t)$	____	____	____	____	____

b. Write a formula for the function $P(t)$.

$P(t) =$___________

c. Graph the function, using the table values to choose an appropriate window.

d. What does the model predict for the perch population in 2025? ____

Answer 1. 6195.2

Answer 2. 2877.08

Answer 3. 2228.01

Answer 4. 1175.79

Answer 5. 620.502

Answer 6. $8000 \cdot 0.88^t$

Answer 7. 327.459

Solution.

a.

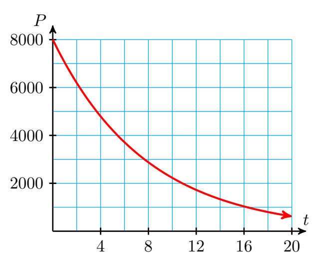
t	2	8	10	15	20
$P(t)$	6195	2877	2228	1176	621

b. $8000 \cdot 0.88^t$

c. A graph is below.

d. 327

Graph for part (c):



Checkpoint 7.9 QuickCheck 2.

a. We can tell whether a linear function is increasing or decreasing by whether its slope is (☐ positive or negative ☐ greater than 1 or less than 1)

b. We can tell whether an exponential function is increasing or decreasing by whether b is (☐ positive or negative ☐ greater than 1 or less than 1)

c. If a linear function has slope 1.5, each time we increase the input by 1 unit, how do we find the new function value from the old one? (☐ Add 1.5 ☐ Multiply by 1.5)

d. If an exponential function has growth factor 1.5, each time we increase the input by 1 unit, how do we find the new function value from the old one? (☐ Add 1.5 ☐ Multiply by 1.5)

Answer 1. positive or negative

Answer 2. greater than 1 or less than 1

Answer 3. Add 1.5

Answer 4. Multiply by 1.5

Solution.

a. positive or negative

b. greater than 1 or less than 1

c. Add 1.5

d. Multiply by 1.5

Percent Increase

Exponential growth is often described as growth by a certain percent increase. Suppose the town of Lakeview had 4000 residents in the year 2000, and grew at a rate of 5% per year. This means that each year we add 5% of last year's population to find the current population, $P(t)$. Thus

In 2000, $P(0) = 4000$
In 2001, $P(1) = 4000 + 0.05(4000) = 4200$ Add 5% of $P(0)$.
In 2002, $P(2) = 4200 + 0.05(4200) = 4410$ Add 5% of $P(1)$.

and so on. Now here is the important observation about percent increase:

Adding 5% of the old population is the same as multiplying the old population by 1.05.

$$4000 + 0.05(4000) = 4000(1 + 0.05) = 4000(1.05)$$ Factor out 4000.

$$4200 + 0.05(4200) = 4200(1 + 0.05) = 4200(1.05)$$ Factor out 4000.

Thus, we can find the current population by multiplying the old population by 1.05. In other words,

Growing by 5% is the same as growing by a factor of 1.05.

A formula for the population of Lakeview t years after 2000 is

$$P(t) = 4000(1.05)^t$$

This formula describes exponential growth with a growth factor of $b = 1.05$. In general, a function that grows at a percent rate r, where r is expressed as a decimal, has a growth factor of $b = 1 + r$.

Growth by a Constant Percent.

The function

$$P(t) = P_0(1 + r)^t$$

describes exponential growth at a constant percent rate of growth, r.

The **initial value** of the function is $P_0 = P(0)$, and $b = 1 + r$ is the **growth factor**.

Many quantities besides population can grow by a fixed percent. For example, an inflation rate gives the percent rate at which prices are rising.

Example 7.10

During a period of rapid inflation, prices rose by 12% each year. At the beginning of this time, a loaf of bread cost \$2.

a Make a table showing the cost of bread over the next four years.

b Write a function that gives the price of a loaf of bread years after inflation began.

c How much did a loaf of bread cost after 6 years? After 30 months?

d Graph the function found in (b).

Solution.

a The percent increase in the cost of bread is 12% every year. Therefore, the growth factor for the cost of bread is $1 + 0.12 = 1.12$ every year. If $P(t)$ represents the price of bread after t years, then $P(0) = 2$, and we multiply the price by 1.12 every year, as shown in the table.

t	$P(t)$	
0	$P(0) = 2.00$	2
1	$P(1) = 2.24$	$2(1.12)$
2	$P(2) = 2.51$	$2(1.12)^2$
3	$P(3) = 2.81$	$2(1.12)^3$
4	$P(4) = 3.15$	$2(1.12)^4$

b After t years of inflation the original price of \$2 has been multiplied t times by a factor of 1.12. Thus,

$$P(t) = 2(1.12)^t$$

c To find the price of bread at any time after inflation began, we evaluate the function at the appropriate value of t.

$$P(\mathbf{6}) = 2(1.12)^{\mathbf{6}} \approx 3.95$$

After 6 years the price was \$3.95. Thirty months is 2.5 years, so we evaluate $P(2.5)$.

$$P(\mathbf{2.5}) = 2(1.12)^{\mathbf{2.5}} \approx 2.66$$

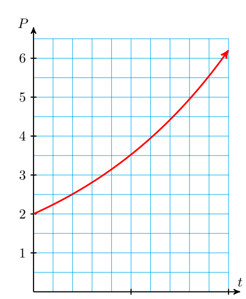
After 30 months the price was \$2.66.

d

To graph the function

$$P(t) = 2(1.12)^t$$

we evaluate it for several values, as shown in the table. We plot the points and connect them with a smooth curve to obtain the graph shown.



Checkpoint 7.11 QuickCheck 3. Fill in the blanks.

a. Increasing by 10% is the same as multiplying by ___.

b. If a population grows by 2% annually, its growth factor is ___.

c. If a population grows by 46% annually, its growth factor is ___.

d. If a population grows by 100% annually, its growth factor is ___.

Answer 1. 1.1

Answer 2. 1.02

Answer 3. 1.46

Answer 4. 2

Solution.

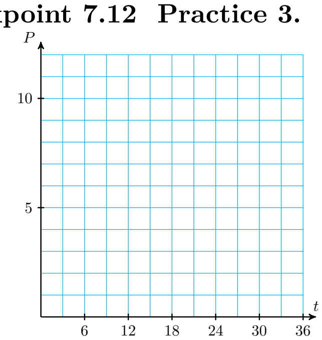
a. 1.10

b. 1.02

c. 1.46

d. 2

Checkpoint 7.12 Practice 3.



Tombstone, Arizona was the most famous "boomtown" during the gold rush in the American west. It was established in December, 1879, after the discovery of a large silver deposit nearby. The original town had 40 dwellings and a population of 100. Over the next two to three years, the population grew at an average rate of 19% per month.

a. What was the population one year later, in December, 1880? ___

b. Write a formula for $P(t)$, the population of Tombstone t months after its founding.

   $P(t) =$_________

c. Complete the table and sketch a graph of $P(t)$. A suggest grid is above.

t	0	5	10	15	20	25
$P(t)$	__	__	__	__	__	__

d. Tombstone's peak population was about 10,000 people. Use your graph to estimate the time it took to reach that figure.

   About __ months

Answer 1. 806.424

Answer 2. $100 \cdot 1.19^t$

Answer 3. 100

Answer 4. 238.635

Answer 5. 569.468

Answer 6. 1358.95

Answer 7. 3242.94

Answer 8. 7738.81

Answer 9. 26.4736

Solution.

a. 806

b. $P(t) = 100(1.19)^t$

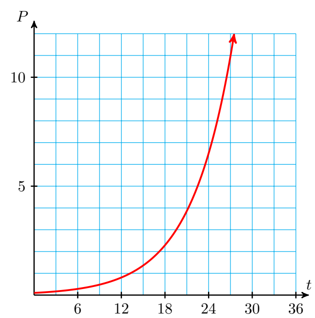
c.

t	0	5	10	15	20	25
$P(t)$	100	239	569	1359	3243	7739

A graph is below.

d. About 26 months

Graph for part (c):



Compound interest is another example of exponential growth. Suppose you deposit a sum of money into an account that pays 5% interest compounded annually. Each year, 5% of your current balance is added to your account as interest, so your balance grows by a factor of 1.05. In general, we have the following formula.

Compound Interest.

If a **principal** of P dollars is invested in an account that pays an interest rate r compounded annually, the **balance** B after t years is given by

$$B = P(1 + r)^t$$

Percent Decrease

We have seen that a percent increase of r (in decimal form) corresponds to a growth factor of $b = 1 + r$. A percent decrease of r corresponds to a decay factor of $b = 1 - r$. For example, if a population declines by 25% each year, then each year the new population is 75% of its previous value. So

$$b = 1 - 0.25 = 0.75$$

and $P(t) = P_0(0.75)^t$. Remember that multiplying by b gives us the population remaining, not the amount of decline.

Example 7.13

According to *Context* magazine: "Computing prices have been falling exponentially for the past 30 years and will probably stay on that curve for another couple of decades." In fact, prices have been falling at a rate of 37% every year. Suppose an accounting firm invests $50,000 in new computer equipment.

a Write a formula for the value $V(t)$ of the equipment t years from now.

b What will the equipment be worth in 5 years?

c Graph the function $V(t)$ for $0 \leq t \leq 20$.

Solution.

a The initial value of the equipment is $V_0 = 50,000$. Every year, the value of the equipment is multiplied by

$$b = 1 - r = 1 - 0.37 = 0.63$$

After t years, the value of the equipment is

$$V(t) = 50,000(0.63)^t$$

b After 5 years, we have

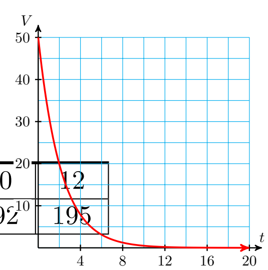
$$V(\mathbf{5}) = 50,000(0.63)^{\mathbf{5}} = 4962.18$$

The value of the equipment after 5 years is $4962.18.

c

We evaluate the function $V(t)$ for several values of t, and plot the points to obtain the graph shown.

t	2	4	6	10	12
$V(t)$	19,845	7876	3126	492	195



Note 7.14 In the preceding Example, the value of the computer equipment decreases by 37% each year, so 63% of the value remains, and the decay factor for the value function is 0.63, not 0.37. The function $V(t)$ gives the value remaining, not the amount that has depreciated.

Checkpoint 7.15 **Practice 4.** The number of butterflies visiting a nature station is declining by 18% per year. In 2012, 3600 butterflies visited the nature station.

a. What is the decay factor in the annual butterfly count?

Answer: ___

b. Write a formula for $B(t)$, the number of butterflies t years after 2012.

$B(t) =$___________

c. Complete the table and sketch a graph of $B(t)$.

t	0	2	4	6	8	10
$B(t)$	__	__	__	__	__	__

Answer 1. 0.82

Answer 2. $3600 \cdot 0.82^t$

Answer 3. 3600

Answer 4. 2420.64

Answer 5. 1627.64

Answer 6. 1094.42

Answer 7. 735.891

Answer 8. 494.813

Solution.

a. 0.82

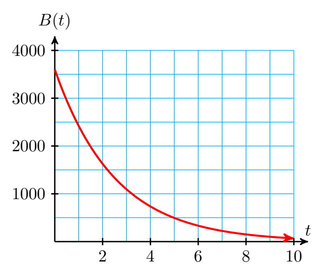
b. $B(t) = 3600 \cdot 0.82^t$

c.

t	0	2	4	6	8	10
$B(t)$	3600	2421	1628	1094	736	495

A graph is below.

Graph for part (c):



Caution 7.16 Compare these two descriptions of exponential decay:

- Each year, the population decreases *by* 25% of its previous value.
- Each year, the population decreases *to* 25% of its previous value.

In the first description, $r = 0.25$ and $b = 1 - r = 0.75$. If the population this year is 100, next year it will be 75. If 25% of the population is gone, 75% remains.

In the second description, only 25% of the population remains, so $b = 0.25$. If the population this year is 100, next year it will be 25.

Checkpoint 7.17 QuickCheck 4.

a. If $b = 0.9$, it represents a decrease of __%.

b. An annual decrease of 15% corresponds to a decay factor of __.

c. Which function decreases more rapidly: $f(x)$, with a decay factor of 0.15, or $g(x)$, with a decay factor of 0.05?

⊙ $f(x)$, with a decay factor of 0.15,

⊙ $g(x)$, with a decay factor of 0.05

d. An exponential function $P(t) = P_0 b^t$ is decreasing if ______.

Answer 1. 10

Answer 2. 0.85

Answer 3. Choice 1

Answer 4. $b < 1$

Solution.

a. 10%

b. 0.85

c. $f(x)$

d. $b < 1$

We summarize our observations about exponential growth and decay functions as follows.

Exponential Growth and Decay.

The function

$$P(t) = P_0 b^t$$

models exponential growth and decay.

- $P_0 = P(0)$ is the **initial value** of P;
- b is the **growth** or **decay factor**.

1. If $b > 1$, then $P(t)$ is increasing, and $b = 1 + r$, where r represents percent increase.
2. If $0 < b < 1$, then $P(t)$ is decreasing, and $b = 1 - r$, where r represents percent decrease.

Checkpoint 7.18 Practice 5. A new car begins to depreciate in value as soon as you drive it off the lot. Some models depreciate linearly, and others depreciate exponentially. Suppose you buy a new car for \$20,000, and 1 year later its value has decreased to \$17,000.

a. If the value decreased linearly, what was its annual rate of decrease?

\$____ per year

b. If the value decreased exponentially, what was its annual decay factor? What was its annual percent depreciation? __%

c. Calculate the value of your car when it is 5 years old under each assumption, linear or exponential depreciation.

Linear: \$____

Exponential: \$____

Answer 1. 3000

Answer 2. 0.85

Answer 3. 15

Answer 4. 5000

Answer 5. $20000 \cdot 0.85^5$

Solution.

a. \$3000 per year

b. 0.85; 15%

c. Linear: \$5000; Exponential: \$8874

Problem Set 7.1

Warm Up

1.

a A parking permit at Huron College cost \$25 last year, but this year the price increased by 12%. What is the price this year?

b If the price of a parking permit increases by 12% again next year, what will the price be then?

c Did the parking permit increase by the same amount each year? Why or why not?

2.

a The computer you want cost \$1200 when it first came on the market, but after 3 months the price was reduced by 15%. What was the price then?

b If the price falls by another 15% next month, what will the price be then?

c Did the price fall by the same amount each month? Why or why not?

3. The value of your stock portfolio fell 10% last year, but this year it increased by 10%. How does the current value of your portfolio compare to what it was two years ago?

4. You got a 5% raise in January, but then in March everyone took a pay cut of 5%. How does your new salary compare to what it was last December?

For Problems 5–10, solve. Round your answers to two places if necessary.

5. $768 = 12b^3$

6. $1875 = 3b^4$

7. $14,929.92 = 5000b^6$

8. $151,875 = 20,000b^5$

9. $1253 = 260(1 + r)^{12}$

10. $56.27 = 78(1 - r)^8$

Skills Practice

11. The population of Summerville is currently 12 hundred people.

a Write a formula for the population if it grows at a constant rate of 1.5 hundred people per year. What is the population after 3 years?

b Write a formula for the population if it has a constant growth factor of 1.5 per year. What is the population after 3 years?

12. Delbert's sports car was worth \$45,000 when he bought it.

a Write a formula for the value of the car if it depreciates at a constant rate of \$7000 per year. What is the value of the car after 4 years?

b Write a formula for the value of the car if it has a constant depreciation factor of 0.70 per year. What is the value of the car after 4 years?

13. Francine's truck was worth \$18,000 when she bought it.

a Write a formula for the value of the truck if it depreciates by \$2000 per year. What is the value of the truck after 5 years?

b Write a formula for the value of the truck if it depreciates by 20% per year. What is the value of the truck after 5 years?

14. The population of Lakeview is currently 150,000 people.

(a) Write a formula for the population if it grows by 6000 people per year. What is the population after 2 years?

(b) Write a formula for the population if grows by 6% per year. What is the population after 2 years?

15. The table shows the growth factor for a number of different populations. For each population, find the percent growth rate.

Population	A	B	C	D	E
Growth factor	1.2	1.02	1.075	2.0	2.15
Percent growth rate					

16. The table shows the decay factor for a number of different populations. For each population, find the percent decay rate.

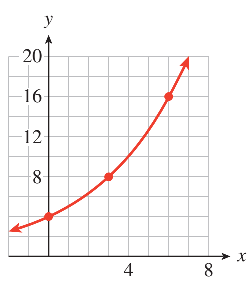
Population	A	B	C	D	E
Decay factor	0.6	0.06	0.96	0.996	0.096
Percent decay rate					

The graphs in Problems 17 and 18 represent exponential growth or decay.

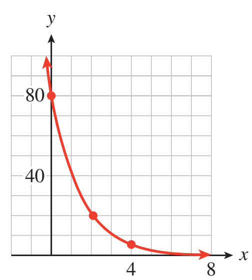
(a) Find the initial value and the growth or decay factor.

(b) Write a formula for the function.

17.



18.



For Problems 19–22,

a Each table describes exponential growth or decay. Find the growth or decay factor.

b Complete the table. Round values to two decimal places if necessary.

19.

x	0	1	2	3	4
Q	20	24			

20.

w	0	1	2	3	4
N	120	96			

21.

t	0	1	2	3	4
C	10		6.4		

22.

n	0	1	2	3	4
B	200			266.2	

Applications

For Problems 23–30,

a Complete the table of values.

b Write a function that describes exponential growth or decay.

c Use your calculator to graph the function.

d Evaluate the function at the given values.

23. Sales of Windsurfers have increased 12% per year since 2010. If Sunsails sold 1500 Windsurfers in 2010, how many did it sell in 2015? How many should it expect to sell in 2022?

Years after 2010	0	1	2	3	4
Windsurfers					

24. Paul bought a house for $200,000 in 1983. For the next 20 years, housing prices rose an average of 5% per year. How much was the house worth in 1995? In 2000?

Years after 1983	0	5	10	15	20
Value of house					

25. A typical beehive contains 20000 insects. The population can increase in size by a factor of 2.5 every 6 weeks. How many bees could there be after 4 weeks? After 20 weeks?

Weeks	0	6	12	18	24
Bees					

26. Otto invests $600 in an account that pays 7.3% interest compounded annually. How much is in Otto's account after 3 years? After 6 years?

Years	0	1	2	3	4
Account balance					

27. During a vigorous spraying program, the mosquito population was reduced to $\frac{3}{4}$ of its previous size every week. If the mosquito population was originally estimated at 250,000, how many mosquitoes remained after 3 weeks of spraying? After 9 weeks?

Weeks	0	2	4	6	8
Mosquitos					

28. Scuba divers find that the water in Emerald Lake filters out 15% of the sunlight for each 4 feet they descend. How much sunlight penetrates to a depth of 20 feet? To a depth of 45 feet?

Feet	0	4	8	12	16
% of light					

29. Arch's motorboat cost $15,000 in 2000 and has depreciated by 11.5% every year. How much was the boat worth in 2009? In 2010?

Years	0	3	6	9	12
Value of boat					

30. Plutonium-238 is a radioactive element that decays over time into a less harmful element at a rate of 0.8% per year. A power plant has 50 pounds of plutonium-238 to dispose of. How much plutonium-238 will be left after 10 years? After 100 years?

Years	0	10	20	30	40
Pounds					

31. Riverside County is the fastest growing county in California. In 2000, the population was 1,545,387.

a Write a formula for the population of Riverside County as a function of time. (You don't know the value of the growth factor b yet.)

b In 2004, the population had grown to 1,871,950. Find the growth factor and the percent rate of growth.

c According to this model, what was the population of Riverside County in 2010?

32. In 2006, a new Ford Focus cost \$15,574. The value of a Focus decreases exponentially over time.

a Write a formula for the value of a Focus as a function of time. (You don't know the value of the decay factor yet.)

b A 2-year old Focus cost \$11,788. Find the decay factor and the percent rate of depreciation.

c According to this model, how much did a 4-year old Focus cost?

33. In the 1940s David Lack undertook a study of the European robin. He tagged 130 one-year-old robins and found that on average 35.6% of the birds survived each year. (Source: Burton, 1998.)

a According to the data, how many robins would have originally hatched to produce 130 one-year-olds?

b Write a formula for the number of the original robins still alive after t years.

c Graph your function on your calculator.

d One of the original robins actually survived for 9 years. How many robins does the model predict will survive for 9 years?

34. Many insects grow by discrete amounts each time they shed their exoskeletons. Dyar's rule says that the size of the insect increases by a constant ratio at each stage. (Source: Burton, 1998.)

a Dyar measured the width of the head of a caterpillar of a swallowtail butterfly at each stage. The caterpillar's head was initially approximately 42 millimeters wide and was 63.84 millimeters wide after its first stage. Find the growth ratio.

b Write an equation for the width of the caterpillar's head at the nth stage.

c Graph your equation on your calculator.

d What head width does the model predict after 5 stages?

35. The world's population of tigers declined from 10,400 in 1980 to 6000 in 1998.

a If the population declined linearly, what was its annual rate of decrease?

b If the population declined exponentially, what was its annual decay factor? What was its annual percent decrease?

c Predict the number of tigers in 2010 under each assumption, linear or exponential decline.

36. On January 10, the college infirmary treated 4 cases of flu. One week later, the total number of flu cases had grown to 6.

a If the number of cases grew linearly, what was its weekly rate of growth?

b If the number of cases grew exponentially, what was its weekly growth factor? What was its weekly percent increase?

c Predict the number of flu cases 6 weeks later under each assumption, linear or exponential growth.

d Write a function for the number of flu cases under each assumption. Graph both functions for 10 weeks from January 10.

37. A researcher starts 2 populations of fruit flies of different species, each with 30 flies. Species A increases by 30% in 6 days, and species B increases by 20% increases in 4 days.

a What was the population of species A after 6 days? Find the daily growth factor for species A.

b What was the population of species B after 4 days? Find the daily growth factor for species B.

c Which species multiplies more rapidly?

38.

a The population of Elmira was 350,000 in 1970 and doubled in 20 years. What was the annual percent increase to the nearest hundredth of a percent?

b If a population doubles in 20 years, does the percent increase depend on the size of the original population?

c The population of Grayling doubled in 20 years. What was the annual percent increase to the nearest hundredth of a percent?

Exponential Functions

Introduction

In Section 7.1, p. 449, we studied functions that describe exponential growth or decay. More formally, we define an **exponential function** as follows.

Definition 7.19 Exponential Function.

An **exponential function** has the form

$$f(x) = ab^x, \qquad \text{where} \quad b > 0 \quad \text{and} \quad b \neq 1, \quad a \neq 0$$

Some examples of exponential functions are

$$f(x) = 5^x, \qquad P(t) = 250(1.7)^t, \qquad \text{and} \qquad g(n) = 2.4(0.3)^n$$

The constant a is the y-intercept of the graph because

$$f(0) = a \cdot b^0 = a \cdot 1 = a$$

For the examples above, we find that the y-intercepts are

$$f(0) = 5^0 = 1$$
$$P(0) = 250(1.7)^0 = 250$$
$$g(0) = 2.4(0.3)^0 = 2.4$$

The positive constant b is called the **base** of the exponential function.

Checkpoint 7.20 QuickCheck 1. Which of the following is an exponential function?

- ⊙ $f(x) = 3x^4$
- ⊙ $f(x) = 3(4)^x$
- ⊙ $f(x) = 2x^{\frac{3}{4}}$
- ⊙ $f(x) = \dfrac{4}{x^3}$

Answer. Choice 2

Solution. $f(x) = 3(4)^x$ is an exponential function,

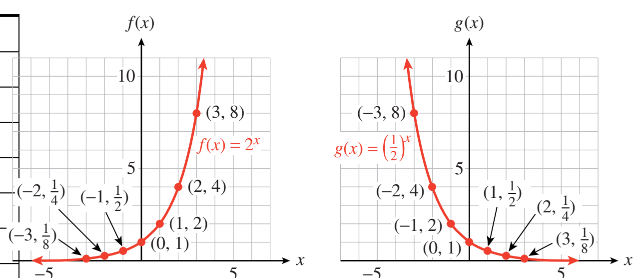
Note 7.21

- We do not allow b to be negative, because if $b < 0$, then b^x is not a real number for some values of x. For example, if $b = -4$ and $f(x) = (-4)^x$, then $f(1/2) = (-4)^{1/2}$ is an imaginary number.
- We also exclude $b = 1$ as a base because $1^x = 1$ for all values of x; hence the function $f(x) = 1^x$ is actually the constant function $f(x) = 1$.

Graphs of Exponential Functions

The graphs of exponential functions have two characteristic shapes, depending on whether the base, b, is greater than 1 or less than 1. As typical examples, consider the graphs of $f(x) = 2^x$ and $g(x) = \left(\dfrac{1}{2}\right)^x$ shown below. Some values for f and g are recorded in the tables.

x	$f(x)$	x	$g(x)$
-3	$\frac{1}{8}$	-3	8
-2	$\frac{1}{4}$	-2	4
-1	$\frac{1}{2}$	-1	2
0	1	0	1
1	2	1	$\frac{1}{2}$
2	4	2	$\frac{1}{4}$
3	8	3	$\frac{1}{8}$



(a) **(b)**

Notice that $f(x) = 2^x$ is an increasing function and $g(x) = \left(\dfrac{1}{2}\right)^x$ is a decreasing function. Both are concave up. In general, exponential functions have the following properties.

Properties of Exponential Functions, $f(x) = ab^x$, $a > 0$.

1. If $b > 1$, the function is increasing and concave up;
   if $0 < b < 1$, the function is decreasing and concave up.
2. The y-intercept is $(0, a)$. There is no x-intercept.

In the table for $f(x)$, you can see that as the x-values decrease toward negative infinity, the corresponding y-values decrease toward zero. As a result, the graph of f decreases toward the x-axis, but never touches it, as we move to the left. The negative x-axis is a **horizontal asymptote** for exponential functions with $b > 1$, as shown in figure (a).

For exponential functions with $0 < b < 1$, the positive x-axis is an asymptote, as illustrated in figure (b).

Checkpoint 7.22 QuickCheck 2. Which statement is true?

- ⊙ An exponential function is not defined for negative inputs.
- ⊙ The outputs of an exponential function cannot be negative.
- ⊙ The y-intercept of the function $f(x) = 2(3)^x$ is $(0, 6)$.
- ⊙ The function $f(x) = 16(0.5)^x$ decreases by 8 each time we increase x by 1.

Answer. Choice 2

Solution. The outputs of an exponential function cannot be negative.

How does the value of b affect the graph? For increasing functions, the larger the value of the base, b, the faster the function grows. In the Example, p. 470 below, we compare two exponential functions with different bases.

Example 7.23

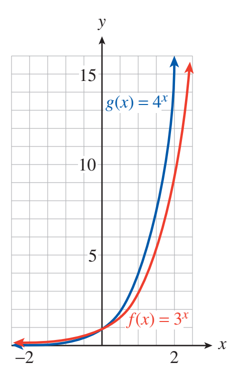
Compare the graphs of $f(x) = 3^x$ and $g(x) = 4^x$.

Solution. First, we evaluate each function for several convenient values, as shown in the table.

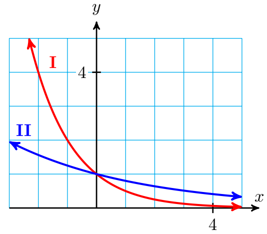
Then we plot the points for each function and connect them with smooth curves. Note that $a = 1$ for both functions, so their graphs have the same y-intercept, $(0, 1)$.

For positive x-values, $g(x)$ is always larger than $f(x)$, and is increasing more rapidly. In the figure, we can see that $g(x) = 4^x$ climbs more rapidly than $f(x) = 3^x$. However, for negative x-values, $g(x)$ is smaller than $f(x)$.

x	$f(x)$	$g(x)$
-2	$\frac{1}{9}$	$\frac{1}{16}$
-1	$\frac{1}{3}$	$\frac{1}{4}$
0	1	1
1	3	4
2	9	16



Checkpoint 7.24 Practice 1.



a. Which of the graphs above is $p(x) = 0.8^x$ (☐ I ☐ II) , and which is $q(x) = 0.5^x$ (☐ I ☐ II) ?

b. Which graph decreases more steeply? (☐ I ☐ II)

Answer 1. II

Answer 2. I

Answer 3. I

Solution.

a. Graph I is q, and graph II is p.

b. q decreases more sreeply

Checkpoint 7.25 QuickCheck 3. True or False.

a. An exponential function $f(x) = b^x$ is always positive. (☐ True ☐ False)

b. The function $f(x) = ab^x$ has a horizontal asymptote at $y = 0$. (☐ True ☐ False)

c. The value of b determines how rapidly the graph of $f(x) = ab^x$ increases or decreases. (☐ True ☐ False)

d. The graph of $f(x) = ab^x$ is decreasing if $b < 0$. (☐ True ☐ False)

Answer 1. True

Answer 2. True

Answer 3. True

Answer 4. False

Solution.

a. True

b. True

c. True

d. False

Comparing Exponential and Power Functions

We have studied several families of functions, including linear, quadratic, and power functions. Exponential functions are useful because they model growth or decline by a constant factor.

Caution 7.26 Exponential functions are not the same as the power functions we studied earlier. Although both involve expressions with exponents, it is the location of the variable that makes the difference. A power function, $h(x) = hx^p$, has a variable base and a constant exponent. An exponential function, $f(x) = ab^x$, has a constant base and a variable exponent.

Power Functions vs Exponential Functions.

	Power Functions	**Exponential Functions**
General formula	$h(x) = kx^p$	$f(x) = ab^x$
Description	variable base and constant exponent	constant base and variable exponent
Example	$h(x) = 2x^3$	$f(x) = 2(3^x)$

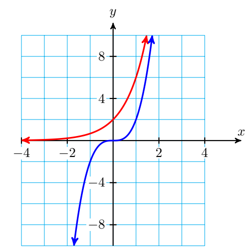
These two families of functions have very different properties, as well.

Example 7.27

Which of the functions shown below is exponential, and which is a cubic power function? Find an equation for each.

x	$h(x)$
-3	-54
-2	-16
-1	-2
0	0
1	2
2	16
3	54

x	$f(x)$
-3	$\frac{2}{27}$
-2	$\frac{1}{4}$
-1	$\frac{2}{3}$
0	2
1	6
2	18
3	54

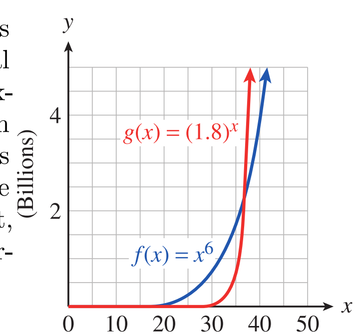


Solution. We can tell that f is the exponential function because its values increase by a factor of 3 for each unit increase in x. The base of the function is $b = 3$. We also see from the table that $a = f(0) = 2$, so $f(x) = 2(3^x)$. Thus, h must be the power function, $h(x) = kx^3$. To find k, we notice that $h(1) = k$, so $k = 2$, and $h(x) = 2x^3$.

Note 7.28 As you can see from the figure, the graphs of the two functions in the previous Example are also quite different. In particular, note that the power function passes through the origin, while the exponential function approaches the negative x-axis as a horizontal asymptote.

From the table, we see that $h(3) = f(3) = 54$, so the two graphs intersect at $x = 3$. (They also intersect at approximately $x = 2.48$.) However, if you compare the values of $h(x) = 2x^3$ and $f(x) = 2(3^x)$ for larger values of x, you will see that eventually the exponential function overtakes the power function.

The relationship in Example 7.27, p. 472 holds true for all increasing power and exponential functions: For large enough values of x, the exponential function will always be greater than the power function, regardless of the parameters in the functions. The figure at left shows the graphs of $f(x) = x^6$ and $g(x) = 1.8^x$. At first, $f(x) > g(x)$, but at around $x = 37$, $g(x)$ overtakes $f(x)$, and $g(x) > f(x)$ for all $x > 37$.



Checkpoint 7.29 Practice 3. Which of the following functions are exponential functions, and which are power functions?

a. $F(x) = 1.5^x$

b. $G(x) = 3x^{1.5}$

c. $H(x) = 3^{1.5x}$

d. $K(x) = (3x)^{1.5}$

Exponential: ☐ \(F(x)= 1.5^x \) ☐ \(G(x)= 3x^{1.5} \) ☐ \(H(x)= 3^{1.5x} \) ☐ \(K(x)= (3x)^{1.5} \) ☐ None of the above

Power: ☐ \(F(x)= 1.5^x \) ☐ \(G(x)= 3x^{1.5} \) ☐ \(H(x)= 3^{1.5x} \) ☐ \(K(x)= (3x)^{1.5} \) ☐ None of the above

Solution. Exponential: (a) and (c); power: (b) and (d)

Checkpoint 7.30 QuickCheck 4. True or False.

a. A power function eventually grows more rapidly then an exponential function. (☐ True ☐ False)

b. To find the base of an exponential function, we can divide any function value of an integer by the previous one. (☐ True ☐ False)

c. Power functions and exponential functions both pass through the origin. (☐ True ☐ False)

d. Exponential functions model growth or decline by a constant factor. (☐ True ☐ False)

Answer 1. False

Answer 2. True

Answer 3. False

Answer 4. True

Solution.

a. False

b. True

c. False

d. True

Exponential Equations

An **exponential equation** is one in which the variable is part of an exponent. For example, the equation

$$3^x = 81$$

is exponential.

Many exponential equations can be solved by writing both sides of the equation as powers with the same base. To solve the equation above, we write

$$3^x = 3^4$$

which is true if and only if $x = 4$.

In general, if two equivalent powers have the same base, then their exponents must be equal also, as long as the base is not 0 or ± 1.

In the next Example, we use the laws of exponents to express both sides of the equation as single powers of a common base.

Example 7.31

Solve the following equations.

a $3^{x-2} = 9^3$

b $27 \cdot 3^{-2x} = 9^{x+1}$

Solution.

a Using the fact that $9 = 3^2$, we write each side of the equation as a power of 3:

$$3^{x-2} = \left(3^2\right)^3 \qquad \text{Simplify the right side.}$$
$$3^{x-2} = 3^6$$

Now we equate the exponents to obtain

$$x - 2 = 6$$

The solution is $x = 8$.

b We write each factor as a power of 3.

$$3^3 \cdot 3^{-2x} = \left(3^2\right)^{x+1} \qquad 27 = 3^3 \text{ and } 9 = 3^2.$$

We use the laws of exponents to simplify each side:

$$3^{3-2x} = 3^{2x+2}$$

Now we equate the exponents to obtain

$$3 - 2x = 2x + 2$$
$$-4x = -1$$

The solution is $x = \frac{1}{4}$.

Checkpoint 7.32 QuickCheck 5. Which is a good strategy for solving $3^{x-2} = 81$?

- ⊙ Divide both sides by 3.
- ⊙ Add 3^2 to both sides.
- ⊙ Simplify the left side.
- ⊙ Write the right side as a power of 3.

Answer. Choice 4

Solution. Write the right side as a power of 3.

Checkpoint 7.33 Practice 3. Solve the equation $2^{x+2} = 128$.

$x =$__

Hint. Write each side as a power of 2.

Equate exponents.

Answer. 5

Solution. $x = 5$

Applications

Exponential equations arise frequently in the study of exponential growth.

Example 7.34

During the summer a population of fleas doubles in number every 5 days.

a If a population starts with 10 fleas, write a formula for the number of fleas present after t days.

b How long will it be before there are 10,240 fleas?

Solution.

a We let P represent the number of fleas present after t days. The original population of 10 fleas doubles not every day, but every 5 days. A table of values for P would look like this.

t	$P(t)$	
0	10	
5	20	$P(\mathbf{5}) = 10 \cdot 2^{\mathbf{1}} = 20$
10	40	$P(\mathbf{10}) = 10 \cdot 2^{\mathbf{2}} = 40$
15	80	$P(\mathbf{15}) = 10 \cdot 2^{\mathbf{3}} = 80$
20	160	$P(\mathbf{20}) = 10 \cdot 2^{\mathbf{4}} = 160$

Notice that the original population is multiplied by another factor of 2 every 5 days. We must divide t by 5 to see how many times the population doubles. The formula for P is thus

$$P(t) = 10 \cdot 2^{t/5}$$

b We set $P = \mathbf{10,240}$ and solve for t:

$$\mathbf{10,240} = 10 \cdot 2^{t/5} \qquad \text{Divide both sides by 10.}$$
$$1024 = 2^{t/5} \qquad \text{Write 1024 as a power of 2.}$$
$$2^{10} = 2^{t/5}$$

We equate the exponents to get $10 = \dfrac{t}{5}$, or $t = 50$. The population will grow to 10,240 fleas in 50 days.

In Example 4a, we can also use the laws of exponents to write the formula as

$$P(t) = 10 \cdot (2^{1/5})^t$$

From this form see that the growth factor for this function is $2^{1/5}$, or about 1.149. The flea population grows at about 14.9% per day.

Checkpoint 7.35 Practice 4. During an advertising campaign in a large city, the makers of Chip-O's corn chips estimate that the number of people who have heard of Chip-O's increases by a factor of 8 every 4 days.

a. If 100 people are given trial bags of Chip-O's to start the campaign, write a function, $N(t)$, for the number of people who have heard of Chip-O's after t days of advertising.

$N(t) =$________

b. Use your calculator to graph the function $N(t)$ on the domain $0 \leq t \leq 15$.

c. How many days should the makers run the campaign in order for Chip-O's to be familiar to 51,200 people? Use algebraic methods to find your answer and verify on your graph.

Answer: __ days

Answer 1. $100 \cdot 8^{\frac{t}{4}}$

Answer 2. 12

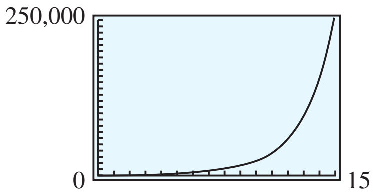
Solution.

a. $N(t) = 100 \cdot 8^{t/4}$

b. A graph is below.

c. 12 days

Graph for part (b):



Checkpoint 7.36 QuickCheck 6.

a. If two equivalent powers have the same nonzero base, $b \neq 1$ or -1, then their (☐ coefficients ☐ exponents) must be equal also.

b. An exponential equation is one in which the variable is part of (☐ an exponent ☐ a base ☐ a radical) .

c. $(3^2)^{x+1}$ simplifies to 3 raised to what power? _____

d. If a population doubles every 3 days, its growth factor is 2 raised to what power? ___

Answer 1. exponents

Answer 2. an exponent

Answer 3. $2x + 2$

Answer 4. $\frac{1}{3}$

Solution.

a. exponents

b. an exponent

c. $2x + 2$

d. $\dfrac{1}{3}$

Graphical Solution of Exponential Equations

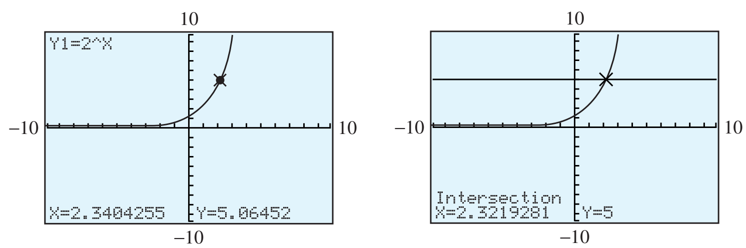
It is not always so easy to express both sides of the equation as powers of the same base. In the following sections, we will develop more general methods for finding exact solutions to exponential equations. But we can use a graphing utility to obtain approximate solutions.

Example 7.37

Use the graph of $y = 2^x$ to find an approximate solution to the equation $2^x = 5$ accurate to the nearest hundredth.

Solution. We enter $Y_1 = 2$ [^] X and use the standard graphing window ([ZOOM] 6) to obtain the graph shown in figure (a). We are looking for a point on this graph with y-coordinate 5.

Using the TRACE feature, we see that the y-coordinates are too small when $x < 2.1$ and too large when $x > 2.4$. The solution we want lies somewhere between $x = 2.1$ and $x = 2.4$, but this approximation is not accurate enough.



(a) **(b)**

To improve our approximation, we will use the **intersect** feature. Set $Y_2 = 5$ and press [GRAPH]. The x-coordinate of the intersection point of the two graphs is the solution of the equation $2^x = 5$ Activating the **intersect** command results in figure (b), and we see that, to the nearest hundredth, the solution is 2.32.

We can verify that our estimate is reasonable by substituting into the equation:

$$2^{2.32} \stackrel{?}{=} 5$$

We enter 2 [^] 2.32 [ENTER] to get 4.993322196. This number is not equal to 5, but it is close, so we believe that $x = 2.32$ is a reasonable approximation to the solution of the equation $2^x = 5$.

Checkpoint 7.38 **Practice 5.** Use the graph of $y = 5^x$ to find an approximate solution to $5^x = 285$, accurate to two decimal places.

Answer: $x \approx$___

Answer. 3.51209

Solution. The point on the graph where $y = 285$ has $x \approx 3.51$

Problem Set 7.2

Warm Up

In Problems 1–4, use the laws of exponents to simplify.

1.

a $3^x\,3^4$ b $(3^x)^4$ c $3^x\,4^x$

2.
a $8^x\, 8^x$ b $8^{x+2}\, 8^{x-1}$ c $\dfrac{8^{2x}}{8^x}$

3.
a $b^{-4t}\, b^{2t}$ b $(b^t)^{1/2}$ c $b^{t-1}\, b^{1-t}$

4.
a $b^{t/2}\, b^{t/2}$ b $\dfrac{b^{2t}}{b}$ c $b^{1/t}\, b^t$

For Problems 5 and 6, solve. Round your answers to two places if necessary.

5. $116,473 = 48,600(1 + r)^{15}$ **6.** $10.56 = 12.4(1 - r)^{20}$

Skills Practice

For Problems 7-14, solve.

7. $5^{x+2} = 25^{4/3}$ **8.** $3^{2x-1} = \dfrac{\sqrt{3}}{9}$

9. $4 \cdot 2^{x-3} = 8^{-2x}$ **10.** $9 \cdot 3^{x+2} = 81^{-x}$

11. $27^{4x+2} = 81^{x-1}$ **12.** $16^{2-3x} = 64^{x+5}$

13. $10^{x^2-1} = 1000$ **14.** $5^{x^2-x-4} = 25$

For Problems 15 and 16,

a graph the function in the window

$$\text{Xmin} = -4.7 \qquad \text{Xmax} = 4.7$$
$$\text{Ymin} = 0 \qquad \text{Ymax} = 10$$

b Use your graph to find an approximate solution accurate to the nearest hundredth.

15.
a $y = 3^{x-1}$
b $3^{x-1} = 4$

16.
a $y = 4^{-x}$
b $4^{-x} = 7$

17. Find the y-intercept of each exponential function and decide whether the graph is increasing or decreasing.
(a) $f(x) = 26(1.4)^x$
(b) $g(x) = 1.2(0.84)^x$
(c) $h(x) = 75\left(\dfrac{4}{5}\right)^x$
(d) $k(x) = \dfrac{2}{3}\left(\dfrac{9}{8}\right)^x$

18. Decide whether each function is an exponential function, a power function, or neither.
a $g(t) = 3t^{0.4}$
b $h(t) = 4(0.3)^t$
c $D(x) = 6x^{1/2}$
d $E(x) = 4x + x^4$

For Problems 19 and 20, graph each pair of functions on the same axes by making a table of values and plotting points by hand. Choose appropriate scales for the axes.

19.
a $f(x) = 3^x$
b $g(x) = \left(\dfrac{1}{3}\right)^x$

20.
a $h(t) = 4^{-t}$
b $q(t) = -4^t$

21. Which of the following functions have identical graphs? Explain why.

a $h(x) = 6^x$

b $k(x) = \left(\dfrac{1}{6}\right)^x$

c $m(x) = 6^{-x}$

d $n(x) = \dfrac{1}{6^x}$

For Problems 22 and 23:

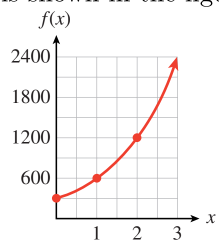
a Use a graphing calculator to obtain the graphs for $-5 \le x \le 5$.

b Find the largest and smallest function values on the interval $-5 \le x \le 5$.

22. $h(x) = 2.4^x$

23. $P(x) = 0.7^x$

24. The graph of $f(x) = P_0 a^x$ is shown in the figure.



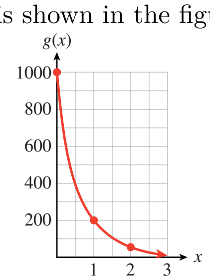
a Read the value of P_0 from the graph.

b Make a short table of values for the function by reading values from the graph. Does your table confirm that the function is exponential?

c Use your table to calculate the growth factor a.

d Using your answers to parts (a) and (c), write a formula for $f(x)$.

25. The graph of $g(x) = P_0 a^x$ is shown in the figure.



a Read the value of P_0 from the graph.

b Make a short table of values for the function by reading values from the graph. Does your table confirm that the function is exponential?

c Use your table to calculate the decay factor a.

d Using your answers to parts (a) and (c), write a formula for $g(x)$.

For Problems 26 and 27, which tables could describe exponential functions? Explain why or why not. If the function is exponential, find its growth or decay factor.

26.

a

h	a
0	70
1	7
2	0.7
3	0.07
4	0.007

b

t	Q
0	0
1	$\frac{1}{4}$
2	1
3	$\frac{9}{4}$
4	4

27.

a

t	y
1	100
2	50
3	$33\frac{1}{3}$
4	25
5	20

b

x	P
1	$\frac{1}{2}$
2	1
3	2
4	4
5	8

Applications

28. A mobile home loses 20% of its value every 3 years.

a A certain mobile home costs \$20,000. Write a function for its value after t years.

b Use your calculator to graph your function on the interval $0 \le t \le 30$.

c How long will it be before a \$20,000 mobile home depreciates to \$12,800? Use algebraic methods to find your answer, and verify it on your graph.

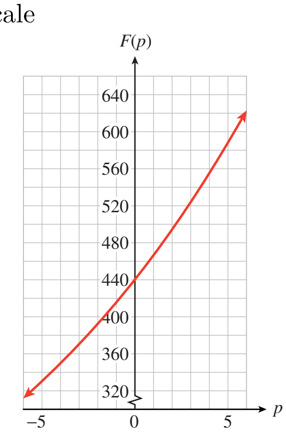
29. Before the advent of antibiotics, an outbreak of cholera might spread through a city so that the number of cases doubled every 6 days.

a Twenty-six cases were discovered on July 5. Write a function for the number of cases of cholera t days later.

b Use your calculator to graph your function on the interval $0 \le t \le 90$.

c When should hospitals expect to be treating 106,496 cases? Use algebraic methods to find your answer, and verify it on your graph.

30. The frequency of a musical note depends on its pitch. The graph shows that the frequency increases exponentially. The function $F(p) = F_0 b^p$ gives the frequency as a function of the number of half-tones, p, above the starting point on the scale



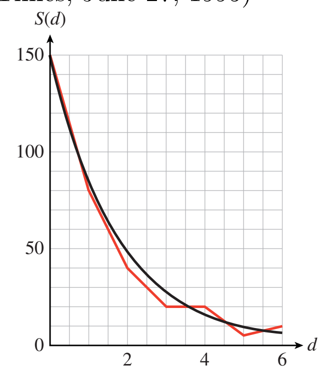
a Read the value of F_0 from the graph. (This is the frequency of the note A above middle C.)

b Find an approximation for the growth factor, b, by comparing two points on the graph. (Some of the points on the graph of $F(p)$ are approximately $(1, 466)$, $(2, 494)$, $(3, 523)$, and $(4, 554)$.)

c Using your answers to (a) and (b), write a formula for $F(p)$.

d The frequency doubles when you raise a note by one octave, which is equivalent to 12 half-tones. Use this information to find an exact value for b.

31. For several days after the Northridge earthquake on January 17, 1994, the area received a number of significant aftershocks. The red graph shows that the number of aftershocks decreased exponentially over time. The graph of the function $S(d) = S_0 b^d$, shown in black, approximates the data. (Source: *Los Angeles Times*, June 27, 1995)



a Read the value of S_0 from the graph.

b Find an approximation for the decay factor, b, by comparing two points on the graph. (Some of the points on the graph of $S(d)$ are approximately $(1, 82)$, $(2, 45)$, $(3, 25)$, and $(4, 14)$.)

c Using your answers to (a) and (b), write a formula for $S(d)$.

For Problems 32 and 33, fill in the tables. Graph each pair of functions in the same window. Then answer the questions below.

a For how many values of x does $f(x) = g(x)$?

b Estimate the value(s) of x for which $f(x) = g(x)$.

c For what values of x is $f(x) < g(x)$?

d Which function grows more rapidly for large values of x?

32.

x	$f(x) = x^2$	$g(x) = 2^x$
-2		
-1		
0		
1		
2		
3		
4		
5		

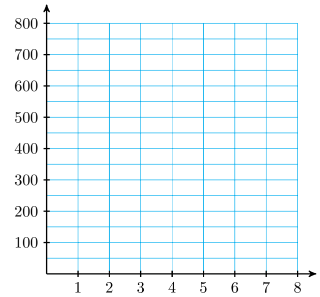
33.

x	$f(x) = x^3$	$g(x) = 3^x$
-2		
-1		
0		
1		
2		
3		
4		
5		

34. Related species living in the same area often evolve in different sizes to minimize competition for food and habitat. Here are the masses of eight species of fruit pigeon found in New Guinea, ranked from smallest to largest. (Source: Burton, 1998)

Size rank	1	2	3	4	5	6	7	8
Mass (grams)	49	76	123	163	245	414	592	802

a Plot the masses of the pigeons against their order of increasing size. What kind of function might fit the data?



b Compute the ratios of the masses of successive sizes of fruit pigeons. Are the ratios approximately constant? What does this information tell you about your answer to part (a)?

c Compute the average ratio to two decimal places. Using this ratio, estimate the mass of a hypothetical fruit pigeon of size rank zero.

d Using your answers to part (c), write an exponential function that approximates the data. Graph this function on top of the data and evaluate the fit.

Logarithms

In this section, we introduce a new mathematical tool called a **logarithm**, which will help us solve exponential equations.

A Logarithm is an Exponent

Suppose that a colony of bacteria doubles in size every day. If the colony starts with 50 bacteria, how long will it be before there are 800 bacteria? We answer questions of this type by writing and solving an exponential equation. The function

$$P(t) = 50 \cdot 2^t$$

gives the number of bacteria present on day t, so we must solve the equation

$$800 = 50 \cdot 2^t$$

We are looking for an unknown exponent on base 2. Dividing both sides by 50 yields

$$16 = 2^t$$

This equation asks the question:

To what power must we raise 2 in order to get 16?

Because $2^4 = 16$, we see that the solution of the equation is 4. You can check that$t = 4$ solves the original problem:

$$P(\mathbf{4}) = 50 \cdot 2^{\mathbf{4}} = 800$$

The unknown exponent that solves the equation $16 = 2^t$ is called the base 2 **logarithm** of 16. The exponent in this case is 4, and we write this fact as

$$\log_2 16 = 4$$

So, we solve an exponential equation by computing a logarithm. We make the following definition.

Definition 7.39 Definition of Logarithm.

(logarithm)For $b > 0, b \neq 1$, the **base** b **logarithm of** x, written $\log_b x$, is the exponent to which b must be raised in order to yield x.

Example 7.40

Compute the logarithms.

a $\log_3 9 = 2$ because $3^2 = 9$

b $\log_5 125 = 3$ because $5^3 = 125$

c $\log_4 \dfrac{1}{16} = -2$ because $4^{-2} = \dfrac{1}{16}$

d $\log_5 \sqrt{5} = \dfrac{1}{2}$ because $5^{1/2} = \sqrt{5}$

The positive constant b is called the **base** of the exponential function.

Checkpoint 7.41 Practice 1. Find each logarithm.

a. $\log_3 81 =$__

b. $\log_{10} \dfrac{1}{1000} =$__

Answer 1. 4

Answer 2. -3

Solution.

a. 4

b. -3

From the definition of a logarithm and the examples above, we see that the following two statements are equivalent.

Logarithms and Exponents: Conversion Equations.

If $b > 0$, $b \neq 1$, and $x > 0$,

$$y = \log_b x \quad \text{if and only if} \quad x = b^y$$

This equivalence tells us that the logarithm, y, is the same as the *exponent* in $x = b^y$. We see again that *a logarithm is an exponent*; it is the exponent to which b must be raised to yield x.

The conversion equations allow us to convert from logarithmic to exponential form, or vice versa. You should memorize the conversion equations, because we will use them frequently.

As special cases of the equivalence above, we can compute the following useful logarithms.

For any base $b > 0, b \neq 1$,

Some Useful Logarithms.

For any base $b > 0, b \neq 1$,

$$\begin{aligned} \log_b b = 1 &\quad \text{because} \quad b^1 = b \\ \log_b 1 = 0 &\quad \text{because} \quad b^0 = 1 \\ \log_b b^x = x &\quad \text{because} \quad b^x = b^x \end{aligned}$$

Example 7.42

a $\log_2 2 = 1$ b $\log_5 1 = 0$ c $\log_3 3^4 = 4$

Checkpoint 7.43 QuickCheck 1.

a. $\log_6 216 =$ __ because 6^__ $= 216$

b. $\log_8 1 =$ __ because 8^__ $= 1$

c. $\log_9 9^5 =$ __

d. If $\log_w 87 = p$, then $87 =$ _____

Answer 1. 3
Answer 2. 3
Answer 3. 0
Answer 4. 0
Answer 5. 5
Answer 6. w^p
Solution.

a. 3

b. 0

c. 5

d. w^p

Logs and Exponential Equations

We use logarithms to solve exponential equations, just as we use square roots to solve quadratic equations. Consider the two equations

$$x^2 = 25 \quad \text{and} \quad 2^x = 8$$

We solve the first equation by taking a square root, and we solve the second equation by computing a logarithm:

$$x = \pm\sqrt{25} = \pm 5 \quad \text{and} \quad x = \log_2 8 = 3$$

The operation of taking a base b logarithm is the inverse operation for raising the base b to a power, just as extracting square roots is the inverse of squaring a number.

Every exponential equation can be rewritten in logarithmic form by using the conversion equations. Thus,

$$3 = \log_2 8 \quad \text{and} \quad 8 = 2^3$$

are equivalent statements, just as

$$5 = \sqrt{25} \qquad \text{and} \qquad 25 = 5^2$$

are equivalent statements. Rewriting an equation in logarithmic form is a basic strategy for finding its solution.

Example 7.44

Rewrite each equation in logarithmic form.

a $2^{-1} = \dfrac{1}{2}$

b $a^{1/5} = 2.8$

c $6^{1.5} = T$

d $M^v = 3K$

Solution. First identify the base b, and then the exponent or logarithm y. Use the conversion equations to rewrite $b^y = x$ in the form $\log_b x = y$.

a The base is 2 and the exponent is -1. Thus, $\log_2 \dfrac{1}{2} = -1$.

b The base is a and the exponent is $\dfrac{1}{5}$. Thus, $\log_a 2.8 = \dfrac{1}{5}$.

c The base is 6 and the exponent is 1.5. Thus, $\log_6 T = 1.5$.

d The base is M and the exponent is v. Thus, $\log_M 3K = v$.

Checkpoint 7.45 Practice 2. Rewrite each equation in logarithmic form.

a. $8^{-1/3} = \dfrac{1}{2}$

The given equation is equivalent to one of the form $\log_b m = n$, where

$b =$___

$m =$___, and

$n =$___

b. $5^x = 46$

The given equation is equivalent to one of the form $\log_b m = n$, where

$b =$___

$m =$___, and

$n =$___

Answer 1. 8

Answer 2. $\frac{1}{2}$

Answer 3. $-\frac{1}{3}$

Answer 4. 5

Answer 5. 46

Answer 6. x

Solution.

a. $\log_8 \left(\dfrac{1}{2}\right) = \dfrac{-1}{3}$

b. $\log_5 46 = x$

Checkpoint 7.46 QuickCheck 2.

a. What is the inverse operation for squaring a number? (☐ Taking a logarithm. ☐ Taking a square root. ☐ Dividing by the number. ☐ Subtracting the number.)

b. What is the inverse operation for raising to a power? (☐ Taking a logarithm. ☐ Taking a square root. ☐ Dividing by the number. ☐ Subtracting the number.)

c. Every exponential equation can be rewritten in logarithmic form by using the (☐ reciprocals ☐ slope ☐ conversion equations) .

d. A logarithm is an unknown (☐ quotient ☐ product ☐ exponent ☐ base) .

Answer 1. Taking a square root.

Answer 2. Taking a logarithm.

Answer 3. conversion equations

Answer 4. exponent

Solution.

a. Taking a square root

b. Taking a logarithm

c. conversion equations

d. exponent

Approximating Logarithms

Now let's consider computing logarithms that are not obvious by inspection. Suppose we would like to solve the equation

$$2^x = 26$$

The solution of this equation is $x = \log_2 26$, but can we find a decimal approximation for this value? There is no integer power of 2 that equals 26, because

$$2^4 = 16 \qquad \text{and} \qquad 2^5 = 32$$

So $\log_2 26$ must be between 4 and 5. We can use trial and error to find the value of $\log_2 26$ to the nearest tenth. Use your calculator to make a table of values for $y = 2^x$, starting with $x = 4$ and using increments of 0.1.

x	2^x
4	$2^4 = 16$
4.1	$2^{4.1} = 17.148$
4.2	$2^{4.2} = 18.379$
4.3	$2^{4.3} = 19.698$
4.4	$2^{4.4} = 21.112$

x	2^x
4.5	$2^{4.5} = 22.627$
4.6	$2^{4.6} = 24.251$
4.7	$2^{\mathbf{4.7}} = 25.992$
4.8	$2^{\mathbf{4.8}} = 27.858$
4.9	$2^{4.9} = 29.857$

From the table we see that 26 is between $2^{4.7}$ and $2^{4.8}$, and is closer to $2^{4.7}$. To the nearest tenth, $\log_2 26 \approx 4.7$.

Trial and error can be a time-consuming process. In the Example, p. 487 below, we illustrate a graphical method for estimating the value of a logarithm.

Example 7.47

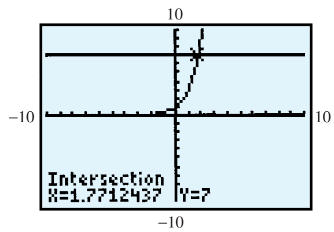
Approximate $\log_3 7$ to the nearest hundredth.

Solution. If $\log_3 7 = x$, then $3^x = 7$. We will use the graph of $y = 3^x$ to approximate a solution to $3^x = 7$.

We graph $Y_1 = 3$ [^] X and $Y_2 = 7$ in the standard window ([ZOOM] 6) to obtain the graph shown below. Next we activate the intersect feature to find that the two graphs intersect at the point $(1.7712437, 7)$. Because this point lies on the graph of $y = 3^x$, we know that

$$3^{1.7712437} \approx 7 \quad , \text{ or } \quad \log_3 7 \approx 1.7712437$$

To the nearest hundredth, $\log_3 7 \approx 1.77$.



Checkpoint 7.48 Practice 3.

a. Rewrite the equation $3^x = 90$ in logarithmic form.

The given equation is equivalent to one of the form $\log_b m = n$, where

$b =$___

$m =$___, and

$n =$___

b. Use a graph to approximate the solution to the equation in part (a). Round your answer to three decimal places.

$x \approx$____

Answer 1. 3
Answer 2. 90
Answer 3. x
Answer 4. 4.0959
Solution.

a. $\log_8(90) = x$

b. $x \approx 4.096$

Base 10 Logarithms

Some logarithms are used so frequently in applications that their values are programmed into scientific and graphing calculators. These are the base 10 logarithms, such as

$$\log_{10} 1000 = 3 \quad \text{and} \quad \log_{10} 0.01 = -2$$

Base 10 logarithms are called **common logarithms**, and the subscript 10 is often omitted, so that $\log x$ is understood to mean $\log_{10} x$.

Checkpoint 7.49 QuickCheck 3. If $\log x = 2.5$, what can you say about x?

- ⊙ $x = 25$
- ⊙ x is between 20 and 30
- ⊙ x is between 100 and 1000
- ⊙ $x = \dfrac{1}{2.5} = 0.4$

Answer. Choice 3

Solution. x is between 100 and 1000

To evaluate a base 10 logarithm, we use the LOG key on a calculator. Many logarithms are irrational numbers, and the calculator gives as many digits as its display allows. We can then round off to the desired accuracy.

Example 7.50

Approximate the following logarithms to 2 decimal places.

a $\log 6.5$ b $\log 256$

Solution.

a The keying sequence LOG 6.5) ENTER produces the display

log (6.5)
.812913566

so $\log 6.5 \approx 0.81$.

b The keying sequence LOG 256) ENTER yields 2.408239965, so $\log 256 \approx 2.41$.

Note 7.51 We can check the approximations found in Example 7.50, p. 488 with our conversion equations. Remember that a logarithm is an exponent, and in this example the base is 10. We find that

$$10^{0.81} \approx 6.45654229$$

and

$$10^{2.41} \approx 257.0395783$$

so our approximations are reasonable, although you can see that rounding a logarithm to 2 decimal places does lose some accuracy.

For this reason, *rounding logarithms to 4 decimal places is customary.*

Checkpoint 7.52 Practice 4. Approximate the logarithms to four decimal places.

a. $\log 0.2 \approx$_____

b. $\log 846,000 \approx$_____

Answer 1. -0.699

Answer 2. 5.9274

Solution.

a. -0.6990

b. 5.9274

Checkpoint 7.53 QuickCheck 4. True or False.

a. The LOG key on a calculator computes logarithms base 2. (☐ True ☐ False)

b. $\log_5 100$ is a number between 2 and 3. (☐ True ☐ False)

c. Rounding a logarithm to two decimal places gives very accurate results. (☐ True ☐ False)

d. The value of $\log_4 392$ is also the solution of $4^x = 392$. (☐ True ☐ False)

Answer 1. False
Answer 2. True
Answer 3. False
Answer 4. True
Solution.

a. False

b. True

c. False

d. True

Solving Exponential Equations

We can now solve any exponential equation with base 10.

Example 7.54

Solve the equation $38 = 95 - 15 \cdot 10^{0.4x}$

Solution. First, we isolate the power of 10: We subtract 95 from both sides of the equation and divide by −15 to obtain

$$-57 = -15 \cdot 10^{0.4x} \qquad \text{Divide by } -15.$$
$$3.8 = 10^{0.4x}$$

Next, we convert the equation to logarithmic form as

$$\log_{10} 3.8 = 0.4x$$

Solving for x yields

$$\frac{\log_{10} 3.8}{0.4} = x$$

We can evaluate this expression on the calculator by entering

LOG 3.8) ÷ 0.4 ENTER

which yields 1.449458992. Thus, to four decimal places, $x \approx 1.4495$.

Caution 7.55 Do not omit the parenthesis when entering the expression in Example 5. Without the parenthesis, you are calculating $\log_{10} \frac{3.8}{0.4}$. You can check that this is not the same as $\frac{\log_{10} 3.8}{0.4}$.

To solve exponential equations involving powers of 10, we can use the following steps.

Steps for Solving Exponential Equations.

1 Isolate the power on one side of the equation.

2 Rewrite the equation in logarithmic form.

3 Use a calculator, if necessary, to evaluate the logarithm.

4 Solve for the variable.

Checkpoint 7.56 QuickCheck 5. What is the first step in solving the equation $5(10)^x = 12$?

- ⊙ Multiply 5 times 10.
- ⊙ Take the log of both sides.
- ⊙ Get zero on one side of the equation.
- ⊙ Divide both sides by 5.

Answer. Divide both sides by 5.

Solution. Divide both sides by 5.

Checkpoint 7.57 Practice 5. Solve the equation $20 + 10^x = 220$.

$x =$ ______

Hint. Isolate the power of 10.
Rewrite in logarithmic form.

Answer. $\log(220 - 20)$

Solution. 2.3010

Application to Exponential Models

We have seen that exponential functions are used to describe some applications of growth and decay, $P(t) = P_0 b^t$. There are two common questions that arise in connection with exponential models:

1. Given a value of t, what is the corresponding value of $P(t)$?
2. Given a value of $P(t)$, what is find the corresponding value of t?

To answer the first question, we evaluate the function $P(t)$ at the appropriate value. To answer the second question, we must solve an exponential equation, and this usually involves logarithms.

Example 7.58

The value of a large tractor originally worth $30,000 depreciates exponentially according to the formula

$$V(t) = 30,000(10)^{-0.04t}$$

where t is in years. When will the tractor be worth half its original value?

Solution. We want to find the value of t for which $V(t) = 15,000$. That is, we want to solve the equation

$$15,000 = 30,000(10)^{-0.04t} \qquad \text{Divide both sides by 30,000.}$$

$$0.5 = 10^{-0.04t}$$

Once we have isolated the power, we convert the equation to logarithmic form.

$$\log_{10} 0.5 = -0.04t \qquad \text{Divide both sides by } -0.04.$$
$$\frac{\log_{10} 0.5}{-0.04} = t$$

To evaluate this expression, we key in

LOG 0.5) ÷ (-) 0.04 ENTER

to find $t \approx 7.525749892$. The tractor will be worth \$15,000 in approximately $7\frac{1}{2}$ years.

Checkpoint 7.59 QuickCheck 6. True or False.

a. The first step in solving an exponential equation is to convert to logarithmic form. (☐ True ☐ False)

b. To solve the equation $10^x = 19$, we divide both sides by 10. (☐ True ☐ False)

c. $\dfrac{\log 15}{6}$ is the same as $\log \dfrac{15}{6}$. (☐ True ☐ False)

d. Logarithms are used in the study of exponential growth and decay. (☐ True ☐ False)

Answer 1. False

Answer 2. False

Answer 3. False

Answer 4. True

Solution.

a. False

b. False

c. False

d. True

Checkpoint 7.60 Practice 6. The percentage of American homes with computers grew exponentially from 1994 to 1999. For $t = 0$ in 1994, the growth law was

$$P(t) = 25.85 \cdot 10^{0.052t}$$

[Source: Los Angeles Times, August 20, 1999]

a. What percent of American homes had computers in 1994?

Answer: ____%

b. If the percentage of homes with computers continued to grow at the same rate, when did 90% of American homes have a computer?

Answer: $t \approx$___, which was the year (☐ 2004 ☐ 2005)

c. Do you think that the function $P(t)$ will continue to model the percentage of American homes with computers? Why or why not?

⊙ Yes: The percentage will continue increasing, faster and faster.

⊙ No: the percent of homes with computers cannot exceed 100%.

Answer 1. 25.85

Answer 2. $\frac{\log\left(\frac{90}{25.85}\right)}{0.052}$

Answer 3. 2004

Answer 4. Choice 2

Solution.

a. 25.85%

b. $t \approx 10.4$ (year 2004)

c. No, the percent of homes with computers cannot exceed 100%.

At this stage, it seems we will only be able to solve exponential equations in which the base is 10. However, we will see in future sections how the properties of logarithms enable us to solve exponential equations with any base.

Problem Set 7.3

Warm Up

1. There were 300 lizards on a small island in 2000, and since then the population has been growing by 15% each year.

a Write a formula for the size of the lizard population t years after 2000.

b Write an equation to calculate how long it will take the lizard population to reach 500.

c Use trial and error to estimate the solution to the equation.

2. Complete each table. Then use the tables to approximate each logarithm between two integers.

a

2^{-5}	
2^{-4}	
2^{-3}	
2^{-2}	
2^{-1}	
2^{0}	
2^{1}	
2^{2}	
2^{3}	
2^{4}	
2^{5}	

$\_\_\_ < \log_2 12 < \_\_\_$

$\_\_\_ < \log_2 \frac{1}{6} < \_\_\_$

$\_\_\_ < \log_2 0.02 < \_\_\_$

b

10^{-5}	
10^{-4}	
10^{-3}	
10^{-2}	
10^{-1}	
10^{0}	
10^{1}	
10^{2}	
10^{3}	
10^{4}	
10^{5}	

$\_\_\_ < \log_{10} 296 < \_\_\_$

$\_\_\_ < \log_{10} 0.095 < \_\_\_$

$\_\_\_ < \log_{10} 3.8 < \_\_\_$

Skills Practice

For Problems 3-8, use the definition to find each logarithm without using a calculator.

3.
a $\log_7 49$ b $\log_5 625$

4.
a $\log_4 4$ b $\log_6 1$

5.
a $\log_3 \sqrt{3}$ b $\log_3 \dfrac{1}{3}$

6.
a $\log_9 9^{-6}$ b $\log_8 8^5$

7.
a $\log_{10} 0.1$ b $\log_{10} 10,000$

8.
a $\log_{10} 0.001$ b $\log_{10} 1000$

For Problems 9 and 10, without using a calculator, estimate the log between two integers.

9.
a $x = \log_{10} 5678$
b $y = \log_{10} 0.25$

10.
a $\log_{10} 137,624$
b $\log_{10} 0.009$

For Problems 11–16, solve. Round your answers to hundredths.

11. $10^{-3x} = 5$

12. $25 \cdot 10^{0.2x} = 80$

13. $12.2 = 2(10^{1.4x}) - 11.6$

14. $16.1 = 28.2 - 4\left(10^{-0.7x}\right)$

15. $3\left(10^{-1.5x}\right) - 14.7 = 17.1$

16. $80\left(1 - 10^{-0.2x}\right) = 65$

For Problems 17 and 18, rewrite each equation in logarithmic form.

17.
a $t^{3/2} = 16$
b $0.8^{1.2} = M$

18.
a $3.7^{2.5} = Q$
b $3^{-0.2t} = 2N_0$

For Problems 19 and 20, rewrite each equation in exponential form.

19.
a $\log_{16} 256 = w$
b $\log_b 9 = -2$

20.
a $\log_{10} a = -2.3$
b $\log_4 36 = 2q - 1$

21. Solve each equation, writing your answer as a logarithm. Then use trial and error to approximate the logarithm to tenths.
a $4^x = 2.5$ b $2^x = 0.2$

For Problems 22–24, use a graph to approximate each logarithm to the nearest hundredth. (Hint: Graph an appropriate function $y = b^x$.)

22. $\log_3 67.9$

23. $\log_5 86.3$

24. $\log_{10} 50$

Applications

25. In 2015, Summit City used 4.2 million kilowatt-hours of electricity, and the demand for electricity has increased every year according to the formula

$$E(t) = 4.2(10^{0.0261t})$$

a When will Summit City need 10 million kilowatt-hours annually?

b What is the annual growth rate in the demand for electricity?

26. Radium, a radioactive element, was used to paint watches and instrument dials until its serious health effects were discovered in the 1920's. Radium-226 is the most stable isotope of radium and decays according to the formula

$$R(t) = R_0(10^{-1.0177t})$$

where t is in years.

a How long will it take for one gram of radium-226 to decay to one-half gram?

b What is the annual decay rate for radium-226?

The atmospheric pressure decreases with altitude above the surface of the Earth, according to the function

$$P(h) = 30(10)^{-0.09h}$$

where h is altitude given in miles, and P is atmospheric pressure in inches of mercury. Graph this function in the window

Xmin = 0	Xmax = 9.4
Ymin = 0	Ymax = 30

Solve Problems 27 and 28 algebraically, then verify with your graph.

27.

a The elevation of Mount Everest, the highest mountain in the world, is 29,028 feet. What is the atmospheric pressure at the top?

Hint: 1 mile = 5280 feet

b The atmospheric pressure at the top of Mount McKinley, the highest peak in the United States, is 13.51 inches of mercury. Estimate the elevation of Mount McKinley.

28.

a What is the atmospheric pressure at sea level ($h = 0$)?

b Find the height above sea level at which the atmospheric pressure is equal to one-half the pressure at sea level.

29. From 1950 to 2000, the population of the state of Nevada increased according to the formula

$$P(t) = 162,000(10)^{0.02191t}$$

where t is measured in years since 1950.

(a) What was the population in 2000?

(b) What was the annual percent growth rate from 1950 to 2000?

(c) According to this model, when would the population of Nevada reach 3,000,000? (In fact the population of Nevada was 3,139,658 in 2020.)

30. From 1970 to 1990, the population of the state of California increased according to the formula

$$P(t) = 19,971,000(10)^{0.0088t}$$

where t is measured in years since 1970.

(a) What was the population in 1990?

(b) What was the annual percent growth rate from 1970 to 1990?

(c) According to this model, when would the population of California reach 40,000,000? (In fact the population of California was 39,747,267 in 2020.)

Equations Practice: For Problems 31 and 32, solve the equation and check your solutions.

31.

a $x^2 = 16$

b $2^x = 16$

c $x^{2/3} = 16$

d $x^3 = 16$

32.

a $2^{3x-2} = 16$

b $(3x-2)^2 = 16$

c $x^2 - 6x = 16$

d $x^2 - 3x = 16$

33.

a Complete the table.

x	x^2	$\log_{10} x$	$\log_{10} x^2$
1			
2			
3			
4			
5			
6			

b Do you notice a relationship between $\log_{10} x$ and $\log_{10} x^2$? State the relationship as an equation.

34.

a Complete the table.

x	$\frac{1}{x}$	$\log_{10} x$	$\log_{10} \frac{1}{x}$
1			
2			
3			
4			
5			
6			

b Do you notice a relationship between $\log_{10} x$ and $\log_{10} \frac{1}{x}$? State the relationship as an equation.

Properties of Logarithms

Logarithms are Exponents

Because logarithms are actually exponents, they have several properties that can be derived from the laws of exponents. Here are the laws we will need at present.

1. To multiply two powers with the same base, add the exponents and leave the base unchanged.

$$a^m \cdot a^n = a^{m+n}$$

2. To divide two powers with the same base, subtract the exponents and leave the base unchanged.

$$\frac{a^m}{a^n} = a^{m-n}$$

3. To raise a power to a power, keep the same base and multiply the exponents.

$$(a^m)^n = a^{mn}$$

Each of these laws corresponds to one of three properties of logarithms.

Properties of Logarithms.

If x, y, and $b > 0$, and $b \neq 1$, then

1 $\log_b xy = \log_b x + \log_b y$

2 $\log_b \dfrac{x}{y} = \log_b x - \log_b y$

3 $\log_b x^k = k \log_b x$

Study the examples below, keeping in mind that a logarithm is an exponent.

1. Property (1):

$$\begin{aligned} \log_2 32 = \log_2 (4 \cdot 8) \quad &= \log_2 4 + \log_2 8 && \text{because } 2^{\mathbf{5}} = 2^{\mathbf{2}} \cdot 2^{\mathbf{3}} \\ \mathbf{5} \quad &= \mathbf{2} + \mathbf{3} && 32 = 4 \cdot 8 \end{aligned}$$

2. Property (2):

$$\begin{aligned} \log_2 8 = \log_2 \frac{16}{2} \quad &= \log_2 16 - \log_2 2 && \text{because } 2^{\mathbf{3}} = \frac{2^{\mathbf{4}}}{2^{\mathbf{1}}} \\ \mathbf{3} \quad &= \mathbf{4} - \mathbf{1} && 8 = \frac{16}{2} \end{aligned}$$

3. Property (3):

$$\begin{aligned} \log_2 64 = \log_2 (4)^3 \quad &= 3 \log_2 4 && \text{because } \left(2^{\mathbf{2}}\right)^{\mathbf{3}} = 2^{\mathbf{6}} \\ \mathbf{6} \quad &= \mathbf{3} \cdot \mathbf{2} && (4)^{\mathbf{3}} = 64 \end{aligned}$$

Checkpoint 7.61 QuickCheck 1. Which statement is true?

- ⊙ $\log(8 + y) = \log 8 + \log y$
- ⊙ $\log(8y) = \log 8 + \log y$
- ⊙ $\log(8y) = \log 8 \cdot \log y$

⊙ $\log\left(\dfrac{8}{y}\right) = \dfrac{\log 8}{\log y}$

Answer. Choice 2

Solution. $\log(8y) = \log 8 + \log y$

Using the Properties of Logarithms

Of course, these properties are useful not so much for computing logs but rather for simplifying expressions that contain variables. We will use them to solve exponential equations. But first, we will practice applying the properties.

Example 7.62

Write $\log_b \sqrt{xy}$ as an expression in simpler logs.

Solution. First, we write $\sqrt{xy}$ using a fractional exponent:

$$\begin{aligned} \log_b xy &= \log_b (xy)^{1/2} && \text{Apply Property 3.} \\ &= \frac{1}{2}\log_b (xy) && \text{Apply Property 1 to the product } xy. \\ &= \frac{1}{2}(\log_b x + \log_b y) \end{aligned}$$

Thus, $\log_b \sqrt{xy} = \dfrac{1}{2}(\log_b x + \log_b y)$.

Checkpoint 7.63 **Practice 1.** Write $\log_b \dfrac{x}{y^2}$ as an expression in simpler logs.

Answer: $\log_{b\_} + \_ \cdot \log_{b\_}$

Answer 1. x

Answer 2. -2

Answer 3. y

Solution. $\log_b x - 2\log_b y$

We can also use the properties of logarithms to combine sums and differences of logarithms into one logarithm.

Example 7.64

Express $3(\log_b x - \log_b y)$ as a single logarithm with a coefficient of 1.

Solution. We begin by applying Property (2) to combine the logs.

$$\begin{aligned} 3(\log_b x - \log_b y) &= 3\log_b \left(\frac{x}{y}\right) && \text{Apply Property 3.} \\ &= \log_b \left(\frac{x}{y}\right)^3 \end{aligned}$$

Thus, $3(\log_b x - \log_b y) = \log_b \left(\dfrac{x}{y}\right)^3$.

Caution 7.65 Be careful when using the properties of logarithms! Compare the statements below:

1. $\log_b (2x) = \log_b 2 + \log_b x$ by Property 1,

but

$\log_b (2 + x) \neq \log_b 2 + \log_b x$

2. $\log_b \left(\dfrac{x}{5}\right) = \log_b x - \log_b 5$ by Property 2,

but

$\log_b \left(\dfrac{x}{5}\right) \neq \dfrac{\log_b x}{\log_b 5}$

Checkpoint 7.66 Practice 2. Express $2 \log_b x + 4 \log_b (x + 3)$ as a single logarithm with a coefficient of 1.

Answer: $\log_b$____________

Answer. $x^2(x + 3)^4$

Solution. $2 \log_b x + 4 \log_b (x + 3) = \log_b x^2(x + 3)^4$

Checkpoint 7.67 QuickCheck 2. True or False.

a. The log of a product is equal to the sum of the logs of the factors. (☐ True ☐ False)

b. The log of a quotient is the equal to the difference of the logs. (☐ True ☐ False)

c. The log of a power is equal to the exponent times the log of the base. (☐ True ☐ False)

d. The distributive law applies to the operation of taking a logarithm of a sum or difference. (☐ True ☐ False)

Answer 1. True

Answer 2. True

Answer 3. True

Answer 4. False

Solution.

a. True

b. True

c. True

d. False

Solving Exponential Equations

By using Property (3), we can now solve exponential equations in which the base is not 10. For example, to solve the equation

$$5^x = 7$$

we could rewrite the equation in logarithmic form to obtain the exact solution

$$x = \log_5 7$$

However, we cannot evaluate $\log_5 7$; there is no log base 5 button on the calculator. If we want a decimal approximation for the solution, we begin by taking the base 10 logarithm of both sides, even though the base of the power is not

10. This gives us

$$\log_{10}(5^x) = \log_{10} 7$$

Then we use Property (3) to rewrite the left side as

$$x \log_{10} 5 = \log_{10} 7$$

and divide both sides by $\log_{10} 5$ to get

$$x = \frac{\log_{10} 7}{\log_{10} 5}$$

On your calculator, enter the sequence
LOG 7) ÷ LOG 5) ENTER
to find that $x \approx 1.2091$.

Note 7.68 Note how using Property (3) allows us to solve the equation above: The variable, x, is no longer in the exponent, and it is multiplied by a constant, $\log_{10} 5$.

Caution 7.69 Do not confuse the expression $\dfrac{\log_{10} 7}{\log_{10} 5}$ with $\log_{10}\left(\dfrac{7}{5}\right)$; they are not the same! Although

$$\log_{10}\left(\frac{7}{5}\right) = \log_{10} 1.4 \approx 0.1416$$

We cannot simplify $\dfrac{\log_{10} 7}{\log_{10} 5}$; we must evaluate it as

$$(\log_{10} 7) \div (\log_{10} 5) \approx 1.2091$$

Example 7.70

Solve $1640 = 80 \cdot 6^{0.03x}$

Solution. First we divide both sides by 80 to isolate the power.

$$20.5 = 6^{0.03x}$$

Next, we take the base 10 logarithm of both sides of the equation and use Property (3) to get

$$\begin{aligned} \log_{10} 20.5 &= \log_{10} 6^{0.03x} && \text{Apply Property 3.} \\ &= 0.03x \log_{10} 6 \end{aligned}$$

On the right side of the equation, x is multiplied by two constants, 0.03 and $\log_{10} 6$. So, to solve for x we must divide both sides of the equation by $0.03 \log_{10} 6$. We use a calculator to evaluate the answer:

$$x = \frac{\log_{10} 20.5}{0.03 \log_{10} 6} \approx 56.19$$

(On your calculator, remember to enclose the denominator, $0.03 \log_{10} 6$, in parentheses.)

Caution 7.71 In the previous example, do not try to simplify

$$80 \cdot 6^{0.03x} \to 480^{0.03x} \quad \text{Incorrect!}$$

Remember that the order of operations tells us to compute the power $6^{0.03x}$

before multiplying by 80. To solve the equation, we must first isolate the power with the variable exponent.

Checkpoint 7.72 Practice 3. Solve $5(1.2)^{2.5x} = 77$

$x =$ ___________

Hint. Divide both sides by 5.
the log of both sides.
Apply Property (3) to simplify the left side.
Solve for x.

Answer. $\frac{\log\left(\frac{77}{5}\right)}{2.5\log(1.2)}$

Solution. $x = \dfrac{\log 15.4}{2.5 \log 1.2} \approx 5.999$

By using the properties of logarithms, we can now solve equations that arise in exponential growth and decay models, no matter what base the exponential function uses.

Example 7.73

The population of Silicon City was 6500 in 1990 and has been tripling every 12 years. When will the population reach 75,000?

Solution. The population of Silicon City grows according to the formula

$$P(t) = 6500 \cdot 3^{t/12}$$

where t is the number of years after. We want to find the value of t for which $P(t) = 75,000$; that is, we want to solve the equation

$$6500 \cdot 3^{t/12} = 75,000$$

To isolate the power, we divide both sides by 6500 to get

$$3^{t/12} = \frac{150}{13}$$

Now we can take the base 10 logarithm of both sides and solve for t.

$$\log_{10}\left(3^{t/12}\right) = \log_{10}\left(\frac{150}{13}\right) \qquad \text{Apply Property (3).}$$

$$\frac{t}{12}\log_{10} 3 = \log_{10}\left(\frac{150}{13}\right) \qquad \text{Divide by } \log_{10} 3\text{; multiply by 12.}$$

$$t = \frac{12\left(\log_{10}\frac{150}{13}\right)}{\log_{10} 3}$$

We use a calculator to evaluate the answer, $t \approx 26.71$. The population of Silicon City will reach 75,000 about 27 years after 1990, or in 2017.

Checkpoint 7.74 Practice 4. The volume of traffic on U.S. highways is growing by 2.7% per year. (Source: *Time*, Jan. 25, 1999)

a. Write a formula for the volume, V, of traffic as a function of time, using V_0 for the current volume. [Note: Enter "V0" to get V_0.]

   $V =$ ___________

b. How long will it take the volume of traffic to double? *Hint:* Find the value of t that gives $V = 2V_0$.

   Answer: about __ years

Answer 1. $V_0 \cdot 1.027^t$

Answer 2. $\frac{\log(2)}{\log(1.027)}$

Solution.

a. $V(t) = V_0(1.027)^t$

b. about 26 years

Checkpoint 7.75 QuickCheck 3. The first step for solving each equation below is *incorrect*. Replace it with the correct step.

a.

$$6(5^{3t}) = 20$$
$$(\log 6)(\log 5^{3t}) = \log 20 \quad \text{(Wrong!)}$$

Instead, the first step should be

- ⊙ $5^{3t} = \dfrac{20}{6}$
- ⊙ $18t = \log_5 20$
- ⊙ $30^{3t} = 20$

b.

$$t \log 8 = \log 53$$
$$t = \log \frac{53}{8} \quad \text{(Wrong!)}$$

Instead, the first step should be

- ⊙ $t = \dfrac{53}{8}$
- ⊙ $t = 8^{53}$
- ⊙ $t = \dfrac{\log 53}{\log 8}$

c.

$$12(10)^{-0.06t} = 192$$
$$120^{-0.06t} = 192 \quad \text{(Wrong!)}$$

Instead, the first step should be

- ⊙ $(\log 12)\left(\log 10^{-0.06t}\right) = \log 192$
- ⊙ $10^{-0.72t} = 192$
- ⊙ $10^{-0.06t} = \dfrac{192}{12}$

d.

$$\log x + \log(x-2) = 3$$
$$x + (x-2) = 10^3 \quad \text{(Wrong!)}$$

Instead, the first step should be

- ⊙ $\log 2x - 2 = 3$
- ⊙ $\log(x(x-2)) = 3$
- ⊙ $\log x + \log x - \log 2 = 3$

Answer 1. Choice 1

Answer 2. Choice 3

Answer 3. Choice 3

Answer 4. Choice 2

Solution.

a. $5^{3t} = \dfrac{20}{6}$

b. $t = \dfrac{\log 53}{\log 8}$

c. $10^{-0.06t} = \dfrac{192}{12}$

d. $\log(x(x-2)) = 3$

Solving Formulas

We can also solve formulas involving exponential or logarithmic expressions for one variable in terms of the others.

Example 7.76

Solve $P = Cb^{kt}$ for t. (C and $k \neq 0$.)

Solution. First, we divide both sides by C to isolate the power.

$$b^{kt} = \frac{P}{C}$$

Then we write the exponential equation in logarithmic form:

$$kt = \log_b \frac{P}{C} \qquad \text{Divide both sides by } k.$$

$$t = \frac{1}{k}\log_b \frac{P}{C}$$

Checkpoint 7.77 Practice 5. Solve $N = N_0 \log_b(ks)$ for s.

$s =$______________

Hint. Hint: Divide both sides by N_0.
Rewrite in exponential form.
Solve for s.

Answer. $\frac{1}{k}b^{\frac{N}{N_0}}$

Solution. $s = \dfrac{1}{k}b^{N/N_0}$

Problem Set 7.4

Warm Up

1. Convert each equation to logarithmic form.

a $8^{-1/3} = \dfrac{1}{2}$ b $5^x = 46$

2. Convert each equation to exponential form.

a $\log_{10} C = -4.5$ b $\log_m n = 5$

For Problems 3-8, follow the instructions, then state which property of logarithms each problem illustrates.

3.

a Simplify $10^2 \cdot 10^6$

b Compute $\log 10^2$, $\log 10^6$, and $\log (10^2 \cdot 10^6)$

4.

a Simplify $\dfrac{10^9}{10^6}$

b Compute $\log 10^9$, $\log 10^6$, and $\log \dfrac{10^9}{10^6}$

5.

a Simplify $b^8 \cdot b^5$

b Compute $\log_b b^8$, $\log_b b^5$, and $\log_b \dfrac{b^8}{b^5}$

6.

a Simplify $b^4 \cdot b^3$

b Compute $\log_b b^4$, $\log_b b^3$, and $\log_b (b^4 \cdot b^3)$

7.

a Simplify $(10^3)^5$

b Compute $\log ((10^3)^5)$, and $\log (10^3)$,

8.

a Simplify $(b^2)^6$

b Compute $\log_b (b^2)^6$, and $\log_b b^2$

For Problems 9-12, evaluate each expression. Which (if any) are equal?

9.

(a) $\log_2(4 \cdot 8)$ (b) $(\log_2 4)(\log_2 8)$ (c) $\log_2 4 + \log_2 8$

10.

(a) $\log_3(27^2)$ (b) $(\log_3 27)^2$ (c) $\log_3 27 + \log_3 27$

11.

(a) $\log_{10}\left(\dfrac{240}{10}\right)$ (b) $\dfrac{\log_{10} 240}{\log_{10} 10}$ (c) $\log_{10}(240 - 10)$

12.

(a) $\log_{10}\left(\dfrac{1}{2} \cdot 80\right)$ (b) $\dfrac{1}{2}\log_{10} 80$ (c) $\log_{10} \sqrt{80}$

Skills Practice

For Problems 13-16, use the properties of logarithms to write each expression in terms of simpler logarithms. All variable expressions denote positive numbers.

13.

a $\log_b 2x$

b $\log_b \dfrac{x}{2}$

c $\log_b x^3$

d $\log_b \sqrt[3]{x^2}$

14.

a $\log_b \dfrac{2x}{x-2}$

b $\log_b x(2x+3)$

c $\log_4 x^2y^3$

d $\log_3 \sqrt[3]{x^2+1}$

15.

a $\log_3(3x^4)$

b $\log_5 1.1^{1/t}$

c $\log_b(4b)^t$

d $\log_2 5(2^x)$

16.

a $\log_{10} \sqrt{\dfrac{2L}{R^2}}$

b $\log_{10} 2\pi\sqrt{\dfrac{l}{g}}$

For Problems 17-20, combine into one logarithm and simplify. Assume all expressions are defined.

17.

a $\log_b 8 - \log_b 2$

b $2\log_4 x + 3\log_4 y$

18.

a $\log_b 5 + \log_b 2$

b $\dfrac{1}{4}\log_5 x - \dfrac{3}{4}\log_5 y$

19.

a $\log 2x + 2\log x - \log\sqrt{x}$

b $\log(t^2-16) - \log(t+4)$

20.

a $\log x^2 + \log x^3 - 5\log x$

b $\log(x^2-x) - \log\sqrt{x^3}$

For Problems 21-24, solve for the unknown value.

21. $\log_5 y = -2$

22. $\log_b 625 = 4$

23. $\log_b 10 = \dfrac{1}{2}$

24. $\log_5(9-4x) = 3$

For Problems 25–30, use base 10 logarithms to solve. Round your answers to hundredths.

25. $2^x = 7$

26. $4^{x^2} = 15$

27. $4.26^{-x} = 10.3$

28. $25 \cdot 3^{2.1x} = 47$

29. $3600 = 20 \cdot 8^{-0.2x}$

30. $3^{2x-3} = 4529$

For Problems 31 and 32, use the three logs below to find the value of each expression.

$$\log_b 2 = 0.6931, \quad \log_b 3 = 1.0986, \quad \log_b 5 = 1.6094$$

(*Hint*: For example, $\log_b 15 = \log_b 3 + \log_b 5$.)

31.

(a) $\log_b 6$

(b) $\log_b \dfrac{2}{5}$

32.

(a) $\log_b 75$

(b) $\log_b \sqrt{50}$

In Problems 33-35, which expressions are equivalent?

33.

(a) $\log\left(\dfrac{x}{4}\right)$

(b) $\dfrac{\log x}{\log 4}$

(c) $\log(x-4)$

(d) $\log x - \log 4$

34.

(a) $\log(x+2)$

(b) $\log x + \log 2$

(c) $\log(2x)$

(d) $(\log 2)(\log x)$

35.

(a) $\log x^3$

(b) $(\log 3)(\log x)$

(c) $3\log x$

(d) $\log 3^x$

36.

a Explain why $\log \dfrac{1}{x}$ is the same as $-\log x$.

b Explain why $\log x^2$ is the same as $2\log x$.

Applications

37. If hamburger is allowed to thaw at 50°F, Salmonella bacteria grows at a rate of 9% per hour.

(a) Write a formula for the amount of Salmonella present after t hours, if the initial amount is S_0.

(b) Health officials advise that the amount of Salmonella initially present in hamburger should not be allowed to increase by more than 50%. How long can hamburger be safely left to thaw at 50°F?

38. Starting in 1998, the demand for electricity in Ireland grew at a rate of 5.8% per year. In 1998, 20,500 gigawatts were used. (Source: Electricity Supply Board of Ireland)

(a) Write a formula for electricity demand in Ireland as a function of time.

(b) If demand continues to grow at the same rate, when would it reach 30,000 gigawatts?

39. The concentration of a certain drug injected into the bloodstream decreases by 20% each hour as the drug is eliminated from the body. The initial dose creates a concentration of 0.7 milligrams per milliliter.

(a) Write a function for the concentration of the drug as a function of time.

(b) The minimum effective concentration of the drug is 0.4 milligrams per milliliter. When should the second dose be administered?

(c) Graph the function, and label the point on the graph that illustrates your answer to part (b).

40. A small pond is tested for pollution and the concentration of toxic chemicals is found to be 80 parts per million. Clean water enters the pond from a stream, mixes with the polluted water, then leaves the pond so that the pollution level is reduced by 10% each month.

(a) Write a function for the concentration of toxic chemicals as a function of time.

(b) How long will it be before the concentration of toxic chemicals reaches a safe level of 25 parts per million?

(c) Graph the function, and label the point on the graph that illustrates your answer to part (b).

41. Sodium-24 is a radioactive isotope that is used in diagnosing circulatory disease. It decays into stable isotopes of sodium at a rate of 4.73% per hour.

(a) Technicians inject a quantity of sodium-24 into a patient's bloodstream. Write a formula for the amount of sodium-24 present in the bloodstream as a function of time.

(b) How long will it take for 75% of the isotope to decay?

42. The population of Afghanistan is growing at 2.6% per year.

(a) Write a formula for the population of Afghanistan as a function of time.

(b) In 2005, the population of Afghanistan was 29.9 million. At the given rate of growth, how long would it take the population to reach 40 million?

For Problems 43-46, solve the formula for the specified variable.

43. $N = N_0 a^{kt}$, for k

44. $Q = Q_0 b^{t/2}$, for t

45. $A = A_0(10^{kt} - 1)$, for t

46. $w = pv^q$, for q

Exponential Models

Fitting an Exponential Function through Two Points

To write a formula for an exponential function, we need to know the initial value, a, and the growth or decay factor, b. We can find these two parameters if we know any two function values.

Example 7.78

Find an exponential function that has the values $f(2) = 4.5$ and $f(5) = 121.5$.

Solution. We would like to find values of a and b so that the given function values satisfy $f(x) = ab^x$. By substituting the function values into the formula, we can write two equations.

$$\begin{aligned} f(2) &= 4.5 \quad \text{means} \quad x = 2, y = 4.5, && \text{so } ab^2 = 4.5 \\ f(5) &= 121.5 \quad \text{means} \quad x = 5, y = 121.5, && \text{so } ab^5 = 121.5 \end{aligned}$$

This is a system of equations in the two unknowns, a and b, but it is not a linear system. We can solve the system by the method of elimination, but instead of adding the equations, we will divide one of the equations by the other.

$$\frac{ab^5}{ab^2} = \frac{121.5}{4.5}$$
$$b^3 = 27$$

Note that by dividing the two equations, we eliminated a, and we can now solve for b.

$$b^3 = 27$$
$$b = \sqrt[3]{27} = 3$$

Next we substitute $b = 3$ into either of the two equations and solve for a.

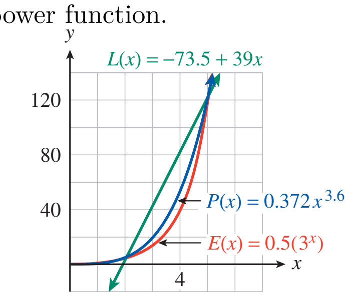
$$a(3)^2 = 4.5$$
$$a = \frac{4.5}{9}$$
$$= 0.5$$

Thus, $a = 0.5$ and $b = 3$, and the function is $f(x) = 0.5(3^x)$.

Caution 7.79 Knowing only two points on the graph of f is not enough to tell us what *kind* of function f is. Through the two points in Example 7.78, p. 506, we can also fit a linear function or a power function.

You can check that the three functions below all satisfy $f(2) = 4.5$ and $f(5) = 121.5$. The graphs of the functions are shown at right.

$$L(x) = -73.5 + 39x$$
$$P(x) = 0.372x^{3.6}$$
$$E(x) = 0.5(3^x)$$



However, if we already know that we are looking for an exponential function, we can follow the steps below to find its formula. This method is sometimes called the **ratio method**. (Of course, if one of the known function values is the initial value, we can find b without resorting to the ratio method.)

To find an exponential function $f(x) = ab^x$ through two points:.

1 Use the coordinates of the points to write two equations in a and b.

2 Divide one equation by the other to eliminate a.

3 Solve for b.

4 Substitute b into either equation and solve for a.

Checkpoint 7.80 Practice 1. Use the ratio method to find an exponential function whose graph includes the points $(1, 20)$ and $(3, 125)$.

$f(x) =$_________

Answer. $8 \cdot 2.5^x$

Solution. $f(x) = 8(2.5)^x$

Checkpoint 7.81 QuickCheck 1. You have written a system of equations to fit an exponential function through two points. What is the next step?

- ⊙ A) Calculate the slope.
- ⊙ B) Subtract one equation from the other.
- ⊙ C) Divide one equation by the other.
- ⊙ D) Take the log of both sides.

Answer. C) Divide ... the other.

Solution. Divide one equation by the other.

We can use the ratio method to find an exponential growth or decay model if we know two function values.

Example 7.82

The unit of currency in Ghana is the cedi, denoted by ¢. Beginning in 1986, the cedi underwent a period of exponential inflation. In 1993, one U.S. dollar was worth ¢720, and in 1996, the dollar was worth about ¢1620. Find a formula for the number of cedi to the dollar as a function of time since 1986. What was the annual inflation rate?

Solution. We want to find a function $C(t) = ab^t$ for the number of cedi to the dollar, where $t = 0$ in 1986. We have two function values, $C(7) = 720$, and $C(10) = 1620$, and with these values we can write two equations.

$$\begin{aligned} ab^7 &= 720 \\ ab^{10} &= 1620 \end{aligned}$$

We divide the second equation by the first to find

$$\begin{aligned} \frac{ab^{10}}{ab^7} &= \frac{1620}{720} \\ b^3 &= 2.25 \end{aligned}$$

Now we can solve this last equation for b to get $b = \sqrt[3]{2.25} \approx 1.31$. Finally, we substitute $b = 1.31$ into the first equation to find a.

$$\begin{aligned} a(1.31)^7 &= 720 \\ a &= \frac{720}{1.317} \\ &= 108.75 \end{aligned}$$

Thus, $C(t) = 108.75(1.31)^t$, and the annual inflation rate was 31%.

Checkpoint 7.83 Practice 2. The number of earthquakes that occur worldwide is a decreasing exponential function of their magnitude on the Richter scale. Between 2000 and 2005, there were 7480 earthquakes of magnitude 5 and 793 earthquakes of magnitude 6. (Source: National Earthquake Information Center, U.S. Geological Survey)

a. Find a formula for the number of earthquakes, $N(m)$, in terms of their magnitude.

$N(m) =$__________________

Do not use any commas: For example, instead of "10,000" enter simply "10000".

b. It is difficult to keep an accurate count of small earthquakes. Use your formula to estimate the number of magnitude 1 earthquakes that occurred between 2000 and 2005. ________

How many earthquakes of magnitude 8 occurred? __

Answer 1. $(5.58527 \times 10^8) \cdot 0.106016^m$

Answer 2. 5.92128×10^7

Answer 3. 8.91283

Solution.

a. $N(m) = 558,526,329(0.106)^m$

b. 59,212,751; 9

Doubling Time

Instead of giving the rate of growth of a population, we can specify its rate of growth by giving the time it takes for the population to double.

Example 7.84

In 2005, the population of Egypt was 74 million and was growing by 2% per year.

a If it continues to grow at the same rate, how long will it take the population of Egypt to double?

b How long will it take the population to double again?

c Illustrate the results on a graph.

Solution.

a The population of Egypt is growing according to the formula

$$P(t) = 74(1.02)^t$$

where t is in years and $P(t)$ is in millions. We would like to know when the population will reach 148 million (twice 74 million), so we solve the equation

$$
\begin{aligned}
74(1.02)^t &= 148 && \text{Divide both sides by 74.}\\
1.02^t &= 2 && \text{Take the log of both sides.}\\
t \log 1.02 &= \log 2 && \text{Divide both sides by log 1.02.}\\
t &= \frac{\log 2}{\log 1.02} \approx 35 \text{ years}
\end{aligned}
$$

It will take the population about 35 years to double.

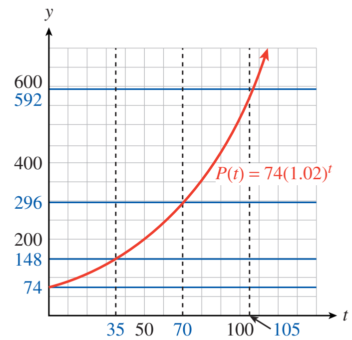
b Twice 148 million is 296 million, so we solve the equation

$$148(1.02)^t = 296 \qquad \text{Divide both sides by 148.}$$

$$1.02^t = 2 \quad \text{Take the log of both sides.}$$
$$t \log 1.02 = \log 2 \quad \text{Divide both sides by log 1.02.}$$
$$t = \frac{\log 2}{\log 1.02} \approx 35 \text{ years}$$

It will take the population about 35 years to double again.

c A graph of $P(t) = 74(1.02)^t$ is shown below. Note that the population doubles every 35 years.



In the Example, p. 509 above, notice that the calculations in parts (a) and (b) are identical after the first step, and give the same result, 35 years. In fact, we can start at any time, and it will take the population 35 years to double. We say that 35 years is the **doubling time** for this population. Every increasing exponential function has a constant doubling time. In fact, if a function has a constant doubling time, it must be exponential.

Checkpoint 7.85 Practice 3. In 2005, the population of Uganda was 26.9 million people and was growing by 3.2% per year.

a. Write a formula for the population of Uganda as a function of years since 2005.

   $P(t) =$___________ million

b. How long will it take the population of Uganda to double?

   __ years

c. Use your formula from part (a) to verify the doubling time for three doubling periods.

Answer 1. $26.9 \cdot 1.032^t$

Answer 2. $\frac{\log(2)}{\log(1.032)}$

Solution.

a. $P(t) = 26.9(1.032)^t$ million

b. 22 years

c. $P(0) = 26.9$; $P(22) \approx 53.8$, so $P(22) \approx 2 \cdot P(0)$; $P(44) \approx 107.6$, so $P(44) \approx 2 \cdot P(22)$; $P(66) \approx 215.1$, so $P(66) \approx 2 \cdot P(44)$

Checkpoint 7.86 QuickCheck 2. Which statement is true?

⊙ A) The doubling time of a population depends on its initial value.

- ⊙ B) An increasing exponential function has a constant doubling time.
- ⊙ C) The doubling time is twice the percent growth rate.
- ⊙ D) The doubling time is half the percent growth rate.

Answer. B) An ... doubling time.

Solution. An increasing exponential function has a constant doubling time.

If we know the doubling time for a population, we can immediately write down its growth law. Because the population of Egypt doubles in 35 years, we can write

$$P(t) = 74 \cdot 2^{t/35}$$

In this form, the growth factor for the population is $2^{1/35}$, and you can check that, to five decimal places, $2^{1/35} = 1.02$.

Doubling Time.

If D is the doubling time for an exponential function $P(t)$, then

$$P(t) = P_0 2^{t/D}$$

So, from knowing the doubling time, we can easily find the growth rate of a population.

Example 7.87

At its current rate of growth, the population of the United States will double in 115.87 years.

a Write a formula for the population of the United States as a function of time.

b What is the annual percent growth rate of the population?

Solution.

a The current population of the United States is not given, so we represent it by P_0. With t expressed in years, the formula is then

$$P(t) = P_0 2^{t/115.87}$$

b We write $2^{t/115.87}$ in the form $\left(2^{1/115.87}\right)^t$ to see that the growth factor is $b = 2^{1/115.87}$, or 1.006. For exponential growth, $b = 1+r$, so $r = 0.006$, or 0.6%.

Checkpoint 7.88 Practice 4. At its current rate of growth, the population of Mexico will double in 36.8 years. What is its annual percent rate of growth?

Answer: ___%

Answer. $100 \cdot \left(2^{\frac{1}{36.8}} - 1\right)$

Solution. 1.9%

Half-Life

The **half-life** of a decreasing exponential function is the time it takes for the output to decrease to half its original value. For example, the half-life of a radioactive isotope is the time it takes for half of the substance to decay. The half-life of a drug is the time it takes for half of the drug to be eliminated from the body. Like the doubling time, the half-life is constant for a particular function; no matter where you start, it takes the same amount of time to reach half that value.

Example 7.89

If you take 200 mg of ibuprofen to relieve sore muscles, the amount of the drug left in your body after t hours is $Q(t) = 200(0.73)^t$.

a What is the half-life of ibuprofen?

b When will 50 mg of ibuprofen remain in your body?

c Use the half-life to sketch a graph of $Q(t)$.

Solution.

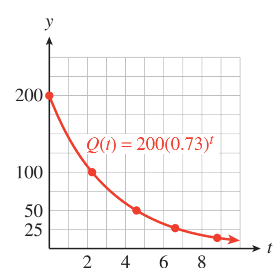
a To find the half-life, we calculate the time elapsed when only half the original amount, or 100 mg, is left.

$$\begin{aligned} 200(0.73)^t &= 100 && \text{Divide both sides by 200.} \\ 0.73^t &= 0.5 && \text{Take the log of both sides.} \\ t\log 0.73 &= \log 0.5 && \text{Divide both sides by } \log 0.73. \\ t &= \frac{\log 0.5}{\log 0.73} = 2.2 \end{aligned}$$

The half-life is 2.2 hours.

b After 2.2 hours, 100 mg of ibuprofen is left in the body. After another 2.2 hours, half of that amount, or 50 mg, is left. Thus, 50 mg remain after 4.4 hours.

c We locate multiples of 2.2 hours on the horizontal axis. After each interval of 2.2 hours, the amount of ibuprofen is reduced to half its previous value. The graph is shown below.



t	0	2.2	4.4	6.6	8.8
$Q(t)$	200	100	50	25	12.5

Checkpoint 7.90 Practice 5. Alcohol is eliminated from the body at a rate of 15% per hour.

a. Write a decay formula for the amount of alcohol remaining in the body, using A_0 for the initial amount of alcohol. [Note: Enter "A" to get A_0.].

$A(t) =$ ______________

b. What is the half-life of alcohol in the body?

___ hours

Answer 1. $A_0 \cdot 0.85^t$

Answer 2. $\frac{\log(\frac{1}{2})}{\log(0.85)}$

Solution.

a. $A(t) = A_0(0.85)^t$

b. 4.3 hours

Checkpoint 7.91 QuickCheck 3. The half-life of DDT is 15 years. This means that:

- ⊙ A) 30 pounds of DDT dissolve in one year.
- ⊙ B) 100 pounds of DDT dissolve in 30 years.
- ⊙ C) After 30 years, 100 pounds of DDT is reduced to 25 pounds.
- ⊙ D) Each half-pound of DDT takes 15 years to dissolve.

Answer. C) After ... 25 pounds.

Solution. After 30 years, 100 pounds of DDT is reduced to 25 pounds.

Just as we can write an exponential growth law in terms of its doubling time, we can use the half-life to write a formula for exponential decay. For example, the half-life of ibuprofen is 2.2 hours, so every 2.2 hours the amount remaining is reduced by a factor of 0.5. After t hours a 200-mg dose will be reduced to

$$Q(t) = 200(0.5)^{t/2.2}$$

Once again, you can check that this formula is equivalent to the decay function given in the previous Example, p. 512.

Half-Life.

If H is the half-life for an exponential function $Q(t)$, then

$$Q(t) = Q_0(0.5)^{t/H}$$

Radioactive isotopes are molecules that decay into more stable molecules, emitting radiation in the process. Although radiation in large doses is harmful to living things, radioactive isotopes are useful as tracers in medicine and industry, and as treatment against cancer. The decay laws for radioactive isotopes are often given in terms of their half-lives.

Example 7.92

Cobalt-60 is used in cold pasteurization to sterilize certain types of food. Gamma rays emitted by the isotope during radioactive decay kill any bacteria present without damaging the food. The half-life of cobalt-60 is 5.27 years.

a Write a decay law for cobalt-60.

b What is the annual decay rate for cobalt-60?

Solution.

a We let $Q(t)$ denote the amount of cobalt-60 left after t years, and let Q_0 denote the initial amount. Every 5.27 years, $Q(t)$ is reduced by a factor of 0.5, so

$$Q(t) = Q_0(0.5)^{t/5.27}$$

b We rewrite the decay law in the form $Q(t) = Q_0(1-r)^t$ as follows:

$$Q(t) = Q_0(0.5)^{t/5.27} = Q_0\left((0.5)^{1/5.27}\right)^t = Q_0(0.8768)^t$$

Thus, $1 - r = 0.8768$, so $r = 0.1232$, or 12.32%.

Checkpoint 7.93 Practice 6. Cesium-137, with a half-life of 30 years, is one of the most dangerous by-products of nuclear fission. What is the annual decay rate for cesium-137?

Answer: ___%

Answer. $100 \cdot \left(1 - \frac{1}{2^{\frac{1}{30}}}\right)$

Solution. 2.28%

Checkpoint 7.94 QuickCheck 4. True or False.

a. Every increasing exponential function has a constant doubling time. (☐ True ☐ False)

b. If the doubling time of a population is 5 years, then its growth factor is given by $2^{1/5}$. (☐ True ☐ False)

c. The half-life of a substance is half the time it takes for all of the substance to decay. (☐ True ☐ False)

d. We can sketch the graph of an exponential decay function if we know its half-life and initial value. (☐ True ☐ False)

Answer 1. True

Answer 2. True

Answer 3. False

Answer 4. True

Solution.

a. True

b. True

c. False

d. True

Problem Set 7.5

Warm Up

For Problems 1-4, solve. Round your answers to hundredths.

1. $4.6(x-3)^{1.8} + 12 = 18$

2. $2x^2 - 4.7x = 3.8$

3. $28(1.65)^{-0.3t} = 2.53$

4. $5(10)^x = 30$

For Problems 5–8, find an equation for the line with the given pooperties.

5. slope $= \dfrac{-2}{3}$, y-intercept is $(0, -1)$

6. slope $= -2$, passes through $(-1, 2)$

7. passes through $(0, 4)$ and $(2, 3)$

8. passes through $(-3, 0)$ and $(0, -5)$

Skills Practice

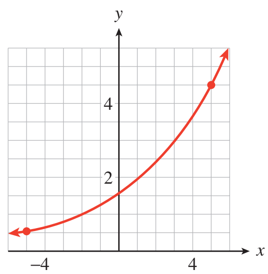
For Problems 9 and 10, find an exponential function that has the given values.

9. $P(0) = 8,\ P(5) = 0.25$

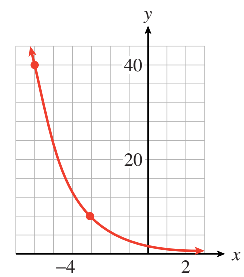
10. $f(2) = 9,\ f(3) = 12$

For Problems 11 and 12, find a formula for the exponential function.

11.



12.



For Problems 13 and 14,

a fit a linear function to the points,

b fit an exponential function to the points,

c graph both functions in the same window.

13. $(0, 2.6)$, $(1, 1.3)$

14. $(-2, 0.75)$, $(4, 6)$

Applications

For Problems 15-18, write a growth or decay formula for the exponential function. Then find the percent growth or decay rate.

15. A population starts with 2000 and has a doubling time of 5 years.

16. You have 10 grams of a radioactive isotope whose half-life is 42 years.

17. A certain medication has a half-life of 18 hours in the body. You are given an initial dose of D_0 mg.

18. The doubling time of a certain financial investment is 8 years. You invest an amount M_0.

19. In 1798, the English political economist Thomas R. Malthus claimed that human populations, unchecked by environmental or social constraints, double every 25 years, regardless of the initial population size.

a Write a growth law for human populations under these conditions.

b What is the growth rate in unconstrained conditions?

20. David Sifry observed in 2005 that over the previous two years, the number of Weblogs, or blogs, was doubling every 5 months. (Source: www.sifry.com/alerts/archives)

a Write a formula for the number of blogs t years after January 2005, assuming it continues to grow at the same rate.

b What is the growth rate for the number of blogs?

21. Radioactive potassium-42, which is used by cardiologists as a tracer, decays at a rate of 5.4% per hour.

a Find the half-life of potassium-42.

b How long will it take for three-fourths of the sample to decay? For seven-eighths of the sample?

c Suppose you start with 400 milligrams of potassium-42. Using your answers to (a) and (b), make a rough sketch of the decay function.

22. Caffeine leaves the body at a rate of 15.6% each hour. Your first cup of coffee in the morning has 100 mg of caffeine.

a How long will it take before you have 50 mg of that caffeine in your body?

b How long will it take before you have 25 mg of that caffeine in your body?

c Using your answers to (a) and (b), make a rough sketch of the decay function.

23. Dichloro-diphenyl-trichloroethane (DDT) is a pesticide that was used in the middle decades of the twentieth century to control malaria. After 1945, it was also widely used on crops in the United States, and as much as one ton might be sprayed on a single cotton field. However, after the toxic effects of DDT on the environment began to appear, the chemical was banned in 1972.

a A common estimate for the half-life of DDT in the soil is 15 years. Write a decay law for DDT in the soil.

b In 1970, many soil samples in the United States contained about 0.5 mg of DDT per kg of soil. The NOAA (National Oceanic and Atmospheric Administration) safe level for DDT in the soil is 0.008 mg/kg. When will DDT content in the soil be reduced to a safe level?

24. In 1986, the inflation rate in Bolivia was 8000% annually. The unit of currency in Bolivia is the boliviano.

a Write a formula for the price of an item as a function of time. Let P_0 be its initial price.

b How long did it take for prices to double? Give both an exact value and a decimal approximation rounded to two decimal places.

c Suppose $P_0 = 5$ bolivianos. Graph your function in the window Xmin = 0, Xmax = 0.94, Ymin = 0, Ymax = 100.

d Use **intersect** to verify that the price of the item doubles from 5 to 10 bolivianos, from 10 to 20, and from 20 to 40 in equal periods of time.

In Problems 25 and 26,

a Write a decay law for the isotope.

b Use the decay law to answer the question. (Round to the nearest ten years.)

25. Carbon-14 occurs in living organisms with a fixed ratio to nonradioactive carbon-12. After a plant or animal dies, the carbon-14 decays into stable carbon with a halflife of 5730 years. When samples from the Shroud of Turin were analyzed in 1988, they were found to have 91.2% of their original carbon-14. How old were those samples in 1988?

26. Rubidium-strontium radioactive dating is used in geologic studies to measure the age of minerals. Rubidium-87 decays into strontium-87 with a half-life of 48.8 billion years. Several meteors were found to have 93.7% of their original rubidium. How old are the meteors?

For Problems 27-30, use the following formula for compound interest. If P dollars is invested at an annual interest rate r (expressed as a decimal) compounded n times yearly, the amount A after t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

27. What rate of interest is required so that \$1000 will yield \$1900 after 5 years if the interest rate is compounded monthly?

28. What rate of interest is required so that \$400 will yield \$600 after 3 years if the interest rate is compounded quarterly?

29. How long will it take a sum of money to triple if it is invested at 10% compounded daily?

30. How long will it take a sum of money to increase by a factor of 5 if it is invested at 10% compounded quarterly?

Chapter 7 Summary and Review

Glossary

- exponential growth
- growth factor
- initial value
- compound interest
- principal
- percent growth rate
- exponential decay
- decay factor
- exponential function
- base
- exponential equation
- logarithm
- common logarithm
- logarithmic scale
- doubling time
- half-life

Key Concepts

1 Exponential growth was modeled by increasing functions of the form $P(t) = P_0 b^t$, where the growth factor, b, is a number greater than 1.

Exponential Growth.

2 The function

$$P(t) = P_0 b^t$$

describes **exponential growth**, where $P_0 = P(0)$ is the **initial value** of the function and b is the **growth factor**.

Growth by a Constant Percent.

3 The function

$$P(t) = P_0(1 + r)^t$$

describes exponential growth at a constant percent rate of growth, r.

The **initial value** of the function is $P_0 = P(0)$, and $b = 1 + r$ is the **growth factor**.

Compound Interest.

4 If a **principal** of P dollars is invested in an account that pays an interest rate r compounded annually, the **balance** B after t years is given by

$$B = P(1 + r)^t$$

Laws of Exponents.

5

I $a^m \cdot a^n = a^{m+n}$

II $\dfrac{a^m}{a^n} = a^{m-n}$

III $(a^m)^n = a^{mn}$

IV $(ab)^n = a^n b^n$

V $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

6 If $0 < b < 1$, then $P(t) = P_0 b^t$ is a decreasing function. In this case $P(t)$ is said to describe exponential decay.

7 A percent increase of r (in decimal form) corresponds to a growth factor of $b = 1 + r$. A percent decrease of corresponds to a decay factor of $b = 1 - r$.

Exponential Growth and Decay.

8 The function

$$P(t) = P_0 b^t$$

models exponential growth and decay.

- $P_0 = P(0)$ is the **initial value** of P;
- b is the **growth** or **decay factor**.

(a) If $b > 1$, then $P(t)$ is increasing, and $b = 1 + r$, where r represents percent increase.

(b) If $0 < b < 1$, then $P(t)$ is decreasing, and $b = 1 - r$, where r represents percent decrease.

Exponential Function.

9 An **exponential function** has the form

$$f(x) = ab^x, \qquad \text{where} \qquad b > 0 \quad \text{and} \quad b \neq 1, \quad a \neq 0$$

Exponential Notation for Radicals.

10 For any integer $n \geq 2$ and for $a \geq 0$,

$$a^{1/n} = \sqrt[n]{a}$$

11 The growth factor of an exponential function is analogous to the slope of a linear function: Each measures how quickly the function is increasing (or decreasing).

12 Suppose $L(t) = a + mt$ is a linear function and $E(t) = ab^t$ is an exponential function. For each unit that t increases, m units are added to the value of $L(t)$, whereas the value of $E(t)$ is multiplied by b.

13 We do not allow the base of an exponential function to be negative, because if $b < 0$, then b^x is not a real number for some values of x.

Properties of Exponential Functions, $f(x) = ab^x$, $a > 0$.

14

(a) If $b > 1$, the function is increasing and concave up;

if $0 < b < 1$, the function is decreasing and concave up.

(b) The y-intercept is $(0, a)$. There is no x-intercept.

15 The negative x-axis is a horizontal asymptote for exponential functions with $b > 1$. For exponential functions with $0 < b < 1$, the positive x-axis is an asymptote.

16 Exponential functions are not the same as the power functions we studied earlier. Although both involve expressions with exponents, it is the location of the variable that makes the difference.

Power Functions vs Exponential Functions.

17

	Power Functions	Exponential Functions
General formula	$h(x) = kx^p$	$f(x) = ab^x$
Description	variable base and constant exponent	constant base and variable exponent
Example	$h(x) = 2x^3$	$f(x) = 2(3^x)$

18 Many exponential equations can be solved by writing both sides of the equation as powers with the same base. If two equivalent powers have the same base, then their exponents must be equal also (as long as the base is not 0 or ± 1).

Definition of Logarithm.

19 For $b > 0, b \neq 1$, the **base b logarithm of x**, written $\log_b x$, is the exponent to which b must be raised in order to yield x.

Logarithms and Exponents: Conversion Equations.

20 If $b > 0$, $b \neq 1$, and $x > 0$,

$$y = \log_b x \quad \text{if and only if} \quad x = b^y$$

Some Useful Logarithms.

21 For any base $b > 0, b \neq 1$,

$$\log_b b = 1 \quad \text{because} \quad b^1 = b$$
$$\log_b 1 = 0 \quad \text{because} \quad b^0 = 1$$
$$\log_b b^x = x \quad \text{because} \quad b^x = b^x$$

22 We use logarithms to solve exponential equations, just as we use square roots to solve quadratic equations. The operation of taking a base b logarithm is the inverse of raising the base b to a power, just as extracting square roots is the inverse of squaring a number.

Steps for Solving Base 10 Exponential Equations.

23

1 Isolate the power on one side of the equation.

2 Rewrite the equation in logarithmic form.

3 Use a calculator, if necessary, to evaluate the logarithm.

4 Solve for the variable.

Properties of Logarithms.

24 If x, y, and $b > 0$, and $b \neq 1$, then

1 $\log_b xy = \log_b x + \log_b y$

2 $\log_b \dfrac{x}{y} = \log_b x - \log_b y$

3 $\log_b x^k = k \log_b x$

Steps for Solving Exponential Equations.

25

1 Isolate the power on one side of the equation.

2 Take the base 10 logarithm of both sides of the equation.

3 Apply the third property of logarithms to simplify.

4 Use a calculator, if necessary, to evaluate the logarithm.

5 Solve for the variable.

Chapter 7 Review Problems

For Problems 1–4,

a Write a function that describes exponential growth or decay.

b Evaluate the function at the given values.

1. The number of computer science degrees awarded by Monroe College has increased by a factor of 1.5 every 5 years since 1984. If the college granted 8 degrees in 1984, how many did it award in 1994? In 2005?

2. The price of public transportation has been rising by 10% per year since 1975. If it cost $0.25 to ride the bus in 1975, how much did it cost in 1985? How much will it cost in the year 2020 if the current trend continues?

3. A certain medication is eliminated from the body at a rate of 15% per hour. If an initial dose of 100 milligrams is taken at 8 a.m., how much is left at 12 noon? At 6 p.m.?

4. After the World Series, sales of T-shirts and other baseball memorabilia decline 30% per week. If $200,000 worth of souvenirs were sold during the Series, how much will be sold 4 weeks later? After 6 weeks?

For Problems 5–8, graph the function.

5. $f(t) = 6(1.2)^t$

6. $g(t) = 35(0.6)^{-t}$

7. $P(x) = 2^x - 3$

8. $R(x) = 2^{x+3}$

For Problems 9-12, solve the equation.

9. $3^{x+2} = 9^{1/3}$

10. $2^{x-1} = 8^{-2x}$

11. $4^{2x+1} = 8^{x-3}$

12. $3^{x^2-4} = 27$

For Problems 13-18, find the logarithm.

13. $\log_2 16$ **14.** $\log_4 2$ **15.** $\log_3 \frac{1}{3}$

16. $\log_7 7$ **17.** $\log_{10} 10^{-3}$ **18.** $\log_{10} 0.0001$

For Problems 19 and 20, write the equation in exponential form.

19. $\log_2 3 = x - 2$ **20.** $\log_n q = p - 1$

For Problems 21 and 22, write the equation in logarithmic form.

21. $0.3^{-2} = x + 1$ **22.** $4^{0.3t} = 3N_0$

For Problems 23-30, solve for the unknown value.

23. $\log_3 \frac{1}{3} = y$ **24.** $\log_3 x = 4$

25. $\log_b 16 = 2$ **26.** $\log_2(3x - 1) = 3$

27. $4 \cdot 10^{1.3x} = 20.4$ **28.** $127 = 2(10^{0.5x}) - 17.3$

29. $3(10^{-0.7x}) + 6.1 = 9$ **30.** $40(1 - 10^{-1.2x}) = 30$

In Problems 31-34, evaluate the expression.

31. $k = \frac{1}{t}(\log N - \log N_0);$ for $t = 2.3,\ N = 12,000$, and $N_0 = 9000$

32. $P = \frac{1}{k}\sqrt{\frac{\log N}{t}}$: for $k = 0.4,\ N = 48$, and $t = 12$

33. $h = k \log\left(\frac{N}{N - N_0}\right)$ for $k = 1.2,\ N = 6400$, and $N_0 = 2000$

34. $Q = \frac{1}{t}\left(\frac{\log M}{\log N}\right);$ for $t = 0.3,\ M = 180$, and $N = 640$

For Problems 35-38, write the expression in terms of simpler logarithms. (Assume that all variables and variable expressions denote positive real numbers.)

35. $\log_b\left(\frac{xy^{1/3}}{z^2}\right)$ **36.** $\log_b \sqrt{\frac{L^2}{2R}}$

37. $\log_{10}\left(x\sqrt[3]{\frac{x}{y}}\right)$ **38.** $\log_{10} \sqrt{(s-a)(s-g)^2}$

For Problems 39-42, write the expression as a single logarithm with coefficient 1.

39. $\frac{1}{3}\left(\log_{10} x - 2\log_{10} y\right)$ **40.** $\frac{1}{2}\log_{10}(3x) - \frac{2}{3}\log_{10} y$

41. $\frac{1}{3}\log_{10} 8 - 2\left(\log_{10} 8 - \log_{10} 2\right)$ **42.** $\frac{1}{2}(\log_{10} 9 + 2\log_{10} 4) + 2\log_{10} 5$

For Problems 43–46, solve the equation by using base 10 logarithms.

43. $3^{x-2} = 7$ **44.** $4 \cdot 2^{1.2x} = 64$

45. $1200 = 24 \cdot 6^{-0.3x}$ **46.** $0.08 = 12 \cdot 3^{-1.5x}$

47. Solve $N = N_0(10^{kt})$ for t.

48. Solve $Q = R_0 + R\log_{10} kt$ for t.

49. The population of Dry Gulch has been declining according to the function

$$P(t) = 3800 \cdot 2^{-t/20}$$

where t is the number of years since the town's heyday in 1910.

(a) What was the population of Dry Gulch in 1990?

(b) In what year did the population dip below 120 people?

50. The number of compact discs produced each year by Delta Discs is given by the function

$$N(t) = 8000 \cdot 3^{t/4}$$

where t is the number of years since discs were introduced in 1980.

(a) How many discs did Delta produce in 1989?

(b) In what year did Delta first produce over 2 million discs?

51.

(a) Write a formula for the cost of a camera t years from now if it costs \$90 now and the inflation rate is 6% annually.

(b) How much will the camera cost 10 months from now?

(c) How long will it be before the camera costs \$120?

52.

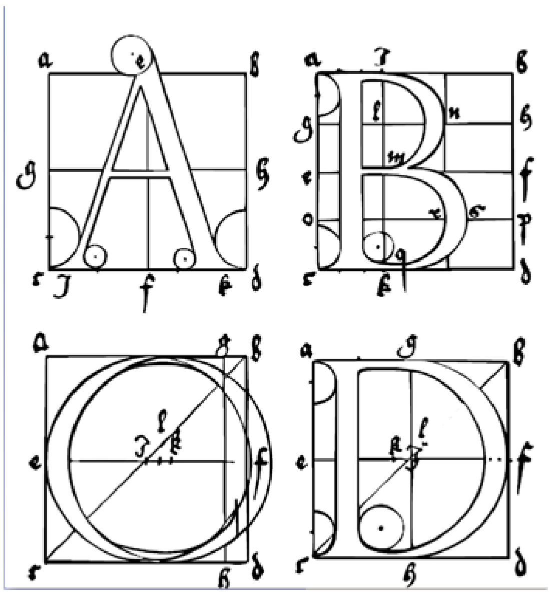
(a) Write a formula for the cost of a sofa t years from now if it costs \$1200 now and the inflation rate is 8% annually.

(b) How much will the sofa cost 20 months from now?

(c) How long will it be before the sofa costs \$1500?

Chapter 8

Polynomial and Rational Functions

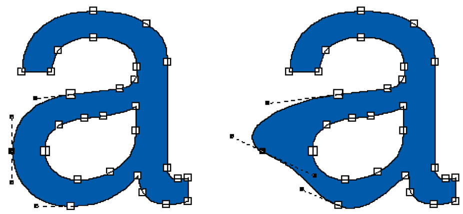


The graphs of linear functions, quadratic functions, power functions, and exponential all have a characteristic shape. On the other hand, the family of polynomial funcions has graphs that represent a huge variety of different shapes.

Ever since Gutenberg's invention of movable type in 1455, artists and printers have been interested in the design of pleasing and practical fonts. In 1525, Albrecht Durer published *On the Just Shaping of Letters*, which set forth a system of rules for the geometric construction of Roman capitals. The letters shown above are examples of Durer's font. Until the twentieth century, a ruler and compass were the only practical design tools, so straight lines and circular arcs were the only geometric objects that could be accurately reproduced.

With the advent of computers, complex curves and surfaces, such as the smooth contours of modern cars, can be defined precisely. In the 1960s the French automobile engineer Pierre Bézier developed a new design tool based on **polynomials**. **Bézier curves** are widely used today in all fields of design, from technical plans and blueprints to the most creative artistic projects.

The study of Bézier curves falls under the general topic of curve fitting, but these curves do not really have a scientific purpose. A scientist does not use Bézier curves to fit a function to data. Rather, Bézier curves have more of an artistic purpose. Computer programs like Illustrator, Freehand, and CorelDraw use cubic Bézier curves. The PostScript printer language and Type 1 fonts also use cubic Bézier curves, and TrueType fonts use quadratic Bézier curves.



Investigation 8.1 Bézier Curves. A Bézier curve is actually a sequence of short curves pieced together. Each piece has two endpoints and (for nonlinear curves) at least one control point. The control points do not lie on the curve itself, but they determine its shape. Two polynomials define the curve, one for the x-coordinate and one for the y-coordinate.

A. Linear Bézier Curves

The linear Bézier curve for two endpoints, (x_1, y_1) and (x_2, y_2), is the straight line segment joining those two points. The curve is defined by the two functions

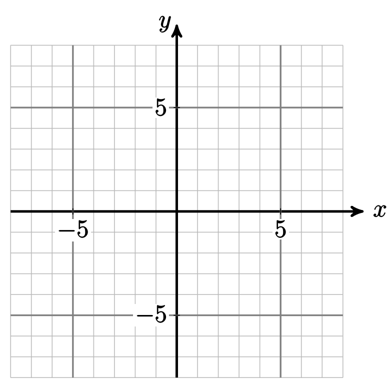
$$x = f(t) = x_1 \cdot (1-t) + x_2 \cdot t$$
$$y = g(t) = y_1 \cdot (1-t) + y_2 \cdot t$$

for $0 \le t \le 1$.

1 Find the functions f and g defining the linear Bézier curve joining the two points $(-4, 7)$ and (20). Simplify the formulas defining each function.

2 Fill in the table of values and plot the curve.

t	0	0.25	0.5	0.75	1
x					
y					



B. Quadratic Bézier Curves: Drawing a Simple Numeral 7

The quadratic Bézier curve is defined by two endpoints, (x_1, y_1) and (x_3, y_3), and a control point, (x_2, y_2).

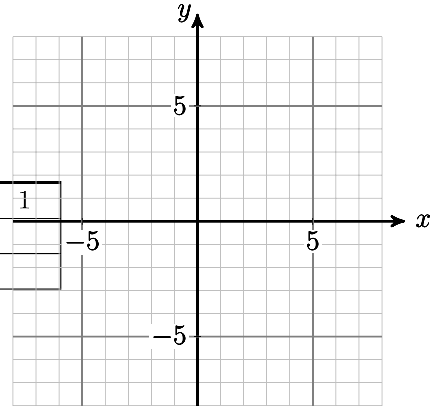
$$x = f(t) = x_1 \cdot (1-t)^2 + 2x_2 \cdot t(1-t) + x_3 \cdot t^2$$
$$y = g(t) = y_1 \cdot (1-t)^2 + 2y_2 \cdot t(1-t) + y_3 \cdot t^2$$

for $0 \le t \le 1$.

1 Find the functions f and g for the quadratic Bézier curve defined by the endpoints $(-4, 7)$ and $(2, 0)$, and the control point $(0, 5)$. Simplify the formulas defining each function.

2 Fill in the table of values and plot the curve.

t	0	0.25	0.5	0.75	1
x					
y					

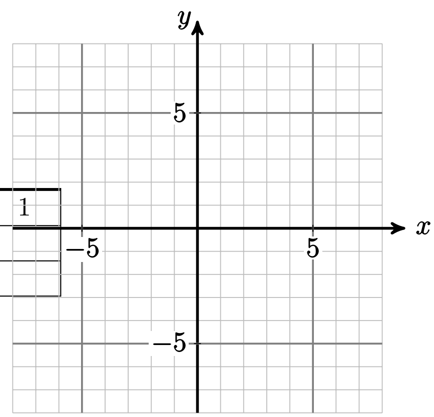


3 Draw a line segment from $(-4, 7)$ to $(4, 7)$ on the grid above to complete the numeral 7.

4 We can adjust the curvature of the diagonal stroke of the 7 by moving the control point. Find the functions f and g for the quadratic Bézier curve defined by the endpoints $(4, 7)$ and $(0, -7)$, and the control point $(0, -3)$. Simplify the formulas defining each function.

5 Fill in the table of values and plot the curve.

t	0	0.25	0.5	0.75	1
x					
y					



6 Draw a line segment from $(-4, 7)$ to $(4, 7)$ on the grid above to complete the numeral 7.

7 On your graphs in steps (5) and (8), plot the three points that defined the curved section of the numeral 7, then connect them in order with line segments. How does the position of the control point change the curve?

C. Cubic Bézier Curves: Drawing a Letter y

A cubic Bézier curve is defined by two endpoints, (x_1, y_1) and (x_4, y_4), and two control points, (x_2, y_2) and (x_3, y_3).

$$x = f(t) = x_1 \cdot (1-t)^2 + 3x_2 \cdot t(1-t)^2 + 3x_3 \cdot t^2(1-t) + x_4 \cdot t^3$$
$$y = g(t) = y_1 \cdot (1-t)^2 + 3y_2 \cdot t(1-t)^2 + 3y_3 \cdot t^2(1-t) + y_4 \cdot t^3$$

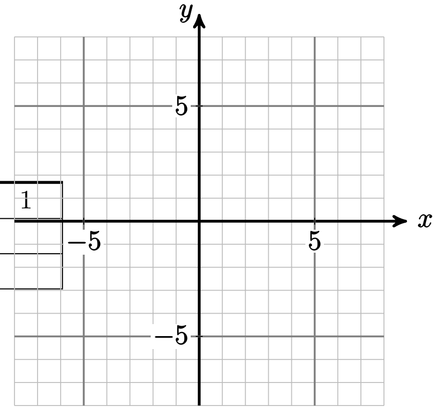
for $0 \le t \le 1$.

1 Find the functions f and g for the cubic Bézier curve defined by the endpoints $(4, 7)$ and $(-4, -5)$, and the control points $(3, 3)$ and $(0, -8)$. Simplify the formulas defining each function.

Fill in the table of values and plot the curve.

2

t	0	0.25	0.5	0.75	1
x					
y					



3 Connect the four given points in order using three line segments. How does the position of the control points affect the curve? Finish the letter y by including the linear Bézier curve you drew for step (2).

Polynomial Functions

Introduction

We have already encountered some examples of polynomial functions. Linear functions,

$$f(x) = ax + b$$

and quadratic functions

$$f(x) = ax^2 + bx + c$$

are special cases of polynomial functions. In general, we make the following definition.

Definition 8.1 Polynomial Function.

A **polynomial function** has the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$$

where a_0, a_1, a_2, …, a_n are constants and $a_n \neq 0$. The coefficient a_n of the highest power term is called the **lead coefficient**.

Some examples of polynomials are

$$f(x) = 6x^3 - 4x^2 + x - 2 \qquad g(x) = 9x^5 - 2$$
$$p(x) = x^4 + x^2 + 1 \qquad q(x) = 2x^{10} - x^7 + 3x^6 + 5x^3 + 3x$$

Each of the polynomials above is written in **descending powers**, which means that the highest-degree term comes first, and the degrees of the terms decrease from largest to smallest. Sometimes it is useful to write a polynomial in **ascending powers**, so that the degrees of the terms increase. For example, the polynomial $f(x)$ above would be written as

$$f(x) = -2 + x - 4x^2 + 6x^3$$

in ascending powers.

Checkpoint 8.2 QuickCheck 1. What is the lead coefficient of a polynomial?

⊙ A) The first coefficient

⊙ B) The constant term

⊙ C) The largest coefficient

⊙ D) The coefficient of the highest power

Answer. D) ... highest power

Solution. The coefficient of the highest power, that is, the coefficient of the largest degree

Products of Polynomials

When we multiply two or more polynomials together, we get another polynomial of higher degree.

Example 8.3

Compute the products.

a $(x+2)(5x^3-3x^2+4)$ b $(x-3)(x+2)(x-4)$

Solution.

a. $(x+2)(5x^3-3x^2+4)$ Apply the distributive law.

$= x(5x^3-3x^2+4)+2(5x^3-3x^2+4)$ Apply the distributive law again.

$= 5x^4-3x^3+4x+10x^3-6x^2+8$ Combine like terms.

$= 5x^4+7x^3-6x^2+4x+8$

b. $(x-3)(x+2)(x-4)$ Multiply two of the factors first.

$= (x-3)(x^2-2x-8)$ Apply the distributive law.

$= x(x^2-2x-8)-3(x^2-2x-8)$ Apply the distributive law again.

$= x^3-2x^2-8x-3x^2+6x+24$ Combine like terms.

$= x^3-5x^2-2x+24$

Checkpoint 8.4 Practice 1. Multiply $(y+2)(y^2-2y+3) =$_______ .

Answer. y^3-y+6

Solution. y^3-y+6

In part (a) of the Example, p. 529 above, we multiplied a polynomial of degree one by a polynomial of degree three, and the product was a polynomial of degree four. In the Example, p. 529 part (b), the product of three first-degree polynomials is a third-degree polynomial.

Degree of a Product.

The degree of a product of non-zero polynomials is the sum of the degrees of the factors. That is:

If $P(x)$ has degree m and $Q(x)$ has degree n, then their product $P(x)Q(x)$ has degree $m+n$.

Checkpoint 8.5 QuickCheck 2. What is the linear term of the product $(x-3)(x+5)$?

- ⊙ x^2
- ⊙ $2x - 15$
- ⊙ -15
- ⊙ $2x$

Answer. Choice 4

Solution. $2x$ is the linear term of $x^2 + 2x - 15$.

Example 8.6

Let $P(x) = 5x^4 - 2x^3 + 6x^2 - x + 2$, and

$$Q(x) = 3x^3 - 4x^2 + 5x + 3.$$

a What is the degree of their product? What is the coefficient of the lead term?

b Find the coefficient of the x^3-term of the product.

Solution.

a The degree of P is 4, and the degree of Q is 3, so the degree of their product is $4 + 3 = 7$. The only degree 7 term of the product is $(5x^4)(3x^3) = 15x^7$, which has coefficient 15.

b In the product, each term of $P(x)$ is multiplied by each term of $Q(x)$. We get degree 3 terms by multiplying together terms of degree 0 and 3, or 1 and 2. For these polynomials, the possible combinations are:

$P(x)$	$Q(x)$	Product
2	$3x^3$	$6x^3$
$-2x^3$	3	$-6x^3$
$-x$	$-4x^2$	$4x^3$
$6x^2$	$5x$	$30x^3$

The sum of the third-degree terms of the product is $34x^3$, with coefficient 34.

Checkpoint 8.7 Practice 2. Find the coefficient of the fourth-degree term of the product of $f(x) = 2x^6 + 2x^4 - x^3 + 5x^2 + 1$ and $g(x) = x^5 - 3x^4 + 2x^3 + x^2 - 4x - 2$.

Answer: __

Answer. 2

Solution. 2 is the coefficient of the x^4 term in the product.

To compute the fourth-degree term in the product, we multiply a term of $f(x)$ of degree 4, 3, 2, or 0 by a term of $g(x)$ of degree 0, 1, 2, or 4 respectively. We add those four products of terms:

$$\left(2x^4 \cdot (-2)\right) + \left((-x^3)(-4x)\right) + \left(5x^2 \cdot x^2\right) + \left(1 \cdot (-3x^4)\right)$$

So the fourth-degree term in the product is $2x^4$, and the coefficient is 2.

Special Products

Recall the following special products involving quadratic expressions.

Special Products of Binomials.

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$
$$(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$$
$$(a+b)(a-b) = a^2 - b^2$$

There are also special products resulting in cubic polynomials. In the Homework problems, you will be asked to verify the following products.

Cube of a Binomial.

1 $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

2 $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

If you become familiar with these general forms, you can use them as patterns to find specific examples of such products.

Example 8.8

Write $(2w-3)^3$ as a polynomial.

Solution. Use product 2, p. 531 above, with x replaced by $\mathbf{2w}$ and y replaced by 3.

$$\begin{aligned} (x-y)^3 &= x^3 - 3x^2y + 3xy^2 - y^3 \\ (\mathbf{2w}-3)^3 &= (\mathbf{2w})^3 - 3(\mathbf{2w})^2(3) + 3(\mathbf{2w})(3)^2 - 3^3 \quad \text{Simplify.} \\ &= 8w^3 - 36w^2 + 54w - 27 \end{aligned}$$

Of course, we can also expand the product in Example 8.8, p. 531 simply by polynomial multiplication and arrive at the same answer.

Checkpoint 8.9 **Practice 3.** Write $\left(5+x^2\right)^3$ as a polynomial.
Answer: ________________

Answer. $125 + 75x^2 + 15x^4 + x^6$

Solution. $125 + 75x^2 + 15x^4 + x^6$

Checkpoint 8.10 **QuickCheck 3.** Fill in the blanks.

a. The coefficient of the highest power term of a polynomial is called the (☐ discriminant ☐ lead coefficient) .

b. The largest exponent in a polynomial is called the (☐ degree ☐ logarithm) of the polynomial.

c. If the degrees of the terms decrease from largest to smallest, the polynomial is written in (☐ ascending powers ☐ descending powers) .

d. The degree of a product of two cubic polynomials is __.

Answer 1. lead coefficient

Answer 2. degree

Answer 3. descending powers

Answer 4. 6

Solution.

a. lead coefficient

b. degree

c. descending powers

d. 6

Factoring Cubics

Another pair of products is useful for factoring cubic polynomials. In the Homework problems, you will be asked to verify the following products:

$$(x+y)(x^2-xy+y^2)=x^3+y^3$$
$$(x-y)(x^2+xy+y^2)=x^3-y^3$$

Viewing these products from right to left, we have the following special factorizations for the sum and difference of two cubes.

Factoring the Sum or Difference of Two Cubes.

1 $x^3+y^3=(x+y)(x^2-xy+y^2)$

2 $x^3-y^3=(x-y)(x^2+xy+y^2)$

When we recognize a polynomial as a sum or difference of two perfect cubes, we then identify the two cubed expressions and apply the formula.

Example 8.11

Factor each polynomial.

a $8a^3+b^3$

b $1-27h^6$

Solution.

a This polynomial is a sum of two cubes. The cubed expressions are $2a$, because $(2a)^3=8a^3$, and b. Use formula 1, p. 532 as a pattern, replacing x with $\mathbf{2a}$, and y with b.

$$\begin{aligned} x^3+y^3 &= (x+y)(x^2-xy+y^2) \\ (\mathbf{2a})3+b3 &= (\mathbf{2a}+b)((\mathbf{2a})^2-(\mathbf{2a})b+b^2) \quad \text{Simplify.} \\ &= (2a+b)(4a^2-2ab+b^2) \end{aligned}$$

b This polynomial is a difference of two cubes. The cubed expressions are 1, because $1^3=1$, and $3h^2$, because $(3h^2)^3=27h^6$. Use formula 2, p. 532 above as a pattern, replacing x by $\mathbf{1}$, and y by $3h^2$:

$$\begin{aligned} x^3-y^3 &= (x-y)(x^2+xy+y^2) \\ \mathbf{1}^3-(3h^2)^3 &= (\mathbf{1}-3h^2)(\mathbf{1}^2+\mathbf{1}(3h^2)+(3h^2)^2) \quad \text{Simplify.} \\ &= (1-3h^2)(1+3h^2+9h^4) \end{aligned}$$

Checkpoint 8.12 **Practice 4.** Factor $125n^3-p^3=$ (________)(________)

Answer 1. $5n-p$

Answer 2. $25n^2 + 5np + p^2$

Solution. $(5n - p)(25n^2 + 5np + p^2)$

Checkpoint 8.13 QuickCheck 4. True or False.

a. We cannot factor the sum of two squares. (☐ True ☐ False)

b. We cannot factor the sum of two cubes. (☐ True ☐ False)

c. $(x + y)^3 = x^3 + y^3$ (☐ True ☐ False)

d. To factor $a^3 - b^3$, we must first identify a and b. (☐ True ☐ False)

Answer 1. True

Answer 2. False

Answer 3. False

Answer 4. True

Solution.

a. True

b. False

c. False

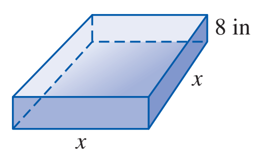
d. True

Modeling with Polynomials

Polynomials model many variable relationships, including volume and surface area.

Example 8.14

A closed box has a square base of length and width x inches and a height of 8 inches, as shown at right.



a Write a polynomial function $S(x)$ that gives the surface area of the box in terms of the dimensions of the base.

b What is the surface area of a box of length and width 18 inches?

Solution.

a The surface area of a box is the sum of the areas of its six faces,

$$S = 2lh + 2wh + 2lw$$

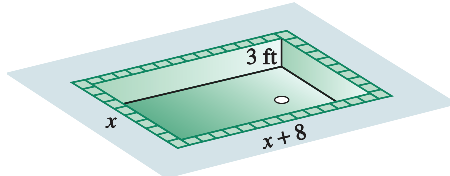
Substituting x for l and w, and 8 for h gives us

$$S(x) = 2(8)x + 2(8)x + 2x^2 = 2x^2 + 32x$$

b We evaluate the polynomial for $x = 18$ to find

$$S(18) = 2(18)^2 + 32(18) = 1224 \text{ square inches}$$

Checkpoint 8.15 Practice 5.



An empty reflecting pool is 3 feet deep. It is 8 feet longer than it is wide, as illustrated above.

a. Write a polynomial function $S(x)$ that gives the surface area of the empty pool.

   $S(x) =$__________________

b. Write a polynomial function $V(x)$ for the volume of the pool.

   $V(x) =$__________________

Answer 1. $x^2 + 20x + 48$

Answer 2. $3x^2 + 24x$

Solution.

a. We add the area of the bottom to the areas of each of the sides of the pool: $S(x) = x^2 + 20x + 48$

b. We multiply depth, length, and width: $V(x) = 3x^2 + 24x$

Cubic polynomials are often used in economics to model cost functions. The cost of producing x items is an increasing function of x, but its rate of increase is usually not constant.

Example 8.16

Pegasus Printing, Ltd. is launching a new magazine. The cost of printing x thousand copies is given by

$$C(x) = x^3 - 24x^2 + 195x + 250$$

a What are the **fixed costs**, that is, the costs incurred before any copies are printed?

b Graph the cost function in the window below and describe the graph.

Xmin = 0	Xmax = 20
Ymin = 0	Ymax = 1500

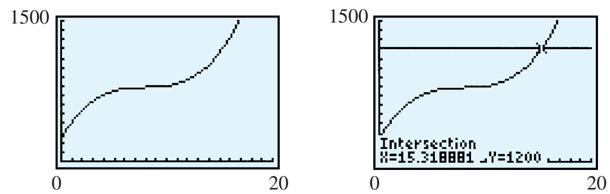
c How many copies can be printed for \$1200?

d What does the concavity of the graph tell you about the cost function?

Solution.

a Fixed costs are given by $C(0) = 250$, or \$250. The fixed costs include expenses like utility bills that must be paid even if no magazines are produced.

b The graph is shown in figure (a). It is increasing from a vertical intercept of 250. The graph is concave down for $x < 8$ approximately, and concave up for $x > 8$.



(a) (b)

c We must solve the equation

$$x^3 - 24x^2 + 195x + 250 = 1200$$

We will solve the equation graphically, as shown in figure (b). Graph $y = 1200$ along with the cost function, and use the *intersect* command to find the intersection point of the graphs, $(15.319, 1200)$. $C(x) = 1200$ when x is about 15.319, so 15,319 copies can be printed for \$1200.

d Although the cost is always increasing, it increases very slowly from about $x = 5$ to about $x = 11$. The flattening of the graph in this interval is a result of economy of scale: By buying supplies in bulk and using time efficiently, the cost per magazine can be minimized. However, if the production level is too large, costs begin to rise rapidly again.

In Example 8.16, p. 534c, we solved a cubic equation graphically. There is a cubic formula, analogous to the quadratic formula, that allows us to solve cubic equations algebraically, but it is complicated and not often used.

Cubic polynomials are also used to model smooth curves connecting given points. Such a curve is called a **cubic spline**.

Checkpoint 8.17 Practice 6. Leon is flying his plane to Au Gres, Michigan. He maintains a constant altitude until he passes over a marker just outside the neighboring town of Omer, when he begins his descent for landing. During the descent, his altitude, in feet, is given by

$$A(x) = 128x^3 - 960x^2 + 8000$$

where x is the number of miles Leon has traveled since passing over the marker in Omer.

a. What is Leon's altitude when he begins his descent?

____ ft

b. Graph $A(x)$ in the window

$$\begin{aligned} \text{Xmin} &= 0 \quad \text{Xmax} = 5 \\ \text{Ymin} &= 0 \quad \text{Ymax} = 8000 \end{aligned}$$

c. Use the *Trace* feature to discover how far from Omer Leon will travel before landing. (In other words, how far is Au Gres from Omer?)

__ mi

d. Verify your answer to part (c) algebraically.
   $A(5) =$__

Answer 1. 8000
Answer 2. 5
Answer 3. 0
Solution.

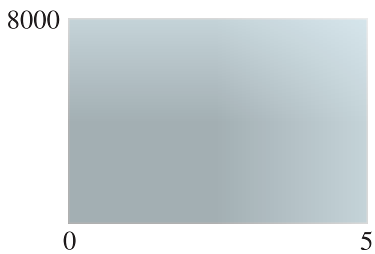
a. 8000 ft

b. A graph is below.

c. 5 mi

d. $A(5) = 0$, that is, Leon is on the ground (at altitude 0) when he is 5 miles past Omer.

Graph for part (b)



Checkpoint 8.18 QuickCheck 5. Fill in the blanks.

a. We find the surface area of an object by calculating the area of each face and then (☐ adding ☐ averaging ☐ folding ☐ multiplying) them.

b. In economics, (☐ cubic polynomials ☐ decreasing lines ☐ demand curves ☐ inverse-square functions) are often used to model cost functions.

c. The flattening of a cost function in its midrange is a result of (☐ consumer surplus ☐ economy of scale ☐ inflation ☐ measurement error) .

d. A smooth curve connecting given points by cubic polynomials is called a (☐ cubic spline ☐ elastic demand ☐ a catenary ☐ an ogive) .

Answer 1. adding
Answer 2. cubic polynomials
Answer 3. economy of scale
Answer 4. cubic spline
Solution.

a. adding

b. cubic polynomials

c. economy of scale

d. cubic spline

Problem Set 8.1

Warm Up

1. Complete the formulas for the special products of quadratic polynomials.
a $(a+b)^2 =$ b $(a-b)^2 =$

2. Factor if possible.
a $a^2 - b^2$ b $a^2 + b^2$

3. Use the formulas to factor the polynomials, if possible.
a $x^2 - 14x + 49$ c $x^2 + 6x + 9$
b $x^2 + 64$ d $x^2 - 64$

4. Use the formulas to write down the products without multiplying:
a $(5x - 6y)^2$ c $(3x - 8y^2)(3x + 8y^2)$
b $(2x^3 + y^5)^2$ d $(x^2 + y^2)^2$

Skills Practice

5. Which of the following are polynomials? If it is not a polynomial, explain why not.
a $5x^4 - 3x^2$ c $\dfrac{1}{2a^2 + 5a - 6}$
b $3x + 1 + \dfrac{2}{x^2}$ d $\dfrac{2}{3}t^2 + \dfrac{1}{4}t^3 + \dfrac{5}{8}$

6. Give the degree of each polynomial.
a $x^2 + 4x - \dfrac{1}{4}$ b $y - 2.8y^7$ c $\dfrac{z}{4} - 3z^4 + 4z^3$

7. Write each polynomial in descending powers of x.
a $x - 1.9x^3 + 6.4$ b $6xy - 2x^2 + 2y^3$

8. Add or subtract the polynomials.
a $(2y^3 - 4y^2 - y) + (6y^2 + 2y + 1)$
b $(5x^3 + 3x^2 - 4x + 8) - (2x^3 - 4x - 3)$

9. Without performing the multiplication, give the degree of each product.
a $(x^2 - 4)(3x^2 - 6x + 2)$
b $(x - 3)(2x - 5)(x^3 - x + 2)$
c $(3x^2 + 2x)(x^3 + 1)(-2x^2 + 8)$

For Problems 10 and 11, multiply.

10.
a $(3x - 2)(4x^2 + x - 2)$ b $(x - 2)(x - 1)(x - 3)$

11.
a $(2a^2 - 3a + 1)(3a^2 + 2a - 1)$ b $(y - 2)(y + 2)(y + 4)(y + 1)$

For Problems 12 and 13, use the formulas for the cube of a binomial to expand the products.

12.
a $(1 + 2z)^3$ b $(1 - x^2)^3$

13.
a $(1 + 5\sqrt{t})^3$ b $(1 - \dfrac{3}{a})^3$

For Problems 14 and 15, write each product as a polynomial and simplify.

14.

a $(2x+1)(4x^2-2x+1)$

b $(3x-1)(9x^2+3x+1)$

15.

a $(3a-2b)(9a^2+6ab+4b^2)$

b $(2a+3b)(4a^2-6ab+9b^2)$

For Problems 16 and 21, factor completely.

16. x^3+27 **17.** a^3-8b^3 **18.** x^3y^6-1

19. $27a^3+64b^3$ **20.** $125a^3b^3-1$ **21.** $64t^9+w^6$

22. Evaluate each polynomial for $n=10$. Try to do the calculations mentally. What do you notice?

a $5n^2+6n+7$

b $5n^3+n^2+3n+3$

c n^3+1

d $8n^4+8n$

Applications

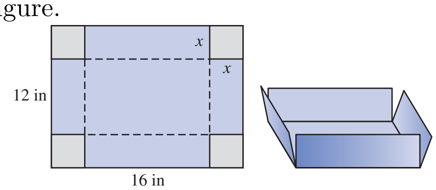
23. A large wooden box (without lid) is 3 feet longer than it is wide, and its height is 2 feet shorter than its width.

a If the width of the box is w, write expressions for its length and its height.

b Write a polynomial for the volume of the box.

c Write a polynomial for the surface area of the box.

24. A paper company plans to make boxes without tops from sheets of cardboard 12 inches wide and 16 inches long. The company will cut out four squares of side x inches from the corners of the sheet and fold up the edges as shown in the figure.



a Write expressions in terms of x for the length, width, and height of the resulting box.

b Write a formula for the volume, V, of the box as a function of x.

c What are the largest and smallest reasonable values for x?

d Make a table of values for $V(x)$ on its domain.

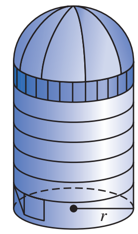
e Graph your function V in the window

Xmin = 0 Xmax = 6
Ymin = 0 Ymax = 200

f Use your graph to find the value of x that will yield a box with maximum possible volume. What is the maximum possible volume?

25.

(a) A grain silo is built in the shape of a cylinder with a hemisphere on top (see the figure). Write an expression for the volume of the silo in terms of the radius and height of the cylindrical portion of the silo.



(b) If the total height of the silo is five times its radius, write a polynomial function $V(r)$ in one variable for its volume.

26. The annual profit, $P(t)$, of the Enviro Company, in thousands of dollars, is given by

$$P(t) = 2t^3 - 152t^2 + 3400t + 30$$

where t is the number of years since 1980, the first year that the company showed a profit.

(a) Graph $P(t)$ in the window

Xmin = 0	Xmax = 94
Ymin = 0	Ymax = 50,000

(b) What was the profit in 1980? In 2000? In 2020?

(c) How did the profit change from 1980 to 1981? From 2000 to 2001? From 2020 to 2021?

(d) During which years did the profit decrease from one year to the next?

27. A doctor who is treating a heart patient wants to prescribe medication to lower the patient's blood pressure. The body's reaction to this medication is a function of the dose administered. If the patient takes x milliliters of the medication, his blood pressure should decrease by $R = f(x)$ points, where

$$f(x) = 3x^2 - \frac{1}{3}x^3$$

a For what values of x is $R = 0$?

b What values of x make sense for this function? Why?

c Graph the function in the window

Xmin = 0	Xmax = 10
Ymin = 0	Ymax = 40

d How much should the patient's blood pressure drop if he takes 2 milliliters of medication?

e What is the maximum drop in blood pressure that can be achieved with this medication?

f There may be risks associated with a large change in blood pressure. How many milliliters of the medication should be administered to produce half the maximum possible drop in blood pressure?

28. The water level (in feet) at a harbor is approximated by the polynomial

$$W(t) = 0.00733t^4 - 0.332t^2 + 9.1$$

where t is the number of hours since the high tide. The approximation is valid for $-4 \leq t \leq 4$. (A negative value of t corresponds to a number of hours before the high tide.)

(a) Graph the polynomial for $-4 \leq t \leq 4$.

(b) What is the water level at high tide?

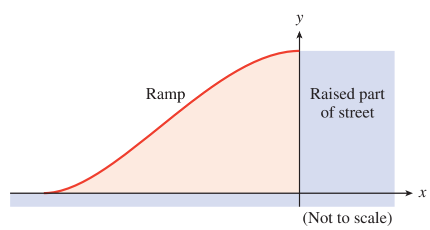
(c) What is the water level 3 hours before high tide?

(d) When is the water level below 8 feet?

(e) When is the water level above 7 feet?

29. During an earthquake, Nordhoff Street split in two, and one section shifted up several centimeters. Engineers created a ramp from the lower section to the upper section. In the coordinate system shown in the figure below, the ramp is part of the graph of

$$y = f(x) = -0.00004x^3 - 0.006x^2 + 20$$



(a) By how much did the upper section of the street shift during the earthquake?

(b) What is the horizontal distance from the bottom of the ramp to the raised part of the street?

30. Expand each expression by removing parentheses. What do you notice?

a $x[x(x+3)+4]+1$

b $x(x[x(x-7)-5]+8)-3$

c Evaluate each expression for $x = 2$. Can you do this mentally? Is it easier to evaluate the expression before or after expanding it?

d Use a calculator to evaluate each expression for $x = 0.8$.

31. Find the first three terms of the product in ascending powers. (Do not compute the entire product!)

a $(2 - x + 3x^2)(3 + 2x - x^2 + 2x^4)$

b $(1 - 2x^2 - x^4)(4 + x^2 - 2x^4)$

32. Find the indicated term in each product. (Do not compute the entire product!)

a $(4 + 2x - x^2)(2 - 3x + 2x^2)$; x^2

b $(3x + x^3 - 7x^5)(1 + 4x - 3x^2)$; x^3

33. Verify the formula for factoring the sum of two cubes:

$$(x + y)(x^2 - xy + y^2) = x^3 + y^3$$

34. Verify the formula for factoring the difference of two cubes:

$$(x-y)(x^2+xy+y^2) = x^3 - y^3$$

Algebraic Fractions

Introduction

An **algebraic fraction** (or **rational expression**, as they are someitmes called,) is a fraction in which both numerator and denominator are polynomials. Here are some examples of algebraic fractions:

$$\frac{3}{x}, \qquad \frac{a^2+1}{a-2}, \qquad \text{and} \qquad \frac{z-1}{2z+3}$$

We can evaluate algebraic fractions just as we evaluate any other algebraic expression.

Caution 8.19 If we try to evaluate the fraction $\frac{a^2+1}{a-2}$ for $a = 2$, we get $\frac{2^2+1}{2-2}$ or $\frac{5}{0}$, which is undefined. When working with fractions, we must exclude any values of the variable that make the denominators equal to zero.

Reducing Fractions

You'll recall from your study of arithmetic that we can **reduce** a fraction if we can divide both numerator and denominator by a **common factor**. In algebra, it is helpful to think of factoring out the common factor first. For example,

$$\frac{27}{36} = \frac{\not{9}\cdot 3}{\not{9}\cdot 4} = \frac{3}{4}$$

where we have divided both numerator and denominator by 9. The new fraction has the same value as the old one, namely 0.75, but it is simpler (the numbers are smaller.) Reducing is an application of the Fundamental Principle of Fractions.

Fundamental Principle of Fractions.

We can multiply or divide the numerator and denominator of a fraction by the same nonzero factor, and the new fraction will be equivalent to the old one.

$$\frac{a\cdot c}{b\cdot c} = \frac{a}{b} \qquad \text{if} \qquad b, c \neq 0$$

Example 8.20

Reduce $\frac{8x^3y}{6x^2y^3}$

Solution. The common factor for numerator and denominator is $2x^2y$. We factor $2x^2y$ from the numerator and denominator, then divide by the common factor.

$$\frac{8x^3y}{6x^2y^3} = \frac{4x\cdot \cancel{2x^2y}}{3y^2\cdot \cancel{2x^2y}} = \frac{4x}{3y^2}$$

Checkpoint 8.21 Practice 1. Reduce $\dfrac{-15ab^4}{25a^2b}$:

$$\frac{\_\_\_\_}{\_\_\_\_}$$

Answer 1. $-3b^3$

Answer 2. $5a$

Solution. $\dfrac{-3b^3}{5a}$

Caution 8.22 When we cancel common factors, we are *dividing*. Because division is the inverse or opposite operation for multiplication, we can cancel common *factors*, but *we cannot cancel common terms*.

Example 8.23

Use your calculator to decide which calculation is correct.

a $\dfrac{12}{8} = \dfrac{4 \cdot 3}{4 \cdot 2} \to \dfrac{3}{2}$

b $\dfrac{7}{6} = \dfrac{4+3}{4+2} \to \dfrac{3}{2}$

Solution. We can cancel the 4's in part (a), because they are factors of numerator and denominator.

We cannot cancel the 4's in part (b), because they are terms.

You can verify that $\dfrac{12}{8} = \dfrac{3}{2}$, but $\dfrac{7}{6} \neq \dfrac{3}{2}$.

Checkpoint 8.24 QuickCheck 1. Fill in the blanks.

a. An algebraic fraction is undefined when the (☐ denominator ☐ index ☐ numerator ☐ radicand) is (☐ even ☐ negatove ☐ positive ☐ zero) .

b. "Canceling" common factors uses the operation of (☐ addition ☐ division ☐ subtraction ☐ squaring) .

c. Terms are expressions that are (☐ added ☐ exponentiated ☐ multiplied ☐ reduced) or (☐ divided ☐ expended ☐ simplified ☐ subtracted) .

d. When we reduce a fraction, the new fraction is (☐ equivalent ☐ inverse ☐ superior ☐ unrelated) to the old one.

Answer 1. denominator

Answer 2. zero

Answer 3. division

Answer 4. added

Answer 5. subtracted

Answer 6. equivalent

Solution.

a. denominator, zero

b. division

c. added, subtracted

d. equivalent

Checkpoint 8.25 Practice 2. Which calculation is correct? Choose a value for x and use your calculator to verify your answer.

⊙ (a) $\dfrac{5x}{8x} = \dfrac{5}{8}$

⊙ (b) $\dfrac{x+5}{x+8} = \dfrac{5}{8}$

Answer. Choice 1

Solution. (a) $\dfrac{5x}{8x} = \dfrac{5}{8}$

If the numerator or denominator of the fraction contains more than one term, it is especially important to *factor before reducing*, so that numerator and denominator are written as products of factors, instead of sums of terms.

Example 8.26

Reduce each fraction.

a $\dfrac{4x+2}{4}$

b $\dfrac{9x^2+3}{6x+3}$

Solution. First we factor the numerator and denominator. Then we divide numerator and denominator by any common factors.

a $$\frac{4x+2}{4} = \frac{\cancel{2}(2x+1)}{\cancel{2}(2)} = \frac{2x+1}{2}$$

b $$\frac{9x^2+3}{6x+3} = \frac{\cancel{3}(3x^2+1)}{\cancel{3}(2x+1)} = \frac{3x^2+1}{2x+1}$$

Caution 8.27 We cannot cancel the 4's in part (a) of the Example, p. 543 above, because 4 is not a factor of the entire numerator. Thus,

$$\frac{4x+2}{4} \neq x+2$$

In the Example, p. 543 part (b), we cannot cancel the 3's, because they are terms and not factors. Thus,

$$\frac{9x^2+3}{6x+3} \neq \frac{9x^2}{6x}$$

Checkpoint 8.28 Practice 3. Explain why each calculation is incorrect.

a. $\dfrac{x-6}{x-9} \to \dfrac{x-2}{x-3}$

⊙ A) 3 is not a factor of numerator or denominator.

⊙ B) More cancelling is required.

b. $\dfrac{2x+5}{2} \to x+5$

⊙ A) 2 is not a factor of the numerator.

⊙ B) 2 should be subtracted from 5.

Answer 1. A) ... denominator.

Answer 2. A) 2 ... the numerator.

Solution.

a. 3 is not a factor of numerator or denominator

b. 2 is not a factor of the numerator

We summarize the procedure for reducing algebraic fractions as follows.

To reduce an algebraic fraction.

1 Factor the numerator and the denominator.

2 Divide the numerator and denominator by any common factors.

Example 8.29

Reduce each fraction.

a $\dfrac{3x+12}{6x+24}$

b $\dfrac{27x^3-1}{9x^2-1}$

Solution.

a We first factor the numerator and denominator completely.

$$\frac{3x+12}{6x+24} = \frac{3(x+4)}{2\cdot 3(x+4)}$$

Then we divide numerator and denominator by the common factors 3 and $(x+4)$. We must cancel the entire expression $(x+4)$ from numerator and denominator (we cannot cancel the x's or the 4's separately!).

$$\frac{3(x+4)}{2\cdot 3(x+4)} = \frac{\cancel{3}\cancel{(x+4)}}{2\cdot \cancel{3}\cancel{(x+4)}} = \frac{1}{2}$$

All the factors are canceled from the numerator, so we replace them by 1, because any expression divided by itself is 1.

b The numerator of the fraction is a difference of two cubes, and the denominator is a difference of two squares. We factor each to obtain

$$\frac{27x^3-1}{9x^2-1} = \frac{(3x-1)(9x^2+3x+1)}{(3x-1)(3x+1)}$$

We cancel the factor $(3x-1)$ from top and bottom, to get

$$\frac{\cancel{(3x-1)}(9x^2+3x+1)}{\cancel{(3x-1)}(3x+1)} = \frac{9x^2+3x+1}{3x+1}$$

Because we cannot factor any further, we cannot reduce the fraction any further.

Checkpoint 8.30 QuickCheck 2. Fill in the blanks.

a. If we want to cancel an expression from a fraction, we must be able to (☐ exponentiate ☐ factor ☐ invalidate ☐ subtract) it from numerator and denominator.

b. When we factor an expression, we write it as a (☐ difference ☐ product ☐ quotient ☐ sum) .

c. If a factor appears in both numerator and denominator, it is called a (☐ common ☐ fallacious ☐ missing ☐ zero) factor.

d. If all the factors in the numerator or denominator cancel out, we replace it by __.

Answer 1. factor

Answer 2. product

Answer 3. common

Answer 4. 1

Solution.

a. factor

b. product

c. common

d. 1

Checkpoint 8.31 Practice 4. Reduce each fraction.

a. $\dfrac{x^2-x-6}{x^2-9}$

- ⊙ $\dfrac{x+6}{9}$
- ⊙ $\dfrac{x+2}{x+3}$
- ⊙ $\dfrac{x-2}{x-3}$

b. $\dfrac{16t^2-4}{8t+4}$

- ⊙ $2t-1$
- ⊙ $\dfrac{t-2}{2}$
- ⊙ $3t^2$

Answer 1. Choice 2

Answer 2. Choice 1

Solution.

a. $\dfrac{x+2}{x+3}$

b. $2t-1$

Keep in mind that the reduced form is equivalent to the original form of the fraction. If we evaluate the original form and the reduced form at the same value of the variable, the results are equal.

Example 8.32

Verify that $\dfrac{x^2-x-6}{x^2-9}$ is equal to $\dfrac{x+2}{x+3}$ for $x=2$.

Solution. We evaluate each fraction at $x = \mathbf{2}$.

$$\frac{x^2-x-6}{x^2-9} = \frac{(\mathbf{2})^2-\mathbf{2}-6}{(\mathbf{2})^2-9} = \frac{-4}{-5} = \frac{4}{5}$$

and

$$\frac{x+2}{x+3} = \frac{\mathbf{2}+2}{\mathbf{2}+3} = \frac{4}{5}$$

Checkpoint 8.33 **Practice 5.** If you evaluate $\frac{3x+12}{6x+24}$ for $x=2$ and for $x=-5$, what answer do you expect to get? (Recall the earlier Example with the same algebraic fraction in part a.)

Answer. $\frac{1}{2}$

Solution. $\frac{1}{2}$

Opposite of a Binomial

Any number (except zero) divided by itself is 1, and any number divided by its opposite is -1. For example,

$$\frac{5}{5} = 1 \qquad \text{and} \qquad \frac{-5}{5} = -1$$

The same is true for binomials and other algebraic expressions. The opposite of an expression can be found by multiplying it by -1. Thus, the opposite of $a-b$ is

$$-(a-b) = -a+b = b-a$$

and so

$$\frac{b-a}{a-b} = \frac{-\cancel{(a-b)}}{\cancel{(a-b)}} = -1$$

Here are some examples of opposites.

$$\begin{array}{lcl} 2a-3b & \text{and} & 3b-2a \\ 2a+3b & \text{and} & -2a-3b \\ -x-1 & \text{and} & x+1 \\ -x+1 & \text{and} & x-1 \end{array}$$

We can cancel opposites when we reduce fractions.

Example 8.34

Reduce $\frac{2x-4y}{6y-3x}$

Solution. We first factor the numerator and denominator.

$$\frac{2x-4y}{6y-3x} = \frac{2(x-2y)}{3(2y-x)}$$

We see that $x-2y$ is the opposite of $2y-x$, that is, $x-2y = -(2y-x)$. Thus,

$$\frac{2(x-2y)}{3(2y-x)} = \frac{-2\cancel{(2y-x)}}{3\cancel{(2y-x)}} = \frac{-2}{3}$$

Checkpoint 8.35 **Practice 6.** Reduce if possible.

a. $\frac{-x+1}{1-x}$ ____

b. $\dfrac{1+x}{1-x}$ ______

c. $\dfrac{2a-3b}{3b-2a}$ ______

d. $\dfrac{2a-3b}{2b-3a}$ ______

Answer 1. 1

Answer 2. cannot be reduced or cbr

Answer 3. -1

Answer 4. cannot be reduced or cbr

Solution.

a. 1

b. cannot be reduced

c. -1

d. cannot be reduced

Rational Functions

A **rational function** is a function defined by an algebraic fraction. That is, it has the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials. A rational function is undefined at any x-values where $Q(x) = 0$.

Example 8.36

Francine is planning a 60-mile training flight through the desert on her cycle-plane, a pedal-driven aircraft. If there is no wind, she can pedal at an average speed of 15 miles per hour, so she can complete the flight in 4 hours.

a If there is a headwind of x miles per hour, it will take Francine longer to fly 60 miles. Express the time it will take to complete the training flight as a function of x.

b Make a table of values for the function.

c Graph the function and explain what it tells you about the time Francine should allot for the flight.

Solution.

a If there is a headwind of x miles per hour, Francine's ground speed will be $15 - x$ miles per hour. Using the fact that time $\text{time} = \dfrac{\text{distance}}{\text{rate}}$, we find that the time needed for the flight will be

$$t = f(x) = \frac{60}{15 - x}$$

b We evaluate the function for several values of x, as shown in the table below.

x	0	3	5	7	9	10
t	4	5	6	7.5	10	12

For example, if the headwind is **5** miles per hour, then

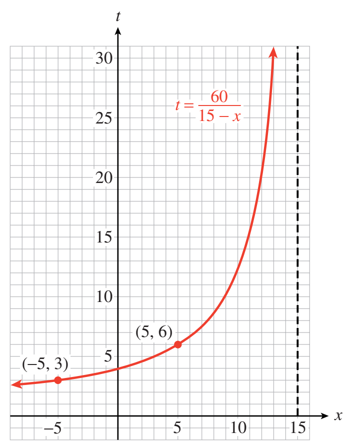
$$t = \frac{60}{15 - \mathbf{5}} = \frac{60}{10} = 6$$

Francine's effective speed is only 10 miles per hour, and it will take her 6 hours to fly the 60 miles. The table shows that as the speed of the headwind increases, the time required for the flight increases also.

c The graph of the function is shown below. You can use your calculator with the window

$$\text{Xmin} = -8.5 \qquad \text{Xmax} = 15$$
$$\text{Ymin} = 0 \qquad \text{Ymax} = 30$$

to verify the graph. In particular, the point $(0, 4)$ lies on the graph. This point tells us that if there is no wind, Francine can fly 60 miles in 4 hours, as we calculated earlier.



The graph is increasing, as indicated by the table of values. In fact, as the speed of the wind gets close to 15 miles per hour, Francine's flying time becomes extremely large. In theory, if the wind speed were exactly 15 miles per hour, Francine would never complete her flight. On the graph, the time becomes infinite at $x = 15$.

What about negative values for x? If we interpret a negative headwind as a tailwind, Francine's flying time should decrease for negative x-values. For example, if $x = -5$, there is a tailwind of 5 miles per hour, so Francine's effective speed is 20 miles per hour, and she can complete the flight in 3 hours. As the tailwind gets stronger (that is, as we move farther to the left in the x-direction), Francine's flying time continues to decrease, and the graph approaches the x-axis.

The vertical dashed line at $x = 15$ on the graph of $t = \dfrac{60}{15 - x}$ is a **vertical asymptote** for the graph. We first encountered asymptotes in Section 5.3,

p. 299 when we studied the graph of $y = \frac{1}{x}$. Locating the vertical asymptotes of a rational function is an important part of determining the shape of the graph.

Checkpoint 8.37 Practice 7. EarthCare decides to sell T-shirts to raise money. The company makes an initial investment of \$100 to pay for the design of the T-shirt and to set up the printing process. After that, the T-shirts cost \$5 each for labor and materials.

a. Express the average cost, C, per T-shirt as a function of the number of T-shirts EarthCare produces.

   $C =$_________

b. Make a table of values for the function.

x	1	2	4	5	10	20
C	___	__	__	__	__	__

c. Graph the function and explain what it tells you about the cost of the T-shirts.

Answer 1. $\frac{100+5x}{x}$

Answer 2. 105

Answer 3. 55

Answer 4. 30

Answer 5. 25

Answer 6. 15

Answer 7. 10

Solution.

a. $C = g(x) = \dfrac{100 + 5x}{x}$

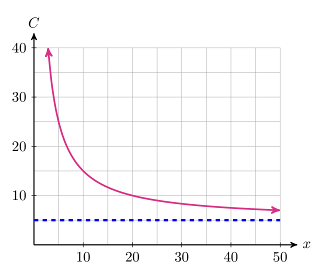
b.

x	1	2	4	5	10	20
C	105	55	30	25	15	10

c. A graph is below.

   As the number of T-shirts increases, the average cost per shirt decreases, but eventually levels off and approaches \$5 per T-shirt.

Graph for part (c):



In Practice 7, p. 549, the horizontal line $C = 5$ is a **horizontal asymptote** for the graph of the function. As x increases, the graph approaches the line $C = 5$ but will never actually meet it. The average price per T-shirt will always be slightly more than \$5. Horizontal asymptotes are also important in sketching the graphs of rational functions.

Checkpoint 8.38 QuickCheck 3. True or False.

a. A rational function is a quotient of polynomials. (☐ True ☐ False)

b. A vertical asymptote occurs where a rational function is undefined. (☐ True ☐ False)

c. A horizontal asymptote is a line that the graph of a rational function approaches as x increases. (☐ True ☐ False)

d. The algebraic fraction $\dfrac{60}{x-15}$ is never equal to zero. (☐ True ☐ False)

Answer 1. True
Answer 2. True
Answer 3. True
Answer 4. True
Solution.

a. True

b. True

c. True

d. True

Problem Set 8.2

Warm Up

For Problems 1–2,

a Evaluate the fraction for the given values of the variable.

b For what values of the variable is the fraction undefined?

c Find a value of x for which the fraction is equal to zero.

1. $\dfrac{x+1}{x-3}$, $x = \dfrac{1}{2},\ -4$

2. $\dfrac{2a-a^2}{a^2+1}$, $a = 3,\ -1$

3. $\dfrac{2x}{x^2-1}$, $x = -2,\ 20$

4. $\dfrac{x-2}{x^2-2x+1}$, $x = -1,\ \dfrac{1}{2}$

For Problems 5–8, reduce the fraction.

5.

a $\dfrac{15}{3x}$

b $\dfrac{24b}{14}$

6.

a $\dfrac{-5z}{6z}$

b $\dfrac{5u}{120uv}$

7.

a $\dfrac{16ab}{-10ab}$

b $\dfrac{3a^2}{27a}$

8.

a $\dfrac{-9y^3z}{42yz}$

b $\dfrac{8u^3v^2}{12v^2w}$

Skills Practice

9. Which of the following is a correct application of the fundamental principle of fractions? Explain why or why not in each case.

a $\dfrac{3x+5}{3x} \to 5$

b $\dfrac{4x+3}{2y} \to \dfrac{2x+3}{y}$

c $\dfrac{2x^2+x-12}{x-12} \to 2x^2$

d $\dfrac{8x^2-9}{2x-3} \to 4x+3$

For Problems 10 and 11, decide whether the fraction is equivalent to 1, to −1, or cannot be reduced.

10.

a $\dfrac{x+4}{x-4}$

b $\dfrac{t+5w}{5w-t}$

c $\dfrac{x+3z}{z+3x}$

d $\dfrac{-(m-1)}{1-m}$

11.

a $\dfrac{2a+b}{2a-b}$

b $\dfrac{-(a-b)}{b-a}$

c $\dfrac{2a^2-1}{2a^2}$

d $\dfrac{-a^2+3)}{a^2+3}$

For Problems 12-15, reduce if possible, and select the correct reduced form.

12. $\dfrac{2x+3}{2x}$

a $\dfrac{x+3}{x}$ b 3 c neither of these

13. $\dfrac{3a+a^2}{3a}$

a a^2 b $\dfrac{3+a}{3}$ c neither of these

14. $\dfrac{y+3}{2y^2+6y}$

a $\dfrac{1}{2y}$ b y c neither of these

15. $\dfrac{a^3}{a^4-a^3}$

a $\dfrac{1}{a-1}$ b $\dfrac{1}{a^4}$ c neither of these

16. Reduce each fraction. Which of the fractions are equivalent to $3b$?

a $\dfrac{9b^2-3b}{3b}$

b $\dfrac{b+2}{3b^2+6b}$

c $\dfrac{3b-9}{9}$

d $\dfrac{9b^2-3b)}{3b-1}$

For Problems 17-28, reduce if possible.

17. $\dfrac{2a^2}{2a^2-6a}$

18. $\dfrac{b-2}{4-2b}$

19. $\dfrac{a-b}{a^2-b^2}$

20. $\dfrac{(3x+2y)^2}{4y^2-9x^2}$

21. $\dfrac{y^2-9x^2}{(3x-y)^2}$

22. $\dfrac{6-6t^2}{(t-1)^2}$

23. $\dfrac{2y^2-8}{2y+4}$

24. $\dfrac{4x^3-36x}{6x^2-18x}$

25. $\dfrac{3a-a^2}{a^2-2a-3}$

26. $\dfrac{2x^2+x-6}{x^2+x-2}$

27. $\dfrac{8z^3-27}{4z^2-9}$

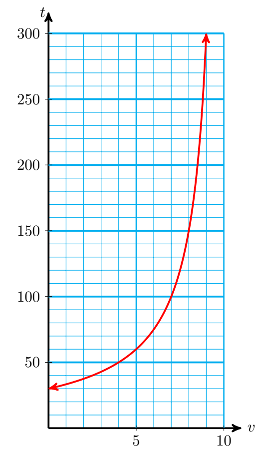
28. $\dfrac{6-2v}{v^3-27}$

Applications

29. The crew team can row at a steady pace of 10 miles per hour in still water. Every afternoon, their training includes a five-mile row upstream on the river. If the current in the river on a given day is v miles per hour, then the time required for this exercise, in minutes, is given by

$$t = \frac{300}{10 - v}$$

Use the graph of this equation shown in the figure to answer questions (a) and (b).



a How long does the exercise take if there is no current in the river?

b How long will it take if the current is 4 miles per hour?

c Find an exact answer for part (b) by using the equation.

d If the exercise took 2 hours, what was the current in the river?

e As the speed of the current increases, what happens to the time needed for the exercise?

30. The eider duck, one of the world's fastest flying birds, can exceed an airspeed of 65 miles per hour. A flock of eider ducks is migrating south at an average airspeed of 50 miles per hour against a moderate headwind. Their next feeding grounds are 150 miles away.

a Express the ducks' travel time, t, as a function of the windspeed, v.

b Complete the table showing the travel time for various windspeeds.

v	0	10	20	30	40	50
t						

What happens to the travel time as the headwind increases?

c Use the table to choose an appropriate window and graph your function $t(v)$. Label the scales on the axes.

d Estimate the wind speed if the travel time was 12 hours. Illustrate your result on the graph.

e Give the equations of any horizontal or vertical asymptotes. What does the vertical asymptote tell us about the problem?

31. The cost, in thousands of dollars, for extracting p percent of a precious ore from a mine is given by the equation

$$C(p) = \frac{360p}{100 - p}$$

a What input values make sense for C?

b Complete the table showing the cost of extracting various percent-ages of the ore. (Note: do not convert percents to decimals.)

p	0	25	50	75	90	100
C						

c Graph the function C in an appropriate window. What percentage of the ore can be extracted if \$540,000 can be spent on the extraction?

d For what values of p is the total cost less than \$1,440,000?

e The graph has a vertical asymptote. What is it? What is its significance in the context of this problem?

32. The total cost in dollars of producing n calculators is approximately $20,000 + 8n$.

a Express the cost per calculator, C, as a function of the number n of calculators produced.

b Complete the table showing the cost per calculator for various production levels.

n	100	400	500	1000	4000	5000
C						

c Graph the function $C(n)$ for the cost per calculator. Use the window

$$\text{Xmin} = 0 \qquad \text{Xmax} = 9400$$
$$\text{Ymin} = 0 \qquad \text{Ymax} = 50$$

d How many calculators should be produced so that the cost per calculator is \$18?

e For what values of n is the cost less than \$12 per calculator?

f Find the horizontal asymptote of the graph. What does it represent in this context?

33. The volume of a test tube is given by its height times the area of its cross-section. A test tube that holds 200 cubic centimeters is $2x-1$ centimeters long.

a What is the area of its cross-section?

b Evaluate your fraction for $x = 13$. What does your answer mean in the context of the problem?

34. Delbert prepares a 25% glucose solution of by mixing 2 ml of glucose with 8 ml of water. If he adds x ml of glucose to the solution, its concentration is given by

$$C(x) = \frac{2+x}{8+x}$$

a How many ml of glucose should Delbert add to increase the concentration to 50%?

b Graph the function for $0 \le x \le 100$

c What is the horizontal asymptote of the graph? What does it tell you about the solution?

35. A computer store sells approximately 300 of its most popular model per year. The manager would like to minimize her annual inventory cost by ordering the optimal number of computers, x, at regular intervals. If she orders x computers in each shipment, the cost of storage will be $6x$ dollars, and the cost of reordering will be $\dfrac{300}{x}(15x + 10)$ dollars. The inventory cost is the sum of the storage cost and the reordering cost.

a Use the distributive law to simplify the expression for the reordering cost. Then express the inventory cost, C, as a function of x.

b Complete the table of values for the inventory cost for various reorder sizes.

x	20	40	60	80	100
C					

c Graph the function C for the cost per calculator. Use the window

$$\text{Xmin} = 0 \qquad \text{Xmax} = 150$$
$$\text{Ymin} = 4500 \qquad \text{Ymax} = 5500$$

Estimate the minimum possible value for C.

d How many computers should the manager order in each shipment so as to minimize the inventory cost? How many orders will she make during the year?

e Graph the function $y = 6x + 4500$ in the same window with the function C. What do you observe?

36. A train whistle sounds higher when the train is approaching you than when it is moving away from you. This phenomenon is known as the Doppler effect. If the actual pitch of the whistle is 440 hertz (this is the A note below middle C), then the note you hear will have the pitch

$$P(v) = \frac{440(332)}{332 - v}$$

where the velocity, v, in meters per second is positive as the train approaches and is negative when the train is moving away. (The number 332 that appears in this expression is the speed of sound in meters per second.)

a Complete the table of values showing the pitch of the whistle at various train velocities.

v	-100	-75	-50	-25	0	25	50	75	100
P									

b Graph the function P in an appropriate window.

c What is the velocity of the train if the note you hear has a pitch of 415 hertz (corresponding to the note A-flat)? A pitch of $553.\overline{3}$ hertz (C-sharp)?

d For what velocities will the pitch you hear be greater than 456.5 hertz?

e The graph has a vertical asymptote (although it is not visible in the suggested window). Where is it and what is its significance in this context?

Operations on Algebraic Fractions

Products of Fractions

To multiply two fractions together, we multiply their numerators together and then multiply their denominators together.

Product of Fractions.

If $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

If a common factor occurs in a numerator and a denominator of either fraction, we can divide it out either before or after multiplying. For example,

$$\frac{3a}{4} \cdot \frac{5}{6a^2} = \frac{\cancel{3} \cdot \cancel{a}}{4} \cdot \frac{5}{\cancel{3} \cdot 2 \cdot \cancel{a} \cdot a} = \frac{5}{8a}$$

or

$$\frac{3a}{4} \cdot \frac{5}{6a^2} = \frac{15a}{24a^2} = \frac{\cancel{3} \cdot 5 \cdot \cancel{a}}{\cancel{3} \cdot 8 \cdot \cancel{a} \cdot a} = \frac{5}{8a}$$

It is usually easier to cancel any common factors before multiplying.

Example 8.39

Multiply $\dfrac{2x-4}{3x+6} \cdot \dfrac{6x+9}{x-2}$

Solution. First, we factor each numerator and denominator. Then we divide numerator and denominator by any common factors.

$$\frac{2x-4}{3x+6} \cdot \frac{6x+9}{x-2} = \frac{2\cancel{(x-2)}}{\cancel{3}(x+2)} \cdot \frac{\cancel{3}(2x+3)}{\cancel{x-2}}$$

$$= \frac{2(2x+3)}{x+2} \quad \text{or} \quad \frac{4x+6}{x+2}$$

Checkpoint 8.40 **Practice 1.** Find the product $\dfrac{4y^2-1}{4-y^2} \cdot \dfrac{y^2-2y}{4y+2}$:

Answer 1. $-y(2y-1)$

Answer 2. $2(y+2)$

Solution. $\dfrac{-y(2y-1)}{2(y+2)}$

To multiply a fraction by a whole number, we write the whole number with 1 as denominator.

$$\frac{2}{3} \cdot \mathbf{4} = \frac{2}{3} \cdot \frac{\mathbf{4}}{\mathbf{1}} = \frac{8}{3}$$

The same applies to the product of an algebraic fraction and any nonfractional expression. For example,

$$\mathbf{6x}\left(\frac{2}{x^2-x}\right) = \frac{\mathbf{6x}}{\mathbf{1}} \cdot \frac{2}{x^2-x} = \frac{6\cancel{x}}{1} \cdot \frac{2}{\cancel{x}(x-1)} = \frac{12}{x-1}$$

We summarize the procedure for multiplying algebraic fractions as follows.

To multiply algebraic fractions:.

1 Factor each numerator and denominator.

2 If any factor appears in both a numerator and a denominator, divide out that factor.

3 Multiply the remaining factors of the numerator and the remaining factors of the denominator.

4 Reduce the product if necessary.

Quotients of Fractions

To divide one fraction by another, we multiply the first fraction by the reciprocal of the second fraction. For example,

$$\frac{m}{2} \div \frac{2p}{3} = \frac{m}{2} \cdot \frac{3}{2p} = \frac{3m}{4p}$$

We express this rule in symbols as follows.

Quotient of Fractions.

If $b, c, d \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

Thus, to divide two algebraic fractions, we take the reciprocal of the divisor and then follow the rules for multiplying algebraic fractions.

Example 8.41

Divide: $\dfrac{a-2}{6a^2} \div \dfrac{4-a^2}{4a^2-2a}$

Solution. First, we change the operation to multiplication by taking the reciprocal of the divisor.

$$\frac{a-2}{6a^2} \div \frac{4-a^2}{4a^2-2a} = \frac{a-2}{6a^2} \cdot \frac{4a^2-2a}{4-a^2}$$

Now we follow the rules for multiplication: First, we factor each numerator and denominator.

$$= \frac{a-2}{3 \cdot 2 \cdot a \cdot a} \cdot \frac{2a(2a-1)}{(2-a)(2+a)} \qquad \text{Divide out common factors.}$$

$$= \frac{-1\cancel{(2-a)}}{3 \cdot \cancel{2} \cdot \cancel{a} \cdot a} \cdot \frac{\cancel{2}a(2a-1)}{\cancel{(2-a)}(2+a)} \qquad \text{Note } a-2 = -1(2-a).$$

$$= \frac{-1(2a-1)}{3a(2+a)} = \frac{1-2a}{3a(2+a)}$$

Note 8.42 When working with algebraic fractions, we often leave the denominators in factored form. This makes it easier to add and subtract fractions, and to check whether they can be reducecd.

Checkpoint 8.43 QuickCheck 1. Fill in the blanks.

a. The first step in multiplying fractions is to (☐ add together the numerators ☐ factor each numerator and denominator ☐ find the reciprocal of each fraction ☐ remove any exponents) .

b. It is usually easier to cancel common factors (☐ after ☐ before) multiplying.

c. To divide two fractions, we multiply the first fraction by the (☐ GCF ☐ LCD ☐ opposite ☐ reciprocal) of the second fraction.

d. We can write a whole number as a fraction with denominator __.

Answer 1. factor each numerator and denominator

Answer 2. before

Answer 3. reciprocal

Answer 4. 1

Solution.

a. factor

b. before

c. reciprocal

d. 1

We summarize the procedure for dividing algebraic fractions as follows.

To divide algebraic fractions:.

1 Take the reciprocal of the second fraction and change the operation to multiplication.

2 Follow the rules for multiplication of fractions.

Checkpoint 8.44 Practice 2. Divide: $\dfrac{x^2-4y^2}{4y^2} \div \dfrac{x^2-3xy+2y^2}{3xy}$

$\dfrac{\rule{2cm}{0.4pt}}{\rule{2cm}{0.4pt}}$

Answer 1. $3x(x+2y)$

Answer 2. $4y(x-y)$

Solution. $\dfrac{3x(x+2y)}{4y(x-y)}$

Adding and Subtracting Like Fractions

Fractions with the same denominator are called **like fractions**. For example,

$$\frac{5}{8} \text{ and } \frac{9}{8}, \qquad \frac{4}{5x} \text{ and } \frac{3}{5x}, \qquad \frac{1}{a-2} \text{ and } \frac{a}{a-2}$$

are like fractions, while

$$\frac{2}{3} \text{ and } \frac{2}{5}, \qquad \frac{5}{x+1} \text{ and } \frac{2x}{x-1}$$

are unlike fractions. We can add or subtract like fractions for the same reason that we can add like terms. Just as

$$3x + 4x = 7x$$

we can think of the sum $\dfrac{3}{5} + \dfrac{4}{5}$ as

$$3\left(\frac{1}{5}\right) + 4\left(\frac{1}{5}\right) = 7\left(\frac{1}{5}\right)$$

or $\dfrac{7}{5}$. The denominators of the terms must be the same because they tell us what kind of quantity we are adding. We can add two quantities if they are the same kind.

When we add like fractions, we add their numerators and keep their denominators the same. For example,

$$\frac{10}{3x} + \frac{4}{3x} = \frac{10+4}{3x} = \frac{14}{3x}$$

The same holds true for all algebraic fractions.

Sum or Difference of Like Fractions.

If $c \neq 0$, then

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$
$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

Example 8.45

Add.

a $\dfrac{2x}{9z^2} + \dfrac{5x}{9z^2}$

b $\dfrac{2x-5}{x+2} + \dfrac{x+4}{x+2}$

Solution.

a Because these are like fractions, we add their numerators and keep the same denominator.

$$\frac{2x}{9z^2} + \frac{5x}{9z^2} = \frac{2x+5x}{9z^2} = \frac{7x}{9z^2}$$

b We combine the numerators over a single denominator.

$$\frac{2x-5}{x+2} + \frac{x+4}{x+2} = \frac{(2x-5)+(x+4)}{x+2} \quad \text{Add like terms in the numerator.}$$
$$= \frac{3x-1}{x+2}$$

We summarize our procedure for adding or subtracting like fractions as follows.

To add or subtract like fractions.

1 Add or subtract the numerators.

2 Keep the same denominator.

3 Reduce the sum or difference if necessary.

Checkpoint 8.46 Practice 3. Add: $\dfrac{2n}{n-3} + \dfrac{n+2}{n-3}$:

$\dfrac{\_\_\_\_\_\_}{\_\_\_\_\_\_}$

Answer 1. $3n + 2$

Answer 2. $n - 3$

Solution. $\dfrac{3n+2}{n-3}$

We must be careful when subtracting algebraic fractions: A subtraction sign in front of a fraction applies to the *entire* numerator.

Example 8.47

Subtract: $\dfrac{x-3}{x-1} - \dfrac{3x-5}{x-1}$

Solution. We combine the numerators over a single denominator. We use parentheses around $3x - 5$ to show that the subtraction applies to the entire numerator.

$$\frac{x-3}{x-1} - \frac{3x-5}{x-1} = \frac{(x-3)-(3x-5)}{x-1} \quad \text{Remove parentheses; distribute negative sign.}$$

$$= \frac{x-3-3x+5}{x-1} \quad \text{Combine like terms in the numerator.}$$

$$= \frac{-2x+2}{x-1}$$

We should always check to see whether the fraction can be reduced. Factor the numerator to find

$$\frac{-2x+2}{x-1} = \frac{-2\cancel{(x-1)}}{\cancel{x-1}} = -2$$

Caution 8.48 In the Example, p. 559 above, the subtraction symbol in the numerator $(x-3)-(3x-5)$ applies to *both* terms of $(3x-5)$. That is why

$$(x-3)-(3x-5) = x-3-3x+5$$

Checkpoint 8.49 QuickCheck 2. True or False.

a. Like fractions have the same denominator. (☐ True ☐ False)

b. To add like fractions, we add their numerators and add their denominators. (☐ True ☐ False)

c. A negative sign in front of a fraction changes the sign of both numerator and denominator. (☐ True ☐ False)

d. When we subtract fractions, the subtraction symbol applies only to the first term of the numerator. (☐ True ☐ False)

Answer 1. True

Answer 2. False

Answer 3. False

Answer 4. False

Solution.

a. True

b. False

c. False

d. False

Checkpoint 8.50 **Practice 4.** Subtract: $\dfrac{3}{x^2+2x+1} - \dfrac{2-x}{x^2+2x+1}$:

____

____

Answer 1. 1

Answer 2. $x+1$

Solution. $\dfrac{1}{x+1}$

Unlike Fractions

Let's review the steps for adding or subtracting unlike fractions. For example, to compute the sum

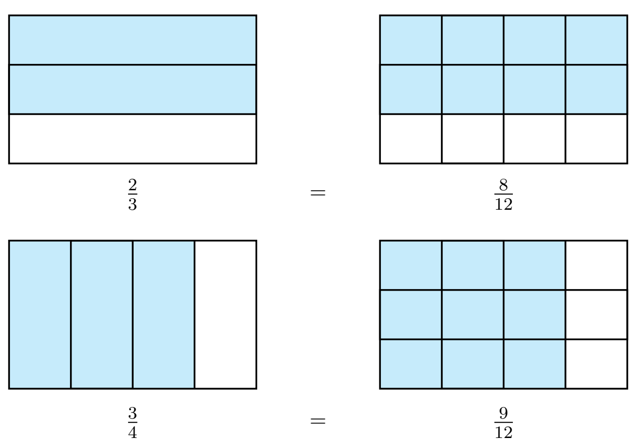
$$\frac{2}{3}+\frac{3}{4}$$

we first find the lowest common denominator, or LCD, for the two fractions: The smallest number that is a multiple of both 3 and 4 is 12.

Next, we use the fundamental principle of fractions to build each fraction to an equivalent one with denominator 12:

$$\frac{2}{3}=\frac{2\cdot\mathbf{4}}{3\cdot\mathbf{4}}=\frac{8}{12}$$
$$\frac{3}{4}=\frac{3\cdot\mathbf{3}}{4\cdot\mathbf{3}}=\frac{9}{12}$$

There is noting mysterious about the building process; we are breaking up each fraction into smaller pieces of the same size so that we can add them together.



Finally, the new fractions are like fractions, and we can add them by combining their numerators.

$$\frac{2}{3}+\frac{3}{4}=\frac{8}{12}+\frac{9}{12}=\frac{17}{12}$$

We add or subtract algebraic fractions with the same three steps.

To add or subtract algebraic fraction.

1 Find the lowest common denominator (LCD) for the fractions.

2 Build each fraction to an equivalent one with the same denominator.

3 Add or subtract the resulting like fractions: Add or subtract their numerators, and keep the same denominator.

4 Reduce the sum or difference if necessary.

Example 8.51

Subtract: $\dfrac{x+2}{6} - \dfrac{x-1}{15}$

Solution. Step 1: We find the LCD: The smallest multiple of both 6 and 15 is 30.

Step 2: We build each fraction to an equivalent one with denominator 30. The building factor for the first fraction is **5**, and for the second fraction the building factor is **2**.

$$\frac{x+2}{6} = \frac{(x+2)\cdot \mathbf{5}}{6\cdot \mathbf{5}} = \frac{5x+10}{30}$$
$$\frac{x-1}{15} = \frac{(x-1)\cdot \mathbf{2}}{15\cdot \mathbf{2}} = \frac{2x-2}{30}$$

Step 3: We subtract the resulting like fractions to obtain

$$\frac{x+2}{6} - \frac{x-1}{15} = \frac{5x+10}{30} - \frac{2x-2}{30}$$ Combine the numerators.

$$= \frac{(5x+10)-(2x-2)}{30}$$ Simplify the numerator.

$$= \frac{5x+10-2x+2}{30} = \frac{3x+12}{30}$$

Step 4: Finally, we reduce the fraction.

$$\frac{3x+12}{30} = \frac{\cancel{3}(x+4)}{\cancel{3}\cdot 10} = \frac{x+4}{10}$$

Checkpoint 8.52 **Practice 5.** Add: $\dfrac{5x}{6} + \dfrac{3-2x}{4}$:

$\dfrac{\_\_\_\_\_\_}{\_\_\_\_\_\_}$

Answer 1. $4x+9$

Answer 2. 12

Solution. $\dfrac{4x+9}{12}$

Checkpoint 8.53 **QuickCheck 3.** Fill in the blanks.

a. To add unlike fractions, we must first convert them into (☐ decimal ☐ factored ☐ like ☐ reduced) fractions.

b. Building a fraction is an application of the (☐ distributive law ☐ fundamental principle of fractions ☐ Pythagorean Theorem ☐ zero factor principle) .

c. The LCD is the smallest (☐ divisor ☐ factor ☐ multiple) . of each denominator.

d. The last step in adding fractions is to (☐ check the solution in the equation ☐ factor the numerator ☐ reduce the fraction) .

Answer 1. like

Answer 2. fundamental principle of fractions

Answer 3. multiple

Answer 4. reduce the fraction

Solution.

a. like

b. fundamental principle of fractions

c. multiple

d. reduce

Finding the Lowest Common Denominator

The **lowest common denominator** for two or more algebraic fractions is the simplest algebraic expression that is a multiple of each denominator. If neither denominator can be factored, then their LCD is just the product of the two expressions.

Example 8.54

Add: $\dfrac{6}{x}+\dfrac{x}{x-2}$

Solution. Step 1: The LCD for these fractions is just the product of their denominators, $x(x-2)$

Step 2: We build each fraction to an equivalent one with denominator $x(x-2)$. The building factor for $\dfrac{6}{x}$ is $\boldsymbol{(x-2)}$ and the building factor for $\dfrac{x}{x-2}$ is $\boldsymbol{x}$. We multiply numerator and denominator of each fraction by its building factor.

$$\frac{6}{x}=\frac{6\boldsymbol{(x-2)}}{x\boldsymbol{(x-2)}}=\frac{6x-12}{x(x-2)}$$
$$\frac{x}{x-2}=\frac{x\boldsymbol{(x)}}{(x-2)\boldsymbol{(x)}}=\frac{x^2}{x(x-2)}$$

Step 3: We combine the resulting like fractions.

$$\frac{6}{x}+\frac{x}{x-2}=\frac{6x-12}{x(x-2)}+\frac{x^2}{x(x-2)} \qquad \text{Combine the numerators.}$$
$$=\frac{x^2+6x-12}{x-2}$$

Step 4: The numerator of this fraction cannot be factored, so the sum cannot be reduced.

Checkpoint 8.55 **Practice 6.** Write as a single fraction in reduced form:

$\dfrac{2}{x+2} - \dfrac{3}{x-2}$

$$\frac{\rule{2em}{0.4pt}}{\rule{2em}{0.4pt}}$$

Answer 1. $-x-10$

Answer 2. x^2-4

Solution. $\dfrac{-x-10}{x^2-4}$

If the denominators contain any common factors, the LCD is not simply their product. For example, the LCD for

$$\frac{5}{12} + \frac{7}{18}$$

is not $12(18) = 216$. It is true that 216 is a multiple of both 12 and 18, but it is not the smallest one! We can find a smaller common denominator by factoring each denominator.

$12 = 2 \cdot 2 \cdot 3$ This denominator has the most factors of 2.

$18 = 2 \cdot 3 \cdot 3$ This denominator has the most factors of 3.

To find a number that both 12 and 18 divide into evenly, we need only enough factors to cover each of them. In this case two 2'sand two 3'sare sufficient, so the LCD is

$$\text{LCD} = 2 \cdot 2 \cdot 3 \cdot 3 = 36$$

You can check that both 12 and 18 divide evenly into 36.

In general, we can find the LCD in the following way.

To Find the LCD.

1 Factor each denominator completely.

2 Include each different factor in the LCD as many times as it occurs in any one of the given denominators.

Example 8.56

Find the LCD for the fractions $\dfrac{2x}{x^2-1}$ and $\dfrac{x+3}{x^2+x}$

Solution. We factor the denominators of each of the given fractions.

$$x^2 - 1 = (x-1)(x+1) \quad \text{and} \quad x^2 + x = x(x+1)$$

The factor $(x-1)$ occurs once in the first denominator, the factor x occurs once in the second denominator, and the factor $(x+1)$ occurs once in each denominator. Therefore we include in our LCD one copy of each of these factors. The LCD is $x(x-1)(x+1)$.

Caution 8.57 In the previous Example, we do not include two factors of $(x+1)$ in the LCD. We need only one factor of $(x+1)$, because $(x+1)$ occurs only once in each denominator. You can check that each original denominator divides evenly into our LCD, $x(x-1)(x+1)$.

Checkpoint 8.58 **Practice 7.** Find the LCD for the fractions $\dfrac{1}{2x^3(x-1)^2}$

and $\dfrac{3}{4x^2-4x}$
LCD= ____________

Answer. $4x^3(x-1)^2$

Solution. $4x^3(x-1)^2$

Adding and Subtracting Unlike Fractions

After finding the LCD, we build each fraction to an equivalent one with the LCD as its denominator. The new fractions will be like fractions, so we can combine their numerators.

Building a fraction is an application of the fundamental principle of fractions,

$$\frac{a}{b}=\frac{a\cdot c}{b\cdot c}, \quad \text{if} \quad c\neq 0$$

It is the opposite of reducing a fraction, because we multiply, rather than divide, the numerator and denominator by an appropriate factor. To find the **building factor**, we compare the factors of the original denominator with those of the desired common denominator.

Example 8.59

Add: $\dfrac{x-4}{x^2-2x}+\dfrac{4}{x^2-4}$

Solution. Step 1: Find the LCD. We factor each denominator completely.

$$x^2-2x=x(x-2)$$
$$x^2-4=(x-2)(x+2)$$

The LCD is $x(x-2)(x+2)$.

Step 2: We build each fraction to an equivalent one with the LCD as its denominator. The building factor for $\dfrac{x-4}{x(x-2)}$ is $\boldsymbol{(x+2)}$. We multiply the numerator and denominator of the first fraction by $\boldsymbol{(x+2)}$:

$$\frac{x-4}{x(x-2)}=\frac{(x-4)\boldsymbol{(x+2)}}{x(x-2)\boldsymbol{(x+2)}}=\frac{x^2-2x+8}{x(x-2)(x+2)}$$

The building factor for $\dfrac{4}{(x-2)(x+2)}$ is $\boldsymbol{x}$.

$$\frac{4}{(x-2)(x+2)}=\frac{4\boldsymbol{x}}{(x-2)(x+2)\boldsymbol{x}}$$

Step 3: The fractions are now like fractions, so we add them by combining their numerators.

$$\frac{x-4}{x^2-2x}+\frac{4}{x^2-4}=\frac{x^2-2x+8}{x(x-2)(x+2)}+\frac{4x}{x(x-2)(x+2)}$$
$$=\frac{x^2+2x+8}{x(x-2)(x+2)}$$

Step 4: Finally, we reduce the fraction.

$$\frac{x^2+2x+8}{x(x-2)(x+2)}=\frac{(x+4)\cancel{(x-2)}}{x\cancel{(x-2)}(x+2)}=\frac{x+4}{x(x+2)}$$

Caution 8.60 Do not reduce the built-up fractions in Step 3 -- you will just get back to the original problem. When adding fractions, we have to make the fractions "harder" before we can combine them. Don't reduce until the last step of the problem.

Checkpoint 8.61 QuickCheck 4. True or False.

a. We can always find the LCD by multiplying together the denominators of the fractions. (☐ True ☐ False)

b. If each denominator includes one factor of $x-3$, we include two factors of $x-3$ in the LCD. (☐ True ☐ False)

c. We find the building factor for each fraction by comparing its denominator to the LCD. (☐ True ☐ False)

d. When adding fractions, it is best to reduce the fractions at each step. (☐ True ☐ False)

Answer 1. False

Answer 2. False

Answer 3. True

Answer 4. False

Solution.

a. False

b. False

c. True

d. False

Checkpoint 8.62 Practice 8. Subtract: $\dfrac{2}{x^2-x-2} - \dfrac{2}{x^2+2x+1}$

$$\frac{\rule{2cm}{0.4pt}}{\rule{2cm}{0.4pt}}$$

Answer 1. 6

Answer 2. $(x+1)^2(x-2)$

Solution. $\dfrac{6}{(x+1)^2(x-2)}$

Applications

Recall that the formulas for rational functions are algebraic fractions. It is often useful to simplify the formula for a function before using it.

Example 8.63

When estimating their travel time, pilots must take into account the prevailing winds. A tail wind adds to the plane's ground speed, while a head wind decreases the ground speed. Skyhigh Airlines is setting up a shuttle service from Dallas to Phoenix, a distance of 800 miles.

a Express the time needed for a one-way trip, without wind, as a function of the speed of the plane.

b Suppose there is a prevailing wind of 30 miles per hour blowing from the west. Write expressions for the flying time from Dallas to Phoenix and from Phoenix to Dallas.

c Write an expression for the round trip flying time with a 30-mile-per-hour wind from the west, as a function of the plane's speed. Simplify your expression.

Solution.

a Recall that time $= \dfrac{\text{distance}}{\text{rate}}$. If we let r represent the speed of the plane in still air, then the time required for a one-way trip is

$$f(r) = \frac{800}{r}$$

b On the trip from Dallas to Phoenix the plane encounters a head wind of 30 miles per hour, so its actual ground speed is $r - 30$. On the return trip the plan enjoys a tail wind of 30 miles per hour, so its actual ground speed is $r + 30$. Therefore, the flying times are

$$\begin{aligned} &\text{Dallas to Phoenix:} && \frac{800}{r-30} \\ &\text{Phoenix to Dallas:} && \frac{800}{r+30} \end{aligned}$$

c The round-trip flying time from Dallas to Phoenix and back is

$$F(r) = \frac{800}{r-30} + \frac{800}{r+30}$$

The LCD for these fractions is $(r-30)(r+30)$. Thus,

$$\begin{aligned} \frac{800}{r-30} + \frac{800}{r+30} &= \frac{800\boldsymbol{(r+30)}}{(r-30)\boldsymbol{(r+30)}} + \frac{800\boldsymbol{(r-30)}}{(r+30)\boldsymbol{(r-30)}} \\ &= \frac{(800r + 24,000) + (800r - 24,000)}{(r+30)(r-30)} \\ &= \frac{1600r}{r^2 - 900} \end{aligned}$$

Checkpoint 8.64 **Practice 9.** A rowing team can maintain a speed of 15 miles per hour in still water. The team's daily training session includes a 5-mile run up the Red Cedar River and the return downstream.

a. Express the team's time on the upstream leg as a function of the speed of the current.

b. Write a function for the team's time on the downstream leg.

c. Write and simplify an expression for the total time for the training run as a function of the current's speed.

Answer 1. $\frac{5}{15-c}$

Answer 2. $\frac{5}{15+c}$

Answer 3. $\frac{150}{225-c^2}$

Solution.

a. $\dfrac{5}{15-c}$

b. $\dfrac{5}{15+c}$

c. $\dfrac{150}{225-c^2}$

Problem Set 8.3

Warm Up

In Problems 1–6, we review the operations on arithmetic fractions.

1. $\dfrac{5}{8}\cdot\dfrac{3}{4}=\dfrac{15}{32}$

Multiply the numerators together; multiply the denominators together.

a $\dfrac{2}{5}\cdot\dfrac{2}{3}$ b $\dfrac{2}{w}\cdot\dfrac{z}{3}$

2. $\dfrac{3}{4}\cdot\dfrac{5}{6}=\dfrac{\cancel{3}}{4}\cdot\dfrac{5}{2\cdot\cancel{3}}=\dfrac{5}{8}$

Divide out common factors first, then multiply.

a $\dfrac{5}{4}\cdot\dfrac{8}{9}$ b $\dfrac{5}{2a}\cdot\dfrac{4a}{9}$

3. $\dfrac{3}{4}\cdot 6=\dfrac{3}{4}\cdot\dfrac{6}{1}=\dfrac{3}{2\cdot\cancel{2}}\cdot\dfrac{\cancel{2}\cdot 3}{1}=\dfrac{9}{2}$

Write 6 as $\dfrac{6}{1}$

a $\dfrac{7}{3}\cdot 12$ b $\dfrac{7}{x}\cdot 4x$

4. $\dfrac{12}{5}\div\dfrac{8}{5}=\dfrac{12}{5}\cdot\dfrac{5}{8}=\dfrac{\cancel{4}\cdot 3}{\cancel{5}}\cdot\dfrac{5\cancel{5}}{\cancel{4}\cdot 2}=\dfrac{3}{2}$

Take the reciprocal of the second fraction; then change to multiplica

a $\dfrac{8}{3}\div\dfrac{2}{9}$ b $\dfrac{4a}{3b}\div\dfrac{2a}{3}$

5. $\dfrac{3}{5}\div 6=\dfrac{3}{5}\cdot\dfrac{1}{6}=\dfrac{\cancel{3}}{5}\cdot\dfrac{1}{\cancel{3}\cdot 2}=\dfrac{1}{15}$

Take the reciprocal of the second fraction; then change to multiplication.

a $\dfrac{2}{3}\div 4$ b $\dfrac{2}{3y}\div(4y)$

6. $\dfrac{5}{4}+\dfrac{3}{4}=\dfrac{5+3}{4}=\dfrac{8}{4}=2$

Combine the numerators; keep the same denominator.

a $\dfrac{13}{6}-\dfrac{5}{6}$ b $\dfrac{13k}{n}-\dfrac{5k}{n}$

Skills Practice

7. Multiply.

a $\dfrac{-24ab}{5b}\cdot\dfrac{15ab}{14}$ b $\dfrac{2b}{3}\cdot\dfrac{4}{b+1}$ c $\dfrac{-v}{v+1}\cdot\dfrac{v}{v-1}$

8. Write each product as a fraction.

a $\dfrac{3}{4}(a-b)$ b $4a^2b\dfrac{2a+b}{6ab}$ c $15x^2y\dfrac{3}{35xy^2}$

For Problems 9–14, multiply.

9. $\dfrac{3y}{4xy-6y^2}\cdot\dfrac{2x-3y}{12x}$

10. $\dfrac{3x-9}{5x-15}\cdot\dfrac{10x-5}{8x-4}$

11. $\dfrac{4a^2-1}{a^2-16}\cdot\dfrac{a^2-4a}{2a+1}$

12. $\dfrac{2x^2-x-6}{3x^2+4x+1}\cdot\dfrac{3x^2+7x+2}{2x^2+7x+6}$

13. $\dfrac{3x^4-48}{x^4-4x^2-32}\cdot\dfrac{4x^4-8x^3+4x^2}{2x^4+16x}$

14. $\dfrac{x^4-3x^3}{x^4+6x^2-27}\cdot\dfrac{x^4-81}{3x^4-81x}$

For Problems 15–24, divide.

15. $\dfrac{12a^4}{21b^4} \div \dfrac{24ab^2}{27b^3}$

16. $6x^2y \div \dfrac{3x}{-2y^3}$

17. $\dfrac{a^2-ab}{ab} \div \dfrac{2a-2b}{3ab}$

18. $\dfrac{x^2+3x}{2y} \div (3x)$

19. $\dfrac{3xy+x}{y^2-y} \div \dfrac{3y+1}{xy}$

20. $\dfrac{6a^2-12a}{3a+9} \div \dfrac{8a^2-4a^3}{15+5a}$

21. $\dfrac{2z^2+3z-2}{2z^2-3z-2} \div \dfrac{2z^3-z^2}{z^2-4}$

22. $\dfrac{a^2-a-6}{a^2+2a-15} \div \dfrac{a^2-4}{a^2+6a+5}$

23. $\dfrac{8x^3-y^3}{x+y} \div \dfrac{2x-y}{x^2-y^2}$

24. $(x^2+5x-4) \div \dfrac{x^2-1}{x^2}$

For Problems 25–28, add or subtract.

25. $\dfrac{x-2y}{3x} + \dfrac{x+3y}{3x}$

26. $\dfrac{m^2+1}{m-1} - \dfrac{2m}{m-1}$

27. $\dfrac{z^2-2}{z+2} - \dfrac{z+4}{2+z}$

28. $\dfrac{b+1}{b^2-2b+1} - \dfrac{5-3b}{b^2-2b+1}$

For Problems 29–36, find the lowest common denominator for the fractions, then add or subtract.

29. $\dfrac{5}{2x} + \dfrac{3}{4x^2}$

30. $\dfrac{2z-3}{8z} + \dfrac{z-2}{6z}$

31. $\dfrac{3}{2a-b} + \dfrac{1}{8a-4b}$

32. $\dfrac{3}{x} - \dfrac{2}{x+1}$

33. $h - \dfrac{3}{h+2}$

34. $\dfrac{3}{n+3} - \dfrac{4n}{n-3}$

35. $\dfrac{2}{x} + \dfrac{x}{x-2} - 2$

36. $\dfrac{1}{x} + \dfrac{x}{y} + \dfrac{x}{z}$

37. $\dfrac{5}{2p-4} - \dfrac{2}{6-3p}$

38. $\dfrac{-2}{m^2-3m} + \dfrac{1}{m^2-9}$

39. $\dfrac{4}{k^2-3k} + \dfrac{1}{k^2+k}$

40. $\dfrac{x-1}{x^2+3x} + \dfrac{x}{x^2+6x+9}$

41. $\dfrac{y}{2y-1} - \dfrac{2y}{y+1}$

42. $\dfrac{y-1}{y+1} - \dfrac{y-2}{2y-3}$

For Problems 43–46, find the lowest common denominator.

43. $\dfrac{5}{6(x+y)^2}, \ \dfrac{4}{3xy^2}$

44. $\dfrac{2a}{a^2+5a+4}, \ \dfrac{2}{(a+1)^2}$

45. $\dfrac{x+2}{x^2-x}, \ \dfrac{x+1}{(x-1)^3}$

46. $\dfrac{n+2}{n^3-n} - \dfrac{n-2}{n^3-n^2}$

Applications

For Problems 47–54, multiply.

47. $\dfrac{4V}{D} \cdot \dfrac{LR}{DV}$

48. $\dfrac{1}{2}MR^2 \cdot \dfrac{a}{R}$

49. $\dfrac{2L}{c}\left(1+\dfrac{V^2}{c^2}\right)$

50. $\dfrac{4\pi}{c^2}\left(\dfrac{c}{4\pi}H + cM\right)$

51. $\dfrac{q}{8\pi}\left(\dfrac{3}{R}-\dfrac{a^2}{R^3}\right)$

52. $\dfrac{a^2}{d}\left(1-\dfrac{at}{2d}\right)$

53. $\dfrac{-2}{t^2}\left(4t^3-\dfrac{t^2}{8}+\dfrac{3t}{2}\right)$

54. $\dfrac{4}{3}v\left(\dfrac{2}{3}v-\dfrac{6}{v^2}-\dfrac{3}{v}\right)$

Write an algebraic expression for each phrase Problems 55–57, and simplify.

55.

a One-half of x

b x divided by one-half

c One-half divided by x

56.

a Two-thirds of y

b y divided by two-thirds

c Two-thirds divided by y

57.

a The reciprocal of $a+b$

b Three-fourths of the reciprocal of $a+b$

c The reciprocal of $a+b$ divided by three-fourths

58. Simplify.

a $\left(\dfrac{1}{c}\cdot\dfrac{1}{5}\right)\div\dfrac{2}{5}$

b $\dfrac{1}{c}\cdot\left(\dfrac{1}{5}\div\dfrac{2}{5}\right)$

c $\dfrac{1}{c}\div\left(\dfrac{1}{5}\div\dfrac{2}{5}\right)$

d $\left(\dfrac{1}{c}\div\dfrac{1}{5}\right)\div\dfrac{2}{5}$

For Problems 59–64, add or subtract.

59. $\dfrac{-H}{RT}+\dfrac{S}{R}$

60. $\dfrac{q}{4\pi r}+\dfrac{qa}{2\pi r^2}$

61. $\dfrac{1}{LC}-\left(\dfrac{R}{2L}\right)^2$

62. $\dfrac{L}{c-V}+\dfrac{L}{c+v}$

63. $\dfrac{2r^2}{a^2}+\dfrac{2r}{a}+1$

64. $\dfrac{q}{r-a}-\dfrac{2q}{r}+\dfrac{q}{r+a}$

For Problems 65–68, write algebraic fractions in simplest.

65. The dimensions of a rectangular rug are $\dfrac{12}{x}$ feet and $\dfrac{12}{x-2}$ feet.

a Write and sinplify an expression for the area of the rug.

b Write and sinplify an expression for the perimeter of the rug.

66. Colonial Airline has a commuter flight between Richmond and Washington, a distance of 100 miles. The plane flies at x miles per hour in still air. Today there is a steady wind from the north at 10 miles per hour.

a How long will the flight from Richmond to Washington take?

b How long will the flight from Washington to Richmond take?

c How long will a round trip take?

d Evaluate your fractions in parts (a)-(c) for $x = 150$

67. Two pilots for the Flying Express parcel service receive packages simultaneously. Orville leaves Boston for Chicago at the same time Wilbur leaves Chicago for Boston. Each pilot selects an air speed of 400 miles per hour for the 900-mile trip. The prevailing winds blow from east to west.

a Express Orville's flying time as a function of the wind speed.

b Write a function for Wilbur's flying time.

c Who reaches his destination first? By how much time (in terms of wind speed)?

68. Francine's cocker spaniel eats a large bag of dog food in d days, and Delbert's sheep dog takes 5 fewer days to eat the same size bag.

a What fraction of a bag of dog food does Francine's cocker spaniel eat in one day?

b What fraction of a bag of dog food does Delbert's sheep dog eat in one day?

c If Delbert and Francine get married, what fraction of a bag of dog food will their dogs eat in one day?

d If $d = 25$, how soon will Delbert and Francine have to buy more dog food?

More Operations on Fractions

Complex Fractions

A fraction that contains one or more fractions in either its numerator or its denominator or both is called a **complex fraction**. For example,

$$\frac{\frac{2}{3}}{\frac{5}{6}} \qquad \text{and} \qquad \frac{x+\frac{3}{4}}{x-\frac{1}{2}}$$

are complex fractions. Like simple fractions, complex fractions represent quotients. For the examples above,

$$\frac{\frac{2}{3}}{\frac{5}{6}} = \frac{2}{3} \div \frac{5}{6}$$

and

$$\frac{x+\frac{3}{4}}{x-\frac{1}{2}} = \left(x+\frac{3}{4}\right) \div \left(x-\frac{1}{2}\right)$$

We can always simplify a complex fraction into a standard algebraic fraction. One way to do this is to treat the fraction as a division.

Example 8.65

Simplify $\dfrac{\frac{2}{3}}{\frac{5}{6}}$

Solution. First, Write the complex fraction as a quotient. Invert the divisor and multiply.

$$\frac{\frac{2}{3}}{\frac{5}{6}} = \frac{2}{3} \div \frac{5}{6} = \frac{2}{3} \cdot \frac{6}{5} = \frac{12}{15} = \frac{4}{5}$$

Checkpoint 8.66 Practice 1. Simplify $\dfrac{\frac{2x}{y^3}}{\frac{x}{3y}}$

____

____

Answer 1. 6

Answer 2. y^2

Solution. $\dfrac{6}{y^2}$

If the numerator or denominator of the complex fraction contains more than one term, there is an easier way to simplify the fraction that takes advantage of the fundamental principle of fractions.

Example 8.67

Simplify $\dfrac{\frac{1}{2x} - \frac{1}{x^2}}{\frac{1}{4} - \frac{1}{2x}}$

Solution. This complex fraction contains the simple fractions $\frac{1}{2x}$, $\frac{1}{x^2}$, $\frac{1}{4}$ and $\frac{1}{2x}$. The LCD of these fractions is $4x^2$. We multiply the numerator and denominator of the complex fraction by $4x^2$. Doing this will not change the value of the fraction, but will clear all the "smaller" fractions inside.

$$\frac{\frac{1}{2x} - \frac{1}{x^2}}{\frac{1}{4} - \frac{1}{2x}} = \frac{\mathbf{4x^2}(\frac{1}{2x} - \frac{1}{x^2})}{\mathbf{4x^2}(\frac{1}{4} - \frac{1}{2x})}$$

Then we apply the distributive law, so that we multiply each term of the numerator and each term of the denominator by $\mathbf{4x^2}$.

$$\frac{\mathbf{4x^2}(\frac{1}{2x} - \frac{1}{x^2})}{\mathbf{4x^2}(\frac{1}{4} - \frac{1}{2x})} = \frac{\frac{\mathbf{4x^2}}{\mathbf{1}} \cdot \frac{1}{2x} - \frac{\mathbf{4x^2}}{\mathbf{1}} \cdot \frac{1}{x^2}}{\frac{\mathbf{4x^2}}{\mathbf{1}} \cdot \frac{1}{4} - \frac{\mathbf{4x^2}}{\mathbf{1}} \cdot \frac{1}{2x}} = \frac{2x - 4}{x^2 - 2x}$$

Finally, we reduce the result to obtain

$$\frac{2x-4}{x^2-2x} = \frac{2\cancel{(x-2)}}{x\cancel{(x-2)}} = \frac{2}{x}$$

We summarize the procedure for multiplying algebraic fractions as follows.

To simplify a complex fraction.

1. Find the LCD of all the fraction contained in the complex fraction.
2. Multiply the numerator and the denominator of the complex fraction by the LCD.
3. Reduce the resulting simple fraction, if possible.

Checkpoint 8.68 **Practice 2.** Simplify $\dfrac{1+\dfrac{b}{a}}{1-\dfrac{bc}{ad}}$

$$\frac{\_\_\_\_}{\_\_\_\_}$$

Answer 1. $ad+bd$

Answer 2. $ad-bc$

Solution. $\dfrac{ad+bd}{ad-bc}$

Negative Exponents

Algebraic fractions are sometimes written using negative exponents.

Example 8.69

Write each expression as a single algebraic fraction.

a $x^{-1}-y^{-1}$

b $(x^{-2}+y^{-2})^{-1}$

Solution.

a We write each power as a fraction, then simplify.

$$\begin{aligned} x^{-1}-y^{-1} &= \frac{1}{x}-\frac{1}{y} && \text{The LCD is } xy. \\ &= \frac{1}{x}\cdot\frac{y}{y}-\frac{1}{y}\cdot\frac{x}{x} && \text{Build to the LCD.} \\ &= \frac{y-x}{xy} \end{aligned}$$

b We simplify the expression inside parentheses first.

$$\begin{aligned} \left(x^{-2}+y^{-2}\right)^{-1} &= \left(\frac{1}{x^2}+\frac{1}{y^2}\right)^{-1} && \text{Add fractions.} \\ &= \left(\frac{y^2+x^2}{x^2y^2}\right)^{-1} = \frac{x^2y^2}{x^2+y^2} \end{aligned}$$

For the last step, remember that a negative exponent indicates the reciprocal of its base.

Caution 8.70 When working with fractions and exponents,it is important to avoid some tempting but *incorrect* algebraic operations.

1 In Example 3a, note that

$$\frac{1}{x} - \frac{1}{y} \neq \frac{1}{x-y}$$

For example, you can check that, for $x = 2$ and $y = 3$,

$$\frac{1}{2} - \frac{1}{3} \neq \frac{1}{2-3} = -1$$

2 In Example 3b, note that

$$(x^{-2} + y^{-2})^{-1} \neq x^2 + y^2$$

The fourth law of exponents does not apply to sums and differences; that is,

$$(a+b)^n \neq a^n + b^n$$

Checkpoint 8.71 Practice 3. Simplify $(1 + x^{-1})^{-1}$

$\underline{\qquad}$

$\underline{\qquad}$

Answer 1. x

Answer 2. $x + 1$

Solution. $\dfrac{x}{x+1}$

Checkpoint 8.72 QuickCheck 1. True or False.

a. $\dfrac{1+\frac{1}{x}}{1-\frac{1}{x^2}} = \left(1+\dfrac{1}{x}\right) \div \left(1-\dfrac{1}{x^2}\right)$ (☐ True ☐ False)

b. $\dfrac{1+\frac{1}{x}}{1-\frac{1}{x^2}} = \dfrac{x+1}{x^2+1}$ (☐ True ☐ False)

c. $\left(a^{-1}+b^{-1}\right)^{-2} = a^{-2} + b^{-2}$ (☐ True ☐ False)

d. $\dfrac{\frac{1}{a}-\frac{1}{b}}{\frac{1}{a}+\frac{1}{b}} = \dfrac{\frac{1}{a-b}}{\frac{1}{a+b}}$ (☐ True ☐ False)

Answer 1. True

Answer 2. False

Answer 3. False

Answer 4. False

Solution.

a. True

b. False

c. False

d. False

Applications

Sometimes mathematics can help us solve problems when our intuition fails us or leads us astray.

Example 8.73

On a weekday afternoon, when traffic is always horrible, Kathy left her home north of Los Angeles and drove south 120 miles along the San Diego Freeway to San Juan Capistrano. Her average speed was 40 miles per hour. She returned home on Saturday, at an average speed of 60 miles per hour. What was her average speed for the round trip?

Solution. If you said that the average speed is 50 miles per hour, you would be wrong! Let's do some calculations. Kathy's average speed for the round trip is given by

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}}$$

The total distance she drove is 240 miles, but to find the total time we must compute the time Kathy drove on each part of the trip. We use the formula $d = rt$ and solve for t.

	d	r	t
Driving south	120	40	3
Driving north	120	60	2

The total time for the round trip was $3 + 2 = 5$ hours, so Kathy's average speed was

$$\frac{240}{5} = 48 \text{ miles per hour}$$

Why does the average speed turn out to be less than 50 miles per hour? Because Kathy spent more time driving at 40 miles per hour (3 hours) than she did driving at 60 miles per hour (2 hours).

By generalizing the problem above we can find an algebraic formula for the average speed on a two-part trip. Suppose the distance traveled on the two parts of the trip are d_1 and d_2, and the corresponding speeds on the two parts are r_1 and r_2. We fill in the table to find the time required for each part.

	Distance	Rate	Time
First part	d_1	r_1	$\frac{d_1}{r_1}$
Second part	d_2	r_2	$\frac{d_2}{r_2}$

The total distance traveled on the trip is d_1+d_2, and the total time required is

$$\frac{d_1}{r_1} + \frac{d_2}{r_2}$$

Thus, the average speed for the entire trip is

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}} = \frac{d_1 + d_2}{\frac{d_1}{r_1} + \frac{d_2}{r_2}}$$

Checkpoint 8.74 Practice 4. Bruce drove for 24 miles in rush-hour traffic at an average speed of 20 miles per hour. Then he drove 126 miles on the highway at an average speed of 70 miles an hour. Use the formula above to find his average speed for the entire trip.

__ (☐ miles ☐ hours ☐ mph ☐ hours/mile)

Answer 1. 50

Answer 2. mph

Solution. 50 mph

Polynomial Division

Consider three improper fractions: $\frac{8}{6}$, $\frac{8}{4}$, and $\frac{8}{3}$. Can these fractions be simplified?

- We can reduce the first fraction: $\frac{8}{6} = \frac{4}{3}$
- The second fraction reduces to a whole number: $\frac{8}{4} = \frac{2}{1} = 2$
- The third fraction does not reduce, but by dividing the denominator into the numerator, we can write it as a whole number plus a proper fraction: $\frac{8}{3} = 2\frac{2}{3}$

An algebraic fraction is "improper" if the degree of the numerator is greater than the degree of the denominator. If it cannot be reduced, we can simplify the expression by treating it as a division of polynomials. The quotient will be the sum of a polynomial and a simpler algebraic fraction.

If the divisor is a monomial, we can simply divide the monomial into each term of the numerator.

Example 8.75

Divide $\dfrac{9x^3 - 6x^2 + 4}{3x}$

Solution. We divide $3x$ into each term of the numerator.

$$\begin{aligned}\frac{9x^3 - 6x^2 + 4}{3x} &= \frac{9x^3}{3x} - \frac{6x^2}{3x} + \frac{4}{3x}\\ &= 3x^2 - 2x + \frac{4}{3x}\end{aligned}$$

Note that the quotient is the sum of a polynomial, $3x^2 - 2x$, and an algebraic fraction, $\frac{4}{3x}$.

Checkpoint 8.76 **Practice 5.** Divide: $\dfrac{6a^3 + 2a^2 - a}{2a^2} =$ ______ + ______

Answer 1. $3a + 1$

Answer 2. $\frac{-1}{2a}$

Solution. $3a + 1 - \dfrac{1}{2a}$

If the denominator is not a monomial, we use a method similar to the long division algorithm used in arithmetic.

Example 8.77

Divide $\dfrac{2x^2 + x - 7}{x + 3}$

Solution. We first write the quotient as a division problem:

$$x+3\,\overline{)\,2x^2+x-7}$$

and divide $2x^2$ (the first term of the numerator) by x (the first term of the denominator) to obtain $2x$. (It may be helpful to write down the division: $\dfrac{2x^2}{2x} = x$.) We write $\mathbf{2x}$ above the quotient bar as the first term of the quotient, as shown below.

Next, we multiply $x+3$ by $2x$ to obtain $2x^2+6x$, and subtract this product from $2x^2+x-7$:$2x^2+x-7$.

$$\begin{array}{r l}
 \mathbf{2x} & \\
x+3\,\overline{)\,2x^2+x-7} & \\
\underline{-(2x^2+6x)} & \text{Multiply } x{+}3 \text{ by } 2x\text{, and} \\
-5x-7 & \text{subtract the result.}
\end{array}$$

Repeating the process, we divide $-5x$ by x to obtain -5. We write $\mathbf{-5}$ as the second term of the quotient. Then we multiply $x+3$ by -5 to obtain $-5x-15$, and subtract:

$$\begin{array}{r l}
2x \;\mathbf{-\;5} & \\
x+3\,\overline{)\,2x^2+x-7} & \\
\underline{-(2x^2+6x)} & \\
-5x-7 & \\
\underline{-(-5x-15)} & \text{Multiply } x{+}3 \text{ by } -5\text{, and} \\
8 & \text{subtract the result.}
\end{array}$$

Because the degree of the remainder, 8, is less than the degree of $x+3$, the division is finished. The quotient is $2x-5$, with a remainder of 8. We write the remainder as a fraction to obtain

$$\frac{2x^2+x-7}{x+3} = 2x-5+\frac{8}{x+3}$$

When using polynomial division, it helps to write the polynomials in descending powers of the variable. If the numerator is missing any terms, we can insert terms with zero coefficients so that like powers will be aligned. For example, to perform the division

$$\frac{3x-1+4x^3}{2x-1}$$

we first write the numerator in descending powers as $4x^3+3x-1$. We insert $0x^2$ between $4x^3$ and $3x$ and set up the quotient as

$$2x-1\,\overline{)\,4x^3+0x^2+3x-1}$$

We then proceed as in Example 8.77, p. 575. You can check that the quotient is

$$2x^2+x+2+\frac{1}{2x-1}$$

Checkpoint 8.78 Practice 6. Divide: $\dfrac{4+8y^2-3y^3}{3y+1} =$ _______ + ______

Answer 1. $-y^2 + 3y - 1$

Answer 2. $\frac{5}{3y+1}$

Solution. $-y^2 + 3y - 1 + \dfrac{5}{3y+1}$

Checkpoint 8.79 QuickCheck 2. True or False.

a. The shortcut for simplifying complex fractions applies the fundamental principle of fractions. (☐ True ☐ False)

b. An improper algebraic fraction is one in which the denominator has higher degree than the numerator. (☐ True ☐ False)

c. If the remainder is zero in polynomial division $\dfrac{p(x)}{q(x)}$, the numerator $p(x)$ can be factored. (☐ True ☐ False)

d. If the divisor in polynomial division is a binomial, we use a process like long division. (☐ True ☐ False)

Answer 1. True

Answer 2. False

Answer 3. True

Answer 4. True

Solution.

a. True

b. False

c. True

d. True

Problem Set 8.4

Warm Up

1. Add $\dfrac{x}{x-1} + \dfrac{x+2}{x}$

2. Subtract $\dfrac{x}{x-1} - \dfrac{x+2}{x}$

3. Multiply $\dfrac{x}{x-1} \cdot \dfrac{x+2}{x}$

4. Divide $\dfrac{x}{x-1} \div \dfrac{x+2}{x}$

Skills Practice

For Problems 5–16, simplify the complex fraction.

5. $\dfrac{\frac{3x}{y}}{\frac{x}{2y^2}}$

6. $\dfrac{\frac{4a}{5b^2}}{\frac{8a^3}{15b}}$

7. $\dfrac{1-\frac{1}{6}}{2+\frac{2}{3}}$

8. $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{3}-\frac{1}{6}}$

9. $\dfrac{\frac{2}{a}+\frac{3}{2a}}{5+\frac{1}{a}}$

10. $\dfrac{1+\frac{2}{a}}{1-\frac{4}{a^2}}$

11. $\dfrac{4-\dfrac{1}{x^2}}{2-\dfrac{1}{x}}$

12. $\dfrac{1}{1-\dfrac{1}{q}}$

13. $\dfrac{n}{\dfrac{p}{q}+1}$

14. $\dfrac{h+\dfrac{h}{m}}{1+\dfrac{1}{m}}$

15. $\dfrac{\dfrac{u}{x}-\dfrac{v}{x}}{v}$

16. $\dfrac{\dfrac{x}{at}-V}{1-V\dfrac{x}{t}}$

For Problems 17–20, divide. Write your answer as the sum of a polynomial and an algebraic fraction.

17. $\dfrac{6x^4-6x^2-4}{12x^2}$

18. $\dfrac{3n^3-3n^2+2n-3}{3n^2}$

19. $\dfrac{2x^2y^2-4xy^2+6xy}{2xy^2}$

20. $\dfrac{15s^{10}-21s^5+6}{-3s^2}$

For Problems 21–24, use polynomial division to write the quotient as the sum of a polynomial and an algebraic fraction.

21. $\dfrac{4y^2+12y+7}{2y+1}$

22. $\dfrac{x^3+2x^2+x+1}{x-2}$

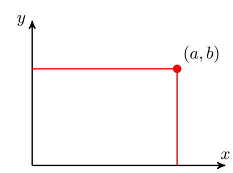
23. $\dfrac{4z^2+5z+8z^4+3}{2z+1}$

24. $\dfrac{x^4-1}{x^2-2}$

Applications

25.

a On the figure at right, locate the points $P\left(\dfrac{a}{2},0\right)$, $Q\left(a,\dfrac{b}{2}\right)$, $R\left(\dfrac{a}{2},b\right)$, and $S\left(0,\dfrac{b}{2}\right)$. Connect the points with line segments in the order $PQRS$ to form a four-sided figure.



b Compute the slopes of each side of the figure.

26.

a Suppose x and y are two positive numbers. Write an expression for their average, and then for the reciprocal of their average.

b For the same numbers x and y, write an expression for the average of their reciprocals.

c Are your expressions in parts (a) and (b) the same? Choose values for x and y and evaluate both expressions.

27. The focal length of a lens is given by the formula

$$\frac{1}{f}=\frac{1}{p}+\frac{1}{q}$$

where f stands for the focal length, p is the distance from the object viewed to the lens, and q is the distance from the image to the lens. Suppose you estimate that the distance from your cat (the object viewed) to your camera lens is 60 inches greater than the distance from the lens to the film inside the camera, where the image forms.

a Express $1/f$ as a single fraction in terms of q.

b Write and simplify an expression for f as a function of q.

28. Andy drives 300 miles to Lake Tahoe at 70 miles per hour and returns home at 50 miles per hour. What is his average speed for the round trip? (It is not 60 miles per hour!)

a Write expressions for the time it takes for each leg of the trip if Andy drives a distance d at speed r_1 and returns at speed r_2.

b Write expressions for the total distance and total time for the trip.

c Write an expression for the average speed for the entire trip.

d Write your answer to part (c) as a simple fraction.

e Use your formula to answer the question stated in the problem.

For Problems 29–36, Write each expression as a single fraction in simplest form.

29. $\left(1-\dfrac{k}{n}\right)\left(1+\dfrac{k}{n}\right)$

30. $m\left(\dfrac{v-V}{t}\right)\left(\dfrac{v+V}{2}\right)$

31. $\dfrac{2d}{c}\cdot\dfrac{1}{1-\left(\dfrac{u}{c}\right)^2}$

32. $\dfrac{1}{\dfrac{1}{n}\left(\dfrac{1}{n}-1\right)}$

33. $\dfrac{\dfrac{L}{F}}{\dfrac{L}{F}-1}\cdot\dfrac{K}{N}$

34. $b\left(\dfrac{a-b}{a+b}\right)+b$

35. $\dfrac{1-\dfrac{2h}{m}}{m-\dfrac{4h^2}{m}}$

36. $\dfrac{\dfrac{6}{h+2}-\dfrac{3}{h}}{\dfrac{4}{h}-\dfrac{3}{h+2}}$

For Problems 37–42, write each expression as a single fraction in simplest form.

37. $x^{-2}+y^{-2}$

38. $2w^{-1}-(2w)^{-2}$

39. $a^{-1}b-ab^{-1}$

40. $(x^{-1}+y^{-1})^{-1}$

41. $(1-xy^{-1})^{-1}$

42. $\dfrac{a^{-1}+b^{-1}}{(ab)^{-1}}$

For Problems 43-48, write each complex fraction as a simple fraction in lowest terms, and rationalize the denominator.

43. $\dfrac{\dfrac{\sqrt{5}}{3}}{\dfrac{1}{\sqrt{3}}}$

44. $\dfrac{1-\dfrac{2}{\sqrt{6}}}{\dfrac{\sqrt{2}}{\sqrt{6}}}$

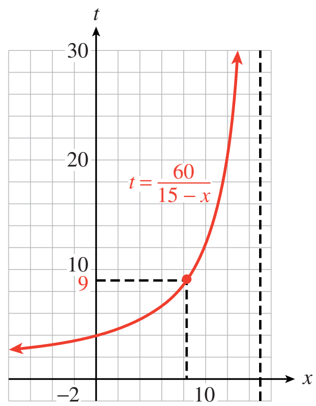
45. $\dfrac{1+\dfrac{1}{\sqrt{2}}}{2\sqrt{3}-\dfrac{\sqrt{3}}{2}}$

46. $\dfrac{\dfrac{1}{2}-\dfrac{1}{\sqrt{3}}}{\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{3}}$

47. $\dfrac{\dfrac{\sqrt{3}}{2}+\dfrac{1}{\sqrt{2}}}{1-\dfrac{\sqrt{3}}{2}\cdot\dfrac{1}{\sqrt{2}}}$

48. $\dfrac{\dfrac{1}{\sqrt{3}}-\dfrac{5}{3}}{1+\dfrac{\sqrt{1}}{\sqrt{3}}\cdot\dfrac{\sqrt{5}}{3}}$

Equations with Fractions



In Example 8.36, p. 547, of Section 8.2, Francine planned a 60-mile training run on her cycle-plane. The time required for the training run, in terms of the windspeed, x, is given by:

$$t = f(x) = \frac{60}{15-x}$$

If it takes Francine 9 hours to cover 60 miles, what is the speed of the wind? We can answer this question by reading values from the graph of f, as shown at left. When $t = 9$, the value of x is between 8 and 9, so the windspeed is between 8 and 9 miles per hour.

Solving Algebraically

If we need a more accurate value for the windspeed, we can solve the equation

$$\frac{60}{15-x} = 9$$

To start, we multiply each side of the equation by the denominator of the fraction. This will clear the fraction and give us an equivalent equation without fractions.

Example 8.80

Solve the equation $\frac{60}{15-x} = 9$

Solution. We multiply both sides of the equation by $\mathbf{15 - x}$ to obtain

$$\mathbf{(15-x)}\frac{60}{15-x} = 9\mathbf{(15-x)}$$

$$60 = 9(15-x) \qquad \text{Apply the distributive law.}$$

From here we can proceed as usual.

$$60 = 135 - 9x \qquad \text{Subtract 135 from both sides.}$$
$$-75 = -9x \qquad \text{Divide by } -9.$$
$$8.\overline{3} = x$$

The windspeed was $8.\overline{3}$, or $8\frac{1}{3}$ miles per hour.

Checkpoint 8.81 Practice 1. Solve $\frac{x^2}{x+4} = 2$

$x =$ ________

Answer. $-2, 4$

Solution. $x = -2, x = 4$

If the equation contains more than one fraction, we can clear all the denominators at once by multiplying both sides by the LCD of the fractions.

Example 8.82

Solve $\dfrac{3}{4} = 8 - \dfrac{2x+11}{x-5}$

Solution. The LCD for the two fractions in the equation is $4(x-5)$. We multiply both sides of the equation by the LCD.

$$4(x-5)\left(\frac{3}{4}\right) = \left(8 - \frac{2x+11}{x-5}\right) \cdot 4(x-5) \qquad \text{Apply the distributive law.}$$

$$\cancel{4}(x-5)\left(\frac{3}{\cancel{4}}\right) = 4(x-5)(8) - 4\cancel{(x-5)}\left(\frac{2x+11}{\cancel{x-5}}\right)$$

$$3(x-5) = 32(x-5) - 4(2x+11)$$

We proceed as usual to complete the solution. First we use the distributive law to remove parentheses.

$$3x - 15 = 32x - 160 - 8x - 44 \qquad \text{Combine like terms.}$$
$$3x - 15 = 24x - 204$$
$$-21x = -189$$
$$x = 9$$

Caution 8.83 We must multiply each term of an equation by the LCD, whether or not the term is a fraction. In the previous Example, p. 581 we multiplied each term by the LCD, including the 8.

Checkpoint 8.84 **Practice 2.** Solve $\dfrac{1}{x-2} + \dfrac{2}{x} = 1$

$x =$______ Enter solutions separated by a comma.

Answer. $4, 1$

Solution. 1, 4

Proportions

A proportion is a statement that two ratios are equal. For example,

$$\frac{7}{5} = \frac{x}{6}$$

To solve this proportion, we multiply both sides by the LCD, 30, to get

$$\mathbf{30}\left(\frac{7}{5}\right) = \left(\frac{x}{6}\right)\mathbf{30}$$
$$42 = 5x \qquad \text{Divide both sides by 5.}$$
$$x = \frac{42}{5} = 8.4$$

There is a short-cut we can use that avoids calculating an LCD. Observe that we can arrive at the equation $42 = 5x$ by **cross-multiplying**:

$$\frac{7}{5} = \frac{x}{6}$$
$$42 = 5x$$

We then proceed as before to complete the solution.
The cross-multiplying shortcut is a fundamental property of proportions.

Property of Proportions.

If $\dfrac{a}{b} = \dfrac{c}{d}$, then $ad = bc$, as long as $b, d \neq 0$

Example 8.85

The scale on a map of Fairfield County says that $\dfrac{3}{4}$ centimeter represents a distance of 10 kilometers. If Eastlake and Kenwood are 6 centimeters apart on the map, what is the distance between the two towns?

Solution. The ratio of the two actual distances is the same as the ratio of the corresponding distances on the map. We let x stand for the distance between Eastlake and Kenwood, and write a proportion.

We must be careful to keep the same order in both ratios. We choose to put the distance between towns in the numerators, and the distances on the scale in the denominator.

$$\frac{\text{distance between towns}}{\text{scale}}: \qquad \frac{x}{10} = \frac{6}{\frac{3}{4}}$$

To solve the propotion, we cross-multiply.

$$\frac{3}{4}x = 10 \cdot 6$$
$$x = \frac{60}{\frac{3}{4}} = 80$$

The two towns are 80 kilometers apart.

Checkpoint 8.86 Practice 3. On a scale model of Fantasy Valley, $1\dfrac{1}{2}$ inches represents 50 yards. If the distance from the water slide to the bungee jump is 20 inches on the model, what is the distance between the two rides?

______ yards

Answer. $666 + \frac{2}{3}$

Solution. $666\dfrac{2}{3}$ yards

Caution 8.87 Do not try to use "cross-multiplying" on equations that are not proportions, or on any other operations involving fractions. The shortcut works only on proportions.

Checkpoint 8.88 QuickCheck 1.

a. To clear fractions from an equation, we multiply each term by the (☐ LCD ☐ average ☐ reciprocal) of all the fractions.

b. True or False: When clearing fraction from an equation, we do not multiply terms that are not fractions. (☐ True ☐ False)

c. A proportion is a statement that two (☐ equations ☐ factors ☐ ratios) are equal.

d. Cross-multiplying works only on (☐ reduced fractions ☐ improper fractions ☐ proportions) .

Answer 1. LCD

Answer 2. False

Answer 3. ratios

Answer 4. proportions

Solution.

a. LCD

b. False

c. ratios

d. proportions

Extraneous Solutions

An algebraic fractions is undefined for any values of x that make its denominator equal zero. These values cannot be solutions to equations involving the fraction. Consider the equation

$$\frac{x}{x-3} = \frac{3}{x-3} + 2$$

When we multiply both sides by the LCD, $x - 3$, we obtain

$$\begin{aligned} \boldsymbol{(x-3)}\frac{x}{x-3} &= \boldsymbol{(x-3)}\frac{3}{x-3} + \boldsymbol{(x-3)} \cdot 2 \\ x &= 3 + 2x - 6 \end{aligned}$$

whose solution is $x = 3$. However, $x = 3$ is *not* a solution of the original equation. Both sides of the equation are undefined at $x = 3$. If you graph the two functions

$$Y_1 = \frac{x}{x-3} \qquad \text{and} \qquad Y_2 = \frac{3}{x-3} + 2$$

you will find that the graphs never intersect, which means that there is no solution to the original equation.

What went wrong with our method of solution? We multiplied both sides of the equation by $x - 3$, which is zero when $x = 3$, so we really multiplied both sides of the equation by zero. Multiplying by zero does not produce an equivalent equation, and false solutions may be introduced.

An apparent solution that does not satisfy the original equation is called an **extraneous solution**. Whenever we multiply an equation by an expression containing the variable, we should check that the solution obtained does not cause any of the fractions to be undefined.

Example 8.89

Solve the equation $\dfrac{6}{x} + 1 = \dfrac{1}{x+2}$.

Solution. We multiply both sides by the LCD, $x(x+2)$. Notice that we multiply *each term* on the left side by the LCD, to get

$$\boldsymbol{x(x+2)}\left(\frac{6}{x} + 1\right) = \boldsymbol{x(x+2)}\frac{1}{x+2}$$

or

$$6(x+2)+x(x+2)=x$$

We use the distributive law to remove the parentheses and write the result in standard form:

$$\begin{aligned} 6x+12+x^2+2x &= x \\ x^2+7x+12 &= 0 \end{aligned}$$

This is a quadratic equation that we can solve by factoring.

$$(x+3)(x+4)=0$$

so the solutions are $x=-3$ and $x=-4$. Neither of these values causes either denominator to equal zero, so they are not extraneous solutions.

Caution 8.90 The following "solution" for the previous Example, p. 583 is incorrect. Do you see why?

$$\begin{aligned} \boldsymbol{x(x+2)}\frac{6}{x}+1 &= \boldsymbol{x(x+2)}\frac{1}{x+2} \\ 6x+12+1 &= x \\ 5x &= -13 \\ x &= \frac{-13}{5} \end{aligned}$$

Checkpoint 8.91 QuickCheck 2.

a. An algebraic fraction is undefined when (☐ the numerator equals zero ☐ the denominator equals zero ☐ it includes decimals ☐ there is a variable in the denominator) .

b. An apparent solution that does not satisfy the original equation is called (☐ an extraneous ☐ a double ☐ a principal ☐ a multiple) solution.

c. We may introduce extraneous solutions if we multiply both sides of an equation by (☐ a nonzero integer ☐ an expression containing the variable ☐ any irrational number) .

d. We must multiply each (☐ term ☐ denominator ☐ fraction only) of the equation by the LCD.

Answer 1. the denominator equals zero

Answer 2. an extraneous

Answer 3. an expression containing the variable

Answer 4. term

Solution.

a. the denominator equals zero

b. an extraneous

c. an expression containing the variable

d. term

Checkpoint 8.92 Practice 4. Solve $\dfrac{9}{x^2+x-2}+\dfrac{1}{x^2-x}=\dfrac{4}{x-1}$

$x =$______

Answer. $\frac{-1}{2}$

Solution. $\dfrac{-1}{2}$

Solving Graphically

We can use graphs to solve the equation in the following Example.

Example 8.93

Solve the equation graphically: $\dfrac{6}{x}+1=\dfrac{1}{x+2}$.

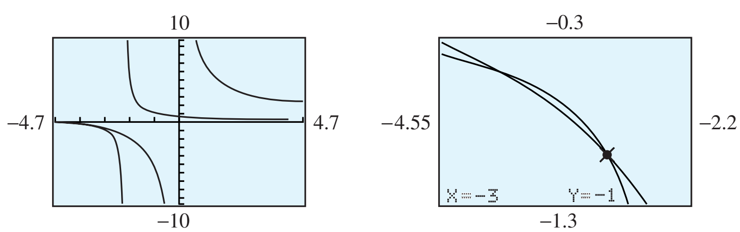
Solution. We graph the two functions

$$Y_1=\frac{6}{x}+1 \qquad \text{and} \qquad Y_2=\frac{1}{x+2}$$

in the window

$$\text{Xmin} = -4.7 \qquad \text{Xmax} = 4.7$$
$$\text{Ymin} = -10 \qquad \text{Ymax} = 10$$

as shown in figure (a).



(a) **(b)**

It appears that the two graphs may intersect in the third quadrant, around $x=-3$. To investigate further, we change the window settings to

$$\text{Xmin} = -4.55 \qquad \text{Xmax} = -2.2$$
$$\text{Ymin} = -1.3 \qquad \text{Ymax} = -0.3$$

and obtain the close-up view shown in figure (b). In this window, we can see that the graphs intersect in two distinct points, and by using the Trace we find that their x-coordinates are $x=-3$ and $x=-4$.

Checkpoint 8.94 Practice 5. The manager of a new health club kept track of the number of active members over the club's first few months of operation. The number, N of active members, in hundreds, t months after the club opened is given by the equation

$$N=\frac{10t}{4+t^2}$$

a. Graph the equation in the window

$$\text{Xmin} = 0 \quad \text{Xmax} = 9.4$$
$$\text{Ymin} = 0 \quad \text{Ymax} = 3$$

b. Use the graph to find out in which months the club had 200 active members.

Months: _____ Enter month numbers separated by a comma.

c. Verify your answer algebraically by solving an equation.

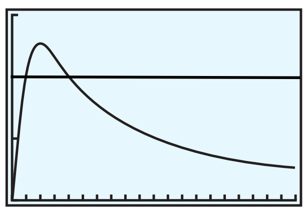
Answer. $1, 4$

Solution.

a. A graph is below.

b. 1, 4

Graph for (a):



Applications

Application problems may lead to equations with algebraic fractions.

Example 8.95

Rani times herself as she kayaks 30 miles down the Derwent River with the help of the current. Returning upstream against the current, she manages only 18 miles in the same amount of time. Rani knows that she can kayak at a rate of 12 miles per hour in still water. What is the speed of the current?

Solution. If we let x represent the speed of the current, we can use the formula

$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

to fill in the following table.

	Distance	Rate	Time
Downstream	30	$12 + x$	$\frac{30}{12+x}$
Upstream	18	$12 - x$	$\frac{18}{12-x}$

Because Rani paddled for equal amounts of time upstream and downstream, we have the equation

$$\frac{30}{12+x} = \frac{18}{12-x}$$

The LCD for the fractions in this equation is $\mathbf{(12+x)(12-x)}$. We multiply both sides of the equation by the LCD to obtain

$$\mathbf{(12+x)(12-x)}\frac{30}{12+x} = \frac{18}{12-x}\mathbf{(12+x)(12-x)}$$
$$30(12-x) = 18(12+x)$$

Solving this equation, we find

$$360 - 30x = 216 + 18x$$
$$144 = 48x$$
$$3 = x$$

The speed of the current is 3 miles per hour.

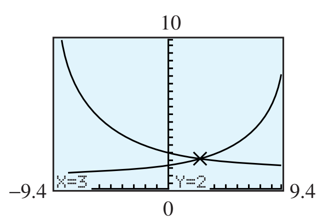
We can solve the equation in Example 8.95, p. 586 graphically by considering two functions, one for each side of the equation. Graph the two functions

$$Y_1 = \frac{30}{12+x} \qquad \text{and} \qquad Y_2 = \frac{30}{12-x}$$

in the window

$$\text{Xmin} = -9.4 \qquad \text{Xmax} = 9.4$$
$$\text{Ymin} = 0 \qquad \text{Ymax} = 10$$

to obtain the graph shown below.



The function Y_1 gives the time it takes Rani to paddle 30 miles downstream, and Y_2 gives the time it takes her to paddle 18 miles upstream. Both of these times depend on the speed of the current, x.

We are looking for a value of x that makes Y_1 and Y_2 equal. This occurs at the intersection point of the two graphs, $(3, 2)$. Thus, the speed of the current is 3 miles per hour, as we found in Example 8.95, p. 586. The y-coordinate of the intersection point gives the time Rani paddled on each part of her trip: 2 hours each way.

Checkpoint 8.96 Practice 6. A cruise boat travels 18 miles downstream and back in $4\frac{1}{2}$ hours. If the speed of the current is 3 miles per hour, what is the speed of the boat in still water?

a. Let x represent the speed of the boat in still water, and fill in the table.

	Distance	Rate	Time
Downstream	__	___	________
Upstream	__	___	________

b. Write an equation to model the problem:

Downstream time + Upstream time = Total trip time

__________________= ___

c. Solve your equation, and answer the question in the problem.

$x =$___: The speed of the boat in still water is __ mph.

Answer 1. 18

Answer 2. $x+3$

Answer 3. $\frac{18}{x+3}$

Answer 4. 18

Answer 5. $x-3$

Answer 6. $\frac{18}{x-3}$

Answer 7. $\frac{18}{x+3}+\frac{18}{x-3}$

Answer 8. $\frac{9}{2}$

Answer 9. 9

Answer 10. 9

Solution.

a.

	Distance	Rate	Time
Downstream	18	$x+3$	$\dfrac{18}{x+3}$
Upstream	18	$x-3$	$\dfrac{18}{x-3}$

b. $\dfrac{18}{x+3}+\dfrac{18}{x-3}=\dfrac{9}{2}$

c. 9 mph

Formulas

Algebraic fractions may appear in formulas that relate several variables. If we want to solve for one variable in terms of the others, we may need to clear the fractions.

Example 8.97

Solve the formula $p=\dfrac{v}{q+v}$ for v.

Solution. Because the variable we want appears in the denominator, we must first multiply both sides of the equation by that denominator, $q+v$.

$$(q+v)p=(q+v)\frac{v}{q+v}$$
$$(q+v)p=v$$

We apply the distributive law on the left side, then collect all terms that involve v on one side of the equation.

$$qp+vp=v \qquad \text{Subtract } vp \text{ from both sides.}$$
$$qp=v-vp$$

We cannot combine the two terms containing v because they are not like terms. However, we can factor out v, so that the right side is written as a single term containing the variable v. We can then complete the solution.

$$qp=v(1-p) \qquad \text{Divide both sides by } 1-p.$$

$$\frac{qp}{1-p} = v$$

Checkpoint 8.98 Practice 7. Solve for a: $\dfrac{2ab}{a+b} = H$

$a =$______________

Answer. $\frac{bH}{2b-H}$

Solution. $a = \dfrac{bH}{2b-H}$

Checkpoint 8.99 QuickCheck 3. True or False.

a. To solve an equation graphically, we graph two functions, $y =$ (each side of the equation.) (☐ True ☐ False)

b. The solutions are the y-coordinates of the intersection points of the two graphs. (☐ True ☐ False)

c. To solve a formula that is linear in the desired variable, we must get all terms including that variable on one side of the equation. (☐ True ☐ False)

d. If two or more terms on one side of the equation include the desired variable, we factor it out. (☐ True ☐ False)

Answer 1. True
Answer 2. False
Answer 3. True
Answer 4. True
Solution.

a. True

b. False

c. True

d. True

Problem Set 8.5

Warm Up

For Problems 1–8, solve.

1. $\dfrac{6}{w+2} = 4$	**2.** $\dfrac{12}{r-7} = 3$	**3.** $9 = \dfrac{h-5}{h-2}$
4. $-3 = \dfrac{v+1}{v-6}$	**5.** $\dfrac{15}{s^2} = 8$	**6.** $\dfrac{3}{m^2} = 5$
7. $4.3 = \sqrt{\dfrac{18}{y}}$	**8.** $6.5 = \dfrac{52}{\sqrt{z}}$	

Skills Practice

For Problems 9–16, solve.

9. $\dfrac{4}{x} - 3 = \dfrac{5}{2x+3}$

10. $\dfrac{4}{x-1} - \dfrac{4}{x+2} = \dfrac{3}{7}$

11. $\dfrac{2}{n^2-2n} + \dfrac{1}{2n} = \dfrac{-1}{n^2+2n}$

12. $\dfrac{3}{x-2} = \dfrac{1}{2} + \dfrac{2x-7}{2x-4}$

13. $\dfrac{4}{x+2} - \dfrac{1}{x} = \dfrac{2x-1}{x^2+2x}$

14. $\dfrac{1}{x-1} + \dfrac{2}{x+1} = \dfrac{x-2}{x^2-1}$

15. $\dfrac{x}{x+2} - \dfrac{3}{x-2} = \dfrac{x^2+8}{x^2-4}$

16. $-3 = \dfrac{-10}{x+2} + \dfrac{10}{x+5}$

17. What is wrong with the solution to the following addition problem:

$$\text{Add} \quad \frac{3}{8} + \frac{1}{2}$$

Solution:

$$\begin{aligned} 8 \cdot \frac{3}{8} + 8 \cdot \frac{1}{2} &= 3 + 4 && \text{Multiply by the LCD, 8.} \\ &= 7 && \text{The answer is 7.} \end{aligned}$$

18. Compare the methods in each calculation with fractions. Explain how the LCD is used in each operation.

a Add: $\dfrac{3}{x} + \dfrac{1}{x+3}$

b Divide: $\dfrac{3}{x} \div \dfrac{1}{x+3}$

c Solve: $\dfrac{3}{x} + \dfrac{1}{x+3} = 3$

d Simplify: $\dfrac{1 + \dfrac{3}{x}}{3 + \dfrac{1}{x+3}}$

For Problems 19–24, solve the formula for the specified variable.

19. $S = \dfrac{a}{1-r}$, for r

20. $m = \dfrac{y-k}{x-h}$, for x

21. $\dfrac{x}{a} + \dfrac{y}{b} = 1$, for x

22. $F = \dfrac{Gm_1m_2}{d^2}$, for d

23. $C = \dfrac{rR}{r+R}$, for R

24. $E = \dfrac{Ff}{(P-x)p}$, for x

Applications

25. Rani went hiking in the Santa Monica mountains last weekend. She drove 10 miles in her car and then walked 4 miles, and arrived at a small lake hours after she left home. If Rani drives 20 times faster than she walks, how fast does she walk?

26. Sam Scholarship and Reginald Privilege each travel the 360 miles to Fort Lauderdale on spring break, but Reginald drives his Porsche while Sam hitches a ride on a vegetable truck. Reginald travels 20 miles per hour faster than Sam and arrives in 3 hours less time. How fast did each travel?

27. Walt and Irma use a tank of fuel oil for their furnace every 25 days during the winter. Last winter it was so cold that they also lit their space heater 10 days after they filled the fuel oil tank. If the space heater uses a tank of fuel oil every 40 days, how much longer will the fuel last with both heaters running?

28. An underground spring fills a small pond in 12 days. Evaporation from the surface of the pond can empty the pond in 28 days. If the pond is completely dry, how long will it take to fill again?

29. The cost, in thousands of dollars, for immunizing p percent of the residents of Emporia against a dangerous new disease is given by the function

$$C(p) = \frac{72p}{100 - p}$$

Write and solve an equation to determine what percent of the population can be immunized for \$168,000.

30. A school of bluefin tuna average 36 miles per hour on a 200-mile trip in still water, but this time they encounter a current.

(a) Express the tuna's travel time, t, as a function of the current speed, v, and graph the function in the window

$$\text{Xmin} = 0 \qquad \text{Xmax} = 36$$
$$\text{Ymin} = 0 \qquad \text{Ymax} = 50$$

(b) Write and solve an equation to find the current speed if the school makes its trip in 8 hours. Label the corresponding point on your graph.

31. During the baseball season so far this year, Pete got hits 44 times out of 164 times at bat.

a What is Pete's batting average so far? (Batting average is the fraction of at-bats that resulted in hits.)

b If Pete gets hits on every one of his next x at-bats, write an expression for his new batting average.

c How many consecutive hits does Pete need to raise his batting average to 0.350 ?

32. The manager of Joe's Burgers discovers that he will sell $\frac{160}{x}$ burgers per day if the price of a burger is x dollars. On the other hand, he can afford to make $6x + 49$ burgers if he charges x dollars apiece for them.

a Graph the demand function, $D(x) = \frac{160}{x}$ and the supply function, $S(x) = 6x+49$, in the same window. At what price$x$does the demand for burgers equal the number that Joe can afford to supply? This value for is called the equilibrium price.

b Write and solve an equation to verify your equilibrium price.

33. A chartered sightseeing flight over the Grand Canyon is scheduled to return to its departure point in 3 hours. The pilot would like to cover a distance of 144 miles before turning around, and he hears on the Weather Service that there will be a headwind of 20 miles per hour on the outward journey.

a Express the time it takes for the outward journey as a function of the airspeed of the plane.

b Express the time it takes for the return journey as a function of the speed of the plane.

c Graph the sum of the two functions and find the point on the graph with y-coordinate 3. Interpret the coordinates of the point in the context of the problem.

d The pilot would like to know what airspeed to maintain in order to complete the tour in 3 hours. Write an equation to describe this situation.

e Solve your equation to find the appropriate airspeed.

34. The cost of wire fencing is \$7.50 per foot. A rancher wants to enclose a rectangular pasture of 1000 square feet with this fencing.

a Express the length of the pasture as a function of its width.

b Express the cost of the fence as a function of its width.

c Graph your function for the cost and find the coordinates of the lowest point on the graph. Interpret those coordinates in the context of the problem.

d The rancher has \$1050 to spend on the fence, and she would like to know what the width of the pasture should be. Write an equation to describe this situation.

e Solve your equation and find the dimensions of the pasture.

35. Distances on a map vary directly with actual distances. The scale on a map of Michigan uses $\dfrac{3}{8}$ inch to represent 10 miles. If Isle Royale is $1\dfrac{11}{16}$ inches long on the map, what is the actual length of the island?

36. The highest point on Earth is Mount Everest in Tibet, with an elevation of 8848 meters. The deepest part of the ocean is the Challenger Deep in the Mariana Trench, near Indonesia, 11,034 meters below sea level.

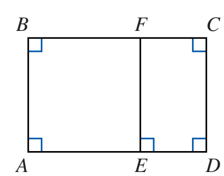
a What is the total height variation in the surface of the Earth?

b What percentage of the Earth's radius, 6400 kilometers, is this variation?

c If the Earth were shrunk to the size of a basketball, with a radius of 4.75 inches, what would be the corresponding height of Mount Everest?

37. The rectangle $ABCD$ is divided into a square and a smaller rectangle, $CDEF$. The two rectangles $ABCD$ and $CDEF$ are similar (their corresponding sides are proportional.) A rectangle $ABCD$ with this property is called a **golden rectangle**, and the ratio of its length to its width is called the golden ratio.

The golden ratio appears frequently in art and nature, and it is considered to give the most pleasing proportions to many figures. We will compute the golden ratio as follows.



a Let $AB = 1$ and $AD = x$. What are the lengths of AE, DE, and CD?

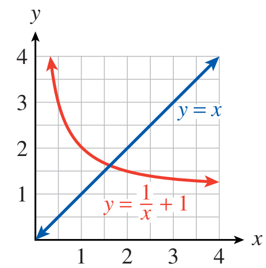
b Write a proportion in terms of x for the similarity of rectangles $ABCD$ and $CDEF$. Be careful to match up the corresponding sides.

c Solve your proportion for x. Find the golden ratio, $\dfrac{AD}{AB} = \dfrac{x}{1}$.

38.

The figure shows the graphs of two equations,

$$y = x \quad \text{and} \quad y = \frac{1}{x} + 1.$$



a Find the x-coordinate of the intersection point of the two graphs.

b Compare your answer to the golden ratio you computed in Problem 37.

39. Find the error in the following "proof" that $1 = 0$: Start by letting $x = 1$.

$x = 1$	Multiply both sides by x.
$x^2 = x$	Subtract 1 from both sides.
$x^2 - 1 = x - 1$	Factor the left side.
$(x-1)(x+1) = x - 1$	Divide both sides by $x - 1$.
$x + 1 = 1$	Subtract 1 from both sides.
$x = 0$	

Because $x = 1$ and $x = 0$, we have "proved" that $1 = 0$.

For Problems 40–42,

a Solve the equation graphically by graphing two functions, one for each side of the equation.

b Solve the equation algebraically.

40. $\dfrac{2x}{x+1} = \dfrac{x+1}{2}$

41. $\dfrac{2}{x+1} = \dfrac{x}{x+1} + 1$

42. $\dfrac{15x}{1+x^2} = 6$

Chapter 8 Summary and Review

Glossary

- polynomial
- degree
- descending powers
- ascending powers
- lead coefficient
- algebraic fraction
- vertical asymptote
- horizontal asymptote
- rational function
- lowest common denominator

- building factor
- like fractions
- common factor
- complex fraction
- proportion
- extraneous solution

Key Concepts

Polynomial Function.

1 A **polynomial function** has the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$$

where a_0, a_1, a_2, ..., a_n are constants and $a_n \neq 0$. The coefficient a_n of the highest power term is called the **lead coefficient**.

Degree of a Product.

2 The degree of a product of non-zero polynomials is the sum of the degrees of the factors. That is:

If $P(x)$ has degree m and $Q(x)$ has degree n, then their product $P(x)Q(x)$ has degree $m+n$.

Special Products of Binomials.

3

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$
$$(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$$
$$(a+b)(a-b) = a^2 - b^2$$

Cube of a Binomial.

4

1 $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

2 $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

Factoring the Sum or Difference of Two Cubes.

5

1 $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

2 $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

Fundamental Principle of Fractions.

6 We can multiply or divide the numerator and denominator of a fraction by the same nonzero factor, and the new fraction will be

equivalent to the old one.

$$\frac{a \cdot c}{b \cdot c} = \frac{a}{b} \qquad \text{if} \qquad b, c \neq 0$$

7 When we cancel common factors, we are *dividing.* Because division is the inverse or opposite operation for multiplication, we can cancel common *factors*, but *we cannot cancel common terms.*

To reduce an algebraic fraction.

8

1 Factor the numerator and the denominator.

2 Divide the numerator and denominator by any common factors.

Exponential Function.

9 An **exponential function** has the form

$$f(x) = ab^x, \qquad \text{where} \quad b > 0 \quad \text{and} \quad b \neq 1, \quad a \neq 0$$

Operations on Fractions.

10

- If $b,\ d \neq 0$, then
$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$
- If $b,\ c,\ d \neq 0$, then
$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$
- If $c \neq 0$, then
$$\frac{a}{b} + \frac{c}{d} = \frac{a+b}{c}$$
$$\frac{a}{b} - \frac{c}{d} = \frac{a-b}{c}$$

To multiply algebraic fractions:.

11

1 Factor each numerator and denominator.

2 If any factor appears in both a numerator and a denominator, divide out that factor.

3 Multiply the remaining factors of the numerator and the remaining factors of the denominator.

4 Reduce the product if necessary.

To divide algebraic fractions:.

12

1 Take the reciprocal of the second fraction and change the operation to multiplication.

2 Follow the rules for multiplication of fractions.

To add or subtract algebraic fractions.

13

1 Find the lowest common denominator (LCD) for the fractions.

2 Build each fraction to an equivalent one with the same denominator.

3 Add or subtract the resulting like fractions: Add or subtract their numerators, and keep the same denominator.

4 Reduce the sum or difference if necessary.

To Find the LCD.

14

1 Factor each denominator completely.

2 Include each different factor in the LCD as many times as it occurs in any one of the given denominators.

To simplify a complex fraction.

15

1 Find the LCD of all the fraction contained in the complex fraction.

2 Multiply the numerator and the denominator of the complex fraction by the LCD.

3 Reduce the resulting simple fraction, if possible.

16 An algebraic fractrion is ”improper” if the degree of the numerator is greater than the degree of the denominator. If it cannot be reduced, we can simplify the expression by treating it as a division of polynomials. The quotient will be the sum of a polynomial and a simpler algebraic fraction.

17 If the equation contains more than one fraction, we can clear all the denominators at once by multiplying both sides by the LCD of the fractions.

Property of Proportions.

18

If $\dfrac{a}{b} = \dfrac{c}{d}$, then $ad = bc$, as long as $b, d \neq 0$

19 Whenever we multiply an equation by an expression containing the variable, we should check that the solution obtained does not cause oany of the fractions to be undefined.

Chapter 8 Review Problems

For Problems 1 and 2, multiply.

1. $(2x - 5)(x^2 - 3x + 2)$

2. $(b^2 - 2b - 3)(2b^2 + b - 5)$

For Problems 3 and 4, factor.

3. $8x^3 - 27z^3$

4. $1 + 125a^3b^3$

5. The expression $\dfrac{n}{6}(n-1)(n-2)$ gives the number of different 3-item pizzas that can be created from a list of n toppings.

a Write the expression as a polynomial.

b If Mitch's Pizza offers 12 different toppings, how many different combinations for 3-item pizzas can be made?

c Use a table or graph to determine how many different toppings are needed in order to be able to have more than 1000 possible combinations for 3-item pizzas.

6. The expression $n(n - 1)(n - 2)$ gives the number of different triple-scoop ice cream cones that can be created from a list of n flavors.

a Write the expression as a polynomial.

b If Zanner's Ice Cream Parlor offers 21 flavors, how many different triple-scoop ice cream cones can be made?

c Use a table or graph to determine how many different flavors are needed in order to be able to have more than 10,000 possible triple-scoop ice cream cones.

7. The radius, r, of a cylindrical can should be one-half its height, h.

a Express the volume, V, of the can as a function of its height.

b What is the volume of the can if its height is 2 centimeters? 4 centimeters?

c Graph the volume as a function of the height and verify your results of part (b) graphically. What is the approximate height of the can if its volume is 100 cubic centimeters?

8. The Twisty-Freez machine dispenses soft ice cream in a cone-shaped peak with a height 3 times the radius of its base. The ice cream comes in a round bowl with base diameter d.

a Express the volume, V, of Twisty-Freez in the bowl as a function of d.

b How much Twisty-Freez comes in a 3-inch diameter dish? A 4-inch dish?

c Graph the volume as a function of the diameter and verify your results of part (b) graphically. What is the approximate diameter of a Twisty-Freez if its volume is 5 cubic inches?

9. A new health club opened up, and the manager kept track of the number of active members over its first few months of operation. The equation below gives the number, N, of active members, in hundreds, t months after the club opened.

$$N = \frac{44t}{40 + t^2}$$

a Use your calculator to graph the function N in the window

Xmin = 0	Xmax = 20
Ymin = 0	Ymax = 4

b How many active members did the club have after 8 months?

c In which months did the club have 200 active members?

d When does the health club have the largest number of active members? What happens to the number of active members as time goes on?

10. A small lake in a state park has become polluted by runoff from a factory upstream. The cost for removing p percent of the pollution from the lake is given, in thousands of dollars, by

$$C = \frac{25p}{100 - p}$$

a Use your calculator to graph the function C on a suitable domain.

b How much will it cost to remove 40% of the pollution?

c How much of the pollution can be removed for \$100,000 ?

d What happens to the cost as the amount of pollution to be removed increases? How much will it cost to remove all the pollution?

11. The Explorer's Club is planning a canoe trip to travel 90 miles up the Lazy River and return in 4 days. Club members plan to paddle for 6 hours each day, and they know that the current in the Lazy River is 2 miles per hour.

a Express the time it will take for the upstream journey as a function of their paddling speed in still water.

b Express the time it will take for the downstream journey as a function of their paddling speed in still water.

c Graph the sum of the two functions in the window

Xmin = 0	Xmax = 18.8
Ymin = 0	Ymax = 50

and find the point on the graph with y-coordinate 24. Interpret the coordinates of the point in the context of the problem.

d The Explorer's Club would like to know what average paddling speed members must maintain in order to complete their trip in 4 days. Write an equation to describe this situation.

e Solve your equation to find the required paddling speed.

12. Pam lives on the banks of the Cedar River and makes frequent trips in her outboard motorboat. The boat travels at 20 miles per hour in still water.

a Express the time it takes Pam to travel 8 miles upstream to the gas station as a function of the speed of the current.

b Express the time it takes Pam to travel 12 miles downstream to Marie's house as a function of the speed of the current.

c Graph the two functions in the window

$$\text{Xmin} = 0 \qquad \text{Xmax} = 10$$
$$\text{Ymin} = 0 \qquad \text{Ymax} = 1$$

and find the coordinates of the intersection point. Interpret those coordinates in the context of the problem.

d Pam traveled to the gas station in the same time it took her to travel to Marie's house. Write an equation to describe this situation.

e Solve your equation to find the speed of the current in the Cedar River.

For Problems 13–18, reduce the fraction to lowest terms.

13. $\dfrac{2a^2(a-1)^2}{4a(a-1)^3}$ **14.** $\dfrac{4y-6}{6}$ **15.** $\dfrac{2x^2y^3-4x^3y}{4x^2y}$

16. $\dfrac{(x-2y)^2}{4y^2-x^2}$ **17.** $\dfrac{a^2-6a+9}{2a^2-18}$ **18.** $\dfrac{4x^2y^2+4xy+1}{4x^2y^2-1}$

For Problems 19–26, write the expression as a single fraction in lowest terms.

19. $\dfrac{2a^2}{3b} \cdot \dfrac{15b^2}{a}$ **20.** $\dfrac{-1}{3}ab^2 \cdot \dfrac{3}{4}a^3b$

21. $\dfrac{4x+6}{2x} \cdot \dfrac{6x^2}{(2x+3)^2}$ **22.** $\dfrac{4x^2-9}{3x-3} \cdot \dfrac{x^2-1}{4x-6}$

23. $\dfrac{a^2-a-2}{a^2-4} \div \dfrac{a^2+2a+1}{a^2-2a}$ **24.** $\dfrac{a^3-8b^3}{a^2b} \div \dfrac{a^2-4ab+4b^2}{ab^2}$

25. $1 \div \dfrac{4x^2-1}{2x+1}$ **26.** $\dfrac{y^2+2y}{3x} \div (4y)$

For Problems 27–30, divide.

27. $\dfrac{36x^6-28x^4+16x^2-4}{4x^4}$ **28.** $\dfrac{y^3+3y^2-2y-4}{y+1}$

29. $\dfrac{x^3-4x^2+2x+3}{x-2}$ **30.** $\dfrac{x^2+2x^3-1}{2x-1}$

For Problems 31–36, write the expression as a single fraction in lowest terms.

31. $\dfrac{x+2}{3x} - \dfrac{x-4}{3x}$ **32.** $\dfrac{5}{6}b - \dfrac{1}{3}b + \dfrac{3}{4}b$

33. $\dfrac{3}{2x-6} - \dfrac{4}{x^2-9}$

34. $\dfrac{1}{y^2+4y+4} + \dfrac{3}{y^2-4}$

35. $\dfrac{2a+1}{a-3} - \dfrac{-2}{a^2-4a+3}$

36. $a - \dfrac{1}{a^2+2a+1} + \dfrac{3}{a^2-1}$

For Problems 37–40, write the complex fraction as a simple fraction in lowest terms.

37. $\dfrac{\frac{3}{4}-\frac{1}{2}}{\frac{3}{4}+\frac{1}{2}}$

38. $\dfrac{y-\frac{2y}{x}}{1+\frac{2}{x}}$

39. $\dfrac{x-4}{x-\frac{16}{x}}$

40. $\dfrac{\frac{1}{x-1}}{1-\frac{1}{x^2}}$

For Problems 41–44, solve.

41. $\dfrac{y+3}{y+5} = \dfrac{1}{3}$

42. $\dfrac{z^2+2}{z^2-2} = 3$

43. $\dfrac{x}{x-2} = \dfrac{2}{x-2} + 7$

44. $\dfrac{3x}{x+1} - \dfrac{2}{x^2+x} = \dfrac{4}{x}$

For Problems 45–48, solve for the indicated variable.

45. $V = C\left(1-\dfrac{t}{n}\right)$, for n

46. $r = \dfrac{dc}{1-ec}$, for c

47. $\dfrac{p}{q} = \dfrac{r}{q+r}$, for q

48. $I = \dfrac{E}{R+\frac{r}{n}}$, for R

For Problems 49–54, write the expression as a single fraction involving positive exponents only.

49. $x^{-3} + y^{-1}$

50. $\dfrac{x^{-1}}{y} - \dfrac{x}{y^{-1}}$

51. $\dfrac{x^{-1}-y}{x-y^{-1}}$

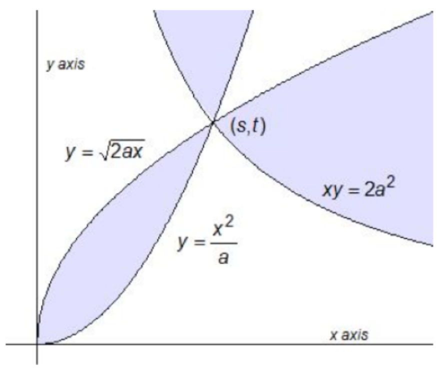
52. $\dfrac{x^{-1}+y^{-1}}{x^{-1}}$

53. $\dfrac{x^{-1}-y^{-1}}{(x-y)^{-1}}$

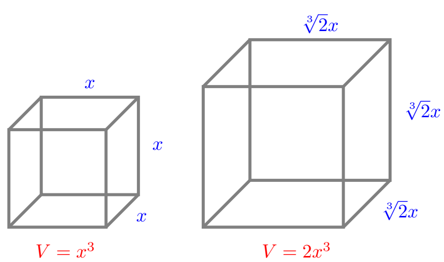
54. $\dfrac{(xy)^{-1}}{x^{-1}-y^{-1}}$

Chapter 9

Equations and Graphs

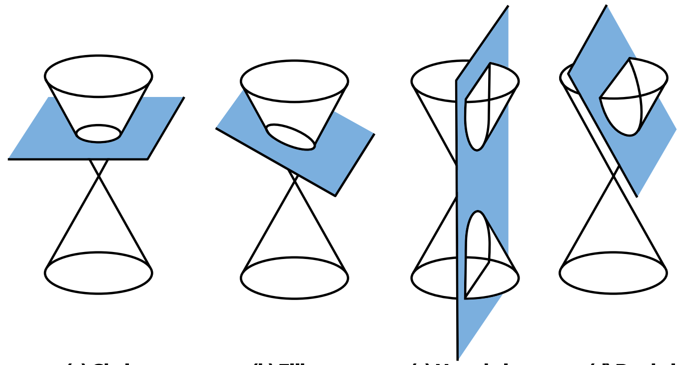


In ancient Athens, around 430 BCE, one quarter of the city's residents perished in a plague. The people of Athens consulted the oracle at Delos, so the story goes, to find a cure for the plague. The oracle replied that they should construct a new cubical altar to the gods, and its volume should be double the volume of the existing cubical altar. Now, if the original altar had a side of length x, you can work out that the new altar should have side length $\sqrt[3]{2}x$, so that the new volume is $2x^3$.



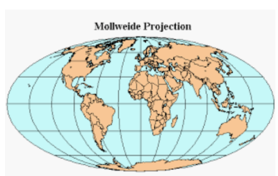
But the ancient Greeks could not use algebra to solve the problem -- it hadn't been invented yet. Instead, they used geometrical methods. The Delian problem of doubling the cube remained unsolved for many years. But sometime around 350 BCE, a mathematician named Menaechmus, who was the tutor to Alexander the Great, solved the problem, and in so doing invented the **conic sections**.

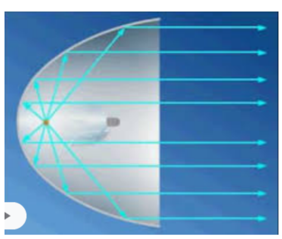
A "section" is a slice, and the conic sections are the curves formed by slicing a cone by a plane. Depending on the angle of the slice, we get four different curves, as shown below.

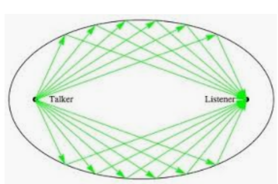


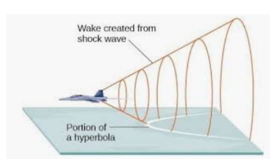
(a) Circle **(b) Ellipse** **(c) Hyperbola** **(d) Parabola**

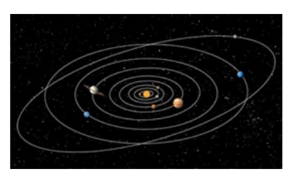
We have already met the parabola, which is described by the quadratic equation $y = ax^2 + bx + c$. It turns out that the other conic sections are also described by quadratic equations. These curves have many interesting properties as abstract objects, but as often happens with mathematical discoveries, they are also useful in applications. Conic sections appear in art and architecture, in medicine, science, and engineering.

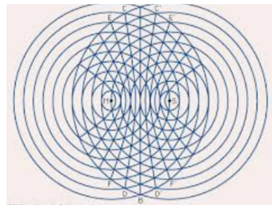
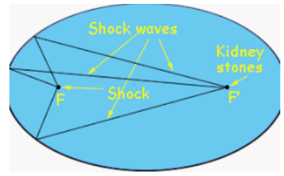










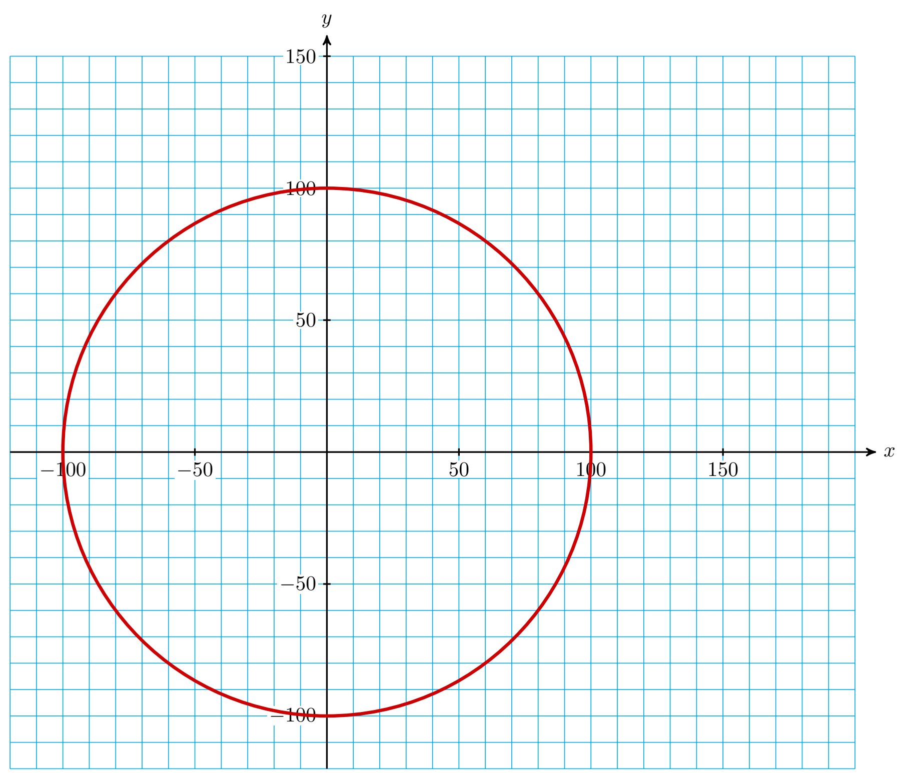




Investigation 9.1 Global Positioning. The Global Positioning System (GPS) is used by hikers, pilots, surveyors, automobiles to determine their location (latitude, longitude, and elevation) anywhere on the surface of the earth. The system depends on a collection of satellites in orbit around the earth. Each GPS satellite transmits its own position and the current time at regular intervals.

A person with a GPS receiver on earth can calculate his or her distance from the satellite by comparing the time of transmission with the time when it receives the signal. Of course, there are many points at the same distance from the satellite—in fact, the set of all points at a certain distance r from the satellite lie on a sphere centered at the satellite. That is why there are several satellites: You calculate your position by finding the intersection point of several such spheres centered on different satellites.

We will consider a simplified, two-dimensional model of a GPS system in which the satellites and the receiver all lie in the xy-plane instead of in three-dimensional space.

In this model we'll need data from two GPS satellites. The satellites are orbiting along a circle of radius 100 meters centered at the origin. You have a receiver inside that circle and would like to know the coordinates of your position within the circle.



To make the computations simpler, we will also assume that the satellite transmissions travel at 5 meters per second.

1 A signal from Satellite A arrives 18 seconds after it was transmitted. How far are you from Satellite A?

2 The signal says that Satellite A was located at $(100, 0)$ at the time of transmission. Use a compass to sketch a graph showing your possible positions relative to Satellite A.

3 Find an equation for the graph you sketched in part (2).

4 A signal from Satellite B arrives 8.4 seconds after it was transmitted. How far are you from Satellite B?

5 The signal says that Satellite B was located at $(28, 96)$ at the time of transmission. Use a compass to sketch a graph showing your possible positions relative to Satellite B.

6 Find an equation for the graph you sketched in part (5).

7 Your position must lie at the intersection point, P, of your two graphs. Estimate the coordinates of your position from the graph. (Remember that you are within the orbits of the satellites.)

8 Later in this chapter you will learn how to find the coordinates of P algebraically by solving a system of equations. Verify that the ordered pairs $(28, 54)$ and $(68.32, 84.24)$ both satisfy the equations you wrote in part (3) and part (6). What are the coordinates of P?

Properties of Lines

Horizontal and Vertical Lines

The general form for a linear equation is

$$Ax + By = C$$

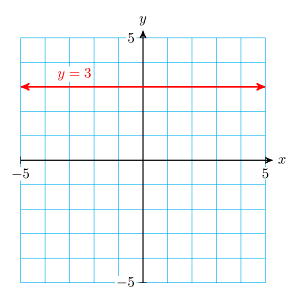
What happens if $A = 0$? For example, what does the graph of

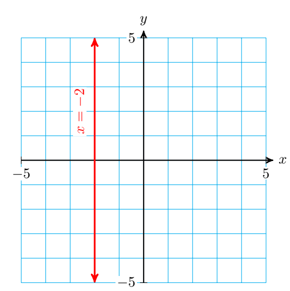
$$0x + y = 3, \qquad \text{or} \qquad y = 3$$

look like? This equation tells us that every point on the line must have y-coordinate 3, but it puts no restrictions on the x-coordinates. For example,

$$(-1, 3), \quad (2, 3), \quad \text{and} \quad (4, 3)$$

are all solutions of the equation. These points lie on a horizontal line, as shown below left.





On the other hand, if $B = 0$, we have an equation such as

$$x + 0y = -2, \qquad \text{or} \qquad x = -2$$

This equation tells us that any point with x-coordinate -2 lies on the graph. For example, $(-2, 3)$ and $(-2, -1)$ are solutions. All the solutions lie on the vertical line shown above right.

Note that the horizontal line $y = 3$ has y-intercept $(0, 3)$ but no x-intercept, and the vertical line $x = -2$ has x-intercept $(-2, 0)$ but no y-intercept.

Horizontal and Vertical Lines.

1 The equation of the **horizontal line** passing through $(0, b)$ is

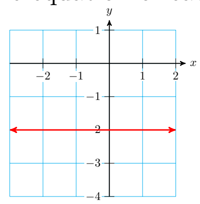
$$y = b$$

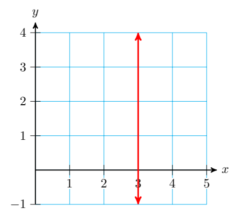
2 The equation of the **vertical line** passing through $(a, 0)$ is

$$x = a$$

Example 9.1

Give the equation of each line.

a 

b 

Solution.

a This is a horizontal line with y-intercept $(0, -2)$, so its equation is $y = -2$.

b This is a vertical line with x-intercept $(3, 0)$, so its equation is $x = 3$.

Now let's compute the slopes of the two lines in the previous Example, p. 605. Choose two points on the graph of $y = -2$, say $(-5, -2)$ and $(4, -2)$. Use these points to compute the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - (-2)}{4 - (-5)} = \frac{0}{9} = 0$$

The slope of the horizontal line $y = -2$ is zero. In fact, the slope of any horizontal line is zero, because the y-coordinates of all the points on the line are equal. Thus

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0$$

On a vertical line, the x-coordinates of all the points are equal. For example, two points on the line $x = 3$ are $(3, 1)$ and $(3, 6)$. Using these points to compute the slope, we find

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 1}{3 - 3} = \frac{5}{0}$$

which is undefined. The slope of any vertical line is undefined because the expression $x_2 - x_1$ equals zero.

Slopes of Horizontal and Vertical Lines.

The slope of a **horizontal** line is **zero**.
The slope of a **vertical** line is **undefined**.

Checkpoint 9.2 Practice 1.

a. What is the slope of any line parallel to the x-axis? (☐ 0 ☐ Undefined)

b. What is the slope of any line parallel to the y-axis? (☐ 0 ☐ Undefined)

Answer 1. 0
Answer 2. Undefined
Solution.

a. 0

b. undefined

Checkpoint 9.3 QuickCheck 1. True or False.

a. A horizontal line has no slope. (☐ True ☐ False)

b. The equation of a vertical line is undefined. (☐ True ☐ False)

c. A vertical line is not the graph of a function. (☐ True ☐ False)

d. The graph of $x = 6$ is a point. (☐ True ☐ False)

Answer 1. False
Answer 2. False
Answer 3. True
Answer 4. False
Solution.

a. False

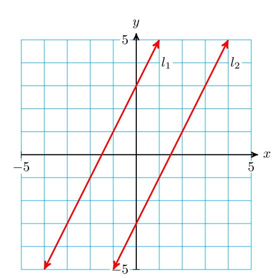
b. False

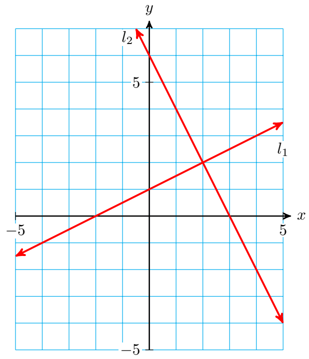
c. True

d. False

Parallel and Perpendicular Lines

Lines that lie in the same plane but never intersect are called **parallel** lines. They have the same "steepness" or inclination, so it is easy to understand that parallel lines have the same slope. You can verify that the parallel lines in the figure at right both have slope $m = 2$.





Perpendicular lines in a plane meet at right angles, or 90°. The relationship between the slopes of perpendicular lines is not so easy to see as the relationship for parallel lines. However, for the perpendicular lines shown at left, you can verify that

$$m_1 = \frac{1}{2} \quad \text{and} \quad m_2 = -2$$

Note that the product of m_1 and m_2 is -1, that is,

$$m_1 m_2 = \frac{1}{2}(-2) = -1$$

This relationship holds for any pair of perpendicular lines. We summarize these results as follows.

Parallel and Perpendicular Lines.

1 Two lines are **parallel** if their slopes are equal, that is, if

$$m_1 = m_2$$

or if both lines are vertical.

2 Two lines are **perpendicular** if the product of their slopes is -1, that is, if

$$m_1 m_2 = -1$$

or if one of the lines is horizontal and one is vertical.

Another way to state the condition for perpendicular lines is

$$m_2 = \frac{-1}{m_1}$$

Because of this relationship, we often say that the slope of one perpendicular line is the **negative reciprocal** of the other.

Example 9.4

Decide whether the lines

$$2x + 3y = 6 \quad \text{and} \quad 3x - 2y = 6$$

are parallel, perpendicular, or neither.

Solution. We could graph the lines, but we can't be sure from a graph if the line sare exactly parallel or exactly perpendicular. A more accurate way to settle the question is to find the slope of each line. To do this we write each equation in slope-intercept form; that is, we solve for y.

$$\begin{aligned} 2x + 3y &= 6 \\ 3y &= -2x + 6 \\ y &= \frac{-2x}{2} + 2 \end{aligned} \qquad \begin{aligned} 3x - 2y &= 6 \\ -2y &= -3x + 6 \\ y &= \frac{-3}{2}x - 3 \end{aligned}$$

The slope of the first line is $m_1 = \dfrac{-2}{3}$, and the slope of the second line is $m_2 = \dfrac{3}{2}$. The slopes are not equal, so the lines are not parallel. However, the product of the slopes is

$$m_1 m_2 = \left(\frac{-2}{3}\right)\left(\frac{3}{2}\right) = -1$$

so the lines are perpendicular.

Checkpoint 9.5 Practice 2.

a. What is the slope of a line that is parallel to $x + 4y = 2$?

   Answer: _____

b. What is the slope of a line that is perpendicular to $x + 4y = 2$?

   Answer: _____

Answer 1. $\frac{-1}{4}$
Answer 2. 4
Solution.

a. $\dfrac{-1}{4}$

b. 4

Checkpoint 9.6 QuickCheck 2. True or false.

a. The best way to test whether two lines are perpendicular is to graph them. (☐ True ☐ False)

b. Two lines are perpendicular if their slopes are reciprocals. (☐ True ☐ False)

c. If two lines are not parallel, they must be perpendicular. (☐ True ☐ False)

d. A vertical line is perpendicular to a horizontal line. (☐ True ☐ False)

Answer 1. False
Answer 2. False
Answer 3. False
Answer 4. True
Solution.

a. False

b. False

c. False

d. True

Applications to Geometry

These relationships for the slopes of parallel and perpendicular lines can help us solve numerous geometric problems.

Example 9.7

Show that the triangle with vertices $A(0,8)$, $B(6,2)$, and $C(-4,4)$ is a right triangle.

Solution. We will show that two of the sides of the right triangle are perpendicular. The line segment $\overline{AB}$ has slope

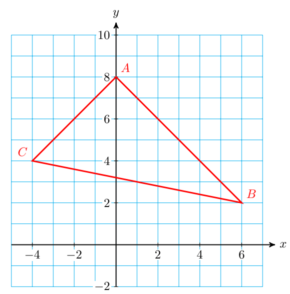
$$m_1 = \frac{2-8}{6-0} = \frac{-6}{6} = -1$$

and the segment $\overline{AC}$ has slope

$$m_2 = \frac{4-8}{-4-0} = \frac{-4}{-4} = 1$$

Because

$$m_1 m_2 = (-1)(1) = -1$$



The sides $\overline{AB}$ and $\overline{AC}$ perpendicular, and therefore the triangle is a right triangle.

Checkpoint 9.8 **Practice 3.** Show that the quadrilateral with vertices $A(-5,4)$, $B(7,-11)$, $C(12,25)$, and $D(0,40)$ is a parallelogram.

(Hint: Show that $\overline{AB}$ is parallel to $\overline{CD}$, and that $\overline{BC}$ is parallel to $\overline{AD}$.)

The slope of $\overline{AB}$ is ____, and the slope of $\overline{CD}$ is ____. These slopes are (☐ equal ☐ different) , so the sides are (☐ parallel ☐ perpendicular) .

The slope of $\overline{BC}$ is ____, and the slope of $\overline{AD}$ is ____. These slopes are (☐ equal ☐ different) , so the sides are (☐ parallel ☐ perpendicular) .

Answer 1. $\frac{-5}{4}$

Answer 2. $\frac{-5}{4}$

Answer 3. equal

Answer 4. parallel

Answer 5. $\frac{36}{5}$

Answer 6. $\frac{36}{5}$

Answer 7. equal

Answer 8. parallel

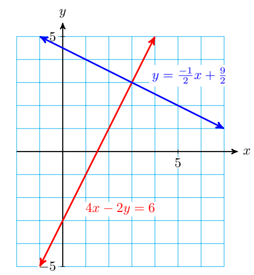
Solution. $m_{AB} = m_{CD} = \dfrac{-5}{4}$; $m_{BC} = m_{AD} = \dfrac{36}{5}$

Example 9.9

Find an equation for the line that passes through the point $(1,4)$ and is perpendicular to the line $4x - 2y = 6$.

Solution.

First we find the slope of the desired line, then use the point-slope formula to write its equation. The line we want is perpendicular to the given line, so its slope is the negative reciprocal of $m_1 = 2$, the slope of the given line. Thus



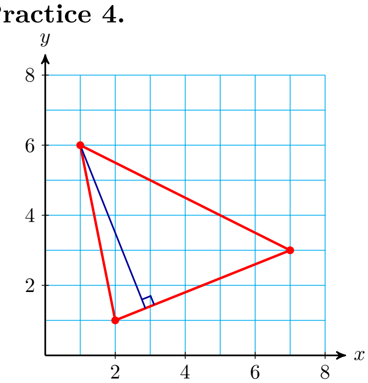
$$m_2 = \frac{-1}{m_1} = \frac{-1}{2}$$

Now we use the point-slope formula with $m = \frac{-1}{2}$ and $(x_1, y_1) = (1, 4)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= \frac{-1}{2}(x - 1) && \text{Apply the distributive law.} \\ y - 4 &= \frac{-1}{2}x + \frac{1}{2} && \text{Add 4 to both sides.} \\ y &= \frac{-1}{2}x + \frac{9}{2} && \frac{1}{2} + 4 = \frac{1}{2} + \frac{8}{2} = \frac{9}{2} \end{aligned}$$

The given line and the perpendicular line are shown in the figure on the above.

Checkpoint 9.10 Practice 4.



Find an equation for the altitude of the triangle shown above . (The altitude of a triangle is perpendicular to its base.) Find the slope of the base.

$m_1 =$ _____

Find the slope of the altitude.

$m_2 =$ _____

Use the point-slope formula. Use m_2 for the slope, and the vertex of the triangle for (x_1, y_1).

$$y - y_1 = m(x - x_1)$$

Write your answer in slope-intercept form.

$y =$________

Answer 1. $\frac{2}{5}$

Answer 2. $\frac{-5}{2}$

Answer 3. $\frac{-5}{2}x + \frac{17}{2}$

Solution. $m_1 = \frac{2}{5}$; $m_2 = \frac{-5}{2}$; $y = \frac{-5}{2}x + \frac{17}{2}$

Problem Set 9.1

Warm Up

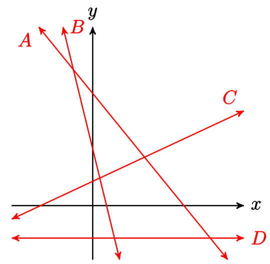
1.

a Decide whether the slope of each line is positive, negative, zero, or undefined.

A ___________ B ___________

C ___________ D ___________

b List the lines in order of increasing slope.



2. Find the slope of each line.

a $5x - 4y = 0$

b $\frac{1}{3}x - \frac{3}{4}y = -2$

c $0.8x + 0.004y = 0.24$

d $y = 12 = 0$

3. In the table below, give the **negative reciprocal** for each number. Then find the product of each number with its negative reciprocal. What do you notice?

Number	Negative reciprocal	Their product
$\frac{2}{3}$	$\frac{-3}{2}$	$\frac{2}{3}\left(\frac{-3}{2}\right) = -1$
$\frac{-5}{2}$		
6		
-4		
-1		

4. Find the slope of the line joining the points $(-6, -2)$ and $(4, 1)$.

Skills Practice

For Problems 5 and 6,

a sketch a rough graph of the equation,

b state the slope and intercept of the line.

5. $2x = 8$

6. $3y = 15$

For Problems 7–10, find an equation for the line described.

7. A vertical line through the point $(-5, 8)$.

8. The x-axis.

9. Perpendicular to $x = -2$ and intersecting it at $(-2, 6)$.

10. Parallel to the y-axis with intercept at $(-3, 0)$.

11. The slopes of several lines are given below. Which of the lines are parallel to the graph of $y = 0.75x + 2$, and which are perpendicular to it?

a $m = \frac{3}{4}$ d $m = \frac{-39}{52}$ g $m = \frac{36}{48}$

b $m = \frac{8}{6}$ e $m = \frac{4}{3}$

c $m = \frac{-20}{15}$ f $m = \frac{-16}{12}$ h $m = \frac{9}{12}$

12. In each part, determine whether the two lines are parallel, perpendicular, or neither.

a $y = \frac{3}{5}x - 7; \quad 3x - 5y = 2$

b $y = 4x + 3; \quad y = \frac{1}{4}x - 3$

c $6x + 2y = 1; \quad x = 1 - 3y$

d $2y = 5; \quad 5y = -2$

13.

a Use your calculator to graph the equations $y = 3x + 8$ and $y = 3.1x + 6$ together in the standard window. Do you think the lines are parallel?

b Find the slope of each line in part (a). Are the lines parallel?

c Find the y-value for each equation when $x = 20$. What do your answers tell you about the two lines?

14.

a Use your calculator to graph the equations

$$y = 0.625x - 3 \quad \text{and} \quad y = -1.6x + 2$$

together in the standard window. Do the lines appear to be perpendicular?

b Find the slope of each line in part (a). Are the lines perpendicular?

Applications

15.

a Sketch the triangle with vertices $P(-1, 3)$, $Q(-3, 8)$, and $R(4, 5)$.

b Show that the triangle is a right triangle. (Hint: Find the slope of each side of the triangle.)

16.

a Sketch the quadrilateral with vertices $P(2, 4)$, $Q(3, 8)$, and $R(5, 1)$, and $S(4, -3)$.

b Show that the quadrilateral is a parallelogram. (Hint: What should be true about the slopes of the opposite sides of the parallelogram?)

17.

a Put the equation $2y - 3x = 5$ into slope-intercept form, and graph the equation.

b What is the slope of any line that is parallel to $2y - 3x = 5$?

c On your graph for part (a), sketch by hand a line that is parallel to $2y - 3x = 5$ and passes through the point $(-3, 2)$.

d Use the point-slope formula to write an equation for the line that is parallel to the graph of $2y - 3x = 5$ and passes through the point $(-3, 2)$.

18.

a Put the equation $2y - 3x = 5$ into slope-intercept form, and graph the equation.

b What is the slope of any line that is perpendicular to $2y - 3x = 5$?

c On your graph for part (a), sketch by hand a line that is perpendicular to $2y - 3x = 5$ and passes through the point $(1, 4)$.

d Use the point-slope formula to write an equation for the line that is perpendicular to the graph of $2y - 3x = 5$ and passes through the point $(1, 4)$.

19. Two of the vertices of rectangle $ABCD$ are $A(-5, 2)$ and $B(-2, -4)$.

a Find an equation for the line that includes side $\overline{AB}$.

b Find an equation for the line that includes side $\overline{BC}$.

20.

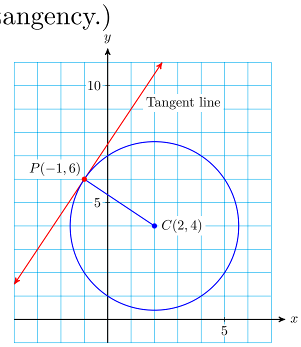
a Sketch the triangle with vertices $A(-5, 12)$, $B(4, -2)$, and $C(-1, 6)$.

b Find the slope of the side $\overline{AC}$.

c Find the slope of the altitude from point B to side $\overline{AC}$.

d Find an equation for the line that includes the altitude from point B to the side $\overline{AC}$.

21. The center of a circle is the point $C(2, 4)$, and $P(-1, 6)$ is a point on the circle. Find the equation of the line tangent to the circle at the point P. (Hint: Recall that the tangent line to a circle is perpendicular to the radius at the point of tangency.)



22. Show that the line passing through the points $A(0, -3)$ and $B\left(3, \frac{1}{2}\right)$ also passes through the point $C(-6, -10)$.

The Distance and Midpoint Formulas

Distance in a Coordinate Plane

Figure (a) shows a line segment joining the two points $(-2, 7)$ and $(6, 3)$. What is the distance between the two points?

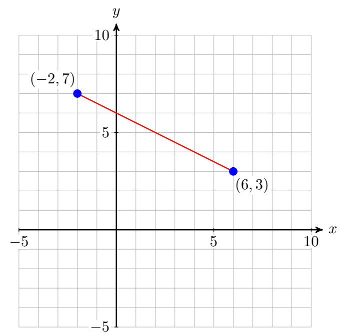


Figure a

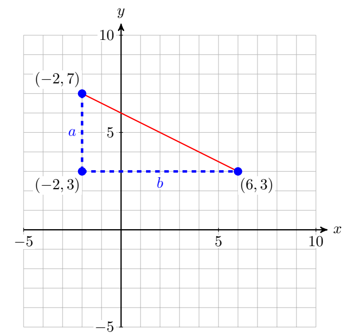


Figure b

The distance between two points is the length of the segment joining them.

If we make a right triangle as shown in Figure (b), we can use the Pythagorean theorem to find its length. First, notice that the coordinates at the right angle are $(-2, 3)$. We can find the lengths of the two legs of the triangle, because they are horizontal and vertical segments.

$$a = |3 - 7| = 4$$
$$b = \big|6 - (-2)\big| = 8$$

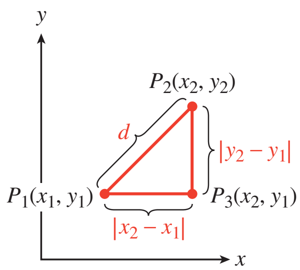
The segment we want is the hypotenuse of the right triangle, so we apply the Pythagorean theorem.

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 4^2 + 8^2 = 80 && \text{Take square roots.} \\ c &= \sqrt{80} \approx 8.9 \end{aligned}$$

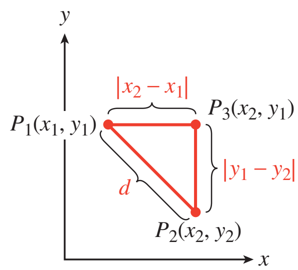
The Distance Formula

We can also use the Pythagorean theorem to derive a formula for the distance between any two points, P_1 and P_2, in terms of their coordinates. We first label a right triangle, as we did in the exmple above. Draw a horizontal line through P_1 and a vertical line through P_2.

These lines meet at a point P_3, as shown in the figure below. The x-coordinate of P_3 is the same as the x-coordinate of P_2, and the y-coordinate of P_3 is the same as the y-coordinate of P_1. Thus, the coordinates of P_3 are (x_2, y_1).



(a)



(b)

The distance between P_1 and P_3 is $|x_2 - x_1|$, and the distance between P_2 and P_3 is $|y_2 - y_1|$. These two numbers are the lengths of the legs of the right triangle. The length of the hypotenuse is the distance between P_1 and P_2, which we'll call d. By the Pythagorean theorem,

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Taking the (positive) square root of each side of this equation gives us the **distance formula**.

Distance Formula.

The **distance** d between points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Checkpoint 9.11 QuickCheck 1. True or false.

a. The distance formula is a version of the Pythagorean theorem. (☐ True ☐ False)

b. $(u - v)^2 = (v - u)^2 = |u - v|^2 = |v - u|^2$ (☐ True ☐ False)

c. We can find the length of a horizontal segment by subtracting the x-coordinates of its endpoints. (☐ True ☐ False)

d. The distance between two points is always a positive number (or zero). (☐ True ☐ False)

Answer 1. True

Answer 2. True

Answer 3. True

Answer 4. True

Solution.

a. True

b. True

c. True

d. True

Example 9.12

Find the distance between $(2, -1)$ and $(4, 3)$.

Solution. Substitute $(2, -1)$ for (x_1, y_1) and $(4, 3)$ for (x_2, y_2) in the distance formula to obtain

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4-2)^2 + \left[3 - (-1)\right]^2} \\ &= \sqrt{4+16} = \sqrt{20} \approx 4.47 \end{aligned}$$

It doesn't matter which point we call P_1 and which is P_2. We obtain the same answer in the previous Example, p. 615 if we switch the two points and use $(4, 3)$ for P_1 and $(2, -1)$ for P_2:

$$\begin{aligned} d &= \sqrt{(2-4)^2 + [(-1) - 3]^2} \\ &= \sqrt{4+16} = \sqrt{20} \end{aligned}$$

Caution 9.13 We cannot simplify $\sqrt{4+16}$ as $\sqrt{4} + \sqrt{16}$. Remember that $\sqrt{a^2+b^2} \neq a + b$. You can easily see this by observing that

$$\sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{5}$$

so it cannot be true that $\sqrt{3^2+4^2}$ equals $3+4$, or 7. For the same reason, we cannot simplify the distance formula to $(x_2 - x_1) + (y_2 - y_1)$.

Checkpoint 9.14 Practice 1.

a. Find the distance between the points $(-5, 3)$ and $(3, -9)$: ________

b. Plot the points, and illustrate how the Pythagorean theorem is used in calculating the distance.

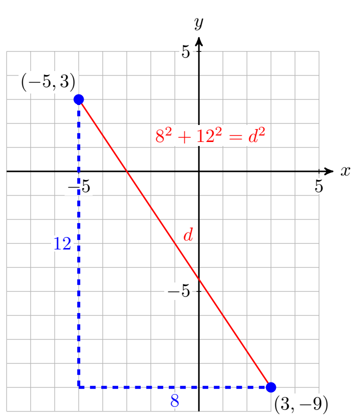
Answer. $\sqrt{208}$

Solution.

a. $\sqrt{208}$

b. A graph is below.

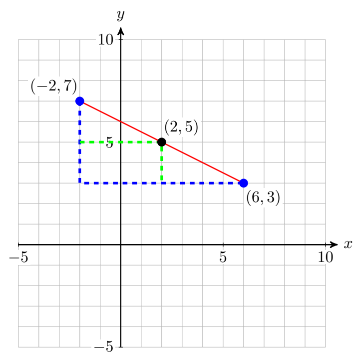
Graph for Practice 1:



Finding the Midpoint

The **midpoint** of a segment is the point halfway between its endpoints, so that the distance from the midpoint to either endpoint is the same. The

x-coordinate of the midpoint is halfway between the x-coordinates of the endpoints, and likewise for the y-coordinate. For the points $(-2, 7)$ and $(6, 3)$ shown below, the x-coordinate of the midpoint is 2, which is halfway between -2 and 6. The y-coordinate is halfway between 7 and 3, or 5. Thus, the midpoint is $(2, 5)$.



The Midpoint Formula

If we know the coordinates of two points, we can calculate the coordinates of the midpoint. Each coordinate of the midpoint is the average of the corresponding coordinates of the two points.

Midpoint Formula.

The **midpoint** of the line segment joining the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is the point $M(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{x_1 + x_2}{2} \qquad \text{and} \qquad \overline{y} = \frac{y_1 + y_2}{2}$$

Example 9.15

Find the midpoint of the line segment joining the points $(-2, 1)$ and $(4, 3)$.

Solution. We substitute $(-2, 1)$ for (x_1, y_1) and $(4, 3)$ for (x_2, y_2) in the midpoint formula to obtain

$$\overline{x} = \frac{x_1 + x_2}{2} = \frac{-2 + 4}{2} = 1$$
$$\overline{y} = \frac{y_1 + y_2}{2} = \frac{1 + 3}{2} = 2$$

The midpoint of the segment is the point $(\overline{x}, \overline{y}) = (1, 2)$.

Checkpoint 9.16 Practice 2.

a. Find the midpoint of the line joining the points $(-5, 3)$ and $(3, -9)$: ____

b. Plot the points and draw a rectangle with the points as opposite vertices. Illustrate that the midpoint is the center of the rectangle.

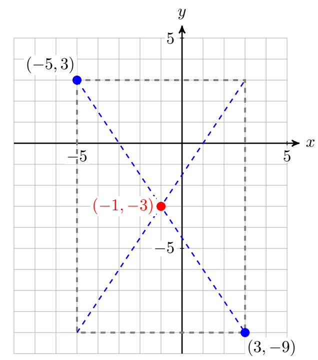
Answer. $(-1, -3)$

Solution.

a. $(-1, -3)$

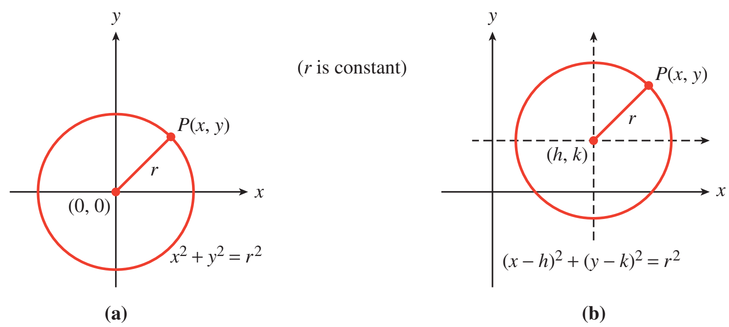
b. A figure is below.

Graph for Practice 2:



Circles

A **circle** is the set of all points in a plane that lie at a given distance, called the **radius**, from a fixed point called the **center**. We can use the distance formula to find an equation for a circle. First consider the circle in Figure (a), whose center is the origin, $(0, 0)$.



The distance from the origin to any point $P(x, y)$ on the circle is r. Therefore,

$$\sqrt{(x-0)^2 + (y-0)^2} = r$$

or, squaring both sides,

$$(x-0)^2 + (y-0)^2 = r^2$$

Thus, the equation for a circle of radius r centered at the origin is

$$x^2 + y^2 = r^2$$

Now consider the circle in Figure (b), whose center is the point (h, k). Every point $P(x, y)$ on the circle lies a distance r from (h, k), so the equation of the circle is given by the following formula.

Standard Form for a Circle.

The equation for a **circle** of radius r centered at the point (h, k) is

$$(x-h)^2 + (y-k)^2 = r^2$$

Checkpoint 9.17 QuickCheck 3.

a. Does the center of a circle lie on the graph of the circle? (☐ Yes ☐ No)

b. What is the radius of the circle $x^2 + y^2 = 81$? __

c. What is the diameter of the circle in part (b)? __

d. Can we simplify the equation in part (b) to $x + y = 9$? (☐ Yes ☐ No)

Answer 1. No
Answer 2. 9
Answer 3. 18
Answer 4. No
Solution.

a. No

b. 9

c. 18

d. No

The equation $(x - h)^2 + (y - k)^2 = r^2$ is the **standard form** for a circle of radius r with center at (h, k). It is easy to graph a circle if its equation is given in standard form.

Example 9.18

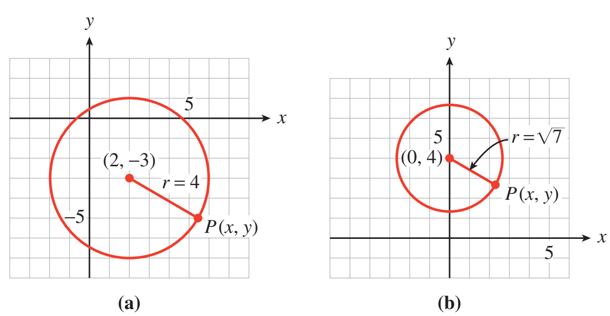
Graph the circles.

a $(x - 2)^2 + (y + 3)^2 = 16$ b $x^2 + (y - 4)^2 = 7$

Solution.

a The graph of $(x - 2)^2 + (y + 3)^2 = 16$ is a circle with radius 4 and center at $(2, -3)$. To sketch the graph, we first locate the center of the circle. (The center is not part of the graph of the circle.)

From the center, we move a distance of 4 units (the radius of the circle) in each of four directions: up, down, left, and right. This locates four points that lie on the circle: $(2, 1)$, $(2, -7)$, $(-2, -3)$, and $(6, -3)$. We sketch the circle through these four points.



b The graph of $x^2 + (y - 4)^2 = 7$ is a circle with radius $\sqrt{7}$ and center at $(0, 4)$. From the center, we move approximately $\sqrt{7}$, or 2.6 units in each of the four coordinate directions to obtain the points $(0, 6.6)$, $(0, 1.4)$, $(-2.6, 4)$, and $(2.6, 4)$. We sketch the circle through these four points.

Checkpoint 9.19 Practice 3.

a. State the center and radius of the circle

$$(x+3)^2 + (y+2)^2 = 16$$

center: ______, radius: __

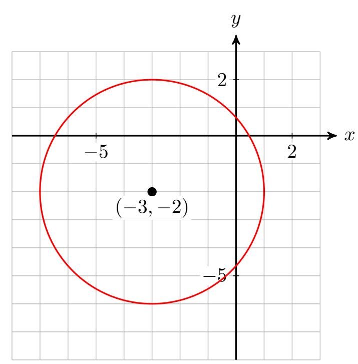
b. Graph the circle.

Answer 1. $(-3, -2)$
Answer 2. 4
Solution.

a. center $(-3, -2)$, radius 4

b. A graph is below.

$(x+3)^2 + (y+2)^2 = 16$



General Form for Circles

The equations of circles often appear in a **general quadratic form**, rather than the standard form described above. For example, we can expand the squares of binomials in part (a) of the previous Example, p. 619,

$$(x-2)^2 + (y+3)^2 = 16$$

to obtain

$$x^2 - 4x + 4 + y^2 + 6y + 9 = 16$$

or

$$x^2 + y^2 - 4x + 6y - 3 = 0$$

This is a quadratic equation in two variables. Such an equation describes a circle if the coefficients of the quadratic, or squared, terms are equal.

Conversely, an equation of the form $x^2 + y^2 + ax + by + c = 0$ can be converted to standard form by completing the square in both variables. Once this is done, the center and radius of the circle can be determined directly from the equation.

Example 9.20

Write the equation of the circle

$$x^2 + y^2 + 8x - 2y + 6 = 0$$

in standard form, and graph the equation.

Solution. We prepare to complete the square in both variables by writing the equation as

$$(x^2 + 8x + \_\_\_\_\_\_) + (y^2 - 2y + \_\_\_\_\_\_) = -6$$

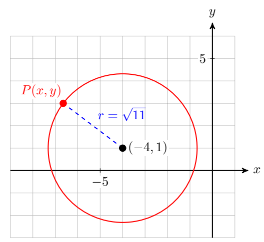
We complete the square in x by adding 16 to each side of the equation, and complete the square in y by adding 1 to each side, to get

$$(x^2 + 8x + \mathbf{16}) + (y^2 - 2y + \mathbf{1}) = -6 + \mathbf{16} + \mathbf{1}$$

from which we obtain the standard form,

$$(x + 4)^2 + (y - 1)^2 = 11$$

Thus, the circle has its center at $(-4, 1)$, and its radius is $\sqrt{11}$, or approximately 3.3. The graph is shown at right.



Checkpoint 9.21 QuickCheck 4. Fill in the blanks.

a. The standard form for a circle is useful because we can see the (☐ area and circumference ☐ center and radius ☐ diameter and arclength) of the circle in the equation.

b. A quadratic equation in two variables describes a circle if the coefficients of the quadratic terms are (☐ squared ☐ opposites ☐ equal ☐ zero) .

c. We convert the equation for a circle into standard form by (☐ completing the square ☐ cross-multiplying ☐ factoring) in both variables.

Answer 1. center and radius

Answer 2. equal

Answer 3. completing the square

Solution.

a. center and radius

b. equal

c. completing the square

Checkpoint 9.22 Practice 4. Write the equation of the circle $x^2 + y^2 - 14x + 4y + 25 = 0$ in standard form:

$$(x - h)^2 + (y - k)^2 = r^2$$

$h =$__, $k =$__, $r =$________,

Answer 1. 7

Answer 2. -2

Answer 3. $\sqrt{28}$

Solution. $(x - 7)^2 + (y + 2)^2 = 28$

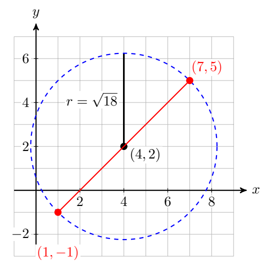
We can write an equation for any circle if we can find its center and radius.

Example 9.23

Find an equation for the circle whose diameter has endpoints $(7, 5)$ and $(1, -1)$.

Solution.

The center of the circle is the midpoint of its diameter. We use the midpoint formula to find the center:



$$h = \overline{x} = \frac{7+1}{2} = 4$$
$$k = \overline{y} = \frac{5-1}{2} = 2$$

Thus, the center is the point $(h, k) = (4, 2)$.

The radius is the distance from the center to either of the endpoints of the diameter, say the point $(7, 5)$. We use the distance formula with the points $(7, 5)$ and $(4, 2)$ to find the radius.

$$r = \sqrt{(7-4)^2 + (5-2)^2}$$
$$= \sqrt{3^2 + 3^2} = \sqrt{18}$$

Finally, we substitute 4 for$h$ and 2 for k (the coordinates of the center) and $\sqrt{18}$ for r (the radius) into the standard form

$$(x-h)^2 + (y-k)^2 = r^2$$

to obtain

$$(x-4)^2 + (y-2)^2 = 18$$

Checkpoint 9.24 Practice 5. Find an equation for the circle whose diameter has endpoint $(-7, -5)$ and $(-1, -1)$.

Answer. $\big(x - (-4)\big)^2 + \big(y - (-3)\big)^2 = 13$

Solution. $(x+4)^2 + (y+3)^2 = 13$

Problem Set 9.2

Warm Up

1. Choose values for x and y to decide whether the statements are true.

a $\sqrt{x^2 + y^2} = x + y$

b $(x+y)^2 = x^2 + y^2$

2. Leanne is sailing 3 miles west and 5 miles south of the harbor. She heads directly towards an island that is 8 miles west and 7 miles north of the harbor.

a How far is Leanne from the island?

b How far will Leanne be from the harbor when she is halfway to the island?

For Problems 3 and 4, complete the table of values, then sketch the graph.

3.

$x^2 + y^2 = 16$

x	-4	-3	-2	-1	0	1	2	3	4
y									

4. $(x-2)^2+(y+2)^2 = 9$

x	-1	0	1	2	3	4	5
y							

For Problems 5 and 6, solve by completing the square.

5. $x^2 - 5x - 2 = 0$

6. $x^2 + 6x = 11$

Skills Practice

For Problems 7–10, find the distance between each of the given pairs of points, and find the midpoint of the segment joining them.

7. $(2, -3)$, $(-2, -1)$

8. $(1, 1)$, $(4, 5)$

9. $(-2, -5)$, $(-2, 3)$

10. $(5, -4)$, $(-1, 1)$

For Problems 11–14, state the center and radius of the circle.

11. $2x^2 + 2y^2 = 50$

12. $x^2 + y^2 = 16$

13. $(x + 3)^2 + y^2 = 10$

14. $(x - 4)^2 + (y + 2)^2 = 9$

Applications

15. Find the perimeter of the triangle with vertices $(-1, 5)$, $(8, -7)$, $(4, 1)$. (Hint: The length of a side is the distance between its endpoints.)

16. Show that the points $(-2, 1)$, $(0, -1)$, and $(\sqrt{3} - 1, \sqrt{3})$ are the vertices of an equilateral triangle. (Hint: Use the distance formula to show that the lengths of the three sides are equal.)

17. The points $(1, 6)$, $(5, 2)$, $(-2, 3)$, and $(2, -1)$ are the vertices of a quadrilateral. Show that its diagonals are of equal length.

18. Two opposite vertices of a square are $A(-9, -5)$ and $C(3, 3)$.

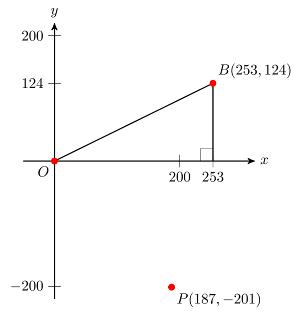
a Find the length of a diagonal of the square.

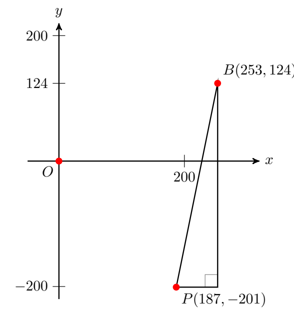
b Find the length of the side of the square.

19. Find the equation of the perpendicular bisector of the line segment joining $A(2, 1)$ and $B(1, 3)$.

20. Brie took off from Oldfield Airport in a small plane and is now 253 miles east and 124 miles north of the airport. She knows that Preston Airport is 187 miles east and 201 miles south of Oldfield Airport. Which airport is closer to Brie's present location?

a Use the diagram and the Pythagorean theorem to find the distance from Brie's position to Oldfield Airport.



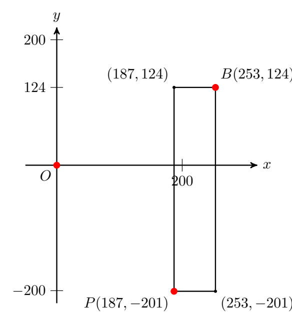


b Use the diagram and the distance formula to find the distance from Brie's position to Preston Airport.

c Answer the question in the problem.

21. Suppose that Brie flies from her present position at (253, 124) towards Preston Airport at $(187, -201)$.

a What are her coordinates when she is halfway to Preston?



b Show that the distance from Brie's present position to the halfway point is the same as the distance from the halfway point to Preston Airport.

For Problems 22-27, graph the equation.

22. $x^2 + y^2 = 25$

23. $4x^2 + 4y^2 = 16$

24. $(x - 4)^2 + (y + 2)^2 = 9$

25. $(x + 3)^2 + y^2 = 10$

26. $(x - 1)^2 + (y - 3)^2 = 16$

27. $x^2 + (y + 4)^2 = 12$

For Problems 28–31, write the equation in standard form. State the center and radius of the circle

28. $x^2 + y^2 + 2x - 4y - 6 = 0$

29. $x^2 + y^2 + 8x = 4$

30. $x^2 + y^2 - 6x + 2y - 4 = 0$

31. $x^2 + y^2 - 10y = 2$

For Problems 32–35, write an equation for the circle with the given properties.

32. Center at $(-2, 5)$, radius $2\sqrt{3}$.

33. Center at $\left(\dfrac{3}{2}, -4\right)$, one point on the circle is $(4, -3)$.

34. Endpoints of a diameter at $(1, 5)$ and $(3, -1)$.

35. Center at $(-3, -1)$, the x-axis is tangent to the circle.

36. Find an equation for the circle that passes through the points $(2,3)$, $(3,2)$, and $(-4,-5)$. (Hint: Find values for a, b, and c so that the three points lie on the graph of $x^2+y^2+ax+by+c=0$.)

Conic Sections: Ellipses

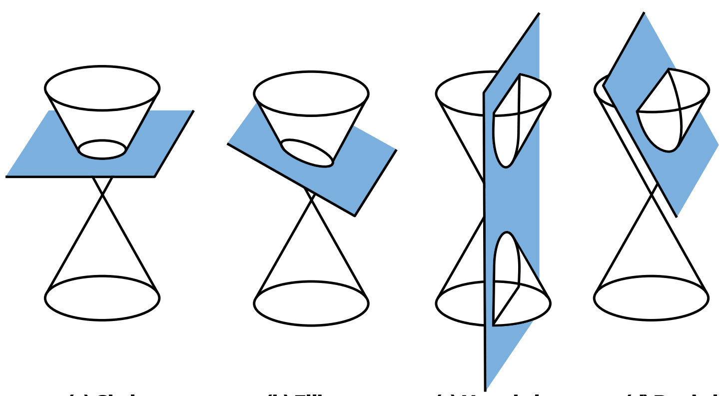
The graph of any first-degree equation in two variables,

$$Ax + By = C$$

is a line (as long as A and B are not both 0). A second-degree equation in two variables has the general form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where A, B, and C cannot all be zero (because in that case the equation would not be second degree). The graphs of such equations are curves called **conic sections** because they are formed by the intersection of a plane and a cone, as illustrated below. Except for a few special cases called **degenerate conics**, the conic sections fall into four categories called **circles**, **ellipses**, **hyperbolas**, and **parabolas**.



(a) Circle **(b) Ellipse** **(c) Hyperbola** **(d) Parabola**

Conic sections whose centers (or vertices, in the case of parabolas) are located at the origin are called **central conics**.

Circles and Ellipses

The circle is the most familiar of the conic sections. Recall that the standard equation for a circle of radius, r, centered at the point (h,k) is:

$$(x-h)^2 + (y-k)^2 = r^2$$

and a circle whose center is the origin has equation

$$x^2 + y^2 = r^2$$

If we divide through by r^2, we can also write this equation in the form

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

Notice that the denominators of both the x^2- and y^2 -terms are r^2. You can check that the x- and y-intercepts of this circle are $(0,\pm r)$ and $(\pm r,0)$.

If the denominators of the x-squared and y-squared terms are not equal, the graph is called an **ellipse**. An ellipse is an elongated circle, or oval. Ellipses appear in a variety of applications. The orbits of the planets and of satellites about the earth are ellipses. The arches in some bridges are elliptical in shape, and whispering domes, such as the ceiling of the Mormon Tabernacle in Salt Lake City, are made from ellipses.

Recall that a circle is the set of all points in a plane that lie at a fixed distance from its center. An ellipse also has a geometric definition.

Definition 9.25 Ellipse.

An **ellipse** is the set of points in the plane, the sum of whose distances from two fixed points (the **foci**) is a constant.

Using the distance formula and the definition above, we can show that the equation of an ellipse centered at the origin has the following standard form.

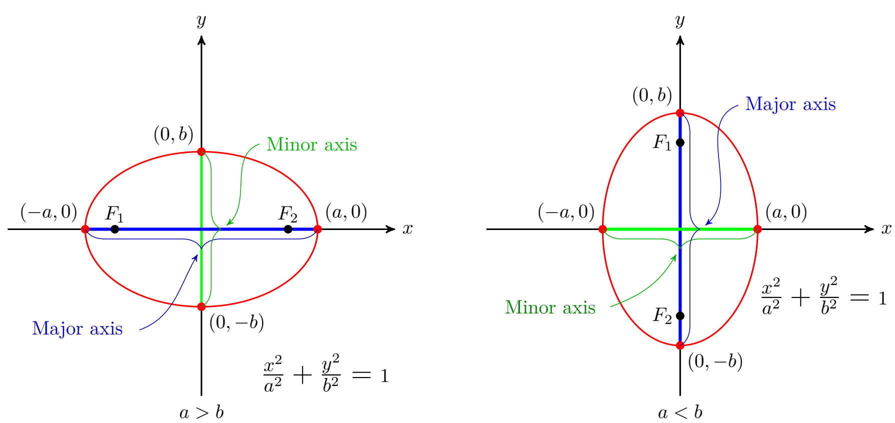
Central Ellipse.

The equation of an **ellipse** centered at the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

By setting y equal to zero in the equation above, we find that the x-intercepts of this ellipse are a and $-a$; by setting x equal to zero, we find that the y-intercepts are b and $-b$.

The line segment that passes through the foci (labeled F_1 and F_2 on the graphs below) and ends on the ellipse is called the **major axis**. If $a > b$, the major axis is horizontal, as shown in the Figure below left. The x-intercepts are the endpoints of the major axis, so its length is $2a$. The vertical segment with length $2b$ is called the **minor axis**. The endpoints of the major axis are the **vertices** of the ellipse and the endpoints of the minor axis are the **covertices**.



If $a < b$, the major axis is vertical and has length $2b$. In this case the endpoints of the major axis are the y-intercepts of the ellipse. (See Figure above right.) The minor axis is horizontal and has length $2a$.

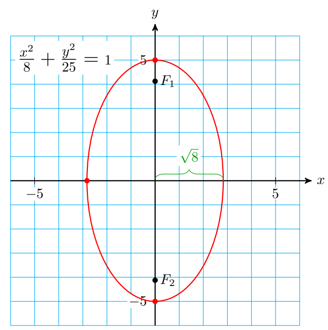
The standard form of the equation for an ellipse gives us enough information to sketch its graph.

Example 9.26

Graph $\dfrac{x^2}{8} + \dfrac{y^2}{25} = 1$

Solution. The graph is an ellipse with major axis on the y-axis. Because $a^2 = 8$ and $b^2 = 25$, the vertices are located at $(0, 5)$ and $(0, -5)$, and the covertices lie $\sqrt{8}$ units to the right and left of the center, or approximately at $(2.8, 0)$ and $(-2.8, 0)$.

To sketch the ellipse, we first locate the vertices and covertices. Then we draw a smooth curve through the points. The graph of $\dfrac{x^2}{8} + \dfrac{y^2}{25} = 1$ is shown below.



Checkpoint 9.27 Practice 1.

a. Find the intercepts of the graph of $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$. Separate different intercepts with a comma.

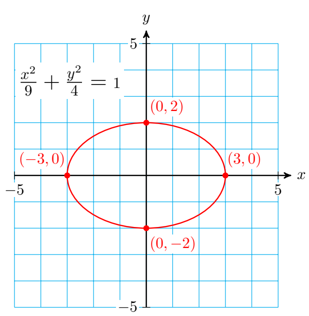
b. Graph the ellipse.

Answer. $(-3, 0), (3, 0), (0, -2), (0, 2)$

Solution.

a. $(3, 0)$, $(-3, 0)$, $(0, 2)$, $(0, -2)$

b. A graph is below.



Checkpoint 9.28 QuickCheck 1.

a. To sketch an ellipse, we locate (☐ two foci ☐ center and radius ☐ vertices and covertices ☐ slope and intercepts) , then draw a smooth curve through them.

b. If $a > b$, is the major axis horizontal or vertical? (☐ horizontal ☐ vertical)

c. The y-intercepts of a central ellipse lie (☐ a ☐ b) units from the center.

d. An ellipse is one of the four (☐ conic sections ☐ quadrants ☐ formulas ☐ solutions) .

Answer 1. vertices and covertices

Answer 2. horizontal

Answer 3. b

Answer 4. conic sections

Solution.

a. vertices and covertices

b. horizontal

c. b

d. conic sections

The equation of any central ellipse may be written as

$$Ax^2 + By^2 = C$$

where A,B, and C the same sign. The features of the graph are easier to identify if we first convert the equation to standard form.

Example 9.29

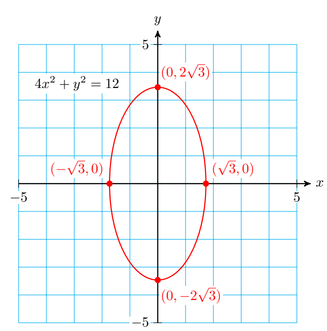
Graph $4x^2 + y^2 = 12$

Solution.

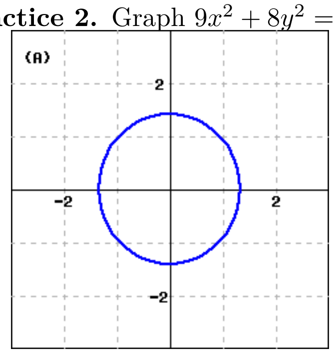
First we convert the equation to standard form: we divide through by the constant term, 12, to obtain

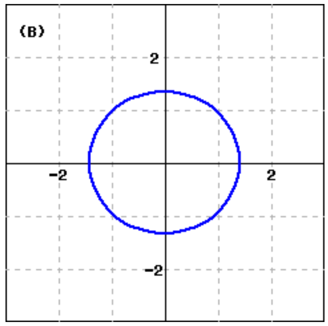
$$\frac{x^2}{3} + \frac{y^2}{12} = 1$$

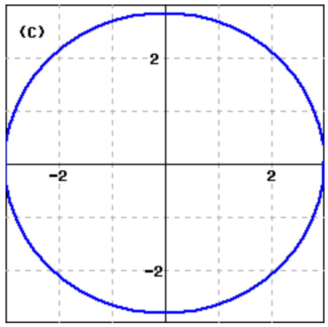
Because $a^2 = 3$ and $b^2 = 12$, the vertices are $(0, \pm 2\sqrt{3})$ and the covertices are $(\pm\sqrt{3}, 0)$. We plot points at about $(0, \pm 3.5)$ and $(\pm 1.7, 0)$, then draw an ellipse through the points, as shown at right.

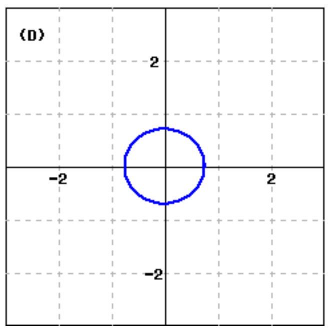


Checkpoint 9.30 **Practice 2.** Graph $9x^2 + 8y^2 = 16$









Which of the above is the best match for the graph?

⊙ (A)

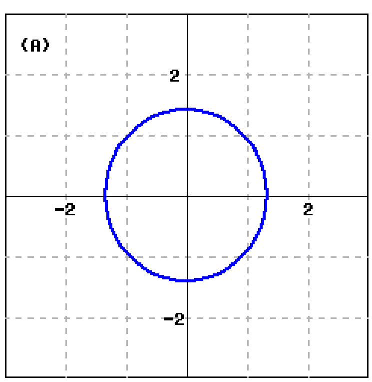
⊙ (B)

⊙ (C)

⊙ (D)

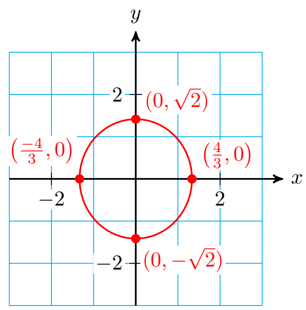
Answer. (A)

Solution.



A graph is also shown below.

$9x^2 + 8y^2 = 16$



We can find coordinates of other points on an ellipse by substituting a value for one variable and solving for the other variable.

Example 9.31

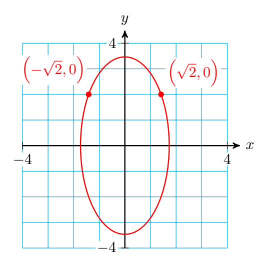
a Find the exact coordinates of any points with y-coordinate 2 on the ellipse $4x^2 + y^2 = 12$. Plot and label those points on the ellipse.

b Solve the equation $4x^2 + y^2 = 12$ when $y = -4$. What do the solutions tell you about the graph of the ellipse?

Solution.

a We substitute $y = \mathbf{2}$ into the equation and solve for x.

$$\begin{aligned} 4x^2 + (\mathbf{2})^2 &= 12 \\ 4x^2 &= 8 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$



There are two points with $y = 2$, namely $\left(\sqrt{2}, 2\right)$ and $\left(-\sqrt{2}, 2\right)$

b We substitute $y = \mathbf{-4}$ into the equation and solve for x.

$$\begin{aligned} 4x^2 + (\mathbf{-4})^2 &= 12 \\ 4x^2 &= -4 \\ x^2 &= -1 \end{aligned}$$

Because there are no real solutions, there are no points on the ellipse with $= -4$.

Checkpoint 9.32 Practice 3. Find the exact coordinates of all points with y-coordinate -1 on the ellipse $9x^2 + 8y^2 = 16$.

Answer. $\left(\frac{2\sqrt{2}}{3}, -1\right), \left(\frac{-2\sqrt{2}}{3}, -1\right)$

Solution. $\left(\dfrac{2\sqrt{2}}{3}, -1\right), \left(\dfrac{-2\sqrt{2}}{3}, -1\right)$

Translated Ellipses

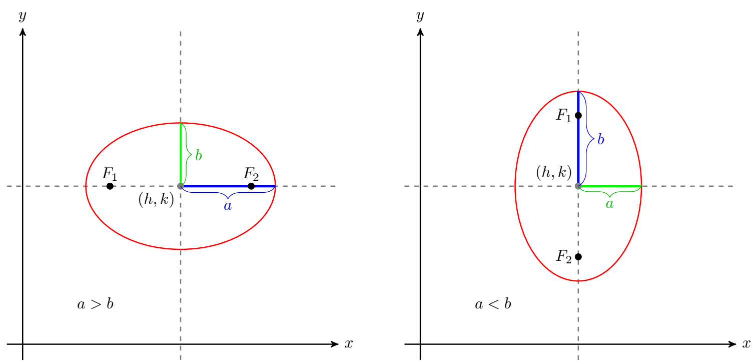
An ellipse whose center is at the point (h, k) instead of the origin is said to be shifted or **translated** to that location.

Ellipse.

The standard equation for an ellipse centered at (h, k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

The horizontal axis of the ellipse has length $2a$, and the vertical axis has length $2b$, the same as for central ellipses. When $a > b$, the major axis is horizontal and the ellipse is short and wide. When $a < b$, the major axis is vertical and the ellipse is tall and narrow, as shown below.



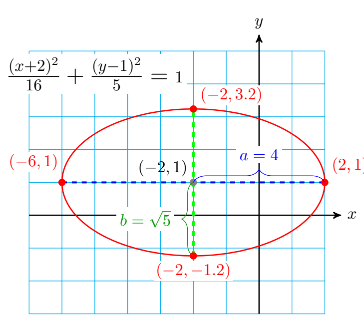
Example 9.33

a Graph $\dfrac{(x+2)^2}{16} + \dfrac{(y-1)^2}{5} = 1$

b Find the exact coordinates of the intercepts of the graph.

Solution.

a The graph is an ellipse with center at $(-2, 1)$. We have $a = 4$ and $b = \sqrt{5}$, and the major axis is parallel to the x-axis because $a > b$. We plot the vertices four units to the left and right of the center, a $(-6, 1)$ and $(2, 1)$. The covertices lie $\sqrt{5}$ units above and below the center, at approximately $(-2, 3.2)$ and $(-2, -1.2)$. The graph is shown below.



b We set $y = \mathbf{0}$ and solve the resulting equation to find the x-

intercepts.

$$\begin{aligned} \frac{(x+2)^2}{16} + \frac{(\mathbf{0}-1)^2}{5} &= 1 && \text{Subtract } \frac{1}{5} \text{ from both sides.} \\ \frac{(x+2)^2}{16} &= \frac{4}{5} && \text{Multiply both sides by 16.} \\ (x+2)^2 &= \frac{64}{5} && \text{Extract roots.} \\ x+2 &= \pm\sqrt{\frac{64}{5}} \\ x &= -2 \pm \frac{8\sqrt{5}}{5} \end{aligned}$$

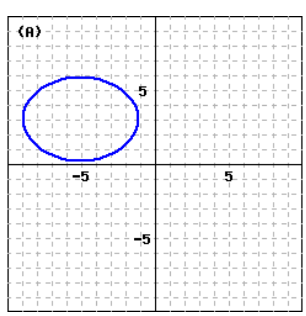
The x-intercepts are $\left(-2 \pm \frac{8\sqrt{5}}{5}, 0\right)$ or approximately $(1.6, 0)$ and $(-5.6, 0)$. We set $x = \mathbf{0}$ to find the y-intercepts.

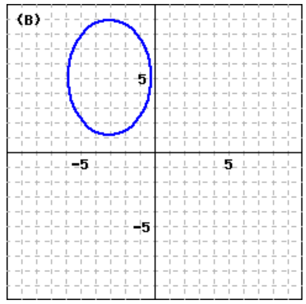
$$\begin{aligned} \frac{(\mathbf{0}+2)^2}{16} + \frac{(y-1)^2}{5} &= 1 && \text{Subtract } \frac{1}{4} \text{ from both sides.} \\ \frac{(y-1)^2}{5} &= \frac{3}{4} && \text{Multiply both sides by 5.} \\ (y-1)^2 &= \frac{15}{4} && \text{Extract roots.} \\ y-1 &= \pm\sqrt{\frac{15}{4}} \\ y &= 1 \pm \frac{\sqrt{15}}{2} \end{aligned}$$

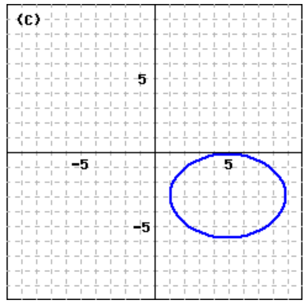
The y-intercepts are $\left(0, 1 \pm \frac{\sqrt{15}}{4}\right)$ or approximately $(0, 2.9)$ and $(0, -0.9)$

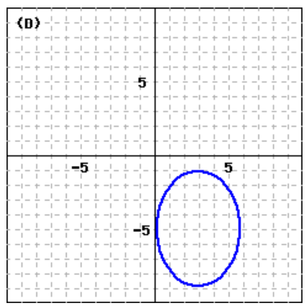
Checkpoint 9.34 Practice 4.

a. Graph $\dfrac{(x-5)^2}{15} + \dfrac{(y+3)^2}{8} = 1$









Which of the above is the best match for the graph?

- ⊙ (A)
- ⊙ (B)
- ⊙ (C)
- ⊙ (D)

b. Find the coordinates of the vertices and covertices. Separate different points with a comma. Use "sqrt(2)" to get $\sqrt{2}$.

vertices: ________________________

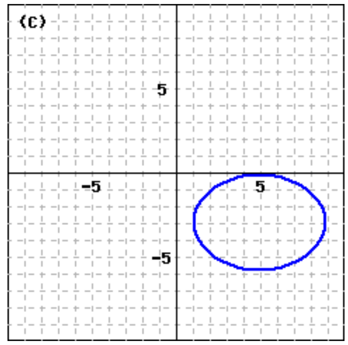
covertices: ________________________

Answer 1. (C)

Answer 2. $(8.87298, -3), (1.12702, -3)$

Answer 3. $(5, -0.171573), (5, -5.82843)$

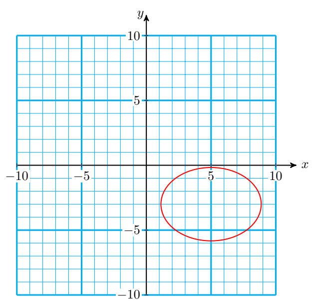
Solution.

a. 

A graph is also shown below.

b. vertices: $(5 - \sqrt{15}, -3)$, $(5 + \sqrt{15}, -3)$
covertices: $(5, -3 + \sqrt{8})$, $(5, -3 - \sqrt{8})$

$\dfrac{(x-5)^2}{15} + \dfrac{(y+3)^2}{8} = 1$:



Checkpoint 9.35 QuickCheck 2.

a. To write the equation $Ax^2 + By^2 = C$ in the standard form for ellipses, we divide through by __

b. The x-coordinates of all points on a central ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ lie between and __

c. In the standard form for an ellipse, (h, k) are the coordinates of the (☐ center ☐ vertex ☐ covertex ☐ intercept) of the ellipse.

d. The vertical axis of an ellipse $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ has length __.

Answer 1. C

Answer 2. $-a$

Answer 3. a

Answer 4. center

Answer 5. $2b$

Solution.

a. C

b. $-a$ and a

c. center

d. $2b$

Writing in Standard Form

Second-degree equations in which the coefficients of x^2 and y^2 have the same sign can be written in one of the standard forms for an ellipse by completing the square. The equation can be graphed easily from the standard form.

Example 9.36

a Write the equation in standard form.

$$4x^2 + 9y^2 - 16x - 18y - 11 = 0$$

b Graph the equation.

Solution.

a We first prepare to complete the square in both x and y. Begin by factoring out the coefficients of x^2 and y^2.

$$\mathbf{4}(x^2 - 4x\_\_\_\_) + \mathbf{9}(y^2 - 2y\_\_\_\_) = 11$$

We complete the square in x by adding **4** to $x^2 - 4x$, and adding $4 \cdot \mathbf{4}$, or 16, to the right side of the equation. We complete the square in y by adding **1** to $y^2 - 2y$, and $9 \cdot \mathbf{1}$, or 9, to the right side.

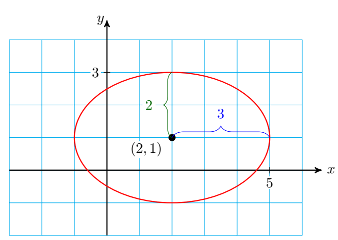
$$4(x^2 - 4x + 4) + 9(y^2 - 2y + 1) = 11 + 16 + 9$$

We write each term on the left side as a perfect square to get

$$4(x-2)^2 + 9(y-1)^2 = 36 \qquad \text{Divide both sides by 36.}$$

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{4} = 1$$

b The graph is an ellipse with center at $(2, 1)$, $a^2 = 9$, and $b^2 = 4$ The vertices lie 3 units to the right and left of the center at $(5, 1)$ and $(-1, 1)$; the covertices lie 2 units above and below the center at $(2, 3)$ and $(2, -1)$. The graph is shown below.



Caution 9.37 When completing the square in the Example above, do not forget the coefficients you factored out in the first step. When we add 4 to complete the square in x, it is multiplied by a factor of 4, so we must add $4 \cdot 4$ or 16 to the right side of the equation. Similarly, we must add $9 \cdot 1$ or 9 to the right side when we complete the square in x.

Checkpoint 9.38 Practice 5.

a. Write the equation $x^2 + 4y^2 + 4x - 16y + 4 = 0$ in standard form: $\dfrac{(x-h)^2}{a^2} +$

$$\frac{(y-k)^2}{b^2} = 1$$

$h = \_\_$, $k = \_\_$, $a^2 = \_\_$, $b^2 = \_\_$

b. Graph the equation.

Answer 1. -2

Answer 2. 2

Answer 3. 16

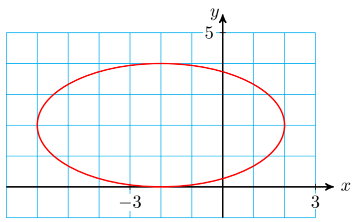
Answer 4. 4

Solution.

a. $\dfrac{(x+2)^2}{16} + \dfrac{(y-2)^2}{4} = 1$

b. A graph is below.

$x^2 + 4y^2 + 4x - 16y + 4 = 0$:



Finding the Equation of an Ellipse

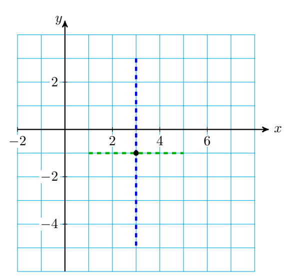
To write the equation of an ellipse from a description of its properties, we must find the center of the ellipse and the lengths of its axes. We can then substitute this information into the standard form.

Example 9.39

Find the equation of the ellipse with vertices at $(3, 3)$ and $(3, -5)$ and covertices at $(1, -1)$ and $(5, -1)$.

Solution.

You may find it helpful to plot the given points to help you visualize the ellipse. The center of the ellipse is the midpoint of the major (or minor) axis.



$$h = \overline{x} = \frac{1+5}{2} = 3$$
$$k = \overline{y} = \frac{3-5}{2} = -1$$

Thus, the center is the point $(3, -1)$. The horizontal axis is shorter, and a is the distance between the center and either covertex, say $(5, -1)$ Thus,

$$a = 5 - 3 = 2$$

The value of b is the distance from the center to one of the vertices, say

(3, 3):

$$b = 3 - (-1) = 4$$

The equation of the ellipse has the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

The equation of the ellipse has $h = 3$, $k = -1$, $a = 2$, $b = 4$. Thus the equation is

$$\frac{(x-3)^2}{4} + \frac{(y+1)^2}{16} = 1$$

If we clear this equation of fractions and expand the powers, we obtain the general form

$$\begin{aligned} 4(x-3)^2 + (y+1)^2 &= 16 && \text{Square each binomial.} \\ 4(x^2 - 6x + 9) + (y^2 + 2y + 1) &= 16 && \text{Apply the distributive law..} \\ 4x^2 - 24x + 36 + y^2 + 2y + 1 &= 16 \\ 4x^2 + y^2 - 24x + 2y + 21 &= 0 \end{aligned}$$

Checkpoint 9.40 Practice 6. Find the equation of the ellipse with vertices at $(-7, 1)$ and $(-1, 1)$ and a minor axis of length 4.

Answer. $\frac{(x-(-4))^2}{9} + \frac{(y-1)^2}{4} = 1$

Solution. $\dfrac{(x+4)^2}{9} + \dfrac{(y-1)^2}{4} = 1$

Problem Set 9.3

Warm Up

1.

a Write the **standard form** for a circle.

b Divide both sides of this equation by r^2.

c Compare the new equation with the standard form for an ellipse. What do you notice?

2. Graph the circle $\dfrac{x^2}{4} + \dfrac{y^2}{4} = 1$

3. Solve the system $\begin{aligned} 3x + 2y &= 6 \\ -4x - 3y &= -10 \end{aligned}$

4. Find all points with x-coordinate 1 on the graph of

$$9(x-3)^2 + 4(y-3)^2 = 36$$

5. Find the x-intercepts and the vertex of the graph of $y = -0.5(x+6)(x-4)$

6. Write an equation for the circle with center $(-4, 5)$ and radius 8.

Skills Practice

For problems 7–12, graph the circle or ellipse.

7. $4x^2 = 16 - 4y^2$ **8.** $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$ **9.** $\dfrac{x^2}{10} + \dfrac{y^2}{25} = 1$

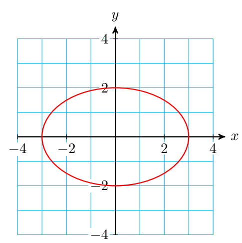
10. $3x^2 + 4y^2 = 36$ **11.** $x^2 = 36 - 9y^2$ **12.** $3y^2 = 30 - 2x^2$

For problems 13 and 14

a Fiind the equation of the ellipse.

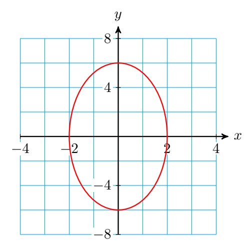
b Use your equation to complete the table.

13.



x	0		−2	
y		0		1

14.



x	0		1	
y		0		−4

For problems 15-18, each equation is a circle or ellipse.

a State the radius or the lengths of the axes of the graph.

b Give the exact coordinates of all points on the graph with the given x- or y-coordinate.

15. $y^2 = 4 - x^2;\quad x = -1$ **16.** $y^2 = 6 - 4x^2;\quad y = 2$

17. $4x^2 = 12 - 2y^2;\quad x = 4$ **18.** $6x^2 = 8 - 6y^2;\quad x = \sqrt{2}$

For problems 19 and 20,

a Graph the ellipse.

b Give the exact coordinates of any four points on the ellipse.

19. $\dfrac{(x-3)^2}{16} + \dfrac{(y-4)^2}{9} = 1$ **20.** $\dfrac{(x+2)^2}{6} + \dfrac{(y-5)^2}{12} = 1$

For Problems 21–24:

a Write the equation in standard form.

b Graph the equation.

21. $9x^2 + 4y^2 - 16y = 20$

22. $9x^2 + 16y^2 - 18x + 96y + 9 = 0$

23. $8x^2 + y^2 - 48x + 4y + 68 = 0$

24. $x^2 + 10y^2 + 4x + 20y + 4 = 0$

For Problems 25–28, write an equation for the ellipse with the properties given.

25. Center at $(1, 6)$, $a = 3$, $b = 2$

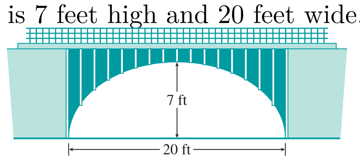
26. Vertices at $(3, 2)$ and $(-7, 2)$, minor axis of length 6

27. Covertices at $(3, 7)$ and $(3, -1)$, major axis of length 10

28. Vertices at $(-4, 9)$ and $(-4, -3)$, covertices at $(-7, 3)$ and $(-1, 3)$

Applications

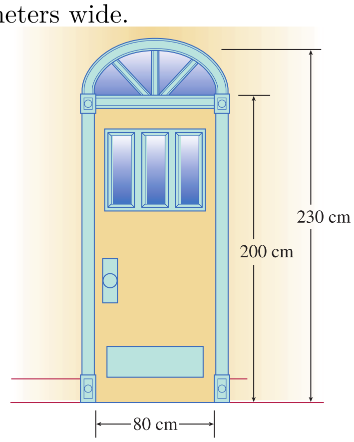
29. The arch of a bridge forms the top half of an ellipse with a horizontal major axis. The arch is 7 feet high and 20 feet wide.



a Find an equation for the ellipse.

b How high is the arch at a distance of 8 feet from the peak?

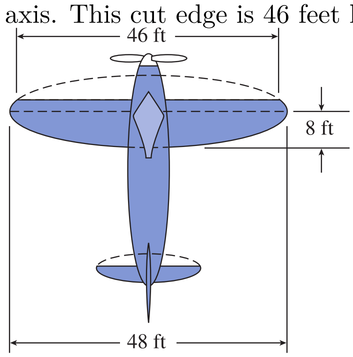
30. A doorway is topped by a semi-elliptical arch. The doorway is 230 centimeters high at its highest point and 200 centimeters high at its lowest point. It is 80 centimeters wide.



a Find an equation for the ellipse.

b How high is the doorway 8 centimeters from the left side?

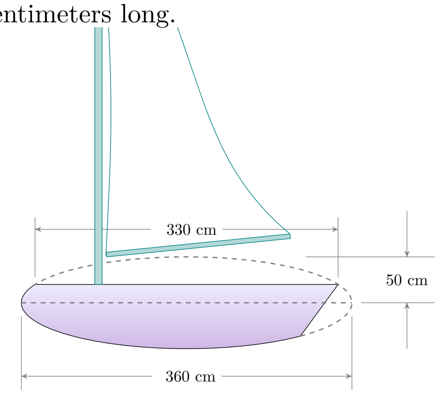
31. The wing of a World War II British Spitfire is an ellipse whose major axis is 48 feet. The minor axis is 16 feet, but part of the ellipse is cut off parallel to the major axis. This cut edge is 46 feet long.



a Find an equation for the ellipse.

b How wide is the wing at its center? Round your answer to two decimal places.

32. The centerline of a sailboat from bow to stern along the bottom (its keel) is elliptical in shape, with a major axis of 360 centimeters. The minor axis of the ellipse is 100 centimeters, but the deck of the sailboat (the top of the ellipse) has been cut off parallel to the major axis. The deck of the sailboat is 330 centimeters long.



a Find an equation for the ellipse.

b What is the maximum distance from the deck to the bottom of the keel? Round your answer to two decimal places.

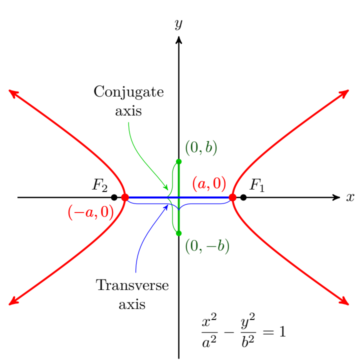
Conic Sections: Hyperbolas

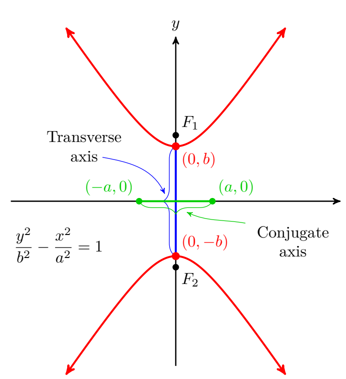
If a cone is cut by a plane parallel to its axis, the intersection is a **hyperbola**, the only conic section made of two separate pieces, or **branches**. Hyperbolas occur in a number of applied settings. The navigational system called LORAN (long-range navigation) uses radio signals to locate a ship or plane at the intersection of two hyperbolas. Satellites moving with sufficient speed will follow an orbit that is branch of a hyperbola; for example, a rocket sent to the moon must be fitted with retrorockets to reduce its speed in order to achieve an elliptical, rather than hyperbolic, orbit about the moon.

The hyperbola is defined as follows.

Definition 9.41 Hyperbola.

A **hyperbola** is the set of points in the plane, the difference of whose distances from two fixed points (the **foci**) is a constant.





If the origin is the center of the hyperbola and the foci (labeled F_1 and F_2

on the graphs above) lie on the axes, we can use the distance formula to derive its equation.

Central Hyperbola.

The equation of a hyperbola with center at the origin is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{or} \qquad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

In the first case, the two branches of the hyperbola open left and right, so the graph has x-intercepts at a and $-a$ but no y-intercepts. (See figure above left.) The segment joining the x-intercepts is the **transverse axis**, and its length is $2a$. The endpoints of the transverse axis are the **vertices** of the hyperbola. The segment of length $2b$ is called the **conjugate axis.**

In the second case, the graph has y-intercepts at b and $-b$ but no x-intercepts—the two branches open up and down. (See figure above right.) Here the y-intercepts are the vertices, so the transverse axis is vertical and has length $2b$. The conjugate axis has length $2a$.

Asymptotes of Hyperbolas

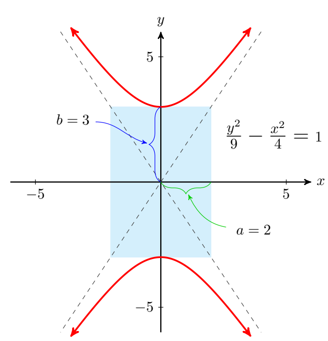
The branches of the hyperbola approach two straight lines that intersect at its center. These lines are **asymptotes** of the graph, and they are useful as guidelines for sketching the hyperbola. We first sketch a rectangle (called the **central rectangle**) whose sides are parallel to the axes and whose dimensions are $2a$ and $2b$. The asymptotes are the diagonals of this rectangle.

Example 9.42

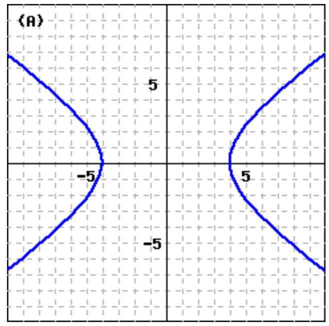
Graph $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$

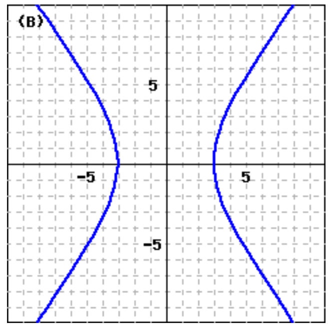
Solution. The graph is a hyperbola with center at the origin. The y^2-term is positive, so the branches of the hyperbola open upward and downward. Because $a^2 = 4$ and $b^2 = 9$, we have $a = 2$ and $b = 3$, and the vertices are $(0, 3)$ and $(0, -3)$. There are no x-intercepts.

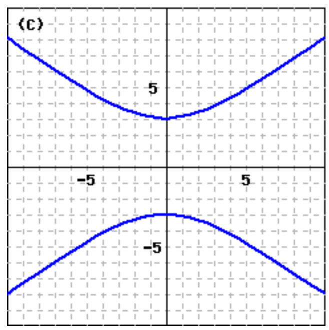
We construct the central rectangle with dimensions $2a = 4$ and $2b = 6$, as shown in the figure. Then we draw the asymptotes through the diagonals of the rectangle. The asymptotes have slopes $\pm\dfrac{3}{2}$. Finally, we sketch the branches of the hyperbola through the vertices and approaching the asymptotes.

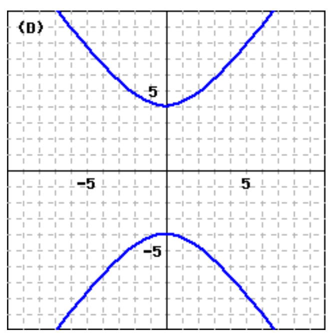


Checkpoint 9.43 Practice 1. Graph $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$







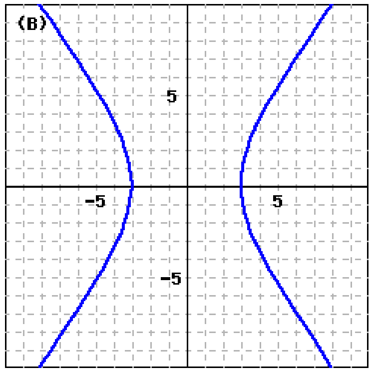


Which of the above is the best match for the graph?

- ⊙ (A)
- ⊙ (B)
- ⊙ (C)
- ⊙ (D)

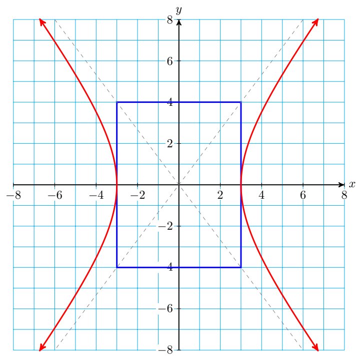
Answer. (B)

Solution.



Another graph is below.

$\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$:



The equation of a central hyperbola may be written as

$$Ax^2 + By^2 = C$$

where A and B have *opposite* signs and $C \neq 0$. As with ellipses, it is best to rewrite the equation in standard form in order to graph it.

Example 9.44

Write the equation $4y^2 - x^2 = 16$ in standard form and describe the important features of its graph.

Solution. We first divide each side by 16 to obtain

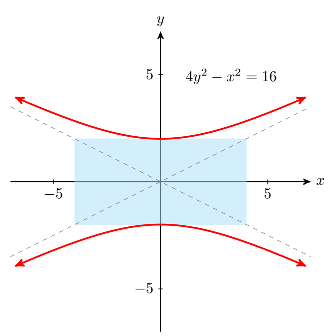
$$\frac{y^2}{4} - \frac{x^2}{16} = 1 \quad \text{or} \quad \frac{y^2}{2^2} - \frac{x^2}{4^2} = 1$$

The graph is a central hyperbola with y-intercepts 2 and -2, as shown in the figure. The slopes of the asymptotes are given by

$$\pm\frac{b}{a} = \pm\frac{2}{4} = \pm\frac{1}{2}$$

so the equations of the asymptotes are

$$y = \frac{1}{2}x \quad \text{and } y = \frac{-1}{2}x$$



Checkpoint 9.45 Practice 2.

a. Write the equation $4x^2 = y^2 + 25$ in standard form.

   $\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = 1$

b. Find the vertices of the graph and the equations of the asymptotes.

   Vertices: __________________ Separate different vertices with a comma.

   Equations of asymptotes: $y =$__ and $y =$__

Answer 1. $\frac{x^2}{\left(\frac{5}{2}\right)^2} - \frac{y^2}{5^2}$

Answer 2. $\left(\frac{-5}{2}, 0\right), \left(\frac{5}{2}, 0\right)$

Answer 3. $2x; -2x$

Solution.

a. $$\frac{x^2}{\left(\frac{5}{2}\right)^2} - \frac{y^2}{5^2} = 1$$

b. $$\left(\frac{5}{2}, 0\right), \left(\frac{-5}{2}, 0\right)$$

   $y = 2x$, $y = -2x$

Checkpoint 9.46 QuickCheck 1.

a. The hyperbola is the only conic section with two separate (☐ vertices ☐ foci ☐ axes ☐ branches) .

b. The endpoints of the (☐ conjugate ☐ transverse ☐ vertical ☐ horizontal) axis are the vertices of the hyperbola.

c. The asymptotes of a hyperbola are the diagonals of the (☐ central rectangle ☐ outer rectangle ☐ regular pentagon ☐ hypotenuse) .

d. If the y^2 term in the standard form is positive, a central hyperbola has no (☐ vertex ☐ asymptotes ☐ x-intercepts ☐ y-intercepts) .

Answer 1. branches

Answer 2. transverse

Answer 3. central rectangle

Answer 4. x-intercepts

Solution.

a. branches

b. transverse

c. central rectangle

d. x-intercepts

We can find exact coordinates of points on a hyperbola by substituting a value for one variable and solving for the other variable.

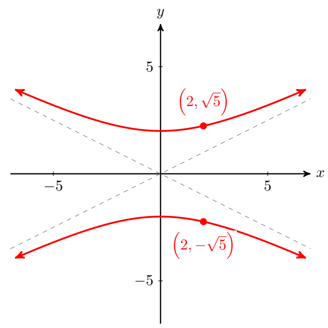
Example 9.47

Find the exact coordinates of any points with x-coordinate $x = 2$ on the hyperbola with equation $4x^2 - y^2 = 16$. Plot and label those points on the hyperbola.

Solution.

We substitute in the given equation.

$$\begin{aligned} 4y^2 - (2)^2 &= 16 \\ 4y^2 &= 20 \\ y^2 &= 5 \\ y &= \pm\sqrt{5} \end{aligned}$$



There are two points with $x = 2$, namely $\left(2, \sqrt{5}\right)$ and $\left(2, -\sqrt{5}\right)$, as shown in the figure.

Checkpoint 9.48 **Practice 3.** Solve the equation $4y^2 - x^2 = 16$ when $y = 1$.

$x =$________

What does this tell you about the graph of the hyperbola?

- ⊙ The line $y = 1$ is one of the asymptotes.
- ⊙ There are no points on the graph with y-coordinate 1.
- ⊙ This is a degenerate conic.
- ⊙ The hyperbola opens left and right.

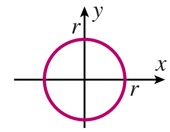
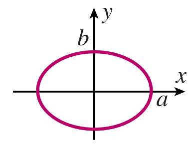
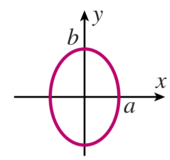
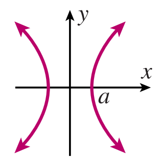
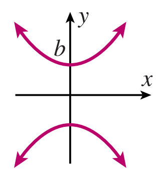
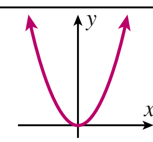
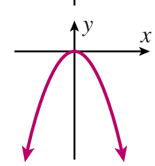
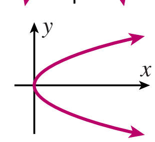
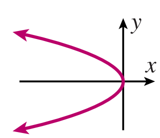
Answer 1. DNE, undefined, does not exist, or none

Answer 2. Choice 2

Solution. There are no real solutions when $y = 1$, so there are no points on the graph with y-coordinate 1.

The Central Conics

The fourth conic section, after circles, ellipses, and hyperbolas, is the parabola. We have already encountered parabolas in our study of quadratic functions. In particular, the graph of $y = ax^2$ has its vertex at the origin and opens up or down, depending on the sign of a. The graph of $x = ay^2$ is a parabola that opens to the left or right. There is also a geometric definition of a parabola, but we will not discuss that here.

Description of conic	Standard form of equation	Graph
circle	$x^2 + y^2 = r^2$	
ellipse (a) major axis on x-axis ("wide and short")	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$	
(b) major axis on y-axis ("tall and narrow")	$(a < b)$	
hyperbola (a) transverse axis on x-axis (opens left and right)	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	
(b) transverse axis on y-axis (opens up and down)	$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$	
parabola (a) opens up	$y = ax^2$ $(a > 0)$	
(b) opens down	$y = ax^2$ $(a < 0)$	
(c) opens right	$x = ay^2$ $(a > 0)$	
(d) opens left	$x = ay^2$ $(a < 0)$	

Example 9.49

Write each equation in standard form and describe its graph.

a $x^2 = 6y^2 + 8$ b $x^2 = \dfrac{4 - y^2}{2}$

Solution.

a The equation $x^2 = 6y^2 + 8$ is equivalent to

$$x^2 - 6y^2 = 8 \qquad \text{or} \qquad \frac{x^2}{\left(\sqrt{8}\right)^2} - \frac{y^2}{\left(\frac{2}{\sqrt{3}}\right)^2} = 1$$

The graph is a hyperbola that opens left and right.

b The equation $x^2 = \dfrac{4 - y^2}{2}$ is equivalent to

$$2x^2 + y^2 = 4 \qquad \text{or} \qquad \frac{x^2}{\left(\sqrt{2}\right)^2} + \frac{y^2}{2^2} = 1$$

The graph is an ellipse with major axis on the y-axis because $2 > \sqrt{2}$.

Checkpoint 9.50 Practice 4. Write each equation in standard form and describe its graph.

a. $4y^2 = x^2 - 8$

Standard form: ______________$= 1$

The graph is (☐ a circle ☐ an ellipse ☐ a hyperbola ☐ a parabola) centered at the point ____, with vertices _____________________. There (☐ are ☐ are not) asymptotes: $y = \pm$___.

b. $4x^2 + y = 0$

Standard form: $y =$______________

The graph is (☐ a circle ☐ an ellipse ☐ a hyperbola ☐ a parabola) with vertex at the point _____, opening (☐ left ☐ right ☐ upward ☐ downward) .

Answer 1. $\frac{x^2}{8} - \frac{y^2}{2}$

Answer 2. a hyperbola

Answer 3. $(0, 0)$

Answer 4. $\left(-2\sqrt{2}, 0\right), \left(2\sqrt{2}, 0\right)$

Answer 5. are

Answer 6. $\frac{1}{2}x$

Answer 7. $-4x^2$

Answer 8. a parabola

Answer 9. $(0, 0)$

Answer 10. downward

Solution.

a. $\dfrac{x^2}{8} - \dfrac{y^2}{2} = 1$. A hyperbola centered at the origin with vertices $\left(2\sqrt{2}, 0\right)$ and $\left(-2\sqrt{2}, 0\right)$, and asymptotes $y = \dfrac{1}{2}x$ and $y = -\dfrac{1}{2}x$.

b. $y = -4x^2$. A parabola opening downward with vertex at the origin.

Translated Hyperbolas

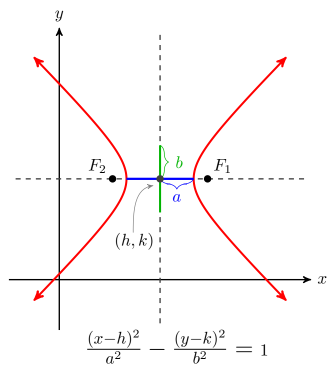
The standard form for the equations of hyperbolas centered at the point (h, k) can be derived using the distance formula and the definition of hyperbola.

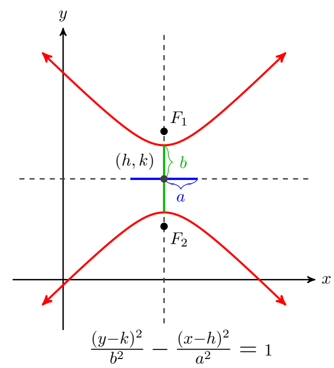
Hyperbolas.

The equation for a hyperbola centered at (h, k) has one of the two standard forms:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$
$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

The first equation describes a hyperbola whose transverse axis is parallel to the x-axis, so that the branches open left and right, and the second equation describes a hyperbola whose transverse axis is parallel to the y-axis, so that the branches open up and down, as shown below.





Example 9.51

a Graph $\frac{(x-3)^2}{8} - \frac{(y+2)^2}{10} = 1$

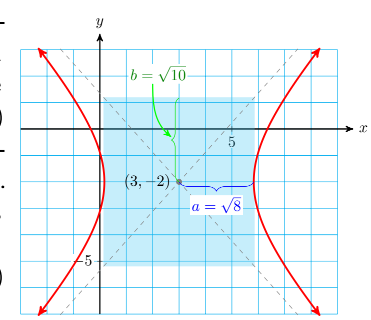
b Find the equations of the asymptotes.

Solution.

a The graph is a hyperbola with

$$(h, k) = (3, -2), \quad a = \sqrt{8} = 2\sqrt{2}, \quad \text{and } b = \sqrt{10}$$

Because the x^2-term is positive, the branches open left and right. The coordinates of the vertices are thus $(3 + 2\sqrt{2}, -2)$ and $(3 - 2\sqrt{2}, -2)$, or approximately $(5.8, -2)$ and $(0.2, -2)$. The ends of the conjugate axis are $(3, -2 + \sqrt{10})$ and $(3, -2 - \sqrt{10})$, or approximately $(3, 1.2)$ and $(3, -5.2)$.



The central rectangle is centered at the point $(3, -2)$ and extends to the vertices in the horizontal direction and to the ends of the conjugate axis in the vertical direction. We draw the asymptotes through the opposite corners of the central rectangle, and sketch the hyperbola through the vertices and approaching the asymptotes to obtain the graph shown below.

b Both asymptotes pass through the center of the hyperbola, $(3, -2)$. Their slopes are

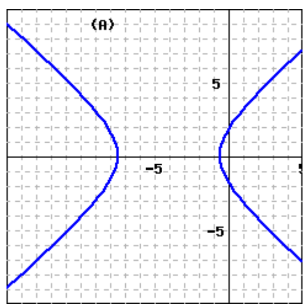
$$\frac{\sqrt{10}}{\sqrt{8}} = \frac{\sqrt{5}}{2} \quad \text{and} \quad \frac{-\sqrt{5}}{2}$$

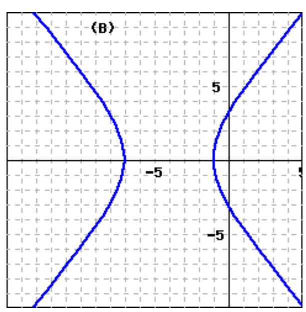
We substitute these values into the point-slope formula to find the equations

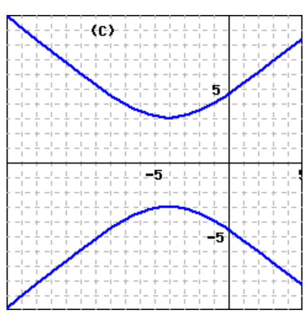
$$y + 2 = \frac{\sqrt{5}}{2}(x - 3) \quad \text{and} \quad y + 2 = \frac{-\sqrt{5}}{2}(x - 3)$$

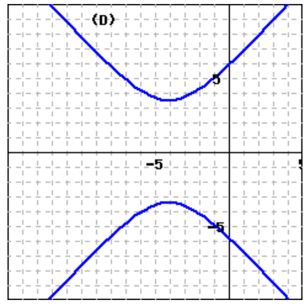
Checkpoint 9.52 Practice 5.

a. Graph $\frac{y^2}{9} - \frac{(x + 4)^2}{12} = 1$









Which of the above is the best match for the graph?

- ⊙ (A)
- ⊙ (B)
- ⊙ (C)
- ⊙ (D)

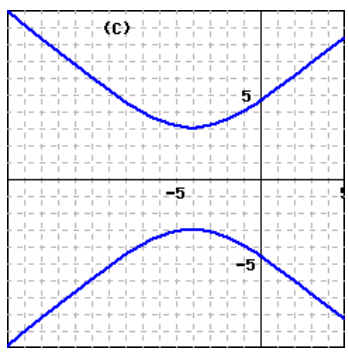
b. Find the equations of the asymptotes. (Enter "sqrt(2)" for $\sqrt{2}$.)
Equations of asymptotes: $y =$____________ and
$y =$____________

Answer 1. (C)

Answer 2. $\frac{\sqrt{3}}{2}(x+4)$

Answer 3. $\frac{-(\sqrt{3})}{2}(x+4)$

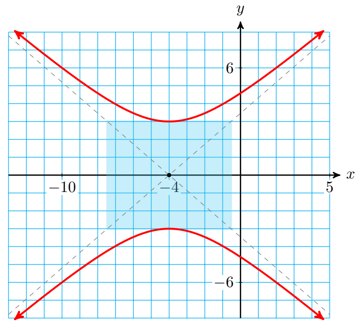
Solution.

a. 

A graph is also shown below.

b. $y = \dfrac{\sqrt{3}}{2}x + 4,\ y = \dfrac{-\sqrt{3}}{2}x + 4$

$\dfrac{y^2}{9} - \dfrac{(x+4)^2}{12} = 1$:



Writing in Standard Form

We can write the equation of a hyperbola in standard form by completing the squares in x and y.

Example 9.53

a Write the equation in standard form.

$$y^2 - 4x^2 + 4y - 8x - 9 = 0$$

b Graph the equation.

Solution.

a We prepare to complete the square by factoring out -4 from the x-terms.

$$(y^2 + 4y \quad \_\_\_\_) - 4(x^2 + 2x \quad \_\_\_\_) = 9$$

We complete the square in y by adding **4** to each side of the equation. To complete the square in x we add 1 to $x^2 + 2x$, so we add $-4 \cdot \mathbf{1}$, or $\mathbf{-4}$ to the right side, to get

$$(y^2 + 4y + \mathbf{4}) - 4(x^2 + 2x + \mathbf{1}) = 9 + \mathbf{4} \ \mathbf{-4}$$

or

$$(y + 2)^2 - 4(x + 1)^2 = 9$$

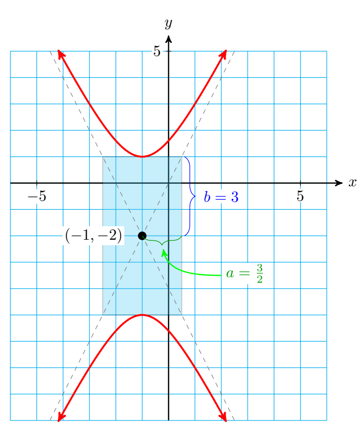
Finally, we divide each side by 9 to obtain the standard form

$$\frac{(y+2)^2}{9} - \frac{(x+1)^2}{\frac{9}{4}} = 1$$

b The graph is a hyperbola with center at $(-1, -2)$. Because the y^2-term is positive, the transverse axis is parallel to the y-axis, and

$$b^2 = 9, \quad a^2 = \frac{9}{4}$$

Thus, $b = 3$ and $a = \dfrac{3}{2}$, and the vertices are $(-1, 1)$ and $(-1, 5)$. The ends of the conjugate axis are $\left(\dfrac{-5}{2}, -2\right)$ and $\left(\dfrac{1}{2}, -2\right)$.



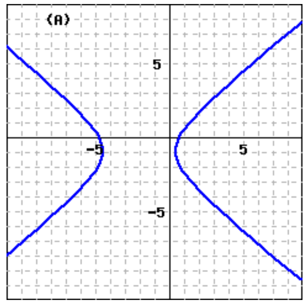
The central rectangle is centered at $(-1, -2)$, as shown in the figure. We draw the asymptotes through the corners of the rectangle, then sketch the hyperbola by starting at the vertices and approaching the asymptotes.

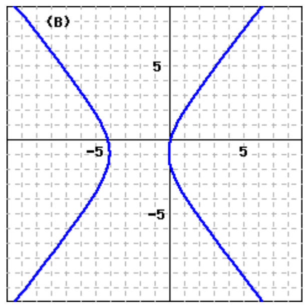
Checkpoint 9.54 Practice 6.

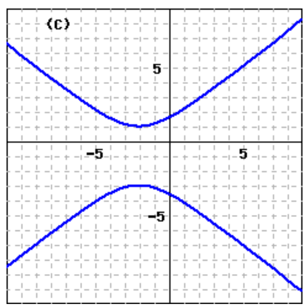
a. Write the equation $4x^2 - 6y^2 + 16x - 12y - 14 = 0$ in standard form.

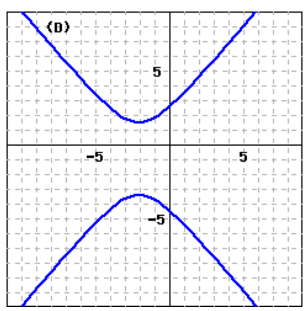
____________________ $= 1$

b. Graph the equation.









Which of the above is the best match for the graph?

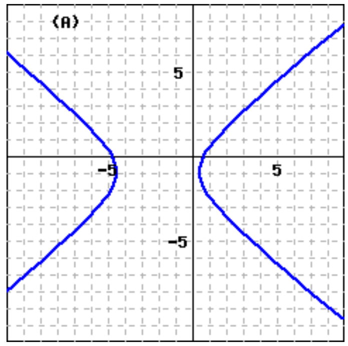
- ⊙ (A)
- ⊙ (B)
- ⊙ (C)
- ⊙ (D)

Answer 1. $\frac{(x+2)^2}{6} - \frac{(y+1)^2}{4}$

Answer 2. (A)

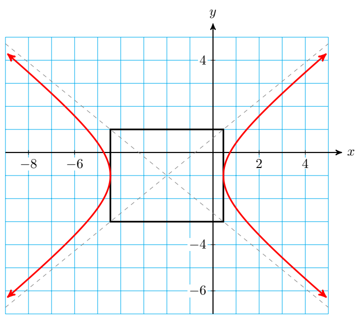
Solution.

a. $\dfrac{(x+2)^2}{6} - \dfrac{(y+1)^2}{4} = 1$

b. 

A graph is also shown below.

$4x^2 - 6y^2 + 16x - 12y - 14 = 0$:



Checkpoint 9.55 QuickCheck 2. True or false.

a. The graph of a general quadratic is a hyperbola if the coefficients of both the x^2-term and the y^2-term are negative. (☐ True ☐ False)

b. The slopes of the asymptotes of a hyperbola are $\pm\dfrac{b}{a}$. (☐ True ☐ False)

c. The asymptotes pass through the vertices of the hyperbola. (☐ True ☐ False)

Answer 1. False

Answer 2. True

Answer 3. False

Solution.

a. False

b. True

c. False

d. True

General Quadratic Equation in Two Variables

We have considered graphs of second-degree equations in two variables,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

for which B, the coefficient of the xy-term, is zero. These graphs are conic sections with axes parallel to one or both of the coordinate axes. If B does not equal zero, the axes of the conic section are rotated with respect to the coordinate axes. The graphing of such equations is taken up in more advanced courses in analytic geometry.

The graph of a second-degree equation can also be a point, a line, a pair of lines, or no graph at all, depending on the values of the coefficients A through F. Such graphs are called **degenerate conics**.

Given an equation of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

we can determine the nature of the graph from the coefficients of the quadratic terms. If the graph is not a degenerate conic, the following criteria apply.

Conic Sections.

The graph of $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is

1 a circle if $A = C$.

2 a parabola if $A = 0$ or $C = 0$ (but not both).

3 an ellipse if A and C have the same sign.

4 a hyperbola if A and C have opposite signs.

Example 9.56

Name the graph of each equation, assuming that the graph is not degenerate.

a $3x^2 + 3y^2 - 2x + 4y - 6 = 0$

b $4y^2 + 8x^2 - 3y = 0$

c $4x^2 - 6y^2 + x - 2y = 0$

d $y + x^2 - 4x + 1 = 0$

Solution.

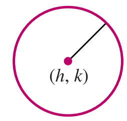
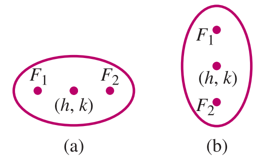
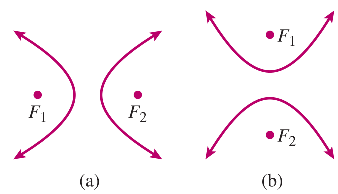
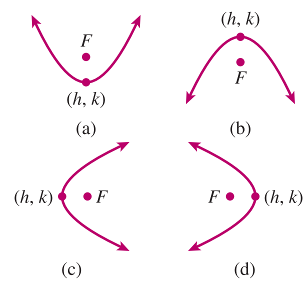
a The graph is a circle because the coefficients of x^2 and y^2 are equal.

b The graph is an ellipse because the coefficients of x^2 and y^2 are both positive.

c The graph is a hyperbola because the coefficients of x^2 and y^2 have opposite signs.

d The graph is a parabola because y is of first degree and x is of second degree.

The standard forms for the conic sections are summarized in the table below. For the parabola, (h, k) is the vertex of the graph, and for the other conics, (h, k) is the center.

Name of curve	Standard form of equation	Graph
Circle	$(x-h)^2+(y-k)^2=r^2$	
Ellipse (a) Major axis parallel to x-axis (b) Major axis parallel to y-axis	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ $a>b$ $a<b$	
Hyperbola (a) Transverse axis parallel to x-axis (b) Transverse axis parallel to y-axis	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$	
Parabola (a) Opens upward (b) Opens downward (c) Opens to the right (d) Opens to the left	$(y-k)=a(x-h)^2,\ a>0$ $(y-k)=a(x-h)^2,\ a<0$ $(x-h)=a(y-k)^2,\ a>0$ $(x-h)=a(y-k)^2,\ a<0$	

The coefficients D, E, and F do not figure in determining the type of conic section the equation represents. They do, however, determine the position of the graph relative to the origin. Once we recognize the form of the graph, we can write the equation in standard form in order to discover more information about the graph.

Checkpoint 9.57 Practice 7. Describe the graph of each equation without graphing.

a. $x^2 = 9y^2 - 9$

(☐ A circle with center ☐ An ellipse with center ☐ A hyperbola with center ☐ A parabola with vertex) at the point _____; vertices: ; asymptotes $y = \pm$_____

b. $x^2 - y = 2x + 4$

(☐ A circle with center ☐ An ellipse with center ☐ A hyperbola with center ☐ A parabola with vertex) at the point _____, opening (☐ left ☐ right ☐ upward ☐ downward)

c. $x^2 + 9y^2 + 4x - 18y + 9 = 0$

(☐ A circle with center ☐ An ellipse with center ☐ A hyperbola with center ☐ A parabola with vertex) at the point _____ with horizontal (☐ major axis ☐ minor axis ☐ transverse axis ☐ conjugate axis) ; $a =$___, and $b =$___

Answer 1. A hyperbola with center

Answer 2. $(0, 0)$

Answer 3. $(0, -1), (0, 1)$

Answer 4. $\frac{1}{3}x$
Answer 5. A parabola with vertex
Answer 6. $(1, -5)$
Answer 7. upward
Answer 8. An ellipse with center
Answer 9. $(-2, 1)$
Answer 10. major axis
Answer 11. 2
Answer 12. $\frac{2}{3}$
Solution.

a. A hyperbola centered at the origin with vertices $(0, 1)$ and $(0, -1)$ and asymptotes $y = \dfrac{1}{3}x$ and $y = \dfrac{-1}{3}x$.

b. A parabola that opens upward from the vertex $(1, -5)$.

c. An ellipse centered at $(-2, 1)$ with a horizontal major axis, $a = 2$ and $b = \dfrac{2}{3}$.

Checkpoint 9.58 QuickCheck 3.

a. A (☐ degenerate ☐ central ☐ translated) conic can be a point, a line, a pair of lines, or no graph at all.

b. The coefficients D, E, and F determine the (☐ type ☐ position) of the conic.

c. The graph is a (☐ circle ☐ ellipse ☐ hyperbola ☐ parabola) if one of A or C is zero.

d. If $A = C$ (and $B = 0$), the graph is a (☐ circle ☐ ellipse ☐ hyperbola ☐ parabola) .

Answer 1. degenerate
Answer 2. position
Answer 3. parabola
Answer 4. circle
Solution.

a. A degenerate conic can be a point, a line, a pair of lines, or no graph at all.

b. The coefficients D, E, and F determine the position of the conic.

c. The graph is a parabola if one of A or C is zero.

d. If $A = C$ (and $B = 0$), the graph is a circle.

Problem Set 9.4

Warm Up

1. Use your calculator to graph $y^2 - x^2 = 1$ in the window $[-6, 6] \times [-4, 4]$. First, solve the equation for y. (You should get two functions.)

$y_1 =$ ____________ and $y_2 =$ ____________

a The graph of this equation is a ____________.

b The graph has **vertices** at _______ and _______.

c Now add to your graph the lines $y = x$ and $y = -x$. These are the asymptotes of the hyperbola.

2. Use the calculator to graph $x^2 - y^2 = 1$ in the window $[-6, 6] \times [-4, 4]$. (Follow the steps above for Problem 1.)

a What are the vertices of this hyperbola?

b What are the asymptotes?

c How does this hyperbola differ from the one in Problem 2?

Skills Practice

For Problems 3–8, graph the hyperbola.

3. $\dfrac{y^2}{12} - \dfrac{x^2}{8} = 1$

4. $\dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$

5. $3x^2 = 4y^2 + 24$

6. $9x^2 - 4y^2 = 36$

7. $y^2 = \dfrac{1}{2}x^2 - 16$

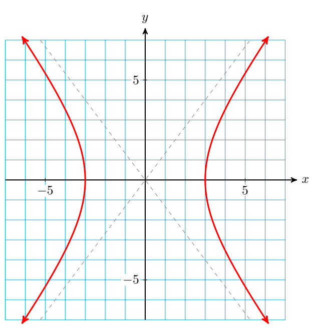
8. $\dfrac{1}{2}x^2 = y^2 - 12$

For Problems 9 and 10:

a Find the equation for the hyperbola.

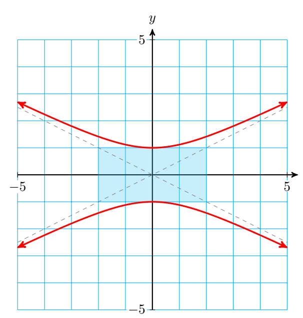
b Use your equation to complete the table.

9.



x	0		5	
y		± 2		-3

10.



x	0		4	
y		0		-2

11.

a Graph $x^2 - y^2 = 4$, $x^2 - y^2 = 1$, and $x^2 - y^2 = 0$ on the same set of axes. What do you observe?

b Graph $4x^2 - y^2 = 16$, $4x^2 - y^2 = 4$, and $4x^2 - y^2 = 0$ on the same set of axes. What do you observe?

12.

a Graph $x^2 - y^2 = 0$. (Hint: Factor the left side of the equation and use the zero-factor principle to write two equations, then graph them.) Describe your graph.

b Graph $4x^2 - y^2 = 0$. Describe your graph.

For problems 13–18:

a Graph the hyperbola.

b Give the exact coordinates of any four points on the hyperbola

13. $\dfrac{(y+2)^2}{6} - \dfrac{(x+2)^2}{10} = 1$

14. $\dfrac{(x-4)^2}{9} - \dfrac{(y+2)^2}{16} = 1$

15. $9y^2 - 8x^2 + 72y + 16x + 64 = 0$

16. $4x^2 - 6y^2 - 32x - 24y + 16 = 0$

17. $10y^2 - 5x^2 + 30x - 95 = 0$

18. $12x^2 - 3y^2 + 24y - 84 = 0$

For Problems 19–22, write an equation for the hyperbola with the properties given.

19. Center at $(6, -2)$, $a = 1$, $b = 4$, opening left and right

20. Center at $(-1, 5)$, $a = 6$, $b = 8$, opening up and down

21. One vertex at $(1, -2)$, one end of the vertical conjugate axis at $(-5, 1)$

22. One vertex at $(-1, 3)$, one end of the horizontal conjugate axis at $(-5, 1)$

For Problems 23–28, name the graph of the equation and descsribe its main features.

23. $x^2 + 2y - 4 = 0$

24. $y^2 = 6 - 4x^2$

25. $y^2 - 4x^2 + 2y - x = 0$

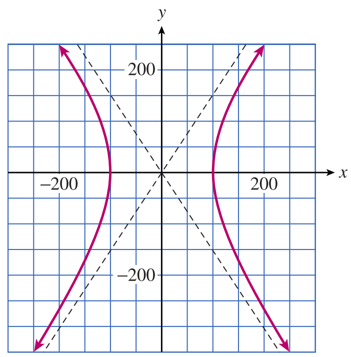
26. $6 + \dfrac{x^2}{4} = y^2$

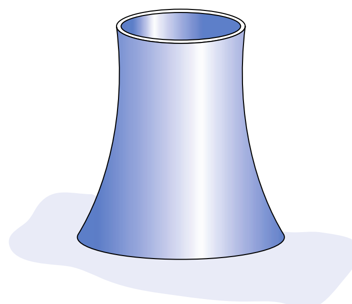
27. $y - 2 = \dfrac{(x+4)^2}{4}$

28. $\dfrac{(x+3)^2}{4} + \dfrac{y^2}{12} = 1$

Applications

Problems 29–32 deal with the cooling tower at an electricity generating facility. The shape of the tower, called a hyperboloid, is obtained by rotating a portion of the hyperbola $\dfrac{x^2}{100^2} - \dfrac{y^2}{150^2} = 1$ around the y-axis.





29. The base of the cooling tower is 360 feet below the center of the hyperbola. What is the diamter of the base?

30. The top of the cooling tower is 200 feet above the center of the hyperbola. What is the diameter of the top?

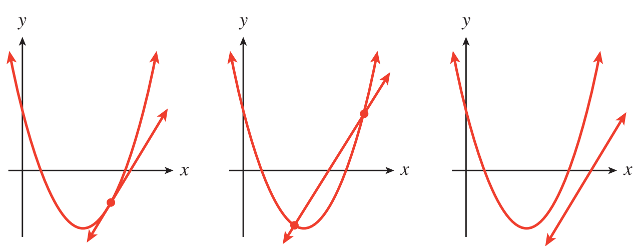
31. The diameter of the tower first decreases with height and then increases again. There are two heights at which the tower's diameter is 250 feet. Find the greater of the two heights.

32. Find the height at which the tower's diameter is 200 feet.

Nonlinear Systems

Systems Involving Quadratic Equations

Recall that the solution to a 2×2 system of linear equations is the intersection point of the graphs of the equations. This is also true of systems in which one or both of the equations is quadratic. The figure below shows the three cases for systems of one quadratic and one linear equation.



One Solution   Two Solutions   No Solution

In Example 9.59, p. 659, we use both graphical and algebraic techniques to solve the system.

Example 9.59

The Pizza Connection calculates that the cost, in dollars, of producing x pizzas per day is given by

$$C = 0.15x^2 + 0.75x + 180$$

The Pizza Connection charges \$15 per pizza, so the revenue from selling x pizzas is

$$R = 15x$$

How many pizzas per day must the Pizza Connection sell in order to break even?

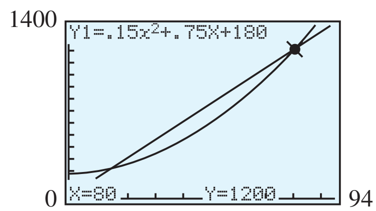
Solution. To break even means to make zero profit. Because

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

the break-even points occur when revenue equals cost. In mathematical terms, we would like to find any values of x for which $R = C$. If we graph the revenue and cost functions on the same axes, these values correspond to points where the two graphs intersect. Use the WINDOW settings

Xmin = 0	Xmax = 94
Ymin = 0	Ymax = 1400

on your calculator to obtain the graph shown below. You can verify that the two intersection points are $(15, 225)$ and $(80, 1200)$.



Thus, the Pizza Connection must sell either 15 or 80 pizzas in order to break even. On the graph we see that revenue is greater than cost for x-values between 15 and 80, so the Pizza Connection will make a profit if it sells between 15 and 80 pizzas.

We can also solve algebraically for the break-even points. The intersection points of the two graphs correspond to the solutions of the system of equations

$$\begin{aligned} y &= 0.15x^2 + 0.75x + 180 \\ y &= 15x \end{aligned}$$

We equate the two expressions for y and solve for x:

$$\begin{aligned} 0.15x^2 + 0.75x + 180 &= 15x && \text{Subtract } 15x \text{ from both sides.} \\ 0.15x^2 - 14.25x + 180 &= 0 && \text{Use the quadratic formula.} \end{aligned}$$

$$\begin{aligned} x &= \frac{14.25 \pm \sqrt{14.25^2 - 4(0.15)(180)}}{2(-0.05)} && \text{Simplify.} \\ &= \frac{14.25 \pm 9.75}{0.3} \end{aligned}$$

The solutions are 15 and 80, as we found from the graph.

Checkpoint 9.60 Practice 1.

a. Solve the system algebraically:

$$\begin{aligned} y &= x^2 - 8x + 17 \\ y + 4x &= 13 \end{aligned}$$

Solution: ______ Note: list solutions as ordered pairs.

b. Graph both equations, and show the solutions on the graph.

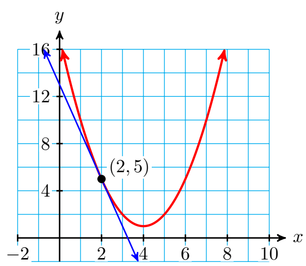
Answer. $(2, 5)$

Solution.

a. $(2, 5)$

b. A graph is below.

Graph of system:



What about a system of two quadratic equations $y = ax^2 + bx + c$? You can sketch some possible systems to convince yourself that two such graphs can intersect in one point, two points, or no points at all.

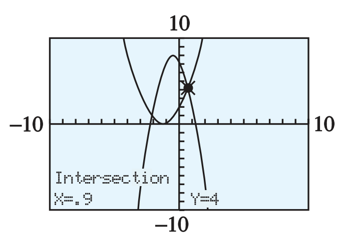
Example 9.61

Solve the system

$$y = (x + 1.1)^2$$
$$y = 7.825 - 2x - 2.5x^2$$

Solution. We graph these two equations in the standard window and use the **intersect** feature to locate one of the solutions, as shown in the figure.

You can check that the point $(0.9, 4)$ is an exact solution to the system by substituting $x = 0.9$ and $y = 4$ into each equation of the system. We find the other solution by moving the bug close to the other intersection point and using the **intersect** feature again. You can verify that the other solution is the point $(-2.1, 1)$.



To solve the system algebraically, we equate the two expressions for y.

$$(x + 1.1)^2 = 7.825 - 2x - 2.5x^2$$

After expanding the left side and collecting like terms, we arrive at a quadratic equation, which we can solve with the quadratic formula, and find two values for x, namely $x = 0.9$ and $x = -2.1$. To find the y-value for each of these intersection points, we substitute the x-coordinate into either of the two original equations. For example, for $x = 0.9$,

$$y = (0.9 + 1.1)^2 = 2^2 = 4$$

Checkpoint 9.62 Practice 2.

a. Solve the system algebraically:

$$y = x^2 - 6x - 7$$
$$y = 13 - x^2$$

Solutions: ___________ Note: list solutions as ordered pairs, and use a comma to different solutions.

b. Graph both equations, and show the solutions on the graph.

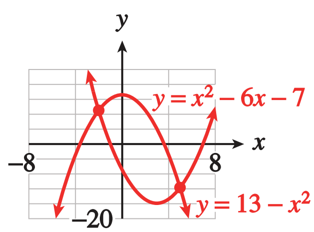
Answer. $(-2, 9), (5, -12)$

Solution.

a. $(-2, 9), (5, -12)$

b. A graph is below.

Graph of system:



Systems Involving Conic Sections

A system of two parabolas can have one, two, or no solutions, depending on the graphs of the two equations. Systems involving other conics may have up to four solutions.

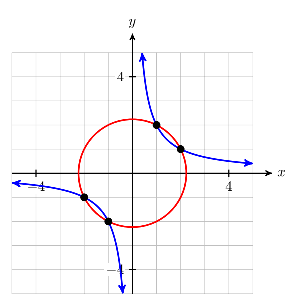
Example 9.63

Find the intersection points of the graphs of

$$x^2 + y^2 = 5$$
$$xy = 2$$

Solution.

We will use substitution to solve the system. We solve the easier of the two equations(the second equation) for y to obtain $y = \dfrac{2}{x}$. Then we substitute $\dfrac{2}{x}$ for y in the first equation to find



$$x^2 + \left(\frac{2}{x}\right)^2 = 5 \quad \text{or} \quad x^2 + \frac{4}{x^2} = 5$$

This equation has only one variable, x, and we solve it by first clearing fractions. We multiply both sides by x^2, and then subtract $5x^2$

to obtain

$$x^4 - 5x^2 + 4 = 0 \qquad \text{Factor the left side.}$$
$$(x^2 - 1)(x^2 - 4) = 0 \qquad \text{Apply the zero-facor principle.}$$
$$x^2 - 1 = 0 \quad \text{or} \quad x^2 - 4 = 0 \qquad \text{Solve each equation.}$$
$$x = 1, \quad x = -1, \quad x = 2, \quad \text{or} \quad x = -2$$

Finally, substitute each of these values into $y = \dfrac{2}{x}$ to find the y-components of each solution. The intersection points of the two graphs are $(1, 2)$, $(-1, -2)$, $(2, 1)$, and $(-2, -1)$. The graph of the system is shown above.

Checkpoint 9.64 Practice 3. Find the intersection points of the graphs of

$$x^2 - y^2 = 35$$
$$xy = 6$$

Solution: _________ Note: list solutions as ordered pairs, and use a comma to different solutions.

Answer. $(6, 1), (-6, -1)$

Solution. $(6, 1)$, $(-6, -1)$

Solving Systems by Elimination

We used substitution in the previous Example to solve the system. If both equations are ofthe form

$$ax^2 + by^2 = c$$

elimination of variables is more efficient.

Example 9.65

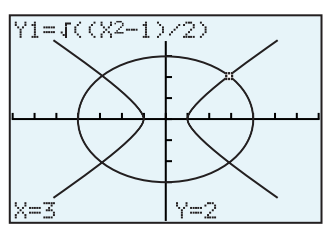
Find the solutions to the following system of equations.

$$x^2 - 2y^2 = 1$$
$$\frac{x^2}{15} + \frac{y^2}{10} = 1$$

Verify the solutions on a graph.

Solution.

We multiply the first equation by 3 and the second by 60 to obtain



$$3x^2 - 6y^2 = 3$$
$$4x^2 + 6y^2 = 60$$

Adding these two equations, we have

$$7x^2 = 63$$
$$x^2 = 9$$

so $x = \pm 3$. We substitute these values for x into either equation and solve for y to find the solutions $(3, 2)$, $(3, -2)$, $(-3, 2)$ and $(-3, -2)$.

The two original equations describe a hyperbola and an ellipse. We can obtain graphs on the calculator by solving each equation for y to get

$$y = \pm\sqrt{\frac{x^2-1}{2}} \quad \text{and} \quad y = \pm\sqrt{10\left(1-\frac{x^2}{15}\right)}$$

Using the window $[-7.05, 7.05] \times [-4.65, 4.65]$, we obtain the graph shown above. The solutions of the system are the intersection points of the graphs.

Checkpoint 9.66 Practice 4. Find the intersection points of the graphs of

$$y^2 - x^2 = 5$$
$$x^2 + y^2 = 13$$

Solution: ______________ Note: list solutions as ordered pairs, and use a comma to different solutions.

Answer. $(2,3), (-2,3), (2,-3), (-2,-3)$

Solution. $(2,3)$, $(-2,3)$, $(2,-3)$, $(-2,-3)$

For some quadratic systems, we use a combination of elimination of variable sand substitution.

Example 9.67

Find the intersection of the circles given by the equations.

$$x^2 - 4x + y^2 + 2y = 20$$
$$x^2 - 12x + y^2 + 10y = -12$$

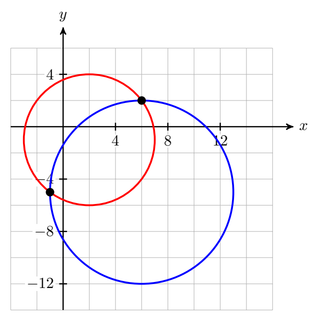
Solution.

We subtract the second equation from the first equation to obtain

$$8x - 8y = 32$$

Solving for x, we have

$$x = y + 4$$



Next, we substitute $\boldsymbol{y+4}$ for x into either of the original equations. We use the first equation to find

$$\begin{aligned}(\boldsymbol{y+4})^2 - 4(\boldsymbol{y+4}) + y^2 + 2y &= 20 && \text{Remove parentheses.}\\ (y^2 + 8y + 16) - 4y - 16 + y^2 + 2y &= 20 && \text{Collect like terms.}\\ 2y^2 + 6y - 20 &= 0 && \text{Divide both sides by 2.}\\ y^2 + 3y - 10 &= 0 && \text{Factor the left side.}\\ (y+5)(y-2) &= 0\end{aligned}$$

Thus, $y = -5$ or $y = 2$. From the equation $x = y + 4$, we find that when $y = -5$, $x = -1$, and when $y = 2$, $x = 6$. Thus the two circles intersect at $(-1, -5)$ and $(6, 2)$, as shown in the figure above.

Checkpoint 9.68 Practice 5. Find the intersection points of the graphs of

$$x^2 - 8x + y^2 + 2y = 23$$
$$x^2 - 12x + y^2 - 6y = -25$$

Solution: _______________ Note: list solutions as ordered pairs, and use a comma to different solutions.

Answer. $(2, 5), (10, 1)$

Solution. $(2, 5)$, $(10, 1)$

Problem Set 9.5

Warm Up

1. Solve the system by substitution.

$$3x - y = 5$$
$$2x - 3y = 8$$

2. Solve the system by elimination.

$$5x + 2y = 5$$
$$4x + 3y = -3$$

3. Solve $x^2 - 19x + 48 = 0$

4. Solve $3x^2 + 18x = 48$

Skills Practice

For Problems 5–10, solve the system algebraically. Then use your calculator to graph both equations and verify your solutions.

5. $y = x^2 - 4x + 7$
$y = 11 - x$

6. $y = -x^2 - 2x + 7$
$y = 2x + 11$

7. $y = x^2 + 8x + 8$
$3y + 2x = -36$

8. $y = x^2 - 9$
$y = -2x^2 + 9x + 21$

9. $y = x^2 - 0.5x + 3.5$
$y = -x^2 + 3.5x + 1.5$

10. $y = x^2 - 4x + 4$
$y = x^2 - 8x + 16$

For problems 11–20, find the intersection points of the graphs by solving a system of equations. Verify your solutions by graphing

11. $xy = 4$
$x^2 + y^2 = 8$

12. $x^2 - 2y^2 = -4$
$xy = -4$

13. $xy = -2$
$x^2 + y^2 = 5$

14. $5y^2 - x^2 = 1$
$x^2 + y^2 = 5$

15. $\dfrac{x^2}{14} + \dfrac{y^2}{35} = 1$
$x^2 + 2y^2 = 54$

16. $\dfrac{x^2}{2} - \dfrac{y^2}{16} = 1$
$y^2 - y^2 = 12$

17. $x^2 - 8y^2 = 4$
$10y^2 - x^2 = 4$

18. $x^2 + y^2 + 2y = 19$
$x^2 - 2x + y^2 + 8y = 33$

19. $(x+1)^2+(y-2)^2=5$
$(x-5)^2+(y+1)^2=50$

20. $(x-1)^2+(y-6)^2=26$
$(x-4)^2+(y-8)^2=65$

Applications

Use a system of equations to solve Problems 21–24.

21. The area of a rectangle is 216 square feet. If the perimeter is 60 feet, find the dimensions of the rectangle.

22. Leon flies his plane 840 miles in the same time that Marlene drives her automobile 210 miles. Suppose that Leon flies 180 miles per hour faster than Marlene drives. Find the rate of each.

23. At a constant temperature, the pressure, P, and the volume, V, of a gas are related by the equation $PV = K$. The product of the pressure (in pounds per square inch) and the volume (in cubic inches) of a certain gas is 30 inch-pounds. If the temperature remains constant as the pressure is increased by 4 pounds per square inch, the volume is decreased by 2 cubic inches. Find the original pressure and volume of the gas.

24. Kristen drove 50 miles to her sister's house, traveling 10 miles in heavy traffic to get out of the city and then 40 miles in less congested traffic. Her average speed in the city was 20 miles per hour less than her speed in light traffic. What was each rate if her trip took 1 hour and 30 minutes?

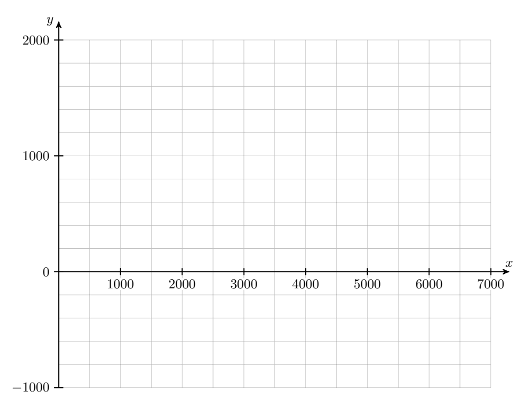
Problems 25 and 26 deal with wildlife management and sustainable yield.

25. The annual increase, I, in the deer population in a national park is given by

$$I = kCx - kx^2$$

where $k = 0.0002$, $C = 6000$, and x is the current population.

a Suppose hunters are allowed to kill 1000 deer per year. Sketch the graph of $H = 1000$ on the same axes with a graph of I.



b What sizes of deer populations will remain stable from year to year if 1000 deer are hunted annually?

c Suppose 1600 deer are killed annually. What sizes of deer populations will remain stable?

d What is the largest annual harvest that still allows for a stable population? (This harvest is called the maximum sustainable yield.) What is the stable population?

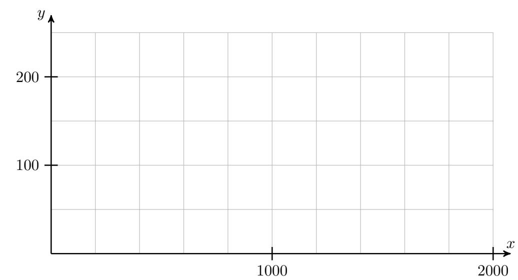
e What eventually happens if the population falls below the stable value but hunting continues at the maximum sustainable yield?

26. The annual increase, N, in a bear population of size x is

$$N = 0.0002x(2000 - x)$$

if the bears are not hunted. The number of bears killed each year by hunters is related to the bear population by the equation $K = 0.2x$. (Note that in this model, hunting limits are adjusted to the size of the bear population.)

a Graph N and K on the same axes for $0 \le x \le 2000$.



b When the bear population is 1200, which is greater, N or K? Will the population increas or decrease in the next year? By how many bears?

c When the bear population is 900, which is greater, N or K? Will the population increas or decrease in the next year? By how many bears?

d What sizes of bear population will remain stable after hunting?

e What sizes of bear population will increase despite hunting? What sizes will decrease?

f Toward what size will the bear population tend over time?

g Suppose hunting limits are raised so that $K = 0.3x$. Toward what size will the bear population tend over time?

For Problems 27–28,

a Find the break-even points by solving a system of equations.

b Graph the equations for revenue and cost in the same window and verify your solutions on the graph.

c Use the fact that

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

to find the value of x for which profit is maximum.

27. Writewell, Inc. makes fountain pens. It costs Writewell

$$C = 0.0075x^2 + x + 2100$$

dollars to manufacture x pens, and the company receives $R = 13x$ dollars in revenue from the sale of the pens. Use the window

Xmin = 0	Ymin = 0
Xmax = 1600	Ymax = 20,000

28. A company can produce x lawn mowers for a cost of

$$C = 0.125x^2 + 100,000$$

dollars. The sale of the lawn mowers will generate $R = 300x$ dollars in revenue. Use the window

Xmin = 0	Ymin = 0
Xmax = 3000	Ymax = 800,000

Chapter 9 Summary and Review

Glossary

- horizontal
- vertical
- parallel
- perpendicular
- conic section
- circle
- ellipse
- parabola
- hyperbola
- central conic
- major axis
- minor axis
- vertices
- covertices
- transverse axis
- conjugate axis
- asymptote
- branch

Key Concepts

Horizontal and Vertical Lines.

1

1 The equation of the **horizontal line** passing through $(0, b)$ is

$$y = b$$

2 The equation of the **vertical line** passing through $(a, 0)$ is

$$x = a$$

Slopes of Horizontal and Vertical Lines.

2 The slope of a **horizontal line** is *zero*.

The slope of a **vertical line** is *undefined*.

Parallel and Perpendicular Lines.

3

1 Two lines are **parallel** if their slopes are equal, that is, if

$$m_1 = m_2$$

or if both lines are vertical.

2 Two lines are **perpendicular** if the product of their slopes is -1, that is, if

$$m_1 m_2 = -1$$

or if one of the lines is horizontal and one is vertical.

4 The distance between two points is the length of the segment joining them.

Distance Formula.

5 The **distance** d between points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

6 The midpoint of a segment is the point halfway between its endpoints, so that the distance from the midpoint to either endpoint is the same.

Midpoint Formula.

7 The **midpoint** of the line segment joining the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is the point $M(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{x_1 + x_2}{2} \qquad \text{and } \overline{y} = \frac{y_1 + y_2}{2}$$

8 A circle is the set of all points in a plane that lie at a given distance, called the radius, from a fixed point called the center.

Circle.

9 The equation for a circle of radius r centered at the point (h, k) is

$$(x - h)^2 + (y - k)^2 = r^2$$

Ellipse.

10 The standard form for an ellipse centered at (h, k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Hyperbola.

11 The equation for a hyperbola centered at the point (h, k) has one of the two standard forms:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

Conic Sections.

12 The graph of $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is

1 a circle of $A = C$.

2 a parabola if $A = 0$ or $C = 0$ (but not both).

3 an ellipse if A and C have the same sign.

4 a hyperbola if A and C have opposite signs.

13 A system of two quadratic equations may have up to four solutions.

Chapter 9 Review Problems

For Problems 1 and 2, decide whether the lines are parallel, perpendicular, or neither.

1. $y = \frac{1}{2}x + 3; \quad x - 2y = 8$

2. $4x - y = 6; \quad x + 4y = -2$

3. Write an equation for the line that is parallel to the graph of $2x + 3y = 6$ and passes through the point $(1, 4)$

4. Write an equation for the line that is perpendicular to the graph of $2x + 3y = 6$ and passes through the point $(1, 4)$

5. Two vertices of the rectangle $ABCD$ are $A(3, 2)$ and $B(7, -4)$. Find an equation for the line that includes side $\overline{BC}$.

6. One leg of the right triangle PQR has vertices $P(-8, -1)$ and $Q(-2, -5)$. Find an equation for the line that includes the leg $\overline{QR}$.

7. Find the perimeter of the triangle with vertices $A(-1, 2)$, $B(5, 4)$, $C(1, -4)$. Is ΔABC a right triangle?

8. Find the midpoint of each side of ΔABC from the previous problem. Join the midpoints to form a new triangle, and find its perimeter.

For Problems 9–18, graph the conic section.

9. $x^2 + y^2 = 9$

10. $\dfrac{x^2}{9} + y^2 = 1$

11. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

12. $\dfrac{y^2}{6} - \dfrac{x^2}{8} = 1$

13. $4(y-2) = (x+3)^2$

14. $(x-2)^2 + (y+3)^2 = 16$

15. $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$

16. $\dfrac{(x+4)^2}{12} + \dfrac{(y-2)^2}{6} = 1$

17. $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$

18. $(x-2)^2 + 4y = 4$

For Problems 19–30

a Write the equation of each conic section in standard form.

b Identify the conic and describe its main features.

19. $x^2 + y^2 - 4x + 2y - 4 = 0$

20. $x^2 + y^2 - 6y - 4 = 0$

21. $4x^2 + y^2 - 16x + 4y + 4 = 0$

22. $8x^2 + 5y^2 + 16x - 20y - 12 = 0$

23. $x^2 - 8x - y + 6 = 0$

24. $y^2 + 6y + 4x + 1 = 0$

25. $x^2 + y = 4x - 6$

26. $y^2 = 2y + 2x + 2$

27. $2y^2 - 3x^2 - 16y - 12x + 8 = 0$

28. $9x^2 - 4y^2 - 72x - 24y + 72 = 0$

29. $2x^2 - y^2 + 6y - 19 = 0$

30. $4y^2 - x^2 + 8x - 28 = 0$

31. An ellipse centered at the origin has a vertical major axis of length 16 and a horizontal minor axis of length 10.

a Find the equation of the ellipse.

b What are the values of y when $x = 4$?

32. An ellipse centered at the origin has a horizontal major axis of length 26 and a vertical minor axis of length 18.

a Find the equation of the ellipse.

b What are the values of y when $x = 12$?

For Problems 33–38, write an equation for the conic section with the given properties.

33. Circle: center at $(-4, 3)$, radius $2\sqrt{5}$

34. Circle: endpoints of a diameter at $(-5, 2)$ and $(1, 6)$

35. Ellipse: center at $(-1, 4)$, $a = 4$, $b = 2$

36. Ellipse: vertices at $(3, 6)$ and $(3, -4)$, covertices at $(1, 1)$ and $(5, 1)$

37. Hyperbola: center $(2, -3)$, $a = 4$, $b = 3$, transverse axis horizontal

38. Hyperbola: one vertex at $(-4, 1)$, one end of the vertical conjugate axis at $(-3, 4)$

For Problems 39–42, graph the system of equations and state the solutions of the system

39. $4x^2 + y^2 = 25$
$x^2 - y^2 = -5$

40. $4x^2 - 9y^2 + 132 = 0$
$x^2 + 4y^2 - 67 = 0$

41. $x^2 + 3y^2 = 13$
$xy = -2$

42. $x^2 + y^2 = 17$
$2xy = -17$

For Problems 43–48, write and solve an equation or a system of equations

43. Moia drives 180 miles in the same time that Fran drives 200 miles. Find the speed of each if Fran drives 5 miles per hour faster than Moia.

44. The perimeter of a rectangle is 26 inches and the area is 12 square inches. Find the dimensions of the rectangle.

45. The perimeter of a rectangle is 34 centimeters and the area is 70 square centimeters. Find the dimensions of the rectangle.

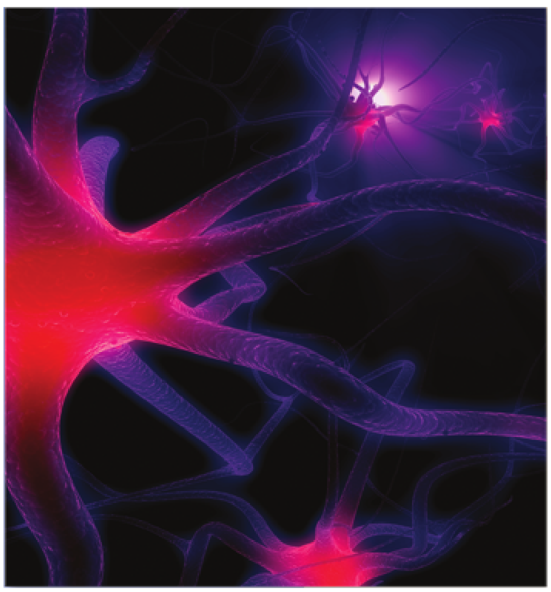
46. A rectangle has a perimeter of 18 feet. If the length is decreased by 5 feet and the width is increased by 12 feet, the area is doubled. Find the dimensions of the original rectangle.

47. Norm takes a commuter train 10 miles to his job in the city. The evening train returns him home at a rate 10 miles per hour faster than the morning traintakes him to work. If Norm spends a total of 50 minutes per day commuting, what is the rate of each train?

48. Hattie's annual income from an investment is \$32. If she had invested \$200 more and the rate had been 1/2% less, her annual income would have been \$35. What are the amount and rate of Hattie's investment?

Chapter 10

Logarithmic Functions

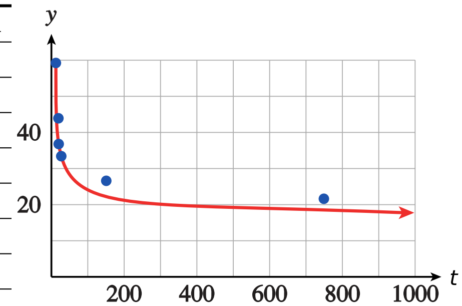


We have used logarithms to solve exponential equations. In this chapter, we consider logarithmic functions as models in their own right. We also study another base for exponential and logarithmic functions, the natural base e, which is the most useful base for many scientific applications.

In 1885, the German philosopher Hermann Ebbinghaus conducted one of the first experiments on memory, using himself as a subject. He memorized lists of nonsense syllables and then tested his memory of the syllables at intervals ranging from 20 minutes to 31 days. After one hour, he remembered less than 50% of the items, but he found that the rate of forgetting leveled off over time. He modeled his data by the function

$$y = \frac{184}{2.88 \log t + 1.84}$$

Time elapsed	Percent remembered
20 minutes	58.2%
1 hour	44.2%
9 hours	35.8%
1 day	33.7%
2 days	27.8%
6 days	25.4%
31 days	21.1%



Ebbinghaus's model uses a logarithmic function. The graph of the data, shown above with t in minutes and y in percents, is called the "forgetting curve." Ebbinghaus's work, including his application of the scientific method to his research, provides part of the foundation of modern psychology.

Investigation 10.1 Interest Compounded Continuously. We learned in Section 7.4, p. 495 that the amount, A (principal plus interest), accumulated in an account with interest compounded n times annually is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where P is the principal invested, r is the interest rate, and t is the time period, in years.

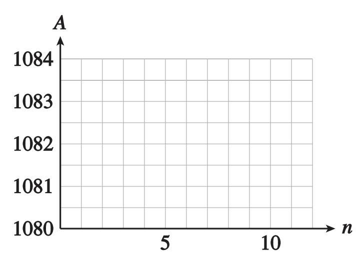
1 Suppose you keep \$1000 in an account that pays 8% interest. How much is the amount A after 1 year if the interest is compounded twice a year? Four times a year?

$$n = \mathbf{2} : A = 1000\left(1 + \frac{0.08}{\mathbf{2}}\right)^{\mathbf{2}(1)} =$$

$$n = \mathbf{4} : A = 1000\left(1 + \frac{0.08}{\mathbf{4}}\right)^{\mathbf{4}(1)} =$$

2 What happens to A as we increase n, the number of compounding periods per year? Fill in the table showing the amount in the account for different values of n.

n	A
1 (annually)	1080
2 (semiannually)	
4 (quarterly)	
6 (bimonthly)	
12 (monthly)	
365 (daily)	
1000	
10,000	



3 Plot the values in the table from $n = 1$ to $n = 12$, and connect them with a smooth curve. Describe the curve: What is happening to the value of A?

4 In part (2), as you increased the value of n, the other parameters in the formula stayed the same. In other words, A is a function of n, given by $A = 1000(1 + \dfrac{0.08}{n})^n$. Use your calculator to graph A on successively larger intervals:

a Xmin $= 0$, Xmax $= 12$; Ymin $= 1080$, Ymax $= 1084$

b Xmin $= 0$, Xmax $= 50$; Ymin $= 1080$, Ymax $= 1084$

c Xmin $= 0$, Xmax $= 365$; Ymin $= 1080$, Ymax $= 1084$

5 Use the **Trace** feature or the **Table** feature to evaluate A for very large values of n. Rounded to the nearest penny, what is the largest value of A that you can find?

6 As n increases, the values of A approach a limiting value. Although A continues to increase, it does so by smaller and smaller increments and will never exceed \$1083.29. When the number of compounding periods increases without bound, we call the limiting result **continuous compounding**.

7 Is there an easier way to compute A under continues compounding? Yes! Compute $1000e^{0.08}$ on your calculator. (Press 2nd LN to enter e^x.) Compare the value to your answer in part (5) for the limiting value. The number e is called the **natural base**. We'll compute its value shortly.

8 Repeat your calculations for two other interest rates, 15% and (an extremely unrealistic) 100%, again for an investment of \$1000 for 1 year. In each case, compare the limiting value of A, and compare to the value of $1000e^r$.

a

$r = 0.15$	
n	A
1	115
2	
4	
6	
12	
3652	
1000	
10,000	

$1000e^{0.15} =$

b

$r = 1$	
n	A
1	200
2	
4	
6	
12	
3652	
1000	
10,000	

$1000e^{1} =$

9 In part (8b), you have computed an approximation for $1000e$. What is the value of e, rounded to 5 decimal places?

10 Complete the table of values. What does $\left(1 + \dfrac{1}{n}\right)^n$ appear to approach as n increases?

n	100	1000	10,000	100,000
$\left(1+\frac{1}{n}\right)^n$				

Logarithmic Functions

In this section we study **logarithmic functions**. For example,

$$f(x) = \log_2 x$$

is a logarithmic function. In order to understand logarithmic functions better, we first investigate how they are related to more familiar functions, the exponential functions.

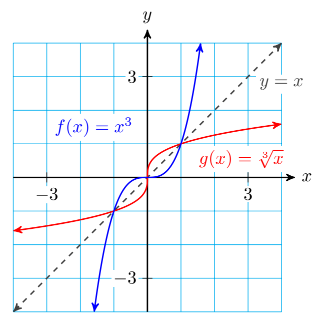
Inverse of a Function

You know that raising to the nth power and taking nth roots are *inverse operations*. For example, if we first cube a number and then take its cube root, we return to the original number.

$$x = \mathbf{5} \quad \rightarrow \quad x^3 = 125 \quad \rightarrow \quad \sqrt[3]{x^3} = \sqrt[3]{125} = \mathbf{5}$$

Cube the number   Take the cube root   Original number

Now consider the functions that describe those operations: The graphs of $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are related in an interesting way, as shown below.



If we place a mirror along the line $y = x$, each graph is the reflection of the other. We say that the graphs are "symmetric about the line $y = x$."

The two functions $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are called **inverse functions**. Look at the tables of values for the two functions.

x	-2	-1	$\frac{-1}{2}$	0	$\frac{1}{2}$	1	2	x	-8	-1	$\frac{-1}{8}$	0	$\frac{1}{8}$	1	8
$f(x)$	-8	-1	$\frac{-1}{8}$	0	$\frac{1}{8}$	1	8	$g(x)$	-2	-1	$\frac{-1}{2}$	0	$\frac{1}{2}$	1	2

By interchanging the rows in the table for $f(x)$, we get the table for $g(x)$. This makes sense when we recall that each function undoes the effect of the other. In fact, we define the cube root function by

$$y = \sqrt[3]{x} \quad \text{if and only if} \quad x = y^3$$

In other words, if we interchange the variables in the cubing function, $y = x^3$, to get $x = y^3$, we have an equivalent formula for the cube root function, $y = \sqrt[3]{x}$.

Checkpoint 10.1 Quickcheck 1. Simplify each expression without a calculator.

a. $\sqrt[4]{17^4} =$___

b. $(\sqrt{213})^2 =$___

c. $(\sqrt[5]{b})^5 =$___

Answer 1. 17

Answer 2. 213

Answer 3. b

Solution.

a. 17

b. 213

c. b

Inverse of an Exponential Function

A similar rule relates the operations of raising a base b to a power and taking a base b logarithm.

$$y = \log_b x \quad \text{if and only if} \quad x = b^y$$

So the function $g(x) = \log_b x$ is the inverse function for $f(x) = b^x$. Each function undoes the effect of the other. For example, let $f(x) = 2^x$ and $g(x) = \log_2 x$. Start with $x = 3$, apply f, and then apply g to the result.

$$x = \mathbf{3} \qquad \rightarrow \qquad f(3) = 2^3 = 8 \qquad \rightarrow \qquad g(8) = \log_2 8 = \mathbf{3}$$

We return to the original number, 3.

We can write these operations in one expression as

$$\log_2(2^{\mathbf{3}}) = \mathbf{3}$$

(Remember the order of operations: do what's inside of parentheses first.) Because the log and the exponential are inverse functions, this identity holds for any value of x and any base $b > 0$. That is

$$\log_b(b^x) = x$$

We can also apply the two functions in the opposite order, so that

$$2^{\log_2 \mathbf{8}} = \mathbf{8}$$

(This one is harder to see, but we compute the exponent, $\log_2 8$, first.) And in general

$$b^{\log_b x} = x$$

Checkpoint 10.2 Practice 1. Simplify each expression.

a. $\log_4 4^6 =$___

b. $4^{\log_4 64} =$___

c. $\log_{10} 10^6 =$___

d. $10^{\log_{10} 1000} =$___

Answer 1. 6
Answer 2. 64
Answer 3. 6
Answer 4. 1000
Solution.

a. 6

b. 64

c. 6

d. 1000

Graphs of Logarithmic Functions

What does the graph of a log function look like? We can use exponential functions to help us.

Example 10.3

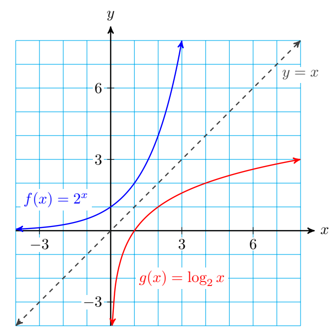
Graph $g(x) = \log_2 x$

Solution. We can make a table of values for $g(x) = \log_2 x$ by interchanging the columns in a table for $f(x) = 2^x$.

x	$f(x) = 2^x$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

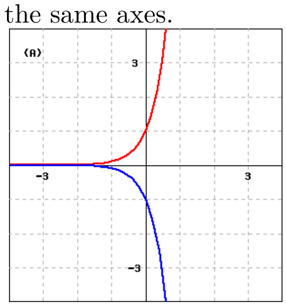
x	$g(x) = \log_2 x$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

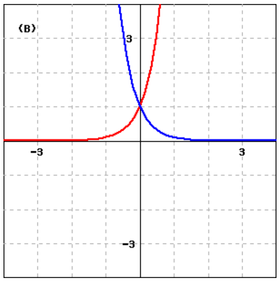
We plot the points for each function, connecting them with smooth curves, as shown below. You can see that the two graphs are symmetric about the line $y = x$.

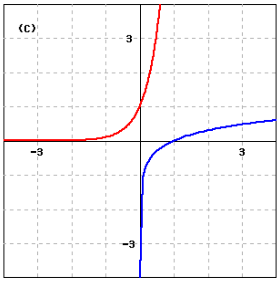


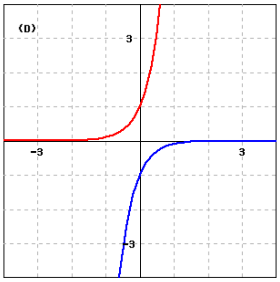
The same technique works for graphing a log function with any base.

Checkpoint 10.4 Practice 2. Graph the function $f(x) = 10^x$ and its inverse function $g(x) = \log_{10} x$ on the same axes.





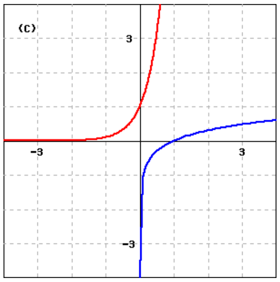




Which of the above is the best match for the graph?

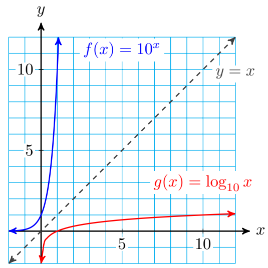
- ⊙ (A)
- ⊙ (B)
- ⊙ (C)
- ⊙ (D)

Answer. (C)
Solution.



A graph is also shown below.

$f(x) = 10^x$ and its inverse function $g(x) = \log_{10} x$:



Checkpoint 10.5 QuickCheck 2. Using a calculator, how could you estimate the value of $\log_5 378$?

- ⊙ Find multiples of 5.
- ⊙ Find the fifth root of 378.
- ⊙ Find powers of 5.
- ⊙ Divide 378 by 5.

Answer. Find powers of 5.

Solution. Find powers of 5.

Did you notice that the graphs of the log functions do not have any points with negative x-coordinates? This is because an exponential function has no negative (or zero) output values, so a log function has no negative or zero input values. In other words,

We cannot take the log of a negative number or zero.

In addition, the logarithmic function has the following properties.

Properties of Log Functions.

For any base $b > 0, b \neq 1$:

1. The logarithmic function $y = \log_b x$ is defined for positive x only.
2. The x-intercept of its graph is $(1, 0)$.
3. The graph has a vertical asymptote at $x = 0$.
4. The graphs of $y = \log_b x$ and $y = b^x$ are symmetric about the line $y = x$.

You can also see that while an exponential growth function increases very rapidly for positive input values, its inverse, the logarithmic function, grows extremely slowly.

Checkpoint 10.6 QuickCheck 3. What is the y-intercept of the graph of $y = \log_5 x$?

- ⊙ $(0, 1)$
- ⊙ $(1, 0)$
- ⊙ $(0, 5)$
- ⊙ There is none.

Answer. Choice 4

Solution. There is none: $\log_5 x$ not defined for $x = 0$ (nor for negative values of x).

Using Logarithmic Functions

We can use the LOG key on a calculator to evaluate the function $f(x) = \log_{10} x$.

Example 10.7

Let $f(x) = \log_{10} x$. Evaluate the following expressions.

a $f(35)$ b $f(-8)$ c $2f(16) + 1$

Solution.

a $f(35) = \log_{10} 35 \approx 1.544$

b $f(-8)$, or $\log_{10}(-8)$, is undefined.

c $2f(16) + 1 = 2(\log_{10} 16) + 1 \approx 2(1.204) + 1 = 3.408$

Checkpoint 10.8 Practice 3. Evaluate $H(t) = 20\log_{10}\dfrac{t}{C_L}$ for $t = 2$, where $C_L = 0.5$.

Answer: ________

Answer. $20\log(4)$

Solution. $20\log_{10} 4 \approx 12.04$

Example 10.9

Evaluate the expression $T = \dfrac{1}{k}\log_{10}\left(\dfrac{M_f}{M_0} + 1\right)$ for $k = 0.028$, $M_f = 1832$, and $M_0 = 15.3$.

Solution. Follow the order of operations and calculate

$$T = \frac{1}{0.028}\log_{10}\left(\frac{1832}{15.3} + 1\right) = \frac{\log_{10} 120.739}{0.028}$$
$$\approx \frac{2.082}{0.028} \approx 74.35$$

A calculator keying sequence for the calculation is

LOG 1832 ÷ 15.3 + 1) ÷ 0.028 ENTER

Checkpoint 10.10 Practice 4. The formula $T = \dfrac{\log 2 \cdot t_i}{3\log(D_f/D_0)}$ is used by X-ray technicians to calculate the doubling time of a malignant tumor. D_0 is the diameter of the tumor when first detected, D_f is its diameter at the next reading, and t_i is the time interval between readings, in days. Calculate the doubling time of the following tumor: its diameter when first detected was 1 cm, and 7 days later its diameter was 1.05 cm.

__ days

Answer. 33.149

Solution. 33 days

Logarithmic functions are useful for modeling increasing functions that slow down as the input increases.

Example 10.11

In 1900, the average life expectancy at birth in the U.S. was 47.3 years. Since then, life expectancy has increased according to the formula

$$L(x) = 18.53 + 28.92 \log x$$

where x is the number of years after 1900.

a Graph the life expectancy function for the years 1900 to 2010.

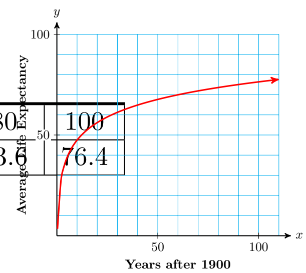
b The life expectancy in 1950 was 68.2 years. What does the function $L(x)$ predict for life expectancy in 1950?

c How much did life expectancy increase between 1920 and 1930? How much did it increase between 1990 and 2000?

Solution.

a We can make a table of values and plot points to obtain the graph below.

x	0	20	40	60	80	100
$L(x)$	47.3	56.2	64.9	70.0	73.6	76.4



b We substitute $x = 50$ into the function to find

$$L(50) = 18.53 + 28.92 \log 50 \approx 67.7$$

The function predicts a life expectancy of 67.7 years in 1950.

c Between 1920 and 1930, life expectancy increased from 56.2 to 61.2, or 5 years. Between 1990 and 2000 it increased from 75.0 to 76.4, or 1.4 years.

Checkpoint 10.12 **Practice 5.** The CDC (Centers for Disease Control and Prevention) provides Growth Charts for the average height and weight of children from age 2 to 20. The average height of girl children is given in centimeters by

$$H(t) = 49.29 + 91.3 \log t$$

where t is in years.

a. Graph the height function for $2 \le t \le 20$.

b. Use the height function to complete the table.

t	2	5	10	15	20
$H(t)$	___	___	___	___	___

a. How much is a girl's height expected to increase between the ages of 5 and 10? ___ cm.

Between the ages of 15 and 20? ___ cm.

Answer 1. 76.774

Answer 2. 113.106

Answer 3. 140.59

Answer 4. 156.667

Answer 5. 168.074

Answer 6. 27.484

Answer 7. 11.4069

Solution.

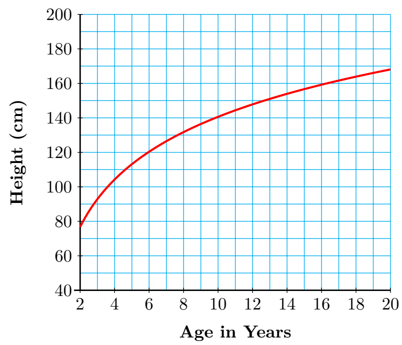
a. A graph is below.

b.

t	2	5	10	15	20
$H(t)$	76.774	113.106	140.59	156.667	168.074

c. 27.484 cm, 11.4069 cm

Graph for part (a):



Checkpoint 10.13 QuickCheck 4. Which function grows most slowly for large values of x ?

- ⊙ $f(x) = 2x$
- ⊙ $f(x) = \log_2 x$
- ⊙ $f(x) = \sqrt{x}$
- ⊙ $f(x) = \dfrac{x}{2}$

Answer. Choice 2

Solution. $f(x) = \log_2 x$

Solving Logarithmic Equations

A **logarithmic equation** is one in which the variable appears inside of a logarithm. For example,

$$\log_4 x = 3$$

is a log equation.

If there is only one log involved, we can use the conversion formula to write the equation in exponential form.

Example 10.14

Solve $\log_4 (2x - 8) = 3$.

Solution. We use the conversion formula to write the equation in exponential form.

$$\begin{aligned} 2x - 8 &= 4^3 && \text{Evaluate the power.} \\ 2x - 8 &= 64 && \text{Solve for } x. \\ x &= 36 \end{aligned}$$

The solution is 36.

Checkpoint 10.15 Practice 6. Solve $\log_5 (5 - 3x) = 3$

$x =$____

Answer. -40

Solution. $x = -40$

Caution 10.16 Remember that we cannot take a logarithm of a negative number or zero, because the output of an exponential function is always positive.

For example suppose that $\log_5 (-25) = x$. Then $5^x = -25$. But this is impossible, because 5^x cannot be negative. Thus, the log of a negative number (or zero) is undefined.

Because of this fact, extraneous solutions can arise when we solve logarithmic equations.

If the equation contains more than one log, we must first combine any expressions involving logs into a single logarithm.

Example 10.17

Solve $\log_{10} (x + 1) + \log_{10} (x - 2) = 1$.

Solution. We use Property (1) of logarithms (see Properties of Logarithms, p. 496) to rewrite the left-hand side as a single logarithm:

$$\log_{10} (x + 1)(x - 2) = 1$$

Once the left-hand side is expressed as a single logarithm, we can rewrite the equation in exponential form as

$$(x + 1)(x - 2) = 10^1$$

Simplifying the right side gives us a quadratic equation to solve.

$$\begin{aligned} x^2 - x - 2 &= 10 && \text{Subtract 10 from both sides.} \\ x^2 - x - 12 &= 0 && \text{Factor the left side.} \\ (x - 4)(x + 3) &= 0 && \text{Apply the zero-factor principle.} \end{aligned}$$

We find $x = 4$ or $x = -3$. But we must check the original equation for extraneous solutions. The number -3 is not a solution of the original equation, because neither $\log_{10} (x + 1)$ nor $\log_{10} (x - 2)$ is defined for $x = -3$. The apparent solution $x = -3$ is extraneous, and the solution of the original equation is 4.

Checkpoint 10.18 Practice 7. Solve $\log_{10} x + \log_{10} 2 = 3$

$x =$____

Hint. Hint: Rewrite the left side as a single logarithm.
Rewrite the equation in exponential form.
Solve for x.
Check for extraneous solutions.

Answer. 500

Solution. $x = 500$

Checkpoint 10.19 QuickCheck 5.

a. We cannot take a logarithm of (☐ a negative number or zero ☐ a fraction ☐ a variable expression) .

b. After solving a logarithmic equation, we must check for (☐ extraneous solutions ☐ irrational numbers) .

c. If an equation contains more than one log, we must first combine them into (☐ a common denominator ☐ a single logarithm) .

d. If there is only one log involved, we write the equation in (☐ standard ☐ general ☐ exponential ☐ point-slope) form.

Answer 1. a negative number or zero

Answer 2. extraneous solutions

Answer 3. a single logarithm

Answer 4. exponential

Solution.

a. a negative number or zero

b. extraneous solutions

c. a single logarithm

d. exponential form

Problem Set 10.1

Warm Up

For Problems 1–4, convert the logarithmic equation into exponential form.

1. $\log_9 729 = y$

2. $\log_b 8 = -3$

3. $\log_{10} C = -4.5$

4. $\log_m n = p$

For Problems 5–8, write the expression as a single log with a coefficient of 1.

5. $\dfrac{1}{2}\log_b 16 + 2\left(\log_b 2 - \log_b 8\right)$

6. $\dfrac{1}{2}\left(\log_5 6 + 2\log_5 4\right) - \log_5 2$

7. $\dfrac{1}{2}\left(\log_{10} y + \log_{10} x - 3\log_{10} z\right)$

8. $\dfrac{1}{3}\left(\log_{10} x - 2\log_{10} y - \log_{10} z\right)$

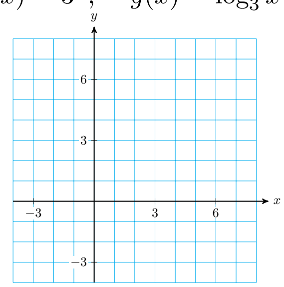
Skills Practice

For Problems 9 and 10,

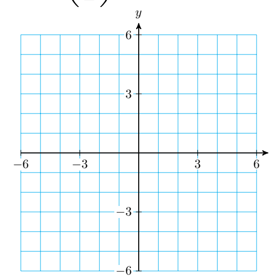
a Make a table of values for each function.

b Graph both functions on the same set of axes.

9. $f(x) = 3^x, \quad g(x) = \log_3 x$



10. $f(x) = \left(\dfrac{1}{2}\right)^x, \; g(x) = \log_{1/2} x$



For Problems 11 and 12, $f(x) = \log_{10} x$. Evaluate the expression, and round the answer to four decimal places.

11.

a $18 - 5f(3)$

b $\dfrac{2}{5 + f(0.6)}$

12.

a $15 - 4f(7)$

b $\dfrac{3}{2 + f(0.2)}$

For Problems 13 and 14, evaluate the expression.

13. $R = \dfrac{1}{L} \log_{10} \left(\dfrac{P}{L - P}\right)$, for $L = 8500$ and $P = 3600$

14. $M = \sqrt{\dfrac{\log_{10} H}{k \log_{10} H_0}}$, for $H = 0.93$, $H_0 = 0.02$, and $k = 0.006$

For Problems 15 and 16, $f(x) = \log_{10} x$. Solve for x.

15.

a $f(x) = 1.41$

b $f(x) = 0.52$

16.

a $f(x) = 0.8$

b $f(x) = -1.3$

For Problems 17 and 18, solve for the unknown variable.

17.

a $\log_2 y = -1$

b $\log_b 0.1 = -1$

18.

a $3(\log_7 x) + 5 = 7$

b $5(\log_2 x) + 6 = -14$

19. Let $f(x) = 3^x$ and $g(x) = \log_3 x$

a Compute $f(4)$

b Compute $g\left[f(4)\right]$

c Compute $\log_3 3^{1.8}$

d Compute $\log_3 3^a$

20. Let $f(x) = \log_2 x$ and $g(x) = 2^x$

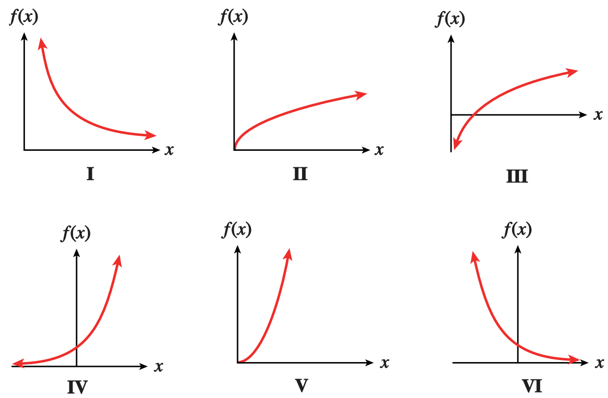
a Compute $f(32)$

b Compute $g\left[f(32)\right]$

c Compute $2^{\log_2 6}$

d Compute $2^{\log_2 Q}$

21. Each figure shows a portion of the graph of one of the following functions. Match each function with its graph.



a $f(x) = 2^x$

b $f(x) = x^2$

c $f(x) = \dfrac{2}{x}$

d $f(x) = \sqrt{x}$

e $f(x) = \log_2 x$

f $f(x) = \left(\dfrac{1}{2}\right)^x$

22. Choose the graph for each function described below.

a The area, A, of a pentagon is a quadratic function of the length l, of its side.

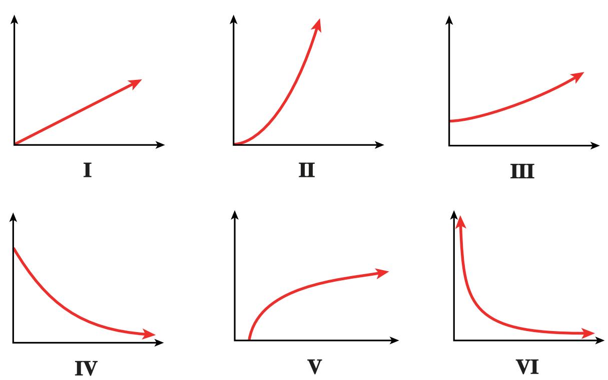
b The strength, F, of a hurricane varies inversely with its speed, s.

c The price of food has increased 3% every year for a decade.

d The magnitude, M, of a star is a logarithmic function of its brightness, I.

e The speed of the train increased at a constant rate.

f if you don't practice a foreign language, you lose $\dfrac{1}{8}$ of the words in your working vocabulary each year.



23.

a How large must x be before the graph of $y = \log_{10} x$ reaches a height of 4?

b How large must x be before the graph of $y = \log_{10} x$ reaches a height of 8?

24.

a How large must x be before the graph of $y = \log_2 x$ reaches a height of 5?

b How large must x be before the graph of $y = \log_2 x$ reaches a height of 10?

For Problems 25–28, solve the logarithmic equation.

25. $\log_{10} x + \log_{10}(x + 21) = 2$

26. $\log_8(x + 5) - \log_8 2 = 1$

27. $\log_{10}(x + 2) + \log_{10}(x - 1) = 1$

28. $\log_3(x - 2) - \log_3(x + 1) = 3$

Applications

29. In a psychology experiment, volunteers were asked to memorize a list of nonsense words, and 24 hours later they were tested to see how many of the words they recalled. On average, the subjects had forgotten 20% of the words. The researchers found that the more lists their volunteers memorized, the larger the fraction of words they were unable to recall. (Source: Underwood, *Scientific American*, vol. 210, no. 3)

Number of lists, n	1	4	8	12	16	20
Percent forgotten, F	20	40	55	66	74	80

(a) Plot the data. What sort of function seems to fit the data points?

(b) Psychologists often describe rates of forgetting by logarithmic functions. Graph the function

$$f(n) = 16.6 + 46.3 \log n$$

on the same graph with your data. Comment on the fit.

(c) What happens to the values of $f(n)$ as n grows increasingly large? Does this behavior accurately reflect the situation being modeled?

30. The water velocity at any point in a stream or river is related to the logarithm of the depth at that point. For the Hoback River near Bondurant, Wyoming,

$$v = 2.63 + 1.03 \log d$$

where v is the velocity of the water, in feet per second, and d is the vertical distance from the stream bed, in feet, at that point. For Pole Creek near Pinedale, Wyoming,

$$v = 1.96 + 0.65 \log d$$

Both streams are 1.2 feet deep at the locations mentioned. (Source: Leopold, Luna, Wolman, and Gordon, 1992)

(a) Complete the table of values for each stream.

Distance from bed (feet)	0.2	0.4	0.6	0.8	1.0	1.2
Velocity, Hoback River, (ft/sec)						
Velocity, Pole Creek (ft/sec)						

(b) If you double the distance from the bed, by how much does the velocity increase in each stream?

(c) Plot both functions on the same graph.

(d) The average velocity of the entire stream can be closely approximated as follows: Measure the velocity at 20% of the total depth of the stream from the surface and at 80% of the total depth, then average these two values. Find the average velocity for the Hoback River and for Pole Creek.

31. In Example 10.11, p. 682 of this section, we considered a formula for the average life expectancy in the U.S. as a function of years after 1900,

$$L(x) = 18.53 + 28.92 \log x$$

According to the formula, in what year was the life expectancy 70 years?

32. In Practice 5 of this section, we considerd a formula for the average height, in centimeters, of girls between the ages of 2 and 20.

$$H(t) = 49.29 + 91.3 \log t$$

According to the formula, at what age should a girl expect to be 152.4 centimeters (5 feet) tall?

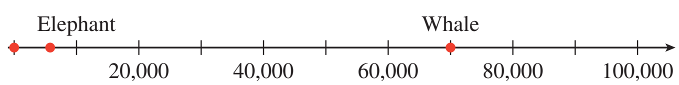
Logarithmic Scales

Making a Log Scale

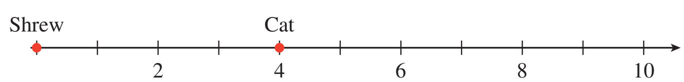
Because $\log x$ grows very slowly as x increases, logarithms are useful for modeling phenomena that take on a very wide range of values. For example, biologists study how metabolic functions such as heart rate are related to an animal's weight, or mass. The table shows the mass in kilograms of several mammals.

Animal	Shrew	Cat	Wolf	Horse	Elephant	Whale
Mass, kg	0.004	4	80	300	5400	70,000

Imagine trying to scale the x-axis to show all of these values. If we set tick marks at intervals of 10,000 kg, as shown below, we can plot the mass of the whale, and maybe the elephant, but the dots for the smaller animals will be indistinguishable.



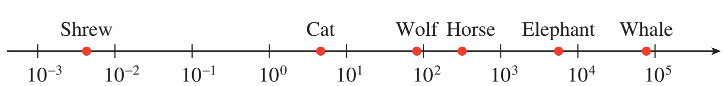
On the other hand, we can plot the mass of the cat if we set tick marks at intervals of 1 kg, but the axis will have to be extremely long to include even the wolf. We cannot show the masses of all these animals on the same scale



To get around this problem, we can plot the log of the mass, instead of the mass itself. The table below shows the base 10 log of each animal's mass, rounded to 2 decimal places.

Animal	Shrew	Cat	Wolf	Horse	Elephant	Whale
Mass, kg	0.004	4	80	300	5400	70,000
Log (mass)	−2.40	0.60	1.90	2.48	3.73	4.85

The logs of the masses range from -2.40 to 4.85. We can easily plot these values on a single scale, as shown below.



The scale above is called a **logarithmic scale**, or log scale. The tick marks are labeled with powers of 10, because, as you recall, a logarithm is actually an exponent. For example, the mass of the horse is 300 kg, and

$$\log 300 = 2.48 \quad \text{so} \quad 10^{2.48} = 300$$

When we plot 2.48 for the horse, we are really plotting the power of 10 that gives its mass, because $10^{2.48} = 300$ kg. The exponents on base 10 are evenly spaced on a log scale, so we plot $10^{2.48}$ about halfway between 10^2 and 10^3.

Example 10.20

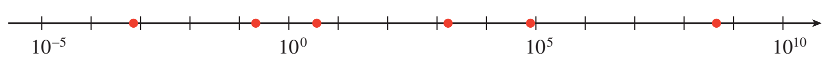
Plot the values on a log scale.

x	0.0007	0.2	3.5	1600	72,000	4×10^8

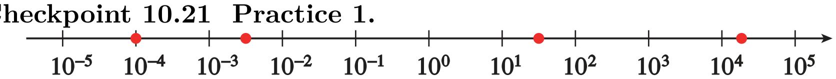
Solution. We actually plot the logs of the values, so we first compute the base 10 logarithm of each number.

x	0.0007	0.2	3.5	1600	72,000	4×10^8
$\log x$	−3.15	−0.70	0.54	3.20	4.86	8.60

Then we plot each logarithm, estimating its position between integer exponents. For example, we plot the first value, −3.15, closer to −3 than to −4. The finished plot is shown below.



Checkpoint 10.21 Practice 1.



Complete the table by estimating the logarithm of each point plotted on the log scale above. Then use a calculator to give a decimal value for each point.

$\log x$	___	___	___	___
x	_____	_____	_____	_____

Answer 1. −4

Answer 2. −2.5

Answer 3. 1.5

Answer 4. 4.25

Answer 5. 0.0001

Answer 6. 0.00316228

Answer 7. 31.6228

Answer 8. 17782.8

Solution.

$\log x$	−4	−2.5	1.5	4.25
x	0.0001	0.00316	31.6	17,782.8

Checkpoint 10.22 QuickCheck 1. Fill in the blanks.

a. We use (☐ logarithms ☐ imaginary numbers) for graphing variables that take on a very wide range of values.

b. The tick marks on a log scale are labeled with (☐ variables ☐ powers of 10) .

c. To plot values on a log scale, we first compute the (☐ reciprocals ☐ logarithms) of the values.

d. A fraction less than 1 is plotted on a log scale as a power with a (☐ negative ☐ positive) exponent.

Answer 1. logarithms

Answer 2. powers of 10

Answer 3. logarithms

Answer 4. negative

Solution.

a. logarithms

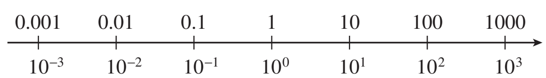
b. powers of 10

c. logs

d. negative

Labeling a Log Scale

Log scales allow us to plot a wide range of values, but there is a trade-off. Equal increments on a log scale do not correspond to equal differences in value, as they do on a linear scale. You can see this more clearly if we label the tick marks with their values, as well as powers of 10. The difference between 10^1 and 10^0 is $10 - 1 = 9$, but the difference between 10^2 and 10^1 is $100 - 10 = 90$.



As we move from left to right on this scale, we *multiply* the value at the previous tick mark by 10. Moving up by equal increments on a log scale does not add equal amounts to the values plotted; it *multiplies* the values by equal *factors*.

If we would like to label the log scale with integers, we get a very different looking scale, one in which the tick marks are not evenly spaced.

Example 10.23

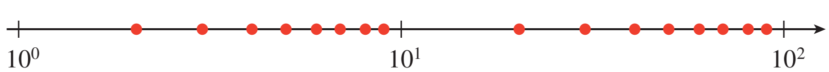
Plot the integer values 2 through 9 and 20 through 90 on a log scale.

Solution. We compute the logarithm of each integer value.

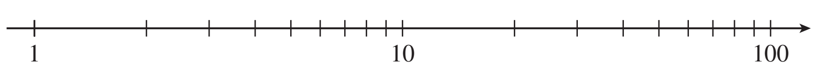
x	2	3	4	5	6	7	8	9
$\log x$	0.301	0.477	0.602	0.699	0.778	0.845	0.903	0.954

x	20	30	40	50	60	70	80	90
$\log x$	1.301	1.477	1.602	1.699	1.778	1.845	1.903	1.954

We plot on a log scale, as shown below.

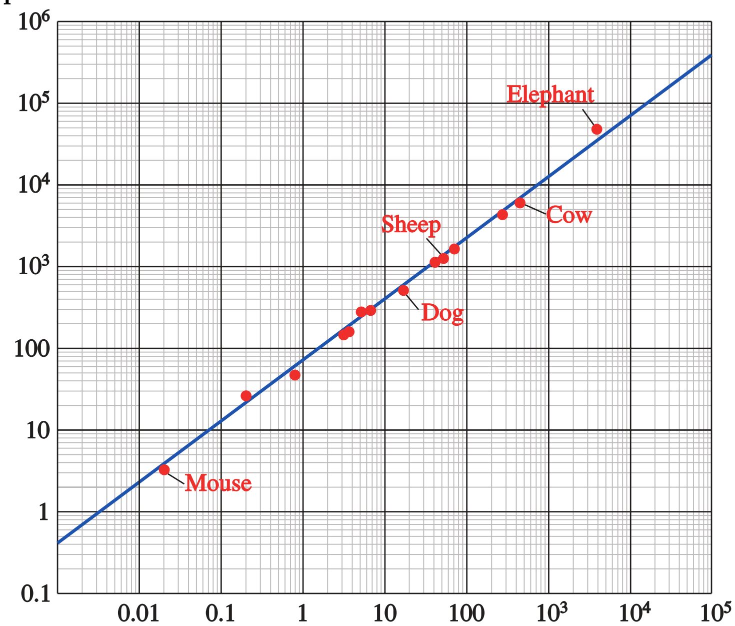


On the log scale in Example 10.23, p. 691, notice how the integer values are spaced: They get closer together as they approach the next power of 10. You will often see log scales labeled not with powers of 10, but with integer values, like this:



In fact, **log-log graph paper** scales both axes with logarithmic scales.

Checkpoint 10.24 Practice 2.



The opening page of Chapter 6 shows the "mouse-to-elephant" curve, a graph of the metabolic rate of mammals as a function of their mass. (The elephant does not appear on that graph, because its mass is too big.) The figure above shows the same function, graphed on log-log paper.

Use this graph to estimate the mass and metabolic rate for the following animals, labeled on the graph.

Animal	Mouse	Dog	Sheep
Mass (kg)	____	____	____
Metabolic rate (kcal/day)	____	____	____

Animal	Cow	Elephant
Mass (kg)	____	____
Metabolic rate (kcal/day)	____	____

Answer 1. 0.02

Answer 2. 15

Answer 3. 50

Answer 4. 3.5

Answer 5. 500

Answer 6. 1500

Answer 7. 500

Answer 8. 4000

Answer 9. 6000

Answer 10. 50000

Solution.

Animal	Mouse	Dog	Sheep	Cow	Elephant
Mass (kg)	0.02	15	50	500	4000
Metabolic rate (kcal/day)	3.5	500	1500	6000	$50,000$

Checkpoint 10.25 QuickCheck 2. True or False.

a. Equal increments on a log scale correspond to equal differences in value. (☐ True ☐ False)

b. Moving up by equal increments on a log scale multiplies the values by equal factors. (☐ True ☐ False)

c. If we label a log scale with integers, the tick marks are evenly spaced. (☐ True ☐ False)

d. On log-log graph paper, both axes are labeled with logarithmic scales. (☐ True ☐ False)

Answer 1. False

Answer 2. True

Answer 3. False

Answer 4. True

Solution.

a. False

b. True

c. False

d. True

Acidity and the pH Scale

You may have already encountered log scales in some everyday applications. A simple example is the **pH scale**, used by chemists to measure the acidity of a substance or chemical compound. This scale is based on the concentration of hydrogen ions in the substance, denoted by $[H^+]$. The pH value is defined by the formula

$$\text{pH} = -\log_{10}[H^+]$$

Values for pH fall between 0 and 14, with 7 indicating a neutral solution. The lower the pH value, the more acidic the substance. Some common substances and their pH values are shown in the table.

Substance	pH	$[H^+]$
Battery acid	1	0.1
Lemon juice	2	0.01
Vinegar	3	0.001
Milk	6.4	$10^{-6.4}$
Baking soda	8.4	$10^{-8.4}$
Milk of magnesia	10.5	$10^{-10.5}$
Lye	13	10^{-13}

A decrease of 1 on the pH scale corresponds to an increase in acidity by a factor of 10. Thus, lemon juice is 10 times more acidic than vinegar, and battery acid is 100 times more acidic than vinegar.

Example 10.26

a Calculate the pH of a solution with a hydrogen ion concentration of 3.98×10^{-5}.

b The water in a swimming pool should be maintained at a pH of 7.5. What is the hydrogen ion concentration of the water?

Solution.

a We use a calculator to evaluate the pH formula with $[H^+] = 3.98 \times 10^{-5}$.

$$\text{pH} = -\log_{10}\left(\mathbf{3.98 \times 10^{-5}}\right) \approx 4.4$$

b We solve the equation

$$\mathbf{7.5} = -\log_{10}[H^+]$$

for $[H^+]$. First, we write

$$-7.5 = \log_{10}[H^+]$$

Then we convert the equation to exponential form to get

$$[H^+] = 10^{-7.5} \approx 3.2 \times 10^{-8}$$

The hydrogen ion concentration of the water is 3.2×10^{-8}.

Checkpoint 10.27 Practice 3. The pH of the water in a tide pool is 8.3. What is the hydrogen ion concentration of the water?

Answer: ___________

Answer. $10^{-8.3}$

Solution. 5.01×10^{-9}

Decibels

The **decibel scale**, used to measure the loudness or intensity of a sound, is another example of a logarithmic scale. The perceived loudness of a sound is measured in decibels, D, by

$$D = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$$

where I is the intensity of its sound waves (in watts per square meter). The table below shows the intensity of some common sounds, measured in watts per square meter.

Sound	Intensity (watts/m^2)	Decibels
Whisper	10^{-10}	20
Background music	10^{-8}	40
Loud conversation	10^{-6}	60
Heavy traffic	10^{-4}	80
Jet airplane	10^{-2}	100
Thunder	10^{-1}	110

Consider the ratio of the intensity of thunder to that of a whisper:

$$\frac{\text{Intensity of thunder}}{\text{Intensity of a whisper}} = \frac{10^{-1}}{10^{-10}} = 10^9$$

Thunder is 10^9, or one billion times more intense than a whisper. It would be impossible to show such a wide range of values on a graph. When we use a log scale, however, there is a difference of only 90 decibels between a whisper and thunder.

Example 10.28

a Normal breathing generates about 10^{-11} watts per square meter at a distance of 3 feet. Find the number of decibels for a breath 3 feet away.

b Normal conversation registers at about 40 decibels. How many times more intense than breathing is normal conversation?

Solution.

a We evaluate the decibel formula with $I = \mathbf{10^{-11}}$ to find

$$\begin{aligned} D &= 10\log_{10}\left(\frac{\mathbf{10^{-11}}}{10^{-12}}\right) && \text{Subtract exponents: } -11-(-12)=1 \\ &= 10\log_{10}10^1 = 10(1) \\ &= 10 \text{ decibels} \end{aligned}$$

The sound of breathing registers at 10 decibels.

b We let I_b stand for the sound intensity of breathing, and I_c stand for the intensity of normal conversation. We are looking for the ratio I_c/I_b. From part (a), we know that

$$I_w = 10^{-11}$$

and we can calculate I_c from the decibelsformula.

$$\begin{aligned} 40 &= 10\log_{10}\left(\frac{I_c}{10^{-12}}\right) && \text{Divide both sides by 10.} \\ 4 &= \log_{10}\left(\frac{I_c}{10^{-12}}\right) && \text{Convert to exponential form.} \\ \frac{I_c}{10^{-12}} &= 10^4 && \text{Multiply both sides by } 10^{-12}. \\ I_c &= 10^4(10^{-12}) = 10^{-8} \end{aligned}$$

Finally, we compute the ratio $\frac{I_c}{I_b}$:

$$\frac{I_c}{I_b} = \frac{10^{-8}}{10^{-11}} = 10^3$$

Normal conversation is 1000 times more intense than breathing.

Checkpoint 10.29 **Practice 4.** The noise of city traffic registers at about 70 decibels.

a. What is the intensity of traffic noise, in watts per square meter?

Answer: $I =$___________ watts/m^2

b. How many times more intense is traffic noise than conversation?

Answer: ___ times

Answer 1. $10^{\frac{70}{10}-12}$

Answer 2. $10^{\frac{70-40}{10}}$

Solution.

a. $I = 10^{-5}$ watts/m^2

b. 1000

Caution 10.30 Both the decibel model and the Richter scale in the next example use expressions of the form $\log\left(\dfrac{a}{b}\right)$. Be careful to follow the order of operations when using these models. We must compute the quotient $\dfrac{a}{b}$ before taking a logarithm. In particular, it is *not* true that $\log\left(\dfrac{a}{b}\right)$ can be simplified to $\dfrac{\log a}{\log b}$.

The Richter Scale

One method for measuring the magnitude of an earthquake compares the amplitude A of its seismographic trace with the amplitude A_0 of the smallest detectable earthquake. The log of their ratio is the Richter magnitude, M. Thus,

$$M = \log_{10}\left(\frac{A}{A_0}\right)$$

Example 10.31

a The Northridge earthquake of January 1994 registered 6.9 on the Richter scale. What would be the magnitude of an earthquake 100 times as powerful as the Northridge quake?

b How many times more powerful than the Northridge quake was the San Francisco earthquake of 1989, which registered 7.1 on the Richter scale?

Solution.

a We let A_N represent the amplitude of the Northridge quake and A_H represent the amplitude of a quake 100 times more powerful. From the Richter model, we have

$$6.9 = \log_{10}\left(\frac{A_N}{A_0}\right)$$

or, rewriting in exponential form,

$$\frac{A_N}{A_0} = 10^{6.9}$$

Now, we want A_H to be 100 times A_N, so

$$A_H = 100A_N$$

$$= 100\left(10^{6.9}A_0\right) = 10^{8.9}A_0$$

Thus, the magnitude of the more powerful quake is

$$\begin{aligned}\log_{10}\left(\frac{A_H}{A_0}\right) &= \log_{10} 10^{8.9}\\ &= 8.9\end{aligned}$$

b We let A_S stand for the amplitude of the San Francisco earthquake. We are looking for the ratio A_S/A_N. First, we use the Richter formula to compute values for A_S and A_N.

$$6.9 = \log_{10}\left(\frac{A_N}{A_0}\right) \quad \text{and} \quad 7.1 = \log_{10}\left(\frac{A_s}{A_0}\right)$$

Rewriting each equation in exponential form, we have

$$\frac{A_N}{A_0} = 10^{6.9} \quad \text{and} \quad \frac{A_S}{A_0} = 10^{7.1}$$

or

$$A_N = 10^{6.9}A_0 \quad \text{and} \quad A_S = 10^{7.1}A_0$$

Now we can compute the ratio we want:

$$\frac{A_S}{A_N} = \frac{10^{7.1}A_0}{10^{6.9}A_0} = 10^{0.2}$$

The San Francisco earthquake was $10^{0.2}$, or approximately 1.58 times as powerful as the Northridge quake.

Checkpoint 10.32 **Practice 5.** In October 2005, a magnitude 7.6 earthquake struck Pakistan. How much more powerful was this earthquake than the 1989 San Francisco earthquake of magnitude 7.1?

Answer: ___

Answer. $10^{7.6-7.1}$

Solution. 3.16

Note 10.33 An earthquake 100, or 10^2, times as strong is only two units greater in magnitude on the Richter scale. In general, a difference of K units on the Richter scale (or any logarithmic scale) corresponds to a factor of 10^K units in the intensity of the quake.

Example 10.34

On a log scale, the weights of two animals differ by 1.6 units. What is the ratio of their actual weights?

Solution. A difference of 1.6 on a log scale corresponds to a factor of $10^{1.6}$ in the actual weights. Thus, the heavier animal is $10^{1.6}$, or 39.8 times as heavy as the lighter animal.

Checkpoint 10.35 **Practice 6.** Two points, labeled A and B, differ by 2.5 units on a log scale. What is the ratio of their decimal values?

Answer: _____

Answer. $10^{2.5}$

Solution. 316.2

Checkpoint 10.36 QuickCheck 3. True or False.

a. An increase of 1 on the pH scale corresponds to an increase in acidity by a factor of 10. (☐ True ☐ False)

b. A ratio of sound intensities of one billion corresponds to a difference of 90 decibels. (☐ True ☐ False)

c. The second property of logs says that $\log\left(\dfrac{a}{b}\right) = \dfrac{\log a}{\log b}$. (☐ True ☐ False)

d. A difference of 3 on a log scale corresponds to a ratio of 10^3, or 1000. (☐ True ☐ False)

Answer 1. False

Answer 2. True

Answer 3. False

Answer 4. True

Solution.

a. False

b. True

c. False

d. True

Problem Set 10.2

Warm Up

For Problems 1 and 2, bound the base 10 log of the number between two integers. Do not use a calculator!

1.

a 8 b 137 c 0.2 d $1,234,567$

2.

a 97 b 0.05 c 1.83 d $26,125$

For Problems 3 and 4, given $\log_{10} n$, find n.

3.

a 3.6 b 0.7 c -3.1 d -0.4

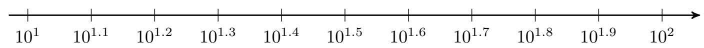
4.

a 1.5 b 5.2 c 0.18 d -2.5

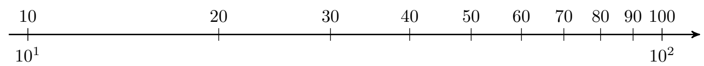
Skills Practice

5.

a The log scale is labeled with powers of 10. Finish labeling the tick marks in the figure with their corresponding decimal values.

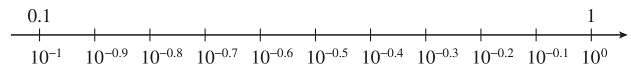


b The log scale is labeled with integer values. Label the tick marks in the figure with the corresponding powers of 10.

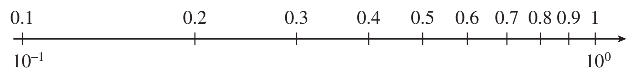


6.

(a) The log scale is labeled with powers of 10. Finish labeling the tick marks in the figure with their corresponding decimal values.



(b) The log scale is labeled with integer values. Label the tick marks in the figure with the corresponding powers of 10.



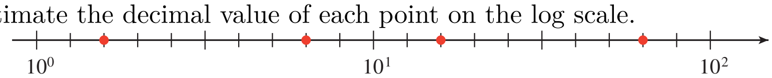
7. Plot the values on a log scale.

x	0.075	1.3	4200	87,000	6.5×10^7

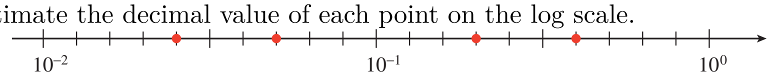
8. Plot the values on a log scale.

x	4×10^{-4}	0.008	0.9	27	90

9. Estimate the decimal value of each point on the log scale.



10. Estimate the decimal value of each point on the log scale.



In Problems 11–18, use the appropriate formulas for logarithmic models.

11. The hydrogen ion concentration of vinegar is about 6.3×10^{-4}. Calculate the pH of vinegar.

12. The hydrogen ion concentration of spinach is about 3.2×10^{-6}. Calculate the pH of spinach.

13. The pH of lime juice is 1.9. Calculate its hydrogen ion concentration.

14. The pH of ammonia is 9.8. Calculate its hydrogen ion concentration.

15. A lawn mower generates a noise of intensity 10^{-2} watts per square meter. Find the decibel level of the sound of a lawn mower.

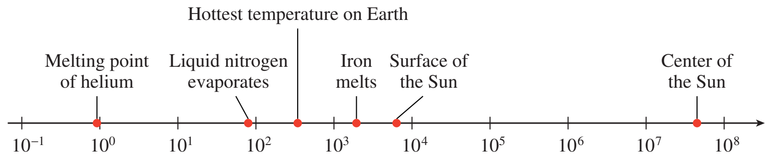
16. A jet airplane generates 100 watts per square meter at a distance of 100 feet. Find the decibel level for a jet airplane.

17. The loudest sound emitted by any living source is made by the blue whale. Its whistles have been measured at 188 decibels and are detectable 500 miles away. Find the intensity of the blue whale's whistle in watts per square meter.

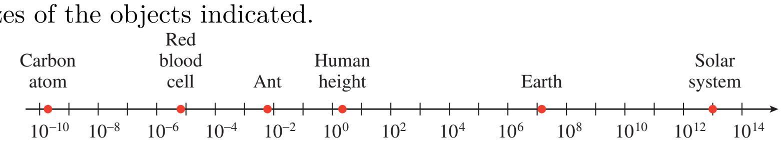
18. The noise of a leaf blower was measured at 110 decibels. What was the intensity of the sound waves?

Applications

19. The log scale shows various temperatures in Kelvins. Estimate the temperatures of the events indicated.



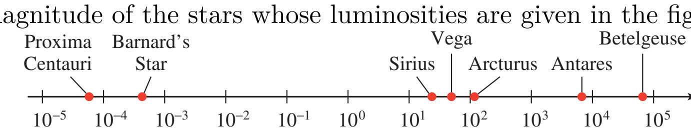
20. The log scale shows the size of various objects, in meters. Estimate the sizes of the objects indicated.



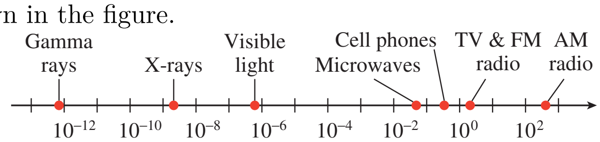
21. The magnitude of a star is a measure of its brightness. It is given by the formula

$$m = 4.83 - 2.5 \log L$$

where L is the luminosity of the star, measured in solar units. Calculate the magnitude of the stars whose luminosities are given in the figure.



22. Estimate the wavelength, in meters, of the types of electromagnetic radiation shown in the figure.



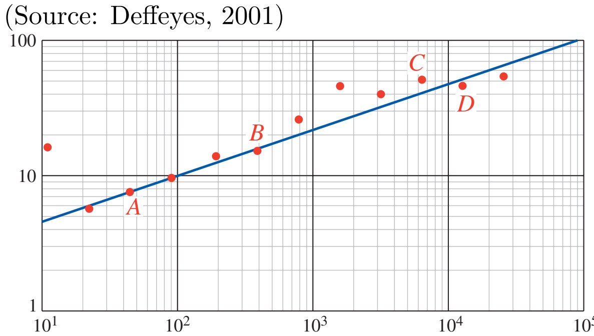
23. Plot the values of $[H^+]$ in the section "Acidity and the pH Scale" on a log scale.

24. Plot the values of sound intensity in the section "Decibels" on a log scale.

25. The distances to two stars are separated by 3.4 units on a log scale. What is the ratio of their distances?

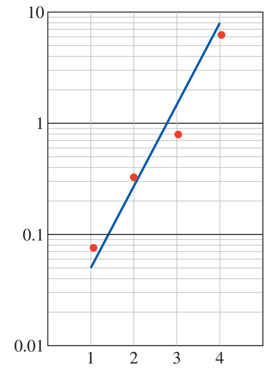
26. The populations of two cities are separated by 2.8 units on a log scale. What is the ratio of their populations?

27. The probability of discovering an oil field increases with its diameter, defined to be the square root of its area. Use the graph to estimate the diameter of the oil fields at the labeled points, and their probability of discovery. (Source: Deffeyes, 2001)



28.

The **order** of a stream is a measure of its size. Use the graph to estimate the drainage area, in square miles, for streams of orders 1 through 4. (Source: Leopold, Wolman, and Miller)



29. The pH of normal rain is 5.6. Some areas of Ontario have experienced acid rain with a pH of 4.5. How many times more acidic is acid rain than normal rain?

30. The pH of normal hair is about 5, the average pH of shampoo is 8, and 4 for conditioner. Compare the acidity of normal hair, shampoo, and conditioner.

31. At a concert by The Who in 1976, the sound level 50 meters from the stage registered 120 decibels. How many times more intense was this than a 90-decibel sound (the threshold of pain for the human ear)?

32. A refrigerator produces 50 decibels of noise, and a vacuum cleaner produces 85 decibels. How much more intense are the sound waves from a vacuum cleaner than those from a refrigerator?

33. In 1964, an earthquake in Alaska measured 8.4 on the Richter scale. An earthquake measuring 4.0 is consideredsmall and causes little damage. How many times stronger was the Alaska quake than one measuring 4.0?

34. On April 30, 1986, an earthquake in Mexico City measured 7.0 on the Richter scale. On September 21, a second earthquake occurred, this one measuring 8.1, hit Mexico City. How many times stronger was the September quake than the one in April?

The Natural Base

There is another base for logarithms and exponential functions that is often used in applications. This base is an irrational number called e, where

$$e \approx 2.71828182845$$

The number e is essential for many advanced topics, and it is often called the **natural base**.

The Natural Exponential Function

The **natural exponential function** is the function $f(x) = e^x$. Values for e^x can be obtained with a calculator using the $\boxed{e^x}$ key (2nd LN on most calculators). For example, you can evaluate e^1 by pressing

2nd LN 1

to confirm the value of e given above.

Checkpoint 10.37 Practice 1. Use your calculator to evaluate the following powers.

a. $e^2 \approx$_____

b. $e^{3.5} \approx$_____

c. $e^{-0.5} \approx$_____

Answer 1. e^2
Answer 2. $e^{3.5}$
Answer 3. $e^{-0.5}$
Solution.

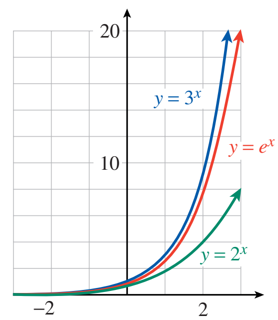
a. $e^2 \approx 7.389$

b. $e^{3.5} \approx 33.115$

c. $e^{-0.5} \approx 0.6065$

Because e is a number between 2 and 3, the graph of $f(x) = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$. Compare the tables of values and the graphs of the three functions below. You can verify the table and graphs on your caclulator.

x	$y = 2^x$	$y = e^x$	$y = 3^x$
-3	0.125	0.050	0.037
-2	0.250	0.135	0.111
-1	0.500	0.368	0.333
0	1	1	1
1	2	2.718	3
2	4	7.389	9
3	8	20.086	27



Checkpoint 10.38 QuickCheck 1. The value of e^2 is closest to

⊙ 3

⊙ 5

⊙ 7

⊙ 9

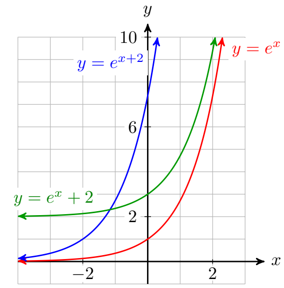
Answer. Choice 3
Solution. 7

Example 10.39

Graph each function. How does each graph differ from the graph of $y = e^x$?

a $g(x) = e^{x+2}$ b $h(x) = e^x + 2$

Solution.



The graph of g is shifted 2 units to the left of $y = e^x$. The graph of h is shifted 2 units up from $y = e^x$. The graphs are shown above.

The Natural Logarithmic Function

The base e logarithm of a number x, or $\log_e x$, is called the **natural logarithm** of x and is denoted by $\ln x$.

The Natural Logarithm.

The natural logarithm is the logarithm base e.

$$\ln x = \log_e x, \qquad x > 0$$

The natural logarithm of x is the exponent to which e must be raised to produce x. For example, the natural logarithm of 10, or $\ln 10$, is the solution of the equation

$$e^y = 10$$

You can verify on your calculator that

$$e^{\mathbf{2.3}} \approx 10 \text{ or } \quad \ln 10 \approx \mathbf{2.3}$$

In general, natural logs obey the same conversion formulas that work for logs to other bases.

Conversion Formulas for Natural Logs.

$$y = \ln x \qquad \text{if and only if} \qquad e^y = x$$

Checkpoint 10.40 QuickCheck 2. Which of the following is equivalent to $e^x = k$?

- ⊙ $e^k = x$
- ⊙ $\ln e = x$
- ⊙ $\ln k = x$
- ⊙ $k^x = e$

Answer. Choice 3

Solution. $\ln k = x$

In particular,

$$\ln e = 1 \quad \text{because} \quad e^1 = e$$
$$\ln 1 = 0 \quad \text{because} \quad e^0 = 1$$

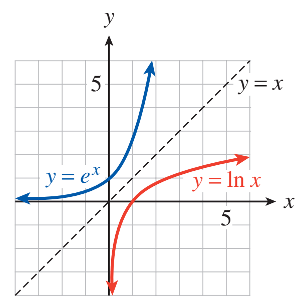
As is the case with exponential and log functions with other bases, the **natural log function**, and the natural exponential function, $f(x) = e^x$, "undo" each other, so they are inverse functions.

Example 10.41

Graph $f(x) = e^x$ and $g(x) = \ln x$ on the same grid.

Solution. We can make a table of values for $g(x) = \ln x$ by interchanging the columns in the table for $f(x) = e^x$. Plotting the points gives us the graph below.

x	$y = \ln x$
0.050	−3
0.135	−2
0.368	−1
1	0
2.718	1
7.389	2
20.086	3



You can see from the graph that the natural log function has only positive numbers as input values. The natural logs of negative numbers and zero are undefined. You can also see that the natural log of a number greater than 1 is positive, while the logs of fractions between 0 and 1 are negative.

Checkpoint 10.42 **Practice 2.** Use your calculator to evaluate each logarithm. Round your answers to four decimal places.

a. $\ln 100 \approx$_____

b. $\ln 0.01 \approx$_____

c. $\ln e^3 \approx$_____

Answer 1. $\ln(100)$
Answer 2. $\ln(0.01)$
Answer 3. $\ln(e^3)$
Solution.

a. $\ln 100 \approx 4.6052$

b. $\ln 0.01 \approx -4.6052$

c. $\ln e^3 = 3$

The three properties of logarithms also apply to base e logarithms.

Properties of Natural Logarithms.

If $x, y > 0$, then

1 $\ln(xy) = \ln x + \ln y$

2 $\ln \frac{x}{y} = \ln x - \ln y$

3 $\ln x^n = n \ln x$

If $x \leq 0$, $\ln x$ is undefined.

And because the functions $y = e^x$ and $y = \ln x$ are inverse functions, the following properties are also true.

The Natural Log and e^x.

$$\ln e^x = x \qquad \text{and} \qquad e^{\ln x} = x$$

Example 10.43

Simplify each expression.

a $\ln e^{0.3x}$

b $e^{2\ln(x+3)}$

Solution.

a The natural log is the log base e, and hence the inverse of e^x. Therefore,

$$\ln e^{0.3x} = 0.3x$$

b First, we simplify the exponent using the third property of logs to get

$$2\ln(x+3) = \ln(x+3)^2$$

Then $e^{2\ln(x+3)} = e^{\ln(x+3)^2} = (x+3)^2$.

Checkpoint 10.44 **Practice 3.** Simplify each expression. Use "sqrt(x)" to get $\sqrt{x}$.

a. $e^{(\ln x)/2} =$______

b. $\ln\left(\frac{1}{e^{4x}}\right) =$______

Answer 1. $\sqrt{x}$

Answer 2. $-4x$

Solution.

a. $\sqrt{x}$

b. $-4x$

Solving Equations

We use the natural logarithm to solve exponential equations with base e. The techniques we've learned for solving other exponential equations also apply to equations with base e.

Example 10.45

Solve each equation for x.

a $e^x = 0.24$

b $\ln x = 3.5$

Solution.

a We convert the equation to logarithmic form and evaluate using a calculator.

$$x = \ln 0.24 \approx -1.427$$

b We convert the equation to exponential form and evaluate.

$$x = e^{3.5} \approx 33.1155$$

Checkpoint 10.46 Practice 4. Solve each equation. Round your answers to four decimal places.

a. $\ln x = -0.2$

$x \approx$______

b. $e^x = 8$

$x \approx$______

Answer 1. $e^{-0.2}$

Answer 2. $\ln(8)$

Solution.

a. 0.8187

b. 2.0794

Checkpoint 10.47 QuickCheck 3. Why is the equation $e^x = 6.5$ easier to solve than $8^x = 6.5$?

- ⊙ 8 is larger than 6.5.
- ⊙ k is a constant.
- ⊙ There is a button for log base e on the calculator, but not a button for log base 8.
- ⊙ Because e is an irrational number.

Answer. Choice 3

Solution. There is a button for log base on the calculator, but not a button for log base 8.

To solve more complicated exponential equations, we isolate the power on one side of the equation before converting to logarithmic form.

Example 10.48

Solve $140 = 20e^{0.4x}$

Solution. First, we divide each side by 20 to obtain

$$7 = e^{0.4x}$$

Then we convert the equation to logarithmic form.

$$0.4x = \ln 7 \qquad \text{Divide both sides by 0.4.}$$

$$x = \frac{\ln 7}{0.4}$$

Rounded to four decimal places, $x \approx 4.8648$.

Note 10.49 We can also solve the equation in the Example above,

$$7 = e^{0.4x}$$

by taking the natural logarithm of both sides. This gives us

$$\ln 7 = \ln e^{0.4x} \qquad \text{Simplify the right side.}$$

$$\ln 7 = 0.4x$$

because $\ln e^a = a$ for any number a. We then proceed with the solution as before.

Checkpoint 10.50 **Practice 5.** Solve

$$80 - 16e^{-0.2x} = 70.3$$

$x =$______________

Answer. $-5\ln\left(\frac{9.7}{16}\right)$

Solution. $x = -5\ln\left(\dfrac{9.7}{16}\right) \approx 2.5023$

Example 10.51

Solve $P = \dfrac{a}{1 + be^{-kt}}$ for t.

Solution. We multiply both sides of the equation by the denominator, $1 + be^{-kt}$, to get

$$P(1 + be^{-kt}) = a$$

Then we isolate the power, e^{-kt}, as follows:

$$P(1 + be^{-kt}) = a \qquad \text{Divide both sides by } P.$$

$$1 + be^{-kt} = \frac{a}{P} \qquad \text{Subtract 1 from both sides.}$$

$$be^{-kt} = \frac{a}{P} - 1 \qquad \text{Rewrite the right side as one fraction.}$$

$$be^{-kt} = \frac{a - P}{P} \qquad \text{Divide both sides by } b.$$

$$e^{-kt} = \frac{a - P}{bP}$$

Next, we take the natural logarithm of both sides to get

$$\ln e^{-kt} = \ln \frac{a - P}{bP}$$

and recall that $\ln e^x = x$ to simplify the left side.

$$-kt = \ln \frac{a - P}{bP}$$

Finally, we divide both sides by $-k$ to solve for t.

$$t = \frac{-1}{k} \ln \frac{a - P}{bP}$$

Checkpoint 10.52 Practice 6. Solve $N = Ae^{-kt}$ for k.

$k =$______________

Hint. Divide both sides by A.
Take the natural log of both sides.
Divide both sides by $-t$.

Answer. $\frac{-\left(\ln\left(\frac{N}{A}\right)\right)}{t}$

Solution. $k = \dfrac{-\ln(N/A)}{t}$

Exponential Growth and Decay

Recall that functions of the form

$$P(t) = P_0 \cdot b^t$$

describe exponential growth when $b > 1$ and exponential decay when $0 < b < 1$. Exponential growth and decay can also be modeled by functions of the form

$$P(t) = P_0 \cdot e^{kt}$$

where we have substituted e^k for the growth factor b, so that

$$\begin{aligned} P(t) &= P_0 \cdot b^t \\ &= P_0 \cdot \left(e^k\right)^t = P_0 \cdot e^{kt} \end{aligned}$$

We can find the value of k by solving the equation $b = e^k$ for k, to get $k = \ln b$.

For instance, consider a colony of bacteria that grows according to the formula

$$P(t) = 100 \cdot \mathbf{3^t}$$

We can express this function in the form $P(t) = 100 \cdot \boldsymbol{e^{kt}}$ if we set

$$3 = e^k \quad \text{or} \quad k = \ln 3 \approx 1.0986$$

Thus, the growth law for the colony of bacteria can be written

$$P(t) \approx 100 \cdot e^{1.0986t}$$

By graphing both functions on your calculator, you can verify that

$$P(t) = 100 \cdot 3^t \quad \text{and} \quad P(t) = 100 \cdot e^{1.0986t}$$

are just two ways of writing the same function.

Example 10.53

From 1990 to 2000, the population of Clark County, Nevada, grew by 6.4% per year.

a What was the growth factor for the population of Clark County from 1990 to 2000? If the population of Clark County was 768,000 in 1990, write a formula for the population t years later.

b Write a growth formula for Clark County using base e.

Solution.

a The growth factor was $b = 1 + r = 1.064$. The population t years later was
$$P(t) = 768,000(1.064)^t$$

b We use the formula $P(t) = P_0 \cdot e^{kt}$, where $e^k = 1.064$. Solving for k, we find
$$k = \ln 1.064 = 0.062$$
so $P(t) = 768,000e^{0.062t}$.

Checkpoint 10.54 Practice 7. From 1994 to 1998, the number of personal computers connected to the Internet grew according to the formula $N(t) = 2.8e^{0.85t}$, where $t = 0$ in 1994 and N is in millions. (Source: Los Angeles Times, September 6, 1999)

a. Evaluate $N(1) =$___ . By what percent did the number of Internet users grow in one year?

About ___%

b. Express the growth law in the form $N(t) = N_0(1 + r)^t$.

$N(t) =$______________

Hint. $e^k = 1 + r$

Answer 1. $2.8e^{0.85}$

Answer 2. $\left(e^{0.85} - 1\right) \cdot 100$

Answer 3. $2.8 \cdot 2.33965^t$

Solution.

a. $N(1) \approx 6.55, \;\; 134\%$

b. $N(t) \approx 2.8(2.3396)^t$

If k is negative, then e^k is a fraction less than 1. For example, if $k = -2$,
$$e^{-2} = \frac{1}{e^2} \approx \frac{1}{7.3891} \approx 0.1353$$

Thus, for negative values of k, the function $P(t) = P_0e^{kt}$ describes exponential decay.

Exponential Growth and Decay.

The function
$$P(t) = P_0e^{kt}$$
describes exponential growth if $k > 0$, and exponential decay if $k < 0$.

Checkpoint 10.55 QuickCheck 4. The natural log of a fraction between 0 and 1 is

⊙ positive.

- ⊙ negative.
- ⊙ undefined.
- ⊙ between e^0 and e^1.

Answer. Choice 2

Solution. negative.

Example 10.56

Express the decay law $N(t) = 60(0.8)^t$ in the form $N(t) = N_0 e^{kt}$.

Solution. For this decay law, $N_0 = 60$ and $b = 0.8$. We would like to find a value for k so that $e^k = b = 0.8$, that is, we must solve the equation

$$\begin{aligned} e^k &= 0.8 && \text{Take natural log of both sides.} \\ \ln e^k &= \ln 0.8 && \text{Simplify.} \\ k &= \ln 0.8 \approx -0.2231 \end{aligned}$$

Replacing b with e^k, we find that the decay law is

$$N(t) \approx 60e^{-0.2231t}$$

Checkpoint 10.57 Practice 8. A scientist isolates 25 grams of krypton-91, which decays according to the formula

$$N(t) = 25e^{-0.07t},$$

where t is in seconds.

a. Complete the table of values showing the amount of krypton-91 left at 10-second intervals over the first minute.

t	0	10	20	30
$N(t)$	___	___	___	___

t	40	50	60
$N(t)$	___	___	___

b. Use the table to choose a suitable window and graph the function $N(t)$.

c. Write and solve an equation to answer the question: How long does it take for 60% of the krypton-91 to decay?

- ⊙ $25e^{-0.07t} = 0.60$
- ⊙ $25e^{-0.07t} = 0.60(25)$
- ⊙ $25e^{-0.07t} = 0.40$
- ⊙ $25e^{-0.07t} = 0.40(25)$
- ⊙ None of the above

Answer: $t =$___________ seconds

Hint. If 60% of the krypton-91 has decayed, 40% of the original 25 grams remains.

Answer 1. 25

Answer 2. 12.4146

Answer 3. 6.16492

Answer 4. 3.06141

Answer 5. 1.52025

Answer 6. 0.754935

Answer 7. 0.374889

Answer 8. Choice 4

Answer 9. $\frac{\ln(0.4)}{-0.07}$

Solution.

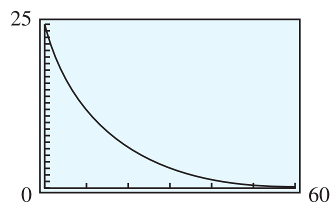
a.

t	0	10	20	30	40	50	60
$N(t)$	25	12.41	6.16	3.06	1.52	0.75	0.37

b. A graph is below.

c. $25e^{-0.07t} = 0.40(25)$; $t = \dfrac{\ln(0.4)}{-0.07} \approx 13.09$ seconds

Graph for part (b)



Continuous Compounding

Some savings institutions offer accounts on which the interest is **compounded continuously**. The amount accumulated in such an account after t years at interest rate r is given by the function

$$A(t) = Pe^{rt}$$

where P is the principal invested.

Example 10.58

Suppose you invest \$500 in an account that pays 8% interest compounded continuously. You leave the money in the account without making any additional deposits or withdrawals.

a Write a formula that gives the value of your account $A(t)$ after t years.

b Make a table of values showing $A(t)$ for the first 5 years.

c Graph the function $A(t)$.

d How much will the account be worth after 10 years?

e How long will it be before the account is worth \$1000?

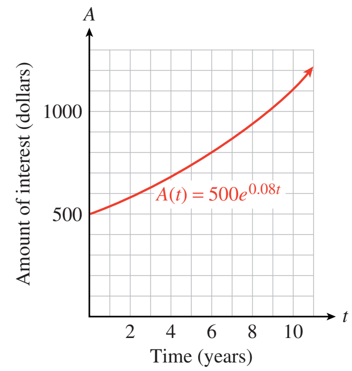
Solution.

a We substitute 500 for P, and 0.08 for r to find

$$A(t) = 500e^{0.08t}$$

b We evaluate the formula for $A(t)$ to obtain a table.

t	$A(t)$
0	500
1	541.64
2	586.76
3	635.62
4	688.56
5	745.91



c The graph of $A(t)$ is shown above.

d We evaluate $A(t)$ for $t = 10$.

$$\begin{aligned} A(10) &= 500e^{0.08(10)} \\ &= 500e^{0.8} \\ &\approx 500(2.2255) = 1112.77 \end{aligned}$$

The account will be worth \$1112.77 after 10 years.

e We substitute 1000 for $A(t)$ and solve the equation.

$$\begin{aligned} 1000 &= 500e^{0.08t} && \text{Divide both sides by 500.} \\ 2 &= e^{0.08t} && \text{Take natural log of both sides.} \\ \ln 2 &= \ln e^{0.08t} = 0.08t && \text{Divide both sides by 0.08.} \\ t &= \frac{\ln 2}{0.08} \approx 8.6643 \end{aligned}$$

The account will be worth \$1000 after approximately 8.7 years.

Checkpoint 10.59 Practice 9. Zelda invested \$1000 in an account that pays 4.5% interest compounded continuously. How long will it be before the account is worth \$2000?

Answer: about ___ years.

Answer. $\frac{\ln(2)}{0.045}$

Solution. About 15.4 years

Homework 10.3

Skills Practice

For Problems 1-4, use your calculator to complete the table for each function. Then choose a suitable window and graph the function.

x	-10	-5	0	5	10	15	20
$f(x)$							

1. $f(x) = e^{0.2x}$

2. $f(x) = e^{0.6x}$

3. $f(x) = e^{-0.3x}$

4. $f(x) = e^{-0.1x}$

For Problems 5 and 6, simplify.

5.

(a) $\ln e^2$ (b) $e^{\ln 5t}$ (c) $e^{-\ln x}$ (d) $\ln \sqrt{e}$

6.

(a) $\ln e^{x^4}$ (b) $e^{3\ln x}$ (c) $e^{\ln x - \ln y}$ (d) $\ln\left(\dfrac{1}{e^{2t}}\right)$

For Problems 7–10, solve for x. Round your answers to two decimal places.

7.

(a) $e^x = 1.9$ (b) $e^x = 45$ (c) $e^x = 0.3$

8.

(a) $e^x = 2.1$ (b) $e^x = 60$ (c) $e^x = 0.9$

9.

(a) $\ln x = 1.42$ (b) $\ln x = 0.63$ (c) $\ln x = -2.6$

10.

(a) $\ln x = 2.03$ (b) $\ln x = 0.59$ (c) $\ln x = -3.4$

For Problems 11-14, express each exponential function in the form $P(t) = P_0 b^t$. Is the function increasing or decreasing? What is its initial value?

11. $P(t) = 20e^{0.4t}$

12. $P(t) = 0.8e^{1.3t}$

13. $P(t) = 6500e^{-2.5t}$

14. $P(t) = 1.7e^{-0.02t}$

15.

(a) Fill in the table, rounding your answers to four decimal places.

x	0	0.5	1	1.5	2	2.5
e^x						

(b) Compute the ratio of each function value to the previous one. Explain the result.

16.

(a) Fill in the table, rounding your answers to four decimal places.

x	0	2	4	6	8	10
e^x						

(b) Compute the ratio of each function value to the previous one. Explain the result.

17.

(a) Fill in the table, rounding your answers to the nearest integer.

x	0	0.6931	1.3863	2.0794	2.7726	3.4657	4.1589
e^x							

(b) Subtract each x-value from the next one. Explain the result.

18.

(a) Fill in the table, rounding your answers to the nearest integer.

x	0	1.0986	2.1972	3.2958	4.3944	5.4931	6.5917
e^x							

(b) Subtract each x-value from the next one. Explain the result.

For Problems 19–26, solve. Round your answers to two decimal places.

19. $6.21 = 2.3e^{1.2x}$

20. $22.26 = 5.3e^{0.4x}$

21. $6.4 = 20e^{0.3x} - 1.8$

22. $4.5 = 4e^{2.1x} + 3.3$

23. $46.52 = 3.1e^{1.2x} + 24.2$

24. $1.23 = 1.3e^{2.1x} - 17.1$

25. $16.24 = 0.7e^{-1.3x} - 21.7$

26. $55.68 = 0.6e^{-0.7x} + 23.1$

For Problems 27-32, solve the equation for the specified variable.

27. $y = e^{kt}$, for t

28. $\dfrac{T}{R} = e^{t/2}$, for t

29. $y = k(1 - e^{-t})$, for t

30. $B - 2 = (A + 3)e^{-t/3}$, for t

31. $T = T_0 \ln(k + 10)$, for k

32. $P = P_0 + \ln 10k$, for k

33.

(a) Fill in the table, rounding your answers to three decimal places.

n	0.39	3.9	39	390
$\ln n$				

(b) Subtract each natural logarithm in your table from the next one. (For example, compute $\ln 3.9 - \ln 0.39$.) Explain the result.

34.

(a) Fill in the table, rounding your answers to three decimal places.

n	0.64	6.4	64	640
$\ln n$				

(b) Subtract each natural logarithm in your table from the next one. (For example, compute $\ln 6.4 - \ln 0.64$.) Explain the result.

35.

(a) Fill in the table, rounding your answers to three decimal places.

n	2	4	8	16
$\ln n$				

(b) Divide each natural logarithm in your table by $\ln 2$. Explain the result.

36.

(a) Fill in the table, rounding your answers to three decimal places.

n	5	25	125	625
$\ln n$				

(b) Divide each natural logarithm in your table by $\ln 5$. Explain the result.

For Problems 37–42,

a Express each growth or decay law in the form $N(t) = N_0 e^{kt}$.

b Check your answer by graphing both forms of the function on the same axes. Do they have the same graph?

37. $N(t) = 100 \cdot 2^t$

38. $N(t) = 50 \cdot 3^t$

39. $N(t) = 1200(0.6)^t$

40. $N(t) = 300(0.8)^t$

41. $N(t) = 10(1.15)^t$

42. $N(t) = 1000(1.04)^t$

Applications

43. The number of bacteria in a culture grows according to the function

$$N(t) = N_0 e^{0.04t}$$

where N_0 is the number of bacteria present at time $t = 0$ and t is the time in hours.

(a) Write a growth law for a sample in which 6000 bacteria were present initially.

(b) Make a table of values for $N(t)$ in 5-hour intervals over the first 30 hours.

(c) Graph $N(t)$.

(d) How many bacteria were present at $t = 24$ hours?

(e) How much time must elapse (to the nearest tenth of an hour) for the original 6000 bacteria to increase to 100,000?

44. Hope invests $2000 in a savings account that pays $5\frac{1}{2}\%$ annual interest compounded continuously.

(a) Write a formula that gives the amount of money $A(t)$ in Hope's account after t years.

(b) Make a table of values for $A(t)$ in 2-year intervals over the first 10 years.

(c) Graph $A(t)$.

(d) How much will Hope's account be worth after 7 years?

(e) How long will it take for the account to grow to $5000?

45. The intensity, I (in lumens), of a light beam after passing through t centimeters of a filter having an absorption coefficient of 0.1 is given by the function

$$I(t) = 1000e^{-0.1t}$$

(a) Graph $I(t)$.

(b) What is the intensity (to the nearest tenth of a lumen) of a light beam that has passed through 0.6 centimeter of the filter?

(c) How many centimeters (to the nearest tenth) of the filter will reduce the illumination to 800 lumens?

46. X-rays can be absorbed by a lead plate so that

$$I(t) = I_0 e^{-1.88t}$$

where I_0 is the X-ray count at the source and $I(t)$ is the X-ray count behind a lead plate of thickness t inches.

(a) Graph $I(t)$.

(b) What percent of an X-ray beam will penetrate a lead plate $\frac{1}{2}$ inch thick?

(c) How thick should the lead plate be in order to screen out 70% of the X-rays?

47. The population of Citrus Valley was 20,000 in 2000. In 2010, it was 35,000.

(a) What is P_0 if $t = 0$ in 2000?

(b) Use the population in 2010 to find the growth factor e^k.

(c) Write a growth law of the form $P(t) = P_0 e^{kt}$ for the population of Citrus Valley.

(d) If it continues at the same rate of growth, what will the population be in 2030?

48. A copy of *Time* magazine cost \$1.50 in 1981. In 1988, the cover price had increased to \$2.00.

(a) What is P_0 if $t = 0$ in 1981?

(b) Use the price in 1988 to find the growth factor e^k.

(c) Find a growth law of the form $P(t) = P_0 e^{kt}$ for the price of *Time*.

(d) In 1999, a copy of *Time* cost \$3.50. Did the price of the magazine continue to grow at the same rate from 1981 to 1999?

49. Cobalt-60 is a radioactive isotope used in the treatment of cancer. A 500-milligram sample of cobalt-60 decays to 385 milligrams after 2 years.

(a) Using $P_0 = 500$, find the decay factor e^k for cobalt-60.

(b) Write a decay law $N(t) = N_0 e^{kt}$ for cobalt-60.

(c) How much of the original sample will be left after 10 years?

50. Weed seeds can survive for a number of years in the soil. An experiment on cultivated land found 155 million weed seeds per acre, and in the following years the experimenters prevented the seeds from coming to maturity and producing new weeds. Four years later, there were 13.6 million seeds per acre. (Source: Burton, 1998)

(a) Find the annual decay factor e^k for the number of weed seeds in the soil.

(b) Write an exponential formula with base e for the number of weed seeds that survived after t years.

Problems 51–58 are about doubling time and half-life.

51. Delbert invests \$500 in an account that pays 9.5% interest compounded continuously.

(a) Write a formula for $A(t)$ that gives the amount of money in Delbert's account after t years.

(b) How long will it take Delbert's investment to double to \$1000?

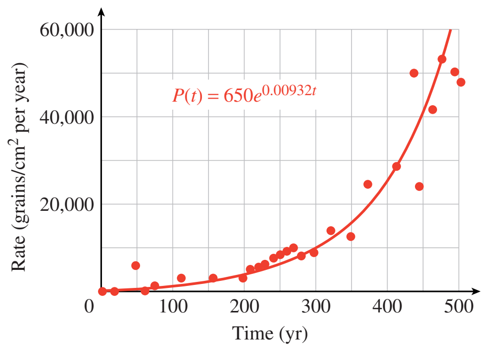
(c) How long will it take Delbert's money to double again, to \$2000?

(d) Graph $A(t)$ and illustrate the doubling time on your graph.

(e) Choose any point (t_1, A_1) on the graph, then find the point on the graph with vertical coordinate $2A_1$. Verify that the difference in the t-coordinates of the two points is the doubling time.

52. The growth of plant populations can be measured by the amount of pollen they produce. The pollen from a population of pine trees that lived more than 9500 years ago in Norfolk, England, was deposited in the layers of sediment in a lake basin and dated with radiocarbon techniques.

The figure shows the rate of pollen accumulation plotted against time, and the fitted curve $P(t) = 650e^{0.00932t}$. (Source: Burton, 1998)



(a) What was the annual rate of growth in pollen accumulation?

(b) Find the doubling time for the pollen accumulation, that is, the time it took for the accumulation rate to double.

(c) By what factor did the pollen accumulation rate increase over a period of 500 years?

53. Technetium-99m (Tc-99m) is an artificially produced radionuclide used as a tracer for producing images of internal organs such as the heart, liver, and thyroid. A solution of Tc-99m with initial radioactivity of 10,000 becquerels (Bq) decays according to the formula

$$N(t) = 10,000e^{-0.1155t}$$

where t is in hours.

(a) How long will it take the radioactivity to fall to half its initial value, or 5000 Bq?

(b) How long will it take the radioactivity to be halved again?

(c) Graph $N(t)$ and illustrate the half-life on your graph.

(d) Choose any point (t_1, N_1) on the graph, then find the point on the graph with vertical coordinate $0.5N_1$. Verify that the difference in the t-coordinates of the two points is the half-life.

54. All living things contain a certain amount of the isotope carbon-14. When an organism dies, the carbon-14 decays according to the formula

$$N(t) = N_0e^{-0.000124t}$$

where t is measured in years. Scientists can estimate the age of an organic object by measuring the amount of carbon-14 remaining.

(a) When the Dead Sea scrolls were discovered in 1947, they had 78.8% of their original carbon-14. How old were the Dead Sea scrolls then?

(b) What is the half-life of carbon-14, that is, how long does it take for half of an object's carbon-14 to decay?

55. The half-life of iodine-131 is approximately 8 days.

(a) If a sample initially contains N_0 grams of iodine-131, how much will it contain after 8 days? How much will it contain after 16 days? After 32 days?

(b) Use your answers to part (a) to sketch a graph of $N(t)$, the amount of iodine-131 remaining, versus time. (Choose an arbitrary height for N_0 on the vertical axis.)

(c) Calculate k, and hence find a decay law of the form $N(t) = N_0e^{kt}$, where $k < 0$, for iodine-131.

56. The half-life of hydrogen-3 is 12.5 years.

(a) If a sample initially contains N_0 grams of hydrogen-3, how much will it contain after 12.5 years? How much will it contain after 25 years?

(b) Use your answers to part (a) to sketch a graph of $N(t)$, the amount of hydrogen-3 remaining, versus time. (Choose an arbitrary height for N_0 on the vertical axis.)

(c) Calculate k, and hence find a decay law of the form $N(t) = N_0e^{kt}$, where $k < 0$, for hydrogen-3.

57. A Geiger counter measures the amount of radioactive material present in a substance. The table shows the count rate for a sample of iodine-128 as a function of time. (Source: Hunt and Sykes, 1984)

Time (min)	0	10	20	30	40	50	60	70	80	90
Counts/sec	120	90	69	54	42	33	25	19	15	13

(a) Graph the data and use your calculator's exponential regression feature to fit a curve to them.

(b) Write your equation in the form $G(t) = G_0e^{kt}$.

(c) Calculate the half-life of iodine-128.

58. The table shows the count rate for sodium-24 registered by a Geiger counter as a function of time. (Source: Hunt and Sykes, 1984)

Time (min)	0	10	20	30	40	50	60	70	80	90
Counts/sec	180	112	71	45	28	18	11	7	4	3

(a) Graph the data and use your calculator's exponential regression feature to fit a curve to them.

(b) Write your equation in the form $G(t) = G_0e^{kt}$.

(c) Calculate the half-life of sodium-24.

Chapter 10 Summary and Review

Glossary

- inverse function
- logarithmic function
- logarithmic equation
- compound interest
- natural base
- natural log function
- natural exponential function
- continuous compounding
- log scale
- pH value
- decibels
- Richter magnitude

Key Concepts

1 Two functions are called *inverse functions* if each function undoes the effects of the other.

2 We can make a table of values for the inverse function by interchanging the columns of a table for f.

3 If we apply the inverse function to the output of f, we return to the original input value.

4 The graphs of f and its inverse function are **symmetric about the line** $y = x$.

5 We define the logarithmic function, $g(x) = \log_b x$, which takes the log base b of its input values. The log function $g(x) = \log_b x$ is the inverse of the exponential function $f(x) = b^x$.

6 Because $f(x) = b^x$ and $g(x) = \log_b x$ are inverse functions for $b > 0,\ b \neq 1$,

$$\log_b b^x = x \quad \text{, for all } x \quad \text{and} \quad b^{\log_b x} = x \quad \text{, for } x > 0$$

7 A **logarithmic equation** is one where the variable appears inside of a logarithm. We can solve logarithmic equations by converting to exponential form.

Steps for Solving Logarithmic Equations.

8

1 Use the properties of logarithms to combine all logs into one log.

2 Isolate the log on one side of the equation.

3 Convert the equation to exponential form.

4 Solve for the variable.

5 Check for extraneous solutions.

9 The natural base is an irrational number called e, where

$$e \approx 2.71828182845$$

10 The **natural exponential function** is the function $f(x) = e^x$. The **natural log function** is the function $g(x) = \ln x = \log_e x$.

Conversion Formulas for the Natural Base.

11
$$y = \ln x \qquad \text{if and only if} \qquad e^y = x$$

Properties of Natural Logarithms.

12 If $x, y > 0$, then

1. $\ln(xy) = \ln x + \ln y$
2. $\ln \dfrac{x}{y} = \ln x - \ln y$
3. $\ln x^k = k \ln x$

also
$$\ln e^x = x \quad \text{for all } x \text{ and} \quad e^{\ln x} = x, \text{ for } x > 0$$

13 We use the natural logarithm to solve exponential equations with base e.

Exponential Growth and Decay.

14 The function
$$P(t) = P_0 e^{kt}$$
describes exponential growth if $k > 0$, and exponential decay if $k < 0$.

15 **Continuous compounding**: The amount accumulated in an account after t years at interest rate r compounded continuously is given by

$$A(t) = Pe^{rt}$$

where P is the principal invested.

16 A **log scale** is useful for plotting values that vary greatly in magnitude. We plot the log of the variable, instead of the variable itself.

17 A log scale is a **multiplicative scale**: Each increment of equal length on the scale indicates that the value is multiplied by an equal amount.

18 The pH value of a substance is defined by the formula

$$\text{pH} = -\log_{10}[H^+]$$

where $[H^+]$ denotes the concentration of hydrogen ions in the substance.

19 The loudness of a sound is measured in decibels, D, by

$$D = 10 \log_{10}\left(\frac{I}{10^{-12}}\right)$$

where I is the intensity of its sound waves (in watts per square meter).

20 The Richter magnitude, M, of an earthquake is given by

$$M = \log_{10}\left(\frac{A}{A_0}\right)$$

where A is the amplitude of its seismographic trace and A_0 is the amplitude of the smallest detectable earthquake.

21 A *difference* of K units on a logarithmic scale corresponds to a *factor* of 10^K units in the value of the variable.

Chapter 10 Review Problems

For Problems 1 and 2, make a table of values for the function and sketch a graph.

1. $f(x) = \log_4 x$

2. $f(x) = \log_{1/4} x$

For Problems 3 and 4, simplify.

3.

a $10^{\log_6 n}$

b $\log_2 4^{x+3}$

4.

a $\log 100^x$

b $3^{2\log_3 t}$

For Problems 5-12, solve.

5. $\log_3 \frac{1}{3} = y$

6. $\log_3 x = 4$

7. $\log_2 y = -1$

8. $\log_5 y = -2$

9. $\log_b 16 = 2$

10. $\log_b 9 = \frac{1}{2}$

11. $\log_4\left(\frac{1}{2}t + 1\right) = -2$

12. $\log_2(3x - 1) = 3$

For Problems 13-16, solve.

13. $\log_3 x + \log_3 4 = 2$

14. $\log_2(x + 2) - \log_2 3 = 6$

15. $\log_{10}(x - 1) + \log_{10}(x + 2) = 1$

16. $\log_{10}(x + 2) - \log_{10}(x - 3) = 1$

For Problems 17-22, solve.

17. $e^x = 4.7$

18. $e^x = 0.5$

19. $\ln x = 6.02$

20. $\ln x = -1.4$

21. $4.73 = 1.2e^{0.6x}$

22. $1.75 = 0.3e^{-1.2x}$

For Problems 23-26, simplify.

23. $e^{(\ln x)/2}$

24. $\ln\left(\frac{1}{e}\right)^{2n}$

25. $\ln\left(\frac{e^k}{e^3}\right)$

26. $e^{\ln(e+x)}$

27. In 1970, the population of New York City was 7,894,862. In 1980, the population had fallen to 7,071,639.

a Write an exponential function using base e for the population of New York over that decade.

b By what percent did the population decline annually?

28. In 1990, the population of New York City was 7,322,564. In 2000, the population was 8,008,278.

a Write an exponential function using base e for the population of New York over that decade.

b By what percent did the population increase annually?

29. You deposit \$1000 in a savings account paying 5% interest compounded continuously.

(a) Find the amount in the account after 7 years.

(b) How long will it take for the original principal to double?

(c) Find a formula for the time t required for the amount to reach A.

30. The voltage, V, across a capacitor in a certain circuit is given by the function

$$V(t) = 100(1 - e^{-0.5t})$$

where t is the time in seconds.

(a) Make a table of values and graph $V(t)$ for $t = 0$ to $t = 10$.

(b) Describe the graph. What happens to the voltage in the long run?

(c) How much time must elapse (to the nearest hundredth of a second) for the voltage to reach 75 volts?

31. Solve for t: $y = 12e^{-kt} + 6$

32. Solve for k: $N = N_0 + 4\ln(k + 10)$

33. Solve for M: $Q = \frac{1}{t}\left(\frac{\log M}{\log N}\right)$

34. Solve for t: $C_H = C_L \cdot 10^k t$

35. Express $P(t) = 750e^{0.32t}$ in the form $P(t) = P_0 b^t$.

36. Express $P(t) = 80e^{-0.6t}$ in the form $P(t) = P_0 b^t$.

37. Express $N(t) = 600(0.4)^t$ in the form $N(t) = N_0 e^{kt}$.

38. Express $N(t) = 100(1.06)^t$ in the form $N(t) = N_0 e^{kt}$.

39. Plot the values on a log scale.

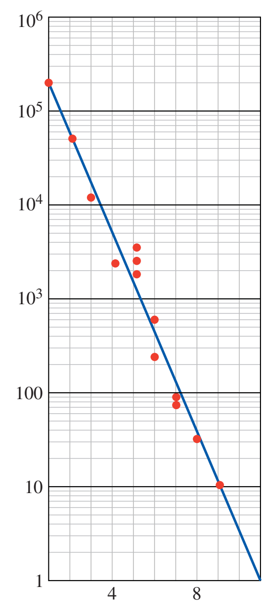
x	0.04	45	1200	560,000

40. Plot the values on a log scale.

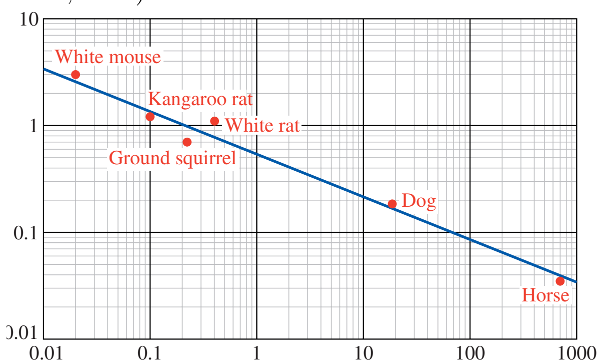
x	0.0007	0.8	3.2	2500

41.

The graph describes a network of streams near Santa Fe, New Mexico. It shows the number of streams of a given order, which is a measure of their size. Use the graph to estimate the number of streams of orders 3, 4, 8, and 9. (Source: Leopold, Wolman, and Miller)



42. Large animals use oxygen more efficiently when running than small animals do. The graph shows the amount of oxygen various animals use, per gram of their body weight, to run 1 kilometer. Estimate the body mass and oxygen use for a kangaroo rat, a dog, and a horse. (Source: Schmidt-Neilsen, 1972)



43. The loudest sound created in a laboratory registered at 210 decibels. The energy from such a sound is sufficient to bore holes in solid material. Find the intensity of a 210-decibel sound.

44. The most powerful earthquake ever recorded occurred in Chile on 22 May 1960. The magnitude of the earthquake was approximately 9.5. What was the amplitude of its seismographic trace.

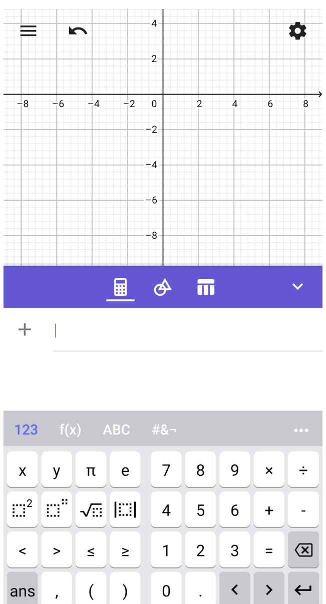
45. In 2004, a magnitude 9.0 earthquake struck Sumatra in Indonesia. How much more powerful was this quake than the 1906 San Francisco earthquake of magnitude 8.3?

46. The sound of rainfall registers at 50 decibels. What is the decibel level of a sound twice as loud?

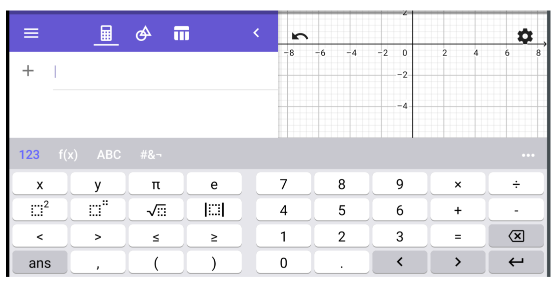
Appendix A

Using a GeoGebra Calculator App

When we open the GeoGebra Graphing Calculator, we see both the "Graphics View" (the region where the graphs are displayed) and the "Algebra View" (the region where we input expressions).



GeoGebra app in portrait orientation



GeoGebra app in landscape orientation

We use the input box next to the large "plus" symbol to enter expressions we wish to evaluate.

- When using the app on a computer, we can type with the physical keyboard or use the app's virtual keyboard. (If the virtual keyboard is not initially displayed, click on the keyboard icon in the lower left corner of the Algebra pane.)

- When using the app on a small screen, tap in the input box and the virtual keyboard will be visible.

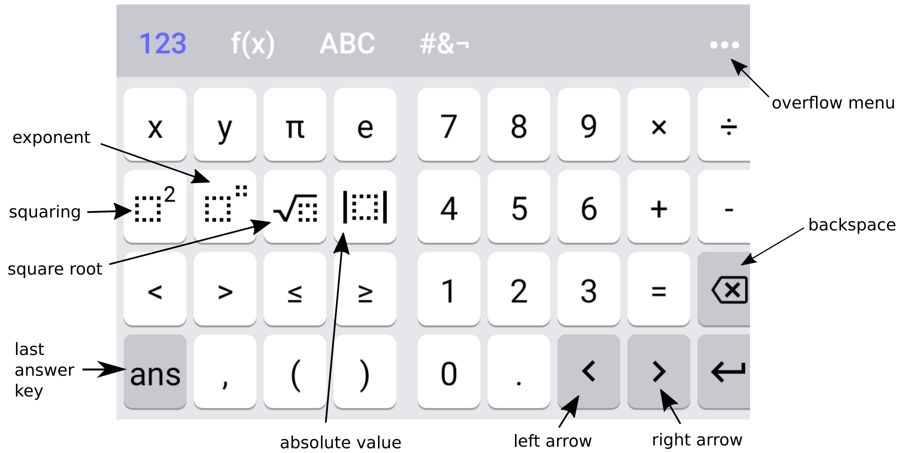


Figure A.1 GeoGebra calculator keyboard

Getting Started

On and Off

If you are reading this textbook in a web browser using a computer with sufficiently wide monitor, you should see button labeled "Calc" just above the text on the page. Clicking on that button will launch a Geogebra calculator applet in the right margin of the book. If the calculator overlaps the text, click on the "Contents" tab above the navigation panel to hide that panel and provide more space for the text. Click again on the "Calc" button to close the calculator app. (And clicking again on the "Contents" tab will re-open the navigation panel.)

Numbers and Operations

Example A.2

Compute $5 - 8$.

Enter 5 [−] 8

and even before we press [↵], GeoGebra shows the result -3 below the input box.

If we do press [↵], GeoGebra shows

$a = 5 - 8$

$\rightarrow -3$

in an output history region above the input box. There is a scroll bar to the right of the output history and the input box.

Example A.3

Compute $-5 + 8$.

Enter 5 [+] 8

to see the result 3

We press [↵] to tell the calculator to store the computation. We can use the scroll bar to see the input box or the output history of previous computations.

The calculator has a key for the value of π.

Example A.4

Compute 2π.
Enter 2 [×] [π] or 2 [π]
to get the approximation 6.2831853071796

Delete and Undo

Press the backspace key (shown in the keyboard image A.1, p. 726) to delete the character before the cursor or any highlighted characters.

To remove an item from the output history, tap the kebab menu icon (three vertically stacked dots) to the right of the item, then tap "Delete".

To remove the most recent item in the output history, tap the *Undo* icon in the Graphics View. (The icon looks like an arched arrow pointing left.)

Entering Expressions

Parentheses

Order of Operations: The calculator follows the standard order of operations.

Example A.5

Compute $2 + 3 \cdot 4$.
Enter 2 [+] 3 [×] 4 [⏎] Ans. 14

Example A.6

Compute $(2 + 3) \cdot 4$.
Enter [(] 2 [+] 3 [)] [×] 4 [⏎] Ans. 20

Fractions

Example A.7

Compute $\dfrac{1+3}{2}$.
Enter [(] 1 [+] 3 [)] [÷] 2 [⏎] Ans. 2

Caution A.8 GeoGebra displays "built-up" fractions like $\dfrac{1}{2}$. Once we enter the numerator and select the [÷] key, the cursor is in the denominator and will stay there until we use an arrow key on the keyboard to move the cursor outside the fraction, or until we press [⏎]. The arrow keys on the virtual keyboard are shown in the keyboard image A.1, p. 726.

Example A.9

Compute $\dfrac{1}{2 \cdot 3}$.
Enter 1 [÷] 2 [×] 3 [⏎] Ans. $\dfrac{1}{6}$

In the output history, there is an approximation icon that looks like the "≈" symbol in a blue square. To get a decimal approximation of the fraction, click on that icon.

Ans. 0.1666666666667

Example A.10

Compute $\frac{1}{2} \cdot 3$.

Enter 1 [÷] 2 [→] [×] 3 [↵] Ans. $\frac{3}{2}$

Exponents and Powers

The key for exponents and the key for squaring are shown in the keyboard image, p. 726.

Example A.11

Evaluate 57^2.

Enter 57, then the squaring key, then [↵]

OR

Enter 57, then the exponent key, then 2 [↵]

Ans. 3249

Example A.12

Evaluate 2^{10}.

Enter 2, and then the exponent key, 10 [↵] Ans. 1024

Caution A.13 GeoGebra nicely displays powers such as 10^7, with the exponent raised like a superscript. Once we enter the base and select the exponent key, the cursor is in the exponent and will stay there until we use the arrow keys to move the cursor outside the power, or until we press [↵]. The arrow keys on the virtual keyboard are shown in the keyboard image, p. 726.

Example A.14

Evaluate $8^{2/3}$.

Enter 8, then the exponent key, then 2 [÷] 3 [↵]

Ans. 4

Example A.15

Evaluate $\frac{8^2}{3}$.

Enter 8, then the exponent key, then 2 [→] [÷] 3 [↵]

Ans. $\frac{64}{3}$

Square Roots

The key for square roots is shown in the keyboard image A.1, p. 726.

Example A.16

Evaluate $\sqrt{2}$.
Select the square root key, then enter 2 Ans. 1.414213562
⏎

Caution A.17 GeoGebra nicely displays square roots such as $\sqrt{9+16}$, with the radicand inside the square root "house". Once we select the square root key, the cursor is in under the radical and will stay there until we use the arrow keys to move the cursor outside, or until we press ⏎. The arrow keys on the virtual keyboard are shown in the keyboard image, p. 726.

Example A.18

Evaluate $\sqrt{9+16}$.
Select the square root key, then enter 9 + 16 ⏎
Ans. 5

Example A.19

Evaluate $\sqrt{9}+16$.
Select the square root key, then enter 9 → + 16 ⏎
Ans. 19

Other Roots

If we click on the "f(x)" on the keyboard in the Algebra View, we see other built-in keys.

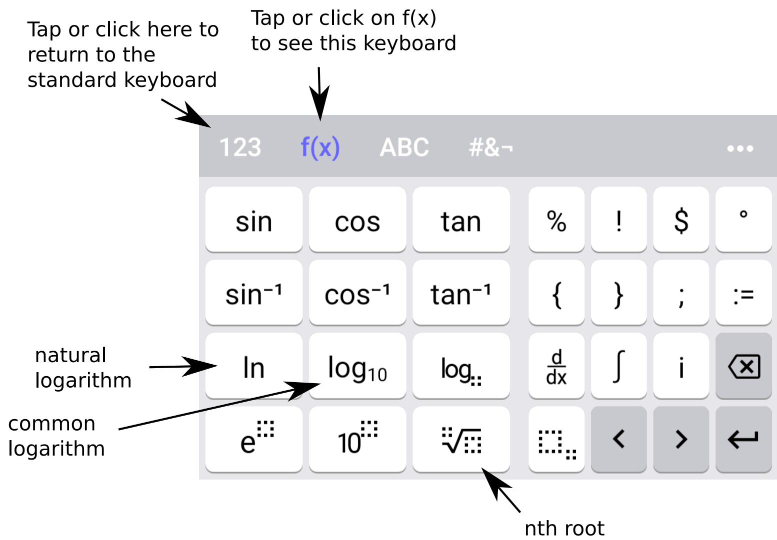


Figure A.20 GeoGebra calculator f(x) keyboard

Example A.21

Compute $\sqrt[3]{1728}$.
Choose the f(x) keyboard, select the nth root key, go back to the standard 123 keyboard, then enter 3 → 1728 ⏎
Ans. 12

Example A.22

Compute $\sqrt[10]{2 \cdot 512}$.

Choose the f(x) keyboard, select the nth root key, go back to the standard 123 keyboard, then enter 10 [→] 2 [×] 512 [↵]

Ans. 2

Absolute Value

GeoGebra nicely displays the vertical bars of the standard absolute value notation, such as in the expression $|21 \cdot 54 - 81|$. Once we select the absolute value key, the cursor is in between the vertical bars and will stay there until we use the arrow keys to move the cursor outside, or until we press [↵]. The arrow keys on the virtual keyboard are shown in the keyboard image, p. 726.

Example A.23

Evaluate $\dfrac{|21 \cdot 54 - 81|}{-9}$.

Select the absolute value key, then enter 21 [×] 54 [−] 81 [→] [÷] −9 [↵]

Ans. −117

Scientific Notation

The GeoGebra calculator does not display scientific notation by default. The calculator can display scientific notation in the Grahics View by using a ScientificText command.

Example A.24

Compute $123,456,789^2$.

Enter 123456789 [x^2] [↵] Ans. 15241578750190522

We will enter ScientificText in the input box.

- On the computer, typing the first three letters (sci) is enough for the full command to appear as an option.

- On a phone app, we tap on the overflow menu (three horizontal dots) on the top right of the virtual keyboard (see keyboard image, p. 726), tap in the Search in All Commands field, then start to input ScientificText until the full command appears as an option.

We click on ScientificText, and the cursor is appropriately positioned, ready for us to enter an expression. We can either repeat the calculation we entered above, or, more efficiently, select the last answer key [ans] and [↵].

The result $1.52415787501905 \times 10^{16}$, appears on the grid in the Graphics View. We can drag the text within the Graphics View, change its color, and/or delete it.

We could also type in the entire command scientifictext using the ABC keyboard, then use the 123 keyboard to enter the parentheses and numbers.

Editing an Entry

We can edit an expression without starting again. The backspace key will remove the character to the left of the cursor. We can move the cursor within an input line using the arrow keys or by clicking in the appropriate place.

We can recall any previous entry by finding it in the output history (scrolling up if necessary), tapping on the kebab menu icon (three vertical dots), and selecting Duplicate. The command will be copied into the current input box, which we can edit before pressing ⏎.

Note A.25 If the menu does not appear. Sometimes when using the GeoGebra app embedded in this textbook, the menu does not appear when we tap on the kebab menu icon. This may occur when we are at a place in the section far from the top of the webpage. Try scrolling within the webpage to the top, and tap on the kebab menu icon again.

If we just want to use the most recent result in the current command, we use the ans key.

Example A.26

Evaluate $5^2 + 12^2$ and then take the square root of the sum.

Enter 5, then the square key, then + 12, the square key again, then ⏎

Ans. 169

Now select the square root key, then ans ⏎

Ans. 13

Graphing an Equation

We can graph equations in the variables x and y. The variable keys are located on the top row of the app 123 keyboard. There are two steps to graphing an equation:

1 Entering the equation

2 Setting the graphing window

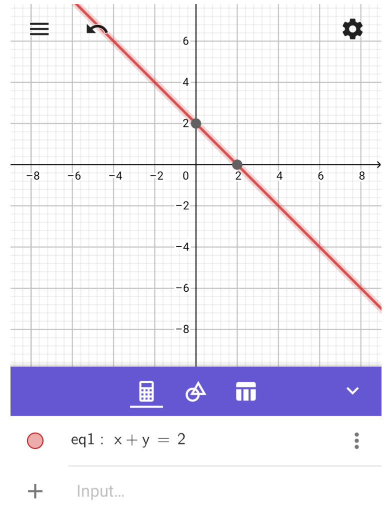
A Basic Graph with intercepts

Example A.27

Graph the equation $x + y = 2$.
In the Algebra View, Press x [+] y [=] 2 [⏎].
The output history shows

$$\text{eq1} : x + y = 2$$

and the Graphics View shows the line passing through the points $(0, 2)$ and $(2, 0)$. If we click on either of these two points, GeoGebra will show a label and the coordinates.



Special Points on a Graph.

GeoGebra automatically highlights "Special Points" with heavy dots on the graph of an equation. GeoGebra will show a label and the coordinates of any Special Point when we clicking on it.

In particular, any intercepts of a graph are Special Points.

The size and appearance of the displayed graph will vary with the display size and orientation of device we use. We can hide the Algebra View to increase the size of the Graphics View by clicking on the arrow key at the right edge of the border between the Algebra View and the Graphics View. Clicking again on the arrow key brings back the Algebra View.

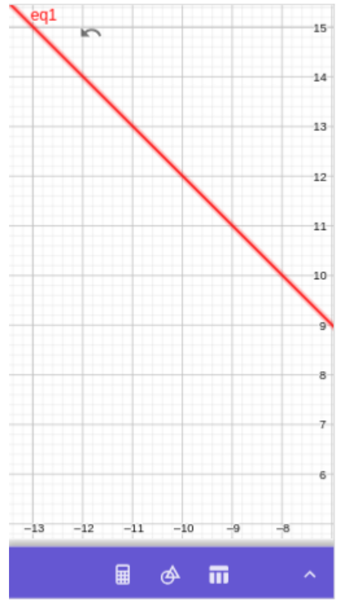
Translate and Zoom

We can adjust "graphing window", called the "Graphics View" in GeoGebra, without entering numerical values.

Example A.28

Continuing from Example 27, p. 732, drag within the Graphics View so that you can see the point on the graph where $y = 15$. Then zoom out until you can see both that point and the origin.

Solution: If we click on the arrow key at the right edge of the border between the Algebra View and the Graphics View, the Algebra View disappears to enlarge the Graphing View. But the point on the line where $y = 15$ is still not visible. We can tap or click on the Graphics View and drag down and to the right until we can see the point $(-13, 15)$. But the origin is no longer visible. We need to zoom out.



- To Zoom out on a touch screen or touch pad, we "pinch in".
- To Zoom using a mouse, click on the Graphics View, then hold the Shift key and rotate the mouse wheel forward.

Note A.29 Retrieving the Algebra View. To access the Algebra View again, click again on the arrow at the right edge of what had previously been the border between the Graphics View and Algebra View.

Graphing a Function, Making a Table, and Zooming One Axis

To graph an equation of the form $y =$ (expression in x), we only need enter the expression in x.

Example A.30

Graph the equation $y = x^2 - 5$ for x-values between -5 and 5.

Clear the output history: In the Algebra View (see the Note, p. 733 above if the Algebra View is closed), tap the kebab menu icon (three vertical dots) in any existing history line, and select Delete. (If the menu does not apppear, you may wish to review the Note, p. 731 from the previous section.)
In the input box, enter x, then tap the squaring key, [$\square^2$] 5 [↵].
The output history shows

$$f(x) = x^2 - 5$$

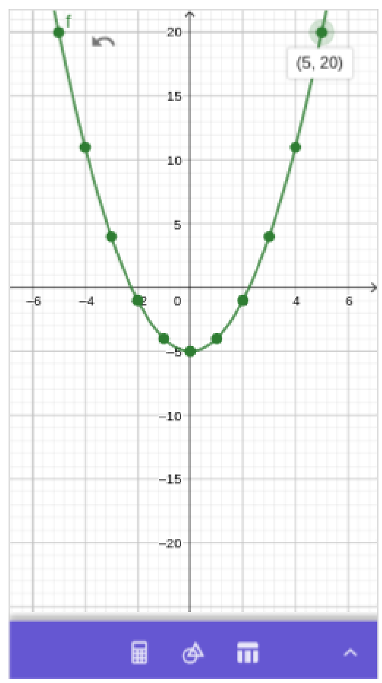
and the Graphics View shows the parabola.
Tap the kebab menu icon (three vertical dots) in the output history line, and select Table of values.

Set the Start value for x to -5. Set the End value for x to 5. Set the Step value to 1, then click [OK]
Click on the arrow key at the right edge of the border between the Algebra View and the Graphics View to hide the Algebra View and enlarge the Graphing View. Tap on any of the dark dots on the graph to see its coordinates.

We zoom out just enough so that both $x = -6$ and $x = 6$ are both visible on the x-axis. Now we zoom out just the y-axis (until $y = 20$ is visible) as follows.

- On a touch screen, we pinch vertically on the y-axis.
- On a computer, we move the cursor above the y-axis until the cursor looks like a hand. Then holding down the [Shift] key, we click (the cursor changes again to an arrow pointing up to a horizontal segment), and drag down.

The points $(-5, 0)$ and $(5, 0)$ are now visible in the Graphics View.



We can also zoom on just the x-axis.

- On a touch screen, we pinch horizontally on the x-axis.

- On a computer, we move the cursor above the x-axis until the cursor looks like a hand. Then holding down the Shift key, we click (the cursor changes again to an arrow pointing right to a vertical segment), and drag left or right.

More graphing

Finding a Suitable Graphing Window

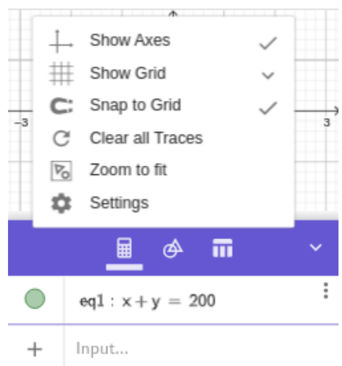
Sometimes the graph of an equation is not visible in the Standard view. The simplest way to modify the graphing window so that at least a part of the graph is visible is to use the Zoom fit option.

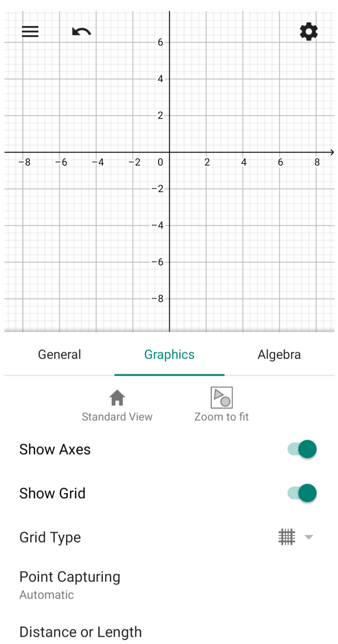
Example A.31

Graph the equation $x + y = 200$
Clear out the output history of any previous graphs, then enter into the input box

$$x + y = 200$$

The graph is not visible in the standard window, so we will try the Zoom fit command. Right-click on the Graphics View and choose Zoom to fit.





But if you use the GeoGebra calculator app on a mobile device, tap the options icon that resembles a gear, located in the upper right corner of the Graphics View, and a slightly different menu apears.
There are icons for both a Standard View and a Zoom to fit.

Once you can see part of the graph, you can further adjust the Graphics View by dragging and/or zooming.

Multiple Graphs and the Intersect Feature

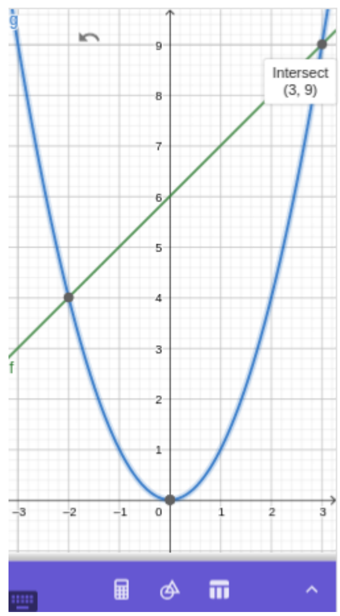
We can display more than one graph at a time. We simply enter multiple equations, pressing ⏎ after each equation to put it into the calculator output history.

The points where two graphs meet are Special Points, and GeoGebra will give a label and coordinates when we click on an intersection point.

To turn off a graph without deleting its equation, we tap in the colored disk to the left of the equation in the output history. We tap the disk again to turn the graph back on.

Example A.32

Find all the intersection points between the graphs of $y = x + 6$ and $y = x^2$.



We enter $x + 6$ ⏎ to graph the first equation, then x , the squaring key, ⏎ to graph the second equation.
We can minimize the Algebra View and then click on each intersection point to see its coordinates. The two intersection points are $(-2, 4)$ and $(3, 9)$.

Regression

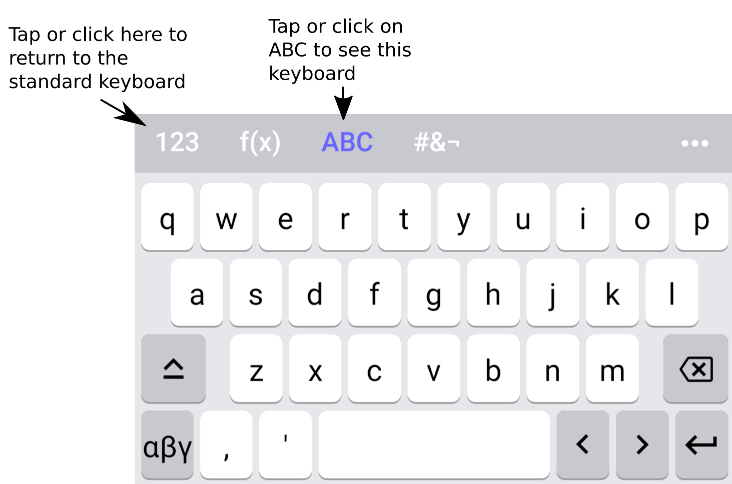


Figure A.33 GeoGebra calculator ABC keyboard

Making a Scatterplot and Finding the Regression Line

We use GeoGebra's FitLine command to produce a regression line.

Example A.34

Make a scatterplot for the points $(10, 12)$, $(11, 14)$, $(12, 14)$, $(12, 16)$, $(14, 20)$. Then find the equation of the lease-squares regression line and plot it on top of the scatterplot.

We enter each of the five ordered pairs. The output history will show each labeled with a capital letter, such as $A = (10, 12)$.

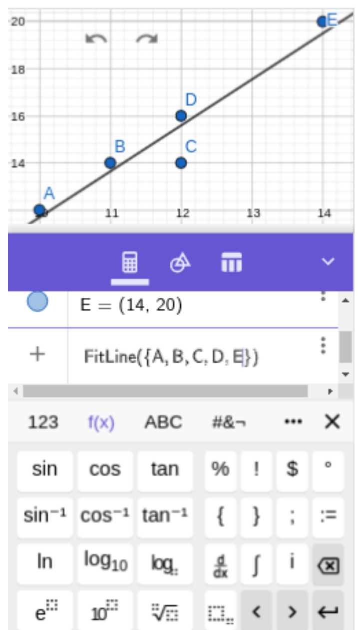
We do a Zoom to fit: right-click on the Graphics View (or click on the gear icon) and select Zoom to fit.

We next enter the FitLine command:

$$\text{fitline}(\{A, B, C, D, E\})$$

using the ABC keyboard shown above for the letters, the 123 keyboard (keyboard image, p. 726) for the parentheses, and the f(x) keyboard (keyboard image, p. 729) for the list curly braces {}.

Important: We must enter capital letters for the names of the points--GeoGebra is case-sensitive regarding labels.



Troubleshooting the GeoGebra App

Issue	Solution
No menu appears after clicking on the kebab menu icon (while using the GeoGebra app embedded in this textbook)	Scroll within the book to the top of the webpage. Click again on the kebab menu icon.
The app has only a graphing window and no input box. (No Algebra View)	Click on the up arrow at the right edge of the blue border on the bottom of the app. (In landscape mode, the border is on the left and the arrow points to the right.)
The graph does not appear even though the equation for the graph is in the history.	Check that the circle next to the equation in the output history is filled with a color—if it is not, click on the circle. If the graph is still not visible, adjust the Graphics View as in Example 31, p. 735
The calculator does not give the decimal form of a fraction.	Tap the approximation key as described in Example 9, p. 727

Figure A.35 Common issues and solutions

Appendix B

Answers to Selected Exercises

1 · Linear Models

1.1 · Linear Models

· Problem Set 1.1

Warm Up

1.

Answer.

a 5.5

b 12

3.

Answer. $I = 200 + 0.09S$

Skills Practice

5.

Answer. $\dfrac{6}{5}$

7.

Answer. $y > \dfrac{6}{5}$

9.

Answer. $R = 50 - 0.4w$

Applications

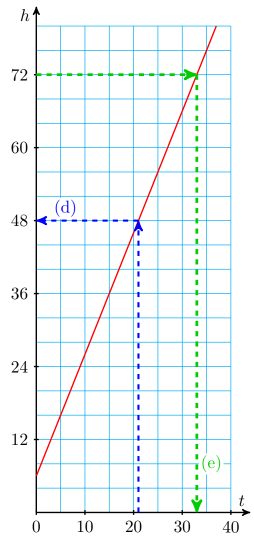
11.

Answer.

a

t (days)	0	5	10	15	20
h (inches)	6	16	26	36	46

b $h = 6 + 2t$

c 

d 48 in

e 33 days

f $h = 6 + 2(21)$; $72 = 6 + 2t$

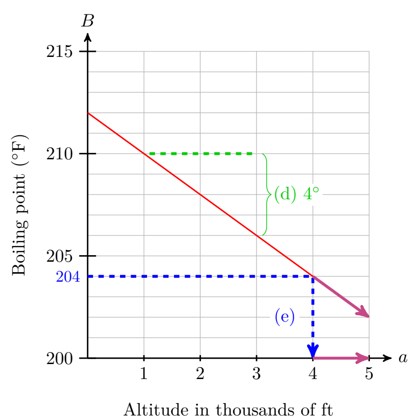
g 14 is the initial height of the plants in inches. 1.5 is the number of inches they grow each day.

13.

Answer.

a

Altitude (1000 ft)	0	1	2	3	4	5
Boiling point (°F)	212	210	208	206	204	202

b 

c 4°F

d Over 4000 feet

1.2 · Graphs and Equations

· Problem Set 1.2

Warm Up

1.

Answer.

a no

b yes

c no

d no

3.

Answer.

a yes

b no

c no

d yes

5.

Answer.

a no

b no

c yes

d yes

7.

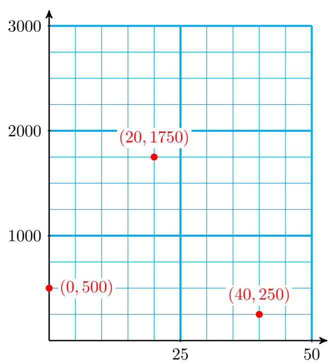
Answer. horizontal: 0.25; vertical: 4

9.

Answer.

a horizontal: 5; vertical: 250

b



Skills Practice

11.

Answer. $\dfrac{-15}{4}$

13.

Answer. 35

17.

Answer.

$x \geq \dfrac{7}{2}$

19.

Answer. $y = \dfrac{2}{3} + 24$

21.

Answer. $y = \dfrac{-7}{13}x + 7$

23.

Answer. $y = \dfrac{2}{9}x + 17$

Applications

25.

Answer.

a
- i $x = -6$
- ii $x < -6$
- iii $x > -6$

b
- i $x = -6$
- ii $x < -6$
- iii $x > -6$

27.

Answer.

a $x = 0.3$

b $x = 0.8$

c $x \leq 0.3$

d $x \geq 0.8$

29.

Answer.

a $x \leq 4$

b $x > 2$

31.

Answer.

a $x = 4$

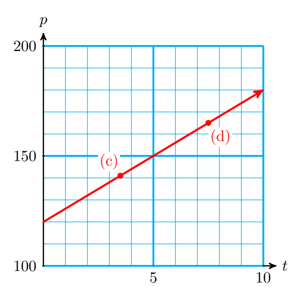
b $x < 22$

33.

Answer.

a $p = 120 + 5t$

b



c 141

d 7.5 min

1.3 · Intercepts

· Problem Set 1.3

Warm Up

1.

Answer.

a $5x$

b $2y$

c 5x+2y=1000

3.

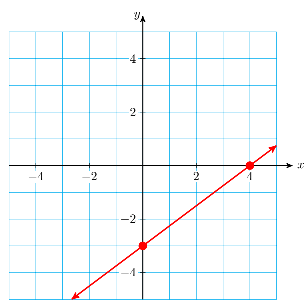
Answer. $y = \dfrac{-3}{5}x + \dfrac{16}{5}$

Skills Practice

5.

Answer.

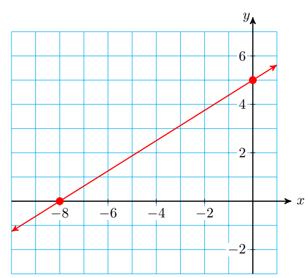
a $(4, 0)$, $(0, -3)$

b 

7.

Answer.

a $(-8, 0)$, $(0, 5)$

b 

9.

Answer.

a $(3, 0), (0, 5)$

b $\left(\dfrac{1}{2}, 0\right), \left(0, \dfrac{-1}{4}\right)$

c $\left(\dfrac{5}{2}, 0\right), \left(0, \dfrac{-3}{2}\right)$

d $(p, 0), (0, q)$

e The value of a is the x-intercept, and the value of b is the y-intercept.

11.

Answer.

a $(0, b)$

b $\left(\dfrac{-b}{m}, 0\right)$

13.

Answer. $-2x + 3y = 2400$

15.

Answer. $3x + 400y = 240$

17.

Answer.

a $22x + 9y = 33$, and

b $y = \frac{11}{3} + \frac{-22}{9}x$

19.

Answer.

a $4x + 12y = 48$, and

b $y = 4 - \frac{1}{3}x$

Applications

21.

Answer.

a $9x$

b $4y$

c $9x + 4y = 1800$

d $(200, 0)$ Delbert must eat 200g of figs daily if he eats no bananas.

e $(0, 450)$ Delbert must eat 450g of bananas daily if he eats no figs.

23.

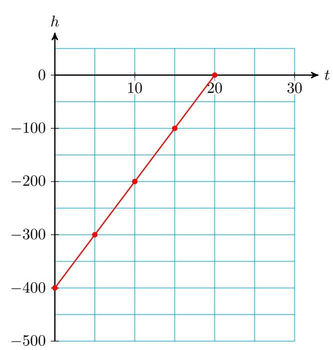
Answer.

a

t	0	5	10	15	20
h	-400	-300	-200	-100	0

b $h = -44 + 20t$

c



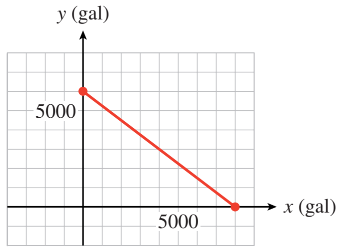
d $(0, -400)$: The diver starts at a depth of 400 feet. $(20, 0)$: The diver surfaces after 20 minutes.

25.

Answer.

a $\$2.40x$, $\$3.20y$

b $2.40x + 3.20y = 19,200$

c 

d The y-intercept, 6000 gallons, is the amount of premium that the gas station owner can buy if he buys no regular. The x-intercept, 8000 gallons, is the amount of regular he can buy if he buys no premium.

1.4 · Slope

· Problem Set 1.4

Warm Up

1.

Answer. Anthony

3.

Answer. Bob's driveway

5.

Answer. -1

7.

Answer. $\frac{-2}{3}$

Skills Practice

9.

Answer.

a $\frac{3}{2}$

b 6, $\frac{3}{2}$

c -9, $\frac{3}{2}$

d 27

11.

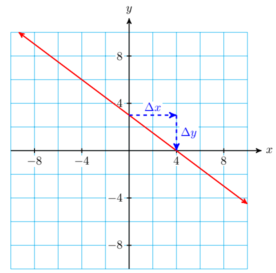
Answer. $\frac{3}{4}$

13.

Answer. -4000

15.

Answer.

a 

b $m = \dfrac{-3}{4}$

17.

Answer. 14.29 ft

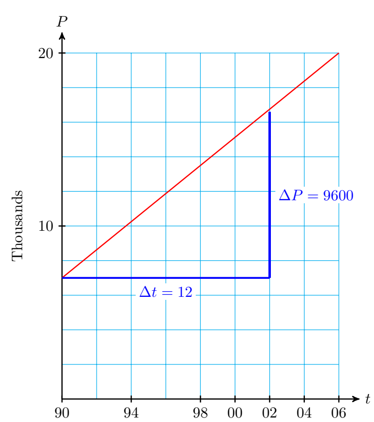
19.

Answer. (a)

Applications

21.

Answer.

a 

b 800

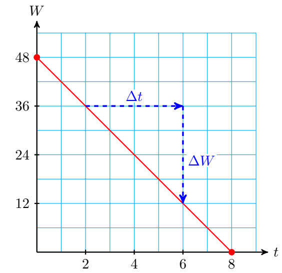
c $P = 7000 + 800t$

23.

Answer.

a -6 liters/day

b The water supply is decreasing at a rate of 6 liters per day.

c 

d $W = 48 - 6t$

25.

Answer.

a Yes, the slope is 0.12

b 0.12 cm/kg: The spring stretches an extra 0.12 cm for each additional 1 kg mass.

27.

Answer.

a The distances to the stations are known.

b 5.7 km/sec

29.

Answer.

a 7.9

b 8.35 km

c 2 hrs 5 min

1.5 · Equations of Lines

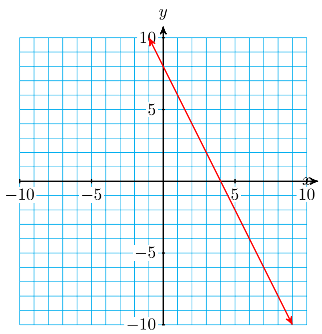
· Problem Set 1.5

Warm Up

1.

Answer.

x	-1	0	2	3	4
y	10	8	4	2	0



a 8

b decreases by 2 units

c The constant term is the y-intercept and the coefficient of x is the slope.

3.

Answer.

a $\Delta h = 22$

b $\Delta r = -57$

Skills Practice

5.

Answer.

a $y = \dfrac{-3}{2}x + \dfrac{1}{2}$

b $m = \dfrac{-3}{2};\ b = \dfrac{1}{2}$

7.

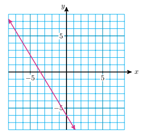
Answer.

a $y = 14x - 22$

b $m = 14;\ b = -22$

9.

Answer.

a 

b $y = \frac{-5}{3}x - 6$

c $x = \frac{-18}{5}$

11.

Answer.

a II

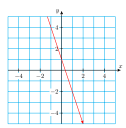
b III

c I

d IV

13.

Answer.

a 

b $y + 5 = -3(x - 2)$

c $y = -3x + 1$

15.

Answer. $y = 4x + 40$

17.

Answer.

a III

b IV

c II

d I

Applications

19.

Answer.

a $M = 7000 - 400w$

b The slope tells us that Tammy's bank account is diminishing at a rate of \$400 per week, the vertical intercept that she had \$7000 (when she lost all sources of income).

21.

Answer. $m = -0.0018$ degree/foot, so the boiling point drops with altitude at a rate of 0.0018 degree per foot. $b = 212$, so the boiling point is 212° at sea level (where the elevation $h = 0$).

23.

Answer.

a

h	8	20
C	645	1425

b $C = 125 + 65h$

c $m = 65$, the lesson rate is 65 dollars per hour

25.

Answer.

a

C	15	-5
F	59	23

b $F = 32 + \frac{9}{5}C$

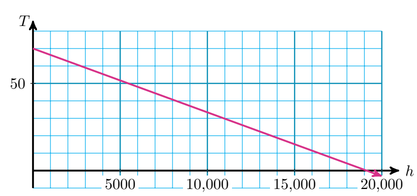
c $m = \frac{9}{5}$, so an increase of 1°C is equivalent to an increase of $\frac{9}{5}^\circ$ F.

27.

Answer.

a 55°F

b 9840 ft

c 

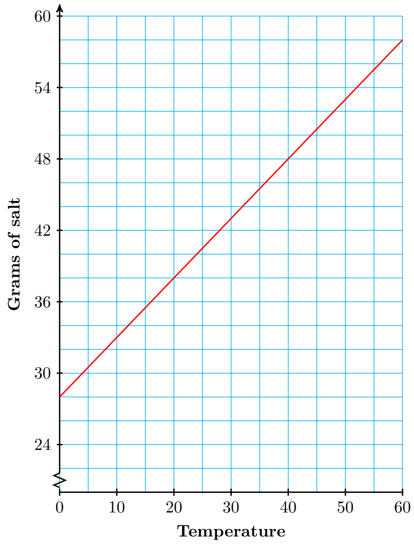
d $m = \frac{-3}{820}$. The temperature decreases 3 degrees for each increase in altitude of 820 feet.

e $(19,133\frac{1}{3}, 0)$. At an altitude of $19,133\frac{1}{3}$ feet, the temperature is 0°F. $(0, 70)$. At an altitude of 0 feet, the temperature is 70°F.

29.

Answer.

a $(0, 28)$



b $m = \frac{1}{2}$

c $y = \frac{1}{2}x + 28$

d 36°C

1.6 · Chapter Summary and Review

· Chapter 1 Review Problems

1.

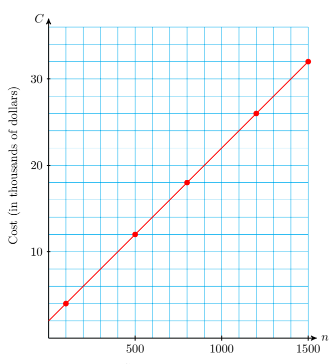
Answer.

a

n	100	500	800	1200	1500
C	4000	12,000	18,000	26,000	32,000

b $C = 20n + 2000$

c



d $22,000

e 400

2.

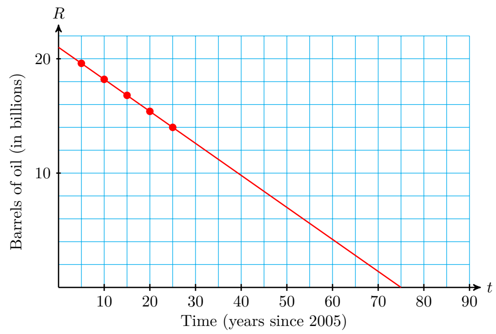
Answer.

a

t	5	10	15	20	25
R	1960	1820	1680	1540	1400

b $R = 2100 - 28t$

c $(75, 0)$, $(0, 2100)$



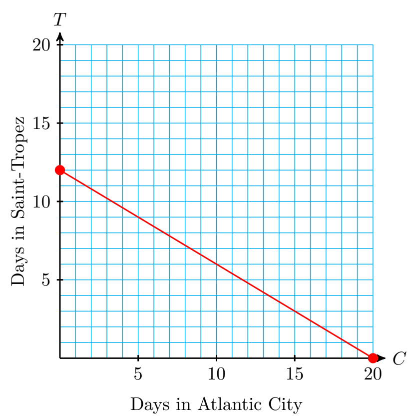
d t-intercept: The oil reserves will be gone in 2080; R-intercept: There were 2100 billion barrels of oil reserves in 2005.

3.

Answer.

a $60C + 100T = 1200$

b $(20, 0)$, $(0, 12)$

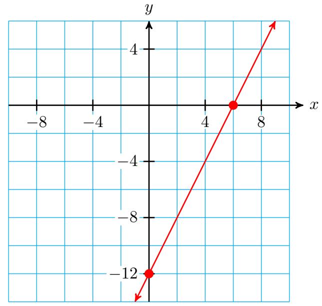


c 6 days

d She can spend 20 days in Atlantic City if she spends no time in Saint-Tropez, or 12 days in Saint-Tropez if she spends no time in Atlantic City.

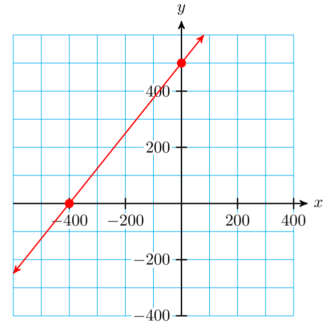
4.

Answer.



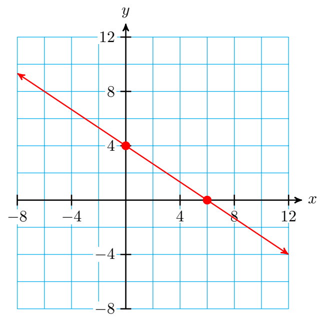
5.

Answer.



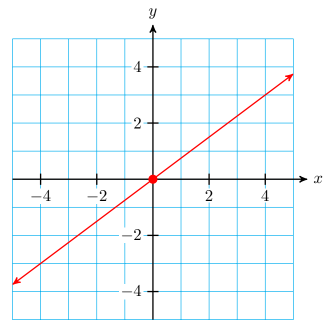
6.

Answer.



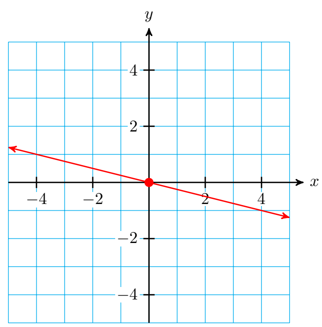
7.

Answer.



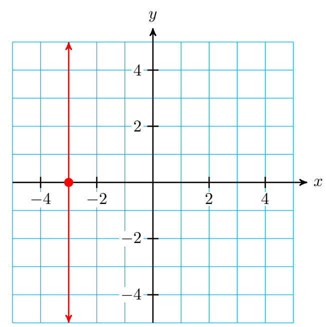
8.

Answer.



9.

Answer.

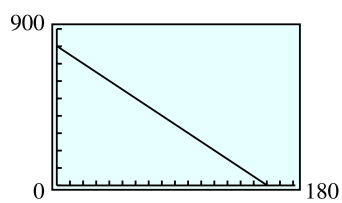


10.

Answer.

a $B = 800 - 5t$

b



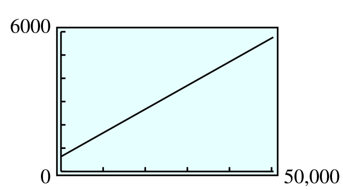
c $m = -5$ thousand barrels/minute: The amount of oil in the tanker is decreasing by 5000 barrels per minute.

11.

Answer.

a $F = 500 + 0.10C$

b



c $m = 0.10$: The fee increases by \$0.10 for each dollar increase in the remodeling job.

12.

Answer. $\dfrac{-3}{2}$

13.

Answer. 2

14.

Answer. -0.4

15.

Answer. -1.75

16.

Answer. neither

17.

Answer. both

18.

Answer.

d	V
-5	-4.8
-2	-3
1	-1.2
6	1.8
10	4.2

19.

Answer.

q	S
-8	-8
-4	36
3	168
5	200
9	264

20.

Answer. 80 ft

21.

Answer. $m = \dfrac{1}{2}$, $b = \dfrac{-5}{4}$

22.

Answer. $m = \dfrac{3}{4}$, $b = \dfrac{5}{4}$

23.

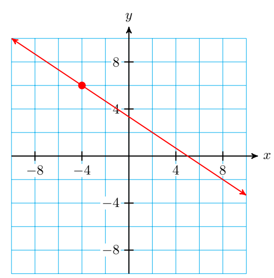
Answer. $m = -4$, $b = 3$

24.

Answer. $m = 0$, $b = 3$

25.

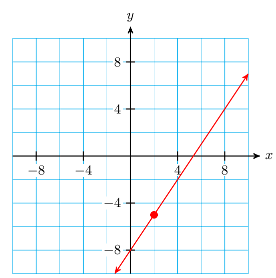
Answer.

author 

buthor $y = \dfrac{-2}{3}x + \dfrac{10}{3}$

26.

Answer.

author 

buthor $y = \dfrac{3}{2}x - 8$

27.

Answer.

a $T = 62 - 0.0036h$

b $-46°$F; $108°$F

c $-71°$F

28.

Answer. $y = \dfrac{-9}{5}x + \dfrac{2}{5}$

29.

Answer. $y = \dfrac{-5}{2}x + 8$

30.

Answer.

a

t	0	15
P	4800	6780

b $P = 4800 + 132t$

c $m = 132$ people/year: the population grew at a rate of 132 people per year.

31.

Answer.

a $m = -2$; $b = 3$

b $y = -2x + 3$

32.

Answer.

a $m = \frac{3}{2}$; $b = -5$

b $y = \frac{3}{2}x - 5$

33.

Answer. $\frac{3}{5}$

34.

Answer.

a $(4, 0)$ and $(0, -6)$

b $\frac{3}{2}$

35.

Answer.

a $\left(\frac{8}{3}, 0\right)$ and $(0, -4)$

b $(4, 3)$; No

2 · Applications of Linear Models

2.1 · Linear Regression

· Problem Set 2.1

Warm Up

1.

Answer. a: II; b: III; c: I; d: IV

2.

Answer. a: III; b: IV; c: II; d: I

3.

Answer.

a

k	70	50
p	154	110

b $p = 2.2k$

c $m = 2.2$ pounds/kg is the conversion factor from kilograms to pounds

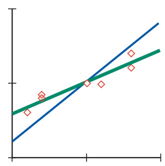
Skills Practice

5.

Answer. The slope is -2.5, which indicates that the snack bar sells 2.5 fewer cups of cocoa for each 1°C increase in temperature. The C-intercept of 52 indicates that 52 cups of cocoa would be sold at a temperature of 0°C. The T-intercept of 20.8 indicates that no cocoa will be sold at a temperature of 20.8°C.

7.

Answer.



9.

Answer.

a 74 feet

b The young whale grows in length about 4.14 foot per month

11.

Answer. 2 min: 21°C; 2 hr: 729°C; The estimate at 2 minutes is reasonable; the estimate at 2 hours is not reasonable.

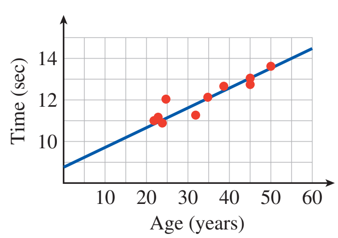
Applications

13.

Answer.

a 12 seconds

b 39

c 

d 11.6 seconds

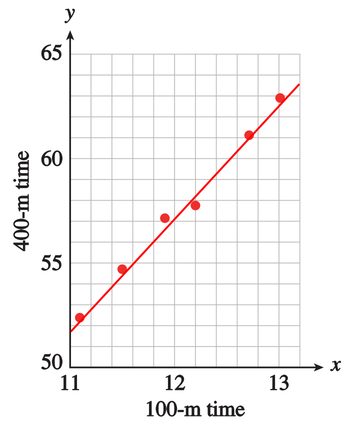
e $y = 8.5 + 0.1x$

f 12.7 seconds; 10.18 seconds; The prediction for the 40-year-old is reasonable, but not the prediction for the 12-year-old.

15.

Answer.

a



b 53 sec, 64 sec

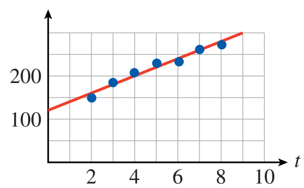
c $y = 5.5x - 8.6$

d 57.95 sec

17.

Answer.

a



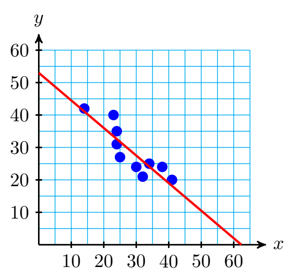
b $y = 121 + 19.86t$

c 419

19.

Answer.

a



b The graph is above.

c The slope tells us that the time it takes for a bird to attract a mate decreases by 0.85 days for every additional song it learns.

d 44.5 days

e The C-intercept tells us that a warbler with a repretoire of 53 songs would acquire a mate immediately. The B-intercept tells us that a warbler with no songs would take 62 days to find a mate. These values make sense in context.

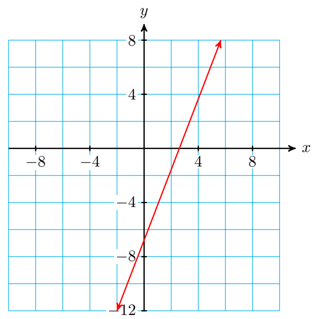
2.2 · Linear Systems

· Problem Set 2.2

Warm Up

1.

Answer.



3.

Answer. 18.75

Skills Practice

5.

Answer. $(-4, 5)$

7.

Answer. $(-2, 3)$

9.

Answer. Inconsistent

11.

Answer. Dependent

Applications

13.

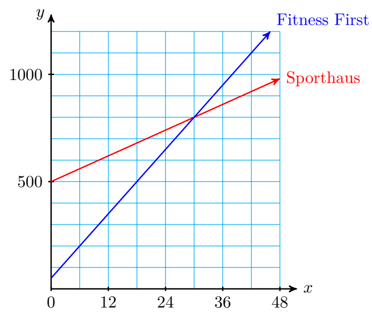
Answer.

a Sporthaus: $y = 500 + 10x$

Fitness First: $y = 50 + 25x$

b

x	Sporthaus	Fitness First
6	560	200
12	620	350
18	680	500
24	740	650
30	800	800
36	860	950
42	920	1100
48	980	1250

c 

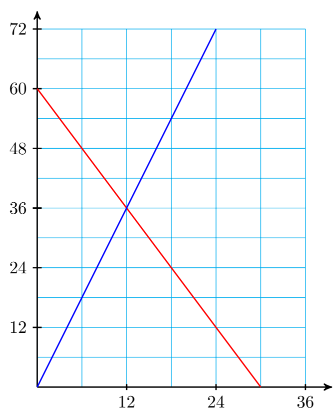
d 30 months

15.

Answer.

a $10x + 5y = 300$

b $y = 3x$

c 

$(12, 36)$

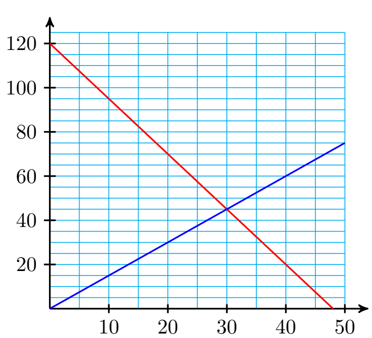
d She should buy 12 hardbacks and 36 paperbacks.

17.

Answer.

a $S = 1.5x$

b $D = 120 - 2.5x$

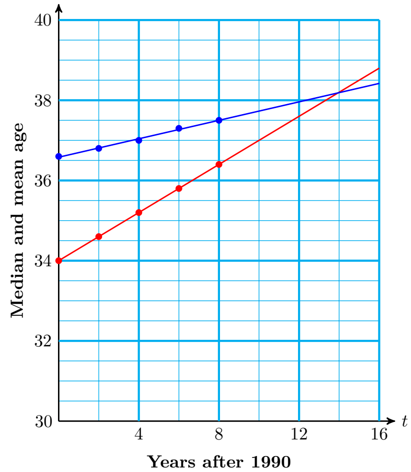
c 

d $x = 30$; $S = 45$

19.

Answer.

a The median age

b 

c 0.3 years of age per year since 1990

d See (b) above

e Slightly less than 14 years since 1990

f More than half the women are older than the mean age of women.

2.3 · Algebraic Solution of Systems · Problem Set 2.3

Warm Up

1.
Answer. $(4, 1)$

Skills Practice

3.
Answer. $(1, 2)$

7.
Answer. $(1, 2)$

9.
Answer. Dependent

Applications

11.
Answer.

a $S = 2.5x$; $D = 350 - 4.5x$

b 50 dollars per machine; 125 machines

13.

Answer.

a

	Pounds	% silver	Amount of silver
First alloy	x	0.45	$0.45x$
Second alloy	y	0.60	$0.60y$
Mixture	40	0.48	0.48(40)

b $x + y = 40$

c $0.45x + 0.60y = 19.2$

d 32 lb

15.

Answer.

a

Rani's speed in still water: x
Speed of the current: y

	Rate	Time	Distance
Downstream	$x+y$	45	6000
Upstream	$x-y$	45	4800

b $45(x + y) = 6000$

c $45(x - y) = 4800$

d Rani's speed in still water is 120 meters per minute, and the speed of the current is $13\frac{1}{3}$ meters per minute.

17.

Answer. 607.5 mi

19.

Answer.

a

	Sports coupes	Wagons	Total
Hours of riveting	3	4	120
Hours of welding	4	5	155

b $3x + 4y = 120$

c $4x + 5y = 155$

d 20 sports coupes and 15 wagons

2.4 · Gaussian Reduction
· Problem Set 2.4

Warm Up

1.
Answer. $(-3, -5)$

3.

Answer.

a

	Principal	Interest rate	Interest
Bonds	x	0.10	$0.10x$
Certificate	y	0.08	$0.08y$
Total	2000	——	184

b $x + y = 2000$

c $0.10x + 0.08y = 184$

Skills Practice

5.
Answer. $(-2, 0, 3)$

7.
Answer. $(2, -2, 0)$

9.
Answer. $(2, -3, 1)$

11.
Answer. $\left(\frac{1}{2}, \frac{2}{3}, -3\right)$

13.
Answer. $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{3}\right)$

15.
Answer. Dependent

Applications

17.
Answer. $x = 40$ in, $y = 60$ in, $z = 55$ in

19.
Answer. $\frac{1}{2}$ lb Colombian, $\frac{1}{4}$ lb French, $\frac{1}{4}$ Sumatran

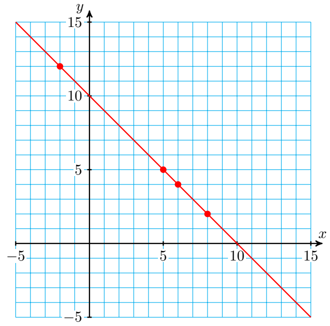
2.5 · Linear Inequalities in Two Variables · Problem Set 2.5

Warm Up

1.
Answer.

a

x	-2	5	6	8
y	12	5	4	2

b 

c $x + y = 10$

d See above.

3.

Answer.

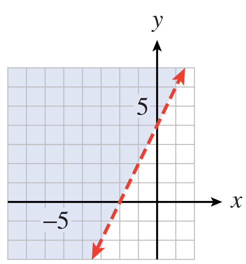
(a) The graph of the equation is a line, and the graph of the inequality is a half-plane. The line is the boundary of the half-plane but is not included in the solution to the inequality.

(b) The graph of $x + y \geq 10,000$ includes both the line $x + y = 10,000$ and the half-plane of the corresponding strict inequality.

Skills Practice

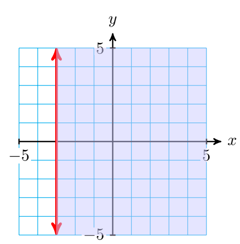
5.

Answer.



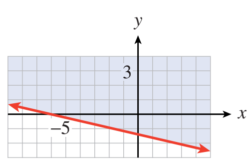
7.

Answer.



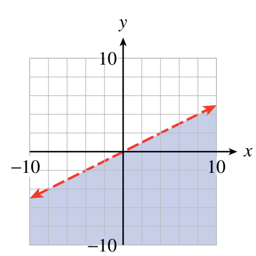
9.

Answer.



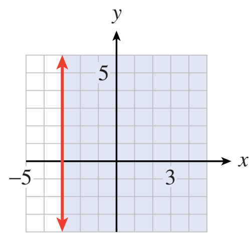
11.

Answer.



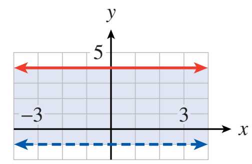
13.

Answer.



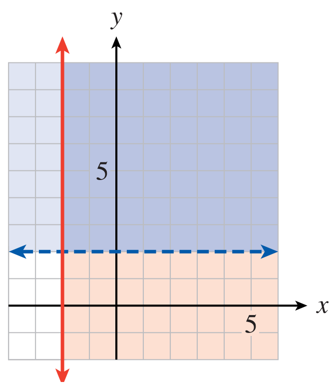
14.

Answer.



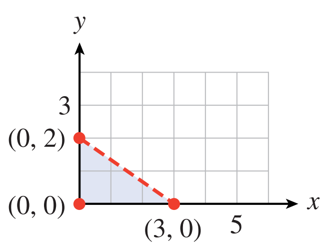
15.

Answer.



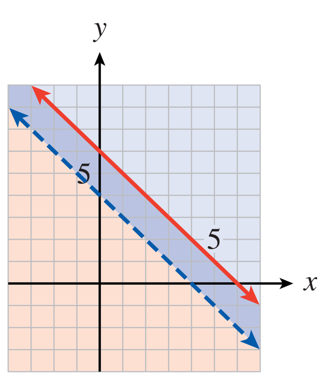
17.

Answer.



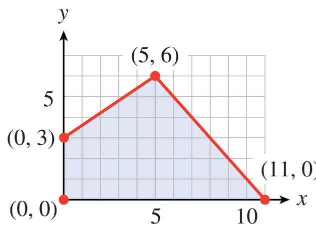
19.

Answer.



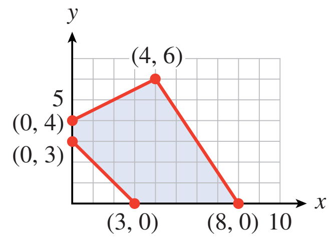
21.

Answer.



23.

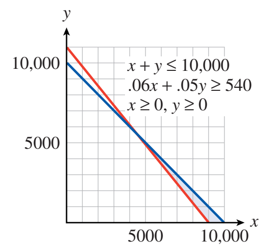
Answer.



Applications

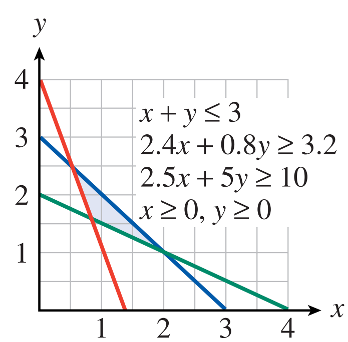
25.

Answer.



27.

Answer.



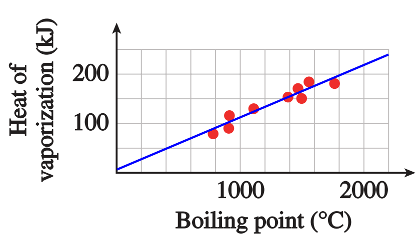
2.6 · Chapter Summary and Review

· Chapter 2 Review Problems

1.

Answer.

a



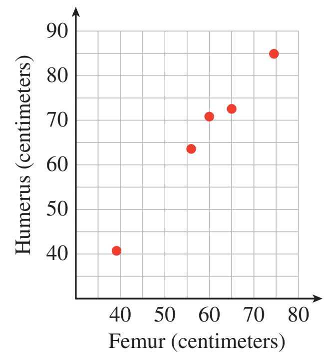
b 235 kilojoules

c $y = 0.106x + 4.6$

d 156.7°C

2.

Answer.



a 45 cm

b 87 cm

c $y = 1.2x - 3$

d 69 cm

e $y = 1.197x - 3.660$; 68.16 cm

3.

Answer. 6

4.

Answer. 26 min

5.

Answer. $(-1, 2)$

6.

Answer. $(1.9, -0.8)$

7.

Answer. $\left(\frac{1}{2}, \frac{7}{2}\right)$

8.

Answer. $(1, 2)$

9.

Answer. $(1, 2)$

10.

Answer. $\left(\frac{1}{2}, \frac{3}{2}\right)$

11.

Answer. Consistent and independent

12.

Answer. Inconsistent

13.

Answer. Dependent

14.

Answer. Consistent and independent

15.

Answer. $(2, 0, -1)$

16.

Answer. $(2, 1, -1)$

17.

Answer. $(2, -5, 3)$

18.

Answer. $\left(2, \frac{3}{2}, -1\right)$

19.

Answer. $(-2, 1, 3)$

20.

Answer. $(2, -1, 0)$

21.

Answer. 26

22.

Answer. 17

23.

Answer. \$3181.82 at 8%, \$1818.18 at 13.5%

24.

Answer. \$4000

25.

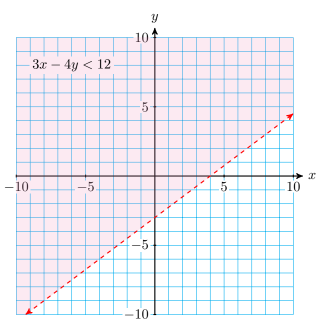
Answer. 5 cm, 12 cm, 13 cm

26.

Answer. 20 to Boston, 25 to Chicago, 10 to Los Angeles

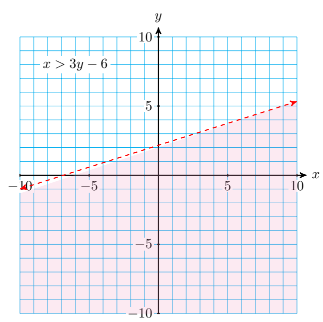
27.

Answer.



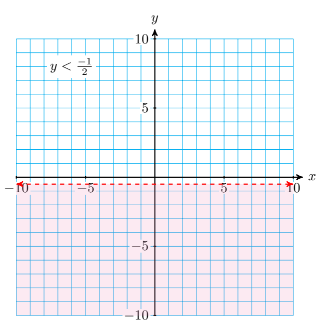
28.

Answer.



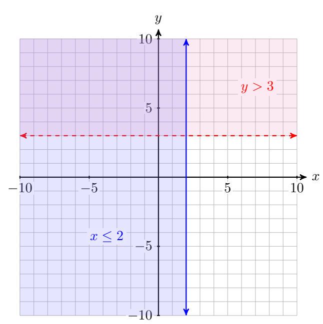
29.

Answer.



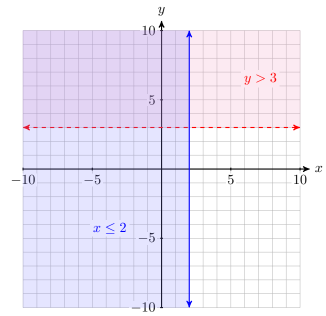
30.

Answer.



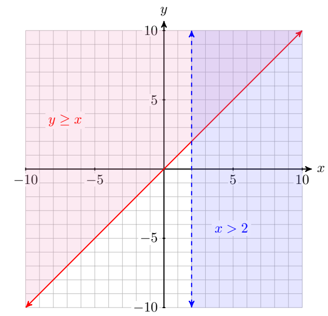
31.

Answer.



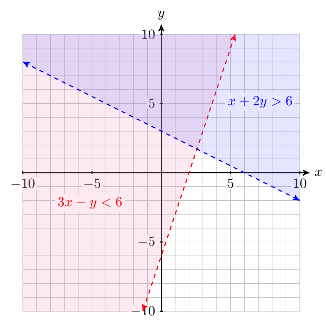
32.

Answer.



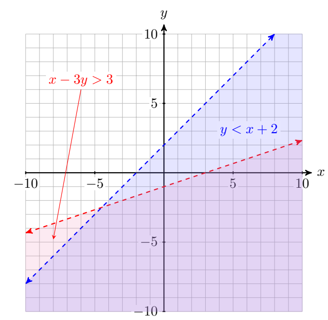
33.

Answer.



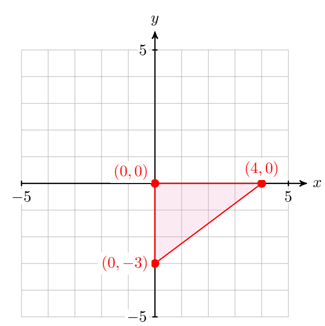
34.

Answer.



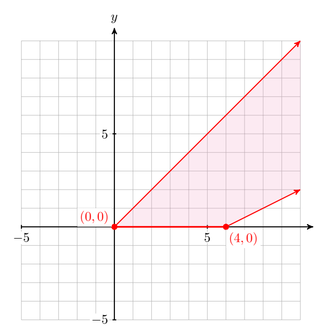
35.

Answer.



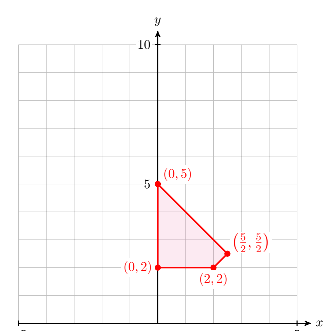
36.

Answer.



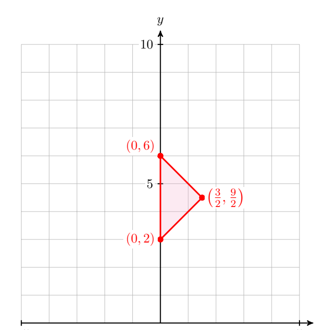
37.

Answer.



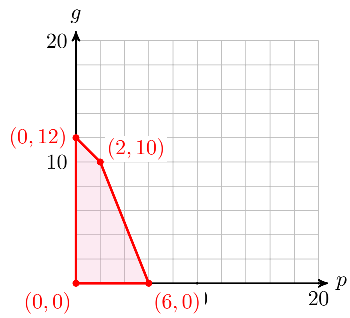
38.

Answer.



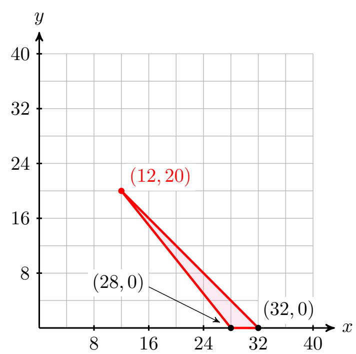
39.

Answer. $20p + 9g \leq 120$, $10p + 10g \leq 120$, $p \geq 0$, $g \geq 0$



40.

Answer. $x + y \leq 32$, $2x + 1.6y \geq 56$, $x \geq 0$, $y \geq 0$, where x represents ounces of tofu, y the ounces of tempeh



3 · Quadratic Models

3.1 · Extraction of Roots

· Problem Set 3.1

Warm Up

1.

Answer.

a -12

b -2

c 9

3.

Answer.

a 29

b 7

c $\sqrt{6}$

4.

Answer.

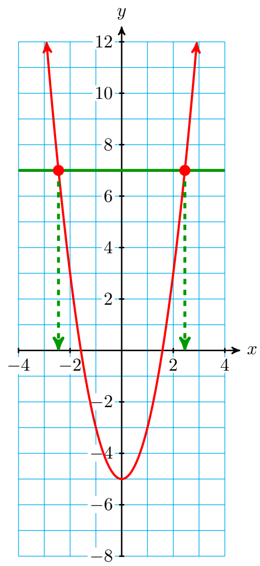
a ± 7

b $\pm \dfrac{5}{2}$

c $\pm\sqrt{5}$

5.

Answer.

a 

b Two solutions. $\approx \pm 2.5$

c $\pm\sqrt{6}$

Skills Practice

7.
Answer. $\pm\sqrt{3}$

9.
Answer. $\frac{5}{2}, \frac{-3}{2}$

11.
Answer. $\frac{2}{3} \pm \frac{\sqrt{5}}{3}$

13.
Answer. $\frac{7}{8}, \pm\frac{\sqrt{8}}{8}$

15.
Answer. ± 5.73

17.
Answer. b. 10, -2

19.
Answer. $\pm\frac{Fr}{m}$

21.
Answer. $\pm pi\sqrt{\frac{L}{8}}$

Applications

23.
Answer. 21 in

25.
Answer.

a $A = 1600(1 + r)^2$

b

r	0.02	0.04	0.06	0.08
A	1664.64	1730.56	1797.76	1866.24

c 11.8%

27.

Answer. 19 ft by 57 ft

29.

Answer.

a $V = 62.8r^2$

b $\frac{1}{4}$

c 1.96 cm

31.

Answer.

a $\pm\sqrt{\frac{bc}{a}}$

b $\pm\sqrt{\frac{ac}{b}}$

3.2 · Intercepts, Solutions, and Factors · Problem Set 3.2

Warm Up

1.

Answer.

a $2b^2 + 9b - 18$

b $12z^2 - 35z + 8$

3.

Answer.

a $6p^3 - 33p^2 + 45p$

b $6v^3 + 16v^2 - 32v$

5.

Answer.

a $(x - 4)(x + 4)$

b $x(x - 16)$

c $(x - 4)(x - 4)$

d cannot be factored

7.

Answer. $(x - 5)(x - 2)$

9.

Answer. $(w - 8)(w + 4)$

11.

Answer. $(x - 5)(x - 2)$

Skills Practice

13.

Answer. $\frac{1}{2},\ -3$

15.

Answer. $0,\ 3$

17.

Answer. $\frac{1}{2},\ 1$

19.

Answer. $-1, 2$

21.

Answer. $\dfrac{7}{8} \pm \dfrac{\sqrt{5}}{8}$

23.

Answer. $\dfrac{b \pm 5}{a}$

25.

Answer. $0.1(x-18)(x+15)$

27.

Answer. All three graphs have the same x-intercepts.

Applications

29.

Answer. c. 306.5 ft at 0.625 sec d. 1.25 sec e. 5 sec

31.

Answer. b. $h^2 + 10^2 = (h+2)^2$ c. 24 ft

33.

Answer.

a $l = x - 4,\ w = x - 4,\ h = 2,\ V = 2(x-4)^2$

b 0 cubic in, 2 cubic in, 8 cubic in, etc.

c $x = 4$

d 9 in by 9 in

e $2(x-4)^2 = 50;\ x = 9$

34.

Answer. a. $l = 6,\ w = 1-2x,\ h = x,\ V = 6x(1-2x)$ e. $6x(1-2x) = \dfrac{3}{4};\ \dfrac{1}{4}$ ft

3.3 · Graphing Parabolas

· Problem Set 3.3

Warm Up

1.

Answer. 13

4.

Answer.

a factoring; $0, 10$

b extraction of roots; $\pm\sqrt{10}$

c factoring; $\dfrac{-1}{2}, 1$

d extraction of roots; $\dfrac{-5}{2}, \dfrac{-5}{2}$

6.

Answer. $2\pi R(R+H)$

8.

Answer. $0, -35$

Skills Practice

9.

Answer.

a ii

b iv

c i

d iii

e vi

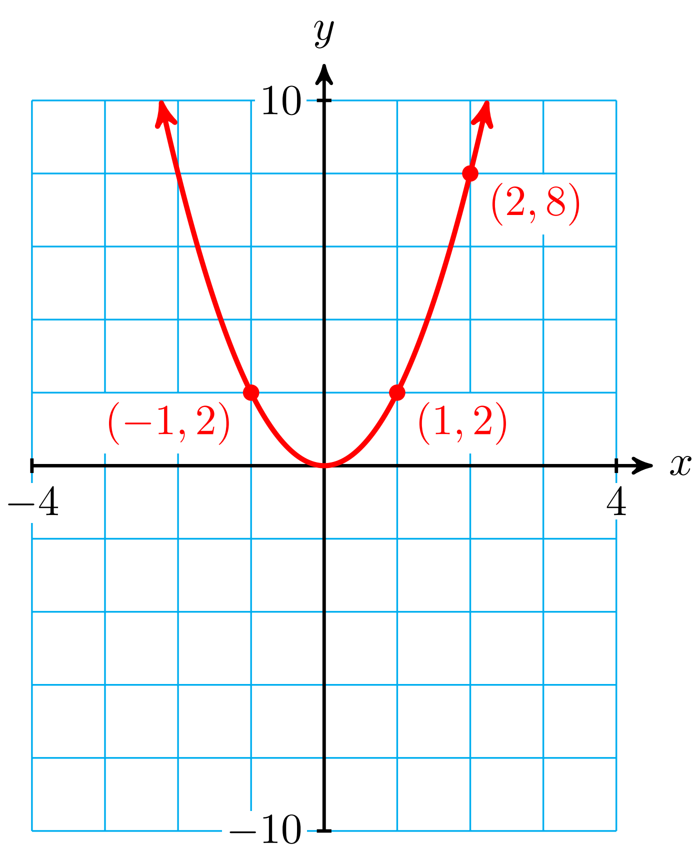
f v

11.

Answer.

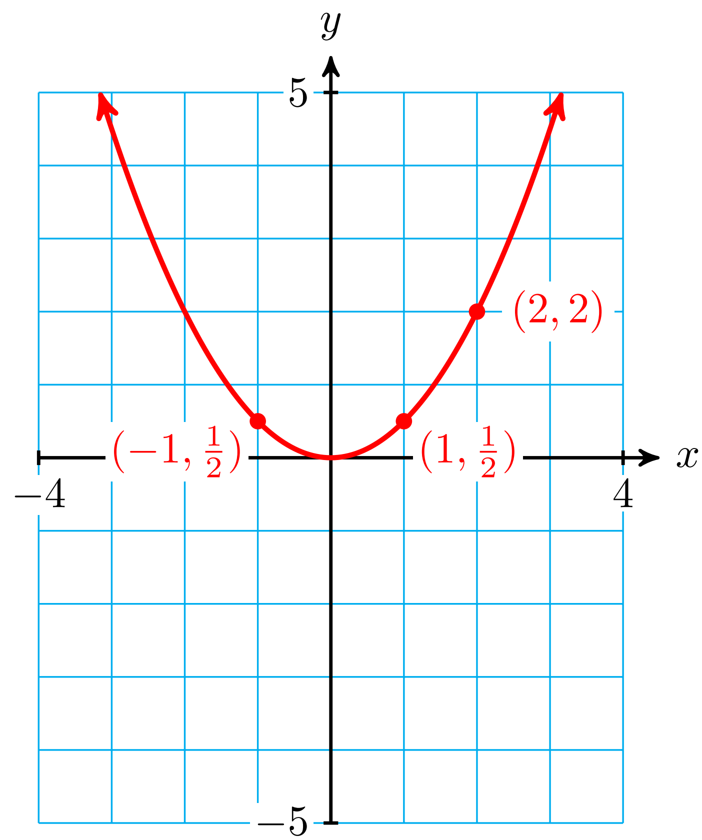
i a Narrower

b



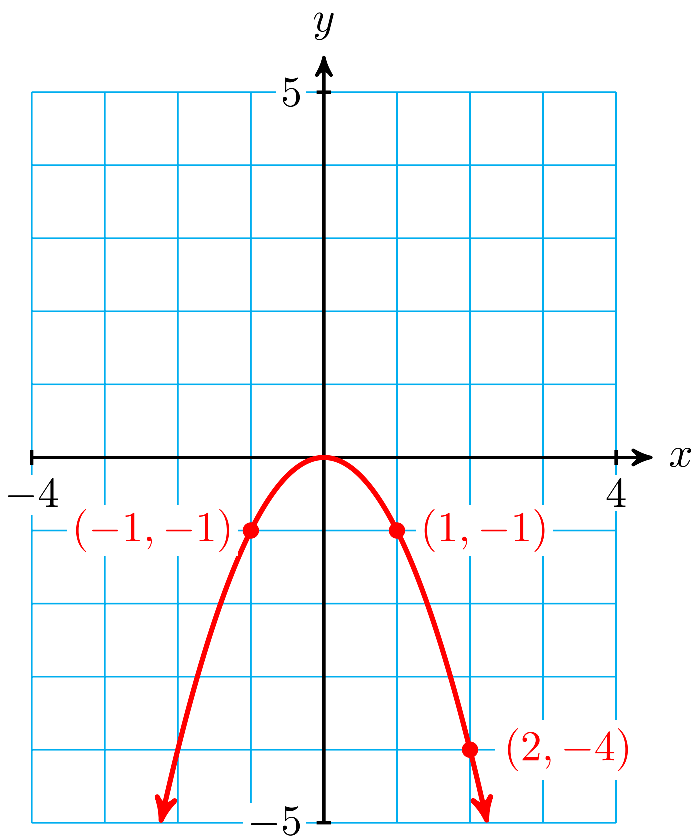
ii a Wider

b



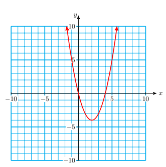
iii a Opens downward

b



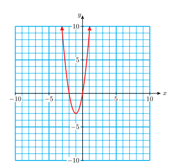
13.

Answer. $(0, 0), (4, 0); (2, -4)$



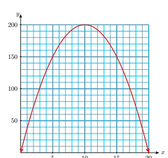
15.

Answer. $(0, 0), (-2, 0); (-1, -3)$



17.

Answer. $(0, 0), (20, 0); (10, 200)$



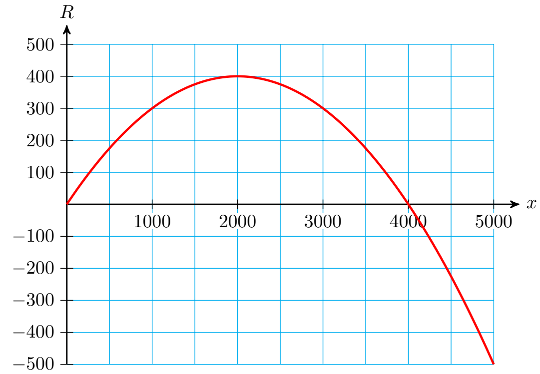
Applications

19.

Answer.

a $(2000, 400)$

b $(0, 0), (4000, 0)$

c 

d $x > 4000$

21.

Answer.

a basic parabola

b narrower

c wider

d narrower and reflected about y-axis

23.

Answer.

a x-intercepts at 0 and 4

b x-intercepts at 0 and -4

c reflection of (a) about y-axis

d reflection of (b) about y-axis

3.4 · Completing the Square

· Problem Set 3.4

Warm Up

1.

Answer. $-4, \dfrac{5}{2}$

3.

Answer.

a $x^2 + 10x + 25$

b $x^2 - 12x + 36$

c $x^2 - 24x + 144$

d $x^2 + 30x + 225$

Skills Practice

5.

Answer. b and c

7.

Answer. $-4, -5$

9.

Answer. $-1 \pm \sqrt{\dfrac{5}{2}}$

11.
Answer. $\frac{-5}{2} \pm \sqrt{\frac{45}{4}}$

13.
Answer. $-5, 8$

15.
Answer. $-3, \frac{4}{3}$

Applications

17.
Answer.

a $(\frac{1}{4} \pm \sqrt{\frac{13}{15}}, 0)$

b $(\frac{1}{4}, -3\frac{1}{4})$

19.
Answer.

a $(-2, 0)$, $(\frac{2}{5}, 0)$

b $(\frac{-4}{5}, -7\frac{1}{5})$

21.
Answer.

a $L^2 + (L-4)^2 = 20^2$

b 12 in by 16 in

23.
Answer.

a $A = \frac{1}{2}(x^2 - y^2)$

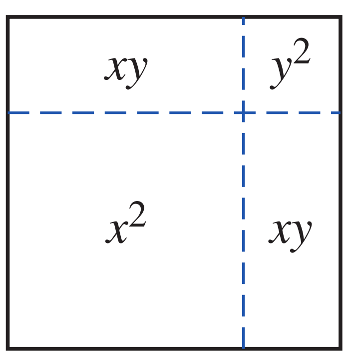
b $A = \frac{1}{2}(x-y)(x+y)$

c 18 sq ft

25.
Answer.

a $A = (x+y)^2$

b $A = x^2 + 2xy + y^2$

c 

27.
Answer. $-1 \pm \sqrt{1-c}$

29.
Answer. $\frac{b}{2} \pm \frac{b^2-4c}{4}$

31.

Answer. $\pm\sqrt{\dfrac{V}{2w} - s^2}$

33.

Answer. $(-15, 0), (15, 0); (0, 255)$

35.

Answer. $\dfrac{-3x \pm 3}{2}$

3.5 · Chapter 3 Summary and Review

· Chapter 3 Review Problems

1.

Answer. $\pm\sqrt{2}$

2.

Answer. $\pm\sqrt{40}$

3.

Answer. $-4 \pm \sqrt{20}$

4.

Answer. $\dfrac{1 \pm \sqrt{15}}{7}$

5.

Answer. $1,\ 4$

6.

Answer. $11.5,\ 23.5$

7.

Answer. $\dfrac{-3}{2},\ 2$

8.

Answer. $2,\ 2$

9.

Answer. $-2,\ 3$

10.

Answer. $-1,\ 1$

11.

Answer. $4x^2 - 29x - 24 = 0$

12.

Answer. $9x^2 - 30x + 25 = 0$

13.

Answer. $y = (x - 3)(x + 2.4)$

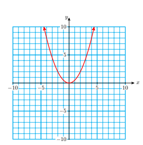
14.

Answer. $y = -(x + 1.3)(x - 2)$

15.

Answer.

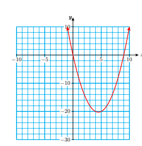
a Vertex: $(0, 0)$, y-int: $(0, 0)$, x-int: $(0, 0)$

b 

17.

Answer.

a Vertex: $\left(\dfrac{9}{2}, \dfrac{-81}{4}\right)$, y-int: $(0, 0)$, x-int: $(0, 0)$, $(9, 0)$

b 

19.

Answer. $2 \pm \sqrt{10}$

20.

Answer. $\dfrac{-3}{2} \pm \sqrt{\dfrac{21}{4}}$

21.

Answer. $\dfrac{3}{2} \pm \sqrt{\dfrac{3}{4}}$

22.

Answer. $\dfrac{1}{3} \pm \sqrt{\dfrac{10}{9}}$

23.

Answer. $v = \pm\sqrt{\dfrac{2K}{m}}$

24.

Answer. $b = \pm\sqrt{c^2 - a^2}$

25.

Answer. $s = \pm\sqrt{\dfrac{3V}{h}}$

26.

Answer. $r = \pm\sqrt{\dfrac{A}{P}} - 1$

27.

Answer. 9

28.

Answer. 13

29.

Answer. 11%

30.

Answer. 8.5%

31.

Answer. $\sqrt{108} \approx 10.4$ in

32.

Answer. 17 in

33.

Answer. 1 sec

34.

Answer. 50 ft by 150 ft

35.

Answer.
$A_1 = x^2 - \left(\dfrac{1}{2}y^2 + \dfrac{1}{2}y^2\right) = x^2 - y^2$; $A_2 = (x+y)(x-y) = x^2 - y^2$

36.

Answer.
$A_1 = \pi(x+y)^2 - \pi x^2 - \pi y^2 = 2\pi xy$; $A_2 = \pi y(2x) = 2\pi xy$

4 · Applications of Quadratic Models

4.1 · Quadratic Formula

· Problem Set 4.1

Warm Up

1.

Answer.

a $8 - 2\sqrt{20}$

b 9

3.

Answer.

a $A = lw,\ P = 2l + 2w$

b
- i $A = 24$ sq ft, $P = 20$ ft
- ii $A = 24$ sq ft, $P = 22$ ft

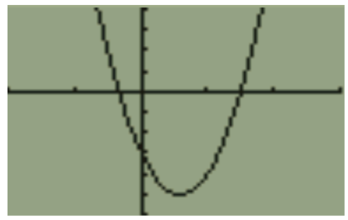
Skills Practice

5.
Answer. 0.618, -1.618

7.
Answer. 0.232, 1.434

9.
Answer. 1.270, 2.480

11.
Answer.

a 

b Approximately $(1.5, 0)$ and $(-0.3, 0)$

c $\frac{3}{2}$ and $\frac{-1}{3}$. These are the x-intercepts of the graph.

13.
Answer.

a $(5, 0)$ and $(1, 0)$; $(3, 0)$; no x-intercepts

b 1 and 5; 3; $\frac{6 \pm i\sqrt{12}}{2}$. The real-valued solutions are the x-intercepts of the graph. If the solutions are complex, the graph has no x-intercepts.

15.
Answer.

a $x^2 - 4x - 1 = 0$

b $x^2 - 8x + 25 = 0$

17.
Answer. $\frac{-4 \pm \sqrt{16 - 64h}}{32}$

19.
Answer. $\frac{v \pm \sqrt{v^2 - 2as}}{a}$

21.
Answer. $\frac{-1 \pm \sqrt{1 + 8S}}{2}$

Applications

23.
Answer. c. 31.77 mph

25.
Answer.

a 357 km

b 7100 m

27.

Answer. b. $10h(2h-6)=2160$ c. 12 ft by 18 ft by 10ft

4.2 · The Vertex

· Problem Set 4.2

Warm Up

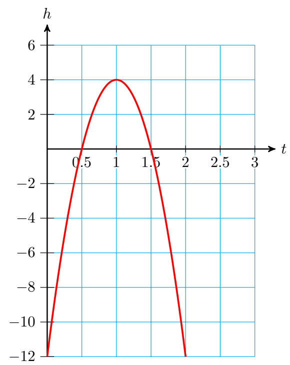
1.

Answer.

a

t	0	0.25	0.5	0.75	1	0.25	1.5
h	-12	-5	0	3	4	3	0

b



c $(1,4)$

d The wrench reaches its greatest height of 4 feet.

e The wrench reaches its greatest height 1 second after Francine throws it.

3.

Answer.

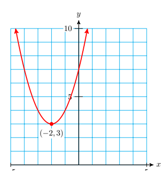
a $y=x^2-3x-4$

b y = -x^2+3x+4

Skills Practice

5.

Answer. $(-2,3)$



7.

Answer.

a $(3, 0)$

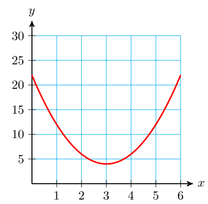
b $(3, 0)$

c $(3, 4)$

9.

Answer.

a $(3, 4)$

b 

c $y = 2x^2 - 12x + 22$

11.

Answer.

a $y = a(x + 2)^2 + 6$

b 3

13.

Answer. $y = (3x + 1)^2 - 5$

15.

Answer.

a IV

b V

c I

d VII

Applications

17.

Answer.

a $l = 40 - w;\ A = 40w - w^2$

b 400 sq yd; 20 yd by 20 yd

19.

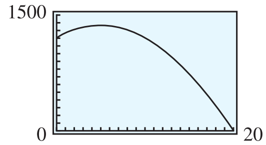
Answer.

a (table)

b $20 + 2x,\ 60 - 3x,\ (20 + 2x)(60 - 3x)$

c (table)

d $x = 20$

e 

f \$24, \$36

g \$1350, \$30, 45 rooms

21.

Answer.

a $h = \frac{-1}{40}(x - 80)^2 + 164$

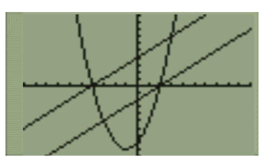
b 160.99 ft

23.

Answer. 5 g; 4 min

25.

Answer.

a 

b $(-4, 0)$, $(0, 2)$

c

	$x < -4$	$-4 < x < 2$	$x > 2$
Y_1	$-$	$+$	$+$
Y_2	$-$	$-$	$+$
Y_3	$+$	$-$	$+$

4.3 · Curve Fitting
· Problem Set 4.3

Warm Up

1.

Answer.
$(4, 10)$, $(0, -22)$, $(4 \pm \sqrt{5}, 0)$

3.

Answer.

a $25 - 2x$

b $30 + 4x$

5.

Answer. $y = -x^2 - 2x + 8$

7.

Answer. $y = -x^2$

Skills Practice

9.
Answer. $(-2, 3, -4)$

11.
Answer. $(1, -4, 7)$

13.
Answer. $a = 3,\ b = 1,\ c = -2$

Applications

15.
Answer.

a $y = a(x-5)^2 - 10 = -0.2x^2 + 11x - 78$

b 42

17.
Answer. $D = \dfrac{1}{2}n^2 - \dfrac{3}{2}n$

19.
Answer.

a $N = -0.59t^2 + 7.33t - 2.54$

b 2000; 7

21.
Answer.

a $(0, 0.14)$

b $y = -1.786x^2 + 0.14$

4.4 · Quadratic Inequalities

· Problem Set 4.4

Warm Up

1.
Answer.

a $(-4, 0),\ (6, 0)$

b up

3.
Answer.

a $(3 \pm \sqrt{12}, 0)$

b down

5.
Answer.

a $[0, 4)$

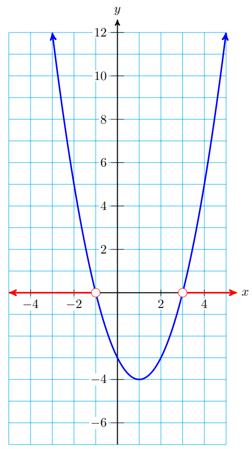
b $(5, 8)$

Skills Practice

7.

Answer.

a



b See graph.

c $x > 3$ or $x < -1$

9.

Answer.

a $-12, 5$

b $x < -12$ or $x > 15$

11.

Answer.

a $(-\infty, -3) \cup (6, \infty)$

b $(-3, 6)$

c $[-2, 5]$

d $(-\infty, 2] \cup [5, \infty)$

13.

Answer. $x < -2$ or $x > 3$

15.

Answer. $0 \le k \le 4$

17.

Answer. $(-6, -3)$

19.

Answer. $(-\infty, \frac{-1}{2}) \cup (4, \infty)$

21.

Answer.
$(-\infty, -2.24) \cup (2.24, \infty)$

23.

Answer. All m

25.

Answer. $-4.8 < x < 6.2$

Applications

27.

Answer. $(-\infty, -4.2) \cup (2.6, \infty)$

29.

Answer. $4t < 16$ sec

31.

Answer. 5 <p <7

33.

Answer.

a $60 - x$; $12 + \frac{1}{2}x$

b $y = (60 - x)(12 + \frac{1}{2}x)$

c 882 bu; 18 trees

d Between 10 and 26, inclusive

4.5 · Chapter 4 Summary and Review

· Chapter 4 Review Problems

1.

Answer. 1, 2

2.

Answer. $\dfrac{3 \pm \sqrt{5}}{2}$

3.

Answer. $-\dfrac{4 \pm \sqrt{8}}{2}$

4.

Answer. $\dfrac{-2 \pm \sqrt{28}}{4}$

5.

Answer. $\dfrac{6 \pm \sqrt{36 - 12h}}{6}$

6.

Answer. $\dfrac{3 \pm \sqrt{9 + 8D}}{2}$

7.

Answer. one repeated real solution

8.

Answer. two complex soutions

9.

Answer. two complex soutions

10.

Answer. two distinct real soutions

11.

Answer.

a $h = 100t - 2.8t^2$

b (graph)

c 893 ft

d $15\frac{5}{7}$ sec and 20 sec

12.

Answer.

a 1.5 sec, 11.025 m

b 7.056 m

c 0.6 sec

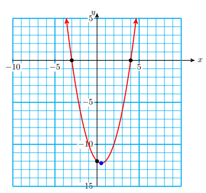
d 0.5 sec, 2.5 sec

e $(0,0)$, $(3,0)$. She leaves the springboard at $t=0$ seconds and returns to the springboard at $t=3$ seconds.

13.

Answer.

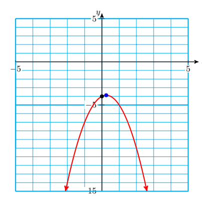
a $\left(\frac{1}{2},\frac{-49}{4}\right)$, $(-3,0)$, $(4,0)$, $(0,-12)$

b 

14.

Answer.

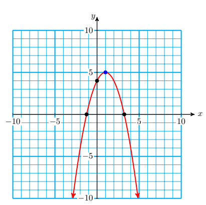
a $\left(\frac{1}{4},\frac{-31}{8}\right)$, no x-intercepts, $(0,-4)$

b 

15.

Answer.

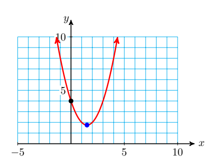
a $(1,5)$, $(-1.24,0)$, $(3.24,0)$, $(0,4)$

b 

16.

Answer.

a $\left(\frac{3}{2},\frac{7}{4}\right)$, no x-intercepts, $(0,4)$

b 

17.

Answer. $y=0.2(x-15)^2-6$

18.

Answer.

a $(1,5)$

b $y=2x^2+4x+7$

19.

Answer.

a 45

b \$810

20.

Answer.

a $R=1200x-80x^2$

b \$7.50

c \$4500

21.

Answer.

a $y = 60(4 + 2x)(32 - 4x)$

b 2

22.

Answer.

a $R = (20 + x)(500 - 10x)$

b \$35

23.

Answer. $a = 1,\ b = -1,\ c = -6$

24.

Answer. $y = -\frac{1}{2}x^2 - 4x + 10$

25.

Answer.

a $h = 36.98t + 5.17$

b 116.1 m, 153.1 m

c (graph)

d $h = -4.858t^2 + 47.67t + 0.89$

e 100.2 m, 113.9 m

f (graph)

g quadratic

26.

Answer.

a $y = -0.05x^2 - 0.003x + 234.2$

b $(-0.03, 234.2)$ The velocity of the debris at its maximum height of 234.2 feet. The velocity there is actually zero.

27.

Answer. $(-\infty, -2) \cup (3, \infty)$

28.

Answer. $[-3, 4]$

29.

Answer. $[-1, \frac{3}{2}]$

30.

Answer. $(-\infty, -\frac{1}{3}) \cup (2, \infty)$

31.

Answer. $[-2, 2]$

32.

Answer. $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

33.

Answer.

a $R = p(220 - \frac{1}{4}p)$

b $400 < p < 480$

34.

Answer.

a $R = p(30 - \frac{1}{2}p)$

b $20 < p < 40$

5 · Functions and Their Graphs

5.1 · Functions

· Problem Set 5.1

Warm Up

1.
Answer. -24

3.
Answer. $\sqrt{20}$

5.
Answer. $-3, \frac{1}{2}$

7.
Answer. $\frac{14}{3}$

Skills Practice

9.

Answer.

a input: v, output: x

b 15

c $-1, \frac{5}{2}$.

11.

Answer.

a $\frac{-1}{3}$

b $\frac{-4}{9}$

c $\frac{-3}{4}$

d -0.530

Applications

13.
Answer. (b), (c), (e), and (f)

15.

Answer.

a 60

b 37.5

c 20

17.
Answer.

a $h(1) = 2$

b 1 month

c 1 month

d 4000

19.
Answer.

a Approximately \$1920

b \$5 or \$15

c $f(12) \approx 1920$; $f(5) = 1500$, $f(15) = 1500$

d $7 < d < 13$

21.
Answer.

a $F(1992) = 7.5\%$

b $F(2000) = 4\%$

c $F(1998+ = 4.5\%$, $F(2001) = 4.5\%$

25.
Answer.

a $N(6000) = 2000$: 2000 cars will be sold at a price of \$6000.

b decrease

c 30,000. At a price of \$30,000, they will sell 400 cars.

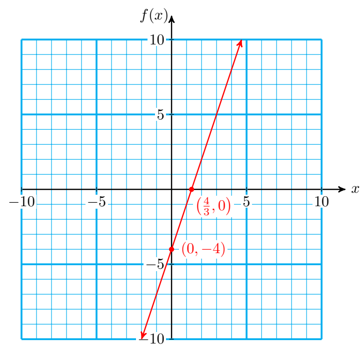
5.2 · Graphs of Functions

· Problem Set 5.2

Warm Up

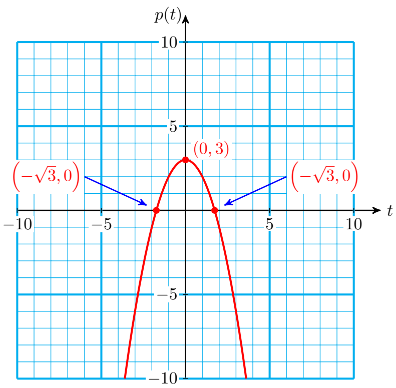
1.

Answer.



3.

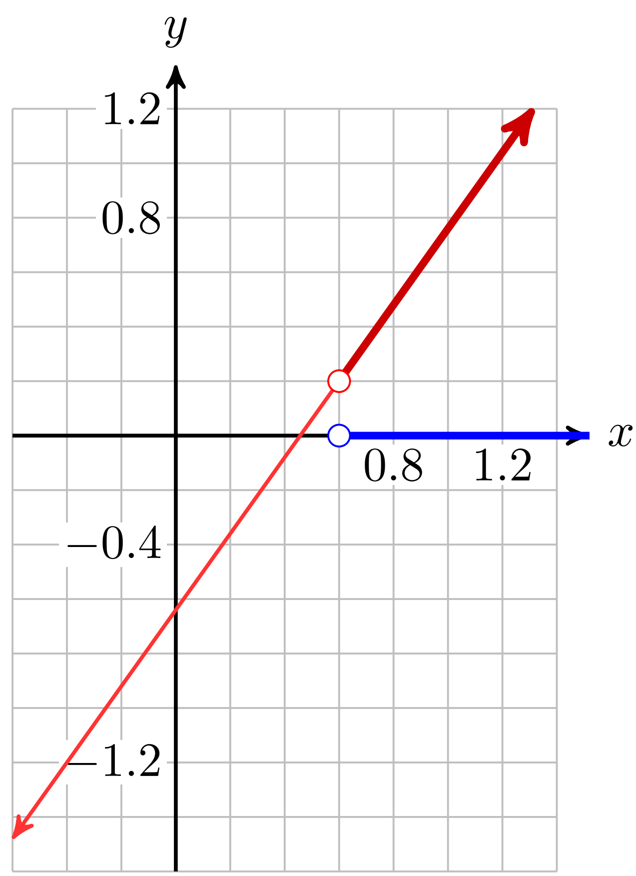
Answer.



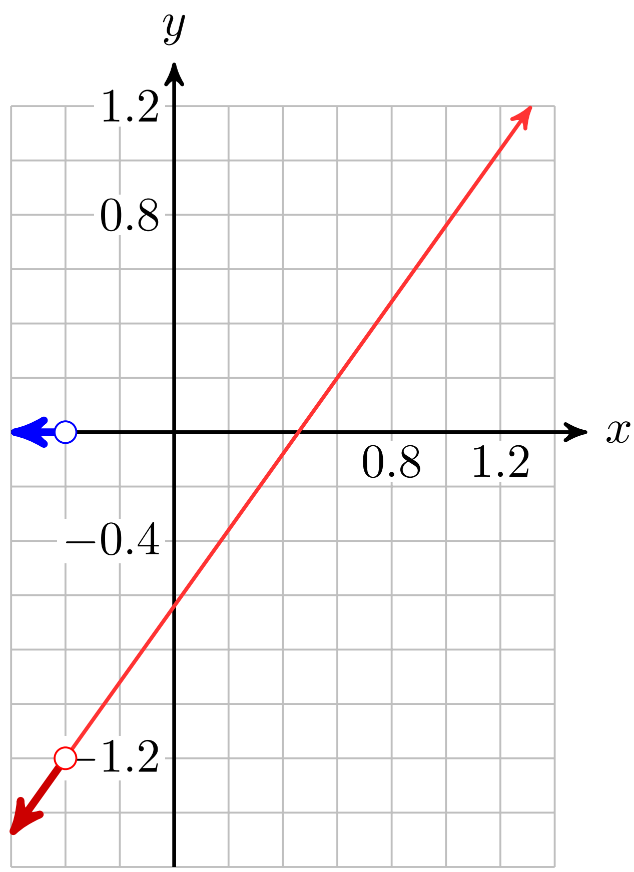
5.

Answer.

a $x > 0.6$



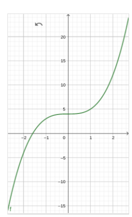
b $x < -0.4$



Skills Practice

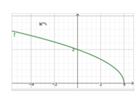
7.

Answer.



9.

Answer.



Applications

11.

Answer.

a $-2, 0, 5$

b 2

c $h(-2) = 0,\ h(1) = 0,\ h(0) = -2$

d 5

e 3

f increasing: $(-4,-2)$ and $(0,3)$; decreasing: $(-2,0)$

13.

Answer.

a $0,\ \frac{1}{2},\ 0$

b $\frac{5}{6}$

c $\frac{-5}{6},\ \frac{-1}{6},\ \frac{7}{6},\ \frac{11}{6}$

d $1,\ -1$

e Max at $x=-1.5,\ 0.5$, min at $x=-0.5,\ 1.5$

15.

Answer.

a $f(1000)=1495$: The speed of sound at a depth of 1000 meters is approximately 1495 meters per second.

b $d=570$ or $d=1070$: The speed of sound is 1500 meters per second at both a depth of 570 meters and a depth of 1070 meters.

c The slowest speed occurs at a depth of about 810 meters and the speed is about 1487 meters per second, so $f(810)=1487$.

d f increases from about 1533 to 1541 in the first 110 meters of depth, then drops to about 1487 at 810 meters, then rises again, passing 1553 at a depth of about 1600 meters.

17.

Answer. (a) and (d)

19.

Answer.

a $-1,1$

b $(-1,1)$

c $[-3,-2]\cup[2,3]$

d $[-5,5]$

21.

Answer.

a $-2,\ 2$

b $-2.8,\ 0,\ 2.8$

c $-2.5<q<-1.25$ and $1.25<q<2.5$

d $-2<q<0$ and $2<q$

23.

Answer.

a $g(-6)=0,\ g(6)=0,\ g(0)=6$

b none

c $g(x)$ is undefined for those x-values

25.
Answer.

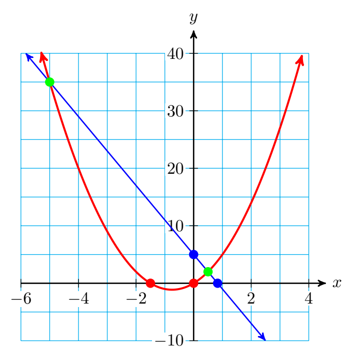
a 0, 5

b 0, $\frac{-3}{2}$

c $\frac{5}{6}$

d -5, $\frac{1}{2}$

e



5.3 · Some Basic Graphs
· Problem Set 5.3

Warm Up

1.
Answer.

a 4

b 2

3.
Answer.

a 2.080

b 6.366

c -0.126

d -1.458

5.
Answer.

a $\frac{1}{2}$

b $\frac{-1}{3}$

7.

Answer.

a -9

b 9

c 9

d -4

9.

Answer.

a -50

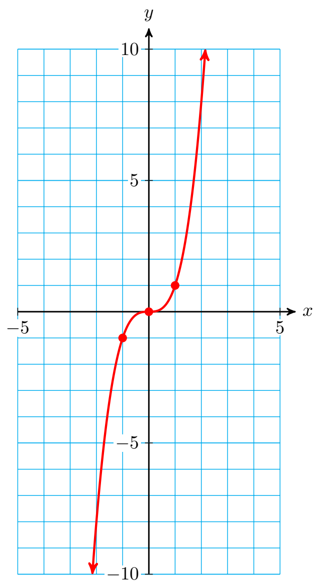
b -43

c 144

Skills Practice

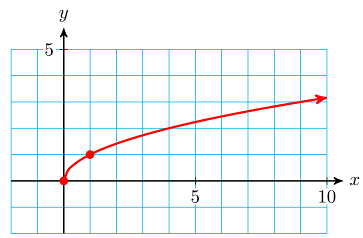
11.

Answer.



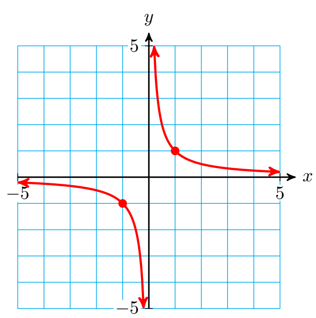
13.

Answer.



15.

Answer.



17.

Answer.

a f

b g

Applications

19.

Answer. $(-\infty, 0) \cup [0.5, \infty)$

21.

Answer. (b): 2 units down, (c): 1 unit up

23.

Answer. (b): 1.5 units left, (c): 1 unit right

25.

Answer. (b): reflected about x-axis, (c): reflected about y-axis

27.

Answer.

i vi

ii ii

iii iv

iv i

v v

vi iii

29.

Answer.

a horizontal shift of square root $y = \sqrt{x}$

b vertical shift of cube root $y = \sqrt[3]{x}$

c vertical shift of absolute value $y = |x|$

d vertical flip of reciprocal $y = \dfrac{1}{x}$

e vertical flip and vertical shift of cube $y = x^3$

f vertical flip and vertical shift of inverse-square $y = \dfrac{1}{x^2}$

31.

Answer.

a $x < 2$

b $x > \dfrac{-2}{3}$

33.

Answer.

a 41

b no solution

c $29 < x \leq 61$

5.4 · Direct Variation

· Problem Set 5.4

Warm Up

1.

Answer.

a -10

b $-16, 20$

Skills Practice

3.

Answer.

a $y = 0.3x$

b

x	2	5	8	12	15
y	-0.6	1.5	2.4	3.6	4.5

c y doubles also

5.

Answer. (b), $k = 0.5$

7.

Answer. (c)

Applications

9.

Answer.

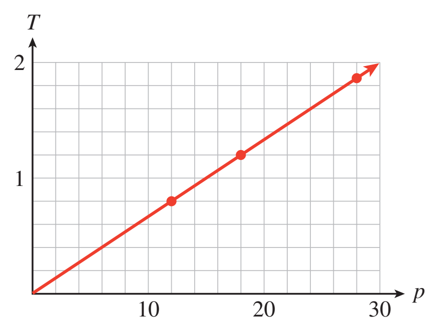
a

Price of item	18	28	12
Tax	1.17	1.82	0.78
Tax/Price	**0.065**	**0.065**	**0.065**

Yes; 6.5%

b $T = 0.065p$

c



11.

Answer.

a $m = 0.165w$

w	50	100	200	400
m	**8.25**	**16.5**	**33**	**66**

b 19.8 lb

c 303.03 lb

d It will double.

13.

Answer.

a $v = 15.306d$

b 3985 million light-years

c 18,979 km/sec

15.

Answer.

a $P = \frac{1825}{8192}w^3 \approx 0.228w^3$

w	10	20	40	80
P	**223**	**1782**	**14, 259**	**114, 074**

b 752 kilowatts

c 33.54 mph

d It is multiplied by 8.

17.

Answer.

a $d = 0.005v^2$

b 50 m

19.

Answer.

a $W = 600d^2$

b 864 newtons

20.

Answer.

a Wind resistance quadruples.

b It is one-ninth as great.

c It is decreased by 19% because it is 81% of the original.

5.5 · Inverse Variation

· Problem Set 5.5

Warm Up

1.

Answer. $R = \frac{1}{3}I$. Not inverse variation.

3.

Answer. $W = 32,000 - d$ Not inverse variation.

Skills Practice

5.

Answer. (c)

7.

Answer.

a $y = \dfrac{120}{x}$

b

x	4	8	20	30	40
y	30	15	6	4	3

c y is divided by 2

9.

Answer. (b), $k = 72$

11.

Answer. (c)

Applications

13.

Answer.

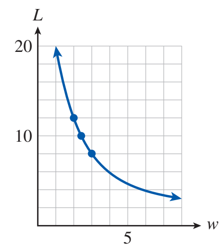
a

Width (feet)	2	2.5	3
Length (feet)	12	9.6	8
Length × width	**24**	**24**	**24**

24 square feet

b $L = \dfrac{24}{w}$

c



15.

Answer.

a $B = \dfrac{88}{d}$

d	1	2	12	24
B	**88**	**44**	**7.3**	**3.7**

b 8.8 milligauss

c More than 20.47 in

d It is one half as strong.

17.

Answer.

a 645

b $D = \dfrac{64,500}{n}$

c 215

18.

Answer.

a $m = \dfrac{8}{p}$

b 0.8 ton

20.

Answer.

a $T = \dfrac{4000}{d}$

b 8°C

22.

Answer.

a It is one-fourth the original illumination.

b It is one-ninth the illumination.

c It is 64% of the illumination.

5.6 · Functions as Models

· Problem Set 5.6

Warm Up

1.

Answer.

a $(0, \infty)$

b none

c $(-infty, 0)$

d none

3.

Answer.

a none

b $(0, \infty)$

c none

d none

5.

Answer.

a $(-infty, 0)$

b none

c $(0, \infty)$

d none

Skills Practice

7.

Answer.

a $s = h(t)$

b After 3 seconds, the duck is at a height of 7 meters.

9.

Answer.

a Increasing

b Concave up

11.

Answer. $y = \dfrac{k}{x}$

Applications

13.

Answer. (b)

15.

Answer. (a)

17.

Answer. (b)

19.

Answer.

a II

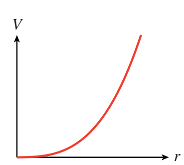
b IV

c I

d III

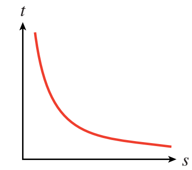
21.

Answer. $y = x^3$ stretched or compressed vertically



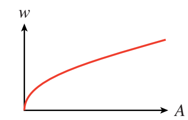
23.

Answer. $y = \dfrac{1}{x}$ stretched or compressed vertically



25.

Answer. $y = \sqrt{x}$



27.

Answer.

a Table (4), Graph (C)

b Table (3), Graph (B)

c Table (1), Graph (D)

d Table (2), Graph (A)

29.

Answer.

a III

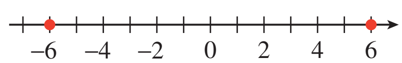
b 3

Absolute Value

1.

Answer.

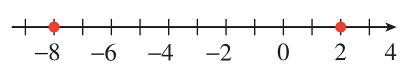
a $|x| = 6$

b 

3.

Answer.

a $|p + 3| = 5$

b 

5.

Answer.

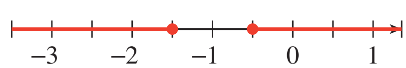
a $|t - 6| < 3$

b 

7.

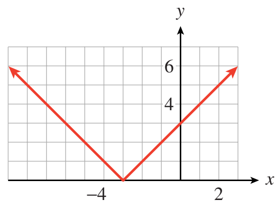
Answer.

a $|b + 1| \geq 0.5$

b 

9.

Answer.



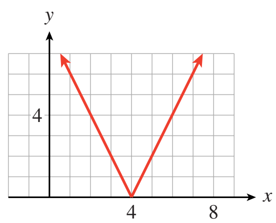
a $x = -5$ or $x = -1$

b $-7 \le x \le 1$

c $x < -8$ or $x > 2$

11.

Answer.



a $x = 4$

b No solution

c No solution

13.

Answer. $x = \frac{-3}{2}$ or $x = \frac{5}{2}$

15.

Answer. $q = \frac{-7}{3}$

17.

Answer. $b = -14$ or $b = 10$

19.

Answer. $w = \frac{13}{2}$ or $w = \frac{15}{2}$

21.

Answer. No solution

23.

Answer. No solution

25.

Answer. $\frac{-9}{2} < x < \frac{-3}{2}$

27.

Answer. $d \le -2$ or $d \ge 5$

29.

Answer. All real numbers

31.

Answer. $1.4 < t < 1.6$

33.

Answer. $T \le 3.2$ or $T \ge 3.3$

35.

Answer. No solution

5.7 · Chapter 5 Summary and Review

· Chapter 5 Review Problems

1.

Answer. A function: Each x has exactly one associated y-value.

2.

Answer. Not a function

3.

Answer. Not a function: The IQ of 98 has two possible SAT scores.

4.

Answer. A function

5.

Answer. $N(10) = 7000$: Ten days after the new well is opened, the company has pumped a total of 7000 barrels of oil.

6.

Answer. $H(16) = 3$: At 16 mph, the trip takes 3 hours.

7.

Answer. $F(0) = 1, \quad F(-3) = \sqrt{37}$

8.

Answer. $G(0) = -2, \quad G(20) = \sqrt[3]{12}$

9.

Answer. $h(8) = -6, \quad h(-8) = -14$

10.

Answer. $m(5) = 6, \quad m(-40) = -4.8$

11.

Answer.

a $P(0) = 5$

b x=5,~x=1

12.

Answer.

a R(0)=2

b $x = 2, \ x = -2$

13.

Answer.

a $f(-2) = 3, \quad f(2) = 5$

b $t = 1, \quad t = 3$

c t-intercepts $(-3, 0), (4, 0)$; $f(t)$-intercept: $(0, 2)$

d Maximum value of 5 occurs at $t = 2$

14.

Answer.

a $P(-3) = -2, \quad P(3) = 3$

b $z = -5, \ \dfrac{-1}{2}, \ 4$

c $(-4, 0), (-1, 0), (5, 0)$; $(0, 3)$

d Maximum value of -3 occurs at $z = -2$

15.

Answer. Function

16.

Answer. not a function

17.

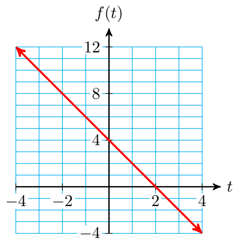
Answer. Not a function

18.

Answer. Function

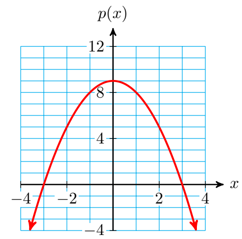
19.

Answer.



21.

Answer.



23.

Answer.

- a $x = \frac{1}{2} = 0.5$
- b $x = \frac{27}{8} \approx 3.4$
- c $x > 4.9$
- d $x \leq 2.0$

24.

Answer.

- a $x = 0.4$
- b $x = 3.2$
- c $0 < x \leq 4.5$
- d $x < 0$ or $x > 0.2$

25.

Answer.

- a $x \approx \pm 5.8$
- b $x = \pm 0.4$
- c $-2.5 < x < 0$ or $0 < x < 2.5$
- d $x \leq -0.5$ or $x \geq 0.5$

26.

Answer.

- a $x = 0.5$
- b $x = 2.9$
- c $0 \leq x < 2.3$
- d $x \geq 1.7$

27.

Answer. $y = 1.2x^2$

28.

Answer. $y = 54x$

29.

Answer. $y = \frac{20}{x}$

30.

Answer. $y = \frac{720}{x^2}$

31.

Answer.

- a $d = 1.75t^2$
- b 63 cm

32.

Answer.

- a $V = \frac{4T}{P}$
- b 32

33.

Answer. 480 bottles

34.

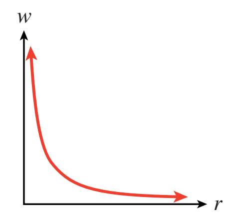
Answer. 14.0625 lumens

35.

Answer.

- a $w = \frac{k}{r^2}$

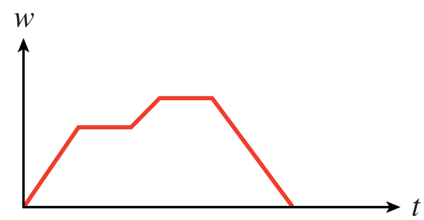
b



c $3960\sqrt{3} \approx 6860$ miles

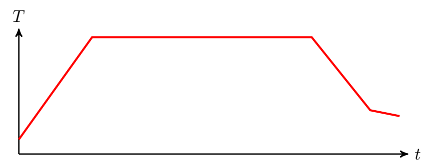
37.

Answer.



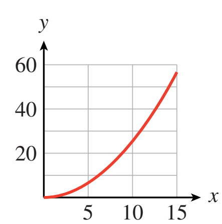
38.

Answer.



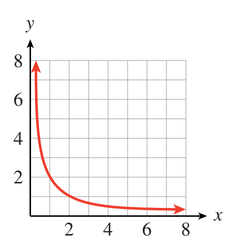
39.

Answer.



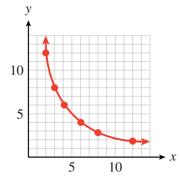
41.

Answer.



43.

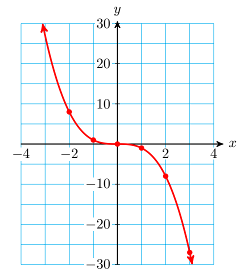
Answer.

a 

b $g(x) = \dfrac{24}{x}$

44.

Answer.

a 

b $F(x) = -x^3$

45.

Answer.

a

x	0	4	8	14	16	22
y	24	20	16	10	8	2

b $y = 24 - x$

46.

Answer.

a

x	0	4	10	12	14	16
y	0	6	15	18	21	24

b $y = \dfrac{3}{2}x$

47.

Answer.

a

x	0	1	4	9	16	25
y	0	1	2	3	4	5

b $y = \sqrt{x}$

48.

Answer.

a

x	0.25	0.50	1.00	1.50	2.00	4.00
y	4.00	2.00	1.00	0.67	0.50	0.25

b $y = \dfrac{1}{x}$

49.

Answer.

a

x	-3	-2	-1	0	1	2
y	5	0	-3	-4	-3	0

b $y = x^2 - 4$

50.

Answer.

a

x	-3	-2	-1	0	1	4
y	0	5	8	9	8	-7

b $y = 9 - x^2$

6 · Powers and Roots

6.1 · Integer Exponents

· Problem Set 6.1

Warm Up

1.

Answer.

a $-2z^2$ b $-24z^4$

3.

Answer.

a $-12x^3y^4$ b $\dfrac{1}{4ab^4}$

5.

Answer.

a $4x^{10}y^{14}$ b x^4y

Skills Practice

7.

Answer.

a 9

b $\frac{1}{9}$

c 9

d $\frac{1}{9}$

e -9

f $\frac{-1}{9}$

9.

Answer.

a 40

b $\frac{5}{8}$

c $\frac{5}{8}$

d $\frac{1}{40}$

e 40

f $\frac{8}{5}$

11.

Answer.

a $\frac{3}{4}$

b $\frac{1}{8}$

c $\frac{1}{2}$

d $\frac{9}{4}$

e $\frac{1}{6}$

f 2

g $\frac{4}{3}$

h 8

13.

Answer.

a $\frac{1}{(m-n)^2}$

b $\frac{1}{y^2}+\frac{1}{y^3}$

c $\frac{2p}{q^4}$

d $\frac{-5x^5}{y^2}$

15.

Answer. 1.25

17.

Answer. 0.2

19.

Answer.

a $\frac{20}{x^3}$

b $\frac{1}{3u^{12}}$

c 5^8t

21.

Answer.

a $\frac{1}{3}x + 3x^{-1}$

b $\frac{1}{4}x^{-2} - \frac{3}{2}x^{-1}$

23.

Answer. $x - 3 + 2x^{-1}$

25.

Answer. $-4 - 2u^{-1} + 6u^{-2}$

27.

Answer. $4x^{-2}(x^4 + 4)$

29.

Answer.

a 2.85×10^2

b 8.372×10^6

c 2.4×10^{-2}

d 5.23×10^{-4}

Applications

31.

Answer.

a

x	1	2	4.5	6.2	9.3
$g(x)$	1	0.125	0.011	0.0042	0.0012

b they decrease

c

x	1.5	0.6	0.1	0.03	0.002
$f(x)$	0.30	4.63	1000	$37,037$	125×10^6

d they increase

33.

Answer.

(a) $P = 0.355v^3$

(b) $v \approx 52.03$ mph

(c) 3.375

35.

Answer.

a $d = 50f^{-1}$

b The are of the aperture decreases by a factor of 0.5 at each f-stop.

36.

Answer.

a 1.905×10^{13}

b $57,552.87

6.2 · Roots and Radicals · Problem Set 6.2

Skills Warm Up

1.

Answer. 13; 4; 3; 10; 6; 7

3.

Answer.

a 1.414

b 4.217

c 1.125

d 0.140

e 2.782

f 3.162

Skills Practice

5.

Answer.

a -3

b *undefined*

c -3

d -3

7.

Answer.

(a) $\sqrt{7}$ (b) $3\sqrt[4]{x}$ (c) $\sqrt[4]{3x}$

9.

Answer.

(a) $5^{1/2}$ (b) $(4y)^{1/3}$ (c) $5x^{1/3}$

11.
Answer. $\frac{1}{4}x^{1/2} - 2x^{-1/2} + \frac{1}{\sqrt{2}}x$

13.
Answer. $x^{0.5} + x^{-0.25} - 1$

15.
Answer. $x = 91.125$

17.
Answer. $x = \frac{19}{2}$

19.
Answer. $x = \pm\sqrt{30}$

21.
Answer. $L = \frac{gT^2}{4\pi^2}$

23.
Answer. $M = \frac{d^3m}{16r^2}$

25.
Answer. $A = \frac{E}{ST^4}$

27.
Answer.

a I

b III

c II

d IV

29.
Answer.

(a) $G(x) = 3.7x^{1/3}$ (b) $H(x) = 85^{1/4}x^{1/4}$ (c) $F(t) = 25t^{-1/5}$

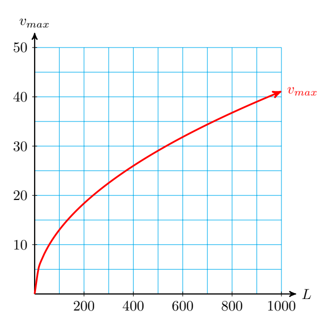
Applications

31.
Answer.

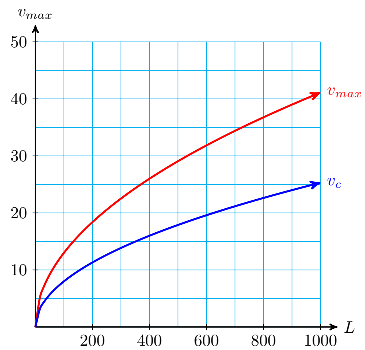
a

L (feet)	200	400	600	800	1000
$v_{\max}$ (knots)	18.4	26	31.8	36.8	41.1

b



c 42.2 knots

d 

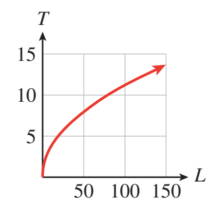
e 26 knots, 62%

33.

Answer.

(a) $T = \dfrac{2\pi}{\sqrt{32}} L^{1/2}$

(b) 10.54 sec

(c) 

35.

Answer.

(a) \$87.68; \$72.00

(b) 1989; 2013

36.

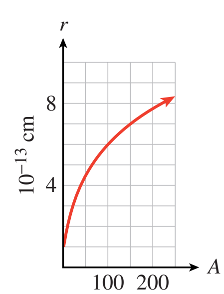
Answer.

(a) 6.5×10^{-13} cm; 1.17×10^{-36} cm^3

(b) $1.8 \times 10^{14} g/\text{cm}^3$

(c)

Element	Carbon	Potassium	Cobalt	Technetium	Radium
Mass number, A	14	40	60	99	226
Radius, r (10^{-13} cm)	3.1	4.4	5.1	6	7.9

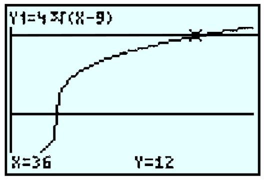
(d) 

37.

Answer. 132.6 km

39.

Answer.

(a) 

(b) $x = 36$

6.3 · Rational Exponents

· Problem Set 6.3

Warm Up

1.

Answer.

a 4; 4

b 15.59; 15.59

c 8; -81

3.

Answer.

(a) $2y^3$

(b) $3x^2$

(c) $2x^2y^9$

(d) $-3a^2b^3$

(e) $4x^2y^6$

(f) $-2x^5y$

Skills Practice

5.

Answer.

(a) $\sqrt[4]{y^3}$

(b) $\dfrac{1}{\sqrt[7]{a^2}}$

(c) $\dfrac{1}{\sqrt[5]{s^3t^3}}$

7.

Answer.

(a) $y^{3/2}$

(b) $6a^{3/5}b^{3/5}$

(c) $-2nq^{-11/8}$

9.

Answer. $4a^2$

11.

Answer. $4w^{3/2}$

13.

Answer. $\dfrac{1}{2k^{1/4}}$

15.

Answer. $x = 64$

17.

Answer. $x = 1.157$

19.

Answer. 29.524

21.

Answer. $\dfrac{13}{3}$

23.
Answer. $2x - x^{2/3}$

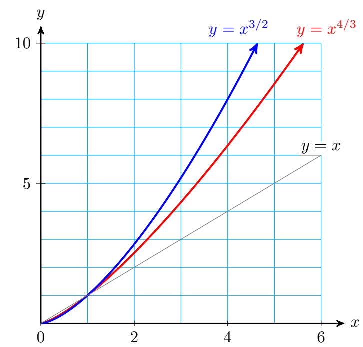
25.
Answer. $2x^{1/2} - x^{1/4} - 1$

27.
Answer. $x(x^{1/2} + 1)$

29.
Answer. $\dfrac{a^{2/3} + a^{1/3} - 1}{a^{1/3}}$

31.
Answer.

x	0	1	2	3	4	5	6
$f(x)$	0	1	2.5	4.3	6.4	8.5	10.9
$g(x)$	0	1	2.8	5.2	8	11.2	14.7



33.
Answer.

(a) 2.83 (b) 3.30

Applications

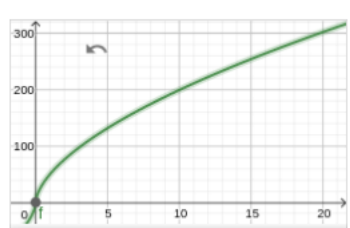
35.
Answer.

a

t	5	10	15	20
$I(t)$	131	199	254	302

b 20 days

c

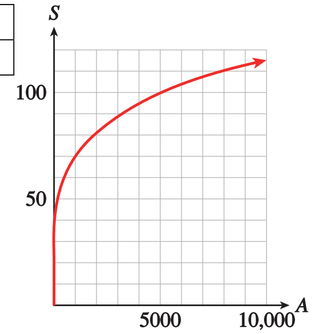


37.
Answer.

(a)

A	10	100	1000	5000	10,000
S	25	42	69	98	115

(b)



(c) 81, 71

(d) 126,000 sq km

39.

Answer.

a 1.62

b 1.62

c 0.84; 0.84

41.

Answer.

a 15 days, 28 days

b $\dfrac{I(m) \times W(m)}{m} = 0.18m^{-0.041}$

c $m^{-0.041}$ is close to 1

43.

Answer.

a $p = K^{1/2}a^{3/2}$

b 1.88 years

6.4 · Working with Radicals

· Problem Set 6.4

Warm Up

1.

Answer. No

3.

Answer. Yes

5.

Answer.

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Skills Practice

7.

Answer.

a $3\sqrt{2}$ b $2\sqrt[3]{3}$ c $-2\sqrt[4]{4}=-2\sqrt{2}$

9.

Answer.

a $x^3\sqrt[3]{x}$ b $3z\sqrt{3z}$ c $2a^2\sqrt[4]{3a}$

11.

Answer.

a $2\sqrt{4-x^2}$ b $A\sqrt[3]{8+A^3}$

13.

Answer.

a $\sqrt[3]{ab^2}$ b $x\sqrt{xy}$

15.

Answer.

a $4\sqrt[3]{3}$ b $-\sqrt[3]{2}$

17.

Answer.

a $6\sqrt{3}+6\sqrt{5}$ b $3k\sqrt{3}-3k^2\sqrt{2}$

19.

Answer.

a $x-9$ b $-4+\sqrt{6}$

21.

Answer.

a $3-\sqrt{5}$ b $\dfrac{-4+\sqrt{2}}{2}$ c $\dfrac{2a-\sqrt{2}}{2a}$

23.

Answer.

a $\dfrac{\sqrt{14x}}{6}$ b $\dfrac{\sqrt{2ab}}{b}$

25.

Answer.

a $-21-\sqrt{3}$ b $\dfrac{x(x+\sqrt{3}}{x^2-3}$

Applications

27.

Answer. $x^2+4x=(4-4\sqrt{5}+5)+(8+4\sqrt{5})=1$

29.

Answer.

a $\dfrac{w\sqrt{3}}{2}$

b $\dfrac{w^2\sqrt{3}}{4}$

6.5 · Radical Equations

· Problem Set 6.5

Warm Up

1.

Answer.

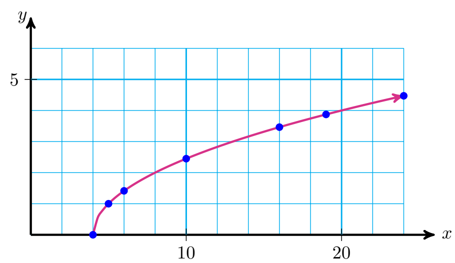
a $z = 4$

b $w = 8$

3.

Answer.

(a)



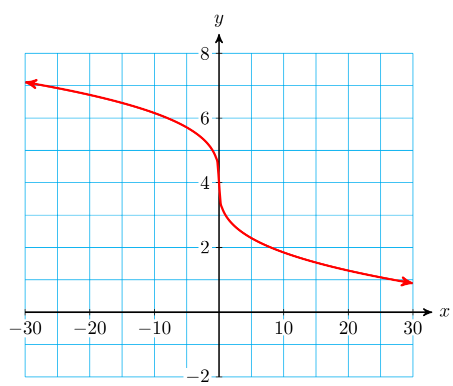
(b) 13

5.

Answer.

a

x	y
-25	6.92
-20	6.71
-15	6.47
-10	6.15
-5	3.21
0	4
5	2.29
10	1.85
15	1.53
20	1.29



b -8

Skills Practice

7.

Answer. $\dfrac{-1}{3}$

9.

Answer. 5

11.

Answer. 5

13.

Answer. 16

15.

Answer. 31

17.

Answer.

a $2|x|$

b $|x-5|$

c $|x-3|$

Applications

19.

Answer.

a 1.25 mi; 6600 ft

b $h = (\frac{d}{89.4})^2$

21.

Answer.

a 339.39 cubic in

b $V = 12.57r^3$

23.

Answer. $b = \pm\sqrt{a^2 - c^2}$

25.

Answer. $W = \dfrac{v}{1-(\frac{D}{S})^3}$

6.6 · Chapter 6 Summary and Review

· Chapter 6 Review Problems

1.

Answer.

a $\frac{1}{81}$

b $\frac{1}{64}$

2.

Answer.

a 9

b 75

3.

Answer.

a $\frac{1}{243m^5}$

b $\frac{-7}{y^8}$

4.

Answer.

a $\frac{1}{a} + \frac{1}{a^2}$

b $\frac{3r^2}{q^9}$

5.

Answer.

a $\frac{2}{c^3}$

b $\frac{99}{z^2}$

6.

Answer.

a $\frac{d^8}{16k^{12}}$

b $\frac{2}{5}w^{14}$

7.

Answer.

a $1.018 \times 10{-}9$ sec, or 0.000 000 001 018 sec

b 8 min 20 sec

8.

Answer. 1,200,000,000,000 hr, or 78,904,109,590 yr

9.

Answer.

a 5000 sec, or 83 min

b $\dfrac{1}{10}$

c 42 yr

10.

Answer.

a

Planet	Density
Mercury	5426
Venus	5244
Earth	5497
Mars	3909
Jupiter	1241
Saturn	620
Uranus	1238
Neptune	1615
Pluto	2355

b Mercury, Venus, Earth, and Mars

11.

Answer.

a $25\sqrt{m}$

b $\dfrac{8}{\sqrt[3]{n}}$

12.

Answer.

a $\sqrt[3]{(13d)^3}$

b $6\sqrt[5]{x^2y^3}$

13.

Answer.

a $\dfrac{1}{\sqrt[4]{27q^3}}$

b $7\sqrt{u^3v^3}$

14.

Answer.

a $\sqrt{a^2+b^2}$

b $\sqrt[4]{16-x^2}$

15.

Answer.

a $2x^{2/3}$

b $\dfrac{1}{4}x^{1/4}$

16.

Answer.

a $z^{5/2}$

b $z^{4/3}$

17.

Answer.

a $6b^{-3/4}$

b $\dfrac{-1}{3}b^{-1/3}$

18.

Answer.

a $-4a^{-1/2}$

b $2a^{-3/2}$

19.

Answer. 112 kg

20.

Answer. Height: 2.673 in; diameter: 5.346 in

21.

Answer.

(a) 480

(b) 498

22.

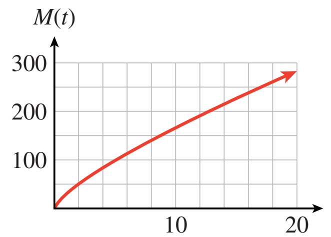
Answer.

(a) 294 sq in

(b) 90 lb

23.

Answer.

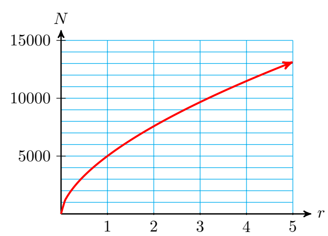
a 

b 283

c 2051

25.

Answer.

a 

b $7114.32

27.

Answer.

a It is the cost of producing the first ship.

b $C = \dfrac{12}{\sqrt[8]{x}}$ million

c About $11 million; about 8.3% ; about 8.3%>

d About 8.3%

29.

Answer.

a 4096

b $\dfrac{1}{8}$

c $36\sqrt{3} \approx 62.35$

d $400,000$

30.

Answer.

a -27

b $\dfrac{-3}{4}$

c $-3\sqrt[3]{400} \approx -22.1$

d -300

31.

Answer. 169

32.

Answer. 49

33.

Answer. 16

34.

Answer. -16

35.

Answer. 1, 4

36.

Answer. 8

37.

Answer. 9

38.

Answer. 12

39.

Answer. 7

40.

Answer. ± 8

41.

Answer. 5

42.

Answer. 4

43.

Answer. $g = \dfrac{2v}{t^2}$

44.

Answer. $r = \dfrac{\pm\sqrt{3q^2 - 6q + 7}}{2}$

45.

Answer. $p = \pm 2\sqrt{R^2 - R}$

46.

Answer. $r = \pm\sqrt{2q^3 - 1}$

47.

Answer.

a $\dfrac{5p^4}{a^2}\sqrt{5p}$

b $\dfrac{2}{w^2}\sqrt[3]{3v^2}$

48.

Answer.

a a^2b

b xy

49.

Answer.

a $2\sqrt[3]{a^3 - 2b^6}$

b $-4ab^2\sqrt[3]{2}$

50.

Answer.

a $2t\sqrt{1 + 6t^4}$

b $4t^4\sqrt{6}$

51.

Answer.

a $x^2 - 4x\sqrt{x} + 4x$

b $x^2 - 4x$

52.

Answer.

a $14 - 4\sqrt{6}$

b $2a - 4b$

53.

Answer.

a $\dfrac{7\sqrt{5y}}{5y}$

b $3\sqrt{2d}$

54.

Answer.

a $\dfrac{\sqrt{33rs}}{11s}$

b $\dfrac{\sqrt{13m}}{m}$

55.

Answer.

a $\dfrac{-3\sqrt{a}+6}{a-4}$

b $\dfrac{-3\sqrt{z}-12}{z-16}$

56.

Answer.

a $\dfrac{2x^2+x\sqrt{3}-3}{x^2-3}$

b $\dfrac{5m^2-7m\sqrt{3}+6}{25m^2-12}$

7 · Exponential Functions

7.1 · Exponential Growth and Decay

· Problem Set 7.1

Warm Up

1.

Answer.

a \$28

b #31.36

c No. It increase by 12% of different amounts.

3.

Answer. Decreased by 1%.

5.

Answer. 4

7.

Answer. ± 1.2

9.

Answer. -2.14; 0.14

Skills Practice

11.

Answer.

a $P = 1200 + 150t$; 1650

b $P = 1200 \cdot 1.5^t$; 4050

13.

Answer.

a $V = 18,000 - 2000t$; \$8000

b $V = 18,000 \cdot 0.8^t$; \$5898.24

15.

Answer. 20%, 2%, 7.5%, 100%, 115%

17.

Answer.

(a) $P_0 = 4,\ b = 2^{1/3}$

(b) $P(t) = 4 \cdot 2^{t/3}$

18.

Answer.

(a) Initial value 80, decay factor $\frac{1}{2}$

(b) $f(x) = 80 \cdot \left(\dfrac{1}{2}\right)^x$

19.

Answer. The growth factor is 1.2.

x	0	1	2	3	4
Q	20	24	28.8	34.56	41.47

20.

Answer. The decay factor is 0.8.

w	0	1	2	3	4
N	120	96	76.8	61.44	49.15

21.

Answer. The decay factor is 0.8.

t	0	1	2	3	4
C	10	8	6.4	5.12	4.10

22.

Answer. The growth factor is 1.1.

n	0	1	2	3	4
B	200	220	242	266.2	292.82

Applications

23.

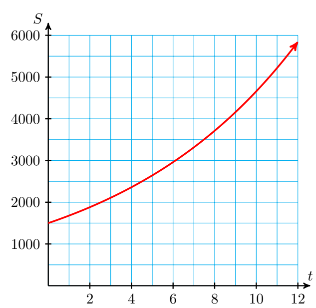
Answer.

a

Years after 2010	0	1	2	3	4
Windsurfers	1500	1680	1882	2107	2360

b $S(t) = 1200(1.12)^t$

c



d 2644; 5844

24.

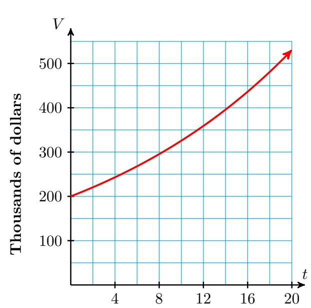
Answer.

a

Years after 1983	0	5	10	15	20
Value of house	20,000	25,526	32,578	41,579	53,066

b $V(t) = 200,000(1.05)^t$

c



d \$359,171.27; \$458,403.66

25.

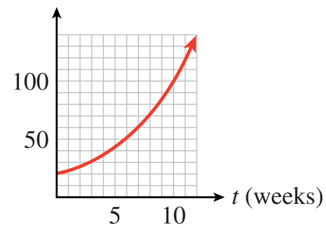
Answer.

a

Weeks	0	6	12	18	24
Bees	2000	5000	12, 500	31, 250	78, 125

b $P(t) = 2000(2.5)^{t/6}$

c



27.

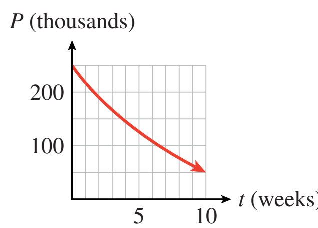
Answer.

a

Weeks	0	2	4	6	8
Mosquitos	250, 000	187, 500	140, 625	105, 469	79, 102

b $P(t) = 250,000(0.75)^{t/2}$

c



d 162,280; 68,504

29.

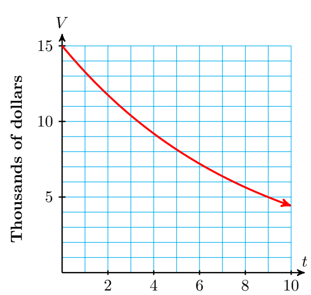
Answer.

a

Years	0	3	6	9	12
Value of boat	15, 000	13, 500	12, 150	10, 935	9841.50

b $V(t) = 15,000(0.885)^t$

c



d $4995.52; $4421.04

31.

Answer.

a $P(t) = 1,545,387b^t$

b $b = 1.049;\ r = 4.9\%$

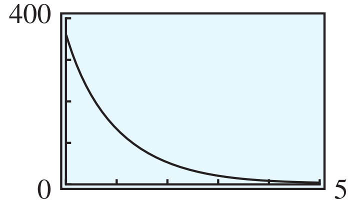
c 3,167,157

33.

Answer.

a 365

b



c $N(t) = 365(0.356)^t$

d 0.03

35.

Answer.

a 2440 tigers per decade

b 0.765; 23.5%

c 3080; 4656

37.

Answer.

a 39; 1.045

b 36; 1.047

c Species B

7.2 · Exponential Functions

· Problem Set 7.2

Warm Up

1.

Answer.

a 3^{x+4} b 3^{4x} c 12^x

3.

Answer.

a b^{-2t} b $b^{t/2}$ c 1

5.

Answer. 0.06

Skills Practice

7.
Answer. $\dfrac{2}{3}$

8.
Answer. $\dfrac{-1}{4}$

9.
Answer. $\dfrac{1}{7}$

11.
Answer. $\dfrac{-5}{4}$

13.
Answer. ± 2

15.
Answer. 2.26

17.
Answer.

(a) $(0, 26)$; increasing

(b) $(0, 1.2)$; decreasing

(c) $(0, 75)$; decreasing

(d) $(0, \frac{2}{3})$; increasing

18.
Answer.

a Power

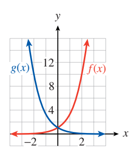
b Exponential

c Power

d Neither

19.
Answer.

x	-3	-2	-1	0	1	2	3
$f(x) = 3^x$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27
$g(x) = \left(\frac{1}{3}\right)^x$	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$



21.
Answer. Because they are defined by equivalent expressions, (b), (c), and (d) have identical graphs.

23.
Answer. $0.17; 5.95$

25.
Answer.

a $P_0 = 800$

b

x	0	1	2
$g(x)$	800	200	50

c $a = \dfrac{1}{4}$

d $g(x) = 800(\frac{1}{4})^x$

27.

Answer.

a Power $y = 100x^{-1}$

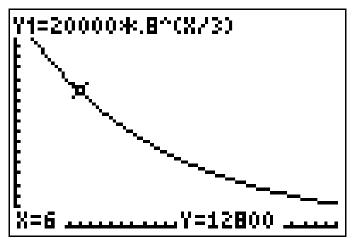
b Exponential $P = \frac{1}{4} \cdot 2^x$

Applications

28.

Answer.

a $V(t) = 20,000(0.8)^{t/3}$

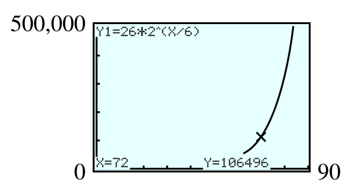
b 

c 6 years

29.

Answer.

a $N(t) = 26(2)^{t/6}$

b 

c 72 days later

30.

Answer.

a $F_0 = 400$

b $b = 1.06$

c $F(p) = 440(1.06)^p$

31.

Answer.

a $S_0 = 150$

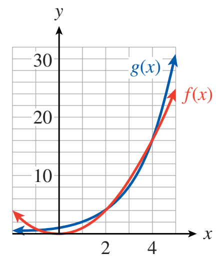
b $b \approx 0.55$

c $S(d) = 150(0.55)^d$

32.

Answer.

x	$f(x) = x^2$	$g(x) = 2^x$
-2	4	$\frac{1}{4}$
-1	1	$\frac{1}{2}$
0	0	1
1	1	2
2	4	4
3	9	8
4	16	16
5	25	32



a *Three*

b $x = -0.77,\ 2,\ 4$

c $(-0.77, 2) \cup (4, \infty)$

d $g(x)$

7.3 · Logarithms

· Problem Set 7.3

Warm Up

1.

Answer.

a $P(t) = 300(1.15)^t$ b $500 = 300(1.15)^t$ c $t \approx 3.65$

Skills Practice

3.

Answer.

a 2 b 4

5.

Answer.

a $\frac{1}{2}$ b -1

7.

Answer.

a -1 b 4

9.

Answer.

a $3 < x < 4$

b $-1 < y < 0$

11.

Answer. -0.23

13.

Answer. 0.77

15.

Answer. 2.53

17.

Answer.

a $\log_t 16 = \frac{3}{2}$

b $\log_{0.8} M = 1.2$

19.

Answer.

a $16^w = 256$

b b^{-2} = 9

21.

Answer.

a $x = \log_4 2.5 \approx 2.7$

b $x = \log_2 0.2 \approx -2.3$

23.

Answer. 2.77

Applications

25.

Answer.

a 2030

b 6.2%

27.

Answer.

a 9.60 in

b 3.85 mi

29.

Answer.

(a) 2,018,436

(b) 5.17%

(c) 2008

31.

Answer.

a 4

b 4

c 64

d $\sqrt[3]{16} \approx 2.52$

7.4 · Properties of Logarithms

· Problem Set 7.4

Warm Up

1.

Answer.

a $\log_8 \dfrac{1}{2} = \dfrac{-1}{3}$

b $\log_5 46 = x$

3.

Answer.

a 10^8

b 2; 6; 8 Property (1)

5.

Answer.

a b^3

b 8; 5; 3 Property (2)

7.

Answer.

a 10^{15}

b 15; 3 Property (3)

9.

Answer.

(a) 5 (b) 6 (c) 5

(a) and (c) are equal.

11.

Answer.

(a) $\log 24 \approx 1.38$ (b) $\log 240 \approx 2.38$ (c) $\log 230 \approx 2.36$

None are equal.

Skills Practice

13.

Answer.

a $\log_t 16 = \dfrac{3}{2}$

b $\log_{0.8} M = 1.2$

c $\log_{3.7} Q = 2.5$

d $\log_3 2N_0 = -0.2t$

15.

Answer.

a $x = \log_4 2.5 \approx 2.7$

b $x = \log_2 0.2 \approx -2.3$

17.

Answer.

(a) $\log_b 4$

(b) $\log_4(x^2y^3)$

19.

Answer.

(a) $\log 2x^{5/2}$

(b) $\log(t-4)$

21.

Answer. $y = \dfrac{1}{25}$

23.

Answer. $b = 100$

25.

Answer. 2.81

27.

Answer. -1.61

29.

Answer. -12.49

31.

Answer.

(a) 1.7918

(b) -0.9163

34.

Answer. (b) and (c)

Applications

37.

Answer.

(a) $S(t) = S_0(1.09)^t$

(b) 4.7 hours

39.

Answer.

(a) $C(t) = 0.7(0.80)^t$

(c)

(b) After 2.5 hours

41.

Answer.

(a) $S(t) = S_0 \cdot 0.9527^t$

(b) 28.61 hours

43.

Answer. $k = \dfrac{\log\left(\dfrac{N}{N_0}\right)}{t \log a}$

45.

Answer. $t = \dfrac{1}{k} \log\left(\dfrac{A}{A_0} + 1\right)$

7.5 · Exponential Models

· Problem Set 7.5

Warm Up

1.

Answer. 4.16

3.

Answer. 16

5.

Answer. $y = \dfrac{-2}{3}x - 1$

7.

Answer. $y = \dfrac{-1}{2}x + 4$

Skills Practice

9.

Answer. $P(x) = 8(0.5)^x$

11.

Answer. $y = 1.5(3)x/5$

13.

Answer.

a $y = 2.6 - 1.3x$

b $y = 2.5(0,5)^x$

c

Applications

15.

Answer. $P(t) = 2000(2^{t/5}$; 14.9%

17.

Answer. $D(t) = D_0(0,5^{t/18}$; 3.8%

19.

Answer.

(a) $P = P_0(2)^{t/25}$

(b) 2.81%

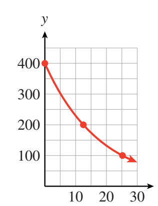
21.

Answer.

a $\dfrac{\log 0.5}{\log 0.946} \approx 12.5$ hours

b 25 hours

c



23.

Answer.

a $D(t) = D_0(0.5)^{t/15}$

b After 89.5 years, or in 2060

25.

Answer.

(a) $A = A_0\left(\dfrac{1}{2}\right)^{t/5730}$

(b) About 760 years old

27.

Answer. 12.9%

29.

Answer. About 11 years

7.6 · Chapter 7 Summary and Review

· Chapter 7 Review Problems

1.

Answer.

a $D = 8(1.5)^{t/5}$

b 18; 44

3.

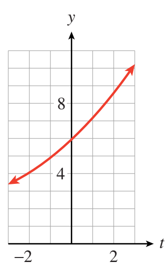
Answer.

a $M = 100(0.85)^t$

b 52.2 mg; 19.7 mg

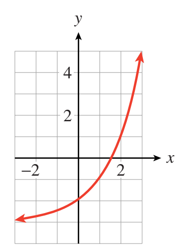
5.

Answer.



7.

Answer.



9.

Answer. $\dfrac{-4}{3}$

11.

Answer. -11

13.

Answer. 4

15.

Answer. -1

17.

Answer. -3

19.

Answer. $2^{x-2} = 3$

20.

Answer. $n^{p-1} = q$

21.

Answer. $\log_{0.3}(x+1) = -2$

23.

Answer. -1

25.

Answer. 4

27.

Answer. $\dfrac{\log 5.1}{1.3} \approx 0.5433$

29.

Answer. $\dfrac{\log(2.9/3)}{-0.7} \approx 0.21$

31.

Answer. 0.054

32.

Answer. 2.959

33.

Answer. 0.195

34.

Answer. 2.823

35.

Answer. $\log_b x + \dfrac{1}{3}\log_b y - 2\log_b z$

37.

Answer. $\dfrac{4}{3}\log x - \dfrac{1}{3}\log y$

39.

Answer. $\log \sqrt[3]{\dfrac{x}{y^2}}$

41.

Answer. $\log \dfrac{1}{8}$

43.

Answer. $\dfrac{\log 63}{\log 3} \approx 3.77$

45.

Answer. $\dfrac{\log 50}{-0.3\log 6} \approx -7.278$

47.

Answer. $\dfrac{\log(N/N_0)}{k}$

49.

Answer.

(a) 238

(b) 2010

51.

Answer.

(a) $C = 90(1.06)^t$

(b) \$94.48

(c) 5 years

8 · Polynomial and Rational Functions

8.1 · Polynomial Functions

· Problem Set 8.1

Warm Up

1.

Answer.

a $a^2 + 2ab + b^2$

b $a^2 - 2ab + b^2$

3.

Answer.

a $(x-7)^2$

b cannot be factored

c $(x+3)^2$

d $(x+8)(x-8)$

Skills Practice

5.

Answer. (b) and (c) are not polynomials; they have variables in a denominator.

7.

Answer.

a $-1.9x^3 + x + 6.4$

b $-2x^2 + 6xy + 2y^3$

9.

Answer.

a 4

b 5

c 7

11.

Answer.

a $6a^4 - 5a^3 - 5a^2 + 5a - 1$

b $y^4 + 5y^3 - 20y - 16$

13.

Answer.

a $1 + 15\sqrt{t} + 75t + 125t\sqrt{t}$

b $1 - \dfrac{9}{a} + \dfrac{27}{a^2} - \dfrac{27}{a^3}$

15.

Answer.

a $27a^3 - 8b^3$

b $8a^3 + 27b^3$

17.

Answer. $(a-2b)(a^2+2ab+4b^2)$

19.

Answer. $(3a + 4b)(9a^2 - 12ab + 16b^2)$

21.

Answer. $(4t^3 + w^2)(16t^6 - 4t^3w^2 + w^4)$

Applications

23.

Answer.

a length: $w + 3$; height: $w - 2$

b $w^3 + w^2 - 6w$

c $5w^2 + w - 12$

25.

Answer.

(a) $\frac{2}{3}\pi r^3 + \pi r^2 h$

(b) $V(r) = \frac{14}{3}\pi r^3$

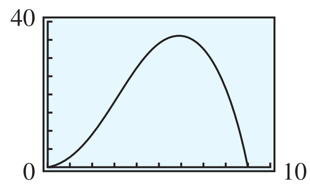
27.

Answer.

(a) $0, 9$

(b) $0 \le x \le 9$; $R \ge 0$ for these values

(c)



(d) $\frac{28}{3}$ points

(e) 36 points

(f) 3 ml or 8.2 ml

29.

Answer.

(a) 20 cm

(b) 100 cm

31.

Answer.

a $6 + x + 5x^2$

b $4 - 7x^2 - 8x^4$

8.2 · Algebraic Fractions

· Problem Set 8.2

Warm Up

1.

Answer.

a $\frac{-3}{5}, \frac{3}{7}$

b 3

c -1

3.

Answer.

a $\frac{-4}{3}, \frac{40}{399}$

b $1, \ -1$

c 0

5.

Answer.

a $\frac{5}{x}$

b $\frac{12b}{7}$

7.

Answer.

a $\frac{-8}{5}$

b $\frac{a}{9}$

Skills Practice

9.

Answer. None are correct

11.

Answer.

a cannot be reduced

b 1

c cannot be reduced

d cannot be reduced

13.

Answer. (b)

15.

Answer. (a)

17.

Answer. $\frac{a}{a-3}$

19.

Answer. $\frac{1}{a+b}$

21.

Answer. $\frac{y+3x}{y-3x}$

23.

Answer. $y-2$

25.

Answer. $\frac{-a}{a+1}$

27.

Answer. $\frac{4z^2+6z+9}{2z+3}$

Applications

29.

Answer.

a 30 min

b 50 min

c 50 min

d 6 mph

e The time increases. If the current is 10 mph, the team will not be able to row upstream

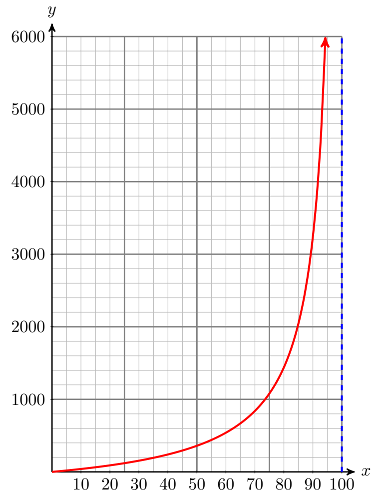
31.

Answer.

a $0 \le p < 100$

b

p	0	25	50	75	90	100
C	0	120	360	1080	3240	$--$

c 

60%

d $p < 80\%$

e $p = 100$: The cost of extracting more ore grows without bound as the amount extracted approaches 100%.

33.

Answer.

(a) $\dfrac{200}{2x-1}$ square centimeters

(b) 8; If $x = 13$, the area of the cross-section is 8 cm^2.

35.

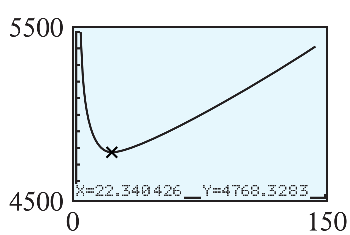
Answer.

a $4500 + \dfrac{3000}{x}$; $C(x) = 6x + 4500 + \dfrac{3000}{x}$

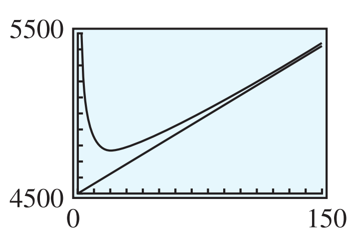
b

x	20	40	60	80	100
C	4770	4815	5018	5130	

c \$4768.33



d 22; 14>

e 

The graph of C approaches the line as an asymptote.

8.3 · Operations on Algebraic Fractions · Problem Set 8.3

Warm Up

1.

Answer.

a $\frac{4}{15}$

b $\frac{2z}{3w}$

3.

Answer.

a 28

b 28

5.

Answer.

a $\frac{1}{6}$

b $\frac{1}{6y^2}$

Skills Practice

7.

Answer.

a $\frac{-36a^2}{7}$

b $\frac{8b}{3b+3}$

c $\frac{v^2}{1-v^2}$

9.

Answer. $\frac{1}{8x}$

11.

Answer. $\frac{a(2a-1)}{a+4}$

13.

Answer. $\frac{6x(x-2)(x-1)^2}{(x^2-8)(x^2-2x+4)}$

15.

Answer. $\frac{9a^3}{14b^3}$

17.

Answer. $\frac{3a}{2}$

19.

Answer. $\frac{x^2}{y-1}$

21.

Answer. $\frac{(z+2)^2}{z^2(2z-1)}$

23.

Answer. $(x-y)(4x^2+2xy+y^2)$

25.

Answer. $\dfrac{2x+y}{3x}$

27.

Answer. $z-3$

29.

Answer. $\dfrac{10x+3}{4x^2}$

31.

Answer. $\dfrac{13}{8a-4b}$

33.

Answer. $\dfrac{h^2+2h-3}{h+2}$

35.

Answer. $\dfrac{6x-x^2-4}{x(x-2)}$

37.

Answer. $\dfrac{19}{6(p-2)}$

39.

Answer. $\dfrac{5k+1}{k(k-3)(k+1)}$

41.

Answer. $\dfrac{3y-3y^2}{(y+1)(2y-1)}$

43.

Answer. $12xy^2(x+y)^2$

45.

Answer. $x(x-1)^3$

Applications

47.

Answer. $\dfrac{4LR}{D^2}$

49.

Answer. $\dfrac{2L}{c}+\dfrac{2LV^2}{c^3}$

51.

Answer. $\dfrac{3q}{8\pi R}-\dfrac{a^2q}{8\pi R^3}$

53.

Answer. $-8t+\dfrac{1}{4}-\dfrac{3}{t}$

55.

Answer.

a $\dfrac{x}{2}$

b $2x$

c $\dfrac{1}{2x}$

57.

Answer.

a $\dfrac{1}{a+b}$

b $\dfrac{3}{4(a+b)}$

c $\dfrac{4}{3(a+b)}$

59.

Answer. $\dfrac{-H+ST}{RT}$

61.

Answer. $\dfrac{4L-R^2C}{4L^2C}$

63.

Answer. $\frac{2r_2^2ra + a^2}{a^2}$

65.

Answer.

a $\frac{144}{x^2 - 2x}$ sq ft

b $\frac{48x - 48}{x^2 - 2x} ft$

67.

Answer.

a $\frac{900}{400 + w}$ hr

b $\frac{900}{400 - w}$ hr

c Orville, by $\frac{1800w}{160,000 - w^2}$ hr

8.4 · More Operations on Fractions · Problem Set 8.4

Warm Up

1.

Answer. $\frac{2x^2 + x - 2}{x(x - 1)}$

3.

Answer. $\frac{x + 2}{x - 1}$

Skills Practice

5.

Answer. $6y$

7.

Answer. $\frac{5}{16}$

9.

Answer. $\frac{7}{10a + 2}$

11.

Answer. $\frac{2x + 1}{x}$

13.

Answer. $\frac{nq}{p + q}$

15.

Answer. $\frac{u - v}{xv}$

17.

Answer. $\frac{1}{2}x^2\frac{1}{2} - \frac{1}{3x^2}$

19.

Answer. $x - 2 + \frac{3}{y}$

21.

Answer. $2y + 5 + \frac{2}{2y + 1)}$

23.

Answer.
$4z^3 - 2z^2 + 3z + 1 + \frac{2}{2z + 1}$

Applications

25.

Answer. $\overline{PQ}$ and $overlineRS$: $\dfrac{b}{a}$; $\overline{QR}$ and $overlineSP$: $\dfrac{b}{a}$

27.

Answer.

(a) $\dfrac{1}{f} = \dfrac{2q+60}{q^2+60q}$

(b) $f = \dfrac{q^2+60q}{2q+60}$

29.

Answer. $\dfrac{n^2-k^2}{n^2}$

31.

Answer. $\dfrac{2cd}{c^2-u^2}$

33.

Answer. $\dfrac{KL}{N(L-F)}$

35.

Answer. $\dfrac{1}{m+2h}$

37.

Answer. $\dfrac{x^2+y^2}{x^2y^2}$

39.

Answer. $\dfrac{b^2-a^2}{ab}$

41.

Answer. $\dfrac{y}{y-x}$

43.

Answer. $\dfrac{\sqrt{15}}{3}$

45.

Answer. $\dfrac{2\sqrt{3}+\sqrt{6}}{9}$

47.

Answer. $\dfrac{6\sqrt{3}+7\sqrt{2}}{5}$

8.5 · Equations with Fractions

· Problem Set 8.5

Warm Up

1.

Answer. $\dfrac{-1}{2}$

3.

Answer. $\dfrac{13}{8}$

5.

Answer. $\pm\sqrt{\dfrac{15}{8}}$

7.

Answer.
$\dfrac{1800}{1849} \approx 0.97$

Skills Practice

9.

Answer. $-2,\ 1$

11.

Answer. -6

13.
Answer. 1

15.
Answer. $\frac{-14}{5}$

17.
Answer. We don't multiply by the LCD in addition problems.

19.
Answer. $r = \frac{S-a}{a}$

21.
Answer. $x = a - \frac{ay}{b}$

23.
Answer. $R = \frac{Cr}{r-C}$

Applications

25.
Answer. 2 mph

27.
Answer. 24 days

29.
Answer. $168 = \frac{72p}{100-p}$; $p = 70\%$

31.
Answer.

a 0.268

b $\frac{44+x}{164+x}$

c 21

33.
Answer.

a $t = \frac{144}{v-20}$

b $t = \frac{144}{v+20}$

c $(100, 3)$

d $t = \frac{144}{v-20} + t = \frac{144}{v+20} = 3$

e 100 mph

35.
Answer. $28\frac{1}{3}$ miles

37.
Answer.

(a) $AE = 1,\ DE = x-1,\ CD = 1$

(b) $\dfrac{1}{x} = \dfrac{x-1}{x}$

(c) $\dfrac{1+\sqrt{5}}{2}$

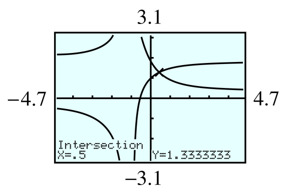
39.

Answer. Because $x = 1$, dividing by $x - 1$ in the fourth step is dividing by 0.

41.

Answer.

(a)



(b) $x = \dfrac{1}{2}$

8.6 · Chapter 8 Summary and Review

· Chapter 8 Review Problems

1.

Answer. $2x^3 - 11x^2 + 19x - 10$

3.

Answer.
$(2x - 3z)(4x^2 + 6xz + 9z^2)$

5.

Answer.

(a) $\dfrac{1}{6}n^3 - \dfrac{1}{2}n^2 + \dfrac{1}{3}n$

(b) 220

(c) 20

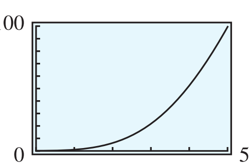
7.

Answer.

(a) $V = \dfrac{\pi h^3}{4}$

(b) 2π cm$^3 \approx 6.28$ cm^3; 16π cm$^3 \approx 50.27$ cm^3

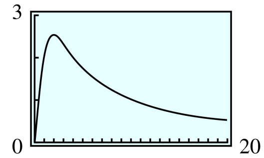
(c)



9.

Answer.

(a)



(b) 338

(c) Months 2 and 02

(d) During month 6. The number of members eventually decreases to zero.

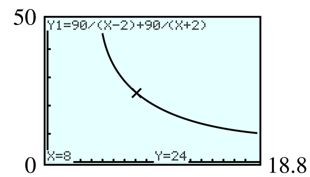
11.

Answer.

(a) $t_1 = \dfrac{90}{v-2}$

(b) $t_2 = \dfrac{90}{v+2}$

(c)



(d) $\dfrac{90}{v-2} + \dfrac{90}{v+2} = 24$

(e) 8 mph

13.
Answer. $\dfrac{a}{2(a-1)}$

15.
Answer. $\dfrac{y^2-2x}{2}$

17.
Answer. $\dfrac{a-3}{2a+6)}$

19.
Answer. $10ab$

21.
Answer. $\dfrac{6x}{2x+3}$

23.
Answer. $\dfrac{a^2-2a}{a^2+3a+2}$

25.
Answer. $\dfrac{1}{2x-1}$

27.
Answer. $9x^2 - 7 + \dfrac{4}{x^2} - \dfrac{1}{x^4}$

29.
Answer. $x^2 - 2x - 2 - \dfrac{1}{x-2}$

31.
Answer. $\dfrac{2}{x}$

33.
Answer. $\dfrac{3x+1}{2(x-3)(x+3)}$

35.
Answer. $\dfrac{2a^2-a+1}{(a-3)(a-1)}$

37.
Answer. $\dfrac{1}{5}$

39.
Answer. $\dfrac{x}{x+4}$

41.

Answer. -2

43.

Answer. No solution

45.

Answer. $n = \dfrac{Ct}{C - V}$

47.

Answer. $q = \dfrac{pr}{r - p}$

49.

Answer. $\dfrac{x^3 + y}{x^3 y}$

51.

Answer. $\dfrac{1 - x^2 y^2}{xy}$

53.

Answer. $\dfrac{-(x - y)^2}{xy}$

9 · Equations and Graphs

9.1 · Properties of Lines

· Problem Set 9.1

Warm Up

1.

Answer.

a A :negative; B : negative; C : positive; D : zero

b B, A, D, C

3.

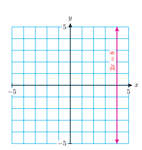
Answer.

Number	Negative reciprocal	Their product
$\frac{2}{3}$	$\frac{-3}{2}$	-1
$\frac{-5}{2}$	$\frac{2}{5}$	-1
6	$\frac{-1}{6}$	-1
-4	$\frac{1}{4}$	-1
-1	1	-1

Skills Practice

5.

Answer.

a 

b m is undefined; $(4, 0)$

7.

Answer. $x = -5$

9.

Answer. $y = 6$

11.

Answer. parallel: a, g, h; perpendicular: c, f

13.

Answer.

a No

b 3 and 3.1; no

c 68. The two lines meet at $(20, 68)$.

Applications

15.

Answer. b. $m_{PQ} = \frac{-5}{2}$; $m_{PR} = \frac{2}{5}$

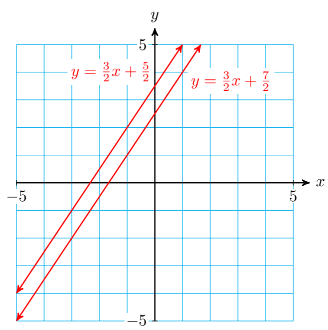
17.

Answer.

a $y = \frac{3}{2}x + \frac{5}{2}$; the graph is below.

b $\frac{3}{2}$

c



d $y = \frac{3}{2}x + \frac{7}{2}$

19.

Answer.

a $y = -2x - 8$

b $y = \frac{1}{2}x - 3$

21.

Answer. $y = \frac{3}{2}x + \frac{15}{2}$

9.2 · The Distance and Midpoint Formulas

· Problem Set 9.2

Warm Up

1.

Answer.

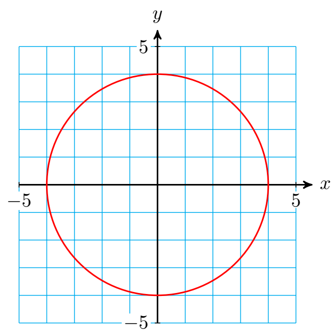
a False

b False

3.

Answer.

x	-4	-3	-2	-1	0	1	2	3	4
y	0	$\pm\sqrt{7}$	$\pm\sqrt{12}$	$\pm\sqrt{15}$	± 4	$\pm\sqrt{15}$	$\pm\sqrt{12}$	$\pm\sqrt{7}$	0



5.

Answer. $\dfrac{5 \pm \sqrt{33}}{2}$

Skills Practice

7.

Answer. distance: $\sqrt{20}$; midpoint: $(0, -2)$

9.

Answer. distance: 8; midpoint: $(-2, -1)$

11.

Answer. center: $(0, 0)$; radius: 5

13.

Answer. center: $(-3, 0)$; radius: $\sqrt{10}$

Applications

15.

Answer. $15 + \sqrt{80} + \sqrt{41} \approx 30.3$

17.

Answer. $\sqrt{50} \approx 7.1$

19.

Answer. $y = \dfrac{1}{2}x + \dfrac{5}{4}$

21.

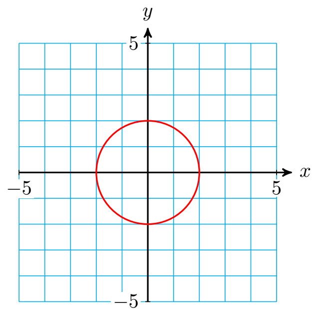
Answer.

a $(220, -38.5)$

b Both distances are 165.8 mi.

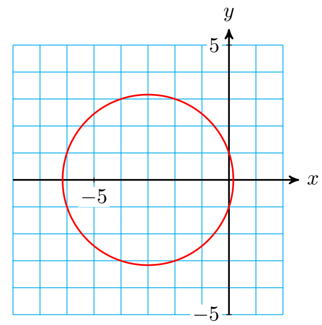
23.

Answer.



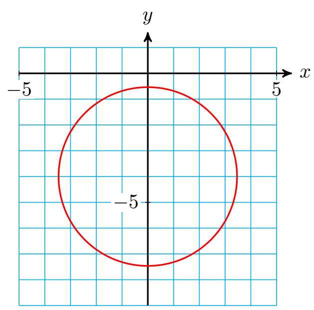
25.

Answer.



27.

Answer.



29.

Answer. $(x+4)^2+y^2=20$; center: $(-4,0)$; radius: $\sqrt{20}$

31.

Answer. $x^2+(y-5)^2=27$; center: $(0,5)$; radius: $\sqrt{27}$

33.

Answer. $\left(x-\frac{3}{2}\right)^2+(y+4)^2=\frac{29}{4}$

35.

Answer. $(x+3)^2+(y+1)^2=1$

9.3 · Conic Sections: Ellipses · Problem Set 9.3

Warm Up

1.

Answer.

a $x^2+y^2=r^2$

b $\dfrac{x^2}{r^2} + \dfrac{y^2}{r^2} = 1$

c $a = b = r$

3.

Answer. $(-2, 6)$

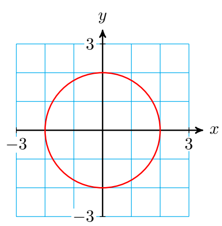
5.

Answer. x-intercepts: $(-6, 0)$, $(4, 0)$, vertex: $\left(-1, \dfrac{25}{2}\right)$

Skills Practice

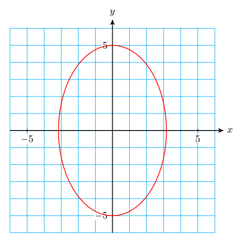
7.

Answer.



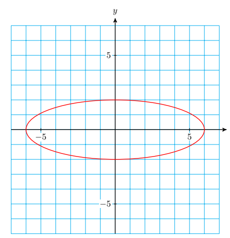
9.

Answer.



11.

Answer.



13.

Answer.

a $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

b

x	0	± 3	-2	$\dfrac{\pm 3\sqrt{3}}{2}$
y	± 2	0	$\dfrac{\pm 2\sqrt{5}}{2}$	1

15.

Answer.

a Radius 2

b $(-1, \pm\sqrt{3})$

17.

Answer.

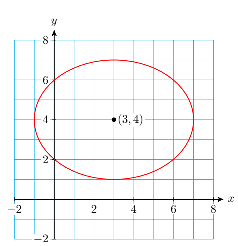
a $a = \sqrt{3},\ b = \sqrt{6}$

b None

19.

Answer.

a

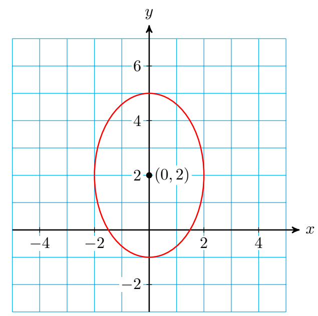


b $(-1, 4)$, $(7, 4)$, $(3, 1)$, $(3, 7)$
(Others are possible.)

21.

Answer.

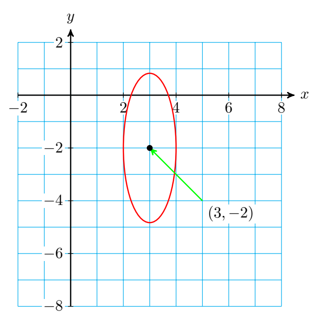
a $\dfrac{x^2}{4}+\dfrac{(y-2)^2}{9}=1$

b 

23.

Answer.

a $\dfrac{(x-3)^2}{1}+\dfrac{(y+2)^2}{8}=1$

b 

25.

Answer. $\dfrac{(x-1)^2}{9}+\dfrac{(y-6)^2}{4}=1$

27.

Answer. $\dfrac{(x-3)^2}{25}+\dfrac{(y-3)^2}{16}=1$

Applications

29.

Answer.

a $\dfrac{x^2}{10^2}+\dfrac{y^2}{7^2}=1$

b 4.2 ft

31.

Answer.

a $\dfrac{x^2}{24^2}+\dfrac{y^2}{8^2}=1$

b 10.29 ft

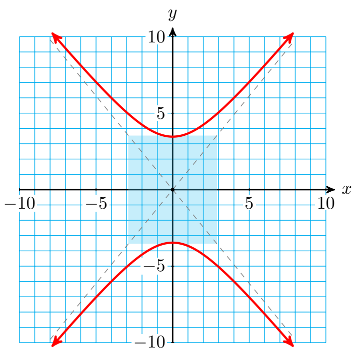
9.4 · Conic Sections: Hyperbolas

· Problem Set 9.4

Skills Practice

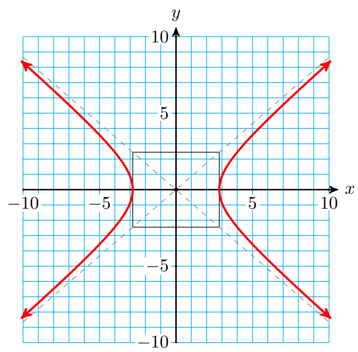
3.

Answer.



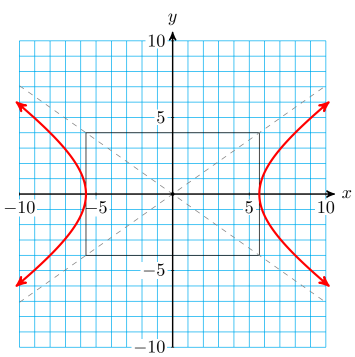
5.

Answer.



7.

Answer.



9.

Answer.

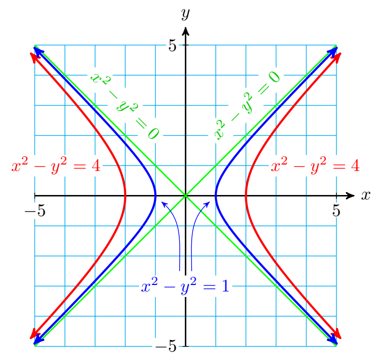
a $\dfrac{x^2}{9}-\dfrac{y^2}{3}=1$

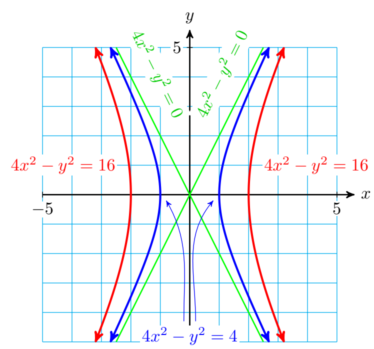
b

x	0	$\dfrac{\pm 3\sqrt{5}}{2}$	5	$\dfrac{\pm 15}{4}$
y	undefined	± 2	$\dfrac{\pm 16}{3}$	-3

11.

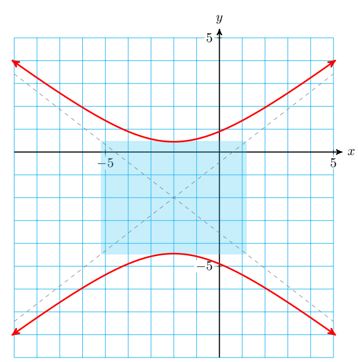
Answer.

a 

b 

13.

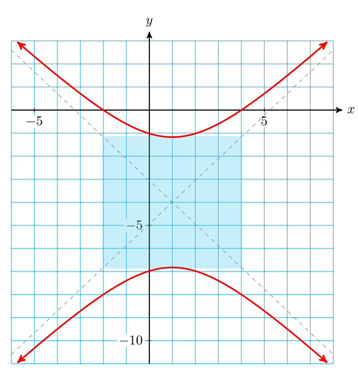
Answer.

a 

b $\left(-2, -2 \pm \sqrt{6}\right)$, $\left(-2 \pm \sqrt{5}, 1\right)$

15.

Answer.

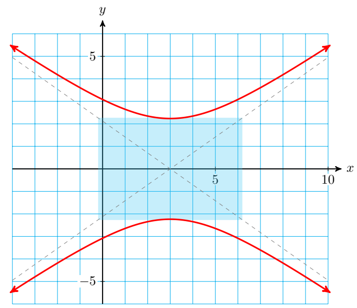
a 

b $\left(1-4\pm 2\sqrt{2}\right)$, $(-2,-8)$, $(4,-8)$

17.

Answer.

a



b $\left(3\pm\sqrt{5}\right)$, $\left(3\pm 2\sqrt{2},3\right)$

19.

Answer. $16x^2-y^2-192x-4y+556=0$

21.

Answer. $x^2-4y^2+10x-16y-27=0$

23.

Answer. Parabola; vertex $(0,2)$, opens downward, $a=\dfrac{-1}{2}$

25.

Answer. Hyperbola; center $\left(\dfrac{-1}{8},-1\right)$, transverse axis vertical, $a^2=\dfrac{15}{65}$, $b^2=\dfrac{15}{16}$

27.

Answer. Parabola; vertex $(-4,2)$, opens upward, $a=\dfrac{1}{4}$

Applications

29.

Answer. 520 ft

31.

Answer. 472.5 ft

9.5 · Nonlinear Systems · Problem Set 9.5

Warm Up

1.

Answer. $(1,-2)$

3.

Answer. $x = 3, 16$

Skills Practice

5.

Answer. $(-1, 12), (4, 7)$

7.

Answer. No solution

9.

Answer. $(1, 4)$

11.

Answer. $(2, 2)$, $(-2, -2)$

13.

Answer. $(2, -1)$, $(-2, 1)$, $(1, -2)$, $(-1, 2)$

15.

Answer. $(\pm 2, -\pm 5)$

17.

Answer. $(\pm 6, -\pm 2)$

19.

Answer. $(0, 4)$, $(-2, 0)$

Applications

21.

Answer. 12 ft by 18 ft

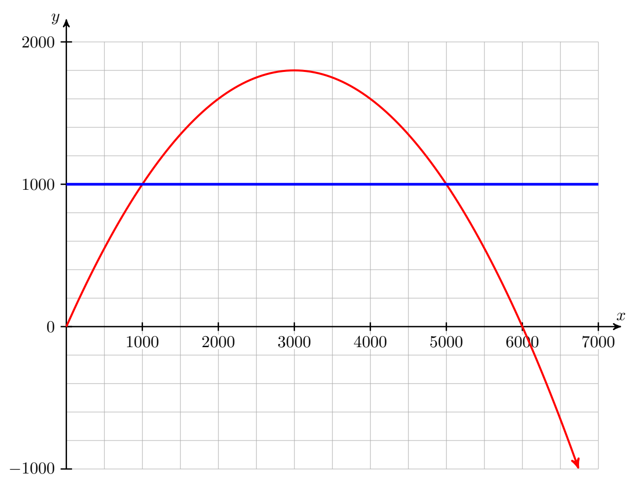
23.

Answer. $P = 6$ lb per sq in; $V = 5$ cu. in.

25.

Answer.

a



b 1000 or 5000

c 2000 or 4000

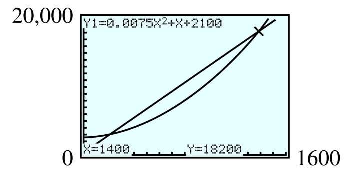
d harvst 1800; stable population 3000

e extinction

27.

Answer.

a $(200,\ 2600), (1400,\ 18,200)$

b 

c $x = 800$

9.6 · Chapter 9 Summary and Review

· Chapter 9 Review Problems

1.

Answer. parallel

2.

Answer. perpendicular

3.

Answer. $y = \dfrac{-2}{3}x + \dfrac{14}{3}$

4.

Answer. $y = \dfrac{3}{2}x + \dfrac{5}{2}$

5.

Answer. $y = \dfrac{2}{3}x - \dfrac{26}{3}$

6.

Answer. $y = \dfrac{3}{2}x - 2$

7.

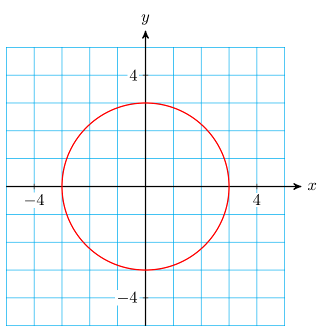
Answer. 21.59; yes

8.

Answer. 10.8

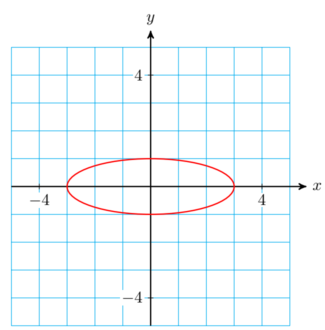
9.

Answer.



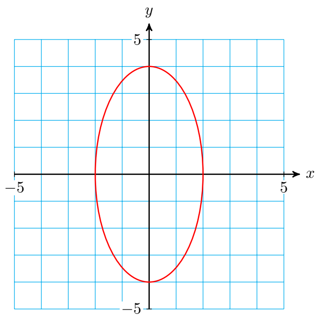
10.

Answer.



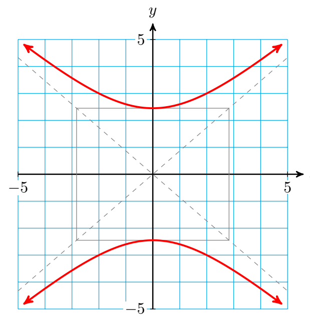
11.

Answer.



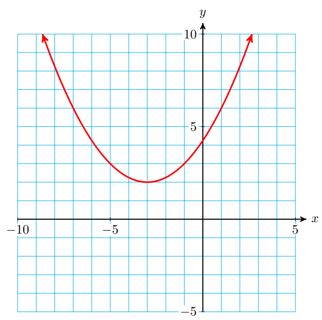
12.

Answer.



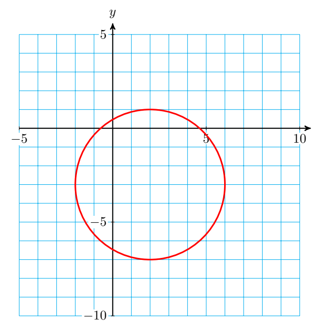
13.

Answer.



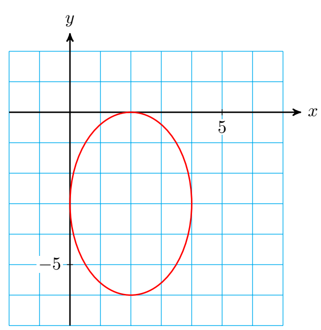
14.

Answer.



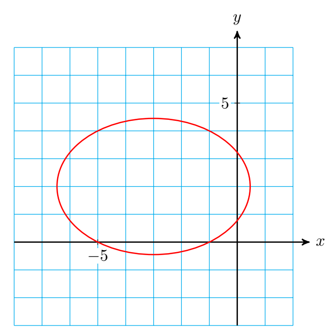
15.

Answer.



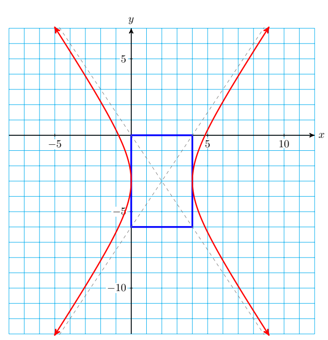
16.

Answer.



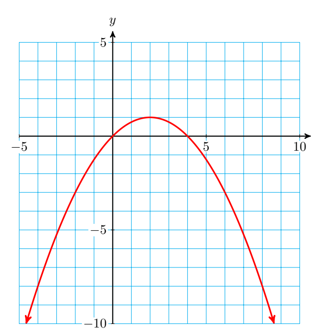
17.

Answer.



18.

Answer.



19.

Answer.

a $(x-2)^2+(y+1)^2=9$

b Circle: center $(2,-1)$, radius 3

20.

Answer.

a $x^2+(y-3)^2=13$

b Circle: center $(0,3)$, radius $\sqrt{13}$

21.

Answer.

a $\dfrac{(x-2)^2}{4}+\dfrac{(y+2)^2}{16}=1$

b Ellipse: center $(2,-2)$, $a=2$, $b=4$

22.

Answer.

a $\dfrac{(x+1)^2}{5}+\dfrac{(y-2)^2}{8}=1$

b Ellipse: center $(-1,2)$, $a=\sqrt{5}$, $b=\sqrt{8}$

23.

Answer.

a $y+10=(x-4)^2$

b Parabola: vertex $(4,10)$, opens upward, $a=1$

24.

Answer.

a $x-2=\dfrac{-1}{4}(y+3)^2$

b Parabola: vertex $(2,-3)$, opens left, $a=\dfrac{-1}{4}$

25.

Answer.

a $y+2=-(x-2)^2$

b Parabola: vertex $(2,-2)$, opens downward, $a=-1$

26.

Answer.

a $x+\dfrac{3}{2}=\dfrac{1}{2}(y-1)^2$

b Parabola: vertex $\left(\dfrac{-3}{2},1\right)$, opens right, $a=\dfrac{1}{2}$

27.

Answer.

a $\dfrac{(y-4)^2}{6}-\dfrac{(x+2)^2}{4}=1$

b Hyperbola: center $(-2,4)$, transverse axis vertical, $a=2$, $b=\sqrt{6}$

28.

Answer.

a $\dfrac{(x-4)^2}{4}-\dfrac{(y+3)^2}{9}=1$

b Hyperbola: center $(4,-3)$, transverse axis horizontal, $a=2$, $b=3$

29.

Answer.

a $\dfrac{x^2}{5}-\dfrac{(y-3)^2}{10}=1$

b Hyperbola: center $(0,3)$, transverse axis horizontal, $a=\sqrt{5}$, $b=\sqrt{10}$

30.

Answer.

a $\dfrac{y^2}{3}-\dfrac{(x-4)^2}{12}=1$

b Hyperbola: center $(4,0)$, transverse axis vertical, $a=2\sqrt{3}$, $b=\sqrt{310}$

31.

Answer.

a $\dfrac{x^2}{25}+\dfrac{y^2}{64}=1$

b $\frac{\pm 24}{5}$

32.

Answer.

a $\frac{x^2}{169}+\frac{y^2}{81}=1$

b $\frac{\pm 45}{13}$

33.

Answer. $(x+4)^2+(y-3)^2=20$

34.

Answer. $(x+2)^2+(y-4)^2=13$

35.

Answer. $\frac{(x+1)^2}{16}+\frac{(y-4)^2}{4}=1$

36.

Answer. $\frac{(x-3)^2}{4}+\frac{(y-1)^2}{25}=1$

37.

Answer. $\frac{(x-2)^2}{16}-\frac{(y+3)^2}{9}=1$

38.

Answer. $(x+3)^2-\frac{(y-1)^2}{9}=1$

39.

Answer. $(\pm 2, \pm 3)$

40.

Answer. $(\pm\sqrt{3}, \pm 4)$

41.

Answer. $(1,-2)$, $(-1,2)$, $\left(2\sqrt{3}, \frac{-1}{\sqrt{3}}\right)$, $\left(-2\sqrt{3}, \frac{1}{\sqrt{3}}\right)$

42.

Answer. $\left(\frac{\sqrt{34}}{2}, \frac{\sqrt{34}}{2}\right)$, $\left(\frac{-\sqrt{34}}{2}, \frac{-\sqrt{34}}{2}\right)$

43.

Answer. Moia: 45 mph, Fran: 50 mph

44.

Answer. 12 in by 1 in

45.

Answer. 7 cm by 10 cm

46.

Answer. 7 ft by 2 ft

47.

Answer. Morning train: 20 mph, evening train: 30 mph

48.

Answer. Amount: \$800, rate: 4%

10 · Logarithmic Functions

10.1 · Logarithmic Functions

· Problem Set 10.1

Warm Up

1.

Answer. $9^y = 729$

3.

Answer. $10^{-4.5} = C$

5.

Answer. $\log_b \frac{1}{4}$

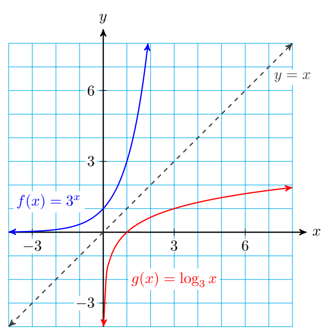
7.

Answer. $\log_{10} \sqrt{\frac{xy}{z^3}}$

Skills Practice

9.

Answer.



11.

Answer.

a 15.6144

b 0.4186

13.

Answer. -1.58×10^{-5}

15.

Answer.

a 25.70

b 3.31

17.

Answer.

a $\frac{1}{2}$

b 01

19.

Answer.

a 81

b 4

c 1.8

d a

21.

Answer.

a IV

b V

c I

d II

e III

f VI

23.

Answer.

a 10,000

b 100,000,000

25.

Answer. 4

27.

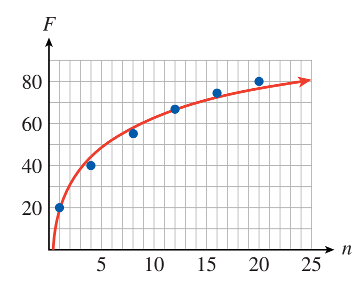
Answer. 3

Applications

29.

Answer.

(a)



(b) The graph resembles a logarithmic function. The function is close to the points but appears too steep at first and not steep enough after $n = 15$. Overall, it is a good fit.

(c) f grows (more and more slowly) without bound. f will eventually exceed 100 per cent, but no one can forget more than 100% of what is learned.

31.

Answer. 1962

10.2 · Logarithmic Scales

· Problem Set 10.2

Warm Up

1.

Answer.

a 0 and 1 b 2 and 3 c −1 and 0 d 6 and 7

3.

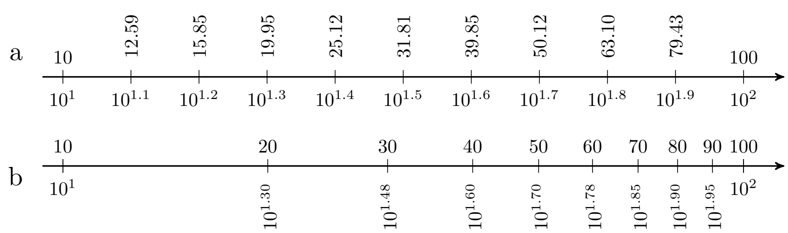
Answer.

a 3981.1 b 5.01 c 0.00079 d 0.398

Skills Practice

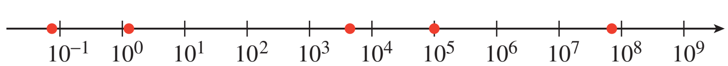
5.

Answer.



7.

Answer.



9.

Answer. 1.58, 6.31, 15.8, 63.1

11.

Answer. 3.2

13.

Answer. 0.0126

15.

Answer. 100

17.

Answer. 6,309,573 watts per square meter

Applications

19.

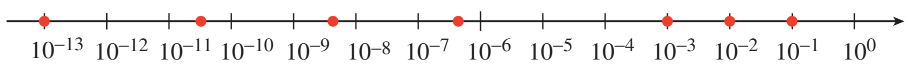
Answer. 1, 80, 330, 1600, 7000, 4×10^7

21.

Answer. Proxima Centauri: 15.5; Barnard: 13.2; Sirius: 1.4; Vega: 0.6; Arcturus: -0.4; Antares: -4.7; Betelgeuse: -7.2

23.

Answer.



25.

Answer. $10^{3.4} \approx 2512$

27.

Answer. A: $a \approx 45$, $p \approx 7.4\%$; B: $a \approx 400$, $p \approx 15\%$; C: $a \approx 6000$, $p \approx 50\%$; D: $a \approx 13000$, $p \approx 45\%$

29.

Answer. 12.6

32.

Answer. 3160

33.

Answer. $\approx 25,000$

10.3 · The Natural Base

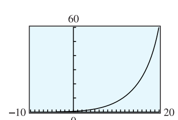
· Homework 10.3

Skills Practice

1.

Answer.

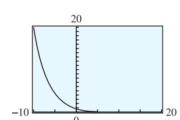
x	-10	-5	0	5	10	15	20
$f(x)$	0.135	0.368	1	2.718	7.389	20.086	54.598



3.

Answer.

x	-10	-5	0	5	10	15	20
$f(x)$	20.086	4.482	1	0.223	0.05	0.011	0.00248



5.

Answer.

(a) 2 (b) $5t$ (c) $\dfrac{1}{x}$ (d) $\dfrac{1}{2}$

7.

Answer.

(a) 0.64 (b) 3.81 (c) −1.20

9.

Answer.

(a) 4.14 (b) 1.88 (c) 0.07

11.

Answer.
$P(t) = 20\left(e^{0.4}\right)^t \approx 20 \cdot 1.492^t$; increasing; initial value 20

13.

Answer.
$P(t) = 6500\left(e^{-2.5}\right)^t \approx 6500 \cdot 0.082^t$; decreasing; initial value 6500

15.

Answer.

(a)

x	0	0.5	1	1.5	2	2.5
e^x	1	1.6487	2.7183	4.4817	7.3891	12.1825

(b) Each ratio is $e^{0.5} \approx 1.6487$: Increasing x-values by a constant $\Delta x = 0.5$ corresponds to multiplying the y-values of the exponential function by a constant factor of $e^{\Delta x}$.

17.

Answer.

(a)

x	0	0.6931	1.3863	2.0794	2.7726	3.4657	4.1589
e^x	1	2	4	8	16	32	64

(b) Each difference in x-values is approximately $\ln 2 \approx 0.6931$: Increasing x-values by a constant $\Delta x = \ln 2$ corresponds to multiplying the y-values of the exponential function by a constant factor of $e^{\Delta x} = e^{\ln 2} = 2$. That is, each function value is approximately equal to double the previous one.

19.

Answer. 0.8277

21.

Answer. −2.9720

23.

Answer. 1.6451

25.

Answer. −3.0713

27.

Answer. $t = \dfrac{1}{k}\ln y$

29.

Answer. $t = \ln\left(\dfrac{k}{k-y}\right)$

31.

Answer. $k = e^{T/T_0} - 10$

33.

Answer.

(a)

n	0.39	3.9	39	390
$\ln n$	−0.942	1.361	3.664	5.966

(b) Each difference in function values is approximately $\ln 10 \approx 2.303$: Multiplying x-values by a constant factor of 10 corresponds to adding a constant value of $\ln 10$ to the y-values of the natural log function.

35.

Answer.

(a)

n	2	4	8	16
$\ln n$	0.693	1.386	2.079	2.773

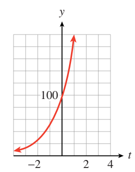
(b) Each quotient equals k, where $n = 2^k$. Because $\ln n = \ln 2^k = k \cdot \ln 2$, $k = \dfrac{\ln n}{\ln 2}$.

37.

Answer.

(a) $N(t) = 100e^{(\ln 2)t} \approx 100e^{0.6931t}$

(b)

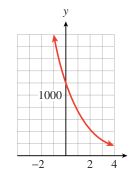


39.

Answer.

(a) $N(t) = 1200e^{(\ln 0.6)t} \approx 1200e^{-0.5108t}$

(b)

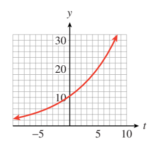


41.

Answer.

(a) $N(t) = 10e^{(\ln 1.15)t} \approx 10e^{0.1398t}$

(b)



Applications

43.

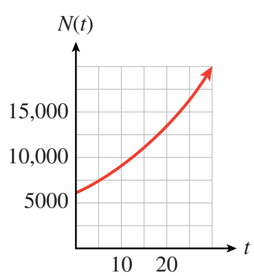
Answer.

(a) $N(t) = 6000e^{0.04t}$

(b)

t	0	5	10	15	20	25	30
$N(t)$	6000	7328	8951	10,933	13,353	16,310	19,921

(c)



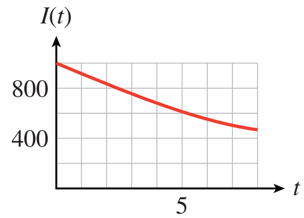
(d) 15,670

(e) 70.3 hrs

45.

Answer.

(a)



(b) 941.8 lumens

(c) 2.2 cm

47.

Answer.

(a) 20,000

(b) $\left(\dfrac{35,000}{20,000}\right)^{1/10} \approx e^{0.056}$

(c) $P(t) = 20,000e^{0.056t}$

(d) 107,188

49.

Answer.

(a) $\left(\dfrac{385}{500}\right)^{1/2} \approx e^{-0.1307}$

(b) $N(t) = 500e^{-0.1307t}$

(c) 135.3 mg

51.

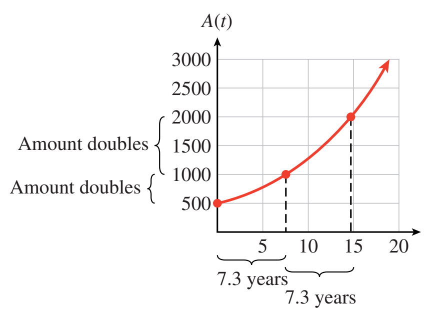
Answer.

(a) $A(t) = 500e^{0.095t}$

(b) 7.3 years

(c) 7.3 years

d–e



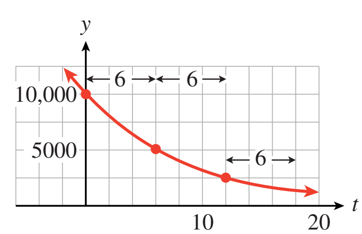
53.

Answer.

(a) 6 hours

(b) 6 hours

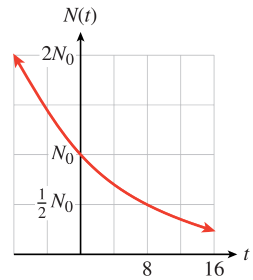
(c)



55.

Answer.

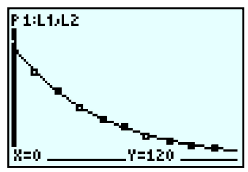
(a) $\frac{1}{2}N_0, \frac{1}{4}N_0, \frac{1}{16}N_0$

(b) 

(c) $N(t) = N_0e^{-0.0866t}$

57.

Answer.

(a) 

$y = 116(0.975)^t$

(b) $G(t) = 116e^{-0.025t}$

(c) 28 minutes

10.4 · Chapter 10 Summary and Review

· Chapter 10 Review Problems

5.
Answer. -1

7.
Answer. $\dfrac{1}{2}$

9.
Answer. 4

11.
Answer. $\dfrac{-15}{8}$

13.
Answer. $\dfrac{9}{4}$

15.
Answer. 3

17.
Answer. $x \approx 1.548$

19.
Answer. $x \approx 411.58$

21.
Answer. $x \approx 2.286$

23.
Answer. $\sqrt{x}$

25.
Answer. $k - 3$

27.
Answer.

(a) $P = 7,894,862e^{-0.011t}$

(b) 1.095%

29.
Answer.

(a) \$1419.07

(b) 13.9 years

(c) $t = 20\ln\left(\dfrac{A}{1000}\right)$

31.

Answer. $t = \dfrac{-1}{k} \ln\left(\dfrac{y-6}{12}\right)$

33.

Answer. $M = N^{Qt}$

35.

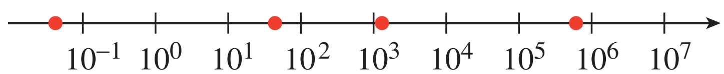
Answer. $P(t) = 750(1.3771)^t$

37.

Answer. $N(t) = 600e^{-0.9163t}$

39.

Answer.



41.

Answer. Order 3: 17,000; Order 4: 5000; Order 8: 40; Order 9: 11

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Colophon

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